Strong Stacks and Classifying Spaces

Chapter · November 2006

DOI: 10.1007/BFb0084222

Joyal Andre
Université du Québec à Montréal

Strong Stacks and Classifying Spaces

André Joyal* and Myles Tierney**

* Département de mathématiques, UQAM, C.P. 8888, Succ 'A', Montréal, Canada, H3C 3P8

** Department of Mathematics, Rutgers University, New Brunswick, NJ 08903

Introduction

Let $\mathcal{E}$ be a Grothendieck topos, and denote by Cat($\mathcal{E}$), respectively Gpd($\mathcal{E}$), the category of categories, respectively groupoids in $\mathcal{E}$. In this paper we show there are Quillen homotopy structures on Cat($\mathcal{E}$) and Gpd($\mathcal{E}$), in which the weak equivalences are the (internal) categorical equivalences, the cofibrations are the functors injective on objects, and the fibrations have the right lifting property with respect to the cofibration weak equivalences. The fibrant objects for these structures are called *strong stacks*, since they represent a strengthening of the notion of *stack* introduced by Grothendieck and Giraud [Giraud 1971]. When the topos $\mathcal{E}$ is the category of simplicial sheaves for a Grothendieck topology, strong stacks have an intimate connection with the theory of classifying spaces for sheaves of simplicial groups, or groupoids.

The paper begins in section 1 with a general discussion of torsors and stacks for groupoids in $\mathcal{E}$.

In section 2 we introduce the concept of a *strong stack* in Gpd($\mathcal{E}$), and treat the problem of strong stack completions by establishing the above-mentioned Quillen homotopy structure on Gpd($\mathcal{E}$). We finish the section with a number of examples.

Section 3 is concerned with strong stacks in Cat($\mathcal{E}$), and the development is parallel to that of section 2. Some of the equivalences of Theorem 3 overlap with results of [Bunge 1979], though the treatment given here is quite different.

Section 4 contains some applications of strong stacks to the problem of the existence of classifying spaces for sheaves of simplicial groupoids. For a full account of this topic, the reader should consult [Joyal-Tierney (to appear)].

1. Torsors and Stacks

Let G be a group in a topos $\mathcal{E}$. A (right) G-*torsor* in $\mathcal{E}$ is a non-empty object E (meaning $E \longrightarrow 1$ is surjective) equipped with a free (right) G-action a: $E \times G \longrightarrow E$, which is transitive. (Free and transitive is expressed by requiring that the map $(x, g) \longmapsto (xg, x)$ from $E \times G$ to $E \times E$ be an isomorphism.) A *mapping* f: $E' \longrightarrow E$ of G-torsors is a function f: $E' \longrightarrow E$ compatible with the G-actions. It is always an isomorphism.

Torsors solve the problem of finding all objects T locally isomorphic to a given object S of $\mathcal{E}$. Recall that T is locally isomorphic to S iff there exists a covering K (meaning K is non-empty) and an isomorphism $K \times T \longrightarrow K \times S$ over K. Equivalently, T is locally isomorphic to S iff Iso(S, T) is non-empty. In fact, letting G = Aut(S), if T is locally isomorphic to S, then

$E = \mathrm{Iso}(S, T)$ is a right G-torsor, and T can be recovered from E since the evaluation mapping defines an isomorphism $E \otimes_G S \longrightarrow T$. Moreover, the correspondence $T \longmapsto \mathrm{Iso}(S, T)$ defines a bijection between the set of isomorphism classes of objects T locally isomorphic to S and the set $H^1(G)$ of isomorphism classes of G-torsors in $\mathcal{E}$. The proofs are clear by localization, but can also be found in [Giraud 1971], which is a general reference for this section.

For a general object X of $\mathcal{E}$, a (right) G-*torsor over X* is a G-torsor in $\mathcal{E}/X$. That is, it is an object $E \longrightarrow X$ over X provided with a free (right) G-action $a: E \times G \longrightarrow E$ over X (free meaning the map $(x, g) \longmapsto (xg, x)$ from $E \times G$ to $E \times E$ is injective), such that $E/G \longrightarrow X$ is an isomorphism. When $\mathcal{E}$ is the category of simplicial sets, a G-torsor over X is a principal G-bundle over X [May 1967]. Let $H^1(X, G)$ be the set of isomorphism classes of G-torsors over X. When S is a fixed object of $\mathcal{E}$, and $G = \mathrm{Aut}(S)$, the above correspondence (interpreted in $\mathcal{E}/X$), yields a bijection between $H^1(X, G)$ and the set of isomorphism classes of objects $T \longrightarrow X$ locally isomorphic (in $\mathcal{E}/X$) to the projection $X \times S \longrightarrow X$. (Such an object in simplicial sets is just a fibre bundle with fibre S.)

In what follows, we shall have to consider not only groups, but also groupoids and their torsors. We recall some fundamental definitions. A *groupoid* $\mathbb{G}$ in $\mathcal{E}$ is a reflexive graph

$$G_0 \xrightarrow{u} G_1 \underset{t}{\overset{s}{\rightrightarrows}} G_0$$

in $\mathcal{E}$, i.e. a diagram such that $su = tu = id$, provided with an associative composition

$$c: G_1 \times_{G_0} G_1 \longrightarrow G_1$$

for which the elements of G_0 are units (via u), and each element of G_1 is invertible. (These statements, of course, are interpreted in the internal language of $\mathcal{E}$.) If $\mathbb{G}$ and $\mathbb{H}$ are groupoids, a functor $f: \mathbb{G} \longrightarrow \mathbb{H}$ is a morphism of reflexive graphs, which respects composition. f is a *categorical equivalence* if it is full, faithful, and representative in the sense that each object of $\mathbb{H}$ is isomorphic to the image of an an object of $\mathbb{G}$ under f. f is a *strong categorical equivalence* if there is a functor $g: \mathbb{H} \longrightarrow \mathbb{G}$ such that $gf \simeq id_{\mathbb{G}}$ and $fg \simeq id_{\mathbb{H}}$. (See [Bunge-Paré 1979] for a discussion of various different notions of equivalence for internal categories.)

Let $\mathbb{G}$ be a groupoid in $\mathcal{E}$. A (right) $\mathbb{G}$-*torsor* is a non-empty object E over G_0 equipped with a free (contravariant) action

$$a: E \times_{G_0} G_1 \longrightarrow E$$

which is transitive. A $\mathbb{G}$-torsor $E \longrightarrow X$ over X is a $\mathbb{G}$-torsor in $\mathcal{E}/X$. The set of isomorphism classes of $\mathbb{G}$-torsors over X is denoted by $H^1(X, \mathbb{G})$ (we write just $H^1(\mathbb{G})$ when X is 1). $H^1(X, \mathbb{G})$ is contravariant in X and covariant in $\mathbb{G}$. In fact, if $g: Y \longrightarrow X$ is a map, and $E \longrightarrow X$ is a $\mathbb{G}$-torsor over X, then $g^*(E)$ is a $\mathbb{G}$-torsor over Y, and if $f: \mathbb{G} \longrightarrow \mathbb{H}$ is a functor then $E \otimes_{\mathbb{G}} \mathbb{H}$ is an $\mathbb{H}$-torsor over X. Note that $H^1(X, \mathbb{G})$ is invariant under categorical equivalence of groupoids.

Torsors for a groupoid solve the problem of finding all objects T locally isomorphic to a member of a given family $S \longrightarrow I$, by which we mean there is a cover K and a mapping $k: K \longrightarrow I$ such that $K \times T \simeq k^*(S)$ in $\mathcal{E}/K$. In fact, let $\mathbb{G}$ be the groupoid $\mathrm{Iso}(S, S) \longrightarrow I \times I$ of isomorphisms of the fibres of $S \longrightarrow I$. Recall that $\mathrm{Iso}(S, S) \longrightarrow I \times I$ is the object $\mathrm{Iso}(S \times I, I \times S)$ in $\mathcal{E}/I \times I$. Since pullback preserves the construction of Iso, $\mathbb{G}$ has the property that given two mappings $f, g: J \longrightarrow I$ in $\mathcal{E}$, there is a mapping $h: J \longrightarrow \mathrm{Iso}(S, S)$ such that

$$\begin{tikzcd} & \mathrm{Iso}(S, S) \arrow[d] \\ J \arrow[ur, "h"] \arrow[r, "(f, g)"'] & I \times I \end{tikzcd}$$

commutes iff $f^*(S) \simeq g^*(S)$ in $\mathcal{E}/J$. Now if T is locally isomorphic to a member of $S \longrightarrow I$ as above, then $E = \mathrm{Iso}(S, T) \longrightarrow I$ is a $\mathbb{G}$-torsor, and evaluation yields an isomorphism $E \otimes_{\mathbb{G}} S \longrightarrow T$. In this way we obtain a bijection between the set of isomorphism classes of objects T locally isomorphic to a member of $S \longrightarrow I$ and $H^1(\mathbb{G})$. When X is an arbitrary object of $\mathcal{E}$, we get a bijection between the set of isomorphism classes of objects T in $\mathcal{E}/X$ locally isomorphic to a member of $S \longrightarrow I$ and $H^1(X, \mathbb{G})$.

Let $\mathbb{G}$ be a groupoid in $\mathcal{E}$. If X is an object of $\mathcal{E}$, we denote by $\hom(X, \mathbb{G})$ the groupoid (in Sets) whose objects are maps $f: X \longrightarrow G_0$. A morphism between f and g is a map $h: X \longrightarrow G_1$ such that

$$\begin{tikzcd} & G_1 \arrow[d, "(s, t)"] \\ X \arrow[ur, "h"] \arrow[r, "(f, g)"'] & G_0 \times G_0 \end{tikzcd}$$

commutes. Let $H(X, \mathbb{G})$ denote the category of $\mathbb{G}$-torsors over X. ($H^1(X, \mathbb{G})$ is the set of connected components of $H(X, \mathbb{G})$.) $t: G_1 \longrightarrow G_0$ is a right $\mathbb{G}$-torsor over G_0, so there is a functor $\hom(X, \mathbb{G}) \longrightarrow H(X, \mathbb{G})$ defined on objects by $f: X \longrightarrow G_0 \longmapsto f^*(G_1)$. It is easy to see that $(s, t): G_1 \longrightarrow G_0 \times G_0$ is canonically isomorphic to $\mathrm{Iso}_{\mathbb{G}}(G_1, G_1) \longrightarrow G_0 \times G_0$, so this function extends in an obvious way to a full and faithful functor. $\mathbb{G}$ is said to be a *stack* if this functor is an equivalence of categories. Clearly, $\mathbb{G}$ is a stack iff the functor is representative. That is, $\mathbb{G}$ is a stack iff for each $\mathbb{G}$-torsor $E \longrightarrow X$ there is a map $f: X \longrightarrow G_0$ such that

$E \simeq f^*(G_1)$. In the example above, $\mathbb{G} = \mathrm{Iso}(S, S) \longrightarrow I \times I$ is a stack iff the family $S \longrightarrow I$ is *complete*, i.e. for any $T \longrightarrow X$ locally isomorphic to a member of $S \longrightarrow I$, there is a map $f\colon X \longrightarrow I$ such that $T \simeq f^*(S)$.

Theorem 1 The following are equivalent for a groupoid $\mathbb{G}$ in $\mathcal{E}$.
(i) $\mathbb{G}$ is a stack.
(ii) For each X in $\mathcal{E}$, every $\mathbb{G}$-torsor $E \longrightarrow X$ has a section $X \longrightarrow E$.
(iii) Every diagram of groupoids

$$\begin{tikzcd}
\mathbb{A} \arrow[r, hook, "e"] \arrow[d] & \mathbb{B} \arrow[dl, dashed] \\
\mathbb{G} &
\end{tikzcd}$$

with e a categorical equivalence injective on objects has a dotted filler making the resulting triangle commute up to isomorphism.
(iv) Every diagram of groupoids

$$\begin{tikzcd}
\mathbb{A} \arrow[r, "f"] \arrow[d] & \mathbb{B} \arrow[dl, dashed] \\
\mathbb{G} &
\end{tikzcd}$$

with f a categorical equivalence has a dotted filler making the resulting triangle commute up to isomorphism.

Proof: (i) $\Leftrightarrow$ (ii): $t\colon G_1 \longrightarrow G_0$ has the section $u\colon G_0 \longrightarrow G_1$, so if every torsor is a pullback of $t\colon G_1 \longrightarrow G_0$, every torsor has a section. On the other hand, let $E \longrightarrow X$ be a $\mathbb{G}$-torsor over X with a section $s\colon X \longrightarrow E$. Then if f denotes the composite of s with the structure map $E \longrightarrow G_0$ of E, $f^*(G_1) \simeq E$.
(ii) $\Rightarrow$ (iii): Categorical equivalences injective on objects are stable under pushout, so to prove (iii) it suffices to show that if (ii) holds, then any categorical equivalence $e\colon \mathbb{G} \hookrightarrow \mathbb{H}$ injective on objects has a retract up to isomorphism. For this, let

$$\begin{tikzcd} E \arrow[r] \arrow[d] & H_1 \arrow[d] \\ G_0 \times H_0 \arrow[r, "e \times 1"'] & H_0 \times H_0 \end{tikzcd}$$

be a pullback. We claim that $E \longrightarrow H_0$ is a $\mathbb{G}$-torsor over H_0. We argue with elements, which can be justified as in [Joyal-Tierney 1984], and will be clearer for the reader. First,

$$E = \{(g, h, e(g) \longrightarrow h) \mid g \in G_0, h \in H_0, \text{ and } e(g) \longrightarrow h \text{ is an arrow in } H_1\}$$

$E \longrightarrow H_0$ is the function $(g, h, e(g) \longrightarrow h) \longmapsto h$, and is surjective, since $e: \mathbb{G} \hookrightarrow \mathbb{H}$ is representative. The action $a: E \times_{G_0} G_1 \longrightarrow E$ of $\mathbb{G}$ on $E \longrightarrow G_0$ is given by composition:

$$a((g, h, e(g) \longrightarrow h), g' \longrightarrow g) = (g', h, e(g') \longrightarrow e(g) \longrightarrow h)$$

It is free by the faithfulness of e. Finally, if $(g', h, e(g') \longrightarrow h)$ and $(g, h, e(g) \longrightarrow h)$ in E both project to $h \in H_0$, then since e is full and faithful, there is a unique $g' \longrightarrow g$ such that $e(g') \longrightarrow h$ is the composite $e(g') \longrightarrow e(g') \longrightarrow h$. Hence, H_0 is the quotient of E by the action of $\mathbb{G}$. By (ii), $E \longrightarrow H_0$ has a section $s: H_0 \longrightarrow E$. Such an s is a choice, for each $h \in H_0$, of an element $s(h) = (g, h, e(g) \longrightarrow h)$ in E. Let $r: H_0 \longrightarrow G_0$ be the composite $H_0 \longrightarrow E \longrightarrow G_0$. Thus, if $h \in H_0$ and $s(h) = (g, h, e(g) \longrightarrow h)$ then $r(h) = g$. Clearly, there is an isomorphism $re \simeq id_{G_0}$. Finally, suppose $h' \longrightarrow h$ is an arrow of H_1. Since $e: \mathbb{G} \hookrightarrow \mathbb{H}$ is full and faithful, there is a unique arrow $rh' \longrightarrow rh$ in G_1 such that

$$\begin{tikzcd} h' \arrow[r] & h \\ e(rh') \arrow[u, "s(h')"] \arrow[r] & e(rh) \arrow[u, "s(h)"'] \end{tikzcd}$$

commutes. Setting $r(h' \longrightarrow h) = rh' \longrightarrow rh$ completes the definition of the retraction r.

(iii) $\Rightarrow$ (iv): Let $f: \mathbb{A} \longrightarrow \mathbb{B}$ be a functor between groupoids in $\mathcal{E}$. Denote by $\mathbb{I}$ the groupoid in $\mathcal{E}$ with two objects 0 and 1, and one isomorphism between them. The *cylinder* on f is defined to be the pushout

$$\begin{tikzcd}
\mathbb{A}\times 0 \arrow[r, "f"] \arrow[d, hook] & \mathbb{B} \arrow[d, hook, "i"] \\
\mathbb{A}\times \mathbb{I} \arrow[r] & \mathbb{C}_f
\end{tikzcd}$$

There is a unique functor r: $\mathbb{C}_f \longrightarrow \mathbb{B}$ given by the identity on $\mathbb{B}$, and the composite $f\pi$ on $\mathbb{A}\times\mathbb{I}$, where π: $\mathbb{A}\times\mathbb{I} \longrightarrow \mathbb{A}$ is the projection. Let e: $\mathbb{A} \hookrightarrow \mathbb{C}_f$ be the composite $\mathbb{A}\times 1 \hookrightarrow \mathbb{A}\times\mathbb{I} \longrightarrow \mathbb{C}_f$. Clearly, f = re, and e is injective on objects. r is a strong categorical equivalence since ri = id and ir$\simeq$ id. It follows that e is a categorical equivalence if f is. Thus, a functor $\mathbb{A} \longrightarrow \mathbb{G}$ can be extended up to isomorphism over e: $\mathbb{A} \hookrightarrow \mathbb{C}_f$ iff it can be extended up to isomorphism over f: $\mathbb{A} \longrightarrow \mathbb{B}$.

(iv) $\Rightarrow$ (i): Let $E \longrightarrow X$ be a $\mathbb{G}$-torsor over X, with action a: $E\times_{G_0}G_1 \longrightarrow E$ and structural map $E \longrightarrow G_0$. The pair

$$(a, \pi_1)\colon E\times_{G_0}G_1 \longrightarrow E\times E$$

is a groupoid $\mathbb{E}$ with objects E (the *category of elements* of the action). It is as well the equivalence relation determined by the surjection $E \longrightarrow X$. The diagram

$$\begin{tikzcd}
E\times_{G_0}G_1 \arrow[r, shift left, "a"] \arrow[r, shift right, "\pi_1"'] \arrow[d, "\pi_2"'] & E \arrow[d] \\
G_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & G_0
\end{tikzcd}$$

is a functor $\mathbb{E} \longrightarrow \mathbb{G}$. Letting disX denote the discrete groupoid in $\mathcal{E}$ determined by the object X, we obtain a categorical equivalence $\mathbb{E} \longrightarrow$disX, and a diagram

$$\begin{tikzcd}
\mathbb{E} \arrow[r] \arrow[d] & \mathrm{disX} \arrow[dl, dashed] \\
\mathbb{G} &
\end{tikzcd}$$

which has a dotted filler up to isomorphism by (iv). The filler disX$\longrightarrow \mathbb{G}$ is a map

$f: X \longrightarrow G_0$, and the isomorphism, on objects, is a map $\varphi: E \longrightarrow G_1$ such that the diagrams

$$\begin{tikzcd} E \arrow[rr, "\varphi"] \arrow[dr] & & G_1 \arrow[dl, "s"] \\ & G_0 & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} E \arrow[r, "\varphi"] \arrow[d] & G_1 \arrow[d, "t"] \\ X \arrow[r, "f"'] & G_0 \end{tikzcd}$$

commute. The statement that φ is a natural transformation is equivalent to the statement that φ is compatible with the right $\mathbb{G}$-actions. It follows that the second diagram above is a pullback, i.e. $E \simeq f^*(G_1)$, and $\mathbb{G}$ is a stack. ■

Let $\mathbb{G}$ be a groupoid in $\mathcal{E}$. A *stack completion* of $\mathbb{G}$ is a categorical equivalence $\mathbb{G} \longrightarrow \mathbb{G}^*$ with $\mathbb{G}^*$ a stack. Stack completions are defined up to strong equivalence of groupoids. We postpone the discussion of their existence until the next section, where a stronger existence result will be proved. Notice, however, that if $\mathbb{G} \longrightarrow \mathbb{G}^*$ is a stack completion of $\mathbb{G}$, then since $H(X, \mathbb{G})$ is equivalent to $H(X, \mathbb{G}^*)$, it follows that $H(X, \mathbb{G})$ is equivalent to $\hom(X, \mathbb{G}^*)$, i.e. $\mathbb{G}^*$ "represents" $H(\ , \mathbb{G})$.

2. Strong Stacks

Definition 1 A groupoid $\mathbb{G}$ in $\mathcal{E}$ is a *strong stack* if condition (iii) of Theorem 1 holds on the nose. That is, if each diagram

$$\begin{tikzcd} \mathbb{A} \arrow[r, hook, "e"] \arrow[d] & \mathbb{B} \arrow[dl, dashed] \\ \mathbb{G} & \end{tikzcd}$$

with e a categorical equivalence injective on objects has a dotted filler making the resulting triangle commute.

Definition 2 Let $\mathbb{G}$ be a groupoid in $\mathcal{E}$. A *strong stack completion* of $\mathbb{G}$ is a categorical equivalence injective on objects $\mathbb{G} \hookrightarrow \mathbb{G}^*$ such that $\mathbb{G}^*$ is a strong stack.

When $\mathcal{E}$ is a Grothendieck topos, every groupoid $\mathbb{G}$ in $\mathcal{E}$ has a strong stack completion. In fact, we prove more: strong stacks are the fibrant objects for a Quillen homotopy structure [Quillen 1969] on the category Gpd($\mathcal{E}$) of groupoids in $\mathcal{E}$.

Recall that a Quillen homotopy structure on a category $\mathcal{C}$ with finite limits and colimits consists of three classes of maps: weak equivalences, cofibrations and fibrations. These are required to satisfy the following axioms.

Q1: (*Composition*) Given two morphisms f: $A \longrightarrow B$ and g: $B \longrightarrow C$ in $\mathcal{C}$, if any two of f, g or gf, are weak equivalences, so is the third.

Q2 : (*Retracts*) Weak equivalences, cofibrations and fibrations are closed under retracts. More precisely, if

$$\begin{tikzcd} A \arrow[r, "s"] \arrow[d, "f"'] & A' \arrow[r, "t"] \arrow[d, "g"] & A \arrow[d, "f"] \\ B \arrow[r, "u"'] & B' \arrow[r, "v"'] & B \end{tikzcd}$$

is a commutative diagram in $\mathcal{C}$ such that ts = id_A and vu = id_B, then if g is a weak equivalence, cofibration or fibration, so is f.

Q3 : (*Lifting*) A commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

in $\mathcal{C}$ such that i is a cofibration and p is a fibration has a dotted filler making both triangles commute if either i or p is a weak equivalence.

Q4 : (*Factorization*) Any morphism f: $X \longrightarrow Y$ in $\mathcal{C}$ can be factored in two ways as

$$\begin{tikzcd} X \arrow[rr, "i"] \arrow[dr, "f"'] & & E \arrow[dl, "p"] \\ & Y & \end{tikzcd}$$

where i is a cofibration and p is a fibration. One in which i is a weak equivalence, and one in which p is a weak equivalence.

Theorem 2 There is a Quillen homotopy structure on Gpd($\mathcal{E}$), in which the weak equivalences are the categorical equivalences, the cofibrations are the functors injective on objects, and the fibrations have the right lifting property with respect to the cofibration weak equivalences.

Before proving the theorem, we prove two lemmas. First, call p: $\mathbb{E} \longrightarrow \mathbb{B}$ in Gpd($\mathcal{E}$) a *trivial fibration* if p has the right lifting property with respect to all cofibrations.

Lemma 1 If p: $\mathbb{E} \longrightarrow \mathbb{B}$ is a trivial fibration, then $E_0 \longrightarrow B_0$ is an injective object in $\mathcal{E}/B_0$, and p is a strong categorical equivalence.

Proof: Let p: $\mathbb{E} \longrightarrow \mathbb{B}$ be a trivial fibration. A diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & E_0 \arrow[d] \\ C \arrow[r] & B_0 \end{tikzcd}$$

in $\mathcal{E}$ is equivalent to a diagram

$$\begin{tikzcd} \mathrm{dis}A \arrow[r] \arrow[d, hook] & \mathbb{E} \arrow[d, "p"] \\ \mathrm{dis}C \arrow[r] \arrow[ur, dashed] & \mathbb{B} \end{tikzcd}$$

in Gpd($\mathcal{E}$), which has a dotted filler since $\mathrm{dis}A \hookrightarrow \mathrm{dis}C$ is a cofibration. Thus the first square has a dotted filler, so $E_0 \longrightarrow B_0$ is an injective object in $\mathcal{E}/B_0$. Let $\mathbb{O}$ denote the empty groupoid. $\mathbb{O} \longrightarrow \mathbb{B}$ is a cofibration, so there is a dotted filler for the square

$$\begin{tikzcd}
\mathbb{O} \arrow[r] \arrow[d] & \mathbb{E} \arrow[d, "p"] \\
\mathbb{B} \arrow[r, "id"'] \arrow[ur, dashed, "s"] & \mathbb{B}
\end{tikzcd}$$

In the commutative square

$$\begin{tikzcd}
(\mathbb{E}\times 0)+(\mathbb{E}\times 1) \arrow[rr, "{(sp, id)}"] \arrow[d, hook] & & \mathbb{E} \arrow[d, "p"] \\
\mathbb{E}\times\mathbb{I} \arrow[r, "\pi"'] \arrow[urr, dashed, "h"] & \mathbb{E} \arrow[r, "p"'] & \mathbb{B}
\end{tikzcd}$$

where π is the first projection, the left-hand vertical mapping is a cofibration, so the square has a dotted filler h, which provides an isomorphism $sp \simeq id$. ∎

Lemma 2 If $p\colon \mathbb{E} \longrightarrow \mathbb{B}$ is such that p is full and faithful, and $E_0 \longrightarrow B_0$ is injective in $\mathcal{E}/B_0$, then p is a trivial fibration.

Proof: Let $\mathbb{A} \hookrightarrow \mathbb{C}$ be a cofibration in Gpd($\mathcal{E}$), and suppose

$$\begin{tikzcd}
\mathbb{A} \arrow[r] \arrow[d, hook] & \mathbb{E} \arrow[d, "p"] \\
\mathbb{C} \arrow[r, "f"'] & \mathbb{B}
\end{tikzcd}$$

is a commutative square. The square

$$\begin{tikzcd} A_0 \arrow[r] \arrow[d, hook] & E_0 \arrow[d, "p"] \\ C_0 \arrow[r, "f"'] \arrow[ur, dashed, "g"] & B_0 \end{tikzcd}$$

has a dotted filler g since $E_0 \longrightarrow B_0$ is injective. Again, we indicate with elements how to extend g to the arrows of $\mathbb{C}$. Namely, suppose $c' \longrightarrow c$ is an arrow in C_1. $f(c' \longrightarrow c) = f(c') \longrightarrow f(c) = pg(c') \longrightarrow pg(c)$. But p is full and faithful, so there is a unique arrow $g(c') \longrightarrow g(c)$ in E_1 such that $p(g(c') \longrightarrow g(c)) = f(c') \longrightarrow f(c)$. We set $g(c' \longrightarrow c) = g(c') \longrightarrow g(c)$. ∎

Proof of Theorem 2: The finite limits and colimits in Gpd($\mathcal{E}$) are clear, as are Q1 and Q2, whose verification we leave to the reader. Let f: $\mathbb{G} \longrightarrow \mathbb{H}$ be an arbitrary functor in Gpd($\mathcal{E}$). Embed $G_0 \longrightarrow H_0$ into an injective $E_0 \longrightarrow H_0$ over H_0.

$$\begin{tikzcd} G_0 \arrow[rr, hook, "i"] \arrow[dr, "f"'] & & E_0 \arrow[dl, "p"] \\ & H_0 & \end{tikzcd}$$

For example, we could take i to be the singleton mapping $f \hookrightarrow \Omega^f$ in $\mathcal{E}/H_0$. Now pull back the arrows of $\mathbb{H}$ over E_0 to obtain a commutative triangle of groupoids

$$\begin{tikzcd} \mathbb{G} \arrow[rr, hook, "i"] \arrow[dr, "f"'] & & \mathbb{E} \arrow[dl, "p"] \\ & \mathbb{H} & \end{tikzcd}$$

i is a cofibration, and p is a trivial fibration by Lemma 2. By Lemma 1, a trivial fibration is a fibration and a categorical equivalence, so we have established half of Q4. Moreover, suppose f: $\mathbb{G} \longrightarrow \mathbb{H}$ is a fibration and a categorical equivalence. Factor f as above into a cofibration i followed by a trivial fibration p. By Q1, i is a cofibration weak equivalence, so the square

$$\begin{tikzcd} \mathbb{G} \arrow[r, "\mathrm{id}"] \arrow[d, hook, "i"'] & \mathbb{G} \arrow[d, "f"] \\ \mathbb{E} \arrow[r, "p"'] \arrow[ur, dashed, "r"] & \mathbb{H} \end{tikzcd}$$

has a dotted filler r, making f: $\mathbb{G} \longrightarrow \mathbb{H}$ a retract of p: $\mathbb{E} \longrightarrow \mathbb{B}$. Since trivial fibrations are closed under retracts, f is a trivial fibration. But this establishes Q3, for the first part holds by definition, and the second is just the statement that a fibration which is a weak equivalence, is a trivial fibration.

It remains to show that an arbitrary f: $\mathbb{G} \longrightarrow \mathbb{H}$ can be factored as a cofibration weak equivalence followed by a fibration. However, cofibration weak equivalences are stable under pushout, and as in [Joyal (to appear)] they have a small set of generators, so we can use "the small object argument" [Quillen 1967] to obtain the desired factorization by repeated pushouts and ordinal colimits. ■

We consider some examples of fibrations, strong stacks and strong stack completions. First, recall that a functor f: $\mathbb{E} \longrightarrow \mathbb{B}$ in Gpd($\mathcal{E}$) is called a *discrete fibration* if the square

$$\begin{tikzcd} E_1 \arrow[r, "t"] \arrow[d, "f_1"'] & E_0 \arrow[d, "f_0"] \\ B_1 \arrow[r, "t"'] & B_0 \end{tikzcd}$$

is a pullback. When this is the case, $\mathbb{E}$ is always the category of elements (as defined in the proof of Theorem 1) for a (contravariant) action of $\mathbb{B}$ on $E_0 \longrightarrow B_0$ [Johnstone 1977]. We claim that a discrete fibration is a fibration for the Quillen homotopy structure on Gpd($\mathcal{E}$). In fact, we prove more.

Proposition 1 A discrete fibration f: $\mathbb{E} \longrightarrow \mathbb{B}$ has the unique right lifting property with respect to the cofibration weak equivalences in Gpd($\mathcal{E}$).

Proof: Let f: $\mathbb{E} \longrightarrow \mathbb{B}$ be a discrete fibration. We want to show that if $\mathbb{A} \hookrightarrow \mathbb{C}$ is a cofibration weak equivalence, then a commutative square

$$\begin{tikzcd} \mathbb{A} \arrow[r] \arrow[d, hook] & \mathbb{E} \arrow[d, "f"] \\ \mathbb{C} \arrow[r] \arrow[ur, dashed] & \mathbb{B} \end{tikzcd}$$

has a unique dotted filler. Since discrete fibrations are stable under pullback, it suffices to show that if i: $\mathbb{A} \hookrightarrow \mathbb{B}$ is a categorical equivalence injective on objects, and if $i^*(\mathbb{E})$ has a section s over $\mathbb{A}$, then there is a unique section t of f over $\mathbb{B}$, which extends s. But the category of discrete fibrations over $\mathbb{B}$ coincides with the category $\mathcal{E}^{\mathbb{B}^{op}}$ of (contravariant) $\mathbb{B}$-actions, and $i^*: \mathcal{E}^{\mathbb{B}^{op}} \longrightarrow \mathcal{E}^{\mathbb{A}^{op}}$ is an (ordinary, external) equivalence of categories. In particular, it is full and faithful. A section s of $i^*(\mathbb{E})$ is a map $id_{\mathbb{A}} \longrightarrow i^*(\mathbb{E})$ in $\mathcal{E}^{\mathbb{A}^{op}}$. Since $id_{\mathbb{A}} = i^*(id_{\mathbb{B}})$, there is a unique t: $id_{\mathbb{B}} \longrightarrow \mathbb{E}$ in $\mathcal{E}^{\mathbb{B}^{op}}$ such that $i^*(t) = s$. ■

Proposition 1 provides a number of examples of strong stacks. In fact, we know that a groupoid $\mathbb{G}$ is a strong stack iff $\mathbb{G}$ is fibrant in the Quillen structure on Gpd($\mathcal{E}$), i.e.iff $\mathbb{G} \longrightarrow \mathbb{1}$ is a fibration, where $\mathbb{1}$ is the terminal groupoid. Fibrations are stable under composition, so if $\mathbb{E} \longrightarrow \mathbb{G}$ is a discrete fibration and $\mathbb{G}$ is a strong stack, it follows that $\mathbb{E}$ is a strong stack. Thus, the category of elements of any $\mathbb{G}$-action is a strong stack. In particular, the category of elements of a $\mathbb{G}$-torsor is a strong stack. Also, for any object X of $\mathcal{E}$ dis$X \longrightarrow \mathbb{1}$ is a discrete fibration, so disX is a strong stack with unique lifting. Another class of examples can be constructed as follows.

Proposition 2 Let p: $E \longrightarrow B$ be a surjective mapping in $\mathcal{E}$, and denote by $\mathbb{E}$ the equivalence relation $E\times_B E \hookrightarrow E\times E$ on E determined by p, considered as a groupoid in $\mathcal{E}$. Then $\mathbb{E}$ is a strong stack iff p: $E \longrightarrow B$ is injective in $\mathcal{E}/B$.

Proof: The categorical equivalence $\mathbb{E} \longrightarrow$ disB induced by p is a trivial fibration in Gpd($\mathcal{E}$) iff p: $E \longrightarrow B$ is injective in $\mathcal{E}/B$ by Lemmas 1 and 2. disB is always a strong stack, so if p: $E \longrightarrow B$ is injective in $\mathcal{E}/B$, $\mathbb{E}$ is a strong stack. On the other hand, suppose $\mathbb{E}$ is a strong stack, $\mathbb{A} \hookrightarrow \mathbb{C}$ is a cofibration weak equivalence, and we are given a commutative square

$$\begin{tikzcd} \mathbb{A} \arrow[r] \arrow[d, hook] & \mathbb{E} \arrow[d] \\ \mathbb{C} \arrow[r] \arrow[ur, dashed] & \mathrm{dis}B \end{tikzcd}$$

The square has a dotted filler making the upper triangle commute since $\mathbb{E}$ is a strong stack. The two functors $\mathbb{C} \longrightarrow \mathrm{dis}B$ agree on $\mathbb{A}$ so they agree on $\mathbb{C}$ by the unique lifting property of disB. It follows that $\mathbb{E} \longrightarrow \mathrm{dis}B$ is a fibration in Gpd($\mathcal{E}$). But it is also a weak equivalence, so it is a trivial fibration and p: $E \longrightarrow B$ is injective in $\mathcal{E}/B$. ■

As an example of Proposition 2, let X be a non-empty object of $\mathcal{E}$, i.e. $X \longrightarrow 1$ is surjective. Then the full equivalence relation id: $X \times X \longrightarrow X \times X$, which we call $\mathbb{X}$ considered as a groupoid of $\mathcal{E}$, is a strong stack iff X is an injective object of $\mathcal{E}$. In general, the strong stack completion of $\mathbb{X}$ is obtained by embedding X into an injective object Y, and taking the groupoid $\mathbb{Y}$. If $\mathbb{G}$ is any groupoid in $\mathcal{E}$, a functor $\mathbb{G} \longrightarrow \mathbb{X}$ is just a mapping $G_0 \longrightarrow X$ in $\mathcal{E}$, and any two such are isomorphic. Thus, it follows from, say, condition (iv) of Theorem 1 that if X has an element $1 \longrightarrow X$, then $\mathbb{X}$ is a stack, which, by the above, is not strong in general.

Proposition 3 A groupoid $\mathbb{G}$ in $\mathcal{E}$ is a strong stack iff for any X in $\mathcal{E}$, each $\mathbb{G}$-torsor $E \longrightarrow X$ is injective in $\mathcal{E}/X$, i.e. for any monomorphism $A \hookrightarrow B$ in $\mathcal{E}$, each commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & E \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

has a dotted filler.

Proof: If $\mathbb{G}$ is a strong stack, and $E \longrightarrow X$ is a $\mathbb{G}$-torsor over X, then the category of elements of the $\mathbb{G}$-action on $E \longrightarrow G_0$ is a strong stack by Proposition 1. But the category of elements is also the equivalence relation on E determined by the surjection $E \longrightarrow X$, so $E \longrightarrow X$ is injective in $\mathcal{E}/X$ by Proposition 2.

In the other direction, suppose each $\mathbb{G}$-torsor $E \longrightarrow X$ is injective in $\mathcal{E}/X$. As before, since categorical weak equivalences injective on objects are stable under pushout, to show that

$\mathbb{G}$ is a strong stack, it suffices to show that each categorical weak equivalence injective on objects e: $\mathbb{G} \hookrightarrow \mathbb{H}$ has a retract. As in Theorem 1, let

$$\begin{tikzcd} E \arrow[r] \arrow[d] & H_1 \arrow[d] \\ G_0 \times H_0 \arrow[r, "e \times 1"'] & H_0 \times H_0 \end{tikzcd}$$

be a pullback. We have seen that $E \longrightarrow H_0$ is a $\mathbb{G}$-torsor over H_0. Furthermore, since e is full and faithful, $e^*(E) \simeq G_1$, i.e. we have a pullback diagram

$$\begin{tikzcd} G_1 \arrow[r, hook] \arrow[d, "t"'] & E \arrow[d] \\ G_0 \arrow[r, hook, "e"'] & H_0 \end{tikzcd}$$

Now t has the section u: $G_0 \longrightarrow G_1$, and since $E \longrightarrow H_0$ is injective over H_0 it has a section s: $H_0 \longrightarrow E$ which extends u. As a result, the retraction r: $\mathbb{G} \longrightarrow \mathbb{H}$ constructed in Theorem 1 satisfies re = id, and we are done. ■

To finish this section, we give an example of a strong stack completion. Namely, let A be an abelian group in $\mathcal{E}$ considered as a groupoid in $\mathcal{E}$ with one object. Embed A in an injective abelian group I and denote the quotient I/A by B. Write p: $I \longrightarrow B$ for the quotient mapping. Let s,t: $B \times I \longrightarrow B$ denote, respectively, the projection, and the map $(x, y) \longmapsto x+p(y)$.

Proposition 4 The groupoid (s, t): $B \times I \longrightarrow B \times B$, with composition given by addition, is a strong stack completion of A.

Proof: There is a functor from A to (s, t): $B \times I \longrightarrow B \times B$ given by sending the single object of A to the element 0 in B, and an arrow a in A to the arrow (0, a) in $B \times I$. This is clearly a categorical equivalence injective on objects, so it remains to show that (s, t): $B \times I \longrightarrow B \times B$ is a strong stack. To do this, recall that, for any abelian group, say C, in $\mathcal{E}$, there is an isomorphism $\mathrm{Ext}^1(\mathbb{Z}X, C) \simeq H^1(X, C)$, where $\mathbb{Z}X$ is the free abelian group on the object X of $\mathcal{E}$. In fact, the correspondence is given as follows: if $0 \longrightarrow C \longrightarrow E \longrightarrow \mathbb{Z}X \longrightarrow 0$ is an extension of $\mathbb{Z}X$ by C, then the pullback $T \longrightarrow X$ of $E \longrightarrow \mathbb{Z}X$ along the inclusion $X \hookrightarrow \mathbb{Z}X$ of the generators is an C-torsor over X. We want to show first that the injective abelian group I is a strong stack considered as a groupoid with one object. (It is a stack by Theorem 1 since $\mathrm{Ext}^1(\mathbb{Z}X, I) = 0$.) We show that I-torsors satisfy the condition of Proposition 3. So, let $T \longrightarrow X$ be an I-torsor over X, which has a splitting s over Y, where $Y \hookrightarrow X$. Let $0 \longrightarrow I \longrightarrow E \longrightarrow \mathbb{Z}X \longrightarrow 0$, be the extension corresponding to $T \longrightarrow X$. The extension corresponding to the restriction of T to Y is the pullback $E' \longrightarrow \mathbb{Z}Y$ of $E \longrightarrow \mathbb{Z}X$ along $\mathbb{Z}Y \hookrightarrow \mathbb{Z}X$. In the diagram

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r] \arrow[d, "\mathrm{id}"'] & E' \arrow[r] \arrow[d, hook] \arrow[l, dashed, bend right, "s"'] & \mathbb{Z}Y \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & I \arrow[r] & E \arrow[r] & \mathbb{Z}X \arrow[r] & 0
\end{tikzcd}$$

the splitting $s: E' \longrightarrow I$ can be extended to a splitting $t: E \longrightarrow I$ since I is injective as an abelian group. It follows that $T \longrightarrow X$ is an injective object in $\mathcal{E}/X$, so I is a strong stack by Proposition 3. Now, notice that I acts on B via $p: I \longrightarrow B$. In fact, the action is just the map $t: B\times I \longrightarrow B$, i.e. $(x, y) \longmapsto x+p(y)$. The category of elements of the action is the groupoid $(s, t): B\times I \longrightarrow B\times B$, which is a strong stack by Proposition 1. ■

3. Strong Stacks in Categories

Until this point, we have only considered stacks and strong stacks for groupoids, since this is where all our applications lie. The theory has a perfectly good extension to arbitrary categories, however, which we now proceed to sketch. We begin with the classical definition of stack, rather than than the one adopted in section 1, so that the reader can more easily compare our treatment with that of [Giraud 1971] or [Bunge 1979].

Thus, let $\mathcal{E}$ be a Grothendieck topos, and suppose $p: \mathfrak{F} \longrightarrow \mathcal{E}$ is a categorical fibration over $\mathcal{E}$. Then $\mathfrak{F}$ is a *stack* (for the canonical topology on $\mathcal{E}$) if

(1) For each set I, and each I-indexed family $\{ X_i \mid i \in I \}$ of objects of $\mathcal{E}$, the canonical functor

$$\mathfrak{F}(\sum_{i\in I} X_i) \to \prod_{i\in I} \mathfrak{F}(X_i)$$

is an equivalence of categories, and

(2) For each surjection $q: X \twoheadrightarrow Y$ of $\mathcal{E}$, the canonical functor $\mathfrak{F}(Y) \longrightarrow \mathrm{des}\mathfrak{F}(X)$ is an equivalence of categories, where $\mathrm{des}\mathfrak{F}(X)$ is the category of objects of $\mathfrak{F}(X)$ provided with descent data relative to the kernel pair of q.

A category $\mathbb{C}$ in $\mathcal{E}$ is called a *stack* if its externalization $\mathcal{F}(X) = \hom(X, \mathbb{C})$ is a stack. Note that condition (1) is automatic in this case. To see more explicitly what condition (2) means, let $q\colon X \twoheadrightarrow Y$ be a surjection of $\mathcal{E}$, and denote by $\mathbb{X}$ the equivalence relation $X{\times_Y}X \hookrightarrow X{\times}X$ considered as a category in $\mathcal{E}$. q determines a categorical equivalence $\mathbb{X} \longrightarrow \mathrm{dis}Y$, and descent data on X with respect to q and $\mathbb{C}$ is simply a functor $\mathbb{X} \longrightarrow \mathbb{C}$. Thus, $\mathbb{C}$ is a stack in the above sense iff for any q, each diagram

$$\begin{tikzcd} \mathbb{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\curvearrowleft"] & \mathrm{dis}Y \arrow[dl, dashed] \\ \mathbb{C} & \end{tikzcd}$$

has a dotted filler making the resulting triangle commute up to isomorphism.

If $\mathbb{C}$ is a category in $\mathcal{E}$, let $\mathrm{Iso}(\mathbb{C})$ denote the groupoid in $\mathcal{E}$ whose objects are those of $\mathbb{C}$ and whose morphisms are the isomorphisms of $\mathbb{C}$. Let $\mathrm{Cat}(\mathcal{E})$ denote the category of categories in $\mathcal{E}$. For any surjection $q\colon X \twoheadrightarrow Y$, both $\mathbb{X}$ and $\mathrm{dis}Y$ are groupoids, so the following proposition is immediate.

Proposition 5 $\mathbb{C}$ is a stack in $\mathrm{Cat}(\mathcal{E})$, iff $\mathrm{Iso}(\mathbb{C})$ is a stack (in the present sense) in $\mathrm{Gpd}(\mathcal{E})$.■

For any category $\mathbb{A}$ in $\mathcal{E}$, let $T(\mathbb{A}) \hookrightarrow A_0$ be defined by

$$T(\mathbb{A}) = \{t \in A_0 \mid \forall a \in A_0\, \exists! a \longrightarrow t \text{ in } A_1\}$$

$T(\mathbb{A})$ is the *collection of terminal objects of* $\mathbb{A}$.

Passing from groupoids to categories in $\mathcal{E}$, torsors are replaced by locally representable functors. In fact, let $\mathbb{C}$ be a category in $\mathcal{E}$, and F an internal, $\mathcal{E}$-valued, contravariant functor on $\mathbb{C}$, with with structural map $F \longrightarrow C_0$, and action $a\colon F{\times_{C_0}}C_1 \longrightarrow F$. As before, the pair

$$(a, \pi_1)\colon F{\times_{C_0}}C_1 \longrightarrow F{\times}F$$

is a category $\mathbb{F}$ in $\mathcal{E}$ with objects F - the *category of elements* of the action. F is said to be *locally representable* if $T(\mathbb{F})$ is non-empty, i.e.$T(\mathbb{F}) \longrightarrow 1$ is a surjection. Note that when $\mathbb{C}$ is a groupoid and the action is free and transitive, then $T(\mathbb{F}) = F$. Thus, if F is a torsor it is locally representable. On the other hand, again when $\mathbb{C}$ is a groupoid, if F is locally representable then the action is free and transitive, so F is a torsor. For a general $\mathbb{C}$, F is locally representable iff $T(\mathbb{F})$ is an $\mathrm{Iso}(\mathbb{C})$-torsor.

An action of $\mathbb{C}$ in Cat($\mathcal{E}$) on a family $F \longrightarrow X$ indexed by X is an action of $X \times \mathbb{C}$, considered as a category in $\mathcal{E}/X$, on $F \longrightarrow X$. Equivalently, this is an action of $\mathbb{C}$ on F, for which the map $F \longrightarrow X$ is constant. Given such an action, we say $F \longrightarrow X$ is a *locally representable family indexed by* X if it is locally representable in $\mathcal{E}/X$. That is, we regard the collection $T(\mathbb{F})_X$ of terminal objects of $\mathbb{F}$ computed fibrewise over X, and we ask that $T(\mathbb{F})_X \longrightarrow X$ be a surjection.
As above, this occurs iff $T(\mathbb{F})_X \longrightarrow X$ is an Iso($\mathbb{C}$)-torsor over X. When $T(\mathbb{F})_X \longrightarrow X$ is a surjection, it follows that $X \simeq \pi_0(\mathbb{F})$, so that F is a locally representable family iff there exists a terminal object (in the internal sense) in each component of the category of elements of F. F is said to be an *X-indexed family of representable functors* if $T(\mathbb{F})_X \longrightarrow X$ splits.

When $\mathbb{C}$ is a category in $\mathcal{E}$, and X is an object of $\mathcal{E}$, the category hom(X, $\mathbb{C}$) is defined as in section 1. Let H(X, $\mathbb{C}$) denote the category of locally representable families indexed by X. $t\colon C_1 \longrightarrow C_0$ is a representable family indexed by C_0, so $(f\colon X \longrightarrow C_0) \longmapsto f^*(C_1)$ induces a full and faithful functor $\hom(X, \mathbb{C}) \longrightarrow H(X, \mathbb{C})$ as before.

Theorem 3 The following are equivalent for a category $\mathbb{C}$ in $\mathcal{E}$.
(i) $\mathbb{C}$ is a stack.
(ii) Every locally representable family of functors is representable.
(iii) For each X in $\mathcal{E}$, $\hom(X, \mathbb{C}) \longrightarrow H(X, \mathbb{C})$ is an equivalence of categories.
(iv) Every diagram of categories

$$\begin{tikzcd} \mathbb{A} \arrow[r, hook, "e"] \arrow[d] & \mathbb{B} \arrow[dl, dashed] \\ \mathbb{C} & \end{tikzcd}$$

with e a categorical equivalence injective on objects has a dotted filler making the resulting triangle commute up to isomorphism.
(v) Every diagram of categories

$$\begin{tikzcd} \mathbb{A} \arrow[r, "f"] \arrow[d] & \mathbb{B} \arrow[dl, dashed] \\ \mathbb{C} & \end{tikzcd}$$

with f a categorical equivalence has a dotted filler making the resulting triangle commute up to isomorphism.

Proof: (i) $\Rightarrow$ (ii): Let $F \longrightarrow X$ be a locally representable family indexed by X. As above, $T(\mathbb{F})_X \longrightarrow X$ is an Iso($\mathbb{C}$)-torsor over X. If $\mathbb{C}$ is a stack, so is Iso($\mathbb{C}$) - in the sense defined here. But this shows that any Iso($\mathbb{C}$)-torsor splits (see (iv) $\Rightarrow$ (i) in the proof of theorem 1). Thus, $T(\mathbb{F})_X \longrightarrow X$ splits, and $F \longrightarrow X$ is a representable family.
(ii) $\Leftrightarrow$ (iii) is similar to (i) $\Leftrightarrow$ (ii) of Theorem 1.
(iii) $\Rightarrow$ (iv) follows the same pattern as (ii) $\Rightarrow$ (iii) of Theorem 1, except that the torsor $E \longrightarrow H_0$ therein defined is now a locally representable family. The fact that this family is then representable allows us to define the retraction in the same way.
(iv) $\Rightarrow$ (v) is the same as (iii) $\Rightarrow$ (iv) of Theorem 1 - this argument does not depend on the fact that the categories involved are groupoids.
(v) $\Rightarrow$ (i) is obvious. ■

Let $\mathbb{C}$ be a category in $\mathcal{E}$. A *stack completion* of $\mathbb{C}$ is a categorical equivalence $\mathbb{C} \longrightarrow \mathbb{C}^*$, such that $\mathbb{C}^*$ is a stack. As before, stack completions are defined up to strong equivalence of categories. To obtain the stack completion of a category $\mathbb{C}$, it is enough to have the stack completion of its groupoid of isomorphisms. In fact, suppose $\mathrm{Iso}(\mathbb{C}) \longrightarrow \mathrm{Iso}(\mathbb{C})^*$ is a stack completion of the groupoid Iso($\mathbb{C}$). Taking the pushout

$$\begin{tikzcd}
\mathrm{Iso}(\mathbb{C}) \arrow[r] \arrow[d, hook] & \mathrm{Iso}(\mathbb{C})^* \arrow[d, hook] \\
\mathbb{C} \arrow[r] & \mathbb{C}^*
\end{tikzcd}$$

provides a stack completion $\mathbb{C} \longrightarrow \mathbb{C}^*$ of $\mathbb{C}$. The proof is immediate by Proposition 5.

Definition 3 A category $\mathbb{C}$ in $\mathcal{E}$ is a *strong stack* if condition (iv) of Theorem 3 holds on the nose. That is, if each diagram of categories

$$\begin{tikzcd} \mathbb{A} \arrow[r, hook, "e"] \arrow[d] & \mathbb{B} \arrow[dl, dashed] \\ \mathbb{C} & \end{tikzcd}$$

with e a categorical equivalence injective on objects has a dotted filler making the resulting triangle commute.

Proposition 6 $\mathbb{C}$ is a strong stack in Cat($\mathcal{E}$) iff Iso($\mathbb{C}$) is a strong stack in Gpd($\mathcal{E}$).

Proof: Clearly, if $\mathbb{C}$ is a strong stack in Cat($\mathcal{E}$), then Iso($\mathbb{C}$) is a strong stack in Gpd($\mathcal{E}$). The other direction follows immediately from the fact that if e: $\mathbb{A} \hookrightarrow \mathbb{B}$ is a categorical equivalence injective on objects, then

$$\begin{tikzcd} \mathrm{Iso}(\mathbb{A}) \arrow[r, hook] \arrow[d, hook] & \mathrm{Iso}(\mathbb{B}) \arrow[d, hook] \\ \mathbb{A} \arrow[r, hook, "e"] & \mathbb{B} \end{tikzcd}$$

is a pushout of categories. ∎

Definition 4 A *strong stack completion* of a category $\mathbb{C}$ in $\mathcal{E}$ is a categorical equivalence $\mathbb{C} \hookrightarrow \mathbb{C}^*$ injective on objects, such that $\mathbb{C}^*$ is a strong stack.

In view of Proposition 6, the strong stack completion of a category $\mathbb{C}$ can be obtained from the strong stack completion of its groupoid of isomorphisms Iso($\mathbb{C}$) as above.

As is the case for groupoids, strong stacks in Cat($\mathcal{E}$) are the fibrant objects for a Quillen homotopy structure on Cat($\mathcal{E}$). Namely, we have

Theorem 4 There is a Quillen homotopy structure on Cat($\mathcal{E}$), in which the weak equivalences are the categorical equivalences, the cofibrations are the functors injective on objects, and the fibrations have the right lifting property with respect to the cofibration weak equivalences.

Proof: There are two ways to prove Theorem 4: either repeat verbatim the proof of Theorem 2, which does not depend on groupoids, or repeat the first part, up to the point of showing that an arbitrary functor f: $\mathbb{C} \longrightarrow \mathbb{D}$ in Cat($\mathcal{E}$) can be factored as a cofibration weak equivalence followed by a fibration. At this point, factor Iso(f) in Gpd($\mathcal{E}$) as

$$\begin{tikzcd} \mathrm{Iso}(\mathbb{C}) \arrow[rr, hook, "i"] \arrow[dr, "\mathrm{Iso}(f)"'] & & \mathbb{E} \arrow[dl, "p"] \\ & \mathrm{Iso}(\mathbb{D}) & \end{tikzcd}$$

where i is a cofibration weak equivalence in Gpd($\mathcal{E}$) and p is a fibration. Now take the pushout in Cat($\mathcal{E}$)

$$\begin{tikzcd} \mathrm{Iso}(\mathbb{C}) \arrow[r, hook, "i"] \arrow[d, hook] & \mathbb{E} \arrow[d, hook] \\ \mathbb{C} \arrow[r, hook] & \mathbb{E}^* \end{tikzcd}$$

and use the obvious generalization of Proposition 6, which states that f is a fibration in Cat($\mathcal{E}$) iff Iso(f) is a fibration in Gpd($\mathcal{E}$). ■

4. Strong Stacks and Classifying Spaces

Let $\mathcal{E}$ be a Grothendieck topos, and let S($\mathcal{E}$) denote the topos of simplicial objects in $\mathcal{E}$. Recall [Joyal (to appear)] that there is a Quillen homotopy structure on S($\mathcal{E}$), in which the weak equivalences are maps f: X $\longrightarrow$ Y inducing isomorphisms on the homotopy sheaves, the cofibrations are the monomorphisms, and the fibrations are maps having the right lifting property with respect to the cofibration weak equivalences, which we call *anodyne extensions*. The *homotopy category* of S($\mathcal{E}$) is obtained by formally inverting the weak equivalences (or just the anodyne extensions) of S($\mathcal{E}$). If X and Y are objects of S($\mathcal{E}$), and Y is fibrant, then the set of maps from X to Y in the homotopy category is in 1-1 correspondence with the set [X,Y] of *homotopy classes* of maps from X to Y. (A homotopy is a mapping $X \times I \longrightarrow Y$ with I the constant simplicial sheaf on the 1-simplex $\Delta[1]$ of S.)

We often use the principle of *boolean localization* to transfer results from the homotopy theory of simplicial sets to simplicial sheaves. Namely, for any Grothendieck topos $\mathcal{E}$ there exists a surjective geometric morphism p: $\mathcal{B} \longrightarrow \mathcal{E}$ such that $\mathcal{B}$ is boolean and satisfies the axiom of choice [Barr 1974]. The inverse image functor p^* provides a faithful embedding of $\mathcal{E}$ into $\mathcal{B}$, whose logic is classical (i.e. $\mathcal{B}$ is a boolean valued model of ZF set theory). Not all constructions are preserved by p^*, but those using only colimits and finite limits are (the so-called *geometric constructions*). For example, p^* preserves the construction of the homotopy groups of a simplicial sheaf. Thus, a mapping $f: X \longrightarrow Y$ of $S(\mathcal{E})$ is a weak equivalence iff $p^*(f)$ is. As a result, geometric constructions yielding weak equivalences in simplicial sets, yield weak equivalences in $S(\mathcal{E})$.

The *nerve* of a groupoid $\mathbb{G}$ in $\mathcal{E}$ is a simplicial sheaf $N\mathbb{G}$ described as follows. $(N\mathbb{G})_n$ is the object G_n of composable strings of length n, $\sigma = x_n \longrightarrow x_{n-1} \longrightarrow \ldots x_1 \longrightarrow x_0$, of arrows of $\mathbb{G}$. Faces and degeneracies are given by $d^0\sigma = x_n \longrightarrow x_{n-1} \longrightarrow \ldots \longrightarrow x_1$, $d^n\sigma = x_{n-1} \longrightarrow \ldots \longrightarrow x_0$, and $d^i\sigma = x_n \longrightarrow \ldots x_{i+1} \longrightarrow x_{i-1} \ldots \longrightarrow x_0$ for $0 < i < n$, where $x_{i+1} \longrightarrow x_{i-1}$ is the composite of the pair $x_{i+1} \longrightarrow x_i \longrightarrow x_{i-1}$. $s^i\sigma = x_n \longrightarrow \ldots \longrightarrow x_i \longrightarrow x_i \longrightarrow \ldots \longrightarrow x_0$, where $x_i \longrightarrow x_i$ is the identity on x_i. In particular, $(N\mathbb{G})_0 = G_0$, the objects of $\mathbb{G}$, $(N\mathbb{G})_1 = G_1$, the morphisms of $\mathbb{G}$, and $d^0(\alpha: x_1 \longrightarrow x_0) = x_1 = s\alpha$, $d^1(\alpha: x_1 \longrightarrow x_0) = x_0 = t\alpha$, and $s^0x = \mathrm{id}: x \longrightarrow x = ux$.

When $\mathbb{G}$ is a groupoid in $S(\mathcal{E})$, $N\mathbb{G}$ is a double simplicial object of $\mathcal{E}$, whose n-th column is the simplicial object G_n. Letting $d: S^2(\mathcal{E}) \longrightarrow S(\mathcal{E})$ denote the *diagonal complex* defined by $d(X)_n = X_{n\,n}$, we write $B\mathbb{G}$ for $d(N\mathbb{G})$. In arguments involving $B\mathbb{G}$, we often use the following, fundamental property of the diagonal complex. Namely, let $f: X \longrightarrow Y$ be a mapping in $S^2(\mathcal{E})$. We call f a *vertical weak equivalence* if $f_{n\,*}: X_{n\,*} \longrightarrow Y_{n\,*}$ is a weak equivalence for each $n \geq 0$, and a *horizontal weak equivalence* if $f_{*\,m}: X_{*\,m} \longrightarrow Y_{*\,m}$ is a weak equivalence for each $m \geq 0$. Then it follows by boolean localization, and the corresponding fact about double simplicial sets, that if $f: X \longrightarrow Y$ is either a vertical or horizontal weak equivalence, $df: dX \longrightarrow dY$ is a weak equivalence.

Proposition 7 Let $\mathbb{G}$ be a groupoid in $S(\mathcal{E})$. If $\mathbb{G} \hookrightarrow \mathbb{G}^*$ is a strong stack completion of $\mathbb{G}$, then G^*_0 is weakly equivalent (in $S(\mathcal{E})$) to $B\mathbb{G}$.

Proof: Proposition 3 says that each $\mathbb{G}^*$ torsor is a trivial fibration (in $S(\mathcal{E})$) over its base. In particular, $t: G^*_1 \longrightarrow G^*_0$ (and therefore also s) is a trivial fibration. Thus, the inclusion $u: G^*_0 \longrightarrow G^*_1$ of the units of $\mathbb{G}^*$ is a weak equivalence. In fact, u defines a functor $\mathrm{dis}G^*_0 \longrightarrow \mathbb{G}^*$, such that $N(\mathrm{dis}G^*_0) \longrightarrow N(\mathbb{G}^*)$ is a vertical weak equivalence. Taking the diagonal complex yields a weak equivalence $G^*_0 \cong B(\mathrm{dis}G^*_0) \longrightarrow B\mathbb{G}^*$. But since $\mathbb{G} \hookrightarrow \mathbb{G}^*$ is a categorical equivalence, $N\mathbb{G} \hookrightarrow N\mathbb{G}^*$ is a horizontal weak equivalence by boolean localization, so $B\mathbb{G} \longrightarrow B\mathbb{G}^*$ is also a weak equivalence, giving the result. ■

Definition 5 A groupoid $\mathbb{G}$ in $S(\mathcal{E})$ is said to be *amenable* if the functor $H^1(\, , \mathbb{G})$ inverts anodyne extensions, i.e. passes to the homotopy category.

Note that amenability is invariant under categorical equivalence of groupoids.

Proposition 8 If $\mathbb{G}$ is an amenable strong stack in $S(\mathcal{E})$, then G_0 is fibrant in the Quillen structure on $S(\mathcal{E})$.

Proof: Let $\mathbb{G}$ be an amenable strong stack in $S(\mathcal{E})$, $i: A \hookrightarrow B$ an anodyne extension, and $f: A \longrightarrow G_0$ a map. The pullback of $t: G_1 \longrightarrow G_0$ along f is a $\mathbb{G}$-torsor $T \longrightarrow A$ over A. Since $\mathbb{G}$ is amenable, there is a $\mathbb{G}$-torsor $S \longrightarrow B$ whose restriction to A is isomorphic to T. Since $\mathbb{G}$ is a strong stack, there is a map $g: B \longrightarrow G_0$ such that $g^*(G_1) \simeq S$. Thus, we obtain a mapping $h: A \longrightarrow G_1$ such that $sh = gi$ and $th = f$. As above, $s: G_1 \longrightarrow G_0$ is a trivial fibration in $S(\mathcal{E})$, so the commutative square

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, hook, "i"'] & G_1 \arrow[d, "s"] \\ B \arrow[r, "g"'] \arrow[ur, dashed, "k"] & G_0 \end{tikzcd}$$

has a dotted filler k. The map tk is the desired extension of f to B. ■

Theorem 5 Let $\mathbb{G}$ be an amenable groupoid in $S(\mathcal{E})$, and $\mathbb{G} \hookrightarrow \mathbb{G}^*$ a strong stack completion of $\mathbb{G}$. Then for any X in $S(\mathcal{E})$, $H^1(X, \mathbb{G}) \cong [X, G^*_0]$. That is, G^*_0 is a classifying space for $\mathbb{G}$-torsors.

Proof: Since $\mathbb{G} \hookrightarrow \mathbb{G}^*$ is a categorical equivalence, $H^1(X, \mathbb{G}) \simeq H^1(X, \mathbb{G}^*)$, $\mathbb{G}^*$ is amenable, and G^*_0 is fibrant by Proposition 8. Since $\mathbb{G}^*$ is a strong stack, $H^1(X, \mathbb{G}^*) \simeq \pi_0(\hom(X, \mathbb{G}^*))$. Thus, we are left with showing that $\pi_0(\hom(X, \mathbb{G}^*)) \simeq [X, G^*_0]$. More precisely, we want to show that for two maps $f,g: X \longrightarrow G^*_0$, the map $(f, g): X \longrightarrow G^*_0 \times G^*_0$ lifts to G^*_1, i.e. the pullback of $t: G^*_1 \longrightarrow G^*_0$ along f is isomorphic to its pullback along g, iff f is homotopic to g. For this, notice that the projection $X \times I \longrightarrow X$ is a weak equivalence. Hence, any $\mathbb{G}^*$-torsor over $X \times I$ is constant in I, by the amenability of $\mathbb{G}^*$. It follows that if f is homotopic to g, then the pullback of $t: G^*_1 \longrightarrow G^*_0$ along f is isomorphic to its pullback along g. In the other direction, consider the diagram

$$\begin{tikzcd}
G^*_0 \arrow[r] \arrow[d, hook, "u"'] & G^{*I}_0 \arrow[d] \\
G^*_1 \arrow[r, "(s,t)"'] \arrow[ur, dashed] & G^*_0 \times G^*_0
\end{tikzcd}$$

The right-hand vertical map is a fibration since G^*_0 is fibrant. u is a cofibration weak equivalence as above, so the diagram has a dotted filler. As a result, if (f, g): $X \longrightarrow G^*_0 \times G^*_0$ lifts to G^*_1, then f is homotopic to g, and the theorem is proved. ■

We remark that in the category of simplicial sets, a locally transitive groupoid $\mathbb{G}$ is amenable, where locally transitive means that $(s,t)\colon G_1 \longrightarrow G_0 \times G_0$ is a Kan fibration. In fact, for simplicial sets the two concepts are equivalent. Thus, Theorem 5 provides a classifying space for any locally transitive simplicial groupoid, e.g. any simplicial group. See [Joyal-Tierney (to appear)] for a full discussion of amenability, local transitivity and classifying spaces.

References

[1] M. Barr, Toposes without points, J. Pure App. Alg. 5, 1974, pp 265-280.

[2] M. Bunge, Stack completions and Morita equivalence for categories in a topos, Cahiers Top. Geom. Diff. xx-4, 1979, pp 401-435.

[3] M. Bunge and R. Pare, Stacks and equivalence of indexed categories, Cahiers Top. Geom. Diff. xx-4, 1979, pp 373-399.

[4] J. Giraud, *Cohomologie non abélienne*, Grundlehren #179, Springer Verlag, 1971.

[5] P. T. Johnstone, *Topos Theory*, L.M.S. Monographs #10, Academic Press, London, 1977.

[6] A. Joyal, Homotopy theory of simplicial sheaves, to appear.

[7] A. Joyal and M. Tierney, An extension of the galois theory of Grothendieck, Memoirs of the AMS vol. 51 No. 309, 1984

[8] ________, Classifying spaces for sheaves of simplicial groupoids, JPAA, to appear.

[9] J. P. May, *Simplicial Objects in Algebraic Topology*, Van Nostrand Math.Studies #11, 1967.

[10] D. G. Quillen, *Homotopical Algebra*, Springer Lecture Notes in Mathematics #43, 1967.

[11] ________, Rational homotopy theory, Annals of Math.90, 1969, pp205-295.

This paper is in final form and will not be published elsewhere.