Cartesian cubical model categories

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Background

- There has recently been work on **cubical** homotopy theory.
- It is related to **homotopy type theory** which is being used for computerized proof checking.
- The cubes used for this are **closed under finite products**.
- This model of homotopy was also proposed by Lawvere who stressed the **tinyness of the geometric interval** $\mathbb{I}$.
- The tinyness of $\mathbb{I}$ is also used in the current theory.

Cartesian cubical sets

The **Cartesian cube category** $\Box$ is the opposite of the category $\mathbb{B}$ of finite, strictly bipointed sets,

$$\Box := \mathbb{B}^{\mathsf{op}}.$$

Thus $\Box$ is the **Lawvere theory of bipointed objects**: the free finite product category with a bipointed object $[0] \rightrightarrows [1]$.

The **Cartesian cubical sets** is the category of presheaves on $\Box$,

$$\mathsf{cSet} = \mathsf{Set}^{\Box^{\mathsf{op}}}.$$

Thus cSet consists of all **covariant** functors $\mathbb{B} \to \mathsf{cSet}$.

The tiny interval $\mathbb{I}$

The 1-cube [1] represents the cubical set that **forgets the points**,

$$\mathbb{I} := \mathbb{B}([1], -) : \mathbb{B} \longrightarrow \mathsf{cSet}\,.$$

It **generates** cSet under finite products and colimits.

The two points $1 \rightrightarrows \mathbb{I}$ have a trivial intersection.

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d] \\ 1 \arrow[r] & \mathbb{I} \end{tikzcd}$$

This is the universal **interval** in a topos.

It provides a **good cylinder** $X + X \rightarrowtail \mathbb{I} \times X$ for every object X, and a **good path object** $X^{\mathbb{I}} \twoheadrightarrow X \times X$ for every **fibrant** object X.

The main result

Theorem (A. 2023)

There is a Quillen model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ *on* cSet *where:*

- *the* **cofibrations** $\mathcal{C}$ *are an axiomatized class of monos,*
- *the* **fibrations** $\mathcal{F}$ *are those* $f : X \to Y$ *for which*

$$(f^{\mathbb{I}} \times \mathbb{I}, \mathrm{eval}) : X^{\mathbb{I}} \times \mathbb{I} \longrightarrow (Y^{\mathbb{I}} \times \mathbb{I}) \times_Y X$$

  lifts on the right against all cofibrations,
- *the* **weak equivalences** $\mathcal{W}$ *are those* $f : X \to Y$ *for which* $K^f : K^Y \longrightarrow K^X$ *is bijective under* π_0 *whenever* K *is fibrant.*

The construction of $(\mathcal{C}, \mathcal{W}, \mathcal{F})$

The **proof** of the theorem

- uses ideas from **type theory**,
- including the **univalence axiom** of Voevodsky,
- is **axiomatized** in terms of:
  1. a classifier $\Phi \hookrightarrow \Omega$ for the cofibrations,
  2. a tiny interval $1 \rightrightarrows \mathbb{I}$,
  3. a universal small map $\dot{\mathsf{V}} \to \mathsf{V}$,
- applies in several different cases.

$(\mathcal{C}, \mathcal{W}, \mathcal{F})$ from $(\Phi, \mathbb{I}, \mathsf{V})$

The model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ is constructed in 3 steps:

1. Φ is used to determine a wfs $(\mathcal{C}, \mathsf{TFib})$,
2. $\mathbb{I}$ is used to determine a wfs $(\mathsf{TCof}, \mathcal{F})$ with $\mathsf{TFib} \subseteq \mathcal{F}$,
3. V is used to show 3-for-2 for $\mathcal{W} := \mathsf{TFib} \circ \mathsf{TCof}$.

1. The cofibration wfs $(\mathcal{C}, \mathsf{TFib})$

The **cofibrations** $\mathcal{C}$ are the monos $C' \rightarrowtail C$ classified by $t : 1 \rightarrowtail \Phi$.

$$\begin{tikzcd}
C' \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[r] \arrow[d, tail, "t"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "\top"] \\
C \arrow[r] & \Phi \arrow[r, hook] & \Omega
\end{tikzcd}$$

The **trivial fibrations** TFib are the maps $T \twoheadrightarrow X$ that lift against the cofibrations.

$$\mathcal{C}^{\pitchfork} =: \mathsf{TFib}$$

$$\begin{tikzcd}
C' \arrow[r] \arrow[d, tail] & T \arrow[d, two heads] \\
C \arrow[r] \arrow[ur, dotted] & X
\end{tikzcd}$$

1. The cofibration wfs $(\mathcal{C}, \mathsf{TFib})$

Proposition

$(\mathcal{C}, \mathsf{TFib})$ *is an algebraic weak factorization system.*

Proof.

The classifier $t : 1 \rightarrowtail \Phi$ determines a fibeblue polynomial monad

$$P_t = \Phi_! t_* : \mathsf{cSet} \longrightarrow \mathsf{cSet}$$

the algebras for which in $\mathsf{cSet}/_X$ are the trivial fibrations. $\square$

2. The fibration wfs $(\mathsf{TCof}, \mathcal{F})$

The **fibrations** $\mathcal{F}$ are defined in terms of the trivial fibrations by

$$(f : F \to X) \in \mathcal{F} \qquad \text{iff} \qquad (\delta \Rightarrow f) \in \mathsf{TFib}$$

where $\delta \Rightarrow f$ is the **gap map** with $\delta : 1 \longrightarrow \mathbb{I}$ in $\mathsf{cSet}/_{\mathbb{I}}$.

$$\begin{tikzcd}
F^{\mathbb{I}} \arrow[rr] \arrow[dr, dotted, "\delta\Rightarrow f"] \arrow[dd] & & F \arrow[d, equal] \\
 & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F \arrow[d, "f"] \\
X^{\mathbb{I}} \arrow[r, equal] & X^{\mathbb{I}} \arrow[r] & X
\end{tikzcd}$$

The **trivial cofibrations** TCof are the maps that lift against $\mathcal{F}$.

$$\mathsf{TCof} := {}^{\pitchfork}\mathcal{F}$$

3. The weak equivalences $\mathcal{W}$

Let $\mathcal{W} := \mathsf{TFib} \circ \mathsf{TCof}$.

Proposition

$(\mathcal{C}, \mathsf{TFib})$ and $(\mathsf{TCof}, \mathcal{F})$ form a Barton **premodel structure**.

$$\mathsf{TCof} = \mathcal{W} \cap \mathcal{C}$$
$$\mathsf{TFib} = \mathcal{W} \cap \mathcal{F}$$

Corollary

If $\mathcal{W}$ satisfies 3-for-2, then $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ is a QMS.

3. The weak equivalences $\mathcal{W}$

We use a **universal fibration** $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$ to show 3-for-2 for $\mathcal{W}$.

(i) there is a **universal small map** $\dot{\mathsf{V}} \to \mathsf{V}$

(ii) U is the **classifying type** for fibration structures on $\dot{\mathsf{V}} \to \mathsf{V}$

(iii) $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$ is **univalent**

(iv) U is **fibrant**

(v) fibrant U implies **3-for-2** for $\mathcal{W}$

The idea of getting a QMS **from** univalence is due to Sattler.

3(i). The universal small map $\dot{\mathsf{V}} \to \mathsf{V}$

The **category of elements** functor $\int_{\mathbb{C}}$

$$
\begin{tikzcd}
\int_{\mathbb{C}} : \widehat{\mathbb{C}} \arrow[r, bend left] & \mathsf{Cat} : \nu_{\mathbb{C}} \arrow[l, bend left]
\end{tikzcd}
$$

always has a right adjoint **nerve** functor $\nu_{\mathbb{C}}$.

Proposition

For any small map $Y \to X$ in $\widehat{\mathbb{C}}$ there is a canonical pullback

$$
\begin{tikzcd}
Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \nu_{\mathbb{C}}\, \dot{\mathsf{set}}^{\mathsf{op}} \arrow[d] \\
X \arrow[r] & \nu_{\mathbb{C}}\, \mathsf{set}^{\mathsf{op}}
\end{tikzcd}
$$

since $\dot{\mathsf{set}}^{\mathsf{op}} \longrightarrow \mathsf{set}^{\mathsf{op}}$ *classifies small discrete fibrations in* Cat*.*

3(i). The universal small map $\dot{\mathsf{V}} \to \mathsf{V}$

The **category of elements** functor $\int_{\mathbb{C}}$

$$\begin{tikzcd} \int_{\mathbb{C}} : \widehat{\mathbb{C}} \arrow[r, bend left] & \mathsf{Cat} : \nu_{\mathbb{C}} \arrow[l, bend left] \end{tikzcd}$$

always has a right adjoint **nerve** functor $\nu_{\mathbb{C}}$.

Proposition

For any small map $Y \longrightarrow X$ in $\widehat{\mathbb{C}}$ there is a canonical pullback

$$\begin{tikzcd} Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \nu_{\mathbb{C}}\, \dot{\mathsf{set}}^{\mathsf{op}} \arrow[r, equal] \arrow[d] & \dot{\mathsf{V}} \arrow[d] \\ X \arrow[r] & \nu_{\mathbb{C}}\, \mathsf{set}^{\mathsf{op}} \arrow[r, equal] & \mathsf{V} \end{tikzcd}$$

since $\dot{\mathsf{set}}^{\mathsf{op}} \to \mathsf{set}^{\mathsf{op}}$ *classifies small discrete fibrations in* Cat*.*

3(ii). The universal fibration $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$

For any $A \to X$ in cSet there is a **classifying type** $\mathrm{Fib}(A) \to X$, the sections of which correspond to fibration structures.

$$\begin{tikzcd} & A \arrow[d, two heads] \\ \mathrm{Fib}(A) \arrow[r] & X \arrow[l, dotted, bend right] \end{tikzcd}$$

3(ii). The universal fibration $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$

The construction of $\mathrm{Fib}(A) \longrightarrow X$ is stable under pullback.

$$\begin{tikzcd}
& f^*A \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & A \arrow[dd] \\
f^*\mathrm{Fib}(A) \arrow[rr] \arrow[dr] & & \mathrm{Fib}(A) \arrow[dr] & \\
& Y \arrow[rr, "f"'] & & X
\end{tikzcd}$$

$$f^*\mathrm{Fib}(A) \;\cong\; \mathrm{Fib}(f^*A)$$

This uses the **root** functor $(-)^{\mathbb{I}} \dashv (-)_{\mathbb{I}}$.

3(ii). The universal fibration $\dot{U} \twoheadrightarrow U$

Let U be the type of fibration structures on $\dot{V} \to V$

$$
\begin{tikzcd}
 & \dot{V} \arrow[d] \\
U := \mathrm{Fib}(\dot{V}) \arrow[r] & V
\end{tikzcd}
$$

then define $\dot{U} \to U$ by pulling back.

$$
\begin{tikzcd}
\dot{U} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{V} \arrow[d] \\
U \arrow[r] & V
\end{tikzcd}
$$

3(ii). The universal fibration $\dot{U} \twoheadrightarrow U$

Since $\mathrm{Fib}(-)$ is stable, the lower square is a pullback.

$$
\begin{tikzcd}
 & \dot{U} \arrow[rr] \arrow[dd] \arrow[dr, phantom, very near start, "\lrcorner"] & & \dot{V} \arrow[dd] \\
\mathrm{Fib}(\dot{U}) \arrow[rr] \arrow[dr] & & \mathrm{Fib}(\dot{V}) \arrow[dr] & \\
 & U \arrow[rr] & & V
\end{tikzcd}
$$

3(ii). The universal fibration $\dot{\mathrm{U}} \twoheadrightarrow \mathrm{U}$

Since $\mathrm{Fib}(-)$ is stable the lower square is also a pullback.

$$\begin{tikzcd}
 & \dot{\mathrm{U}} \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & \dot{\mathrm{V}} \arrow[dd] \\
\mathrm{Fib}(\dot{\mathrm{U}}) \arrow[rr, crossing over] \arrow[dr] & & \mathrm{Fib}(\dot{\mathrm{V}}) \arrow[dr] & \\
 & \mathrm{U} \arrow[rr] \arrow[ul, dotted, bend left, "\Delta_U"] & & \mathrm{V}
\end{tikzcd}$$

But since $\mathrm{U} = \mathrm{Fib}(\dot{\mathrm{V}})$ there is a section of $\mathrm{Fib}(\dot{\mathrm{U}})$.
So $\dot{\mathrm{U}} \to \mathrm{U}$ is a fibration.

3(ii). The universal fibration $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$

A fibration structure α on a small map $A \to X$ determines a factorization (a, α) of its classifying map $a : X \to \mathsf{V}$.

$$
\begin{tikzcd}
 & A \arrow[rr] \arrow[dd, two heads] & & \dot{\mathsf{V}} \arrow[dd] \\
\mathrm{Fib}(A) \arrow[rr, crossing over] \arrow[dr] & & \mathrm{Fib}(\dot{\mathsf{V}}) \arrow[dr] & \\
 & X \arrow[rr, "a"'] \arrow[ul, dotted, bend left, "\alpha"] \arrow[ur, dotted, "{(a,\alpha)}"'] & & \mathsf{V}
\end{tikzcd}
$$

3(ii). The universal fibration $\dot{\mathrm{U}} \twoheadrightarrow \mathrm{U}$

A fibration structure α on a small map $A \to X$ determines a factorization (a, α) of its classifying map $a : X \to \mathrm{V}$,

$$
\begin{tikzcd}
 & A \arrow[rr] \arrow[dr, dotted] \arrow[ddd, two heads] & & \dot{\mathrm{V}} \arrow[ddd] \\
 & & \dot{\mathrm{U}} \arrow[ur] \arrow[d, two heads] & \\
\mathrm{Fib}(A) \arrow[rr] \arrow[dr] & & \mathrm{Fib}(\dot{\mathrm{V}}) \arrow[dr] & \\
 & X \arrow[rr, "a"'] \arrow[ul, dotted, bend left, "\alpha"] \arrow[ur, dotted, "{(a,\alpha)}"'] & & \mathrm{V}
\end{tikzcd}
$$

which classifies $A \twoheadrightarrow X$ **as a fibration** since $\mathrm{Fib}(\dot{\mathrm{V}}) = \mathrm{U}$.

3(iii). $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$ is univalent

The universal fibration $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$ is **univalent** if the type

$$\mathsf{Eq}_B \;=\; \Sigma_B \mathsf{Eq}(-, B) \longrightarrow \mathsf{U}$$

of **based equivalences** is always a trivial fibration.

$$\begin{tikzcd} C' \arrow[r] \arrow[d, tail] & \mathsf{Eq}_B \arrow[d] \\ C \arrow[r, "A"'] \arrow[ur, dotted, "A \simeq B"'] & \mathsf{U} \end{tikzcd} \tag{*}$$

Remark

In HoTT this implies $(A = B) \simeq (A \simeq B)$.

3(iii). $\dot{\mathrm{U}} \twoheadrightarrow \mathrm{U}$ is univalent

Unwinding $(*)$ gives the **equivalence extension property**: weak equivalences extend along cofibrations $C' \rightarrowtail C$.

$$
\begin{tikzcd}
A' \arrow[rr, dotted, tail] \arrow[dr, "w'", "\sim"'] \arrow[dd, two heads] & & \color{blue}{A} \arrow[dr, dotted, "w", "\sim"'] \arrow[dd, dotted, two heads] & \\
& B' \arrow[rr] \arrow[dl, two heads] & & B \arrow[dl, two heads] \\
C' \arrow[rr, tail] & & C &
\end{tikzcd}
$$

3(iii). $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$ is univalent

Proposition

The universal fibration $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$ *is univalent.*

Voevodsky proved this **classically** for Kan fibrations in sSet.

Coquand gave a constructive proof in **type theory** for cSet.

We have generalized Coquand's proof to cartesian cubical sets.

3(iv). U is fibrant

Univalence of $\dot{\mathrm{U}} \twoheadrightarrow \mathrm{U}$ implies that U is fibrant.

Proposition

The universe U *is fibrant.*

Voevodsky proved this for Kan sSets using **minimal fibrations**.

Shulman proved it using **3-for-2** for $\mathcal{W}$.

Coquand proved it from univalence without 3-for-2 using **Kan composition** for cSets in type theory.

We give a general proof from univalence without using 3-for-2.

3(v). From fibrant U to 3-for-2

Finally, we can apply the following.

Proposition (Sattler)

$\mathcal{W}$ satisfies 3-for-2 if fibrations extend along trivial cofibrations.

$$\begin{tikzcd} A \arrow[r, dotted] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & A' \arrow[d, dotted, two heads] \\ X \arrow[r, tail, "\sim"'] & X' \end{tikzcd}$$

This is called the **fibration extension property**.

3(v). From fibrant U to 3-for-2 for $\mathcal{W}$

Lemma

Given a universal fibration $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$ *the FEP holds if* U *is fibrant.*

$$\begin{tikzcd}
A \arrow[r, dotted] \arrow[drr] \arrow[dd, two heads] & A' \arrow[dr, dotted] \arrow[dd, dotted, two heads] & \\
 & & \dot{\mathsf{U}} \arrow[dd, two heads] \\
X \arrow[r, tail, "\sim"] \arrow[drr] & X' \arrow[dr, dotted] & \\
 & & \mathsf{U}
\end{tikzcd}$$

References

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- C. Cohen, et al., Cubical type theory: A constructive interpretation of the univalence axiom, 2016.
- C. Kapulkin and P. LeFanu Lumsdaine, The simplicial model of univalent foundations (after Voevodsky), 2021.
- C. Sattler, The equivalence extension property and model structures, 2017.
- M. Shulman, All $(\infty, 1)$-toposes have strict univalent universes, 2019.

Appendix: U is fibrant (sketch)

It suffices to show the following.

Proposition

Evaluation at the **generic point** $\mathsf{U}^{\mathbb{I}} \longrightarrow \mathsf{U}$ *is a trivial fibration.*

Proof.

We need a diagonal filler for any cofibration c.

$$\begin{tikzcd} C' \arrow[r, "a"] \arrow[d, tail, "c"'] & \mathsf{U}^{\mathbb{I}} \arrow[d, "\mathsf{U}^{\delta}"] \\ C \arrow[r, "b"'] \arrow[ur, dotted] & \mathsf{U} \end{tikzcd}$$

Appendix: U is fibrant (sketch)

Transposing by $\mathbb{I}$ and using the classifying property of U gives the following equivalent problem.

$$
\begin{tikzcd}
 & A_0 \arrow[rr] \arrow[d, two heads] \arrow[ddl] & & A \arrow[d, two heads] \arrow[ddl, dotted] \\
 & C' \arrow[rr, "c'_0"'] \arrow[ddl, tail, "c"] & & C' \times \mathbb{I} \arrow[ddl, tail, "c\times\mathbb{I}"] \\
B \arrow[rr, dotted] \arrow[d, two heads] & & D \arrow[d, dotted, two heads] & \\
C \arrow[rr, "c_0"'] & & C \times \mathbb{I} &
\end{tikzcd}
$$

Appendix: U is fibrant (sketch)

Apply the functor $(-) \times \mathbb{I}$ to the left face to get:

$$
\begin{tikzcd}
 & A_0 \arrow[rr] \arrow[d, two heads] \arrow[ddl] & & A \arrow[d, two heads] \arrow[ddl, dotted] & & A_0 \times \mathbb{I} \arrow[dll, two heads] \arrow[ddl] \\
 & C' \arrow[rr, "C'_0"'] \arrow[ddl, tail, "c"] & & C' \times \mathbb{I} \arrow[ddl, tail] & & \\
B \arrow[rr, dotted] \arrow[d, two heads] & & D \arrow[d, dotted, two heads] & & B \times \mathbb{I} \arrow[dll, two heads] & \\
C \arrow[rr, "C_0"'] & & C \times \mathbb{I} & & &
\end{tikzcd}
$$

Appendix: U is fibrant (sketch)

Apply the functor $(-) \times \mathbb{I}$ to the left face to get:

$$
\begin{tikzcd}
 & A_0 \arrow[rr] \arrow[d, two heads] \arrow[ddl] & & A \arrow[rr, blue, "e", "\sim"'] \arrow[d, two heads] \arrow[ddl, dotted] & & A_0 \times \mathbb{I} \arrow[dll] \arrow[ddl] \\
 & C' \arrow[rr, "C_0'"'] \arrow[ddl, tail, "c"] & & C' \times \mathbb{I} \arrow[ddl, tail] & & \\
B \arrow[rr, dotted] \arrow[d, two heads] & & D \arrow[d, dotted, two heads] & & B \times \mathbb{I} \arrow[dll, two heads] & \\
C \arrow[rr, "C_0"'] & & C \times \mathbb{I} & & &
\end{tikzcd}
$$

There is a weak equivalence $e : A \xrightarrow{\sim} A_0 \times \mathbb{I}$ to which we can apply the EEP.

Appendix: U is fibrant (sketch)

Apply the functor $(-) \times \mathbb{I}$ to the left face to get:

$$
\begin{tikzcd}
 & A_0 \arrow[rr] \arrow[ddl] \arrow[d, two heads] & & A \arrow[rr, "e", "\sim"'] \arrow[ddl] \arrow[d, two heads] & & A_0 \times \mathbb{I} \arrow[dll, two heads] \arrow[ddl] \\
 & C' \arrow[rr, "C_0'"'] \arrow[ddl, tail, "c"] & & C' \times \mathbb{I} \arrow[ddl, tail] & & \\
B \arrow[rr, dotted] \arrow[d, two heads] & & D \arrow[rr, "\sim"'] \arrow[d, two heads] & & B \times \mathbb{I} \arrow[dll, two heads] & \\
C \arrow[rr, "C_0"'] & & C \times \mathbb{I} & & &
\end{tikzcd}
$$

There is a weak equivalence $e : A \simeq A_0 \times \mathbb{I}$ to which we can apply the EEP. This produces the required fibration $D \twoheadrightarrow Z \times \mathbb{I}$. $\square$