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Flat distributive lattices are trivial

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For any algebras A and B in a variety $\mathcal{V}$ the tensor product $A \otimes B$ can be constructed, and has the defining property that any bi-homomorphism from $A \times B$ to any algebra in $\mathcal{V}$ factors uniquely through $A \otimes B$ (see [8]). An algebra $A \in \mathcal{V}$ is called *flat* if the functor $A\otimes -$ preserves monomorphisms in $\mathcal{V}$ (see [2], [3], [5], [6]). In this note we investigate flatness in the variety $\mathcal{D}$ of distributive lattices.

Suppose A and B are distributive lattices. A subset F of $A \times B$ will be called a *bi-filter* of $A \times B$ if, considering A and B as meet-semilattices, it is a bi-filter in the sense of Kimura (see [7]). A non-empty proper bi-filter P of $A \times B$ is called a *prime bi-filter* of $A \times B$ if $(a, b \vee b') \in P$ implies $(a, b) \in P$ or $(a, b') \in P$ ($a \in A$, b, $b' \in B$), and $(a \vee a', b) \in P$ implies $(a, b) \in P$ or $(a', b) \in P$ ($a, a' \in A$, $b \in B$). It is an easy matter to check that there is a 1–1 correspondence between the class of all bi-homomorphisms from $A \times B$ to the 2-element distributive lattice and the class consisting of all prime bi-filters on $A \times B$, together with $A \times B$ itself and the empty set. Furthermore, for a_i, $a_j' \in A$ and b_i, $b_j' \in B$ $(1 \leq i \leq m, 1 \leq j \leq n)$ $\bigwedge_{i=1}^m a_i \otimes b_i \leq \bigvee_{j=1}^n a_j' \otimes b_j'$ in $A \otimes B$ iff, for every prime bi-filter P of $A \times B$, $(a_i, b_i) \in P$ for all i implies $(a_j', b_j') \in P$ for some j (see [4]).

THEOREM. *A distributive lattice A is flat iff $|A| = 1$.*

Proof. Clearly every one-element distributive lattice A is flat since $A \otimes B \cong B$ for all $B \in \mathcal{D}$.

Now assume $A \in \mathcal{D}$ and $|A| \geq 2$. Then there exist a, $a' \in A$ with $a < a'$. Let B denote the three-element chain $\{0, b, 1\}$ with $0 < b < 1$ and let D represent the diamond $\{0, b, c, 1\}$ with b and c incomparable. We shall show that the map $A \otimes B \to A \otimes D$ induced by the embedding $B \hookrightarrow D$ is not a monomorphism. (In

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[4], Fraser shows this for the case in which A is the two-element chain.) Now, $(a'\otimes b)\wedge(a\otimes 1)\leq(a'\otimes 0)\vee(a\otimes b)$ holds in $A\otimes D$. Indeed, if P is a prime bi-filter of $A\times D$ containing (a', b) and $(a, 1)$ then $(a, b)\vee(a, c)=(a, 1)\in P$. Hence either $(a, b)\in P$ or $(a, c)\in P$. In the latter case $(a', c)\in P$, since $a'>a$, and so $(a', b)\wedge(a', c)=(a', 0)\in P$. Thus either $(a, b)\in P$ or $(a', 0)\in P$ as desired. On the other hand, the inequality $(a'\otimes b)\wedge(a\otimes 1)\leq(a'\otimes 0)\vee(a\otimes b)$ does *not* hold in $A\otimes B$. To see this, let π be a prime filter of A which contains a' but excludes a. Then, it is not difficult to verify that $A\times\{1\}\cup\pi\times\{b\}$ is a prime bi-filter of $A\times B$ which contains both (a', b) and $(a, 1)$, but excludes both $(a', 0)$ and (a, b). Therefore the induced map $A\otimes B\to A\otimes D$ is not a monomorphism and A is not flat.

Remarks. (1) The (dual of the) prime ideal theorem for distributive lattices ([1], p. 70) was employed in the above proof. There is no analogue of this result in the bi-filter situation. Let A be the two-element chain $\{0, 1\}$ and let D be the diamond $\{0, b, c, 1\}$ as above. Then $I=\{(0,0), (1,0), (0,b)\}$ is a bi-ideal of $A\times D$ and $F=\{(0,1), (1,b), (1,1)\}$ is a bi-filter of $A\times D$ with $I\cap F=\emptyset$. However there is no prime bi-filter of $A\times D$ which contains F and is disjoint from I.

(2) The tensor product construction employed in this note and in the references is of little interest in the variety $\mathcal{D}_{01}$ of bounded distributive lattices with bound preserving homomorphisms. Indeed, if $A, B, C\in\mathcal{D}_{01}$ and $\phi: A\times B\to C$ is a bi-homomorphism, then $|C|=1$ because $0=\phi(1,0)=1$ (since $\phi(1,-)$ and $\phi(-,0)$ are bound preserving). Thus $|A\otimes B|=1$ for every A, $B\in\mathcal{D}_{01}$. Every algebra in $\mathcal{D}_{01}$ is therefore (trivially) flat. Similar considerations also apply to the variety of Boolean Algebras.

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