Notes on Kan composition versus Kan filling for cartesian cubical sets

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1 The fibration weak factorization system

We first recall from [Awo21] the basic definitions of the *fibration weak factorization system* for cartesian cubical sets, defined in terms of (partial) open box filling. As a warm-up, we recall the specification from [GS17] for Dedekind cubes, where the interval I has connections, which we call "biased filling". In [AGH21] it is shown that these fibrations agree with those specified in the "logical style" of [CCHM16, OP17]. We then define an "unbiased" version that results in more trivial cofibrations and is more appropriate to the cartesian setting.

In the following, recall from [Awo21] that a $+$-*algebra* is an algebra for the pointed polynomial endofunctor on cSet determined by the cofibration classifier $1 \to \Phi$, vis.

$$X^+ = \sum_{\varphi:\Phi} X^{\{\varphi\}}\,,$$

and a map $Y \to X$ is a (relative) +-algebra if it is a +-algebra in $\mathsf{cSet}/_X$ for the relative version of the same. The relative +-algebras form the right class of the cofibration-trivial fibration (algebraic) weak factorization system.

1.1 Partial box filling (biased version)

A *generating class of biased trivial cofibrations* are all maps of the form

$$c \otimes \delta_\epsilon : D \to Z \times \mathrm{I}\,, \tag{1}$$

where:

1. $c : C \rightarrowtail Z$ is an arbitrary cofibration,
2. $\delta_\epsilon : 1 \to \mathrm{I}$ is one of the two "endpoint inclusions" where, recall, $1 = y[0]$, and $I = y[1]$, and for $\epsilon = 0, 1$, we have the maps $\delta_\epsilon : 1 \to \mathrm{I}$ corresponding to the two bipointed maps $0, 1 : \{0, x, 1\} \to \{0, 1\}$.
3. $c \otimes \delta_\epsilon$ is the pushout-product (resp. "Leibniz tensor") of the cofibration $c : C \rightarrowtail Z$ and an endpoint $\delta_\epsilon : 1 \to \mathrm{I}$, as indicated in the following diagram (in which the unlabelled maps are the expected ones).

$$\begin{tikzcd}
C \times 1 \arrow[r] \arrow[d] & C \times \mathrm{I} \arrow[d] \arrow[ddr, bend left] & \\
Z \times 1 \arrow[r] \arrow[drr, bend right] & Z +_C (C \times \mathrm{I}) \arrow[dr, "c\otimes\delta_\epsilon"'] & \\
 & & Z \times \mathrm{I}
\end{tikzcd} \tag{2}$$

4. $D = Z +_C (C \times \mathrm{I})$ is the indicated pushout, the domain of $c \otimes \delta_\epsilon$.

Recall that we are assuming the axioms for cofibrations:

(C1) All isomorphisms are cofibrations.

(C2) The composite of two cofibrations is a cofibration.

(C3) Cofibrations are monomorphisms.

(C4) Any pullback of a cofibration is a cofibration.

(C5) The endpoint inclusions $\delta_\epsilon : 1 \to \mathrm{I}$ are cofibrations.

(C6) The cofibrations are closed under pushout-products.

Note that since δ_0 and δ_1 are disjoint, by (C5) we have that $0 \to 1$ is a cofibration, and hence that $0 \to A$ is a cofibration, for all objects A.

In place of (C6), we could equivantly require the cofibrations to be closed under the join operation $A \vee B$ in the lattice of subobjects of an object (as is done in [CCHM16, OP17]).

Fibrations (biased version). Let

$$\mathcal{C} \otimes \delta_\epsilon \;=\; \{c \otimes \delta_\epsilon : D \rightarrowtail Z \times \mathrm{I} \mid c \in \mathcal{C},\ \epsilon = 0, 1\}$$

be the class of all such pushout-products of arbitrary cofibrations $c : C \rightarrowtail Z$ with endpoint inclusions $\delta_\epsilon : 1 \rightarrowtail \mathrm{I}$. The *biased fibrations* are defined to be the right class of these generating trivial cofibrations,

$$(\mathcal{C} \otimes \delta_\epsilon)^{\pitchfork} \;=\; \mathcal{F}\,.$$

Thus a map $f : Y \to X$ is a biased fibration if for every commutative square of the form

$$\begin{tikzcd} Z +_C (C \times \mathrm{I}) \arrow[r] \arrow[d, tail, "c\otimes\delta_\epsilon"'] & Y \arrow[d, "f"] \\ Z \times \mathrm{I} \arrow[r] \arrow[ur, dotted, "j"] & X \end{tikzcd} \tag{3}$$

with a generating trivial cofibration on the left, there is a diagonal filler j as indicated. This condition can be seen as a generalized homotopy lifting property.

To relate this notion of fibration to the cofibration weak factorization system, fix any map $u : A \to B$, and recall (e.g. from [**?**]) that the pushout-product with u is a functor on the arrow category

$$(-)\otimes u : \mathsf{cSet}^{\mathbb{2}} \to \mathsf{cSet}^{\mathbb{2}}\,.$$

This functor has a right adjoint, the *pullback-hom* (or "Leibniz exponential"), which for a map $f : X \to Y$ we will write as

$$(u \Rightarrow f) : Y^B \to (X^B \times_{X^A} Y^A)\,.$$

The pullback-hom is determined as indicated in the following diagram (in which the unlabelled maps are the expected ones).

$$
\begin{tikzcd}
Y^B \arrow[dr, "u \Rightarrow f"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X^B \times_{X^A} Y^A \arrow[r] \arrow[d] & Y^A \arrow[d] \\
& X^B \arrow[r] & X^A
\end{tikzcd} \tag{4}
$$

The $\otimes \dashv \Rightarrow$ adjunction on the arrow category has the following useful relation to weak factorization systems (cf. [GS17, Rie14, ?]). For any maps $a : A \to B$ and $f : X \to Y$ we write

$$a \pitchfork f$$

to mean that for every solid square of the form

$$
\begin{tikzcd}
A \arrow[r] \arrow[d, "a"'] & X \arrow[d, "f"] \\
B \arrow[r] \arrow[ur, dotted] & Y
\end{tikzcd} \tag{5}
$$

there exists a dotted diagonal filler as indicated.

Lemma 1. *For any maps $a : A_0 \longrightarrow A_1, b : B_0 \longrightarrow B_1, c : C_0 \longrightarrow C_1$ in cSet, there is a logical equivalence between the diagonal filling* conditions*,*

$$(a \otimes b) \pitchfork c \;\Leftrightarrow\; a \pitchfork (b \Rightarrow c).$$

Moreover, there is a natural isomorphism of the associated sets of witnesses,

$$\{j : (a \otimes b) \pitchfork c\} \;\cong\; \{k : a \pitchfork b \Rightarrow c\}\,,$$

which we will not spell out (cf. [Rie14])

Proposition 2. *An object X is fibrant if and only if both of the endpoint projections $X^{\mathrm{I}} \to X$ from the pathspace are (relative) $+$-algebras (over X). More generally, a map $f : Y \to X$ is a fibration iff both of the maps*

$$(\delta_\epsilon \Rightarrow f) : Y^I \to X^I \times_X Y$$

are $+$-algebras (for $\epsilon = 0, 1$).

Proof. The first statement follows from the second, since the pathspace projections $X^{\mathrm{I}} \to X$ are just the maps

$$(\delta_\epsilon \Rightarrow !_X) : X^I \to (1^I \times_1 X) \cong X\,,$$

for $!_X : X \to 1$.

By definition, $f : X \to Y$ is a fibration iff every square of the form

$$\begin{tikzcd} Z +_C (C \times \mathrm{I}) \arrow[r] \arrow[d, tail, "c \otimes \delta_\epsilon"'] & Y \arrow[d, "f"] \\ Z \times \mathrm{I} \arrow[r] \arrow[ur, dotted, "j"] & X\,, \end{tikzcd} \tag{6}$$

with a generating trivial cofibration $c \otimes \delta_\epsilon$ on the left, has is a diagonal filler j as indicated. Briefly,

$$(c \otimes \delta_\epsilon) \pitchfork f \qquad (\text{for } c \in \mathcal{C},\ \epsilon = 0, 1).$$

By the $\otimes \dashv \Rightarrow$ adjunction, this is equivalent to the condition

$$c \pitchfork (\delta_\epsilon \Rightarrow f) \qquad (\text{for } c \in \mathcal{C},\ \epsilon = 0, 1).$$

That is, for every square

$$\begin{tikzcd} C \arrow[r] \arrow[d, tail, "c"'] & Y^I \arrow[d, "\delta_\epsilon \Rightarrow f"] \\ Z \arrow[r] \arrow[ur, dotted, "k"] & X^I \times_X Y\,, \end{tikzcd}$$

with an arbitrary cofibration $c : C \rightarrowtail Z$ on the left, there is a diagonal filler k as indicated, for $\epsilon = 0, 1$. But this is just to say that the maps $\delta_\epsilon \Rightarrow f$ are in the right class of the cofibrations, which is equivalent to their being $+$-algebras, as claimed. □

Fibration structure. The $\otimes \dashv \Rightarrow$ adjunction determines the fibrations in terms of the trivial fibrations, which in turn can be determined by *uniform* lifting against a *set* of basic cofibrations, by proposition **??**. We can similarly determine the fibrations by uniform lifting against a *set* of trivial cofibrations, consisting of all those $c \otimes \delta_\epsilon$ in $\mathcal{C} \otimes \delta_\epsilon$ where $c : C \rightarrowtail Z$ has a representable codomain $Z = \mathrm{I}^n$. Call these maps the *basic (biased) trivial cofibrations*, and let

$$\mathcal{B} \otimes \delta_\epsilon = \{c \otimes \delta_\epsilon : B \rightarrowtail \mathrm{I}^{n+1} \mid c : C \rightarrowtail \mathrm{I}^n,\ \epsilon = 0, 1,\ n \geq 0\}, \tag{7}$$

where the pushout-product $c \otimes \delta_\epsilon$ now takes the simpler form

$$\begin{tikzcd} C \arrow[r] \arrow[d, tail] & C \times \mathrm{I} \arrow[d] \arrow[ddr, bend left] & \\ \mathrm{I}^n \arrow[r] \arrow[drr, bend right] & \mathrm{I}^n +_C (C \times \mathrm{I}) \arrow[dr, tail, "c\otimes\delta_\epsilon"'] & \\ & & \mathrm{I}^n \times \mathrm{I} \end{tikzcd} \tag{8}$$

for a cofibration $c : C \rightarrowtail \mathrm{I}^n$, an endpoint $\delta_\epsilon : 1 \to \mathrm{I}$, and with domain $B = \big(\mathrm{I}^n +_C (C \times \mathrm{I})\big)$. These subobjects $B \rightarrowtail \mathrm{I}^{n+1}$ can be seen geometrically as generalized open box inclusions.

For any map $f : Y \to X$ a *(uniform, biased) fibration structure* on f is a choice of diagonal fillers $j_\epsilon(c, x, y)$,

$$\begin{tikzcd} \mathrm{I}^n +_C (C \times \mathrm{I}) \arrow[r, "x"] \arrow[d, "c\otimes\delta_\epsilon"'] & X \arrow[d, "f"] \\ \mathrm{I}^n \times \mathrm{I} \arrow[r, "y"'] \arrow[ur, dotted, "j_\epsilon(c,x,y)"'] & Y, \end{tikzcd} \tag{9}$$

for each basic trivial cofibration $c \otimes \delta_\epsilon : B = (\mathrm{I}^n +_C (C \times \mathrm{I})) \rightarrowtail \mathrm{I}^{n+1}$ and maps $x : B \to X$ and $y : \mathrm{I}^{n+1} \to Y$, which is *uniform in* I^n in the following sense: given any cubical map $u : \mathrm{I}^m \to \mathrm{I}^n$, the pullback $u^*c : u^*C \rightarrowtail \mathrm{I}^m$ of $c : C \rightarrowtail \mathrm{I}^n$ along u determines another basic trivial cofibration

$$u^*c \otimes \delta_\epsilon : B' = (\mathrm{I}^m +_{u^*C} (u^*C \times \mathrm{I})) \rightarrowtail \mathrm{I}^{m+1},$$

which fits into a commutative diagram of the form

$$\begin{tikzcd} \mathrm{I}^m +_{u^*C} (u^*C \times \mathrm{I}) \arrow[r, "(u\times\mathrm{I})'"] \arrow[d, "u^*c\otimes\delta_\epsilon"'] & \mathrm{I}^n +_C (C \times \mathrm{I}) \arrow[r, "x"] \arrow[d, "c\otimes\delta_\epsilon"'] & X \arrow[d, "f"] \\ \mathrm{I}^m \times \mathrm{I} \arrow[r, "u\times\mathrm{I}"'] & \mathrm{I}^n \times \mathrm{I} \arrow[r, "y"'] \arrow[ur, dotted, "j_\epsilon(c,x,y)"'] & Y, \end{tikzcd} \tag{10}$$

by applying the functor $(-) \otimes \delta_\epsilon$ to the pullback square relating u^*c to c. Now for the outer rectangle in (10) there is a chosen diagonal filler

$$j_\epsilon(u^*c, x(u \times \mathrm{I})', y(u \times \mathrm{I})) : \mathrm{I}^m \times \mathrm{I} \to X$$

and for this map we require that

$$j_\epsilon(u^*c, x(u \times \mathrm{I})', y(u \times \mathrm{I})) = j_\epsilon(c, x, y) \circ (u \times \mathrm{I}). \tag{11}$$

This is a reformulation of the logical specification given in [CCHM16, OP17] (cf. [AGH21]).

Definition 3. A *(uniform, biased) fibration structure* on a map $f : Y \to X$ is a choice of fillers $j_\epsilon(c, x, y)$ as in (12) and satisfying (14) for all maps $u : \mathrm{I}^m \to \mathrm{I}^n$.

Essentially the same argument as that given for Proposition 2 also yields the following sharper formulation in terms of fibration structure.

Corollary 4. *Fibration structure on a map $f : Y \to X$ is equivalent to a pair of $+$-algebra structures on the maps*

$$(\delta_\epsilon \Rightarrow f) : Y^I \to X^I \times_X Y$$

for $\epsilon = 0, 1$.

Finally, we have the analogue of proposition **??** for fibrant objects; we omit the analogous statement of proposition **??** for fibrations, as well as the entirely analogous proof.

Corollary 5. *For any object X in* cSet *the following are equivalent:*

1. *X is* fibrant*, i.e. every partial map to X with a generating trivial cofibration $D \rightarrowtail Z \times \mathrm{I}$ as domain of definition extends to a total map $Z \times \mathrm{I} \to X$,*
$$\mathcal{C} \otimes \delta_\epsilon \pitchfork f$$
2. *The canonical maps*
$$(\delta_\epsilon \Rightarrow X) : X^I \to X,$$
for $\epsilon = 0, 1$, can be equipped with $+$-algebra structures.
3. *$X \to 1$ has a* (uniform, biased) fibration structure*. Explicitly, for each basic trivial cofibration $c \otimes \delta_\epsilon : B \rightarrowtail \mathrm{I}^{n+1}$ and map $x : B \to X$, there is given an extension $j_\epsilon(c, x)$,*

$$\begin{tikzcd} B \arrow[r, "x"] \arrow[d, tail, "c\otimes\delta_\epsilon"'] & X, \\ \mathrm{I}^{n+1} \arrow[ur, dotted, "j_\epsilon(c,x)"'] & \end{tikzcd} \tag{12}$$

and the choice is uniform in I^n *in the sense: given any cubical map* $u : \mathrm{I}^m \to \mathrm{I}^n$*, the pullback* $u^*c \otimes \delta_\epsilon : B' \rightarrowtail \mathrm{I}^m \times \mathrm{I}$ *fits into a commutative diagram of the form*

$$\begin{tikzcd} B' \arrow[r, "(u\times\mathrm{I})'"] \arrow[d, tail, "u^*c\otimes\delta_\epsilon"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[r, "x"] \arrow[d, tail, "c\otimes\delta_\epsilon"'] & X. \\ \mathrm{I}^m \times \mathrm{I} \arrow[r, "u\times\mathrm{I}"'] & \mathrm{I}^n \times \mathrm{I} \arrow[ur, dotted, "j(c,x)"'] & \end{tikzcd} \tag{13}$$

Then for the pair $(u^*c \otimes \delta_\epsilon,\ x(u \times \mathrm{I})')$ *in* (13) *the chosen extension*

$$j(u^*c \otimes \delta_\epsilon, x(u \times \mathrm{I})') : \mathrm{I}^m \times \mathrm{I} \to X$$

is equal to $j(c, x) \circ (u \times \mathrm{I})$,

$$j(u^*c \otimes \delta_\epsilon, x(u \times \mathrm{I})') = j(c, x)(u \times \mathrm{I}). \tag{14}$$

1.2 Partial box filling (unbiased version)

We now eliminate the "bias" on a choice of endpoint $\delta_\epsilon : 1 \to \mathrm{I}$, expressed by the indexing $\epsilon = 0, 1$. This will have the effect of adding more trivial cofibrations, and thus more weak equivalences, to our model structure. Consider first the simple path-lifting condition, which is a special case of (3) with $c = ! : 0 \rightarrowtail 1$, since $! \otimes \delta_\epsilon = \delta_\epsilon$:

$$\begin{tikzcd} 1 \arrow[r] \arrow[d, tail, "\delta_\epsilon"'] & Y \arrow[d, "f"] \\ \mathrm{I} \arrow[r] \arrow[ur, dotted, "j_\epsilon"'] & X\,. \end{tikzcd}$$

(Note that $0 \rightarrowtail 1$ is a cofibration by axioms C4 and C5).

In toplogical spaces, rather than requiring lifts j_ϵ for each of the endpoints $\epsilon = 0, 1$, we could instead (and equivalently!) require a lift j_i for each point $i : 1 \to \mathrm{I}$ in the real interval $\mathrm{I} = [0, 1]$. Such "unbiased path-lifting" can be formulated in cSet by introducing a "generic point" $\delta : 1 \to \mathrm{I}$ by passing to cSet/I, and then requiring path-lifting with respect to δ. The following specification implements that idea, while also adding partiality in the sense of the foregoing section. We then need to strengthen axiom C5 to the following.

(C5') The diagonal map $\delta : \mathrm{I} \to \mathrm{I} \times \mathrm{I}$ is a cofibration.

Definition 6 (Fibration)**.** Let $\delta : \mathrm{I} \to \mathrm{I} \times \mathrm{I}$ be the diagonal map.

1. An object X is *(unbiased) fibrant* if the map

$$(\delta \Rightarrow X) = \langle \mathsf{eval}, p_2 \rangle : X^{\mathrm{I}} \times \mathrm{I} \to X \times \mathrm{I}$$

is a $+$-algebra.

2. A map $f : Y \to X$ is an *(unbiased) fibration* if the map

$$(\delta \Rightarrow f) = \langle f^{\mathrm{I}} \times \mathrm{I}, \langle \mathsf{eval}, p_2 \rangle \rangle : Y^{\mathrm{I}} \times \mathrm{I} \to (X^{\mathrm{I}} \times \mathrm{I}) \times_{(X \times \mathrm{I})} (Y \times \mathrm{I})$$

is a $+$-algebra.

Condition (1) above says that evaluation at the generic point $\delta : 1 \to \mathrm{I}$, i.e. the map $X^{\delta} : X^{\mathrm{I}} \to X$ constructed in the slice category cSet/I, is a $+$-algebra. Condition (2) says that the pullback-hom of the generic point $\delta : 1 \to \mathrm{I}$ with $\mathrm{I}^* f$, constructed in the slice category cSet/I, is a $+$-algebra. The latter can be reformulated as follows.

Proposition 7. *A map $f : Y \to X$ is an (unbiased) fibration if and only if in the following diagram the canonical map c to the pullback is a $+$-algebra.*

$$\begin{tikzcd}
Y^{\mathrm{I}} \times \mathrm{I} \arrow[dr, dotted, "c"] \arrow[drr, bend left, "\mathsf{eval}"] \arrow[ddr, bend right, "f^{\mathrm{I}} \times \mathrm{I}"'] & & \\
& Y_{\mathsf{eval}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "f"] \\
& X^{\mathrm{I}} \times \mathrm{I} \arrow[r, "\mathsf{eval}"'] & X.
\end{tikzcd} \tag{15}$$

Proof. We interpolate another pullback into the rectangle in (15) to obtain

$$\begin{tikzcd}
Y_{\mathsf{eval}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \times \mathrm{I} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "f"] \\
X^{\mathrm{I}} \times \mathrm{I} \arrow[r] & X \times \mathrm{I} \arrow[r] & X
\end{tikzcd} \tag{16}$$

with the evident maps. The lefthand square is therefore a pullback, so we indeed have that

$$Y_{\mathsf{eval}} \;=\; (X^{\mathrm{I}} \times \mathrm{I}) \times_{(X \times \mathrm{I})} (Y \times \mathrm{I})$$

and $c = (\delta \Rightarrow f)$. $\square$

Now we can run the proof of Proposition 2 backwards in order to determine a class of generating trivial cofibrations for the unbiased case. We consider pairs of maps $c : C \rightarrowtail Z$ and $z : Z \to \mathrm{I}$, where the former is a cofibration and the latter is regarded as an "I-indexing", so that

$$\begin{tikzcd} C \arrow[r, tail, "c"] \arrow[dr] & Z \arrow[d, "z"] \\ & \mathrm{I} \end{tikzcd}$$

can be regarded as an I-indexed family of cofibrations. Let

$$\mathsf{Gph}(z) : Z \to Z \times \mathrm{I}\,,$$

be the graph of $z : Z \to \mathrm{I}$, i.e. $\mathsf{Gph}(z) = \langle 1_Z, z \rangle$, and then let

$$c \otimes_z \delta := [\mathsf{Gph}(z), c \times \mathrm{I}] : Z +_C (C \times \mathrm{I}) \to Z \times \mathrm{I}\,,$$

which is easily seen to be well-defined on the indicated pushout.

$$\begin{tikzcd} C \arrow[r, "\mathsf{Gph}(zc)"] \arrow[d, tail, "c"'] & C \times \mathrm{I} \arrow[d] \arrow[ddr, bend left, "c\times\mathrm{I}"] & \\ Z \arrow[r] \arrow[drr, bend right, "\mathsf{Gph}(z)"'] & Z +_C (C \times \mathrm{I}) \arrow[dr, dotted, "c\otimes_z\delta"'] & \\ & & Z \times \mathrm{I}\,. \end{tikzcd} \tag{17}$$

This specification differs from the similar (2) by using $\mathsf{Gph}(z)$ for the inclusion $Z \rightarrowtail Z \times \mathrm{I}$, rather than one of the "face maps" associated to the endpoint inclusions $\delta_\epsilon : 1 \to \mathrm{I}$. (Note that a graph is always a cofibration by pulling back a diagonal.) The subobject $c \otimes_z \delta \rightarrowtail Z \times \mathrm{I}$ is the join of the subobjects $\mathsf{Gph}(z) \rightarrowtail Z \times \mathrm{I}$ and the cylinder $C \times \mathrm{I} \rightarrowtail Z \times \mathrm{I}$.

Observe that the endpoints $\delta_\epsilon : 1 \to \mathrm{I}$ are of the form $c \otimes_z \delta$ by taking $Z = 1$ and $z = \delta_\epsilon$ and $c = \,! : 0 \to 1$, so that biased filling is subsumed.

The maps of the form $c \otimes_z \delta : Z +_C (C \times \mathrm{I}) \rightarrowtail Z$ now form a *class of generating trivial cofibrations* in the expected sense. Let

$$\mathcal{C} \otimes \delta \;=\; \{c \otimes_z \delta : D \rightarrowtail Z \times \mathrm{I} \mid c : C \rightarrowtail Z, z : Z \to \mathrm{I}\}\,, \tag{18}$$

then the fibrations are exactly the right class of these,

$$\mathcal{F} = (\mathcal{C} \otimes \delta)^{\pitchfork}.$$

Proposition 8. *A map $f : Y \to X$ is an (unbiased) fibration iff for every pair of maps $c : C \rightarrowtail Z$ and $z : Z \to \mathrm{I}$, where the former is a cofibration, every commutative square of the following form has a diagonal filler, as indicated.*

$$
\begin{tikzcd}
Z +_C (C \times \mathrm{I}) \arrow[r] \arrow[d, tail, "c \otimes_z \delta"'] & Y \arrow[d, "f"] \\
Z \times \mathrm{I} \arrow[r] \arrow[ur, dotted, "j"] & X.
\end{tikzcd}
\tag{19}
$$

Proof. Suppose that for all $c : C \rightarrowtail Z$ and $z : Z \to \mathrm{I}$, we have $(c \otimes_z \delta) \pitchfork f$ in cSet. Pulling f back over I, this is equivalent to the condition $c \otimes \delta \pitchfork \mathrm{I}^* f$ in cSet/I, for all cofibrations $c : C \rightarrowtail Z$ over I, which is equivalent to $c \pitchfork (\delta \Rightarrow \mathrm{I}^* f)$ in cSet/I for all cofibrations $c : C \rightarrowtail Z$. But this in turn means that $\delta \Rightarrow \mathrm{I}^* f$ is a $+$-algebra, which by definition means that f is a fibration. □

Unbiased fibration structure. As in the biased case, the fibrations can also be determined by *uniform* right-lifting against a generating *set* of trivial cofibrations, now consisting of all those $c \otimes_z \delta$ in $\mathcal{C} \otimes \delta$ for which $c : C \rightarrowtail Z$ has a representable codomain $Z = \mathrm{I}^n$. Call these maps the *basic (unbiased) trivial cofibrations*, and let

$$
\mathcal{B} \otimes \delta = \{c \otimes_z \delta : B \rightarrowtail \mathrm{I}^{n+1} \mid c : C \rightarrowtail \mathrm{I}^n,\ z : \mathrm{I}^n \to \mathrm{I},\ n \geq 0\}\,, \tag{20}
$$

where the pushout-product $c \otimes_z \delta$ now has the form

$$
\begin{tikzcd}
C \arrow[r, "\mathsf{Gph}(zc)"] \arrow[d, tail, "c"'] & C \times \mathrm{I} \arrow[d] \arrow[ddr, bend left, "c \times \mathrm{I}"] & \\
\mathrm{I}^n \arrow[r] \arrow[drr, bend right, "\mathsf{Gph}(z)"'] & \mathrm{I}^n +_C (C \times \mathrm{I}) \arrow[dr, dotted, tail, "c \otimes_z \delta"] & \\
& & \mathrm{I}^n \times \mathrm{I}\,.
\end{tikzcd}
\tag{21}
$$

for a cofibration $c : C \rightarrowtail \mathrm{I}^n$, an indexing map $z : \mathrm{I}^n \to \mathrm{I}$, and with domain $B = \big(\mathrm{I}^n +_C (C \times \mathrm{I})\big)$. These subobjects $B \rightarrowtail \mathrm{I}^{n+1}$ can again be seen geometrically as "generalized open box" inclusions, but now the floor or lid of the open box may be replaced by a "cross-section" given by the graph of a map $z : \mathrm{I}^n \to \mathrm{I}$.

For any map $f : Y \to X$ a (uniform, unbiased) fibration structure on f is a choice of diagonal fillers $j(c, z, x, y)$,

$$\begin{tikzcd} B \arrow[r, "x"] \arrow[d, tail, "c\otimes_z\delta"'] & X \arrow[d, "f"] \\ \mathrm{I}^n \times \mathrm{I} \arrow[r, "y"'] \arrow[ur, dotted, "j(c,z,x,y)"'] & Y, \end{tikzcd} \tag{22}$$

for each basic trivial cofibration $c \otimes_z \delta : B \rightarrowtail \mathrm{I}^{n+1}$, which is *uniform* in I^n in the following sense: given any cubical map $u : \mathrm{I}^m \to \mathrm{I}^n$, the pullback $u^*c : u^*C \rightarrowtail \mathrm{I}^m$ and the reindexing $zu : \mathrm{I}^m \to \mathrm{I}^n \to \mathrm{I}$ determine another basic trivial cofibration $u^*c \otimes_{zu} \delta : B' = (\mathrm{I}^m +_{u^*C} (u^*C \times \mathrm{I})) \rightarrowtail \mathrm{I}^{m+1}$ which fits into a commutative diagram of the form

$$\begin{tikzcd} B' \arrow[r, "(u\times\mathrm{I})'"] \arrow[d, tail, "u^*c\otimes_{zu}\delta"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[r, "x"] \arrow[d, tail, "c\otimes_z\delta"'] & X \arrow[d, "f"] \\ \mathrm{I}^m \times \mathrm{I} \arrow[r, "u\times\mathrm{I}"'] & \mathrm{I}^n \times \mathrm{I} \arrow[r, "y"'] \arrow[ur, dotted, "j(c,z,x,y)"'] & Y. \end{tikzcd} \tag{23}$$

For the outer rectangle in (23) there is a chosen diagonal filler

$$j(u^*c, zu, x(u \times \mathrm{I})', y(u \times \mathrm{I})) : \mathrm{I}^m \times \mathrm{I} \to X,$$

and for this map we require that

$$j(u^*c, zu, x(u \times \mathrm{I})', y(u \times \mathrm{I})) = j(c, z, x, y) \circ (u \times \mathrm{I}). \tag{24}$$

Definition 9. A *(uniform, unbiased) fibration structure* on a map

$$f : Y \to X$$

is a choice of fillers $j(c, z, x, y)$ as in (22) satisfying (24) for all $u : \mathrm{I}^m \to \mathrm{I}^n$.

In these terms, we have following analogue of corollary 5.

Proposition 10. *For any object X in* cSet *the following are equivalent:*

1. *the canonical map $X^{\mathrm{I}} \times \mathrm{I} \to X \times \mathrm{I}$ is a trivial fibration.*
2. *X has the right lifting property with respect to all generating trivial cofibrations,*

$$(\mathcal{C} \otimes_z \delta) \pitchfork X.$$

3. X has a uniform fibration structure in the sense of Definition 9.

Proof. The equivalence between (1) and (2) is proposition 8. Suppose (1), i.e. that the map

$$(\delta \Rightarrow X) : X^{\mathrm{I}} \times \mathrm{I} \to X \times \mathrm{I}$$

is a relative $+$-algebra over $X \times \mathrm{I}$. By proposition **??**, this means that $(\delta \Rightarrow X)$, as an object of $\mathsf{cSet}/(X \times \mathrm{I})$, has a uniform filling structure with respect to all cofibrations $c : C \rightarrowtail \mathrm{I}^n$ over $(X \times \mathrm{I})$. Transposing by the $\otimes \dashv \Rightarrow$ adjunction and unwinding gives, equivalently, a uniform fibration structure on X. $\square$

A statement analogous to the foregoing also holds for maps $f : Y \to X$ in place of objects X. Indeed, as before, we have the following sharper formulation.

Corollary 11. *Fibration structures on a map $f : Y \to X$ correspond uniquely to $+$-algebra structures on the map $(\delta \Rightarrow f)$ (cf. definition 6),*

$$(\delta \Rightarrow f) : Y^{I} \times \mathrm{I} \to (X^{I} \times \mathrm{I}) \times_{(X \times \mathrm{I})} (Y \times \mathrm{I})\,.$$

2 Kan composition

A novelty of the type-theoretic notion of fibration is the method (due to Coquand and introduced in [CCHM16]) of reducing the (type-theoretically specified) notion of *fibration structure* to the apparently weaker notion of a *composition structure*. Composition structure is more easily shown to be preserved by the type-forming operations like Σ and Π (see [CCHM16, OP17]). We give a reformulation into diagrammatic language of those type-theoretic definitions and constructions. Again we start with the case of an interval I with connections.

2.1 Composition with connections

Definition 12. For points $p, q : 1 \to \mathrm{I}$, a cubical set X *has composition from p to q* if for every object Z and cofibration $c : C \rightarrowtail Z$ and commutative square

$$\begin{tikzcd} C \arrow[r] \arrow[d, tail, "c"'] & X^{\mathrm{I}} \arrow[d, "\epsilon_p"] \\ Z \arrow[r] & X, \end{tikzcd} \tag{25}$$

where $\epsilon_p = X^p : X^{\mathrm{I}} \to X$ is the *evaluation at p* map, there is a diagonal arrow $k : Z \to X \times X$ making both subdiagrams below commute,

$$\begin{tikzcd}
C \arrow[r] \arrow[dd, tail, "c"'] & X^{\mathrm{I}} \arrow[d, "\langle \epsilon_p, \epsilon_q \rangle"] \\
& X \times X \arrow[d, "\pi_1"] \\
Z \arrow[r] \arrow[ur, dotted, "k"] & X,
\end{tikzcd} \tag{26}$$

where $\epsilon_q : X^{\mathrm{I}} \to X$ is evaluation at q.

Proposition 13. *In cubical sets with connections, if an object X has composition from δ_0 to δ_1 and back, then X has filling for all trivial cofibrations $c \otimes \delta : B \rightarrowtail Z \times \mathrm{I}$, where $c : C \rightarrowtail Z$ is any cofibration and $\delta = \delta_0, \delta_1 : 1 \to \mathrm{I}$.*

An object X has filling for all trivial cofibrations $c \otimes \delta : B \rightarrowtail Z \times \mathrm{I}$ iff for all cofibrations $c : C \rightarrowtail Z$ and squares as below there is a diagonal filler

$$\begin{tikzcd}
C \arrow[r] \arrow[d, tail, "c"'] & X^{\mathrm{I}} \arrow[d, "\delta \Rightarrow X"] \\
Z \arrow[r] \arrow[ur, dotted] & X
\end{tikzcd}$$

where the Leibniz exponential $\delta \Rightarrow X : X^{\mathrm{I}} \to X$ is "evaluation at the endpoint $\delta : 1 \to \mathrm{I}$" (and we require the condition for both endpoints $\delta = \delta_0, \delta_1$). Clearly if X has filling then it has composition, since there is then a diagonal filler k making both subdiagrams commute in

$$\begin{tikzcd}
C \arrow[r] \arrow[dd, tail, "c"'] & X^{\mathrm{I}} \arrow[d, "\partial \Rightarrow X"] \\
& X \times X \arrow[d] \\
Z \arrow[r] \arrow[ur, dotted, "k"] & X
\end{tikzcd}$$

where $(\partial \Rightarrow X) : X^{\mathrm{I}} \to X \times X$ is the Leibniz exponential of X by the boundary map $\partial : 1 + 1 \to \mathrm{I}$, and we require the condition for both projections $X \times X \to X$.

Conversely, we can obtain filling for e.g. open 2-boxes from composition as follows: to fill the following open 2-box in X:

$$\begin{tikzcd}
A' & B' \\
A \arrow[u, "a"] \arrow[r, "p"'] & B \arrow[u, "b"']
\end{tikzcd}$$

First make a higher-dimensional composition problem using the connections on the right and left sides, and identities on the front and bottom:

$$\begin{tikzcd}
A' & & & B' \\
& A \arrow[ul, "a"] \arrow[r, "p"] & B \arrow[ur, "b"'] & \\
& A \arrow[u, "="] \arrow[r, "p"'] \arrow[dl, "="'] & B \arrow[u, "="'] \arrow[dr, "="] & \\
A \arrow[uuu, "a"] \arrow[rrr, "p"'] & & & B \arrow[uuu, "b"']
\end{tikzcd}$$

Then since X has composition, the indicated (partial) open 3-box has a top face, which is then a filler for the original open 2-box.

$$\begin{tikzcd}
A' & & & B' \\
& A \arrow[ul, "a"] \arrow[r, "p"] & B \arrow[ur, "b"'] & \\
& A \arrow[u, "="] \arrow[r, "p"'] \arrow[dl, "="'] & B \arrow[u, "="'] \arrow[dr, "="] & \\
A \arrow[uuu, "a"] \arrow[rrr, "p"'] & & & B \arrow[uuu, "b"']
\end{tikzcd}$$

For a general, algebraic proof, first use the connections to get maps in $\mathcal{E}^2$ of the form

$$\begin{tikzcd}
\delta & \delta \arrow[l, "="'] \arrow[d] \\
\delta \otimes \delta \arrow[u] & i \otimes \delta \arrow[l]
\end{tikzcd}$$

where $i : 1 \to 1 + 1$.

Applying the functor $(-) \Rightarrow X$ gives the top square in:

$$\begin{tikzcd}
\delta \Rightarrow X \arrow[r, "="] \arrow[d] & \delta \Rightarrow X \\
\delta \otimes \delta \Rightarrow X \arrow[r] \arrow[d, "\cong"'] & i \otimes \delta \Rightarrow X \arrow[u] \\
\delta \Rightarrow (\delta \Rightarrow X) \arrow[r] & \delta \Rightarrow (i \Rightarrow X) \arrow[u, "\cong"']
\end{tikzcd}$$

while the bottom one is by the $\otimes \dashv \Rightarrow$ adjunction.

So for any cofibration $c : C \rightarrowtail Z$ and filling problem

$$\begin{tikzcd}
C \arrow[r] \arrow[d, tail, "c"'] & X^{\mathrm{I}} \arrow[d, "\delta \Rightarrow X"] \\
Z \arrow[r] & X
\end{tikzcd}$$

we can extend on the right as follows.

$$\begin{tikzcd}
C \arrow[r] \arrow[d, tail, "c"'] & X^{\mathrm{I}} \arrow[r] \arrow[d, "\delta \Rightarrow X"] \arrow[rrr, bend left, "="] & \cdot \arrow[r] \arrow[d, "\delta \Rightarrow (\delta \Rightarrow X)"] & \cdot \arrow[r] \arrow[d, "\delta \Rightarrow (i \Rightarrow X)"] & X^{\mathrm{I}} \arrow[d, "\delta \Rightarrow X"] \\
Z \arrow[r] & X \arrow[r] \arrow[rrr, bend right, "="'] & \cdot \arrow[r] & \cdot \arrow[r] & X
\end{tikzcd}$$

Transposing the left three squares yields

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d, tail, "c \otimes \delta"'] & \cdot \arrow[r] \arrow[d, tail, "(\delta \Rightarrow X) \otimes \delta"'] & X^{\mathrm{I}} \arrow[r] \arrow[d, "\delta \Rightarrow X"] & X^{1+1} \arrow[d, "i \Rightarrow X"] \\
\cdot \arrow[r] & \cdot \arrow[r] & X \arrow[r, "="'] & X
\end{tikzcd}$$

which has a diagonal filler by composition, since $c \otimes \delta$ is also a cofibration.

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d, tail, "c \otimes \delta"'] & \cdot \arrow[r] \arrow[d, tail, "(\delta \Rightarrow X) \otimes \delta"'] & X^{\mathrm{I}} \arrow[r] \arrow[d, "\delta \Rightarrow X"] & X^{1+1} \arrow[d, "i \Rightarrow X"] \\
\cdot \arrow[r] \arrow[rrru, dotted] & \cdot \arrow[r] & X \arrow[r, "="'] & X
\end{tikzcd}$$

Transposing back thus gives a diagonal filler

$$\begin{tikzcd}
C \arrow[r] \arrow[d, tail, "c"'] & X^{\mathrm{I}} \arrow[r] \arrow[d, "\delta\Rightarrow X"] \arrow[rrr, bend left, "="] & \cdot \arrow[r] \arrow[d, "\delta\Rightarrow(\delta\Rightarrow X)"] & \cdot \arrow[r] \arrow[d, "\delta\Rightarrow(i\Rightarrow X)"] & X^{\mathrm{I}} \arrow[d, "\delta\Rightarrow X"] \\
Z \arrow[r] \arrow[urrr, dotted] & X \arrow[r] \arrow[rrr, bend right, "="'] & \cdot \arrow[r] & \cdot \arrow[r] & X
\end{tikzcd}$$

which provides a filler for the original problem

$$\begin{tikzcd}
C \arrow[r] \arrow[d, tail, "c"'] & X^{\mathrm{I}} \arrow[d, "\delta\Rightarrow X"] \\
Z \arrow[r] \arrow[ur, dotted] & X
\end{tikzcd}$$

□

2.2 Composition without connections

Composition for an object

If X is fibrant in the *unbiased* sense of section 1.2 then X will have composition over I from the generic point $\delta : 1 \to \mathrm{I}$ to either of δ_0, δ_1, in the obvious sense, for the same reason as before: the map $(\delta \Rightarrow X) = X^{\delta} : X^{\mathrm{I}} \to X$ is a trivial fibration over I, by the definition of fibrancy. In order to arrive at a notion of composition equivalent to unbiased filling, we use a generic form of composition from δ to a second generic point δ'. This further point δ' is obtained by pulling back along the (say, second) projection $\pi : \mathrm{I} \times \mathrm{I} \to \mathrm{I}$ to work in $\mathsf{cSet}/(\mathrm{I} \times \mathrm{I})$, where in addition to $\delta, \delta_0, \delta_1$ we now also have a point $\delta' : 1 \to \mathrm{I}$, given by the additional diagonal map over $\mathrm{I} \times \mathrm{I}$,

$$\begin{tikzcd}
\mathrm{I}\times\mathrm{I} \arrow[rr, "\langle\pi_1,\pi_2,\pi_1\rangle"] \arrow[dr, "\mathsf{id}"'] & & \mathrm{I}\times\mathrm{I}\times\mathrm{I} \arrow[dl, "\langle\pi_1,\pi_2\rangle"] \\
& \mathrm{I}\times\mathrm{I} & .
\end{tikzcd} \tag{27}$$

Observe that in $\mathsf{cSet}/\mathrm{I}\times\mathrm{I}$ the (binary) diagonal $\Delta : \mathrm{I} \rightarrowtail \mathrm{I}\times\mathrm{I}$ is a subobject of the terminal object $\Delta \rightarrowtail 1$, with associated base change

$$\begin{tikzcd}
\mathsf{cSet}/\mathrm{I} \arrow[r, shift right, "\Delta_*"'] & \mathsf{cSet}/\mathrm{I}\times\mathrm{I} \arrow[l, shift right, "\Delta^*"'] .
\end{tikzcd} \tag{28}$$

For any object X in $\mathsf{cSet}/\mathrm{I}\times\mathrm{I}$, let

$$\eta_X : X \to X^{\Delta}$$

be the unit of $\Delta^* \dashv \Delta_*$. Given objects and arrows $f, g : X \rightrightarrows Y$ in $\mathsf{cSet}/\mathrm{I}\times\mathrm{I}$, observe that $\Delta^* f = \Delta^* g$ in cSet/I if and only if the composites with η_Y are equal,

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "\eta_Y"] & Y^{\Delta}. \end{tikzcd} \tag{29}$$

Indeed, consider the double naturality square

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, "\eta_X"'] & Y \arrow[d, "\eta_Y"] \\ X^{\Delta} \arrow[r, shift left, "f^{\Delta}"] \arrow[r, shift right, "g^{\Delta}"'] & Y^{\Delta}. \end{tikzcd} \tag{30}$$

Then

$$\begin{aligned} \eta_Y f = \eta_Y g \quad &\text{iff} \quad f^{\Delta}\eta_X = g^{\Delta}\eta_X : X \to Y^{\Delta} \\ &\text{iff} \quad \Delta^* f = \Delta^* g : \Delta^* X \to \Delta^* Y \end{aligned}$$

the latter by transposition, since $Y^{\Delta} = \Delta_*\Delta^* Y$. (A dual condition involving the counit $X \times \Delta \to X$ also holds.)

We will define (unbiased) composition for an object X by saying that, over $\mathrm{I}\times\mathrm{I}$, it has composition from δ to δ' in the sense of Definition 12, but we need to be careful about what happens when δ and δ' "coincide". Thus we require that the maps $k_1, k_2 : Z \rightrightarrows X$ where $\langle k_1, k_2\rangle = k : Z \to X \times X$, are coequalized by the unit $\eta_X : X \to X^{\Delta}$. We reformulate this condition equivalently as follows.

Definition 14. An object X in cSet has *(unbiased) composition* if in cSet/I for every cofibration $c : C \rightarrowtail Z$ and every commutative square of the form

$$\begin{tikzcd} C \arrow[r, "y"] \arrow[d, tail, "c"'] & X^{\mathrm{I}} \arrow[d, "\epsilon_\delta"] \\ Z \arrow[r, "x"'] & X, \end{tikzcd}$$

upon pulling back along $\pi : \mathrm{I}\times\mathrm{I} \to \mathrm{I}$ to $\mathsf{cSet}/\mathrm{I}\times\mathrm{I}$, there is a map $x' : Z \to X$ making the following commute,

$$\begin{tikzcd} C \arrow[r, "y"] \arrow[d, tail, "c"'] & X^{\mathrm{I}} \arrow[r, "\epsilon_{\delta'}"] \arrow[d, "\epsilon_\delta"'] & X \arrow[d, "\eta_X"] \\ Z \arrow[r, "x"'] \arrow[urr, dotted, "x'"] & X \arrow[r, "\eta_X"'] & X^{\Delta} \end{tikzcd} \tag{31}$$

where $\epsilon_{\delta'} : X^{\mathrm{I}} \to X$ is evaluation at the second generic point $\delta' : 1 \to \mathrm{I}$.

Proposition 15. *Composition implies filling; that is, every cubical set X with composition is fibrant.*

Proof. Let X be a cubical set with composition, and suppose given a filling problem in cSet/I of the form

$$\begin{tikzcd} C \arrow[r, "y"] \arrow[d, tail, "c"'] & X^{\mathrm{I}} \arrow[d, "\epsilon_\delta"] \\ Z \arrow[r, "x"'] & X. \end{tikzcd} \tag{32}$$

Pulling back to $\mathsf{cSet}/(\mathrm{I}\times\mathrm{I})$ (but omitting the π^* everywhere), since X has composition there is a map $x' : Z \to X$ making the following commute,

$$\begin{tikzcd} C \arrow[r, "y"] \arrow[d, tail, "c"'] & X^{\mathrm{I}} \arrow[r, "\epsilon_{\delta'}"] \arrow[d, "\epsilon_\delta"'] & X \arrow[d, "\eta_X"] \\ Z \arrow[r, "x"'] \arrow[urr, dotted, "x'"] & X \arrow[r, "\eta_X"'] & X^{\Delta} . \end{tikzcd} \tag{33}$$

Transposing by the adjunction $\pi^* \dashv \pi_*$ results in a commutative diagram in cSet/I of the form

$$\begin{tikzcd} C \arrow[r, "y"] \arrow[d, tail, "c"'] & X^{\mathrm{I}} \arrow[r, "\widetilde{\epsilon_{\delta'}}"] \arrow[d, "\epsilon_\delta"'] & \pi_*\pi^*X \arrow[d, "\pi_*\eta_X"] \\ Z \arrow[r, "x"'] \arrow[urr, dotted, "\widetilde{x'}"] & X \arrow[r, "\widetilde{\eta_X}"'] & \pi_*X^{\Delta}, \end{tikzcd} \tag{34}$$

where $\widetilde{\epsilon_{\delta'}} : X^{\mathrm{I}} \to \pi_*\pi^*X$ is the adjoint transpose of $\epsilon_{\delta'}$, and similarly for $\widetilde{x'}$ and $\widetilde{\eta_X}$. To compute these transpositions, we factor them through the unit maps η^π of the adjunction $\pi^* \dashv \pi_*$,

$$\begin{tikzcd}
C \arrow[r, "y"] \arrow[d, tail, "c"'] & X^{\mathrm{I}} \arrow[r, "\eta^\pi_{X^{\mathrm{I}}}"] \arrow[d, "\epsilon_\delta"] & (X^{\mathrm{I}})^{\mathrm{I}} \arrow[r, "\pi_*\epsilon_{\delta'}"] \arrow[d, "(\epsilon_\delta)^{\mathrm{I}}"] & \pi_*\pi^*X \arrow[d, "\pi_*\eta_X"] \\
Z \arrow[r, "x"'] \arrow[urrr, dotted, "\widetilde{x'}" near end] & X \arrow[r, "\eta^\pi_X"'] & X^{\mathrm{I}} \arrow[r, "\pi_*\eta_X"'] & \pi_*X^\Delta .
\end{tikzcd} \tag{35}$$

Next, observe that $\pi_*\pi^*X = X^{\mathrm{I}}$ and, up to the iso $(X^{\mathrm{I}})^{\mathrm{I}} \cong X^{\mathrm{I}\times\mathrm{I}}$, the map $\pi_*\epsilon_{\delta'}$ is

$$\pi_*\epsilon_{\delta'} = X^{(\Delta:\mathrm{I}\rightarrowtail\mathrm{I}\times\mathrm{I})} : X^{\mathrm{I}\times\mathrm{I}} \to X^{\mathrm{I}}\,,$$

which we write as $\Delta^* : X^{\mathrm{I}\times\mathrm{I}} \to X^{\mathrm{I}}$ to avoid confusion with the exponential object X^Δ. The map Δ^* is plainly a retraction of

$$\eta^\pi_{X^{\mathrm{I}}} = X^{(\pi:\mathrm{I}\times\mathrm{I}\to\mathrm{I})} : X^{\mathrm{I}} \to (X^{\mathrm{I}})^{\mathrm{I}} \cong X^{\mathrm{I}\times\mathrm{I}}\,.$$

The last diagram (35) now becomes

$$\begin{tikzcd}
C \arrow[r, "y"] \arrow[d, tail, "c"'] & X^{\mathrm{I}} \arrow[r, "\eta^\pi_{X^{\mathrm{I}}}"] \arrow[d, "\epsilon_\delta"] \arrow[rr, bend left, "="] & X^{\mathrm{I}\times\mathrm{I}} \arrow[r, "\Delta^*"] \arrow[d, "(\epsilon_\delta)^{\mathrm{I}}"] & X^{\mathrm{I}} \arrow[d, "\pi_*\eta_X"] \\
Z \arrow[r, "x"'] \arrow[urrr, dotted, "\widetilde{x'}" near end] & X \arrow[r, "\eta^\pi_X"'] & X^{\mathrm{I}} \arrow[r, "\pi_*\eta_X"'] & \pi_*X^\Delta .
\end{tikzcd} \tag{36}$$

Finally, we claim that $\pi_*X^\Delta \cong X$, and that, up to this iso,

$$\pi_*\eta_X = \epsilon_\delta : X^{\mathrm{I}} \to X,$$

which will finish the proof, since this is a retraction of $\eta^\pi_X : X \to X^{\mathrm{I}}$. Indeed, writing out the object π_*X^Δ explicitly, in terms of the two adjunctions $\pi^* \dashv \pi_*$ and $\Delta^* \dashv \Delta_*$, we have

$$\pi_*X^\Delta = \pi_*\Delta_*\Delta^*\pi^*X \cong (\pi\circ\Delta)_*(\pi\circ\Delta)^*X \cong X,$$

since $(\pi\circ\Delta) = 1$.

To see that $\pi_*\eta_X = \epsilon_\delta$, first let us make the base change

$$\mathrm{I}^* : \mathsf{cSet} \to \mathsf{cSet}/\mathrm{I}$$

explicit, so that ϵ_δ is the counit of the adjunction $\mathrm{I}^* \dashv \mathrm{I}_*$ at I^*X,

$$\epsilon_\delta = \epsilon_{(\mathrm{I}^*X)} : X^{\mathrm{I}} \times \mathrm{I} \to X \times \mathrm{I} \quad \text{(over I)}.$$

By a triangle law, this map has the inverse $\mathrm{I}^*(\eta^{\mathrm{I}}_X) : X \times \mathrm{I} \to X^{\mathrm{I}} \times \mathrm{I}$, where

$$\eta^{\mathrm{I}}_X : X \to X^{\mathrm{I}}$$

is the unit of $\mathrm{I}^* \dashv \mathrm{I}_*$ at X. It suffices to show that $\mathrm{I}^*(\eta^{\mathrm{I}}_X)$ is also an inverse for $\pi_*\eta_X$ which, more explicitly is:

$$\pi_*\eta_X = \pi_*(\eta^{\Delta}_{\pi^*(\mathrm{I}^*X)}),$$

where η^{Δ} is the unit of $\Delta^* \dashv \Delta_*$. Since I^* preserves exponentials, we have

$$\mathrm{I}^*(\eta^{\mathrm{I}}_X) = \eta^{\pi}_{(\mathrm{I}^*X)} : \mathrm{I}^*X \to (\mathrm{I}^*X)^{\mathrm{I}^*\mathrm{I}} \cong \pi_*\pi^*(\mathrm{I}^*X) \quad \text{(over I)}.$$

Now, for the composition of the $\pi^* \dashv \pi_*$ and $\Delta^* \dashv \Delta_*$ adjunctions,

$$\begin{tikzcd} \mathsf{cSet}/\mathrm{I} \arrow[r, shift right, "\Delta_*"'] & \mathsf{cSet}/\mathrm{I}\times\mathrm{I} \arrow[l, shift right, "\Delta^*"'] \arrow[r, shift right, "\pi_*"'] & \mathsf{cSet}/\mathrm{I} \arrow[l, shift right, "\pi^*"'] \end{tikzcd} \tag{37}$$

the units η^{π} and η^{Δ} satisfy the well-known law

$$\eta^{\pi\circ\Delta} = \pi_*(\eta^{\Delta}_{\pi^*}) \circ (\eta^{\pi})\,.$$

But $\eta^{\pi\circ\Delta} = \mathsf{id}$, since $\pi \circ \Delta = 1$. We therefore have

$$\pi_*(\eta^{\Delta}_{\pi^*(\mathrm{I}^*X)}) \circ \eta^{\pi}_{(\mathrm{I}^*X)} = \eta^{(\pi\circ\Delta)}_{(\mathrm{I}^*X)} = \mathsf{id}_{(\mathrm{I}^*X)},$$

as required. □

Remark 16. There must be a simpler proof than that!

Composition for a map

We generalize the notion of compostion for an object X to composition for a map $f : Y \to X$. First consider biased fibrations in the sense of section 1.1; recall from Corollary 4 that a (biased) fibration structure on a map $f : Y \to X$ is the same thing as a pair of $+$-algebra structures on the maps

$$(\delta_\epsilon \Rightarrow f) : Y^{\mathrm{I}} \to X^{\mathrm{I}} \times_X Y$$

for $\epsilon = 0, 1$. The construction of $\delta_0 \Rightarrow f$ is recalled from (4) in the pullback diagram below, in which $X^{\delta_0} : X^{\mathrm{I}} \to X$ is the evaluation map at $\delta_0 : 1 \to \mathrm{I}$.

$$\begin{tikzcd}
Y^{\mathrm{I}} \arrow[dr, "\delta_0\Rightarrow f"] \arrow[drr, bend left, "Y^{\delta_0}"] \arrow[ddr, bend right, "f^{\mathrm{I}}"'] & & \\
& X^{\mathrm{I}} \times_X Y \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\
& X^{\mathrm{I}} \arrow[r, "X^{\delta_0}"'] & X
\end{tikzcd} \tag{38}$$

Given another point $\delta_1 : 1 \to \mathrm{I}$, we have an analogous construction

$$\begin{tikzcd}
Y^{\mathrm{I}} \arrow[dr, "\delta_1\Rightarrow f"] \arrow[drr, bend left, "Y^{\delta_1}"] \arrow[ddr, bend right, "f^{\mathrm{I}}"'] & & \\
& X^{\mathrm{I}} \times_X Y \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\
& X^{\mathrm{I}} \arrow[r, "X^{\delta_1}"'] & X.
\end{tikzcd} \tag{39}$$

But note that now the pullback object $X^{\mathrm{I}} \times_X Y$ is a different one, with fiber over $p : \mathrm{I} \to X$ being the fiber of f over $p\delta_1$ rather than over $p\delta_0$. Let us call these two different pulled-back maps $f_0 : Y_0 \to X^{\mathrm{I}}$ and $f_1 : Y_1 \to X^{\mathrm{I}}$ and write $f^\epsilon := (\delta_\epsilon \Rightarrow f)$ for $\epsilon = 0, 1$. There is then a commutative diagram,

$$\begin{tikzcd}
Y^{\mathrm{I}} \arrow[r, "f^1"] \arrow[d, "f^0"'] \arrow[dr, "f^{\mathrm{I}}" description] & Y_1 \arrow[d, "f_1"] \\
Y_0 \arrow[r, "f_0"'] & X^{\mathrm{I}},
\end{tikzcd} \tag{40}$$

We will say that $f : Y \to X$ *has composition from 0 to 1* if for every cofibration $c : C \rightarrowtail Z$ and maps $y_0 : Z \to Y_0$ and $y : C \to Y^{\mathrm{I}}$ making the square on the left below commute, there is a map $y_1 : Z \to Y_1$ making the following commute.

$$\begin{tikzcd}
C \arrow[r, "y"] \arrow[d, tail, "c"'] & Y^{\mathrm{I}} \arrow[r, "f^1"] \arrow[d, "f^0"'] & Y_1 \arrow[d, "f_1"] \\
Z \arrow[r, "y_0"'] \arrow[urr, dotted, "y_1" near end] & Y_0 \arrow[r, "f_0"'] & X^{\mathrm{I}}
\end{tikzcd} \tag{41}$$

To define *unbiased* composition, we begin with $f : Y \to X$ in cSet and then move to cSet/I, where we have the generic point $\delta : 1 \to \mathrm{I}$. Now we consider an arbitrary cofibration $c : C \rightarrowtail Z$ and maps $y_\delta : Z \to Y_\delta$ and $y : C \to Y^{\mathrm{I}}$ making the square below commute

$$\begin{tikzcd} C \arrow[r, "y"] \arrow[d, tail, "c"'] & Y^{\mathrm{I}} \arrow[d, "f^\delta"] \\ Z \arrow[r, "y_\delta"'] & Y_\delta \end{tikzcd} \tag{42}$$

where Y_δ and f^δ are defined in terms of $\delta : 1 \to \mathrm{I}$ just as were Y_0 and f^0 in terms of $\delta_0 : 1 \to \mathrm{I}$. Passing to $\mathsf{cSet}/(\mathrm{I} \times \mathrm{I})$ by a further pullback, as before we have another point $\delta' : 1 \to \mathrm{I}$, as well as a subobject $\Delta \rightarrowtail 1$, determined by the further diagonals.

Definition 17. The map $f : Y \to X$ *has (unbiased) composition* if, in cSet/I, for any cofibration $c : C \rightarrowtail Z$ and maps $y_\delta : Z \to Y_\delta$ and $y : C \to Y^{\mathrm{I}}$ as on the left below, there is in $\mathsf{cSet}/(\mathrm{I} \times \mathrm{I})$ a map $y_{\delta'} : Z \to Y_{\delta'}$ making the following commute

$$\begin{tikzcd} C \arrow[r, "y"] \arrow[d, tail, "c"'] & Y^{\mathrm{I}} \arrow[r, "f^{\delta'}"] \arrow[d, "f^\delta"'] & Y_{\delta'} \arrow[d, "\eta_{Y_{\delta'}}"] \\ Z \arrow[r, "y_\delta"'] \arrow[urr, dotted, "y_{\delta'}"] & Y_\delta \arrow[r, "\eta_{Y_\delta}"'] & Y^\Delta , \end{tikzcd} \tag{43}$$

where $Y_{\delta'}$ and $f^{\delta'}$ are defined in terms of $\delta' : 1 \to \mathrm{I}$, and Y^Δ is $Y_\delta^\Delta = Y_{\delta'}^\Delta$, since $\Delta^*\delta = \Delta^*\delta'$.

Proposition 18. *Composition implies filling for maps; that is, every $f : Y \to X$ with composition is a fibration.*

Proof. One should first simplify the proof of Proposition 15 before writing out the proof of Proposition 18. □

Remark 19. One can also promote the *property* of an object or map "having composition" to that of a *composition structure*, which is defined with respect to cofibrations $c : C \rightarrowtail \mathrm{I}^n$ with representable codomains and a requirement of uniformity in I^n.

3 Fibrancy of $\mathcal{U}$

An example of the use of composition rather than filling is the following proof of fibrancy of the universe (due to Coquand). It uses the version of composition for cubes with connections.

Proposition 20. *The universe $\mathcal{U}$ is fibrant.*

By the reduction of filling to composition (Proposition 15), it suffices to show:

Lemma 21. *The universe $\mathcal{U}$ has composition.*

Proof. Consider a composition problem

$$
\begin{tikzcd}
C \arrow[r] \arrow[dd, tail, "c"'] & \mathcal{U}^{\mathrm{I}} \arrow[d] \\
 & \mathcal{U} \times \mathcal{U} \arrow[d] \\
Z \arrow[r] \arrow[ur, dotted, "k"] & \mathcal{U}
\end{tikzcd}
$$

We claim that the canonical map $\mathcal{U}^{\mathrm{I}} \to \mathcal{U} \times \mathcal{U}$ factors over $\mathcal{U} \times \mathcal{U}$ through the object Eq of equivalences, via a map i as indicated below.

$$
\begin{tikzcd}
C \arrow[r] \arrow[ddd, tail, "c"'] & \mathcal{U}^{\mathrm{I}} \arrow[d, "i"] \\
 & \mathsf{Eq} \arrow[d] \\
 & \mathcal{U} \times \mathcal{U} \arrow[d] \\
Z \arrow[r] \arrow[uur, dotted, "j"] \arrow[ur, dotted, "k"'] & \mathcal{U}
\end{tikzcd}
\tag{44}
$$

Since the projection $\mathsf{Eq} \to \mathcal{U}$ is a trivial fibration by the equivalence extension property, there is a diagonal filler j as indicated. Composing then gives the required k.

The claimed map i is the one known as $\mathsf{IdtoEq} : \mathcal{U}^{\mathrm{I}} \to \mathsf{Eq}$, and is defined in type theory by path induction, which apparently requires $\mathcal{U}^{\mathrm{I}} \to \mathcal{U} \times \mathcal{U}$ to be a fibration. It can also be defined explicitly, however, using the classifying property of $\mathcal{U}$. $\square$

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