A model structure on the cartesian cubical sets

1 The cartesian cube category

We consider the cartesian cube category \mathbb{C} , defined as the free finite product category on an interval $\delta_0, \delta_1 : 1 \rightrightarrows I$. As a classifying category for an algebraic theory $\mathbb{T} = \{0,1\}$, \mathbb{C} has a covariant presentation by Lawvere duality, namely as the dual of the full subcategory of finitely-generated, free \mathbb{T} -algebras $\mathsf{Alg}(\mathbb{T})_{\mathrm{fg}}$. In this case, the algebras are simply bipointed sets (A, a_0, a_1) , and the free ones are the strictly bipointed sets $a_0 \neq a_1$. Thus $\mathsf{Alg}(\mathbb{T})_{\mathrm{fg}}$ consists of the finite, strictly bipointed sets and all bipointed maps between them.

Definition 1. The objects of the cartesian cube category $\mathbb C$ are themselves called cubes, and will be written

$$[n] = \{x_1, ..., x_n\},\$$

where the x_i may be regarded as coordinate axes. The arrows,

$$f: [n] \longrightarrow [m],$$

are then taken to be m-tuples of elements drawn from the set

$$[n]^+ = \{0, x_1, ..., x_n, 1\},\$$

regarded as formal terms representing composites of faces, degeneracies, permutations, and diagonals (alternately, the arrows are arbitrary bipointed maps $[m]^+ \longrightarrow [n]^+$).

2 Cubical sets

The category cSet of *cubical sets* is the category of presheaves on the cartesian cube category \mathbb{C} . It is generated by the representable presheaves y([n]), which will be written $I^n = y([n])$ and called the *standard n-cubes*.

3 Partial map classification and the +-algebra weak factorization system

Cofibrations, partial map classification, the functor X^+ , the awfs of +-algebras.

4 Partial path lifting (biased version)

We first recall the specification of the trivial-cofibration/fibration WFS from [?], and show that the resulting fibrations are equivalent to those specified in the "logical style" given in [?, ?].

The generating trivial cobrations are all maps of the form

$$m \otimes \delta_{\epsilon} : U \longrightarrow \mathbf{I}^{n+1}$$
, (1)

where:

- 1. $n \ge 0$,
- 2. $\delta_{\epsilon}: 1 \longrightarrow I$ is one of the two endpoint inclusions, where $\epsilon = 0, 1$,
- 3. $m \otimes \delta_{\epsilon}$ is the push-out product, resp. "Leibniz tensor", of any cofibration $m: M \longrightarrow I^n$ and a $\delta_{\epsilon}: 1 \longrightarrow I$,
- 4. U is $I^n +_M (M \times I)$, the domain of $m \otimes \delta_{\epsilon}$.

Let $\mathcal{C} \otimes \delta_{\epsilon}$ be the set of all such maps; the *fibrations* are defined to be the elements of the right class of these,

$$\mathcal{F} = (\mathcal{C} \otimes \delta_{\epsilon})^{\pitchfork}$$
 .

A fibration structure on a map $f: Y \longrightarrow X$ is a choice of diagonal fillers,

$$\begin{array}{ccc}
I^{n} +_{M} (M \times I) & \longrightarrow X \\
\downarrow^{m \otimes \delta_{\epsilon}} & \downarrow^{f} \\
I^{n} \times I & \longrightarrow Y.
\end{array} \tag{2}$$

that is uniform with respect to arbitrary pullbacks of the cofibration m, as in the case of the +algebra factorization system.

Fixing the argument δ_{ϵ} , the Leibniz tensor functor

$$(-)\otimes \delta_{\epsilon}:\widehat{\mathbb{C}}^2\longrightarrow\widehat{\mathbb{C}}^2$$

has a right adjoint, the "Leibniz exponential", which for a map $f: X \longrightarrow Y$ we will write as,

$$(\delta_{\epsilon} \Rightarrow f) : X^{\mathrm{I}} \longrightarrow (Y^{\mathrm{I}} \times_{Y} X).$$

Using this adjunction on arrow categories, one can easily show the following:

Proposition 2. An object X is fibrant if and only if both of the pathspace projections $X^{\delta_{\epsilon}}: X^{\mathrm{I}} \longrightarrow X$ are +algebras.

An analogous statement also holds for maps $f: X \longrightarrow Y$ in place of objects X.

4.1 Local partial path lifting

To make the connection to the logical style of presentation used in [?, ?], suppose we want to describe a (uniform) filling structure on an arbitrary $f: X \longrightarrow Y$ with respect to all generating trivial cofibrations $m \otimes \delta_{\epsilon}: I^n +_M (M \times I) \longrightarrow I^{n+1}$,

$$\begin{array}{ccc}
I^{n} +_{M} (M \times I) & \longrightarrow X \\
\downarrow^{m \otimes \delta_{\epsilon}} & \downarrow^{f} \\
I^{n} \times I & \longrightarrow Y.
\end{array} \tag{3}$$

By pulling back along c, it suffices to consider the case $Y = I^n \times I$ and c the identity map. Moreover, since we shall internalize the quantification over all cofibrations $m: M \to I^n$ using the classifier Φ , it suffices to consider just the following case internally,

$$\begin{array}{ccc}
1 +_{[\phi]} ([\phi] \times I) & \xrightarrow{[a_0, s]} X \\
\downarrow & & \downarrow \\
1 \times I & \xrightarrow{\simeq} & I
\end{array}$$
(4)

where the cofibration $[\phi] \rightarrow 1$ is classified by $\phi : 1 \rightarrow \Phi$.

Using a universe Set in the internal language of $\widehat{\mathbb{C}}$, we can regard the family $X \longrightarrow I$ internally as a map $P: I \to Set$ (switching notation from X to P to agree with [?]). Thus we arrive at the following local specification, expressed logically in the internal language of $\widehat{\mathbb{C}}$, of the object of "(0-directed) lifting structures" $L^0(P)$ on a family $P: I \to Set$:

$$L^{0}(P) = \prod_{\phi:\Phi} \prod_{s:\prod_{i:I}(P_{i})^{\phi}} \prod_{a_{0}:P_{0}} a_{0}|_{\phi} = s_{0} \longrightarrow \sum_{a:\prod_{i:I}P_{i}} (a_{0} = a_{0}) \times (a|_{\phi} = s).$$
(5)

Here the variables $s: \prod_{i:I} (Pi)^{\phi}$ and $a_0: P0$, and the condition $a_0|_{\phi} = s0$, give the domain $1 +_{[\phi]} ([\phi] \times I)$ of the arrow $[a_0, s]$ in (4), and $a: \prod_{i:I} Pi$ is the diagonal filler, with $(a0 = a_0) \times (a|_{\phi} = s)$ expressing the commutativity of the top triangle.

There is an analogous condition $L^1(P)$ in which 1 replaces 0 everywhere, describing ("directed") filling from the other end of the interval. Note that [?, ?] derive the "filling" conclusion

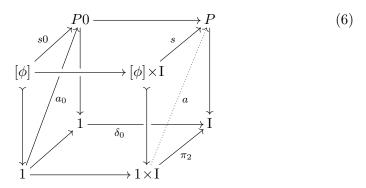
$$\sum_{a:\prod_{i:I}Pi}(a0=a_0)\times(a|_{\phi}=s)$$

from (connections on I and) a weaker "composition operation"

$$\sum_{a_1:P1} a_1|_{\phi} = s_1 \,,$$

but we will not take this approach.

The specification of the type $L^0(P)$ of (5) can also be represented diagrammatically as follows:



Here the left-hand vertical square is determined as a pullback of the right-hand one along the endpoint $\delta_0: 1 \longrightarrow I$.

Now write

$$\widetilde{P} = \prod_{i:I} Pi$$

for the type of sections of the projection $P = \sum_{i:I} Pi \longrightarrow I$, and write

$$\pi_0: \widetilde{P} \longrightarrow P0$$

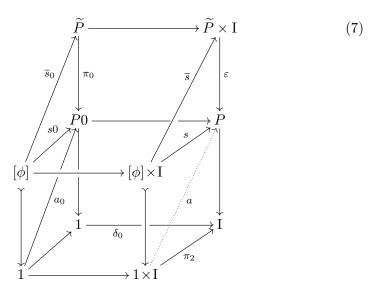
for the 0^{th} -projection (i.e. the evaluation of $P: I \longrightarrow \mathsf{Set}$ at 0: I).

Then the (0-directed) lifting structures on P correspond to +-algebra structures on the projection $\pi_0: \widetilde{P} \longrightarrow P0$, as follows.

Proposition 3. For any $P : Set^{I}$, there is an isomorphism

$$L^0(P) \cong {}^+ Alg(\pi_0 : \widetilde{P} \longrightarrow P0)$$
.

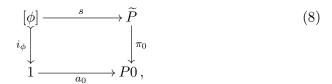
Proof. Consider the following diagram,



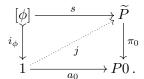
which is (6), extended by the counit (evaluation) $\varepsilon : \widetilde{P} \times I \longrightarrow P$ over I on the right, and with 1 still representing the domain of a variable to reason internally. The pullback of ε over I along δ_0 is then the map $\pi_0 : \widetilde{P} \longrightarrow P0$ that we are interested in.

Given an $L^0(P)$ -structure, reasoning internally we construct a ⁺Alg-structure on $\pi_0: P \longrightarrow P0$ as follows: for any cofibration $i_\phi: [\phi] \rightarrowtail 1$

and any commutative square,



we require a diagonal filler,



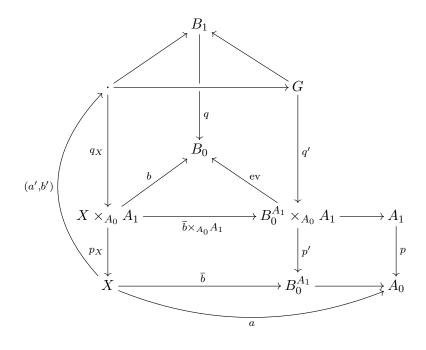
Transposing the top left span in (8) formed by i_{ϕ} and s along the adjunction $I^* \dashv \prod_{I}$ gives the right-hand square in (7), and the commutative square in (8) formed by a_0 and π_0 gives the rest of the data in (7). Thus the assumed $L^0(P)$ -structure gives an $a: 1 \times I \longrightarrow P$ as indicated in (7). But then a lifts uniquely across ε to a map $\overline{a}: 1 \times I \longrightarrow \widetilde{P} \times I$ over I, by the universal property of $\varepsilon: \widetilde{P} \times I \longrightarrow P$. We can therefore set

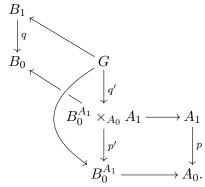
$$j = \delta_0^*(\overline{a}) : 1 \longrightarrow \widetilde{P}$$
.

Suppose conversely that we have a ⁺Alg-structure on $\pi_0: \widetilde{P} \longrightarrow P0$, and we want to build a (0-directed) lifting structure on P. Take any ϕ, s, a_0 as indicated in (7), and we require an $a: 1 \times I \longrightarrow P$ over I. From s we get \overline{s} by the universal property of ε , and we therefore have \overline{s}_0 by pullback. From \overline{s}_0 and a_0 and the ⁺Alg structure on π_0 we obtain a map $j: 1 \longrightarrow \widetilde{P}$ over P0 which is a diagonal filler of the indicated square formed by $i_{\phi}, \overline{s}_0, a_0$ and π_0 . Finally, we obtain the required map $a: 1 \times I \longrightarrow P$ over I as the $(I^* \dashv \prod_I)$ -transpose of j,

$$a = \varepsilon \circ (j \times I)$$
.

We leave to the reader the verification that these assignments are mutually inverse. \Box





5 Unbiased partial path lifting

6 A left-induced model structure on the Cartesian cubical sets

We make use of the Sattler model structure [?] on the *Dedekind cubical* sets $\widehat{\mathbb{D}} = \mathsf{Set}^{\mathbb{D}^{\mathrm{op}}}$, where \mathbb{D} is the category of *Dedekind cubes*, defined as

the Lawvere theory of distributive lattices. The unique product-preserving functor

$$i: \mathbb{C} \longrightarrow \mathbb{D}$$

classifying the Dedekind interval $I_{\mathbb{D}} \in \mathbb{D}$ induces an adjunction,

$$i_! \dashv i^* \dashv i_* : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{D}},$$

where $i^*(Q) = Q \circ i$, for $Q \in \mathbb{D}$.

Lemma 4. Observe that $i_!$ is left exact since the Dedekind interval $I_{\mathbb{D}}$ is strict, $0 \neq 1 : 1 \rightrightarrows I_{\mathbb{D}}$. Thus we have geometric morphisms:

$$(i_! \dashv i^*): \widehat{\mathbb{D}} \longrightarrow \widehat{\mathbb{C}},$$

classifying the bipointed object $i_!(I_{\mathbb{C}}) = I_{\mathbb{D}}$,

$$(i^* \dashv i_*): \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{D}},$$

classifying the dLat $i^*(I_{\mathbb{D}}) := \mathbb{I}$, where $\eta : I_{\mathbb{C}} \longrightarrow \mathbb{I}$ can be described pointwise as the distributive lattice completion of the corresponding bipointed set.

Also, since i is faithful so is $i_!$, and since i is surjective on objects i^* is also faithful.

It follows that:

- $\widehat{\mathbb{C}}$ is $(i_! \circ i^*)$ -coalgebras on $\widehat{\mathbb{D}}$,
- $\widehat{\mathbb{D}}$ is $(i^* \circ i_*)$ -coalgebras on $\widehat{\mathbb{C}}$,
- $\widehat{\mathbb{D}}$ is $(i^* \circ i_!)$ -algebras on $\widehat{\mathbb{C}}$.

We will use the following transfer theorem for QMSs from \cite{MSs} from $\cite{M$

Theorem ([?, ?]). Suppose $\widehat{\mathbb{D}}$ has a (cofibrantly generated) model structure $(\mathcal{C}_{\mathbb{D}}, \mathcal{W}_{\mathbb{D}}, \mathcal{F}_{\mathbb{D}})$. Given an adjunction

$$i_! \dashv i^* : \widehat{\mathbb{D}} \longrightarrow \widehat{\mathbb{C}},$$

there is a left-induced model structure on $\widehat{\mathbb{C}}$ if the following acyclicity condition holds:

$$(i_!^{-1}\mathcal{C}_{\mathbb{D}})^{\pitchfork} \subset i_!^{-1}\mathcal{W}_{\mathbb{D}}.$$

For the left-induced model structure $(\mathcal{C}_{\mathbb{C}}, \mathcal{W}_{\mathbb{C}}, \mathcal{F}_{\mathbb{C}})$ on $\widehat{\mathbb{C}}$ we then have:

$$\mathcal{C}_{\mathbb{C}} = i_{!}^{-1} \mathcal{C}_{\mathbb{D}},$$
 $\mathcal{W}_{\mathbb{C}} = i_{!}^{-1} \mathcal{W}_{\mathbb{D}}.$

The Sattler model structure on $\widehat{\mathbb{D}}$ is given as follows (for a constructive treatment a smaller class of "pointwise decidable cofibrations" is used, but we consider the classical case first):

$$\begin{array}{ll} \mathcal{C} &=& \text{monomorphisms} \,, \\ \mathcal{W} &=& \left\{ f \mid f = p \circ i, \ p \in \mathcal{F} \cap \mathcal{W}, \ i \in \mathcal{C} \cap \mathcal{W} \right\}, \\ \mathcal{F} &=& \left(\mathcal{C} \otimes \delta \right)^{\pitchfork}. \end{array}$$

where $\delta: 1 \longrightarrow I$ is either endpoint inclusion.

For the left-induced model structure on $\widehat{\mathbb{C}}$ we therefore have the following specification:

$$\mathcal{C} = \text{monomorphisms},$$

$$\mathcal{W} = \{ f \mid i_! f = p \circ i, \ p \in \mathcal{F}_{\mathbb{D}} \cap \mathcal{W}_{\mathbb{D}}, \ i \in \mathcal{C}_{\mathbb{D}} \cap \mathcal{W}_{\mathbb{D}} \},$$

$$\mathcal{F} = (\mathcal{C} \cap \mathcal{W})^{\pitchfork}.$$

The determination of C follows from the fact that $i_!: \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{D}}$ is conservative. To check the acyclicity condition,

$$(i_!^{-1}\mathcal{C}_{\mathbb{D}})^{\pitchfork} \subset i_!^{-1}\mathcal{W}_{\mathbb{D}},$$

we know that $i_!^{-1}\mathcal{C}_{\mathbb{D}}$ consists of the monos in \mathbb{C} , so take $f: Y \longrightarrow X$ in $(i_!^{-1}\mathcal{C}_{\mathbb{D}})^{\pitchfork}$, apply $i_!$, and factor the result as $i_!f = p \circ m: i_!Y \longrightarrow Z \longrightarrow i_!X$ with $p \in \mathcal{F}_{\mathbb{D}} \cap \mathcal{W}_{\mathbb{D}}$ and $m: i_!Y \longrightarrow Z$ monic. We then need to show that m is in $\mathcal{W}_{\mathbb{D}}$.

We can apply Theorem 2.2.1 of [?], with $K = \widehat{\mathbb{C}}$, $M = \widehat{\mathbb{D}}$, $V = i_!$, $k = i^*$, and:

- 1. QX = X and $\epsilon = 1_X : X \longrightarrow X$, so that $i_! 1_X = 1_{i_!X}$ and therefore in $\mathcal{W}_{\mathbb{D}}$, while all objects are cofibrant,
- 2. Qf = f for any $f: X \longrightarrow Y$ in $\widehat{\mathbb{C}}$, so that the naturality condition is similarly trivial,
- 3. factor the codiagonal $X + X \longrightarrow X$ as $\pi_2 \circ j : X + X \longrightarrow I \times X \longrightarrow X$ with $j = (\partial I \times X) : X + X \longrightarrow I \times X$.

It remains only to show that $i_!p: i_!(I \times X) \longrightarrow i_!X$ is in $\mathcal{W}_{\mathbb{D}}$ and $i_!j: i_!(X + X) \longrightarrow i_!(I \times X)$ is in $\mathcal{C}_{\mathbb{D}}$. The latter is clear, since j is monic. To show the former, observe that for any $D \in \widehat{\mathbb{D}}$, the projection $\pi_2: I_{\mathbb{D}} \times D \longrightarrow D$ is in $\mathcal{W}_{\mathbb{D}}$ by 3-for-2, since the "cylinder end" inclusion $D \longrightarrow I_{\mathbb{D}} \times D$, as a pullback of an endpoint inclusion, is a cofibration, and a strong deformation retract (using the connection on I), and hence is in $\mathcal{W}_{\mathbb{D}}$ by [?].

Thus we have shown:

Theorem 5. There is a Quillen model structure (C, W, \mathcal{F}) on the category $\widehat{\mathbb{C}}$ of cartesian cubical sets, in which

$$\mathcal{C} = monomorphisms,$$

$$\mathcal{W} = \{ f \mid i_! f = p \circ i, \ p \in \mathcal{F}_{\mathbb{D}} \cap \mathcal{W}_{\mathbb{D}}, \ i \in \mathcal{C}_{\mathbb{D}} \cap \mathcal{W}_{\mathbb{D}} \},$$

$$\mathcal{F} = (\mathcal{C} \cap \mathcal{W})^{\pitchfork}.$$

where $i_!: \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{D}}$ is the left adjoint of precomposition along the canonical map $i: \mathbb{C} \longrightarrow \mathbb{D}$ from Cartesian cubes to Dedekind cubes, and $(\mathcal{C}_{\mathbb{D}}, \mathcal{W}_{\mathbb{D}}, \mathcal{F}_{\mathbb{D}})$ is the Sattler model structure on $\widehat{\mathbb{D}}$.

References:

- Gambino-Sattler
- Sattler
- Hess, Kedziorek, Riehl, Shipley
- Garner, Kedziorek, Riehl