A model structure on the cartesian cubical sets

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1 The cartesian cube category

We consider the cartesian cube category $\mathbb{C}$, defined as the free finite product category on an interval $\delta_0, \delta_1 : 1 \rightrightarrows \mathrm{I}$. As a classifying category for an algebraic theory $\mathbb{T} = \{0, 1\}$, $\mathbb{C}$ has a covariant presentation by Lawvere duality, namely as the dual of the full subcategory of finitely-generated, free $\mathbb{T}$-algebras $\mathsf{Alg}(\mathbb{T})_{\mathrm{fg}}$. In this case, the algebras are simply *bipointed sets* (A, a_0, a_1), and the free ones are the *strictly* bipointed sets $a_0 \neq a_1$. Thus $\mathsf{Alg}(\mathbb{T})_{\mathrm{fg}}$ consists of the finite, strictly bipointed sets and all bipointed maps between them.

Definition 1. The objects of the cartesian cube category $\mathbb{C}$ are themselves called cubes, and will be written

$$[n] = \{x_1, ..., x_n\},$$

where the x_i may be regarded as coordinate axes. The arrows,

$$f : [n] \longrightarrow [m]\,,$$

are then taken to be m-tuples of elements drawn from the set

$$[n]^+ = \{0, x_1, ..., x_n, 1\},$$

regarded as formal terms representing composites of faces, degeneracies, permutations, and diagonals (alternately, the arrows are arbitrary bipointed maps $[m]^+ \longrightarrow [n]^+$).

2 Cubical sets

The category cSet of *cubical sets* is the category of presheaves on the cartesian cube category $\mathbb{C}$. It is generated by the representable presheaves $y([n])$, which will be written $\mathrm{I}^n = y([n])$ and called the *standard n-cubes*.

3 Partial map classification and the +-algebra weak factorization system

Cofibrations, partial map classification, the functor X^+, the awfs of +-algebras.

4 Partial path lifting (biased version)

We first recall the specification of the trivial-cofibration/fibration WFS from [**?**], and show that the resulting fibrations are equivalent to those specified in the "logical style" given in [**?**, **?**].

The generating trivial cobrations are all maps of the form

$$m \otimes \delta_\epsilon : U \longrightarrow \mathrm{I}^{n+1}\,, \tag{1}$$

where:

1. $n \geq 0$,
2. $\delta_\epsilon : 1 \longrightarrow \mathrm{I}$ is one of the two endpoint inclusions, where $\epsilon = 0, 1$,
3. $m \otimes \delta_\epsilon$ is the push-out product, resp. "Leibniz tensor", of any cofibration $m : M \longrightarrow \mathrm{I}^n$ and a $\delta_\epsilon : 1 \longrightarrow \mathrm{I}$,
4. U is $\mathrm{I}^n +_M (M \times \mathrm{I})$, the domain of $m \otimes \delta_\epsilon$.

Let $\mathcal{C} \otimes \delta_\epsilon$ be the set of all such maps; the *fibrations* are defined to be the elements of the right class of these,

$$\mathcal{F} = (\mathcal{C} \otimes \delta_\epsilon)^{\pitchfork}\,.$$

A *fibration structure* on a map $f : Y \longrightarrow X$ is a choice of diagonal fillers,

$$\begin{tikzcd}
\mathrm{I}^n +_M (M \times \mathrm{I}) \arrow[r] \arrow[d, "m\otimes\delta_\epsilon"'] & X \arrow[d, "f"] \\
\mathrm{I}^n \times \mathrm{I} \arrow[r] \arrow[ur, dotted] & Y.
\end{tikzcd} \tag{2}$$

that is uniform with respect to arbitrary pullbacks of the cofibration m, as in the case of the +algebra factorization system.

Fixing the argument δ_ϵ, the Leibniz tensor functor

$$(-) \otimes \delta_\epsilon : \widehat{\mathbb{C}}^2 \longrightarrow \widehat{\mathbb{C}}^2$$

has a right adjoint, the "Leibniz exponential", which for a map $f : X \longrightarrow Y$ we will write as,

$$(\delta_\epsilon \Rightarrow f) : X^{\mathrm{I}} \longrightarrow (Y^{\mathrm{I}} \times_Y X)\,.$$

Using this adjunction on arrow categories, one can easily show the following:

Proposition 2. *An object X is fibrant if and only if both of the pathspace projections $X^{\delta_\epsilon} : X^{\mathrm{I}} \longrightarrow X$ are +algebras.*

An analogous statement also holds for maps $f : X \longrightarrow Y$ in place of objects X.

4.1 Local partial path lifting

To make the connection to the logical style of presentation used in [**?**, **?**], suppose we want to describe a (uniform) filling structure on an arbitrary $f : X \longrightarrow Y$ with respect to all generating trivial cofibrations $m \otimes \delta_\epsilon : \mathrm{I}^n +_M (M \times \mathrm{I}) \longrightarrow \mathrm{I}^{n+1}$,

$$\begin{tikzcd} \mathrm{I}^n +_M (M \times \mathrm{I}) \arrow[r] \arrow[d, "m\otimes\delta_\epsilon"'] & X \arrow[d, "f"] \\ \mathrm{I}^n \times \mathrm{I} \arrow[r, "c"'] \arrow[ur, dotted] & Y. \end{tikzcd} \tag{3}$$

By pulling back along c, it suffices to consider the case $Y = \mathrm{I}^n \times \mathrm{I}$ and c the identity map. Moreover, since we shall internalize the quantification over all cofibrations $m : M \rightarrowtail \mathrm{I}^n$ using the classifier Φ, it suffices to consider just the following case internally,

$$\begin{tikzcd} 1 +_{[\phi]} ([\phi] \times \mathrm{I}) \arrow[r, "{[a_0,s]}"] \arrow[d, "\phi\otimes\delta_\epsilon"'] & X \arrow[d] \\ 1 \times \mathrm{I} \arrow[r, "\cong"'] \arrow[ur, dotted, "a"] & \mathrm{I} \end{tikzcd} \tag{4}$$

where the cofibration $[\phi] \rightarrowtail 1$ is classified by $\phi : 1 \to \Phi$.

Using a universe Set in the internal language of $\widehat{\mathbb{C}}$, we can regard the family $X \longrightarrow \mathrm{I}$ internally as a map $P : \mathrm{I} \to \mathsf{Set}$ (switching notation from X to P to agree with [?]). Thus we arrive at the following local specification, expressed logically in the internal language of $\widehat{\mathbb{C}}$, of the object of "(0-directed) lifting structures" $L^0(P)$ on a family $P : \mathrm{I} \to \mathsf{Set}$:

$$L^0(P) \;=\; \prod_{\phi:\Phi} \;\prod_{s:\prod_{i:I}(Pi)^\phi} \;\prod_{a_0:P0} a_0|_\phi = s0 \;\longrightarrow\; \sum_{a:\prod_{i:I} Pi} (a0 = a_0) \times (a|_\phi = s)\,. \tag{5}$$

Here the variables $s : \prod_{i:I}(Pi)^\phi$ and $a_0 : P0$, and the condition $a_0|_\phi = s0$, give the domain $1 +_{[\phi]} ([\phi] \times \mathrm{I})$ of the arrow $[a_0, s]$ in (4), and $a : \prod_{i:I} Pi$ is the diagonal filler, with $(a0 = a_0) \times (a|_\phi = s)$ expressing the commutitivity of the top triangle.

There is an analogous condition $L^1(P)$ in which 1 replaces 0 everywhere, describing ("directed") filling from the other end of the interval. Note that [?, ?] derive the "filling" conclusion

$$\sum_{a:\prod_{i:I} Pi} (a0 = a_0) \times (a|_\phi = s)$$

from (connections on I and) a weaker "composition operation"

$$\sum_{a_1:P1} a_1|_\phi = s_1\,,$$

but we will not take this approach.

The specification of the type $L^0(P)$ of (5) can also be represented diagrammatically as follows:

$$\begin{tikzcd}
 & P0 \arrow[rr] \arrow[dd] & & P \arrow[dd] \\
{[\phi]} \arrow[ur, "s0"] \arrow[rr] \arrow[dd, tail] & & {[\phi]\times\mathrm{I}} \arrow[ur, "s"] \arrow[dd, tail] & \\
 & 1 \arrow[rr, "\delta_0"'] & & \mathrm{I} \\
1 \arrow[uuur, "a_0"'] \arrow[ur] \arrow[rr] & & 1\times\mathrm{I} \arrow[ur, "\pi_2"'] \arrow[uuur, dotted, "a"'] &
\end{tikzcd} \tag{6}$$

Here the left-hand vertical square is determined as a pullback of the right-hand one along the endpoint $\delta_0 : 1 \longrightarrow \mathrm{I}$.

Now write

$$\widetilde{P} \;=\; \prod_{i:I} Pi$$

for the type of sections of the projection $P = \sum_{i:I} Pi \longrightarrow \mathrm{I}$, and write

$$\pi_0 : \widetilde{P} \longrightarrow P0$$

for the 0^{th}-projection (i.e. the evaluation of $P : \mathrm{I} \longrightarrow \mathsf{Set}$ at $0 : \mathrm{I}$).

Then the (0-directed) lifting structures on P correspond to $+$-algebra structures on the projection $\pi_0 : \widetilde{P} \longrightarrow P0$, as follows.

Proposition 3. *For any $P : \mathsf{Set}^{\mathrm{I}}$, there is an isomorphism*

$$L^0(P) \;\cong\; {}^{+}\mathsf{Alg}(\pi_0 : \widetilde{P} \longrightarrow P0)\,.$$

Proof. Consider the following diagram,

$$\begin{tikzcd}
 & \widetilde{P} \arrow[rr] \arrow[d, "\pi_0"] & & \widetilde{P}\times\mathrm{I} \arrow[d, "\varepsilon"] \\
 & P0 \arrow[rr] \arrow[dd] & & P \arrow[dd] \\
{[\phi]} \arrow[uur, "\overline{s}_0"] \arrow[ur, "s0"] \arrow[rr] \arrow[dd, tail] & & {[\phi]}\times\mathrm{I} \arrow[uur, "\overline{s}"] \arrow[ur, "s"] \arrow[dd, tail] & \\
 & 1 \arrow[rr, "\delta_0"] & & \mathrm{I} \\
1 \arrow[uuur, "a_0"] \arrow[ur] \arrow[rr] & & 1\times\mathrm{I} \arrow[ur, "\pi_2"] \arrow[uuur, dotted, "a"] &
\end{tikzcd} \tag{7}$$

which is (6), extended by the counit (evaluation) $\varepsilon : \widetilde{P} \times \mathrm{I} \longrightarrow P$ over I on the right, and with 1 still representing the domain of a variable to reason internally. The pullback of ε over I along δ_0 is then the map $\pi_0 : \widetilde{P} \longrightarrow P0$ that we are interested in.

Given an $L^0(P)$-structure, reasoning internally we construct a ${}^{+}\mathsf{Alg}$-structure on $\pi_0 : \widetilde{P} \longrightarrow P0$ as follows: for any cofibration $i_\phi : [\phi] \rightarrowtail 1$

and any commutative square,

$$\begin{tikzcd}
[\phi] \arrow[r, "s"] \arrow[d, tail, "i_\phi"'] & \widetilde{P} \arrow[d, "\pi_0"] \\
1 \arrow[r, "a_0"'] & P0\,,
\end{tikzcd} \tag{8}$$

we require a diagonal filler,

$$\begin{tikzcd}
[\phi] \arrow[r, "s"] \arrow[d, tail, "i_\phi"'] & \widetilde{P} \arrow[d, "\pi_0"] \\
1 \arrow[r, "a_0"'] \arrow[ur, dotted, "j"'] & P0\,.
\end{tikzcd}$$

Transposing the top left span in (8) formed by i_ϕ and s along the adjunction $\mathrm{I}^* \dashv \prod_{\mathrm{I}}$ gives the right-hand square in (7), and the commutative square in (8) formed by a_0 and π_0 gives the rest of the data in (7). Thus the assumed $L^0(P)$-structure gives an $a : 1 \times \mathrm{I} \longrightarrow P$ as indicated in (7). But then a lifts uniquely across ε to a map $\overline{a} : 1 \times \mathrm{I} \longrightarrow \widetilde{P} \times \mathrm{I}$ over I, by the universal property of $\varepsilon : \widetilde{P} \times \mathrm{I} \longrightarrow P$. We can therefore set

$$j = \delta_0^*(\overline{a}) : 1 \longrightarrow \widetilde{P}\,.$$

Suppose conversely that we have a ${}^{+}\mathsf{Alg}$-structure on $\pi_0 : \widetilde{P} \longrightarrow P0$, and we want to build a (0-directed) lifting structure on P. Take any ϕ, s, a_0 as indicated in (7), and we require an $a : 1 \times \mathrm{I} \longrightarrow P$ over I. From s we get $\overline{s}$ by the universal property of ε, and we therefore have $\overline{s}_0$ by pullback. From $\overline{s}_0$ and a_0 and the ${}^{+}\mathsf{Alg}$ structure on π_0 we obtain a map $j : 1 \longrightarrow \widetilde{P}$ over $P0$ which is a diagonal filler of the indicated square formed by $i_\phi, \overline{s}_0, a_0$ and π_0. Finally, we obtain the required map $a : 1 \times \mathrm{I} \longrightarrow P$ over I as the $(\mathrm{I}^* \dashv \prod_{\mathrm{I}})$-transpose of j,

$$a = \varepsilon \circ (j \times \mathrm{I})\,.$$

We leave to the reader the verification that these assignments are mutually inverse. $\square$

$$
\begin{tikzcd}
 & & B_1 \arrow[dd, "q"] & & \\
\cdot \arrow[urr] \arrow[rrr] \arrow[dd, "q_X"'] & & & G \arrow[ul] \arrow[dd, "q'"] & \\
 & & B_0 & & \\
X \times_{A_0} A_1 \arrow[urr, "b"] \arrow[rrr, "\bar{b} \times_{A_0} A_1"'] \arrow[d, "p_X"'] & & & B_0^{A_1} \times_{A_0} A_1 \arrow[ul, "ev"'] \arrow[r] \arrow[d, "p'"] & A_1 \arrow[d, "p"] \\
X \arrow[rrr, "\bar{b}"] \arrow[rrrr, bend right, "a"'] \arrow[uuu, bend left, "{(a',b')}"] & & & B_0^{A_1} \arrow[r] & A_0
\end{tikzcd}
$$

$$
\begin{tikzcd}
B_1 \arrow[d, "q"] & & \\
B_0 & G \arrow[ul] \arrow[d, "q'"] \arrow[dd, bend right] & \\
 & B_0^{A_1} \times_{A_0} A_1 \arrow[ul] \arrow[r] \arrow[d, "p'"] & A_1 \arrow[d, "p"] \\
 & B_0^{A_1} \arrow[r] & A_0.
\end{tikzcd}
$$

5 Unbiased partial path lifting

6 A left-induced model structure on the Cartesian cubical sets

We make use of the Sattler model structure [**?**] on the *Dedekind cubical sets* $\widehat{\mathbb{D}} = \mathsf{Set}^{\mathbb{D}^{\mathrm{op}}}$, where $\mathbb{D}$ is the category of *Dedekind cubes*, defined as the Lawvere theory of distributive lattices. The unique product-preserving functor

$$i : \mathbb{C} \longrightarrow \mathbb{D}$$

classifying the Dedekind interval $\mathrm{I}_{\mathbb{D}} \in \mathbb{D}$ induces an adjunction,

$$i_! \dashv i^* \dashv i_* \ : \ \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{D}} \,,$$

where $i^*(Q) = Q \circ i$, for $Q \in \mathbb{D}$.

Lemma 4. *Observe that $i_!$ is left exact since the Dedekind interval $\mathrm{I}_{\mathbb{D}}$ is* strict, $0 \neq 1 : 1 \rightrightarrows \mathrm{I}_{\mathbb{D}}$. *Thus we have geometric morphisms:*

$$(i_! \dashv i^*) : \ \widehat{\mathbb{D}} \longrightarrow \widehat{\mathbb{C}} \,,$$

classifying the bipointed object $i_!(\mathrm{I}_{\mathbb{C}}) = \mathrm{I}_{\mathbb{D}}$,

$$(i^* \dashv i_*) : \ \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{D}} \,,$$

classifying the dLat $i^*(\mathrm{I}_{\mathbb{D}}) := \mathbb{I}$, where $\eta : \mathrm{I}_{\mathbb{C}} \longrightarrow \mathbb{I}$ can be described pointwise as the distributive lattice completion of the corresponding bipointed set.

Also, since i is faithful so is $i_!$, and since i is surjective on objects i^* is also faithful.

It follows that:

- *$\widehat{\mathbb{C}}$ is $(i_! \circ i^*)$-coalgebras on $\widehat{\mathbb{D}}$,*
- *$\widehat{\mathbb{D}}$ is $(i^* \circ i_*)$-coalgebras on $\widehat{\mathbb{C}}$,*
- *$\widehat{\mathbb{D}}$ is $(i^* \circ i_!)$-algebras on $\widehat{\mathbb{C}}$.*

We will use the following transfer theorem for QMSs from [**?**, **?**]:

Theorem ([**?**, **?**])**.** *Suppose $\widehat{\mathbb{D}}$ has a (cofibrantly generated) model structure $(\mathcal{C}_{\mathbb{D}}, \mathcal{W}_{\mathbb{D}}, \mathcal{F}_{\mathbb{D}})$. Given an adjunction*

$$i_! \dashv i^* \ : \ \widehat{\mathbb{D}} \longrightarrow \widehat{\mathbb{C}} \,,$$

there is a left-induced model structure *on $\widehat{\mathbb{C}}$ if the following* acyclicity condition *holds:*

$$(i_!^{-1}\mathcal{C}_{\mathbb{D}})^{\pitchfork} \ \subset \ i_!^{-1}\mathcal{W}_{\mathbb{D}} \,.$$

For the left-induced model structure $(\mathcal{C}_{\mathbb{C}}, \mathcal{W}_{\mathbb{C}}, \mathcal{F}_{\mathbb{C}})$ on $\widehat{\mathbb{C}}$ we then have:

$$\begin{aligned} \mathcal{C}_{\mathbb{C}} \ &= \ i_!^{-1}\mathcal{C}_{\mathbb{D}} \,, \\ \mathcal{W}_{\mathbb{C}} \ &= \ i_!^{-1}\mathcal{W}_{\mathbb{D}} \,. \end{aligned}$$

The Sattler model structure on $\widehat{\mathbb{D}}$ is given as follows (for a constructive treatment a smaller class of "pointwise decidable cofibrations" is used, but we consider the classical case first):

$$\begin{aligned}
\mathcal{C} &= \text{monomorphisms}\,, \\
\mathcal{W} &= \{f \mid f = p \circ i,\ p \in \mathcal{F} \cap \mathcal{W},\ i \in \mathcal{C} \cap \mathcal{W}\}\,, \\
\mathcal{F} &= (\mathcal{C} \otimes \delta)^{\pitchfork}\,.
\end{aligned}$$

where $\delta : 1 \longrightarrow \mathrm{I}$ is either endpoint inclusion.

For the left-induced model structure on $\widehat{\mathbb{C}}$ we therefore have the following specification:

$$\begin{aligned}
\mathcal{C} &= \text{monomorphisms}, \\
\mathcal{W} &= \{f \mid i_! f = p \circ i,\ p \in \mathcal{F}_{\mathbb{D}} \cap \mathcal{W}_{\mathbb{D}},\ i \in \mathcal{C}_{\mathbb{D}} \cap \mathcal{W}_{\mathbb{D}}\}\,, \\
\mathcal{F} &= (\mathcal{C} \cap \mathcal{W})^{\pitchfork}\,.
\end{aligned}$$

The determination of $\mathcal{C}$ follows from the fact that $i_! : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{D}}$ is conservative.

To check the acyclicity condition,

$$(i_!^{-1}\mathcal{C}_{\mathbb{D}})^{\pitchfork} \ \subset\ i_!^{-1}\mathcal{W}_{\mathbb{D}}\,,$$

we know that $i_!^{-1}\mathcal{C}_{\mathbb{D}}$ consists of the monos in $\mathbb{C}$, so take $f : Y \longrightarrow X$ in $(i_!^{-1}\mathcal{C}_{\mathbb{D}})^{\pitchfork}$, apply $i_!$, and factor the result as $i_! f = p \circ m : i_! Y \longrightarrow Z \longrightarrow i_! X$ with $p \in \mathcal{F}_{\mathbb{D}} \cap \mathcal{W}_{\mathbb{D}}$ and $m : i_! Y \longrightarrow Z$ monic. We then need to show that m is in $\mathcal{W}_{\mathbb{D}}$.

We can apply Theorem 2.2.1 of [?], with $\mathsf{K} = \widehat{\mathbb{C}}$, $\mathsf{M} = \widehat{\mathbb{D}}$, $V = i_!$, $k = i^*$, and:

1. $QX = X$ and $\epsilon = 1_X : X \longrightarrow X$, so that $i_! 1_X = 1_{i_! X}$ and therefore in $\mathcal{W}_{\mathbb{D}}$, while all objects are cofibrant,
2. $Qf = f$ for any $f : X \longrightarrow Y$ in $\widehat{\mathbb{C}}$, so that the naturality condition is similarly trivial,
3. factor the codiagonal $X + X \longrightarrow X$ as $\pi_2 \circ j : X + X \longrightarrow \mathrm{I} \times X \longrightarrow X$ with $j = (\partial \mathrm{I} \times X) : X + X \longrightarrow \mathrm{I} \times X$.

It remains only to show that $i_! p : \mathrm{i}_!(\mathrm{I} \times X) \longrightarrow i_! X$ is in $\mathcal{W}_{\mathbb{D}}$ and $i_! j : i_!(X + X) \longrightarrow i_!(\mathrm{I} \times X)$ is in $\mathcal{C}_{\mathbb{D}}$. The latter is clear, since j is monic. To show the former, observe that for any $D \in \widehat{\mathbb{D}}$, the projection $\pi_2 : \mathrm{I}_{\mathbb{D}} \times D \longrightarrow D$ is in $\mathcal{W}_{\mathbb{D}}$ by 3-for-2, since the "cylinder end" inclusion $D \longrightarrow \mathrm{I}_{\mathbb{D}} \times D$, as a pullback of an endpoint inclusion, is a cofibration, and a strong deformation retract (using the connection on I), and hence is in $\mathcal{W}_{\mathbb{D}}$ by [?].

Thus we have shown:

Theorem 5. *There is a Quillen model structure* $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ *on the category* $\widehat{\mathbb{C}}$ *of cartesian cubical sets, in which*

$$
\begin{aligned}
\mathcal{C} &= \textit{monomorphisms}, \\
\mathcal{W} &= \{ f \mid i_! f = p \circ i,\ p \in \mathcal{F}_{\mathbb{D}} \cap \mathcal{W}_{\mathbb{D}}\,,\ i \in \mathcal{C}_{\mathbb{D}} \cap \mathcal{W}_{\mathbb{D}} \}\,, \\
\mathcal{F} &= (\mathcal{C} \cap \mathcal{W})^{\pitchfork}\,.
\end{aligned}
$$

where $i_! : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{D}}$ *is the left adjoint of precomposition along the canonical map* $i : \mathbb{C} \longrightarrow \mathbb{D}$ *from Cartesian cubes to Dedekind cubes, and* $(\mathcal{C}_{\mathbb{D}}, \mathcal{W}_{\mathbb{D}}, \mathcal{F}_{\mathbb{D}})$ *is the Sattler model structure on* $\widehat{\mathbb{D}}$.

References:
- Gambino-Sattler
- Sattler
- Hess, Kedziorek, Riehl, Shipley
- Garner, Kedziorek, Riehl