Cubical model structures

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Abstract

Add an abstract.

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Introduction

[*need to fix epi arrow command*]

Add an introduction.

1 Cubical sets

There are now many treatments of cubical sets in the literature, inclusing [?, ?, ?, ?, ?, ?, ?, ?]. Our construction is intended to work in *all* of these, insofar as the axioms in Definition **??** below are satisfied. For the sake of concreteness, however, we shall consider what may be called the *cartesian* cube category $\square$, defined as the free finite product category on an interval $\delta_0, \delta_1 : 1 \rightrightarrows \mathrm{I}$.

Definition 1. The objects of the cartesian cube category $\square$, called n-cubes, are finite sets of the form

$$[n] = \{0, x_1, ..., x_n, 1\},$$

where $x_1, ..., x_n$, are formal *generators*. The arrows,

$$f : [n] \to [m],$$

may be taken to be m-tuples of elements drawn from the set $\{0, x_1, ..., x_n, 1\}$, regarded as formal terms representing composites of faces, degeneracies, permutations, and diagonals. Equivalently, the arrows $[n] \to [m]$ are arbitrary bipointed maps $[m] \to [n]$.

See [?] for further details.

Definition 2. The category cSet of *cubical sets* is the category of presheaves on the cartesian cube category $\square$,

$$\mathsf{cSet} \;=\; \mathsf{Set}^{\square^{\mathrm{op}}}.$$

It is of course generated by the representable presheaves $\mathsf{y}[n]$, to be written

$$\mathrm{I}^n = \mathsf{y}[n]$$

and called the *geometric n-cubes.*

Note that the representables I^n are closed under finite products, $\mathrm{I}^n \times \mathrm{I}^m = \mathrm{I}^{n+m}$. We of course write I for I^1 and 1 for I^0. We will need the following basic fact about the cubes I^n in cSet.

Proposition 3. *For each n, the n-cube I^n is* tiny*, in the sense that the exponential (or "internal* Hom*") functor $(-)^{\mathrm{I}^n} : \mathsf{cSet} \longrightarrow \mathsf{cSet}$ has a right adjoint.*

Proof. It clearly suffices to prove the claim for $n = 1$. For any cubical set X, the exponential X^{I} is a "shift by one dimension",

$$X^{\mathrm{I}}(n) \cong \mathrm{Hom}(\mathrm{I}^n, X^{\mathrm{I}}) \;\cong\; \mathrm{Hom}(\mathrm{I}^{n+1}, X) \cong\; X(n+1).$$

Thus X^{I} is given by precomposition with the "successor" functor $\square \to \square$ with $[n] \mapsto [n+1]$. Precomposition always has a right adjoint, which in this case we write as

$$X^{\mathrm{I}} \dashv X_{\mathrm{I}}$$

and calculate to be:

$$\begin{aligned}
X_{\mathrm{I}}(n) &\cong \mathrm{Hom}(\mathrm{I}^n, X_{\mathrm{I}}) \\
&\cong \mathrm{Hom}((\mathrm{I}^n)^{\mathrm{I}}, X) \\
&\cong \mathrm{Hom}((\mathrm{I}^{\mathrm{I}})^n, X) \\
&\cong \mathrm{Hom}((\mathrm{I}+1)^n, X) \\
&\cong \mathrm{Hom}\Big(\sum_{k=0}^{n} \binom{n}{k} \mathrm{I}^k, X\Big) \\
&\cong \prod_{k=0}^{n} X(k)^{\binom{n}{k}},
\end{aligned}$$

using the fact that $\mathrm{I}^{\mathrm{I}} \cong (\mathrm{I}+1)$ as in [**?**]. $\square$

2 The cofibration weak factorization system

Definition 4 (Cofibration)**.** The *cofibrations*, written

$$c : A \rightarrowtail B\,,$$

are any class $\mathcal{C}$ of monomorphisms in cSet satisfying the following axioms:

(C0) The map $0 \to C$ is always a cofibration.

(C1) All isomorphisms are cofibrations.

(C2) The composite of two cofibrations is a cofibration.

(C3) Any pullback of a cofibration is a cofibration.

We also require the cofibrations to be classified by a subobject $\Phi \hookrightarrow \Omega$ of the standard subobject classifier $\top : 1 \to \Omega$ of cSet:

(C4) There is a terminal object $t : 1 \rightarrowtail \Phi$ in the category of cofibrations and cartesian squares.

Note that we permit the case $\Phi = \Omega$ where all monos are cofibrations.

The cofibrant partial map classifier. We shall write

$$X^+ := \sum_{\varphi:\Phi} X^{[\varphi]} \;=\; \Phi_!\, t_*(X)\,,$$

for the polynomial endofunctor $\mathsf{cSet} \longrightarrow \mathsf{cSet}$ determined by the cofibration classifier $t : 1 \rightarrowtail \Phi$ (see [**?**]). Observe that by the definition of X^+ there is a pullback square,

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X^+ \arrow[d, "t_* X"] \\ 1 \arrow[r, "t"'] & \Phi \end{tikzcd}$$

since t is monic. Let $\eta : X \rightarrowtail X^+$ be the indicated top horizontal map; we call this map the *cofibrant partial map classifier* of X.

Proposition 5. *The map $\eta : X \rightarrowtail X^+$ classifies partial maps into X with cofibrant domain, in the following sense.*

1. *The map $\eta : X \rightarrowtail X^+$ is a cofibration.*

2. *For any object Z and any partial map $(s, g) : Z \leftarrowtail S \to X$, with $s : S \rightarrowtail Z$ a cofibration, there is a unique $f : Z \to X^+$ making a pullback square as follows.*

$$\begin{tikzcd} S \arrow[r, "g"] \arrow[d, tail, "s"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, tail, "\eta"] \\ Z \arrow[r, "f"'] & X^+ \end{tikzcd}$$

Proof. The map $\eta : X \rightarrowtail X^+$ is a cofibration since it is a pullback of $t : 1 \to \Phi$. Observe that $(\eta, 1_X) : X^+ \leftarrowtail X \to X$ is therefore a partial map into X with cofibrant domain. The second statement is the universal property of X^+ as a polynomial (see [**?**], prop. 7). $\square$

The +-monad.

Proposition 6. *The pointed endofunctor determined by $\eta_X : X \rightarrowtail X^+$ has a natural multiplication $\mu_X : X^{++} \to X^+$ making it a monad.*

Proof. Since the cofibrations are closed under composition, the monad structure on X^+ follows as in [?], proposition nm. Explicitly, μ_X is determined by proposition 5 as the unique map making the following a pullback diagram.

$$\begin{tikzcd}
X \arrow[rr, "="] \arrow[d, tail, "\eta_X"'] & & X \arrow[dd, tail, "\eta"] \\
X^+ \arrow[d, tail, "\eta_{X^+}"'] & & \\
X^{++} \arrow[rr, dotted, "\mu"'] & & X^+
\end{tikzcd}$$

□

Relative partial map classifier. For any object $X \in \mathsf{cSet}$ the usual pullback functor

$$X^* : \mathsf{cSet} \to \mathsf{cSet}/_X\,,$$

taking any A to the second projection $A \times X \to X$, not only preserves the subobject classifier Ω, but also the cofibration classifier $\Phi \hookrightarrow \Omega$, where a map in $\mathsf{cSet}/_X$ is defined to be a cofibration if it is one in cSet. Thus in $\mathsf{cSet}/_X$ the *(relative) cofibration classifier* is the map

$$t \times X : 1 \times X \to \Phi \times X \quad \text{over } X$$

which we may also write $t_X : 1_X \to \Phi_X$. Like $t : 1 \to \Phi$, this map determines a polynomial endofunctor

$$+_X : \mathsf{cSet}/_X \to \mathsf{cSet}/_X\,,$$

which commutes (up to natural isomorphism) with $+ : \mathsf{cSet} \to \mathsf{cSet}$ and $X^* : \mathsf{cSet} \to \mathsf{cSet}/_X$ in the evident way:

$$\begin{tikzcd}
\mathsf{cSet}/_X \arrow[r, "+_X"] & \mathsf{cSet}/_X \\
\mathsf{cSet} \arrow[u, "X^*"] \arrow[r, "+"'] & \mathsf{cSet} \arrow[u, "X^*"']
\end{tikzcd} \tag{1}$$

The endofunctor $+_X$ is also pointed $\eta : Y \to Y^+$ and has a monad multiplication $\mu_Y : Y^{++} \to Y^+$, for any $Y \to X$, for the same reason that $+$ has this structure. Summarizing, we may say that *the polynomial monad* $+ : \mathsf{cSet} \to \mathsf{cSet}$ *is fibered over* cSet.

Definition 7. A $+$-*algebra* in cSet is a cubical set A together with a retraction $\alpha : A^+ \to A$ of $\eta_A : A \to A^+$, i.e. an algebra for the pointed endofunctor $(+ : \mathsf{cSet} \to \mathsf{cSet},\ \eta : 1 \to +)$. Algebras for the monad $(+, \eta, \mu)$ will be referred to specifically as $(+, \eta, \mu)$-*algebras*, or $+$-*monad algebras*.

A *relative* $+$-*algebra* in cSet is a map $A \to X$ together with an algebra structure for the pointed endofunctor $+_X : \mathsf{cSet}/_X \to \mathsf{cSet}/_X$.

The cofibration weak factorization system.

Proposition 8. *There is an (algebraic) weak factoriation system on* cSet *with the cofibrations as the left class and as the right class, the maps underlying the relative* $+$*-algebras. Thus a right map is one* $f : A \to X$ *for which there is a retract* $\alpha : A' \to A$ *over* X *of the canonical map* $\eta_f : A \to A'$*,*

$$\begin{tikzcd} A \arrow[r, "\eta_f"] \arrow[rr, bend left, "="] \arrow[dr, "f"'] & A' \arrow[r, "\alpha"] \arrow[d, "f^+"] & A \arrow[dl, "f"] \\ & X. & \end{tikzcd}$$

Proof. The factorization of a map $f : Y \to X$ is given by applying the relative $+$-functor over the codomain,

$$\begin{tikzcd} Y \arrow[r, tail, "\eta_f"] \arrow[dr, "f"'] & Y' \arrow[d, "f^+"] \\ & X. \end{tikzcd}$$

We know by proposition 5 that the unit η_f is always a cofibration, and since f^+ is the free algebra for the $+$-monad of proposition 6, it is in particular a $+$-algebra.

For the lifting condition, consider a cofibration $c : B \rightarrowtail C$, a right map $A \to X$, with $+_X$-algebra structure map $\alpha : A^+ \to A$ over X, and a commutative square as indicated in the following.

$$\begin{tikzcd}
B \arrow[r, "g"] \arrow[dd, tail, "c"'] & A \arrow[dd] \arrow[dr, "\eta"'] & \\
& & A^+ \arrow[ul, bend right, "\alpha"'] \arrow[dl] \\
C \arrow[r, "f"'] & X &
\end{tikzcd}$$

Thus over X, we have the situation

$$\begin{tikzcd}
B \arrow[r, "g"] \arrow[d, tail, "c"'] & A \arrow[d, "\eta"] \\
C \arrow[ur, dotted, "d"'] & A^+ \arrow[u, bend right=60, "\alpha"']
\end{tikzcd}$$

and we seek a diagonal filler d as indicated. Since $(c, g) : B \leftarrowtail C \to A$ is a cofibrant partial map into A, there is a map $\varphi : C \to A^+$ (over X) making a (pullback) square,

$$\begin{tikzcd}
B \arrow[r, "g"] \arrow[d, tail, "c"'] & A \arrow[d, "\eta"] \\
C \arrow[r, dotted, "\varphi"'] & A^+ \arrow[u, bend right=60, "\alpha"']
\end{tikzcd}$$

We thus have $d := \alpha \circ \varphi : C \to A$ as the required diagonal filler.

The closure of the cofibrations under retracts follows from their classification by a universal object $t : 1 \to \Phi$, and the closure of the right maps under retracts follows from their being the algebras for a pointed endofunctor underlying a monad (cf. [**?**]). Algebraicity of this weak factorization system follows directly, since $+$ is a monad. $\square$

Summarizing, we have an algebraic weak factorization system $(\mathcal{L}, \mathcal{R})$ on the category cSet of cubical sets, in which:

$$\begin{aligned}
\mathcal{L} &= \text{the cofibrations } \mathcal{C} \\
\mathcal{R} &= \text{the maps underlying the relative } +\text{-algebras}
\end{aligned}$$

We shall call this the *cofibration weak factorization system.* The right maps will be called *trivial fibrations* and denoted by

$$\mathsf{TFib} = \mathcal{C}^{\pitchfork}.$$

The cofibration algebraic weak factorization system is a refinement of the one defined in [**?**] and mentioned in [**?**].

Uniform filling structure. It is convenient to relate +-algebra structure with the more familiar diagonal filling condition of cofibrantly generated weak factorization systems, and specifically a special form occuring in [**?**] under the name *uniform filling structure*.

Consider a generating sub*set* of cofibrations consisting of those $c : C \rightarrowtail \mathrm{I}^n$ with representable codomain, and call these the *basic cofibrations*.

$$\mathsf{BCof} = \{c : C \rightarrowtail \mathrm{I}^n \mid c \in \mathcal{C}, n \geq 0\}. \tag{2}$$

Proposition 9. *For any object X in* cSet *the following are equivalent:*

1. *X admits a* +-algebra structure: *a retraction $\alpha : X^+ \to X$ of the unit $\eta : X \to X^+$.*

2. *$X \to 1$ is a* trivial fibration: *it has the right lifting property with respect to all cofibrations,*
$$\mathcal{C} \pitchfork X.$$

3. *X can be given a* uniform filling structure: *for each basic cofibration $c : C \rightarrowtail \mathrm{I}^n$ and map $x : C \to X$ there is given an extension $j(c, x)$,*
$$\begin{tikzcd} C \arrow[r, "x"] \arrow[d, tail, "c"'] & X, \\ \mathrm{I}^n \arrow[ur, dotted, "j(c,x)"'] & \end{tikzcd} \tag{3}$$
and the choice is uniform in I^n *in the following sense: given any cubical map $u : \mathrm{I}^m \to \mathrm{I}^n$, the pullback $u^*c : u^*C \rightarrowtail \mathrm{I}^m$, which is again a basic cofibration, fits into a commutative diagram of the form*
$$\begin{tikzcd} u^*C \arrow[r, "c^*u"] \arrow[d, tail, "u^*c"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[r, "x"] \arrow[d, tail, "c"'] & X. \\ \mathrm{I}^m \arrow[r, "u"'] & \mathrm{I}^n \arrow[ur, dotted, "j(c,x)"'] & \end{tikzcd} \tag{4}$$
For the pair $(u^*c,\ x \circ c^*u)$ in (4), the chosen extension $j(u^*c, x \circ c^*u) : \mathrm{I}^m \to X$, is equal to $j(c, x) \circ u$,
$$j(u^*c, x \circ c^*u) = j(c, x) \circ u. \tag{5}$$

Proof. Let (X, α) be a $+$-*algebra* and suppose given the span (c, x) as below, with c a cofibration.

$$\begin{tikzcd} C \arrow[r, "x"] \arrow[d, tail, "c"'] & X \\ Z & \end{tikzcd}$$

Let $\chi(c, x) : Z \to X^+$ be the classifying map of the evident partial map $(c, x) : Z \rightharpoonup X$, so that we have a pullback square as follows.

$$\begin{tikzcd} C \arrow[rr, "x"] \arrow[d, tail, "c"'] \arrow[drr, phantom, very near start, "\lrcorner"] & & X \arrow[d, "\eta"] \\ Z \arrow[rr, "\chi(c,x)"'] & & X^+ \end{tikzcd} \tag{6}$$

Then set

$$j = \alpha \circ \chi(c, x) : Z \to X \tag{7}$$

to get a filler,

$$\begin{tikzcd} C \arrow[rr, "x"] \arrow[d, tail, "c"'] & & X \arrow[d, "\eta"] \\ Z \arrow[rr, "\chi(c,x)"'] \arrow[urr, dotted, "j"] & & X^+ \arrow[u, bend right, "\alpha"'] \end{tikzcd} \tag{8}$$

since

$$j \circ c = \alpha \circ \chi(c, x) \circ c = \alpha \circ \eta \circ x = x.$$

Thus (1) implies (2). To see that it also implies (3), observe that in the case where $Z = I^n$ and we specify, in (7), that

$$j(c, x) = \alpha \circ \chi(c, x) : \mathrm{I}^n \to X, \tag{9}$$

then the assignment is natural in I^n. Indeed, given any $u : \mathrm{I}^m \to \mathrm{I}^n$, we have

$$j(c', xu') = \alpha \circ \chi(c', xu') = \alpha \circ \chi(c, x) \circ u = j(c, x)u, \tag{10}$$

by the uniqueness of classifying maps.

It is clear that (2) implies (1), since if $\mathcal{C} \pitchfork X$ then we can take as an algebra structure $\alpha : X^+ \to X$ any filler for the span

$$\begin{tikzcd} X \arrow[r, "="] \arrow[d, tail, "\eta"'] & X. \\ X^+ \arrow[ur, dotted, "\alpha"'] & \end{tikzcd}$$

To see that (3) implies (1), suppose that X has a uniform filling structure j and we want to define an algebra structure $\alpha : X^+ \to X$. By Yoneda, for every $y : \mathrm{I}^n \to X^+$ we need a map $\alpha(y) : \mathrm{I}^n \to X$, naturally in I^n, in the sense that for any $u : \mathrm{I}^m \to \mathrm{I}^n$, we have

$$\alpha(yu) = \alpha(y)u. \tag{11}$$

Moreover, to ensure that $\alpha\eta = 1_X$, for any $x : \mathrm{I}^n \to X$ we must have $\alpha(\eta \circ x) = x$. So take $y : \mathrm{I}^n \to X^+$ and let

$$\alpha(y) = j(y^*\eta, y'),$$

as indicated on the right below.

$$\begin{tikzcd}
u^*C \arrow[r, "u'"] \arrow[d, tail, "u^*y^*\eta"'] \arrow[dr, phantom, very near start, "\lrcorner"] & C \arrow[r, "y'"] \arrow[d, tail, "y^*\eta"'] \arrow[dr, phantom, very near start, "\lrcorner"] & X. \arrow[d, "\eta"] \\
\mathrm{I}^m \arrow[r, "u"'] & \mathrm{I}^n \arrow[r, "y"'] \arrow[ur, dotted, "j(y^*\eta, y')"'] & X^+
\end{tikzcd} \tag{12}$$

Then for any $u : \mathrm{I}^m \to \mathrm{I}^n$, we indeed have

$$\alpha(yu) = j\big((yu)^*\eta, y'u'\big) = j(y^*\eta, y') \circ u = \alpha(y)u,$$

by the uniformity of j. Finally, if $y = \eta \circ x$ for some $x : \mathrm{I}^n \to X$ then

$$\alpha(\eta x) = j\big((\eta x)^*\eta, (\eta x)'\big) = j(1_X, x) = x,$$

because the defining diagram for $\alpha(\eta x)$, i.e. the one on the right in (12), then factors as

$$\begin{tikzcd}
\mathrm{I}^n \arrow[r, "x"] \arrow[d, tail, "="'] \arrow[dr, phantom, very near start, "\lrcorner"] & X \arrow[r, "="] \arrow[d, tail, "="'] \arrow[dr, phantom, very near start, "\lrcorner"] & X, \arrow[d, "\eta"] \\
\mathrm{I}^n \arrow[r, "x"'] & X \arrow[r, "\eta"'] & X^+
\end{tikzcd} \tag{13}$$

and the only possible extension $j(1_X, x)$ for the span $(1_{\mathrm{I}^n}, x)$ is x itself. $\square$

The relative version of the foregoing is entirely analogous, since the +-functor is fibered over cSet in the sense of diagram (1). We can therefore omit the entirely analogous proof. The statement is as follows.

Proposition 10. *For any map $f : Y \to X$ in cSet the following are equivalent:*

1. $f : Y \to X$ *admits a* (relative) $+$-algebra structure (over X), *i.e. there is a retraction* $\alpha : Y' \to Y$ *over* X *of the unit* $\eta : Y \to Y'$*, where* $f^+ : Y' \to X$ *is the result of the relative* $+$*-functor applied to* f*, as in definition 7.*

2. $f : Y \to X$ *is a* trivial fibration*,*
$$\mathcal{C} \pitchfork f.$$

3. $f : Y \to X$ *admits a* uniform filling structure*: for each basic cofibration* $c : C \rightarrowtail \mathrm{I}^n$ *and maps* $x : C \to X$ *and* $y : \mathrm{I}^n \to Y$ *making the square below commute, there is given a diagonal filler* $j(c, x, y)$*,*
$$\begin{tikzcd} C \arrow[r, "x"] \arrow[d, tail, "c"'] & X \arrow[d, "f"] \\ \mathrm{I}^n \arrow[r, "y"'] \arrow[ur, dotted, "j(c,x,y)"'] & Y, \end{tikzcd} \tag{14}$$
and the choice is uniform in I^n *in the following sense: given any cubical map* $u : \mathrm{I}^m \to \mathrm{I}^n$*, the pullback* $u^*c : u^*C \rightarrowtail \mathrm{I}^m$ *is again a basic cofibration and fits into a commutative diagram of the form*
$$\begin{tikzcd} u^*C \arrow[r, "c^*u"] \arrow[d, tail, "u^*c"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[r, "x"] \arrow[d, tail, "c"'] & X \arrow[d, "f"] \\ \mathrm{I}^m \arrow[r, "u"'] & \mathrm{I}^n \arrow[r, "y"'] \arrow[ur, dotted, "j(c,x,y)"'] & Y. \end{tikzcd} \tag{15}$$
For the evident triple $(u^*c,\, x \circ c^*u, y \circ u)$ *in (15) the chosen diagonal filler*
$$j(u^*c, x \circ c^*u, y \circ u) : \mathrm{I}^m \to X$$
is equal to $j(c, x, y) \circ u$*,*
$$j(u^*c, x \circ c^*u, y \circ u) = j(c, x, y) \circ u. \tag{16}$$

We next collect some basic facts about trivial fibrations: they have sections, they are closed under composition and retracts, and they are closed under pullback and pushforward along all maps.

Corollary 11. 1. *Every trivial fibration* $A \to X$ *has a section* $s : X \to A$*.*

2. *If $f : Y \to X$ is a trivial fibration and $g : Z \to Y$ is a trivial fibration, then $f \circ g : Z \to X$ is a trivial fibration.*

3. *If $f : Y \to X$ is a trivial fibration and $f' : Y' \to X'$ is a retract of f in the arrow category, then f' is a trivial fibration.*

4. *For any map $f : Y \to X$ and any trivial fibration $A \to X$, the pullback $f^*A \to Y$ is a trivial fibration.*

5. *For any map $f : Y \to X$ and any trivial fibration $A \to Y$, the pushforward $f_*A \to X$ is a trivial fibration.*

Proof. These are all standard. $\square$

3 The fibration weak factorization system

We now specify a second weak factorization system, with a restricted class of "trivial" cofibrations on the left, and an expanded class of right maps, the *fibrations*. For comparison, we first recall the trivial-cofibration/fibration weak factorization system from [**?**], which makes use of connecitons on the interval I, an assumption that we do without (in [**?**] it is shown that the fibrations of [**?**] agree with those specified in the "logical style" of [**?**, **?**]).

3.1 Partial box filling (biased version)

A *generating class of biased trivial cofibrations* are all maps of the form

$$c \otimes \delta_\epsilon : D \rightarrowtail Z \times \mathrm{I}\,, \tag{17}$$

where:

1. $c : C \rightarrowtail Z$ is an arbitrary cofibration,

2. $\delta_\epsilon : 1 \to \mathrm{I}$ is one of the two *endpoint inclusions*, for $\epsilon = 0, 1$.

3. $c \otimes \delta_\epsilon$ is the pushout-product indicated in the following diagram (in

which the unlabelled maps are the expected ones).

$$\begin{tikzcd}
C \times 1 \arrow[r] \arrow[d] & C \times \mathrm{I} \arrow[d] \arrow[ddr, bend left] & \\
Z \times 1 \arrow[r] \arrow[drr, bend right] & Z +_C (C \times \mathrm{I}) \arrow[dr, "c\otimes\delta_\epsilon"'] & \\
 & & Z \times \mathrm{I}
\end{tikzcd} \tag{18}$$

4. $D = Z +_C (C \times \mathrm{I})$ is the indicated domain of $c \otimes \delta_\epsilon$.

In order to ensure that such maps are indeed cofibrations, we assume two further axioms:

(C5) The endpoint inclusions $\delta_\epsilon : 1 \to \mathrm{I}$ are cofibrations.

(C6) The cofibrations are closed under pushout-products.

Note that if we assume δ_0 and δ_1 are disjoint (as they are in most categories of cubical sets), then by (C5) we have that $0 \to 1$ is a cofibration, and hence that $0 \to A$ is a cofibration, for all objects A, so that (C0) is not required in that case. In place of (C6), we could require the cofibrations to be closed under the join operation $A \vee B$ in the lattice of subobjects of an object (as is done in [**?**, **?**]).

Fibrations (biased version). Let

$$\mathcal{C} \otimes \delta_\epsilon \;=\; \{c \otimes \delta_\epsilon : D \rightarrowtail Z \times \mathrm{I} \mid c \in \mathcal{C},\ \epsilon = 0, 1\}$$

be the class of all such generating biased trivial cofibrations. The *biased fibrations* are defined to be the right class of these maps,

$$(\mathcal{C} \otimes \delta_\epsilon)^{\pitchfork} \;=\; \mathcal{F}\,.$$

Thus a map $f : Y \to X$ is a biased fibration if for every commutative square of the form

$$\begin{tikzcd}
Z +_C (C \times \mathrm{I}) \arrow[r] \arrow[d, tail, "c\otimes\delta_\epsilon"'] & Y \arrow[d, "f"] \\
Z \times \mathrm{I} \arrow[r] \arrow[ur, dotted, "j"] & X
\end{tikzcd} \tag{19}$$

with a generating biased trivial cofibration on the left, there is a diagonal filler j as indicated.

To relate this notion of fibration to the cofibration weak factorization system, fix any map $u : A \to B$, and recall (e.g. from [**?**]) that the pushout-product with u is a functor on the arrow category

$$(-) \otimes u : \mathsf{cSet}^{\mathbb{2}} \to \mathsf{cSet}^{\mathbb{2}} .$$

This functor has a right adjoint, the *pullback-hom*, which for a map $f : X \to Y$ we write as

$$(u \Rightarrow f) : Y^B \to (X^B \times_{X^A} Y^A) .$$

The pullback-hom is determined as indicated in the following diagram (in which the unlabelled maps are the expected ones).

$$\begin{tikzcd}
Y^B \arrow[dr, "u \Rightarrow f"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X^B \times_{X^A} Y^A \arrow[r] \arrow[d] & Y^A \arrow[d] \\
& X^B \arrow[r] & X^A
\end{tikzcd} \tag{20}$$

The $\otimes \dashv \Rightarrow$ adjunction on the arrow category has the following useful relation to weak factorization systems (cf. [**?**, **?**, **?**]), where, as usual, for any maps $a : A \to B$ and $f : X \to Y$ we write

$$a \pitchfork f$$

to mean that for every solid square of the form

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "a"'] & X \arrow[d, "f"] \\
B \arrow[r] \arrow[ur, dotted, "j"'] & Y
\end{tikzcd} \tag{21}$$

there exists a diagonal filler j as indicated.

Lemma 12. *For any maps $a : A_0 \to A_1, b : B_0 \to B_1, c : C_0 \to C_1$ in* cSet*,*

$$(a \otimes b) \pitchfork c \quad \textit{iff} \quad a \pitchfork (b \Rightarrow c) .$$

The following is now a direct corollary.

Proposition 13. *An object X is fibrant if and only if both of the endpoint projections $X^{\mathrm{I}} \to X$ from the pathspace are trivial fibrations. More generally, a map $f : Y \to X$ is a fibration iff both of the maps*

$$(\delta_\epsilon \Rightarrow f) : Y^I \to X^I \times_X Y$$

are trivial fibrations (for $\epsilon = 0, 1$).

Fibration structure (biased version). The $\otimes \dashv \Rightarrow$ adjunction determines the fibrations in terms of the trivial fibrations, which in turn can be determined by *uniform* lifting against a *set* of basic cofibrations, by proposition 10. We can similarly determine the fibrations by uniform lifting against a *set* of basic trivial cofibrations, consisting of all those $c \otimes \delta_\epsilon$ in $\mathcal{C} \otimes \delta_\epsilon$ where $c : C \rightarrowtail \mathrm{I}^n$, with a representable codomain. Call these maps the *basic biased trivial cofibrations*, and let

$$\mathcal{B} \otimes \delta_\epsilon = \{c \otimes \delta_\epsilon : B \rightarrowtail \mathrm{I}^{n+1} \mid c : C \rightarrowtail \mathrm{I}^n,\ \epsilon = 0, 1,\ n \geq 0\}, \tag{22}$$

where the pushout-product $c \otimes \delta_\epsilon$ now takes the simpler form

$$\begin{tikzcd}
C \arrow[r] \arrow[d, tail] & C \times \mathrm{I} \arrow[d] \arrow[ddr, bend left] & \\
\mathrm{I}^n \arrow[r] \arrow[drr, bend right] & \mathrm{I}^n +_C (C \times \mathrm{I}) \arrow[dr, tail, "c \otimes \delta_\epsilon"'] & \\
& & \mathrm{I}^n \times \mathrm{I}
\end{tikzcd} \tag{23}$$

for a cofibration $c : C \rightarrowtail \mathrm{I}^n$, an endpoint $\delta_\epsilon : 1 \to \mathrm{I}$, and with domain $B = \big(\mathrm{I}^n +_C (C \times \mathrm{I})\big)$. These subobjects $B \rightarrowtail \mathrm{I}^{n+1}$ can be seen geometrically as generalized open box inclusions.

For any map $f : Y \to X$ a *uniform, biased fibration structure* on f is a choice of diagonal fillers $j_\epsilon(c, x, y)$,

$$\begin{tikzcd}
\mathrm{I}^n +_C (C \times \mathrm{I}) \arrow[r, "x"] \arrow[d, "c \otimes \delta_\epsilon"'] & X \arrow[d, "f"] \\
\mathrm{I}^n \times \mathrm{I} \arrow[r, "y"'] \arrow[ur, dotted, "j_\epsilon(c,x,y)"'] & Y,
\end{tikzcd} \tag{24}$$

for each basic biased trivial cofibration $c \otimes \delta_\epsilon : B = (\mathrm{I}^n +_C (C \times \mathrm{I})) \rightarrowtail \mathrm{I}^{n+1}$ and maps $x : B \to X$ and $y : \mathrm{I}^{n+1} \to Y$, which is *uniform in* I^n in the following sense: given any cubical map $u : \mathrm{I}^m \to \mathrm{I}^n$, the pullback $u^*c : u^*C \rightarrowtail \mathrm{I}^m$ of $c : C \rightarrowtail \mathrm{I}^n$ along u determines another basic biased trivial cofibration

$$u^*c \otimes \delta_\epsilon : B' = (\mathrm{I}^m +_{u^*C} (u^*C \times \mathrm{I})) \rightarrowtail \mathrm{I}^{m+1},$$

which fits into a commutative diagram of the form

$$\begin{tikzcd}
\mathrm{I}^m +_{u^*C} (u^*C \times \mathrm{I}) \arrow[r, "(u\times \mathrm{I})'"] \arrow[d, "u^*c\otimes\delta_\epsilon"'] & \mathrm{I}^n +_C (C \times \mathrm{I}) \arrow[r, "x"] \arrow[d, "c\otimes\delta_\epsilon"'] & X \arrow[d, "f"] \\
\mathrm{I}^m \times \mathrm{I} \arrow[r, "u\times \mathrm{I}"'] & \mathrm{I}^n \times \mathrm{I} \arrow[r, "y"'] \arrow[ur, dotted, "j_\epsilon(c,x,y)"'] & Y,
\end{tikzcd} \tag{25}$$

by applying the functor $(-) \otimes \delta_\epsilon$ to the pullback square relating u^*c to c. Now for the outer rectangle in (27) there is a chosen diagonal filler

$$j_\epsilon(u^*c, x \circ (u \times \mathrm{I})', y \circ (u \times \mathrm{I})) : \mathrm{I}^m \times \mathrm{I} \to X$$

and for this map we require that

$$j_\epsilon(u^*c, x \circ (u \times \mathrm{I})', y \circ (u \times \mathrm{I})) = j_\epsilon(c, x, y) \circ (u \times \mathrm{I}). \tag{26}$$

This can be seen to be a reformulation of the logical specification given in [?] (see [?]).

Definition 14. A *uniform, biased fibration structure* on a map $f : Y \to X$ is a choice of fillers $j_\epsilon(c, x, y)$ as in (24) satisfying (26) for all maps $u : \mathrm{I}^m \to \mathrm{I}^n$.

Finally, we have the analogue of proposition 9 for fibrant objects; we omit the analogous statement of proposition 10 for fibrations, as well as the entirely analogous proof.

Corollary 15. *For any object X in* cSet *the following are equivalent:*

1. *X is* biased fibrant, *i.e. every partial map to X with a generating biased trivial cofibration $D \rightarrowtail Z \times \mathrm{I}$ as domain of definition extends to a total map $Z \times \mathrm{I} \to X$,*

$$\mathcal{C} \otimes \delta_\epsilon \ \pitchfork\ X$$

2. *The canonical maps $(\delta_\epsilon \Rightarrow X) : X^I \to X$ are trivial fibrations.*

3. $X \to 1$ *admits a* uniform biased fibration structure*. Explicitly, for each basic biased trivial cofibration* $c \otimes \delta_\epsilon : B \rightarrowtail \mathrm{I}^{n+1}$ *and map* $x : B \to X$*, there is given an extension* $j_\epsilon(c, x)$*,*

$$\begin{tikzcd} B \arrow[r, "x"] \arrow[d, tail, "c\otimes\delta_\epsilon"'] & X, \\ \mathrm{I}^{n+1} \arrow[ur, dotted, "j_\epsilon(c,x)"'] & \end{tikzcd} \tag{27}$$

and the choice is uniform in I^n *in the following sense: given any cubical map* $u : \mathrm{I}^m \to \mathrm{I}^n$*, the pullback* $u^*c \otimes \delta_\epsilon : B' \rightarrowtail \mathrm{I}^m \times \mathrm{I}$ *fits into a commutative diagram of the form*

$$\begin{tikzcd} B' \arrow[r, "(u\times\mathrm{I})'"] \arrow[d, tail, "u^*c\otimes\delta_\epsilon"'] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[r, "x"] \arrow[d, tail, "c\otimes\delta_\epsilon"'] & X. \\ \mathrm{I}^m \times \mathrm{I} \arrow[r, "u\times\mathrm{I}"'] & \mathrm{I}^n \times \mathrm{I} \arrow[ur, dotted, "j(c,x)"'] & \end{tikzcd} \tag{28}$$

For the pair $(u^*c \otimes \delta_\epsilon,\ x \circ (u \times \mathrm{I})')$ *in* (28) *the chosen extension*

$$j(u^*c \otimes \delta_\epsilon, x \circ (u \times \mathrm{I})') : \mathrm{I}^m \times \mathrm{I} \to X$$

is equal to $j(c, x) \circ (u \times \mathrm{I})$*,*

$$j(u^*c \otimes \delta_\epsilon, x \circ (u \times \mathrm{I})') = j(c, x)(u \times \mathrm{I}). \tag{29}$$

3.2 Partial box filling (unbiased version)

Rather than building a weak factorization system based on the foregoing notion of biased fibration (as is done in [**?**]), we shall first eliminate the bias on a choice of endpoint $\delta_\epsilon : 1 \to \mathrm{I}$, expressed by the indexing $\epsilon = 0, 1$. This will have the effect of adding more trivial cofibrations, and thus more weak equivalences, to our model structure. Consider first the simple path-lifting condition for a map $f : Y \to X$, which is a special case of (19) with $c = \,! : 0 \rightarrowtail 1$, since $! \otimes \delta_\epsilon = \delta_\epsilon$:

$$\begin{tikzcd} 1 \arrow[r] \arrow[d, tail, "\delta_\epsilon"'] & Y \arrow[d, "f"] \\ \mathrm{I} \arrow[r] \arrow[ur, dotted, "j_\epsilon"] & X \end{tikzcd}$$

In topological spaces, for instance, rather than requiring lifts j_ϵ for each of the endpoints $\epsilon = 0, 1$ of the real interval $\mathrm{I} = [0, 1]$, one could instead require that there be a lift j_i *for each point* $i : 1 \to \mathrm{I}$. Such "unbiased path-lifting" can be formulated in cSet by introducing a "generic point" $\delta : 1 \to \mathrm{I}$ by passing to cSet/I via the pullback functor $\mathrm{I}^* : \mathsf{cSet} \to \mathsf{cSet}/\mathrm{I}$, and then requiring path-lifting for $\mathrm{I}^* f$ with respect to δ. The following specification implements that idea, while also adding partiality in the sense of the foregoing section. We first replace axiom (C5) with the following stronger assumption.

(C7) The diagonal map $\delta : \mathrm{I} \to \mathrm{I} \times \mathrm{I}$ is a cofibration.

Th unbiased notion of a fibration is now as follows.

Definition 16 (Fibration)**.** Let $\delta : \mathrm{I} \to \mathrm{I} \times \mathrm{I}$ be the diagonal map.

1. An object X is *fibrant* if the map
$$(\delta \Rightarrow X) = \langle \mathsf{eval}, p_2 \rangle : X^{\mathrm{I}} \times \mathrm{I} \to X \times \mathrm{I}$$
is a trivial fibration.

2. A map $f : Y \to X$ is an *fibration* if the map
$$(\delta \Rightarrow f) = \langle f^{\mathrm{I}} \times \mathrm{I}, \langle \mathsf{eval}, p_2 \rangle \rangle : Y^{\mathrm{I}} \times \mathrm{I} \to (X^{\mathrm{I}} \times \mathrm{I}) \times_{(X \times \mathrm{I})} (Y \times \mathrm{I})$$
is a trivial fibration.

Condition (1) above, which is of course a special case of (2), says that evaluation at the generic point $\delta : 1 \to \mathrm{I}$, i.e. the map $X^\delta : X^{\mathrm{I}} \to X$ constructed in the slice category cSet/I, is a trivial fibration. Condition (2) says that the pullback-hom of the generic point $\delta : 1 \to \mathrm{I}$ with $\mathrm{I}^* f$, constructed in the slice category cSet/I, is a trivial fibration. The latter can be reformulated as follows.

Proposition 17. *A map $f : Y \to X$ is a fibration if and only if the canonical map c to the pullback, in the following diagram in* cSet*, is a trivial fibration.*

$$\begin{tikzcd}
Y^{\mathrm{I}} \times \mathrm{I} \arrow[dr, dotted, "c"'] \arrow[drr, bend left, "\mathsf{eval}"] \arrow[ddr, bend right, "f^{\mathrm{I}} \times \mathrm{I}"'] & & \\
& Y_{\mathsf{eval}} \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & Y \arrow[d, "f"] \\
& X^{\mathrm{I}} \times \mathrm{I} \arrow[r, "\mathsf{eval}"'] & X.
\end{tikzcd} \tag{30}$$

Proof. We interpolate another pullback into the rectangle in (30) to obtain

$$\begin{tikzcd} Y_{\mathsf{eval}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \times \mathrm{I} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "f"] \\ X^{\mathrm{I}} \times \mathrm{I} \arrow[r] & X \times \mathrm{I} \arrow[r] & X \end{tikzcd} \tag{31}$$

with the evident maps. The left hand square is therefore a pullback, so we indeed have that

$$Y_{\mathsf{eval}} \;=\; (X^{\mathrm{I}} \times \mathrm{I}) \times_{(X\times\mathrm{I})} (Y \times \mathrm{I})$$

and $c = (\delta \Rightarrow f)$. □

Now we can run the proof of Proposition 13 backwards in order to determine a class of generating trivial cofibrations for the unbiased case. Consider pairs of maps $c : C \rightarrowtail Z$ and $z : Z \to \mathrm{I}$, where the former is a cofibration and the latter is regarded as an "I-indexing", so that

$$\begin{tikzcd} C \arrow[r, tail, "c"] \arrow[dr] & Z \arrow[d, "z"] \\ & \mathrm{I} \end{tikzcd}$$

is regarded as an I-indexed family of cofibrations. Let

$$\mathsf{Gph}(z) : Z \to Z \times \mathrm{I}\,,$$

be the graph of $z : Z \to \mathrm{I}$, i.e. $\mathsf{Gph}(z) = \langle 1_Z, z\rangle$, and define

$$c \otimes_z \delta := [\mathsf{Gph}(z), c \times \mathrm{I}] : Z +_C (C \times \mathrm{I}) \to Z \times \mathrm{I}\,,$$

which is easily seen to be well-defined on the indicated pushout.

$$\begin{tikzcd} C \arrow[r, "\mathsf{Gph}(zc)"] \arrow[d, tail, "c"'] & C \times \mathrm{I} \arrow[d] \arrow[ddr, bend left, "c\times\mathrm{I}"] & \\ Z \arrow[r] \arrow[drr, bend right, "\mathsf{Gph}(z)"'] & Z +_C (C \times \mathrm{I}) \arrow[dr, dotted, "c\otimes_z\delta"'] & \\ & & Z \times \mathrm{I}\,. \end{tikzcd} \tag{32}$$

This specification differs from the similar (18) by using $\mathsf{Gph}(z)$ for the inclusion $Z \rightarrowtail Z \times \mathrm{I}$, rather than one of the "face maps" associated to the endpoint inclusions $\delta_\epsilon : 1 \to \mathrm{I}$. (Note that a graph is always a cofibration by pulling back a diagonal.) The subobject $c \otimes_z \delta \rightarrowtail Z \times \mathrm{I}$ is the join of the subobjects $\mathsf{Gph}(z) \rightarrowtail Z \times \mathrm{I}$ and the cylinder $C \times \mathrm{I} \rightarrowtail Z \times \mathrm{I}$.

Observe that the endpoints $\delta_\epsilon : 1 \to \mathrm{I}$ are of the form $c \otimes_z \delta$ by taking $Z = 1$ and $z = \delta_\epsilon$ and $c = \,! : 0 \to 1$, so that biased filling is subsumed.

The maps of the form $c \otimes_z \delta : Z +_C (C \times \mathrm{I}) \rightarrowtail Z$ now form a *class of generating trivial cofibrations* in the expected sense. Let

$$\mathcal{C} \otimes \delta \;=\; \{c \otimes_z \delta : D \rightarrowtail Z \times \mathrm{I} \mid c : C \rightarrowtail Z, z : Z \to \mathrm{I}\}\,. \tag{33}$$

The fibrations are exactly the right class of these,

$$(\mathcal{C} \otimes \delta)^{\pitchfork} = \mathcal{F}.$$

Proposition 18. *A map $f : Y \to X$ is a fibration iff for every pair of maps $c : C \rightarrowtail Z$ and $z : Z \to \mathrm{I}$, where the former is a cofibration, every commutative square of the following form has a diagonal filler, as indicated.*

$$\begin{tikzcd}
Z +_C (C \times \mathrm{I}) \arrow[r] \arrow[d, tail, "c \otimes_z \delta"'] & Y \arrow[d, "f"] \\
Z \times \mathrm{I} \arrow[r] \arrow[ur, dotted, "j"'] & X.
\end{tikzcd} \tag{34}$$

Proof. Suppose that for all $c : C \rightarrowtail Z$ and $z : Z \to \mathrm{I}$, we have $(c \otimes_z \delta) \pitchfork f$ in cSet. Pulling f back over I, this is equivalent to the condition $c \otimes \delta \pitchfork \mathrm{I}^* f$ in cSet/I, for all cofibrations $c : C \rightarrowtail Z$ over I, which is equivalent to $c \pitchfork (\delta \Rightarrow \mathrm{I}^* f)$ in cSet/I for all cofibrations $c : C \rightarrowtail Z$. But this in turn means that $\delta \Rightarrow \mathrm{I}^* f$ is a trivial fibration, which by definition means that f is a fibration. $\square$

Unbiased fibration structure. As in the biased case, the fibrations can also be determined by *uniform* right-lifting against a generating *set* of trivial cofibrations, now consisting of all those $c \otimes_z \delta$ in $\mathcal{C} \otimes \delta$ for which $c : C \rightarrowtail \mathrm{I}^n$, with a representable codomain. Call these maps the *basic (unbiased) trivial cofibrations*, and let

$$\mathcal{B} \otimes \delta = \{c \otimes_z \delta : B \rightarrowtail \mathrm{I}^{n+1} \mid c : C \rightarrowtail \mathrm{I}^n,\ z : \mathrm{I}^n \to \mathrm{I},\ n \geq 0\}\,, \tag{35}$$

where the pushout-product $c \otimes_z \delta$ now has the form

$$
\begin{tikzcd}
C \arrow[r, "\mathsf{Gph}(zc)"] \arrow[d, tail, "c"'] & C \times \mathrm{I} \arrow[d] \arrow[ddr, bend left, "c \times \mathrm{I}"] & \\
\mathrm{I}^n \arrow[r] \arrow[drr, bend right, "\mathsf{Gph}(z)"'] & \mathrm{I}^n +_C (C \times \mathrm{I}) \arrow[dr, dotted, "c \otimes_z \delta"'] & \\
 & & \mathrm{I}^n \times \mathrm{I}
\end{tikzcd}
\tag{36}
$$

for a cofibration $c : C \rightarrowtail \mathrm{I}^n$, an indexing map $z : \mathrm{I}^n \to \mathrm{I}$, and with domain $B = \big(\mathrm{I}^n +_C (C \times \mathrm{I})\big)$. These subobjects $B \rightarrowtail \mathrm{I}^{n+1}$ can again be seen geometrically as "generalized open box inclusions", but now the floor or lid of the open box may be replaced by a "cross-section" given by the graph of a map $z : \mathrm{I}^n \to \mathrm{I}$.

For any map $f : Y \to X$ a (uniform, unbiased) fibration structure on f is a choice of diagonal fillers $j(c, z, x, y)$,

$$
\begin{tikzcd}
B \arrow[r, "x"] \arrow[d, tail, "c \otimes_z \delta"'] & X \arrow[d, "f"] \\
\mathrm{I}^n \times \mathrm{I} \arrow[r, "y"'] \arrow[ur, dotted, "j(c,z,x,y)"] & Y,
\end{tikzcd}
\tag{37}
$$

for each basic trivial cofibration $c \otimes_z \delta : B \rightarrowtail \mathrm{I}^{n+1}$, which is *uniform* in I^n in the following sense: given any cubical map $u : \mathrm{I}^m \to \mathrm{I}^n$, the pullback $u^*c : u^*C \rightarrowtail \mathrm{I}^m$ and the reindexing $zu : \mathrm{I}^m \to \mathrm{I}^n \to \mathrm{I}$ determine another basic trivial cofibration $u^*c \otimes_{zu} \delta : B' = (\mathrm{I}^m +_{u^*C} (u^*C \times \mathrm{I})) \rightarrowtail \mathrm{I}^{m+1}$ which fits into a commutative diagram of the form

$$
\begin{tikzcd}
B' \arrow[r, "(u\times \mathrm{I})'"] \arrow[d, "u^*c \otimes_{zu} \delta"'] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[r, "x"] \arrow[d, "c \otimes_z \delta"'] & X \arrow[d, "f"] \\
\mathrm{I}^m \times \mathrm{I} \arrow[r, "u \times \mathrm{I}"'] & \mathrm{I}^n \times \mathrm{I} \arrow[r, "y"'] \arrow[ur, dotted, "j(c,z,x,y)"] & Y.
\end{tikzcd}
\tag{38}
$$

For the outer rectangle in (38) there is a chosen diagonal filler

$$j(u^*c, zu, x(u \times \mathrm{I})', y(u \times \mathrm{I})) : \mathrm{I}^m \times \mathrm{I} \to X,$$

and for this map we require that

$$j(u^*c, zu, x(u \times \mathrm{I})', y(u \times \mathrm{I})) = j(c, z, x, y) \circ (u \times \mathrm{I}). \tag{39}$$

Definition 19. A *(uniform, unbiased) fibration structure* on a map

$$f : Y \to X$$

is a choice of fillers $j(c, z, x, y)$ as in (37) satisfying (39) for all $u : \mathrm{I}^m \to \mathrm{I}^n$.

In these terms, we have following analogue of corollary 15.

Proposition 20. *For any object X in cSet the following are equivalent:*

1. *the canonical map $X^{\mathrm{I}} \times \mathrm{I} \to X \times \mathrm{I}$ is a trivial fibration.*
2. *X has the right lifting property with respect to all generating trivial cofibrations,*
$$(\mathcal{C} \otimes_z \delta) \pitchfork X.$$
3. *X has a uniform fibration structure in the sense of Definition 19.*

Proof. The equivalence between (1) and (2) is proposition 18. Suppose (1), i.e. that the map

$$(\delta \Rightarrow X) : X^{\mathrm{I}} \times \mathrm{I} \to X \times \mathrm{I}$$

is a relative +-algebra over $X \times \mathrm{I}$. By proposition 9, this means that $(\delta \Rightarrow X)$, as an object of $\mathsf{cSet}/(X \times \mathrm{I})$, has a uniform filling structure with respect to all cofibrations $c : C \rightarrowtail \mathrm{I}^n$ over $(X \times \mathrm{I})$. Transposing by the $\otimes \dashv \Rightarrow$ adjunction and unwinding gives, equivalently, a uniform fibration structure on X. $\square$

A statement analogous to the foregoing also holds for maps $f : Y \to X$ in place of objects X. Indeed, as before, we have the following sharper formulation.

Corollary 21. *Fibration structures on a map $f : Y \to X$ correspond uniquely to +-algebra structures on the map $(\delta \Rightarrow f)$ (cf. definition 16),*

$$(\delta \Rightarrow f) : Y^I \times \mathrm{I} \to (X^I \times \mathrm{I}) \times_{(X \times \mathrm{I})} (Y \times \mathrm{I})$$

3.3 Factorization

Definition 22. Summarizing the foregoing definitions and results, we have the following classes of maps:

- The *generating trivial cofibrations* were determined in (33) to be

$$\mathcal{C} \otimes \delta = \{c \otimes_z \delta : D \rightarrowtail Z \times \mathrm{I} \mid c : C \rightarrowtail Z,\ z : Z \to \mathrm{I}\}\,, \tag{40}$$

  where the pushout-product $c \otimes_z \delta$ has the form

$$\begin{tikzcd}
C \arrow[r, "\mathsf{Gph}(zc)"] \arrow[d, tail, "c"'] & C \times \mathrm{I} \arrow[d] \arrow[ddr, bend left, "c \times \mathrm{I}"] & \\
Z \arrow[r] \arrow[drr, bend right, "\mathsf{Gph}(z)"'] & Z +_C (C \times \mathrm{I}) \arrow[dr, dotted, tail, "c \otimes_z \delta"'] & \\
 & & Z \times \mathrm{I}\,.
\end{tikzcd} \tag{41}$$

  for any cofibration $c : C \rightarrowtail Z$ and indexing map $z : Z \to \mathrm{I}$, with domain $D = \big(Z +_C (C \times \mathrm{I})\big)$.

- The class $\mathcal{F}$ of *fibrations*, written $f : Y \twoheadrightarrow X$, may be characterized as the right-lifting class of the generating trivial cofibrations,

$$(\mathcal{C} \otimes \delta)^{\pitchfork} \;=\; \mathcal{F}.$$

- The class of *trivial cofibrations* is defined to be left class of the fibrations,

$$\mathsf{TCof} \;=\; {}^{\pitchfork}\mathcal{F}.$$

It follows that the classes TCof and $\mathcal{F}$ are mutually weakly orthogonal,

$$\mathsf{TCof} \pitchfork \mathcal{F},$$

and are closed under retracts. Thus to have a weak factorization system $(\mathsf{TCof}, \mathcal{F})$ it just remains to show that every map $f : X \to Y$ can be factored as $f = g \circ h$ with $g \in \mathcal{F}$ and $h \in \mathsf{TCof}$.

Proposition 23. *Every map $f : X \to Y$ in cSet can be factored as $f = p \circ i$,*

$$\begin{tikzcd}
X \arrow[r, tail, "i"] \arrow[dr, "f"'] & Y' \arrow[d, two heads, "p"] \\
 & Y
\end{tikzcd} \tag{42}$$

with $i : X \rightarrowtail Y'$ a trivial cofibration and $p : Y' \twoheadrightarrow Y$ a fibration.

Proof. This is a standard argument (cf. [**?**, **?**]), which is simpified in this case by the fact that the codomians of the generating trivial cofibrations are not just representable, but tiny. The reader is referred to [**?**] for the details (in a similar case). □

Proposition 24. *There is a weak factorization system on the category* cSet *in which the right maps are the fibrations and the left maps are the trivial cofibrations, both as specified in definition 22.*

This will be called the *fibration weak factorization system*. The following observation will be of use later on, the proof can be found in [**?**, **?**].

Corollary 25. *The construction of the* fibrant replacement*,*

$$f' = \varinjlim_{n} f_n$$

is functorial in f*, and the canonical trivial cofibrations* $i : X \rightarrowtail Y'$ *are natural in* X*.*

4 Weak equivalences

Definition 26 (Weak equivalence)**.** A map $f : X \to Y$ in cSet is a *weak equivalence* if it can be factored as $f = g \circ h$,

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[dr, "f"'] & W \arrow[d, "g"] \\ & Y \end{tikzcd}$$

with $h : X \to W$ a trivial cofibration and $g : W \to Y$ a trivial fibration. Let

$$\mathcal{W} = \{f : X \to Y \mid f = g \circ h \text{ for } g \in \mathsf{TFib} \text{ and } h \in \mathsf{TCof}\}$$

be the class of weak equivalences.

Observe that every trivial fibration $f \in \mathcal{C}^{\pitchfork}$ is indeed a fibration, because the generating trivial cofibrations are indeed cofibrations; moreover, every trivial fibration is also a weak equivalence, since the identity maps are trivial cofibrations. Thus we have

$$\mathsf{TFib} \subseteq (\mathcal{F} \cap \mathcal{W}).$$

Thus, because the trivial fibrations are fibrations, every trivial cofibration $g \in {}^{\pitchfork}\mathcal{F}$ is a cofibration; moreover, every trivial cofibration is also a weak equivalence, since the identity maps are also trivial fibrations. Thus we also have

$$\mathsf{TCof} \subseteq (\mathcal{C} \cap \mathcal{W}).$$

Lemma 27. $(\mathcal{C} \cap \mathcal{W}) \subseteq \mathsf{TCof}$.

Proof. Let $c : A \rightarrowtail B$ be a cofibration with a factorization

$$c = tf \circ tc : A \to W \to B$$

where $tc \in \mathsf{TCof}$ and $tf \in \mathsf{TFib}$. Let $f : Y \twoheadrightarrow X$ be a fibration and consider a commutative diagram,

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, tail, "c"'] & Y \arrow[d, two heads, "f"] \\
B \arrow[r, "b"'] & X.
\end{tikzcd}$$

Inserting the factorization of c, we have $j : W \to Y$ as indicated, with $j \circ tc = a$ and $f \circ j = b \circ tf$, since $tc \pitchfork f$.

$$\begin{tikzcd}
A \arrow[rr, "a"] \arrow[dr, "tc"'] \arrow[dd, tail, "c"'] & & Y \arrow[dd, two heads, "f"] \\
& W \arrow[ur, dotted, "j"'] \arrow[dl, "tf"] & \\
B \arrow[rr, "b"'] & & X.
\end{tikzcd}$$

Moreover, since $c \pitchfork tf$ there is an $i : B \to W$ as indicated, with $i \circ c = tc$ and $tf \circ i = 1_B$.

$$\begin{tikzcd}
A \arrow[rr, "a"] \arrow[dr, "tc"'] \arrow[dd, tail, "c"'] & & Y \arrow[dd, two heads, "f"] \\
& W \arrow[ur, dotted, "j"'] \arrow[dl, "tf"] & \\
B \arrow[ur, dotted, bend right, "i"'] \arrow[rr, "b"'] & & X.
\end{tikzcd}$$

Let $k = j \circ i$. Then $k \circ c = j \circ i \circ c = j \circ tc = a$, and $f \circ k = f \circ j \circ i = b \circ tf \circ i = b$. $\square$

The proof of the following is dual:

Lemma 28. $(\mathcal{F} \cap \mathcal{W}) \subseteq \mathsf{TFib}$.

Proposition 29. *For the three classes of maps $\mathcal{C}, \mathcal{W}, \mathcal{F}$ in* cSet*, we have*

$$\begin{aligned} \mathcal{F} \cap \mathcal{W} &= \mathsf{TFib}, \\ \mathcal{C} \cap \mathcal{W} &= \mathsf{TCof}, \end{aligned}$$

and therefore two weak factorization systems:

$$(\mathcal{C},\, \mathcal{W} \cap \mathcal{F}) \quad , \quad (\mathcal{C} \cap \mathcal{W},\, \mathcal{F}).$$

Corollary 30. *The following are equivalent for a map $f : X \to Y$.*

1. *$f : X \to Y$ is a weak equivalence*
2. *the first factor $\eta : X \to X'$ of the cofibration factorization of f is a trivial cofibration.*
3. *the second factor $p : Y' \to Y$ of the fibration factorization of f is a trivial fibration.*

It thus remains only to prove that the weak equivalences satisfy the 3-for-2 property. Proving this will take the rest of the paper!

Weak homotopy equivalence

Definition 31. By a *homotopy* between parallel maps $f, g : X \rightrightarrows Y$, written $\vartheta : f \sim g$, we shall mean a map from the *cylinder of* X built using the (representable) interval I,

$$\vartheta : \mathrm{I} \times X \to Y,$$

and such that $\vartheta \circ \iota_0 = f$ and $\vartheta \circ \iota_1 = g$,

$$\begin{tikzcd} X \arrow[r, "\iota_0"] \arrow[dr, "f"'] & \mathrm{I} \times X \arrow[d, "\vartheta"] & X, \arrow[l, "\iota_1"'] \arrow[dl, "g"] \\ & Y & \end{tikzcd}$$

where we write the canonical inclusions into the ends of the cylinder as

$$\iota_\epsilon = \mathsf{Gph}(\delta_\epsilon !) : X \to \mathrm{I} \times X\,, \qquad \epsilon = 0, 1.$$

Proposition 32. *If K is fibrant, then the relation of homotopy $f \sim g$ between maps $f, g : X \rightrightarrows K$ is an equivalence relation. Moreover, it is compatible with pre- and post-composition.*

Proof. For $f, g : X \rightrightarrows Y$, a homotopy $f \overset{\vartheta}{\sim} g : X \times \mathrm{I} \to Y$ is equivalent, under exponential transposition, to a path in the function space $\vartheta : \mathrm{I} \to Y^X$ with endpoints $\vartheta_0 = \vartheta \circ \delta_0 = f : 1 \to Y^X$ and $\vartheta_1 = g$. Note that Y^X is fibrant if Y is fibrant, so we can use box-filling in Y^X.

The reflexivity of homotopy $f \sim f$ is witnessed by $\rho : \mathrm{I} \to 1 \xrightarrow{f} Y^X$.

For symmetry $f \sim g \Rightarrow g \sim f$ take $\vartheta : \mathrm{I} \to Y^X$ with $\vartheta_0 = f$ and $\vartheta_1 = g$ and we want to build $\vartheta' : \mathrm{I} \to Y^X$ with $\vartheta'_0 = g$ and $\vartheta'_1 = f$. Take an open 2-box in Y^X of the form

$$\begin{tikzcd} g & f \\ f \arrow[u, "\vartheta"] \arrow[r, "\rho"'] & f \arrow[u, "\rho"'] \end{tikzcd}$$

This box is a map $b : \mathrm{I} +_1 \mathrm{I} +_1 \mathrm{I} \to Y^X$ with the indicated components, and it has a filler $c : \mathrm{I} \times \mathrm{I} \to Y^X$, i.e. an extension along the canonical map $\mathrm{I} +_1 \mathrm{I} +_1 \mathrm{I} \rightarrowtail \mathrm{I} \times \mathrm{I}$, which is a trivial cofibration. Let $t : \mathrm{I} \to \mathrm{I} \times \mathrm{I}$ be the evident missing top face of the 2-cube. Then we can set $\vartheta' = ct : \mathrm{I} \to Y^X$ to get a homotopy $\vartheta' : \mathrm{I} \to Y^X$ with required endpoints.

For transitivity, $f \overset{\vartheta}{\sim} g \;\&\; g \overset{\varphi}{\sim} h \Rightarrow f \sim h$, an analogous filling construction is used with the open box:

$$\begin{tikzcd} f & h \\ f \arrow[u, "\rho"] \arrow[r, "\vartheta"'] & g \arrow[u, "\varphi"'] \end{tikzcd}$$

Compatibility under pre- and post-composition is shown by representing homotopy by mapping into the pathspace, for precomposition, and out of the cylinder, for post-composition. $\square$

Definition 33 (Connected components)**.** The functor

$$\pi_0 : \mathsf{cSet} \to \mathsf{Set}$$

is defined, for any cubical set X, to be the coequalizer

$$X_1 \rightrightarrows X_0 \to \pi_0 X,$$

where the two parallel arrows are the maps $X_{\delta_0}, X_{\delta_1} : X_1 \rightrightarrows X_0$ induced by the endpoints $\delta_0, \delta_1 : 1 \rightrightarrows \mathrm{I}$. For any Kan complex K we therefore have $\pi_0 K = \mathrm{Hom}(1, K)/{\sim}$, that is, $\pi_0 K$ is the set of points $1 \to K$, modulo the homotopy equivalence relation on them.

One can show that in fact $\pi_0 X = \varinjlim_n X_n$, the colimit being left adjoint to the constant presheaf functor $\Delta : \mathsf{Set} \to \mathsf{cSet}$. Since the category $\mathbb{B}$ of finite strictly bipointed sets is sifted, we have:

Corollary 34. *The functor $\pi_0 : \mathsf{cSet} \to \mathsf{Set}$ preserves finite products.*

As usual, a map $f : X \to Y$ in cSet will be called a *homotopy equivalence* if there is a *quasi-inverse* $g : Y \to X$ and homotopies $\vartheta : 1_X \sim g \circ f$ and $\varphi : 1_Y \sim f \circ g$.

Definition 35 (Weak homotopy equivalence)**.** A map $f : X \to Y$ will be called a *weak homotopy equivalence* if for every fibrant object K, the "internal precomposition" map $K^f : K^Y \to K^X$ is bijective on connected components, i.e.

$$\pi_0 K^f : \pi_0 K^Y \to \pi_0 K^X$$

is a bijection of sets.

Lemma 36. *A homotopy equivalence is a weak homotopy equivalence.*

Proof. If $f : X \to Y$ is a homotopy equivalence, then so is $K^f : K^Y \to K^X$ for any K, since homotopy respects composition. Since K^X is always fibrant when K is, π_0 is well defined, and it clearly takes homotopy equivalences to isomorphisms of sets. $\square$

Lemma 37. *The weak homotopy equivalences $f : X \to Y$ satisfy the 3-for-2 condition.*

Proof. Follows from the corresponding fact about bijections of sets. $\square$

Our goal of showing that the weak equivalences satisfy 3-for-2 is now reduced to showing that a map is a weak equivalence (WE) if and only if it is a weak homotopy equivalence (WHE). This will be proved in four cases, showing that a (co)fibration is a WE if and only if it is a WHE.

Lemma 38. *A map $f : X \to Y$ is a* weak homotopy equivalence *iff it satisfies the following two conditions.*

1. *For every fibrant object K and every map $x : X \to K$ there is a map $y : Y \to K$ such that $y \circ f \sim x$,*

$$\begin{tikzcd} X \arrow[r, "x"] \arrow[d, "f"'] & K. \\ Y \arrow[ur, dotted, "y"'] \arrow[u, phantom, "\sim", shift left=6] & \end{tikzcd}$$

   We say that x "extends along f up to homotopy".

2. *For every fibrant object K and maps $y, y' : Y \to K$ such that $yf \sim y'f$, there is a homotopy $y \sim y'$,*

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & K^{\mathrm{I}} \arrow[d] \\ Y \arrow[r, "\langle y, y' \rangle"'] \arrow[ur, dotted] & K \times K. \end{tikzcd}$$

Proof. Unwind the definition. □

Lemma 39. *A cofibration $c : A \rightarrowtail B$ that is a WE is a WHE.*

Proof. A cofibration $c : A \rightarrowtail B$ that is a WE is a trivial cofibration by proposition 29. So the result follows from Lemma 38, and the fact that $K^{\mathrm{I}} \to K \times K$ is always a fibration when K is fibrant. □

Lemma 40. *A fibration $p : Y \twoheadrightarrow X$ that is a WE is a WHE.*

Proof. A fibration weak equivalence $f : Y \twoheadrightarrow X$ is a trivial fibration by proposition 29, and therefore has a section $s : X \rightarrowtail Y$, by the lifting problem

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, tail] & Y \arrow[d, two heads, "f"] \\ X \arrow[r, "="'] & X, \end{tikzcd}$$

since $0 \to X$ is always a cofibration. Moreover, there is a homotopy $\vartheta : sf \sim 1_Y$, resulting from the lifting problem

$$\begin{tikzcd} Y + Y \arrow[r, "{[sf,1]}"] \arrow[d, tail, "{[\iota_0,\iota_1]}"'] & Y \arrow[d, two heads, "f"] \\ \mathrm{I} \times Y \arrow[r, "f\pi_2"'] & X. \end{tikzcd}$$

Thus f is a homotopy equivalence, and so a WHE by lemma 36. □

Lemma 41. *If K is fibrant, then any fibration $f : Y \twoheadrightarrow K$ that is a HE is a WE.*

Proof. This is a standard argument, which we just sketch. It suffices to show that any diagram of the form

$$\begin{tikzcd} C \arrow[r, "y"] \arrow[d, tail, "c"'] & Y \arrow[d, two heads, "f"] \\ K \arrow[r, "="'] & K, \end{tikzcd} \tag{43}$$

with $c : C \rightarrowtail X$ a cofibration, has a diagonal filler. Since f is a HE it has a quasi-inverse $s : X \to Y$ with $\vartheta : fs \sim 1_K$, which we can correct to a section $s' : K \to Y$. Indeed, consider

$$\begin{tikzcd} K \arrow[r, "s"] \arrow[d, tail, "\iota_0"'] & Y \arrow[d, two heads, "f"] \\ K \times \mathrm{I} \arrow[r, "\vartheta"'] \arrow[ur, dotted, "\vartheta'"] & K \\ K \arrow[u, tail, "\iota_1"] \arrow[ur, "="'] & \end{tikzcd}$$

where ϑ' results from $\iota_0 \pitchfork f$. Let $s' = \vartheta'\iota_1$, so that $\vartheta' : s \sim s'$ and $fs' = 1_K$.

Thus we can assume that $s = s' : K \to Y$ is a section, which fills the diagram (43) up to a homotopy in the upper triangle.

$$\begin{tikzcd} C \arrow[r, "y"] \arrow[d, tail, "c"'] \arrow[dr, phantom, "\sim"] & Y \arrow[d, two heads, "f"] \\ K \arrow[r, "="'] \arrow[ur, "s"'] & K, \end{tikzcd}$$

Now we can correct $s : K \to Y$ to a homotopic $t : K \to Y$ over f by using the homotopy $\varphi : sc \sim y$ to get a map $\varphi : C \to Y^{\mathrm{I}}$ over f. Since f is a fibration, the projections $p_0, p_1 : Y^{\mathrm{I}} \to Y$ over f are trivial fibrations, and so there is a lift $\varphi' : K \to Y^{\mathrm{I}}$ for which $t := p_1\varphi'$ has $tc = y$ and $ft = 1_K$, and so is a filler for (43). $\square$

Lemma 42. *If K is fibrant, then any fibration $f : Y \twoheadrightarrow K$ that is a WHE is a WE.*

Proof. Since K is fibrant, so is Y, and since f is a WHE, there is a map $s : K \to Y$ and a homotopy $\theta : sf \sim 1_Y$ by lemma 38(1). Thus, applying f again, we have a homotopy $f\vartheta : fsf \sim f$, forming the outer commutative square in

$$\begin{tikzcd} Y \arrow[r, "f\vartheta"] \arrow[d, "f"'] & K^{\mathrm{I}} \arrow[d] \\ K \arrow[r, "\langle fs, 1_K \rangle"'] \arrow[ur, dotted, "\varphi"] & K \times K. \end{tikzcd}$$

By lemma 38(2) there is a diagonal filler $\varphi : fs \sim 1_K$, and so f is a HE. Now apply lemma 41. □

Lemma 43. *If K is fibrant, then any cofibration $c : A \rightarrowtail K$ that is a WHE is a WE.*

Proof. Let $c : A \rightarrowtail K$ be a cofibration WHE and factor it into a trivial cofibration $i : A \rightarrowtail Z$ followed by a fibration $p : Z \twoheadrightarrow K$. By lemma 38, it is clear that a trivial cofibration is a WHE. So both c and i are WHE, and therefore so is p by 3-for-2 for WHEs. Since K is fibrant, p is a trivial fibration by lemma 42, and thus c is a WE. □

Lemma 44 ([**?**], x.n.m)**.** *A cofibration $c : A \rightarrowtail B$ WHE lifts against all fibrations $f : Y \twoheadrightarrow K$ with fibrant codomain.*

Proof. Let $c : A \rightarrowtail B$ be a cofibration WHE and $f : Y \twoheadrightarrow K$ a fibration with fibrant codomain K, and consider a lifting problem

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, tail, "c"'] & Y \arrow[d, two heads, "f"] \\ B \arrow[r, "b"'] & K. \end{tikzcd}$$

Let $\eta : B \rightarrowtail B'$ be a fibrant replacement of B, since K is fibrant, b extends along η to give $b' : B' \to K$ as in:

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, tail, "c"'] & Y \arrow[d, two heads, "f"] \\ B \arrow[r, "b"'] \arrow[d, "\eta"'] & K \\ B' \arrow[ur, "b'"'] & \end{tikzcd}$$

Since η is a trivial cofibration, it is a WHE. So the composite ηc is also a WHE. But since B' is fibrant, ηc is then a trivial cofibration by lemma 43. Thus there is a lift $j : B' \to Y$, and therefore also one $k = j\eta : B \to Y$. $\square$

To complete the proof that a cofibration WHE is a WE we will use the fact that the fibration weak factorization system satisfies the fibration extension property, the proof of which is deferred to section 8.

Definition 45 (Fibration extension)**.** The fibration weak factorization system is said to satisfy the *fibration extension property* (FEP) if the following holds: Given a fibration $f : Y \twoheadrightarrow X$ and a trivial cofibration $\eta : X \to X'$, there is a fibration $f' : Y' \twoheadrightarrow X'$ such that f is a pullback of f' along η.

$$\begin{tikzcd} Y \arrow[r] \arrow[d, two heads, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y' \arrow[d, two heads, "f'"] \\ X \arrow[r, tail, "\eta"'] & X'. \end{tikzcd} \tag{44}$$

Lemma 46. *Assuming the FEP, a cofibration that lifts against every fibration $f : Y \twoheadrightarrow K$ with fibrant codomain is a WE.*

Proof. Let $c : A \rightarrowtail B$ be a cofibration and consider a lifting problem against an arbitrary fibration $f : Y \twoheadrightarrow X$,

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, tail, "c"'] & Y \arrow[d, two heads, "f"] \\ B \arrow[r, "b"'] & X. \end{tikzcd} \tag{45}$$

Let $\eta : X \to X'$ be a fibrant replacement, so η is a trivial cofibration and X' is fibrant. By the fibration extension property of definition 45, there is a fibration $f' : Y' \twoheadrightarrow X'$ such that f is a pullback of f' along η. So we can extend diagram (45) to obtain the following, in which the righthand square is a pullback.

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, tail, "c"'] & Y \arrow[r, "y"] \arrow[d, two heads, "f"] \arrow[dr, phantom, "\lrcorner", very near start] & Y' \arrow[d, two heads, "f'"] \\ B \arrow[r, "b"'] & X \arrow[r, "\eta"'] & X'. \end{tikzcd} \tag{46}$$

By assumption, there is a lift $j' : B \to Y'$ with $f'j' = \eta b$ and $j'c = yb$. Therefore, since f is a pullback, there is a map $j : B \to Y$ with $fj = b$ and $yj = j'$.

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, tail, "c"'] & Y \arrow[r, "y"] \arrow[d, two heads, "f"] \arrow[dr, phantom, "\lrcorner", very near start] & Y' \arrow[d, two heads, "f'"] \\
B \arrow[r, "b"'] \arrow[ur, dotted, "j"] \arrow[urr, dotted, "j'"'] & X \arrow[r, "\eta"'] & X'.
\end{tikzcd} \tag{47}$$

Thus $yjc = j'c = ya$. But as a trivial cofibration, η is monic, and as a pullback of η, y is also monic. So $jc = a$. □

Combining the previous two lemmas 44 and 46 we now have the following.

Corollary 47. *Assuming the FEP, a cofibration $c : A \rightarrowtail B$ that is a WHE is a WE.*

The following is not required, but we state it anyway for the record:

Lemma 48. *Assuming the FEP, a fibration $f : Y \twoheadrightarrow X$ that is a WHE is a WE.*

Proof. Factor $f : Y \twoheadrightarrow X$ into a cofibration $i : Y \rightarrowtail Z$ followed by a trivial fibration $p : Z \twoheadrightarrow X$. Then f is a trivial fibration if $i \pitchfork f$, for then f is a retract of p. Since p is a trivial fibration, it is a WHE by lemma 40. Since f is also a WHE, so is i by 3-for-2. Thus i is a trivial cofibration by corollary 47. Since f is a fibration, $i \pitchfork f$ as required. □

Proposition 49. *Assuming the FEP, a map $f : X \rightarrowtail Y$ is a WHE if and only if it is a WE. Thus the weak equivalences $\mathcal{W}$ satisfy the 3-for-2 condition.*

Proof. Let $f : X \twoheadrightarrow Y$ be a WE and factor it into a trivial cofibration $i : X \rightarrowtail Z$ followed by a trivial fibration $p : Z \twoheadrightarrow Y$. Then both i and p are WHE, whence so is f. Conversely, let f be a WHE and factor it into a cofibration $i : X \rightarrowtail Z$ followed by a trivial fibration $p : Z \twoheadrightarrow Y$. Since p is then a WHE, as is f, it follows that i is as well. Thus i is also a WE, by lemma 47, hence a trivial fibration. So f is a WE. □

We summarize the results to this point in the following.

Theorem 50. *If the fibration weak factorization system of Definition 22 satisfies the fibration extension property of Definition 45, then the weak equivalences $\mathcal{W}$ have the 3-for-2 property, and so by Proposition 29, the three classes $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ form a Quillen model structure.*

The proof of the fibration extension property will occupy the remainder of this paper. It requires several intermediate results: the equivalence extension property (Section 7), a universal fibration (section 6), and the Frobenius condition (section 5), to which we now turn.

5 The Frobenius condition

In this section, we show that the fibration WFS from section 3 has the *Frobenius property*: the left maps are stable under pullback along the right maps (see [?]). This will imply the *right properness* of our model structure: the weak equivalences are preserved by pullback along fibrations. The Frobenius property is also needed in the proof of the equivalence extension property in the next section. A proof of Frobenius in a related setting of cubical sets with connections can be found in [?]; however the type theoretic approach of [?, ?] can be applied without connections and is also more direct. This approach proves the "dual" fact that the *pushforward* operation, which is right adjoint to pullback, and which always exists in a topos, when applied along any *fibration* $f : Y \twoheadrightarrow X$ preserves fibrations. This corresponds to the type-theoretic Π-formation rule.

Recall that a map $f : A \to X$ is a fibration if (in the slice cSet/I, where there is a generic point $\delta : 1 \to \mathrm{I}$) the map $\delta \Rightarrow f$ admits a $+$-algebra structure (and so is a trivial fibration), where the definition of $\delta \Rightarrow f$ is recalled below.

$$\begin{tikzcd} A^{\mathrm{I}} \arrow[dr, "\delta\Rightarrow f"] \arrow[drr, bend left, "A^\delta"] \arrow[ddr, bend right, "f^{\mathrm{I}}"'] & & \\ & X^{\mathrm{I}} \times_X A \arrow[r] \arrow[d] & A \arrow[d, "f"] \\ & X^{\mathrm{I}} \arrow[r, "X^\delta"'] & X. \end{tikzcd} \tag{48}$$

Let us write this condition schematically as follows:

$$\begin{tikzcd} A^{\mathrm{I}} \arrow[r, "+" marking] & A_\epsilon \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ & X^{\mathrm{I}} \arrow[r, "\epsilon"'] & X, \end{tikzcd} \tag{49}$$

where $\epsilon = X^\delta$, $A_\epsilon = X^{\mathrm{I}} \times_X A$, and the struck-through arrow indicates that it admits a $+$-algebra structure.

Lemma 51. *Let $g : Y \to X$ be any map and $f : A \to X$ a fibration, then the pullback $g^*f : g^*A \to Y$ is also a fibration.*

Proof. This is clear, since fibrations are the right class of a weak factorization system, but let us see how the "algebraic" specification (52) is also stable under pullback ... □

Lemma 52. *Let $\alpha : A \to X$ and $\beta : B \to A$ be fibrations, then the composite $\alpha \circ \beta : B \to X$ is also a fibration.*

Proof. We have the following for the fibration structures on $B \to A$ and $A \to X$ (with obvious notation).

$$
\begin{tikzcd}
B^{\mathrm{I}} \arrow[r, "\shortmid" marking] & B_{\epsilon_A} \arrow[rr] \arrow[d] \arrow[drr, phantom, "\lrcorner", very near start] & & B \arrow[d] \\
& A^{\mathrm{I}} \arrow[r, "\shortmid" marking] & A_{\epsilon_X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d] \\
& & X^{\mathrm{I}} \arrow[r, "\epsilon_X"'] & X,
\end{tikzcd}
\tag{50}
$$

Pulling back $B \to A$ in two steps we therefore obtain,

$$
\begin{tikzcd}
B^{\mathrm{I}} \arrow[r, "\shortmid" marking] & B_{\epsilon_A} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B_{\epsilon_X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d] \\
& A^{\mathrm{I}} \arrow[r, "\shortmid" marking] & A_{\epsilon_X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d] \\
& & X^{\mathrm{I}} \arrow[r, "\epsilon_X"'] & X,
\end{tikzcd}
\tag{51}
$$

Now use the fact that trivial fibrations are closed under pullback along all maps, and under composition, to infer that the indicated composite map $B^{\mathrm{I}} \to B_{\epsilon_X}$ is also a trivial fibration, as required. □

Proposition 53 (Frobenius)**.** *Let $\alpha : A \to X$ and $\beta : B \to A$ be fibrations, then the pushforward $\alpha_*\beta : \Pi_A B \to X$ is also a fibration.*

Proof. Given the fibrations $\alpha : A \to X$ and $\beta : B \to A$, let $p : A^{\mathrm{I}} \to A_\epsilon$ and $q : B^{\mathrm{I}} \to p^*B_\epsilon$ be the associated +-algebras, so that we have the following situation, with all squares pullbacks.

$$\begin{tikzcd}
B^{\mathrm{I}} \arrow[r, "q"', "\shortmid" marking] & p^*B_\epsilon \arrow[r] \arrow[d] & B_\epsilon \arrow[r] \arrow[d] & B \arrow[d] \\
& A^{\mathrm{I}} \arrow[r, "p"', "\shortmid" marking] & A_\epsilon \arrow[r] \arrow[d] & A \arrow[d] \\
& & X^{\mathrm{I}} \arrow[r, "\epsilon"'] & X,
\end{tikzcd} \tag{52}$$

Adding (some composites and) the relevant pushforward underneath, we have

$$\begin{tikzcd}
B^{\mathrm{I}} \arrow[r, "q"', "\shortmid" marking] \arrow[dr] & p^*B_\epsilon \arrow[r] \arrow[d] & B_\epsilon \arrow[r] \arrow[d] & B \arrow[d] \\
& A^{\mathrm{I}} \arrow[r, "p"', "\shortmid" marking] \arrow[dr] & A_\epsilon \arrow[r] \arrow[d] & A \arrow[d] \\
& & X^{\mathrm{I}} \arrow[r, "\epsilon"'] & X \\
& (\Pi_A B)^{\mathrm{I}} \arrow[r, "r"'] \arrow[ur] & (\Pi_A B)_\epsilon \arrow[r] \arrow[u] & \Pi_A B \arrow[u]
\end{tikzcd} \tag{53}$$

and we wish to show that the indicated map $r : (\Pi_A B)^{\mathrm{I}} \to (\Pi_A B)_\epsilon$ admits a +-algebra structure. We will do so by showing that it is a retract of a known +-algebra.

Indeed, let us apply the pushforward, along the indicated canonical map $\alpha^{\mathrm{I}} : A^{\mathrm{I}} \to X^{\mathrm{I}}$, to the +-algebra $q : B^{\mathrm{I}} \to p^*B_\epsilon$, regarded as an arrow over A^{I}. We obtain an arrow over X^{I} of the form

$$\Pi_{A^{\mathrm{I}}}\, q : \Pi_{A^{\mathrm{I}}}\, B^{\mathrm{I}} \to \Pi_{A^{\mathrm{I}}}\, p^*B_\epsilon$$

which is a +-algebra, because these are preserved by pushforward, according to Lemma **??**.

Now observe that by the Beck-Chevalley condition, we have an isomorphism

$$(\Pi_A B)_\epsilon \cong \Pi_{A_\epsilon} B_\epsilon\,.$$

Moreover, since $\Pi_{A^{\mathrm{I}}} \cong \Pi_{A_\epsilon} \circ p_*$, we have

$$\Pi_{A^{\mathrm{I}}}\, p^* B_\epsilon \cong \Pi_{A_\epsilon} p_* p^* B_\epsilon\,.$$

Thus the image of the unit $\eta : B_\epsilon \to p_* p^* B_\epsilon$ under Π_{A_ϵ} is a map $s = \Pi_{A_\epsilon}\eta$ over X^{I} of the form:

$$\begin{tikzcd}
& X^{\mathrm{I}} \\
(\Pi_A B)^{\mathrm{I}} \arrow[ur] \arrow[r, "r"'] & \Pi_{A_\epsilon} B_\epsilon \arrow[u] \arrow[d, "s"] \\
\Pi_{A^{\mathrm{I}}} B^{\mathrm{I}} \arrow[r, "\Pi_{A^{\mathrm{I}}}\, q"'] & \Pi_{A_\epsilon} p_* p^* B_\epsilon
\end{tikzcd} \tag{54}$$

Our objective is now to fill in the further arrows φ, ψ, ϑ indicated below in order to exhibit r as a retract of $\Pi_{A^{\mathrm{I}}}\, q$ in the arrow category over X^{I}.

$$\begin{tikzcd}
& X^{\mathrm{I}} \\
(\Pi_A B)^{\mathrm{I}} \arrow[ur] \arrow[r, "r"'] \arrow[d, dotted, "\varphi"'] & \Pi_{A_\epsilon} B_\epsilon \arrow[u] \arrow[d, "s"] \\
\Pi_{A^{\mathrm{I}}} B^{\mathrm{I}} \arrow[r, "\Pi_{A^{\mathrm{I}}}\, q"'] \arrow[d, dotted, "\psi"'] & \Pi_{A_\epsilon} p_* p^* B_\epsilon \arrow[d, dotted, "\vartheta"] \\
(\Pi_A B)^{\mathrm{I}} \arrow[r, "r"'] & \Pi_{A_\epsilon} B_\epsilon
\end{tikzcd} \tag{55}$$

- For φ, we require a map

$$\varphi : (\Pi_A B)^{\mathrm{I}} \to \Pi_{A^{\mathrm{I}}} B^{\mathrm{I}} \qquad \text{over } X^{\mathrm{I}}.$$

Consider the following diagram, which is based on (53).

$$\begin{tikzcd}
B^{\mathrm{I}} \arrow[r, "q"'] \arrow[dr] & p^*B_\epsilon \arrow[r] \arrow[d] & B_\epsilon \arrow[r] \arrow[d] & B \arrow[d] & \\
 & A^{\mathrm{I}} \arrow[r, "p"'] \arrow[dr] & A_\epsilon \arrow[r] \arrow[d] & A \arrow[d] & \Pi_A B \times_X A \arrow[l] \arrow[d] \arrow[ul, dotted, "c"'] \\
(\Pi_A B \times_X A)^{\mathrm{I}} \arrow[uu, dotted, "c^{\mathrm{I}}"] \arrow[ur] \arrow[dr] & & X^{\mathrm{I}} \arrow[r, "\epsilon"'] & X & \Pi_A B \arrow[l] \\
 & (\Pi_A B)^{\mathrm{I}} \arrow[ur] \arrow[r, dotted, "\varphi"'] & \Pi_{A^{\mathrm{I}}} B^{\mathrm{I}} \arrow[u] & &
\end{tikzcd} \tag{56}$$

The map c is the counit at $B \to A$ of the pullback-pushforward adjunction along $A \to X$. The right-hand side of the diagram, including c and the associated pullback square, reappear on the left under the functor $(-)^{\mathrm{I}}$, which preserves the pullback. Thus we can take φ to be the transpose of c^{I} under the pullback-pushforward adjunction along $A^I \to X^{\mathrm{I}}$,

$$\varphi = \widetilde{c^{\mathrm{I}}}.$$

A diagram chase involving the pullback-pushforward adjunction along $A_\epsilon \to X^{\mathrm{I}}$ shows that the upper square in (55) commutes.

• For ϑ: referring to the diagram (53), since $p : A^{\mathrm{I}} \to A_\epsilon$ is a trivial fibration, it has a section $o : A_\epsilon \to A^{\mathrm{I}}$ by lemma 11. Pulling $p^*B_\epsilon \to A^{\mathrm{I}}$ back along o results in an iso over A_ϵ,

$$o^*p^*B_\epsilon \cong B_\epsilon\,,$$

and so by the adjunction $o^* \dashv o_*$ there is a map over A^{I},

$$p^*B_\epsilon \to o_*B_\epsilon\,,$$

to which we can apply p_* to obtain a map,

$$\rho : p_*p^*B_\epsilon \to p_*o_*B_\epsilon \cong B_\epsilon \quad \text{over } A_\epsilon.$$

This is a retraction of the unit $\eta : B_\epsilon \to p_*p^*B_\epsilon$ over A_ϵ. Applying the functor Π_{A_ϵ} therefore gives the desired retraction

$$\vartheta = \Pi_{A_\epsilon}\rho : \Pi_{A_\epsilon}p_*p^*B_\epsilon \to \Pi_{A_\epsilon}B_\epsilon$$

of s.

- For ψ, we require a map

$$\psi : \Pi_{A^{\mathrm{I}}} B^{\mathrm{I}} \to (\Pi_A B)^{\mathrm{I}} \qquad \text{over } X^{\mathrm{I}}.$$

Consider the following diagram resulting from combining (53) and (55), and in which all solid arrows are those already introduced. The dotted arrow labelled α is the evident composite.

$$\begin{tikzcd}
 & X^{\mathrm{I}} \arrow[r, "\epsilon"] & X \\
(\Pi_A B)^{\mathrm{I}} \arrow[ur] \arrow[r] \arrow[d] & \Pi_{A_\epsilon} B_\epsilon \arrow[u] \arrow[r] \arrow[d] & \Pi_A B \arrow[u] \arrow[dd, "="] \\
\Pi_{A^{\mathrm{I}}} B^{\mathrm{I}} \arrow[r] \arrow[drr, dotted, "\alpha"] & \Pi_{A_\epsilon} p_* p^* B_\epsilon \arrow[d] & \\
(\Pi_A B)^{\mathrm{I}} \arrow[r] & \Pi_{A_\epsilon} B_\epsilon \arrow[r] & \Pi_A B
\end{tikzcd} \tag{57}$$

Now recall that we are working in the slice category over I, and the objects $\Pi_{A^{\mathrm{I}}} B^{\mathrm{I}}$, $\Pi_A B$, and $(\Pi_A B)^{\mathrm{I}}$ are in the image of the base change I^*, and so are actually of the form $\mathrm{I}^* \Pi_{A^{\mathrm{I}}} B^{\mathrm{I}}$, $\mathrm{I}^* \Pi_A B$, and $\mathrm{I}^*((\Pi_A B)^{\mathrm{I}})$. Indeed, the latter is

$$\mathrm{I}^*((\Pi_A B)^{\mathrm{I}}) = \mathrm{I}^* \mathrm{I}_* \mathrm{I}^* \Pi_A B\,.$$

Since the lower horizontal map is the counit ε of the base change $\mathrm{I}^* \dashv \mathrm{I}_*$, the map α factors as $\varepsilon \circ \mathrm{I}^* \tilde{\alpha}$, where $\tilde{\alpha}$ is the adjoint transpose of α, as shown in the following.

$$\begin{tikzcd}
\mathrm{I}^* \Pi_{A^{\mathrm{I}}} B^{\mathrm{I}} \arrow[r] \arrow[d, "\mathrm{I}^* \tilde{\alpha}"'] \arrow[drr, dotted, "\alpha"] & \Pi_{A_\epsilon} p_* p^* B_\epsilon \arrow[d] & \\
\mathrm{I}^* \mathrm{I}_* \mathrm{I}^* \Pi_A B \arrow[r] \arrow[rr, bend right, "\varepsilon"'] & \Pi_{A_\epsilon} B_\epsilon \arrow[r] & \mathrm{I}^* \Pi_A B
\end{tikzcd} \tag{58}$$

We set $\psi = \mathrm{I}^* \tilde{\alpha}$, making the square commute.

We have now defined all the maps below, the squares involving φ and ψ commute, and the composite of ϑ and s is the identity.

$$
\begin{tikzcd}
 & X^{\mathrm{I}} \arrow[r] & X \\
(\Pi_A B)^{\mathrm{I}} \arrow[ur] \arrow[r] \arrow[d, "\varphi"'] & \Pi_{A_\epsilon} B_\epsilon \arrow[u] \arrow[r] \arrow[d, "s"] & \Pi_A B \arrow[u] \arrow[dd, "="] \\
\Pi_{A^{\mathrm{I}}} B^{\mathrm{I}} \arrow[r] \arrow[d, "\psi"'] & \Pi_{A_\epsilon} p_* p^* B_\epsilon \arrow[d, "\vartheta"] & \\
(\Pi_A B)^{\mathrm{I}} \arrow[r] \arrow[rr, bend right, "\varepsilon"'] & \Pi_{A_\epsilon} B_\epsilon \arrow[r] & \Pi_A B
\end{tikzcd}
\tag{59}
$$

To see that $\psi \circ \varphi = 1$, observe that each map is in the image of I^*, say:

$$
\begin{aligned}
\varphi &= \mathrm{I}^* f \\
\psi &= \mathrm{I}^* g\,,
\end{aligned}
$$

where $g = \tilde{\alpha}$. Recall that in general the unit ε satisfies,

$$\varepsilon \circ \mathrm{I}^*(h) = \tilde{h}$$

for any map $h : X \to \mathrm{I}_* Y$. Thus

$$
\begin{aligned}
\varepsilon \circ \psi \circ \varphi &= \varepsilon \circ \mathrm{I}^* g \circ \mathrm{I}^* f \\
&= \varepsilon \circ \mathrm{I}^*(g \circ f) \\
&= \widetilde{(g \circ f)}.
\end{aligned}
$$

On the other hand, a diagram chase on (59) shows that

$$\varepsilon \circ \psi \circ \varphi = \varepsilon.$$

Therefore $g \circ f = \tilde{\varepsilon} = 1$, so $\psi \circ \varphi = \mathrm{I}^* g \circ \mathrm{I}^* f = \mathrm{I}^*(g \circ f) = \mathrm{I}^* 1 = 1$.

□

6 The universe

In this section, we define a universal small fibration $\dot{\mathcal{U}} \to \mathcal{U}$. In section 8 we will show that the base $\mathcal{U}$ is a fibrant object, using the equivalence extension property of section 7.

6.1 Classifying families

Let κ be an inaccessible cardinal number, and call the sets of size strictly less than κ *small*. Write Set_κ for the category of small sets and $\mathsf{cSet}_\kappa = \mathsf{Set}_\kappa^{\square^{\mathrm{op}}}$ for the category of small set valued presheaves on the cube category $\square$. By a *small fibration* we mean a fibration in the category of small cubical sets, which we identify with the evident subcategory $\mathsf{cSet}_\kappa \subseteq \mathsf{cSet}$. Finally, let $\dot{\mathsf{Set}}_\kappa$ be the category of small pointed sets, i.e. the coslice category $1/\mathsf{Set}_\kappa$. There is an evident forgetful functor $U : \dot{\mathsf{Set}}_\kappa \to \mathsf{Set}_\kappa$.

Definition 54. The (κ-)*universe* $p : \dot{\mathcal{V}} \to \mathcal{V}$ in cSet is defined:

1. $\mathcal{V}_n = \{A : \square/[n] \to \mathsf{Set}_\kappa^{op}\}$, the *set* of small presheaves on $\square/[n]$.

   The action of a map $h : [m] \to [n]$ in $\square$ is given by precomposition with postcomposition: from $h : [m] \to [n]$ we have $\square/h : \square/[m] \to \square/[n]$, which we precompose with any $A : \square/[n] \to \mathsf{Set}_\kappa^{op}$ to get $A.h = A \circ \square/h$,

$$\begin{tikzcd} {[n]} & \square/[n] \arrow[r, "A"] & \mathsf{Set}_\kappa^{op} \\ {[m]} \arrow[u, "h"] & \square/[m] \arrow[u, "\square/h"] \arrow[ur, "A.h"'] & \end{tikzcd} \tag{60}$$

2. $\dot{\mathcal{V}}_n = \{a : \square/[n] \to \dot{\mathsf{Set}}_\kappa^{op}\}$, the *set* of small pointed presheaves on $\square/[n]$, with the corresponding action.

3. For $a \in \dot{\mathcal{V}}_n$, let $p_n(a) = U(a) \in \mathcal{V}_n$, where $U : \dot{\mathsf{Set}}_\kappa \to \mathsf{Set}_\kappa$.

Functoriality of $\mathcal{V}$ and $\dot{\mathcal{V}}$ and naturality of $p : \dot{\mathcal{V}} \to \mathcal{V}$ are immediate.

Lemma 55. *For each $A : \mathrm{I}^n \to \mathcal{V}$ there is a canonical choice of a small family $p_A : E_A \to \mathrm{I}^n$ and a map $q_A : E_A \to \dot{\mathcal{V}}$ making a pullback square as follows.*

$$\begin{tikzcd} E_A \arrow[r, "q_A"] \arrow[d, "p_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathcal{V}} \arrow[d, "p"] \\ \mathrm{I}^n \arrow[r, "A"'] & \mathcal{V} \end{tikzcd} \tag{61}$$

Proof. Since $\mathrm{I}^n \cong y[n]$ is representable, there is a distinguished associated presheaf $A : (\square/[n])^{op} \to \mathsf{Set}_\kappa$. Define $p_A : E_A \to \mathrm{I}^n$ by

$$(E_A)_k = \coprod_{h \in \square(k,n)} A(h) \qquad \ni (h, a)$$

with first projection $(p_A)_k(h, a) = h$. Note that $(E_A)_k$ is small. Then let $q_A : E_A \to \dot{\mathcal{V}}$ be defined on $(h, a) : \mathrm{I}^k \to E_A$ by

$$(q_A) \circ (h, a) = a \in Ah$$

as illustrated below.

$$\begin{tikzcd}
 & E_A \arrow[r, dotted, "q_A"] \arrow[d, "p_A"] & \dot{\mathcal{V}} \arrow[d, "p"] \\
\mathrm{I}^k \arrow[ur, "{(h,a)}"] \arrow[urr, "a"'] \arrow[r, "h"'] & \mathrm{I}^n \arrow[r, "A"'] & \mathcal{V}
\end{tikzcd} \tag{62}$$

The proof that the square is a pullback is left to the reader. □

Lemma 56. *For each small family $p_E : E \to \mathrm{I}^n$ there is a canonical map $\chi_E : \mathrm{I}^n \to \mathcal{V}$ and a map $q_E : E \to \dot{\mathcal{V}}$ making a pullback square as follows.*

$$\begin{tikzcd}
E \arrow[r, "q_E"] \arrow[d, "p_E"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \dot{\mathcal{V}} \arrow[d, "p"] \\
\mathrm{I}^n \arrow[r, "\chi_E"'] & \mathcal{V}
\end{tikzcd} \tag{63}$$

Proof. It suffices to give a small set $(\chi_E)_k(h)$ for each $h : [k] \to [n]$ in a way that is functorial in $h \in \square/[n]$ and natural in $[k]$. Thus let

$$(\chi_E)_k(h) := \Gamma(h, E) \;=\; \{e : \mathrm{I}^k \to E \mid p_E \circ e = h\}.$$

$$\begin{tikzcd}
 & E \arrow[d, "p_E"] \\
\mathrm{I}^k \arrow[ur, dotted, "e"] \arrow[r, "h"'] & \mathrm{I}^n
\end{tikzcd}$$

which is small if each E_k is.

To define $q_E : E \to \dot{\mathcal{V}}$, take any $e : \mathrm{I}^k \to E$ and first compose with p_E and observe that $e \in \Gamma(p_E e, E)$. Thus the assignment gives a map $\dot{e} : \mathrm{I}^k \to \dot{\mathcal{V}}$

making the solid arrows in the following commute.

$$
\begin{tikzcd}
& E \arrow[r, dotted, "q_E"] \arrow[d, "p_E"] & \dot{\mathcal{V}} \arrow[d, "p"] \\
\mathrm{I}^k \arrow[ur, "e"] \arrow[urr, "\dot{e}"'] \arrow[r, "p_E e"'] & \mathrm{I}^n \arrow[r, "E"'] & \mathcal{V}
\end{tikzcd}
\tag{64}
$$

Since the assignment of $\dot{e}$ to e is natural in $[k]$, we get the required map $q_E : E \to \dot{\mathcal{V}}$. The proof that the square is a pullback is again left to the reader. $\square$

Corollary 57. *Given a small family $p_E : E \to \mathrm{I}^n$ there is a unique isomorphism $E \cong E_{\chi_E}$ over I^n making a commutative diagram as follows.*

$$
\begin{tikzcd}
E \arrow[r, "\cong"] \arrow[rr, bend left, "q_E"] \arrow[dr, "p_E"'] & E_{\chi_E} \arrow[r, "q_{\chi_E}"'] \arrow[d, "p_{\chi_E}"] & \dot{\mathcal{V}} \arrow[d, "p"] \\
& \mathrm{I}^n \arrow[r, "\chi_E"'] & \mathcal{V}
\end{tikzcd}
\tag{65}
$$

Proposition 58. *For any cubical set X and any small family $p_E : E \to X$ there are canonical maps $\chi_E : X \to \mathcal{V}$ and $q_E : E \to \dot{\mathcal{V}}$ making a pullback square as follows.*

$$
\begin{tikzcd}
E \arrow[r, "q_E"] \arrow[d, "p_E"'] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathcal{V}} \arrow[d, "p"] \\
X \arrow[r, "\chi_E"'] & \mathcal{V}
\end{tikzcd}
\tag{66}
$$

Moreover, χ_E and q_E are uniquely determined by the equations (68) below.

Proof. Write $X = \varinjlim_x \mathrm{I}^n$ as a colimit of a cocone of maps $x : \mathrm{I}^n \to X$ from representables, over the canonical index category $([n], x) \in \int_\Box X$. Form the family of pullback squares below, where the arrows with a dot represent cocones, and the cocone consisting of the $q_x : E_x \to E$ is determined by taking pullbacks along p_E, and is therefore also a colimit.

$$
\begin{tikzcd}
E_x \arrow[r, "q_x"] \arrow[d, "p_{E_x}"'] \arrow[dr, phantom, very near start, "\lrcorner"] \arrow[rr, bend left, "q_{E_x}"] & E \arrow[r, dotted, "q_E"] \arrow[d, "p_E"'] & \dot{\mathcal{V}} \arrow[d, "p"] \\
\mathrm{I}^n \arrow[r, "x"'] \arrow[rr, bend right, "\chi_{E_x}"'] & X \arrow[r, dotted, "\chi_E"'] & \mathcal{V}
\end{tikzcd}
\tag{67}
$$

The maps χ_{E_x} and q_{E_x} are determined by lemma 56, since the families p_{E_x} are small if $p_E : E \to X$ is. Thus we can define the indicated maps χ_E and q_E from the colimits as those uniquely determined by the equations:

$$\chi_E \circ x = \chi_{E_x} \tag{68}$$

$$q_E \circ q_x = q_{E_x} \tag{69}$$

The square on the right is a pullback because the outer squares are all pullbacks, the family of left-hand squares are pullbacks, and the family of maps $x : \mathrm{I}^n \to X$ covers X. $\square$

Remark 59. Note that the classification operation

$$\chi : \mathsf{cSet}_\kappa/X \to \mathsf{cSet}(X, \mathcal{V})$$

again has the evident "pullback of $p : \dot{\mathcal{V}} \to \mathcal{V}$" operation

$$E : \mathsf{cSet}(X, \mathcal{V}) \to \mathsf{cSet}_\kappa/X$$

as a left (quasi-)inverse $E \cong E_{\chi_E}$, which is (pseudo-)natural in X. But there is no corresponding uniqueness of classifying maps, relating $A : X \to \mathcal{V}$ and $\chi_{E_A} : X \to \mathcal{V}$. (This is what is provided, in a suitable sense, by the *univalence* of the universe $\dot{\mathcal{U}} \to \mathcal{U}$ of fibrations, to be established in section 7.)

6.2 Classifying trivial fibrations

Recall from section 2 that (uniform) trivial fibration structures on a map $A \to X$ correspond bijectively to relative $+$-algebra structures over X (definition 7). A relative $+$-algebra structure on $A \to X$ is an algebra structure for the pointed endofunctor $+_X : \mathsf{cSet}/X \to \mathsf{cSet}/X$, where

$$A^+ = \sum_{\varphi : \Phi} A^\varphi \quad \text{over } X.$$

A $+$-algebra structure is then a retract $\alpha : A^+ \to A$ over X of the canonical map $\eta_A : A \to A^+$,

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[rr, bend left, "="] \arrow[dr] & A^+ \arrow[r, "\alpha"] \arrow[d] & A \arrow[dl] \\ & X. & \end{tikzcd}$$

In more detail, let us write $A \to X$ as a family $\sum_{x:X} A_x \to X$ over X. Since the $+$-functor acts fiberwise, the A^+ in the diagram above is then the indexing projection

$$\sum_{x:X} A_x^+ \to X.$$

Working in the slice cSet/X, we make the (relative) exponentials (internal Hom's) $[A^+, A]$ and $[A, A]$ with the "precomposition by η_A" map $[\eta_A, A]$, which fit into the following pullback diagram

$$\begin{tikzcd} +\mathsf{Alg}(A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & {[A^+, A]} \arrow[d, "{[\eta_A, A]}"] \\ 1 \arrow[r, "'id_A'"'] & {[A, A]}. \end{tikzcd}$$

The constructed object $+\mathsf{Alg}(A) \to X$ over X is the *object of $+$-algebra structures on $A \to X$*, in the sense that sections $X \to +\mathsf{Alg}(A)$ correspond isomorphically to $+$-algebra structures on $A \to X$. Moreover, $+\mathsf{Alg}(A) \to X$ is stable under pullback in the sense that for any $f : Y \to X$, we have two pullback squares,

$$\begin{tikzcd}
f^*A \arrow[r] \arrow[d] & A \arrow[d] \\
Y \arrow[r, "f"'] & X \\
+\mathsf{Alg}(f^*A) \arrow[r] \arrow[u] & +\mathsf{Alg}(A) \arrow[u]
\end{tikzcd} \tag{70}$$

because the +-functor, exponentials and pullbacks occurring in the construction of $+\mathsf{Alg}(A) \to X$ are themselves all stable.

It follows from proposition 58 that if $A \to X$ is small, then $+\mathsf{Alg}(A) \to X$ is itself a pullback of the analogous object $+\mathsf{Alg}(\dot{\mathcal{V}}) \to \mathcal{V}$ constructed from the universal small family $\dot{\mathcal{V}} \to \mathcal{V}$,

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & \dot{\mathcal{V}} \arrow[d] \\
X \arrow[r, "\chi_A"'] & \mathcal{V} \\
+\mathsf{Alg}(A) \arrow[r] \arrow[u] & +\mathsf{Alg}(\dot{\mathcal{V}}) \arrow[u]
\end{tikzcd} \tag{71}$$

Proposition 60. *There is a* universal small trivial fibration

$$\dot{\mathsf{TFib}} \to \mathsf{TFib}.$$

Every small trivial fibration $A \to X$ is a pullback of $\dot{\mathsf{TFib}} \to \mathsf{TFib}$ along a canonically determined classifying map $X \to \mathsf{TFib}$.

$$\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathsf{TFib}} \arrow[d] \\
X \arrow[r] & \mathsf{TFib}
\end{tikzcd} \tag{72}$$

Proof. We can take $\mathsf{TFib} = +\mathsf{Alg}(\dot{\mathcal{V}})$, which comes with its projection $+\mathsf{Alg}(\dot{\mathcal{V}}) \to \mathcal{V}$ as in diagram (71). Now define $p_t : \dot{\mathsf{TFib}} \to \mathsf{TFib}$ by pulling back the universal small family,

$$\begin{tikzcd}
\dot{\mathsf{TFib}} \arrow[r] \arrow[d, "p_t"'] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathcal{V}} \arrow[d, "p"] \\
\mathsf{TFib} \arrow[r] & \mathcal{V}.
\end{tikzcd}$$

Consider the following diagram, in which all the squares (including the distorted ones) are pullbacks, with the outer one coming from proposition 58 and the lower one from (71).

$$
\begin{tikzcd}
 & A \arrow[rr, "q_A"] \arrow[dddd] \arrow[dr, dotted] & & \dot{\mathcal{V}} \arrow[dddd, "p"] \\
 & & \dot{\mathsf{TFib}} \arrow[ur] \arrow[dd, "p_t"'] & \\
 & & & \\
\mathsf{TFib}(A) \arrow[rr, crossing over] \arrow[dr] & & \mathsf{TFib} \arrow[dr] & \\
 & X \arrow[rr, "\chi_A"'] \arrow[ur, dotted, "\alpha'"'] \arrow[ul, dotted, bend left, "\alpha"] & & \mathcal{V}.
\end{tikzcd}
\tag{73}
$$

A trivial fibration structure α on $A \to X$ is a section the object of $+$-algebra structures on A, occurring in the diagram as $\mathsf{TFib}(A)$, the pullback of TFib. Such sections correspond uniquely to factorizations α' of χ_A as indicated, which in turn induce pullback squares of the required kind (72).

Note that the map $p_t : \dot{\mathsf{TFib}} \to \mathsf{TFib}$ has a canonical trivial fibration structure. Indeed, consider the following diagram, in which both squares are pullbacks.

$$
\begin{tikzcd}
\dot{\mathsf{TFib}} \arrow[r] \arrow[d, "p_t"'] & \dot{\mathcal{V}} \arrow[d] \\
\mathsf{TFib} \arrow[r] & \mathcal{V} \\
\mathsf{TFib}(\dot{\mathsf{TFib}}) \arrow[r] \arrow[u] & \mathsf{TFib}(\dot{\mathcal{V}}) \arrow[u]
\end{tikzcd}
\tag{74}
$$

$\mathsf{TFib}(\dot{\mathcal{V}})$ is the object of trivial fibration structures on $\dot{\mathcal{V}} \to \mathcal{V}$, and its pullback $\mathsf{TFib}(\dot{\mathsf{TFib}})$ is therefore the object of trivial fibration structures on $p_t : \dot{\mathsf{TFib}} \to \mathsf{TFib}$. Thus we seek a section of $\mathsf{TFib}(\dot{\mathsf{TFib}}) \to \mathsf{TFib}$. But recall that $\mathsf{TFib} = \mathsf{TFib}(\dot{\mathcal{V}})$ by definition, so the lower pullback square is the pullback of $\mathsf{TFib}(\dot{\mathcal{V}}) \to \mathcal{V}$ against itself, which does indeed have a distinguished section, namely the diagonal

$$\Delta : \mathsf{TFib}(\dot{\mathcal{V}}) \to \mathsf{TFib}(\dot{\mathcal{V}}) \times_{\mathcal{V}} \mathsf{TFib}(\dot{\mathcal{V}}).$$

□

6.3 Classifying fibrations

In order to classify fibrations $A \to X$, we shall proceed as for trivial fibrations by constructing an object $\mathsf{Fib}(A) \to X$ of fibration structures on $A \to X$ which, moreover, is stable under pullback. We then apply the construction to the universal small family $\dot{\mathcal{V}} \to \mathcal{V}$ to get a universal small fibration.

The construction of $\mathsf{Fib}(A) \to X$ is a bit more involved than that of $\mathsf{TFib}(A) \to X$. Recall from section **??** the characterization of (uniform, unbiased) fibration structures on a map $p_A : A \to X$ in terms of $+$-algebra structures:

1. First, pass to cSet/I where there is a generic point $\delta : 1 \to \mathrm{I}$,

2. Form the pullback-hom $\delta \Rightarrow p_A : A^{\mathrm{I}} \to X^{\mathrm{I}} \times_X A$ as indicated in the following diagram.

$$\begin{tikzcd}
A^{\mathrm{I}} \arrow[dr, "\delta \Rightarrow p_A"] \arrow[drr, bend left, "A^{\delta}"] \arrow[ddr, bend right, "(p_A)^{\mathrm{I}}"'] & & \\
& X^{\mathrm{I}} \times_X A \arrow[r] \arrow[d] & A \arrow[d, "p_A"] \\
& X^{\mathrm{I}} \arrow[r, "X^{\delta}"'] & X
\end{tikzcd} \tag{75}$$

3. A fibration structure on $p_A : A \to X$ is then a relative $+$-algebra structure on $\delta \Rightarrow p_A$ in the slice category over the codomain.

   We next construct an object $\mathsf{Fib}(A) \to X$ classifying such structures. For convenience, let us relabel the objects and arrows in the previous diagram as follows:

$$\begin{aligned}
p &:= p_A \\
\epsilon &:= X^{\delta} : X^{\mathrm{I}} \to X \\
A_{\epsilon} &:= X^{\mathrm{I}} \times_X A \\
\epsilon^A &:= \delta \Rightarrow p_A
\end{aligned}$$

so that (the working part of) our diagram becomes:

$$
\begin{tikzcd}
A^{\mathrm{I}} \arrow[dr, "\epsilon^A"'] & & \\
& A_\epsilon \arrow[r] \arrow[d, "p_\epsilon"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "p"] \\
& X^{\mathrm{I}} \arrow[r, "\epsilon"'] & X
\end{tikzcd}
\tag{76}
$$

4. A $+$-algebra structure on ϵ^A is a retract α over A_ϵ of the unit η as indicated below, where D is the domain of the map $(\epsilon^A)^+$:

$$
\begin{tikzcd}
A^{\mathrm{I}} \arrow[r, "\eta"'] \arrow[dr, "\epsilon^A"'] & D \arrow[l, bend right, dotted, "\alpha"'] \arrow[d, "(\epsilon^A)^+"] & \\
& A_\epsilon \arrow[r] \arrow[d, "p_\epsilon"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "p"] \\
& X^{\mathrm{I}} \arrow[r, "\epsilon"'] & X
\end{tikzcd}
\tag{77}
$$

5. Thus, as in the previous section, there is an object $+\mathsf{Alg}(\epsilon^A)$ over A_ϵ of $+$-algebra structures on ϵ^A, the sections of which correspond uniquely to $+$-algebra structures on ϵ^A (and thus fibration structures on A).

$$
\begin{tikzcd}
& A^{\mathrm{I}} \arrow[r, "\eta"'] \arrow[d, "\epsilon^A"'] & D \arrow[l, bend right, dotted, "\alpha"'] \arrow[dl, "(\epsilon^A)^+"] \\
+\mathsf{Alg}(\epsilon^A) \arrow[r] & A_\epsilon \arrow[r] \arrow[d, "p_\epsilon"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "p"] \\
& X^{\mathrm{I}} \arrow[r, "\epsilon"'] & X
\end{tikzcd}
\tag{78}
$$

6. Sections of $+\mathsf{Alg}(\epsilon^A) \to A_\epsilon$ correspond to sections of its push-forward along p_ϵ, which we shall call F_A:

$$F_A := (p_\epsilon)_*(+\mathsf{Alg}(\epsilon^A)) \,.$$

$$
\begin{tikzcd}
 & A^{\mathrm{I}} \arrow[r, "\eta"'] \arrow[d, "\epsilon^A"'] & D \arrow[l, dotted, bend right, "\alpha"'] \arrow[dl, "(\epsilon^A)^+"] \\
+\mathsf{Alg}(\epsilon^A) \arrow[r] & A_\epsilon \arrow[r] \arrow[d, "p_\epsilon"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "p"] \\
F_A \arrow[r] & X^{\mathrm{I}} \arrow[r, "\epsilon"'] & X
\end{tikzcd}
\tag{79}
$$

7. We might now think of taking another pushforward of $F_A \to X^{\mathrm{I}}$ along $\epsilon : X^{\mathrm{I}} \to X$ to get the object $\mathsf{Fib}(A) \to X$ that we seek, but unfortunately, this would not be stable under pullback along arbitrary maps $Y \to X$, because $\epsilon : X^{\mathrm{I}} \to X$ is not stable in this way. Instead we will use the *root* functor, i.e. the "amazing right adjoint" to the pathspace (see [**?**]).

$$(-)^{\mathrm{I}} \dashv (-)_{\mathrm{I}}$$

Let $f : F_A \to X^{\mathrm{I}}$ be the map indicated in (79), and let $\eta_X : X \to (X^{\mathrm{I}})_{\mathrm{I}}$ be the unit of the root adjunction. Then define $\mathsf{Fib}(A) \to X$ as the following pullback.

$$
\begin{tikzcd}
\mathsf{Fib}(A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (F_A)_{\mathrm{I}} \arrow[d, "f_{\mathrm{I}}"] \\
X \arrow[r, "\eta"'] & (X^{\mathrm{I}})_{\mathrm{I}}
\end{tikzcd}
\tag{80}
$$

By adjointness, sections of $\mathsf{Fib}(A) \to X$ correspond uniquely to sections of $f : F_A \to X^{\mathrm{I}}$.

8. Finally, we are still working in the slice cSet/I and need to get back to cSet by applying the functor $\mathrm{I}_* : \mathsf{cSet}/\mathrm{I} \to \mathsf{cSet}$. Call the map $\mathsf{Fib}(A) \to X$ constructed over I in the last step $\mathsf{Fib}(A)_i \to \mathrm{I}^*X$ and apply I_* to get,

$$\mathrm{I}_*(\mathsf{Fib}(A)_i) = \Pi_{i:\mathrm{I}}\mathsf{Fib}(A)_i \to X^{\mathrm{I}}$$

in cSet. We then define the desired map $\mathsf{Fib}(A) \to X$ as the pullback along the unit $\rho : X \to X^{\mathrm{I}}$ of $\mathrm{I}^* \dashv \mathrm{I}_*$, as indicated below.

$$\begin{tikzcd}
\mathsf{Fib}(A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Pi_{i:\mathrm{I}}\mathsf{Fib}(A)_i \arrow[d] \\
X \arrow[r, "\rho"'] & X^{\mathrm{I}}
\end{tikzcd} \tag{81}$$

It follows directly from the adjunction $\mathrm{I}^* \dashv \mathrm{I}_*$ that sections of $\mathsf{Fib}(A) \to X$ correspond bijectively to sections of $\mathsf{Fib}(A)_i \to \mathrm{I}^*X$ over I.

Proposition 61. *The map*

$$\dot{\mathsf{Fib}} \to \mathsf{Fib}$$

just constructed is a universal small fibration. *Every small fibration $A \to X$ is a pullback of $\dot{\mathsf{Fib}} \to \mathsf{Fib}$ along a canonically determined classifying map $X \to \mathsf{Fib}$.*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathsf{Fib}} \arrow[d] \\
X \arrow[r] & \mathsf{Fib}
\end{tikzcd} \tag{82}$$

Proof. First, we need to show that the construction of $\mathsf{Fib}(A) \to X$ as the object of fibration structures on a map $A \to X$ is stable under pullback along all maps $f : Y \to X$. The relevant parts of the construction diagram (83) are repeated below,

$$\begin{tikzcd}
 & A^{\mathrm{I}} \arrow[d, "\epsilon^A"'] & \\
+\mathsf{Alg}(\epsilon^A) \arrow[r] & A_\epsilon \arrow[r] \arrow[d, "p_\epsilon"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "p"] \\
F_A \arrow[r] & X^{\mathrm{I}} \arrow[r, "\epsilon"'] & X
\end{tikzcd} \tag{83}$$

Now consider the following in which the front face of the central cube is a pullback.

$$\begin{tikzcd}
 & & B^{\mathrm{I}} \arrow[rrr] \arrow[d, "\epsilon^B"'] & & & A^{\mathrm{I}} \arrow[d, "\epsilon^A"] & \\
+\mathsf{Alg}(\epsilon^B) \arrow[rr] & & B_\epsilon \arrow[d] \arrow[dr] \arrow[rrr, dotted] & & & A_\epsilon \arrow[d] \arrow[dl] & & +\mathsf{Alg}(\epsilon^A) \arrow[ll] \\
F_B \arrow[rr] & & Y^{\mathrm{I}} \arrow[dr] \arrow[rrr, dotted, bend right] & B \arrow[r] \arrow[d] & A \arrow[d] & X^{\mathrm{I}} \arrow[dl] & & F_A \arrow[ll] \\
 & & \mathsf{Fib}(B) \arrow[r] & Y \arrow[r, "f"'] & X & \mathsf{Fib}(A) \arrow[l] & &
\end{tikzcd} \tag{84}$$

On the left side we will repeat the construction with $B \to Y$ in place of $A \to X$. The left face is thus a pullback, whence the back (dotted) face is a pullback. The two-story square in back is the image of the front square under the right adjoint $(-)^{\mathrm{I}}$ and is therefore a pullback, therefore the top rectangle in the back is a pullback. It follows that $+\mathsf{Alg}(\epsilon^B)$ is a pullback of $+\mathsf{Alg}(\epsilon^A)$ along the upper dotted arrow, as in diagram (70), and so the pushforward F_B is a pullback of the corresponding F_A, along the lower dotted arrow (which is f^{I}), by the Beck-Chevalley condition. Thus we have shown

$$F_B \cong (f^{\mathrm{I}})^* F_A. \tag{85}$$

It remains to show that $\mathsf{Fib}(B)$ is a pullback of $\mathsf{Fib}(A)$ along $f : Y \to X$, and now it is good that we did not take these to be pushforwards of F_B and F_A, because the floor of the cube is not a pullback, and so the Beck-Chavalley condition would not apply. Instead, consider the following diagram.

$$\begin{tikzcd}
 & \mathsf{Fib}(B) \arrow[r] \arrow[dl] \arrow[dd] & \mathsf{Fib}(A) \arrow[dr] \arrow[dd] & \\
(F_B)_{\mathrm{I}} \arrow[rrr] \arrow[dd] & & & (F_A)_{\mathrm{I}} \arrow[dd] \\
 & Y \arrow[r, "f"'] \arrow[dl, "\eta"'] & X \arrow[dr, "\eta"] & \\
(Y^{\mathrm{I}})_{\mathrm{I}} \arrow[rrr, "(f^{\mathrm{I}})_{\mathrm{I}}"'] & & & (X^{\mathrm{I}})_{\mathrm{I}}
\end{tikzcd} \tag{86}$$

The sides of the cube are pullbacks by the construction of $\mathsf{Fib}(A)$ and $\mathsf{Fib}(B)$. The front face is the root of the pullback (85) and is thus also a pullback, since the root is a right adjoint. The base commutes by naturality of the unit, and so the back face is also a pullback as required. Finally, the base change along $\mathrm{I}_* : \mathsf{cSet}/\mathrm{I} \to \mathsf{cSet}$ in step 8 above clearly also preserves the pullback.

Now we can take $\mathsf{Fib} = \mathsf{Fib}(\dot{\mathcal{V}})$, which comes with its projection $\mathsf{Fib}(\dot{\mathcal{V}}) \to \mathcal{V}$, and define the universal small fibration $\dot{\mathsf{Fib}} \to \mathsf{Fib}$ by pulling back the universal small family,

$$\begin{tikzcd}
\dot{\mathsf{Fib}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathcal{V}} \arrow[d, "p"] \\
\mathsf{Fib} \arrow[r] & \mathcal{V}.
\end{tikzcd}$$

The remainder of the proof is just as for proposition 60. $\square$

Definition 62. Write $\dot{\mathcal{U}} \to \mathcal{U}$ for the universal small fibration $\dot{\mathsf{Fib}} \to \mathsf{Fib}$ constructed in proposition 61.

7 The equivalence extension property

The equivalence extension property (EEP) is closely related to the *univalence* of the universal fibration $\dot{\mathcal{U}} \to \mathcal{U}$ constructed in section 6 (see [?]). We shall use it in section 8 to show that the base object $\mathcal{U}$ is fibrant, which implies the fibration extension property. Our proof of the EEP is a reformulation of a type-theoretic argument due to Coquand [?], which in turn is a modification of the original argument of Voevodsky [?]. See [?] for another reformulation.

We first recall some basic facts and make some simple observations. For any map $f : X \to Y$, recall the *pathspace factorization* $f = t \circ s$ indicated below.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "s"] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dd, bend right=40, "="'] & Y \arrow[d, "r"] \arrow[dr, "="] & \\
P_f \arrow[r, "p_0^* f"] \arrow[d, "p_f"] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[rr, dotted, bend right=20, "t"'] & Y^{\mathrm{I}} \arrow[r, "p_1"] \arrow[d, "p_0"] & Y \\
X \arrow[r, "f"'] & Y &
\end{tikzcd} \tag{87}$$

Here p_0, p_1 are the evaluations $Y^{\delta_0}, Y^{\delta_1}$ at the endpoints $\delta_0, \delta_1 : 1 \to \mathrm{I}$, and let $r := Y^!$ for $! : \mathrm{I} \to 1$. Note that $p_0 r = p_1 r = 1_Y$. Then let $p_f := f^* p_0 : P_f \to Y$, the pullback of p_0 along f, and $s := f^* r : X \to P_f$ (over X). Finally, let $t := p_1 \circ p_0^* f : P_f \to Y$ be the indicated horizontal composite.

We make the following well-known observations.

1. If $f : X \to Y$ is over a base Z, then the factorization $t \circ s : X \to P_f \to Y$ is stable under pullback along any map $g : Z' \to Z$, in the sense that $g^* P_f = P_{g^* f} : g^* X \to g^* Y$, and similarly for $g^* s$ and $g^* t$. Note that in this case we form the pathspace Y^{I} as an exponential in the slice category over Z.

2. The retraction $p_0 \circ r = 1_Y$ pulls back along f to a retraction $p_f \circ s = 1_X$.

3. If Y is fibrant (either as an object, or over a base $Y \to Z$), then $p_0 : Y^{\mathrm{I}} \to Y$ is a trivial fibration (as is p_1). In that case, its pullback $p_f : P_f \to X$ is also a trivial fibration.

4. If X and Y are both fibrant, then $t = p_1 \circ p_0^* f : P_f \to Y$ is a fibration. This can be seen by factoring the maps $p_0, p_1 : Y^{\mathrm{I}} \rightrightarrows Y$ through the product projections as $\pi_0 \circ p, \pi_1 \circ p : Y^{\mathrm{I}} \to Y \times Y \rightrightarrows Y$, with $p = (p_0, p_1)$, and then interpolating the pullback along the map $(f, 1_X) : X \times Y \to Y \times Y$ into (87) as indicated below.

$$
\begin{tikzcd}
 & X \arrow[r, "f"] \arrow[d, "s"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "r"] & \\
 & P_f \arrow[r, "p_0^* f"] \arrow[d, "f^* p"] \arrow[dl, "t"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y^{\mathrm{I}} \arrow[d, "p"] \arrow[dr, "p_1"] & \\
Y & X \times Y \arrow[l, "\pi_1'"] \arrow[r, "(f, 1_Y)"'] \arrow[d, "\pi_0'"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \times Y \arrow[r, "\pi_1"'] \arrow[d, "\pi_0"] & Y \\
 & X \arrow[r, "f"'] & Y &
\end{tikzcd}
\tag{88}
$$

The second factor $t = p_1 \circ p_0^* f : P_f \to Y$ now appears also as $\pi_1 \circ (f, 1_Y) \circ f^* p$, which is the pullback $f^* p : P_f \to X \times Y$ followed by the second projection $\pi_1' : X \times Y \to Y$ (which is not a pullback). But if Y is fibrant, then $p : Y^{\mathrm{I}} \to Y \times Y$ is a fibration, and then so is $f^* p$.

And if X is fibrant, then the projection $\pi_1' : X \times Y \to Y$ is a fibration. Thus in this case, $t = \pi_1' \circ f^*p : P_f \to Y$ is a fibration, as claimed.

5. Summarizing (1)-(4), for any map $f : X \to Y$, we have a stable factorization $f = t \circ s : X \to P_f \to Y$, in which s has a retraction p_f, which is a trivial fibration when Y is fibrant, and t is a fibration when both X and Y are fibrant.

$$\begin{tikzcd}
X \arrow[r, "s"] \arrow[dr, "f"'] & P_f \arrow[l, dotted, bend right, "p_f"'] \arrow[d, "t"] \\
& Y
\end{tikzcd} \tag{89}$$

Note that the retraction $p_f : P_f \to X$ is not over Y.

The following simple fact concerning just the cofibration weak factorization system will also be needed.

Lemma 63. *Let $p : E \twoheadrightarrow B$ be a trivial fibration and $c : C \rightarrowtail B$ a cofibration. Then the unit $\eta : E \to c_*c^*E$ of the base change $c^* \dashv c_*$ along c is a trivial fibration.*

Proof. The unit map $\eta : E \to c_*c^*E$ is the pullback-hom $c \Rightarrow p$, as is easily checked. By lemma 12, for any map $a : A \to Z$ we have the equivalence of diagonal filling conditions,

$$a \pitchfork c \Rightarrow p \quad \text{iff} \quad a \otimes c \pitchfork p.$$

But since $c : C \rightarrowtail B$ is a cofibration, $a \otimes c$ is also a cofibration if $a : A \rightarrowtail Z$ is one by axiom (C6), which says that cofibrations are closed under pushout-products. So $a \otimes c \pitchfork p$ indeed holds, since p is a trivial fibration. $\square$

Proposition 64 (EEP)**.** *Weak equivalences extended along cofibrations in the following sense: given a cofibration $c : C' \rightarrowtail C$ and fibrations $A' \twoheadrightarrow C'$ and $B \twoheadrightarrow C$, and a weak equivalence $w' : A' \simeq c^*B$ over C',*

$$\begin{tikzcd}
A' \arrow[rr, dotted] \arrow[dr, "w'", "\sim"'] \arrow[dd, two heads] & & A \arrow[dr, dotted, "w", "\sim"'] \arrow[dd, dotted, two heads] & \\
& c^*B \arrow[rr] \arrow[dl] & & B \arrow[dl, two heads] \\
C' \arrow[rr, tail, "c"'] & & C &
\end{tikzcd} \tag{90}$$

there is a fibration $A \twoheadrightarrow C$ and a weak equivalence $w : A \simeq B$ over C that pulls back along $c : C' \rightarrowtail C$ to w', so that $c^*w = w'$.

Proof. Call the given fibration $q : B \to C$ and let $b := q^*c : c^*B \to B$ be the indicated pullback, which is thus also a cofibration. Let $w := b_*w' : A \to B$ be the pushforward of w' along b. Composing with q gives the map $p := q \circ w : A \to C$. Since b is monic, we indeed have $b^*w = w'$, thus filling in all the dotted arrows in (90). Note moreover that $c^*w = b^*w = w'$, as required. It remains to show that $p : A \to C$ is a fibration and $w : A \to B$ is a weak equivalence.

$$\begin{tikzcd}
A' \arrow[rr, tail] \arrow[dr, "w'", "\sim"'] \arrow[dd, two heads] & & A \arrow[r, "s"] \arrow[dr, "w"] \arrow[dd, "p"' near end] & P_w \arrow[d, "t"] \arrow[l, bend right, dotted, "p_w"'] \\
& c^*B \arrow[rr, tail, "b"' near start] \arrow[dl, two heads] & & B \arrow[dl, two heads, "q"] \\
C' \arrow[rr, tail, "c"'] & & C &
\end{tikzcd} \tag{91}$$

Let us write $p' := c^*p : A' \to C'$ and $B' := c^*B$ and $q' := c^*q$. Now let $w = t \circ s$ be the pathspace factorization (87) of w, as a map over C. Since $q : B \to C$ is a fibration, by the foregoing remarks on pathspace factorizations, we know that $s : A \to P_w$ has a retraction $p_w : P_w \to A$ which is a trivial fibration. The retraction p_w is a map over C.

The pathspace factorization $w = t \circ s : A \to P_w \to B$ is stable under pullback along c, providing a pathspace factorization $w' = t' \circ s' : A' \to P_{w'} \to B'$ over C'. Since both p' and q' are fibrations, the retract $p_{w'} : P_{w'} \to A'$ is a trivial fibration, and now $t' : P_{w'} \to B'$ is a fibration.

$$\begin{tikzcd}
A' \arrow[r, "s'"] \arrow[dr, "w'", "\sim"'] \arrow[dd, two heads, "p'"'] & P_{w'} \arrow[l, dotted, bend right, two heads, "p_{w'}"'] \arrow[d, two heads, "t'"] \\
 & B' \arrow[dl, two heads, "q'"] \\
C' &
\end{tikzcd} \tag{92}$$

Thus the composite $q' \circ t' : P_{w'} \to B' \to C'$ is a fibration and therefore, by the retraction over C' with the trivial fibration $p_{w'}$, we have that $s' : A' \to P_{w'}$ is a weak equivalence, by 3-for-2 for weak equivalences between fibrations. For the same reason, t' is then a weak equivalence, and therefore a trivial fibration.

Since $t' = c^*t = b^*t$ is a trivial fibration, its pushforward b_*b^*t along b is also one. Moreover, $b_*b^*t : b_*b^*P_w \to B$ admits a unit $\eta : P_w \to b_*b^*P_w$ (over B).

$$\begin{tikzcd}
A' \arrow[rr, tail] \arrow[dr, "w'", "\sim"'] \arrow[dd, two heads] & & A \arrow[r, "s"] \arrow[dr, "w"] \arrow[dd, "p"'] & P_w \arrow[l, dotted, bend right, two heads, "p_w"'] \arrow[r, "\eta"] \arrow[d, "t"] & b_*b^*P_w \arrow[dl, "b_*b^*t"] \\
 & B' \arrow[rr, tail, "b"'] \arrow[dl, two heads] & & B \arrow[dl, two heads, "q"] & \\
C' \arrow[rr, tail, "c"'] & & C & &
\end{tikzcd} \tag{93}$$

We now *claim* that $\eta : P_w \to b_*b^*P_w$ is a trivial fibration. Given that, the composite $t = b_*b^*t \circ \eta$ is also a trivial fibration, whence $q \circ t : P_w \to C$ is a fibration, and so its retract $p : A \to C$ is a fibration. Moreover, since s is a section of the trivial fibration $p_w : P_w \to A$ between fibrations, as before it is also a weak equivalence. Thus $w = t \circ s$ is a weak equivalence, and we are finished.

To prove the remaining claim that $\eta : P_w \to b_* b^* P_w$ is a trivial fibration, we shall use lemma 63. But it does not apply directly since $t : P_w \to B$ is not yet known to be a trivial fibration. Instead, we show that η is a pullback of the corresponding unit at the trivial fibration $p_1 : B^{\mathrm{I}} \to B$.

Consider the following cube (viewed with $b : B' \to B$ at the front).

$$\begin{tikzcd}
P_{w'} \arrow[rr, tail, "\overline{a}"] \arrow[dr, "(p_0')^* w'"] \arrow[dd, two heads, "p_{w'}"'] & & P_w \arrow[dr, "p_0^* w"] \arrow[dd, "p_w" near start] & \\
& B'^{\mathrm{I}} \arrow[rr, tail, "\overline{b}" near end] \arrow[dd, two heads, "p_0'" near end] & & B^{\mathrm{I}} \arrow[dd, two heads, "p_0"] \\
A' \arrow[rr, tail, "a"' near end] \arrow[dr, "w'"'] & & A \arrow[dr, "w"] & \\
& B' \arrow[rr, tail, "b"'] & & B
\end{tikzcd} \tag{94}$$

The right hand face is a pullback by definition, and the remainder results from pulling the right face back along b. Thus all faces are pullbacks. The base is also a pushforward, $b_* w' = w$, again by definition. Thus the top is also a pushforward, $\overline{b}_*((p_0')^* w') = p_0^* w$. Indeed, since the front face is a pullback, the Beck-Chevalley condition applies and we have $\overline{b}_*(p_0')^*(w') = p_0^*\, b_*(w') = p_0^* w$.

Now consider the following, in which the top square remains the same as in (94), but p_0 has been relaced by $p_1 : B^{\mathrm{I}} \to B$, so the composite at right is by definition $t = p_1 \circ p_0^* w$.

$$\begin{tikzcd}
P_{w'} \arrow[rr, tail, "\overline{a}"] \arrow[dr, "(p_0')^* w'"] \arrow[ddr, two heads, "t'"'] & & P_w \arrow[dr, "p_0^* w"] \arrow[ddr, two heads, "t" near start] & \\
& B'^{\mathrm{I}} \arrow[rr, tail, "\overline{b}"] \arrow[d, two heads, "p_1'"] & & B^{\mathrm{I}} \arrow[d, two heads, "p_1"] \\
& B' \arrow[rr, tail, "b"'] & & B
\end{tikzcd} \tag{95}$$

The horizontal direction is still pullback along b; let us rename $p_0^* w =: u$ so that $(p_0')^* w' = b^* u$ and $t' = b^* t$ and $p_1' = b^* p_1$ to make this clear. We then add the pushforward along b on the right, in order to obtain the two units η.

$$\begin{tikzcd}
b^*P_w \arrow[rr, tail, "\overline{a}"] \arrow[dr, "b^*u"] \arrow[ddr, two heads, "b^*t"'] & & P_w \arrow[rr, "\eta_t"] \arrow[dr, "u"] \arrow[ddr, two heads, "t"'] & & b_*b^*P_w \arrow[dr, "b_*b^*u"] \arrow[ddr, "b_*b^*t"'] & \\
& b^*B^{\mathrm{I}} \arrow[rr, tail, "\overline{b}"] \arrow[d, two heads, "b^*p_1"] & & B^{\mathrm{I}} \arrow[rr, "\eta_{p_1}"] \arrow[d, two heads, "p_1"] & & b_*b^*B^{\mathrm{I}} \arrow[d, "b_*b^*p_1"] \\
& B' \arrow[rr, tail, "b"'] & & B \arrow[rr, "="'] & & B
\end{tikzcd} \tag{96}$$

By the usual calculation of pushforwards in slice categories, $\overline{b}_* \cong \eta^*_{p_1} \circ b_*$, and so for b^*u we have $\overline{b}_* b^*u = \eta^*_{p_1} b_* b^*u$. But as we just determined in (94) the top left square is already a pushforward, and therefore $u = \eta^*_{p_1} b_* b^* u$, so the top right naturality square is a pullback.

To finish the proof as planned, $p_1 : B^{\mathrm{I}} \to B$ is a trivial fibration because $q : B \to C$ is a fibration, and $b : B' \rightarrowtail B$ is a cofibration because it is a pullback of $c : C' \rightarrowtail C$. Thus by lemma 63, we have that $\eta_{p_1} : B^{\mathrm{I}} \to b_*b^*B^{\mathrm{I}}$ is a trivial fibration, and so its pullback $\eta_t : P_w \to b_*b^*P_w$ is a trivial fibration, as claimed. $\square$

Remark 65. Note that $p : A \to C$ is small if $q : B \to C$ is small.

[*Expected to use alignment here. Also expected to need closure of cofibrations under* Π_{I}*.*]

8 The fibration extension property

In the presence of a universal fibration $\dot{\mathcal{U}} \to \mathcal{U}$, as was constructed in section 6, the fibration extension property (Definition 45) is closely related to the statement that the base object $\mathcal{U}$ is fibrant. For Kan simplicial sets, Voevodsky proved the latter directly using minimal fibrations [?]. Shulman [?] gives a proof from univalence (in the form of the equivalence extension property as stated in section 7) in a more general setting, but it uses the 3-for-2 property for weak equivalences, which is what we are trying to prove. In [?], Coquand uses the equivalence extension property to prove that $\mathcal{U}$ is fibrant, without assuming 3-for-2 for weak equivalences, by a neat argument using a reduction of general box-filling to a condition called "Kan-composition". We shall prove that $\mathcal{U}$ is fibrant using the equivalence extension property via a different argument than that in [**?**], avoiding the reduction of filling to composition, which we therefore do not require.

Returning to the relation between the fibration extension property and the condition that the base object $\mathcal{U}$ is fibrant, it is easily seen that the latter implies the former. Indeed, let $t : X \rightarrowtail X'$ be a trivial cofibration and $Y \twoheadrightarrow X$ a fibration. To extend Y along t, take a classifying map $y : X \to \mathcal{U}$, so that $Y \cong y^*\dot{\mathcal{U}}$ over X. If $\mathcal{U}$ is fibrant then we can extend y along $t : X \rightarrowtail X'$ to get $y' : X' \to \mathcal{U}$ with $y = y' \circ t$. The pullback $Y' = (y')^*\dot{\mathcal{U}} \twoheadrightarrow X'$ is then a (small) fibration such that $t^*Y' \cong t^*(y')^*\dot{\mathcal{U}} \cong y^*\dot{\mathcal{U}} \cong Y$ over X. Thus for the record.

Proposition 66. *If the base object $\mathcal{U}$ of the universal fibration $\dot{\mathcal{U}} \to \mathcal{U}$ is fibrant, then the fibration weak factorization system has the fibration extension property.*

Conversely, we merely note that the FEP clearly implies the fibrancy of $\mathcal{U}$, given the following "alignment" lemma, which will also be required below (cf.(2') of [**?**]).

Lemma 67. *Given a fibration $Y \twoheadrightarrow X$ with classifying map $y : X \to \mathcal{U}$, a cofibration $t : X \rightarrowtail X'$, and a (small) fibration $Y' \twoheadrightarrow X'$ with $Y \cong t^*Y'$ over X, there is a classifying map $y' : X' \to \mathcal{U}$ for Y' with $y' \circ t = y$.*

$$\begin{tikzcd}
Y \arrow[rr] \arrow[dr] \arrow[dd, two heads] & & \dot{\mathcal{U}} \arrow[dd, two heads] \\
& Y' \arrow[ur, dotted] \arrow[dd, two heads] & \\
X \arrow[rr, "y"'] \arrow[dr, tail, "t"'] & & \mathcal{U} \\
& X' \arrow[ur, dotted, "y'"'] &
\end{tikzcd}$$

Proof. This is routine using Yoneda and assuming that cofibrations $A \rightarrowtail B$ have *complemented* monos as components $A[n] \rightarrowtail B[n]$.[1] In more detail,

[1]This of course holds if the base category Set is classical; otherwise, one needs to take this as a further axiom on $\mathcal{C}$.

since $Y' \twoheadrightarrow X'$ is small, there is a classifying map $z : X' \to \mathcal{U}$, perhaps not commuting with t. Nonetheless, we can use z to define the desired $y' : X' \to \mathcal{U}$ objectwise as follows: Take any map from a representable $x' : \mathrm{I}^n \to X'$ and consider whether it factors through t, say as $x' = t \circ x$ for some (necessarily unique) $x : \mathrm{I}^n \to X$. If x' does factor, set $y' \circ x' = y \circ x$; if not, set $y' \circ x' = z \circ x'$. This specification is clearly natural in I^n, so it defines $y' : X' \to \mathcal{U}$, and the specification ensures that $y' \circ t = y$, and that the pullback of $\dot{\mathcal{U}}$ along y' is the same as that along z, namely $Y' \twoheadrightarrow X'$. $\square$

Corollary 68. *The fibration extension property implies that the base $\mathcal{U}$ of the universal fibration is a fibrant object.*

Now we use the equivalence extension property to show that $\mathcal{U}$ is fibrant.

Proposition 69. *The base $\mathcal{U}$ of the universal fibration $\dot{\mathcal{U}} \to \mathcal{U}$ constructed in section 6 is a fibrant object.*

Proof. We need to solve the following filling problem for an arbitrary cofibration $c : C \rightarrowtail Z$, thus showing that $\mathcal{U}^{\delta_0} : \mathcal{U}^{\mathrm{I}} \to \mathcal{U}$ is a trivial fibration (for, say, the point $\delta_0 : 1 \to \mathrm{I}$).[*say something about using δ_0*]

$$\begin{tikzcd} C \arrow[r, "\tilde{a}"] \arrow[d, tail, "c"'] & \mathcal{U}^{\mathrm{I}} \arrow[d, "\mathcal{U}^{\delta_0}"] \\ Z \arrow[r, "b"'] \arrow[ur, dotted] & \mathcal{U} \end{tikzcd} \tag{97}$$

Transposing $\tilde{a}$ to $a : C \times \mathrm{I} \to \mathcal{U}$ and taking pullbacks of $\dot{\mathcal{U}} \twoheadrightarrow \mathcal{U}$ along a and b to get corresponding fibrations $A \twoheadrightarrow C \times \mathrm{I}$ and $B \twoheadrightarrow Z$, we have the following equivalent condition. Letting

$$C_0 : C \cong C \times 1 \to C \times \mathrm{I}$$

be the evident inclusion of the 0-end of the cylinder, let $A_0 = (C_0)^* A \twoheadrightarrow C$ be the "slice of A over C_0". We then have $c^* B \cong A_0$ over C by the outer square of (97).

$$\begin{tikzcd}
 & A_0 \arrow[rr] \arrow[d, two heads] \arrow[ddl] & & A \arrow[d, two heads] \\
 & C \arrow[rr, "C_0"'] \arrow[ddl, "c"] & & C \times \mathrm{I} \arrow[ddl] \\
B \arrow[d, two heads] & & & \\
Z \arrow[rr, "Z_0"'] & & Z \times \mathrm{I} &
\end{tikzcd}$$

The diagonal filler in (97) corresponds, again by transposition and pullback of $\dot{\mathcal{U}} \twoheadrightarrow \mathcal{U}$, to a fibration $D \twoheadrightarrow Z \times \mathrm{I}$ with $(c \times \mathrm{I})^*D \cong A$ over $C \times \mathrm{I}$ and $(Z_0)^*D \cong B$ over Z, as indicated below.

$$\begin{tikzcd}
 & A_0 \arrow[rr] \arrow[d, two heads] \arrow[ddl] & & A \arrow[d, two heads] \arrow[ddl, dotted] \\
 & C \arrow[rr, "C_0"'] \arrow[ddl, "c"] & & C \times \mathrm{I} \arrow[ddl, "c \times \mathrm{I}"] \\
B \arrow[d, two heads] \arrow[rr, dotted] & & D \arrow[d, dotted, two heads] & \\
Z \arrow[rr, "Z_0"'] & & Z \times \mathrm{I} &
\end{tikzcd}$$

We construct $D \twoheadrightarrow Z \times \mathrm{I}$ by the equivalence extension property as follows. Apply the functor $(-) \times \mathrm{I}$ to the left (pullback) face of the above cube to get the following with a new pullback square on the right, with the indicated fibrations.

$$\begin{tikzcd}
 & A_0 \arrow[rr] \arrow[d, two heads] \arrow[ddl] & & A \arrow[d, two heads] \arrow[ddl, dotted] & & A_0 \times \mathrm{I} \arrow[dll, two heads] \arrow[ddl] \\
 & C \arrow[rr, "C_0"'] \arrow[ddl, "c"] & & C \times \mathrm{I} \arrow[ddl, "c \times \mathrm{I}"] & & \\
B \arrow[d, two heads] \arrow[rr, dotted] & & D \arrow[d, dotted, two heads] & & B \times \mathrm{I} \arrow[dll, two heads] & \\
Z \arrow[rr, "Z_0"'] & & Z \times \mathrm{I} & & &
\end{tikzcd} \tag{98}$$

We claim there is a weak equivalence $e : A \simeq A_0 \times \mathrm{I}$ over $C \times \mathrm{I}$, from which follow by the EEP:

(i) a fibration $D \twoheadrightarrow Z \times \mathrm{I}$ with $(c \times \mathrm{I})^* D \cong A$ over $C \times \mathrm{I}$, and

(ii) a weak equivalence $f : D \simeq B \times \mathrm{I}$ over $Z \times \mathrm{I}$ with $(c \times \mathrm{I})^* f \cong e$ over $C \times \mathrm{I}$.

It then remains only to show that $B \cong (Z_0)^* D$ over Z.

To get e, consider the following square, in which the top map is $A_0 \times \delta_0$ (after $A_0 \cong A_0 \times 1$) and the others are those from the previous diagram.

$$\begin{tikzcd} A_0 \arrow[r, tail] \arrow[d, tail] & A_0 \times \mathrm{I} \arrow[d, two heads] \\ A \arrow[r, two heads] & C \times \mathrm{I} \end{tikzcd} \tag{99}$$

The square is easily seen to commute, and the maps with A_0 as domain are trivial cofibrations by Frobenius (proposition 53), because each is the pullback of a trivial cofibration along a fibration. Applying a simple lemma (given below as 70) gives the required weak equivalence $e : A \simeq A_0 \times \mathrm{I}$ over $C \times \mathrm{I}$.

To see that $B \cong (Z_0)^* D$ over Z, recall from the proof of the EEP that the map $f : B \cong (Z_0)^* D$ is the pushforward of $e : A \simeq A_0 \times \mathrm{I}$ along the cofibration $d_0 \times \mathrm{I} : A_0 \times \mathrm{I} \rightarrowtail B \times \mathrm{I}$, calling the evident map $d_0 : A_0 \rightarrowtail B$ in (98). That is, by construction $f = (d_0 \times \mathrm{I})_* e$. We can apply the Beck-Chevalley condition for the pushforward using the pullback square on the left below.

$$\begin{tikzcd} A_0 \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & A_0 \times \mathrm{I} \arrow[d, two heads] & A \arrow[l, "e"'] \\ B \arrow[r, tail] & B \times \mathrm{I} & D \arrow[l, "f"] \end{tikzcd} \tag{100}$$

The pullback of e along the top of the square is the identity on A_0, as can be seen by pulling back e as a map over $C \times \mathrm{I}$ along $C_0 : C \to C \times \mathrm{I}$. Thus the same is true (up to isomorophism) for the pullback of f along the bottom.

An application of the alignment lemma 67 along the trivial cofibration $c \otimes \delta_0$ completes the proof. $\square$

Lemma 70. *Suppose the following square commutes and the indicated cofibrations are trivial.*

$$
\begin{tikzcd}
A \arrow[r, tail] \arrow[d, tail] & C \arrow[d, two heads] \\
B \arrow[r, two heads] & D
\end{tikzcd}
\tag{101}
$$

Then there is a weak equivalence $e : B \simeq C$ over D (and under A).

Proof. Use the fact that any two diagonal fillers are homotopic to get a homotopy equivalence $e : B \simeq C$ filling the square. □

Applying proposition 66 therefore yields the following.

Corollary 71. *The fibration weak factorization system has the fibration extension property (definition 45).*

By Theorem 50, finally, we have the following.

Theorem 72. *There is a Quillen model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ on the category of cubical sets, where:*

1. *the cofibrations $\mathcal{C}$ are any class of monomorphisms satisfying axioms (C0)-(C6),*
2. *the fibrations $\mathcal{F}$ are the maps $f : Y \to X$ for which the canonical map*

$$(f^{\mathrm{I}} \times \mathrm{I}, \mathrm{eval}_Y) : Y^{\mathrm{I}} \times \mathrm{I} \to (X^{\mathrm{I}} \times \mathrm{I}) \times_X Y$$

   lifts on the right against $\mathcal{C}$.