# A Quillen model structure on the category of cartesian cubical sets

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## 1 The cartesian cube category

In contrast to some other treatments of cubical sets [?, ?, ?, ?, ?, ?, ?, ?], we consider what may be termed the *cartesian* cube category $\mathbb{C}$, defined as the free finite product category on an interval $\delta_0, \delta_1 : 1 \rightrightarrows \mathrm{I}$. As a classifying category for an algebraic theory with two constant symbols $\mathbb{T} = \{0, 1\}$, the category $\mathbb{C}$ is dual to the full subcategory of finitely-generated, free $\mathbb{T}$-algebras $\mathsf{Alg}(\mathbb{T})_{\mathrm{fg}}$ (by Lawvere duality). In this case, the algebras are thus simply *bipointed sets* $(A, a_0, a_1)$, and the free ones are the *strictly* bipointed sets $a_0 \neq a_1$. Thus $\mathsf{Alg}(\mathbb{T})_{\mathrm{fg}}$ consists of the finite, strictly bipointed sets and all bipointed maps between them. We will use the following specific presentation.

**Definition 1.** The objects of the cartesian cube category $\mathbb{C}$, called $n$-cubes, will be written

$$[n] = \{0, x_1, ..., x_n, 1\}\,.$$

The arrows,

$$f : [n] \longrightarrow [m]\,,$$

maybe taken to be $m$-tuples of elements drawn from the set $\{0, x_1, ..., x_n, 1\}$ regarded as formal terms representing composites of faces, degeneracies, permutations, and diagonals. Equivalently, the arrows $[n] \longrightarrow [m]$ are arbitrary bipointed maps $[m] \longrightarrow [n]$.

See [?] for further details.

## 2 Cubical sets

The category $\mathsf{cSet}$ of *cubical sets* is the category of presheaves on the cartesian cube category $\mathbb{C}$,

$$\mathsf{cSet} \;=\; \mathsf{Set}^{\mathbb{C}^{\mathrm{op}}}\,.$$

It is thus generated by the representable presheaves $y([n])$, which will be written

$$\mathrm{I}^n = y([n])$$

and called the *standard $n$-cubes.*

## 3 The cofibration weak factorization system

**Cofibrations.** The *cofibrations* are a class $\mathcal{C}$ of maps in $\mathsf{cSet}$, written

$$c : A \rightarrowtail B\,,$$

and are assumed to satisy the following axioms:

(C1) All isomorphisms are cofibrations.

(C2) The composite of two cofibrations is a cofibration.

(C3) Cofibrations are monomorphisms.

(C4) Any pullback of a cofibration is a cofibration.

By conditions (C3-4), the cofibrations are classified by a (pointed) subobject $\Phi \hookrightarrow \Omega$ (a not necessarily cofibrant mono) of the standard subobject classifier $\top : 1 \longrightarrow \Omega$ of $\mathsf{cSet}$. We shall call the canonical factorization $t : 1 \longrightarrow \Phi$ the *cofibration classifier.* Note that we permit the case where $\Phi = \Omega$, i.e. all monos are cofibrant.

**Cofibrant partial map classifier.** The polynomial endofunctor [**?**] determined by the cofibration classifier $t : 1 \longrightarrow \Phi$ is defined on objects by

$$X \;\mapsto\; \Phi_!\, t_*(X) \;=\; \sum_{\phi:\Phi} X^{\phi}\,.$$

We shall write $X^+ := \sum_{\phi:\Phi} X^{\phi}$.

Observe that by the definition of $X^+$ there is a pullback square,

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X^+ \arrow[d, "t_* X"] \\ 1 \arrow[r, "t"'] & \Phi \end{tikzcd}$$

since $t$ is monic. Let $\eta : X \rightarrowtail X^+$ be the indicated top horizontal map; we call this map the *cofibrant partial map classifier* of $X$.

**Proposition 2.** *The map $\eta : X \rightarrowtail X^+$ classifies partial maps with cofibrant domain, in the following sense.*

1. *The map $\eta : X \rightarrowtail X^+$ is a cofibration.*
2. *For any object $Z$ and any partial map $(s, g) : Z \leftarrowtail S \longrightarrow X$, with $s : S \rightarrowtail Z$ cofibrant, there is a unique $f : Z \longrightarrow X^+$ making a pullback square,*

$$\begin{tikzcd} S \arrow[r, "g"] \arrow[d, tail, "s"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, tail, "\eta"] \\ Z \arrow[r, "f"'] & X^+ . \end{tikzcd}$$

*Proof.* $\eta : X \rightarrowtail X^+$ is a cofibration since it is a pullback of $t : 1 \longrightarrow \Phi$. The second statement follows directly from the definition of $X^+$ as a polynomial (see [**?**], prop. 7). □

**The +-Monad.**

**Proposition 3.** *The pointed endofunctor determined by $\eta_X : X \rightarrowtail X^+$ has a natural multiplication $\mu_X : X^{++} \longrightarrow X^+$ making it a monad.*

*Proof.* Since the cofibrations are closed under composition, the monad structure on $X^+$ follows as in [**?**], proposition nm. Explicitly, $\mu_X$ is determined as the unique map making the following a pullback diagram.

$$\begin{tikzcd} X \arrow[r, "="] \arrow[d, tail, "\eta_X"'] & X \arrow[dd, tail, "\eta"] \\ X^+ \arrow[d, tail, "\eta_{X^+}"'] & \\ X^{++} \arrow[r, dotted, "\mu"'] & X^+ \end{tikzcd}$$

□

**Relative partial map classifier.** For any object $X \in \mathsf{cSet}$ the usual pullback functor

$$X^* : \mathsf{cSet} \longrightarrow \mathsf{cSet}/X\,,$$

taking any $A$ to the second projection $A \times X \longrightarrow X$, not only preserves the subobject classifier $\Omega$, but also the cofibration classifier $\Phi \hookrightarrow \Omega$, where a map in $\mathsf{cSet}/X$ is defined to be a cofibration if it is one in $\mathsf{cSet}$. Thus in $\mathsf{cSet}/X$ the *(relative) cofibration classifier* is the map

$$t \times X : 1 \times X \longrightarrow \Phi \times X \quad \text{over } X$$

which we may also write $t_X : 1_X \longrightarrow \Phi_X$. Like $t : 1 \longrightarrow \Phi$, this map determines a polynomial endofunctor

$$+_X : \mathsf{cSet}/X \longrightarrow \mathsf{cSet}/X\,,$$

which commutes (up to natural isomorphism) with $+ : \mathsf{cSet} \longrightarrow \mathsf{cSet}$ and $X^* : \mathsf{cSet} \longrightarrow \mathsf{cSet}/X$ in the evident way:

$$\begin{tikzcd} \mathsf{cSet}/X \arrow[r, "+_X"] & \mathsf{cSet}/X \\ \mathsf{cSet} \arrow[u, "X^*"] \arrow[r, "+"'] & \mathsf{cSet} \arrow[u, "X^*"'] \end{tikzcd} \tag{1}$$

The endofunctor $+_X$ is also pointed $\eta : Y \longrightarrow Y^+$ and has a monad multiplication $\mu_Y : Y^{++} \longrightarrow Y^+$, for any $Y \longrightarrow X$, for the same reason that $+$ has this structure. Summarizing, we may say that *the polynomial monad* $+ : \mathsf{cSet} \longrightarrow \mathsf{cSet}$ *is fibered over* $\mathsf{cSet}$.

**Definition 4.** A $+$-*algebra* in $\mathsf{cSet}$ is a cubical set $A$ together with a retraction $\alpha : A^+ \longrightarrow A$ of $\eta_A : A \longrightarrow A^+$, i.e. an algebra for the pointed endofunctor $(+ : \mathsf{cSet} \longrightarrow \mathsf{cSet},\ \eta : 1 \longrightarrow +)$. Algebras for the monad $(+, \eta, \mu)$ will be referred to specifically as $(+, \eta, \mu)$-*algebras*, or $+$-*monad algebras.*

A *relative* $+$-*algebra* in $\mathsf{cSet}$ is a map $A \longrightarrow X$ together with an algebra structure for the pointed endofunctor $+_X : \mathsf{cSet}/X \longrightarrow \mathsf{cSet}/X$.

**The factorization system.**

**Proposition 5.** *There is an (algebraic) weak factoriation system on* $\mathsf{cSet}$ *given by taking as the left class the cofibrations and as the right class the (maps underlying) the relative* $+$-*algebras. Thus a right map is one* $f$ :

*$A \longrightarrow X$ for which there is a retract $\alpha : A' \longrightarrow A$ over $X$ of the canonical map $\eta_f : A \longrightarrow A'$ over $X$,*

$$\begin{tikzcd}
A \arrow[r, "\eta_f"] \arrow[rr, bend left, "="] \arrow[dr, "f"'] & A' \arrow[r, "\alpha"] \arrow[d, "f^+"] & A \arrow[dl, "f"] \\
& X. &
\end{tikzcd}$$

*Proof.* The factorization of any map $f : Y \longrightarrow X$ is given simply by applying the (relative) +-functor

$$\begin{tikzcd}
Y \arrow[r, "\eta_f"] \arrow[dr, "f"'] & Y' \arrow[d, "f^+"] \\
& X.
\end{tikzcd}$$

We know that the unit $\eta_f$ is always a cofibration, and since $f^+$ is the free algebra for the +-monad, it is in particular a +-algebra.

For the lifting condition, consider a cofibration $c : B \rightarrowtail C$, a right map $A \longrightarrow X$, with a $+_X$-algebra structure map $\alpha : A^+ \longrightarrow A$ over $X$, and a commutative square as indicated in the following.

$$\begin{tikzcd}
B \arrow[r, "g"] \arrow[d, tail, "c"'] & A \arrow[d] \arrow[dr, "\eta"] & \\
C \arrow[r, "f"'] & X & A^+ \arrow[l] \arrow[ul, bend right, "\alpha"']
\end{tikzcd}$$

Thus over $X$, we have the situation

$$\begin{tikzcd}
B \arrow[r, "g"] \arrow[d, tail, "c"'] & A \arrow[d, "\eta"] \\
C \arrow[ur, dotted, "d"] & A^+ \arrow[u, bend right=60, "\alpha"']
\end{tikzcd}$$

and we seek a diagonal filler as indicated. Since $(c, g) : B \leftarrowtail C \longrightarrow A$ is a cofibrant partial map into $A$, there is a map $\varphi : C \longrightarrow A^+$ (over X) making a (pullback) square,

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, tail, "c"'] & A \arrow[d, "\eta"] \\ C \arrow[r, dotted, "\varphi"'] & A^+ \arrow[u, bend right, "\alpha"'] \end{tikzcd}$$

We thus have $d := \alpha \circ \varphi : C \longrightarrow A$ as the required diagonal filler.

The closure of the cofibrations under retracts follows from their classification by a universal object $t : 1 \longrightarrow \Phi$, and the closure of the right maps under retracts follows from their being the algebras for a pointed endofunctor underlying a monad (cf. [**?**]). Algebraicity of this weak factorization system also follows directly, since $+$ is a monad. $\square$

Summarizing, we have a weak factorization system $(\mathcal{L}, \mathcal{R})$ on the category $\mathsf{cSet}$ of cubical sets, in which:

$$\begin{aligned} \mathcal{L} &= \mathcal{C} \quad \text{(the cofibrations)} \\ \mathcal{R} &= +\mathsf{Alg} \quad \text{(the relative +-algebras)} \end{aligned}$$

We shall call this the *cofibration weak factorization system*. As here, we will sometimes say that an object (or map) is a (relative) +-algebra when it can be equipped with a (relative) +-algebra structure; such maps will also be called *trivial fibrations* and the class of all such is denoted $\mathsf{TrivFib}$,

$$\mathsf{TrivFib} = \mathcal{C}^{\pitchfork}.$$

**Uniform filling structure.** It will be convenient to relate +-algebra structure with the more familiar diagonal filling condition of weak factorization systems, and specifically a special form of the latter that occurs in [**?**] under the name *uniform filling structure*.

Consider a generating sub*set* of cofibrations, consisting of all those cofibrations $c : C \rightarrowtail Z$ where $Z$ is representable, $Z = \mathrm{I}^n$. Call these maps the *basic cofibrations*, and let

$$\mathsf{BCof} = \{c : C \rightarrowtail \mathrm{I}^n \mid c \in \mathcal{C}, n \geq 0\}. \tag{2}$$

**Proposition 6.** *For any object $X$ in* $\mathsf{cSet}$ *the following are equivalent:*

1. *$X$ is a* +-algebra*, i.e. there is a retraction $\alpha : X^+ \longrightarrow X$ of the unit $\eta : X \longrightarrow X^+$.*

2. $X$ *is* contractible *in the sense that it has the right lifting property with respect to all cofibrations,*
$$\mathcal{C} \pitchfork X.$$

3. $X$ *has a* uniform filling structure*: for each basic cofibration* $c : C \rightarrowtail \mathrm{I}^n$ *and map* $x : C \longrightarrow X$ *there is given an extension* $j(c,x)$*,*
$$\begin{tikzcd} C \arrow[r, "x"] \arrow[d, tail, "c"'] & X, \\ \mathrm{I}^n \arrow[ur, dotted, "j(c,x)"'] & \end{tikzcd} \tag{3}$$
*and the choice is* uniform in $\mathrm{I}^n$ *in the following sense: given any cubical map* $u : \mathrm{I}^m \longrightarrow \mathrm{I}^n$*, the pullback* $u^*c : u^*C \rightarrowtail \mathrm{I}^m$ *is again a basic cofibration and fits into a commutative diagram of the form*
$$\begin{tikzcd} u^*C \arrow[r, "u'"] \arrow[d, tail, "c'"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[r, "x"] \arrow[d, tail, "c"] & X. \\ \mathrm{I}^m \arrow[r, "u"'] & \mathrm{I}^n \arrow[ur, dotted, "j(c,x)"'] & \end{tikzcd} \tag{4}$$
*For the pair* $(c',\, xu')$ *in* (28) *the chosen extension* $j(c', xu') : \mathrm{I}^m \longrightarrow X$*, is equal to* $j(c,x) \circ u$*,*
$$j(c', xu') = j(c,x)u. \tag{5}$$

*Proof.* Let $(X, \alpha)$ be a $+$-*algebra* and suppose given the span $(c, x)$ as below, with $c$ a cofibration.
$$\begin{tikzcd} C \arrow[r, "x"] \arrow[d, tail, "c"'] & X \\ Z & \end{tikzcd}$$
Let $\chi(c,x) : Z \longrightarrow X^+$ be the classifying map of the evident partial map $(c,x) : Z \rightharpoonup X$, so that we have a pullback square as follows.
$$\begin{tikzcd} C \arrow[r, "x"] \arrow[d, tail, "c"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\eta"] \\ Z \arrow[r, "\chi(c,x)"'] & X^+ \end{tikzcd} \tag{6}$$
Then set
$$j = \alpha \circ \chi(c,x) : Z \longrightarrow X \tag{7}$$

to get a filler,

$$\begin{tikzcd} C \arrow[r, "x"] \arrow[d, tail, "c"'] & X \arrow[d, "\eta"] \\ Z \arrow[r, "\chi(c,x)"'] \arrow[ur, dotted, "j"] & X^+ \arrow[u, bend right, "\alpha"'] \end{tikzcd} \tag{8}$$

since $j \circ c = \alpha \circ \chi(c, x) \circ c = \alpha \circ \eta \circ x = x$. Thus (1) implies (2). To see that it also implies (3), observe that in the case where $Z = I^n$ and we specify, in (7), that

$$j(c, x) = \alpha \circ \chi(c, x) : \mathrm{I}^n \longrightarrow X, \tag{9}$$

then the assignment is natural in $\mathrm{I}^n$. Indeed, given any $u : \mathrm{I}^m \longrightarrow \mathrm{I}^n$, we have

$$j(c', xu') = \alpha \circ \chi(c', xu') = \alpha \circ \chi(c, x) \circ u = j(c, x)u, \tag{10}$$

by the uniqueness of classifying maps.

It is clear that (2) implies (1), since if $\mathcal{C} \pitchfork X$ then we can take as an algebra structure $\alpha : X^+ \longrightarrow X$ any filler for the span

$$\begin{tikzcd} X \arrow[r, "="] \arrow[d, tail, "\eta"'] & X. \\ X^+ \arrow[ur, dotted, "\alpha"'] & \end{tikzcd}$$

To see that (3) implies (1), suppose that $X$ has a uniform filling structure $j$ and we want to define an algebra structure $\alpha : X^+ \longrightarrow X$. By Yoneda, for every $y : \mathrm{I}^n \longrightarrow X^+$ we need a map $\alpha(y) : \mathrm{I}^n \longrightarrow X$, naturally in $\mathrm{I}^n$, in the sense that for any $u : \mathrm{I}^m \longrightarrow \mathrm{I}^n$, we have

$$\alpha(yu) = \alpha(y)u. \tag{11}$$

Moreover, to ensure that $\alpha\eta = 1_X$, for any $x : \mathrm{I}^n \longrightarrow X$ we must have $\alpha(\eta \circ x) = x$. So take $y : \mathrm{I}^n \longrightarrow X^+$ and let

$$\alpha(y) = j(y^*\eta, y'),$$

as indicated on the right below.

$$\begin{tikzcd} u^*C \arrow[r, "u'"] \arrow[d, tail, "u^*y^*\eta"'] \arrow[dr, phantom, very near start, "\lrcorner"] & C \arrow[r, "y'"] \arrow[d, tail, "y^*\eta"'] \arrow[dr, phantom, very near start, "\lrcorner"] & X. \arrow[d, "\eta"] \\ \mathrm{I}^m \arrow[r, "u"'] & \mathrm{I}^n \arrow[r, "y"'] \arrow[ur, dotted, "j(y^*\eta, y')"'] & X^+ \end{tikzcd} \tag{12}$$

Then for any $u : \mathrm{I}^m \longrightarrow \mathrm{I}^n$, we indeed have

$$\alpha(yu) = j\big((yu)^*\eta, y'u'\big) = j(y^*\eta, y') \circ u = \alpha(y)u,$$

by the uniformity of $j$. Finally, if $y = \eta \circ x$ for some $x : \mathrm{I}^n \longrightarrow X$ then

$$\alpha(\eta x) = j\big((\eta x)^*\eta, (\eta x)'\big) = j(1_X, x) = x,$$

because the defining diagram for $\alpha(\eta x)$, i.e. the one on the right in (12), then factors as

$$\begin{tikzcd} \mathrm{I}^n \arrow[r, "x"] \arrow[d, tail, "="'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[r, "="] \arrow[d, tail, "="'] \arrow[dr, phantom, "\lrcorner", very near start] & X, \arrow[d, "\eta"] \\ \mathrm{I}^n \arrow[r, "x"'] & X \arrow[r, "\eta"'] & X^+ \end{tikzcd} \tag{13}$$

and the only possible extension $j(1_X, x)$ for the span $(1_{\mathrm{I}^n}, x)$ is $x$ itself. $\square$

The relative version of the foregoing is entirely analogous, since the +-functor is fibered over cSet in the sense of diagram (1). We can therefore omit the entirely analogous proof. The statement is as follows.

**Proposition 7.** *For any map $f : Y \longrightarrow X$ in* cSet *the following are equivalent:*

1. *$f : Y \to X$ is a* (relative) +-algebra (over $X$)*, i.e. there is a retraction $\alpha : Y' \to Y$ over $X$ of the unit $\eta : Y \to Y'$ over $X$, where $f^+ : Y' \to X$ is the result of the relative $+_X$-functor applied to $f$, as in definition 4.*

2. *$f : Y \to X$ is* trivial fibration *in the sense that it has the right lifting property with respect to all cofibrations,*

$$\mathcal{C} \pitchfork f.$$

3. *$f : Y \to X$ has a* uniform filling structure*: for each basic cofibration $c : C \rightarrowtail \mathrm{I}^n$ and maps $x : C \to X$ and $y : \mathrm{I}^n \to Y$ making the square below commute, there is given a diagonal filler $j(c, x, y)$,*

$$\begin{tikzcd} C \arrow[r, "x"] \arrow[d, tail, "c"'] & X \arrow[d, "f"] \\ \mathrm{I}^n \arrow[r, "y"'] \arrow[ur, dotted, "j(c,x,y)"'] & Y, \end{tikzcd} \tag{14}$$

*and the choice is* uniform in $\mathrm{I}^n$ *in the following sense: given any cubical map* $u : \mathrm{I}^m \to \mathrm{I}^n$*, the pullback* $u^*c : u^*C \rightarrowtail \mathrm{I}^m$ *is again a basic cofibration and fits into a commutative diagram of the form*

$$\begin{tikzcd} u^*C \arrow[r, "c^*u"] \arrow[d, tail, "u^*c"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[r, "x"] \arrow[d, tail, "c"'] & X \arrow[d, "f"] \\ \mathrm{I}^m \arrow[r, "u"'] & \mathrm{I}^n \arrow[r, "y"'] \arrow[ur, dotted, "j(c,x,y)"'] & Y. \end{tikzcd} \tag{15}$$

*For the evident triple* $(u^*c,\, xc^*u, yu)$ *in* (15) *the chosen diagonal filler*

$$j(u^*c, xc^*u, yu) : \mathrm{I}^m \longrightarrow X$$

*is equal to* $j(c, x, y) \circ u$*,*

$$j(u^*c, xc^*u, yu) = j(c, x, y)u. \tag{16}$$

# 4 Partial path lifting (biased version)

Our next goal is the specification of a second weak factorization system (the *fibration weak factorization system*) with a restricted class of "trivial" cofibrations on the left, and an expanded class of right maps, the fibrations.

As a warm-up, we first recall the specification of the trivial-cofibration/fibration WFS from [?]. (In an appendix we show that these fibrations agree with those specified in the "logical style" of [?, ?]). In the subsequent section we shall modify the specification of fibrations in order to arrive at an "unbiased" version that is more appropriate for the cartesian setting.

A *generating class of (biased) trivial cofibrations* are all maps of the form

$$c \otimes \delta_\epsilon : D \longrightarrow Z \times \mathrm{I}\,, \tag{17}$$

where:

1. $c : C \rightarrowtail Z$ is an arbitrary cofibration,

2. $\delta_\epsilon : 1 \longrightarrow \mathrm{I}$ is one of the two "endpoint inclusions" where, recall, $1 = y[0]$, and $I = y[1]$, and for $\epsilon = 0, 1$, we have the maps $\delta_\epsilon : 1 \longrightarrow \mathrm{I}$ corresponding to the two bipointed maps $0, 1 : \{0, x, 1\} \longrightarrow \{0, 1\}$.

3. $c \otimes \delta_\epsilon$ is the pushout-product (resp. "Leibniz tensor") of the cofibration $c : C \rightarrowtail Z$ and an endpoint $\delta_\epsilon : 1 \longrightarrow \mathrm{I}$, as indicated in the following

diagram (in which the unlabelled maps are the expected ones).

$$
\begin{tikzcd}
C \times 1 \arrow[r] \arrow[d] & C \times \mathrm{I} \arrow[d] \arrow[ddr, bend left] & \\
Z \times 1 \arrow[r] \arrow[drr, bend right] & Z +_C (C \times \mathrm{I}) \arrow[dr, "c\otimes\delta_\epsilon"'] & \\
 & & Z \times \mathrm{I}
\end{tikzcd}
\tag{18}
$$

4. $D = Z +_C (C \times \mathrm{I})$ is the indicated pushout, the domain of $c \otimes \delta_\epsilon$.

In order to insure that such maps are indeed cofibrations, we assume two further axioms:

(C5) The endpoint inclusions $\delta_\epsilon : 1 \longrightarrow \mathrm{I}$ are cofibrations.

(C6) The cofibrations are closed under pushout-products.

In place of (C6), we could require that cofibrations be closed under the join operation $A \vee B$ in the lattice of subobjects of an object.

**Fibrations (biased version).** Let

$$\mathcal{C} \otimes \delta_\epsilon \;=\; \{c \otimes \delta_\epsilon : D \rightarrowtail Z \times \mathrm{I} \mid c \in \mathcal{C},\ \epsilon = 0,1\}$$

be the class of all such pushout-products of arbitrary cofibrations $c : C \rightarrowtail Z$ with endpoint inclusions $\delta_\epsilon : 1 \rightarrowtail \mathrm{I}$. The *(biased) fibrations* are defined to be the right class of these generating trivial cofibrations,

$$(\mathcal{C} \otimes \delta_\epsilon)^{\pitchfork} \;=\; \mathcal{F}\,.$$

Thus a map $f : Y \longrightarrow X$ is a (biased) fibration if for every commutative square of the form

$$
\begin{tikzcd}
Z +_C (C \times \mathrm{I}) \arrow[r] \arrow[d, tail, "c\otimes\delta_\epsilon"'] & Y \arrow[d, "f"] \\
Z \times \mathrm{I} \arrow[r] \arrow[ur, dotted, "j"] & X
\end{tikzcd}
\tag{19}
$$

with a generating trivial cofibration on the left, there is a diagonal filler $j$ as indicated. This condition can be seen as a generalized homotopy lifting property.

To relate this notion of fibration to the cofibration weak factorization system, fix any map $u : A \longrightarrow B$, and recall (e.g. from [?]) that the pushout-product with $u$ is a functor on the arrow category

$$(-) \otimes u : \mathsf{cSet}^{\mathbb{2}} \longrightarrow \mathsf{cSet}^{\mathbb{2}}\,.$$

This functor has a right adjoint, the *pullback-hom* (or "Leibniz exponential"), which for a map $f : X \longrightarrow Y$ we will write as

$$(u \Rightarrow f) : Y^B \longrightarrow (X^B \times_{X^A} Y^A)\,.$$

The pullback-hom is determined as indicated in the following diagram (in which the unlabelled maps are the expected ones).

$$\begin{tikzcd}
Y^B \arrow[dr, "u \Rightarrow f"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X^B \times_{X^A} Y^A \arrow[r] \arrow[d] & Y^A \arrow[d] \\
& X^B \arrow[r] & X^A
\end{tikzcd} \tag{20}$$

Using the $\otimes \dashv \Rightarrow$ adjunction on the arrow category, we can now show the following (cf. [?], prop. n.m).

**Proposition 8.** *An object $X$ is fibrant if and only if both of the endpoint projections $X^{\mathrm{I}} \longrightarrow X$ from the pathspace are (relative) $+$-algebras (over $X$). More generally, a map $f : Y \longrightarrow X$ is a fibration iff both of the maps*

$$(\delta_\epsilon \Rightarrow f) : Y^I \longrightarrow X^I \times_X Y$$

*are $+$-algebras (for $\epsilon = 0, 1$).*

*Proof.* The first statement follows from the second, since the pathspace projections $X^{\mathrm{I}} \longrightarrow X$ are just the maps

$$(\delta_\epsilon \Rightarrow !_X) : X^I \longrightarrow (1^I \times_1 X) \cong X\,,$$

for $!_X : X \longrightarrow 1$.

By definition, $f : X \longrightarrow Y$ is a fibration iff every square of the form

$$\begin{tikzcd}
Z +_C (C \times \mathrm{I}) \arrow[r] \arrow[d, tail, "c \otimes \delta_\epsilon"'] & Y \arrow[d, "f"] \\
Z \times \mathrm{I} \arrow[r] \arrow[ur, dotted, "j"] & X\,,
\end{tikzcd} \tag{21}$$

with a generating trivial cofibration $c \otimes \delta_\epsilon$ on the left, has is a diagonal filler $j$ as indicated. Briefly,

$$(c \otimes \delta_\epsilon) \pitchfork f \qquad (\text{for } c \in \mathcal{C},\ \epsilon = 0,1).$$

By the $\otimes \dashv \Rightarrow$ adjunction, this is equivalent to the condition

$$c \pitchfork (\delta_\epsilon \Rightarrow f) \qquad (\text{for } c \in \mathcal{C},\ \epsilon = 0,1).$$

That is, for every square

$$\begin{tikzcd}
C \arrow[r] \arrow[d, tail, "c"'] & Y^I \arrow[d, "\delta_\epsilon \Rightarrow f"] \\
Z \arrow[r] \arrow[ur, dotted, "k"] & X^I \times_X Y\,,
\end{tikzcd}$$

with an arbitrary cofibration $c : C \rightarrowtail Z$ on the left, there is a diagonal filler $k$ as indicated, for $\epsilon = 0, 1$. But this is just to say that the maps $\delta_\epsilon \Rightarrow f$ are in the right class of the cofibrations, which is equivalent to their being $+$-algebras, as claimed. $\square$

**Fibration structure.** The $\otimes \dashv \Rightarrow$ adjunction determines the fibrations in terms of the trivial fibrations, which in turn can be determined by *uniform* lifting against a *set* of basic cofibrations, by proposition 7. We can similarly determine the fibrations by uniform lifting against a *set* of trivial cofibrations, consisting of all those $c \otimes \delta_\epsilon$ in $\mathcal{C} \otimes \delta_\epsilon$ where $c : C \rightarrowtail Z$ has a representable codomain $Z = \mathrm{I}^n$. Call these maps the *basic (biased) trivial cofibrations*, and let

$$\mathcal{B} \otimes \delta_\epsilon = \{c \otimes \delta_\epsilon : B \rightarrowtail \mathrm{I}^{n+1} \mid c : C \rightarrowtail \mathrm{I}^n,\ \epsilon = 0,1,\ n \geq 0\}, \tag{22}$$

where the pushout-product $c \otimes \delta_\epsilon$ now takes the simpler form

$$\begin{tikzcd}
C \arrow[r] \arrow[d, tail] & C \times \mathrm{I} \arrow[d] \arrow[ddr, bend left] & \\
\mathrm{I}^n \arrow[r] \arrow[drr, bend right] & \mathrm{I}^n +_C (C \times \mathrm{I}) \arrow[dr, tail, "c \otimes \delta_\epsilon"'] & \\
& & \mathrm{I}^n \times \mathrm{I}
\end{tikzcd} \tag{23}$$

for a cofibration $c : C \rightarrowtail \mathrm{I}^n$, an endpoint $\delta_\epsilon : 1 \longrightarrow \mathrm{I}$, and with domain $B = \big(\mathrm{I}^n +_C (C \times \mathrm{I})\big)$. These subobjects $B \rightarrowtail \mathrm{I}^{n+1}$ can be seen geometrically as generalized open box inclusions.

For any map $f : Y \longrightarrow X$ a (uniform, biased) fibration structure on $f$ is a choice of diagonal fillers $j_\epsilon(c, x, y)$,

$$\begin{tikzcd} \mathrm{I}^n +_C (C \times \mathrm{I}) \arrow[r, "x"] \arrow[d, "c \otimes \delta_\epsilon"'] & X \arrow[d, "f"] \\ \mathrm{I}^n \times \mathrm{I} \arrow[r, "y"'] \arrow[ur, dotted, "j_\epsilon(c,x,y)"'] & Y, \end{tikzcd} \tag{24}$$

for each basic trivial cofibration $c \otimes \delta_\epsilon : B = (\mathrm{I}^n +_C (C \times \mathrm{I})) \rightarrowtail \mathrm{I}^{n+1}$ and maps $x : B \longrightarrow X$ and $y : \mathrm{I}^{n+1} \longrightarrow Y$, which is *uniform in* $\mathrm{I}^n$ in the following sense: given any cubical map $u : \mathrm{I}^m \longrightarrow \mathrm{I}^n$, the pullback $u^*c : u^*C \rightarrowtail \mathrm{I}^m$ of $c : C \rightarrowtail \mathrm{I}^n$ along $u$ determines another basic trivial cofibration

$$u^*c \otimes \delta_\epsilon : B' = (\mathrm{I}^m +_{u^*C} (u^*C \times \mathrm{I})) \rightarrowtail \mathrm{I}^{m+1},$$

which fits into a commutative diagram of the form

$$\begin{tikzcd} \mathrm{I}^m +_{u^*C} (u^*C \times \mathrm{I}) \arrow[r, "(u \times \mathrm{I})'"] \arrow[d, "u^*c \otimes \delta_\epsilon"'] & \mathrm{I}^n +_C (C \times \mathrm{I}) \arrow[r, "x"] \arrow[d, "c \otimes \delta_\epsilon"'] & X \arrow[d, "f"] \\ \mathrm{I}^m \times \mathrm{I} \arrow[r, "u \times \mathrm{I}"'] & \mathrm{I}^n \times \mathrm{I} \arrow[r, "y"'] \arrow[ur, dotted, "j_\epsilon(c,x,y)"'] & Y, \end{tikzcd} \tag{25}$$

by applying the functor $(-) \otimes \delta_\epsilon$ to the pullback square relating $u^*c$ to $c$. Now for the outer rectangle in (25) there is a chosen diagonal filler

$$j_\epsilon(u^*c, x(u \times \mathrm{I})', y(u \times \mathrm{I})) : \mathrm{I}^m \times \mathrm{I} \longrightarrow X$$

and for this map we require that

$$j_\epsilon(u^*c, x(u \times \mathrm{I})', y(u \times \mathrm{I})) = j_\epsilon(c, x, y) \circ (u \times \mathrm{I}). \tag{26}$$

This is a reformulation of the logical specification given in [**?**] (see the appendix).

**Definition 9.** A *(uniform, biased) fibration structure* on a map $f : Y \to X$ is a choice of fillers $j_\epsilon(c, x, y)$ as in (27) satisfying (29) for all maps $u : \mathrm{I}^m \to \mathrm{I}^n$.

Essentially the same argument as that given for Proposition 8 also yields the following sharper formulation in terms of fibration structure.

**Corollary 10.** *Fibration structure on a map $f : Y \to X$ is equivalent to a pair of $+$-algebra structures on the maps*

$$(\delta_\epsilon \Rightarrow f) : Y^I \longrightarrow X^I \times_X Y$$

*for $\epsilon = 0, 1$.*

Finally, we have the analogue of proposition 6 for fibrant objects; we omit the analogous statement of proposition 7 for fibrations, as well as the entirely analogous proof.

**Corollary 11.** *For any object $X$ in* cSet *the following are equivalent:*

1. *$X$ is* fibrant*, i.e. every partial map to $X$ with a generating trivial cofibration $D \rightarrowtail Z \times \mathrm{I}$ as domain of definition extends to a total map $Z \times \mathrm{I} \to X$,*

$$\mathcal{C} \otimes \delta_\epsilon \pitchfork f$$

2. *There are $+$-algebra structures on the canonical maps*

$$(\delta_\epsilon \Rightarrow X) : X^I \longrightarrow X,$$

*for $\epsilon = 0, 1$.*

3. *$X \to 1$ has a* (uniform, biased) fibration structure*. Explicitly, for each basic trivial cofibration $c \otimes \delta_\epsilon : B \rightarrowtail \mathrm{I}^{n+1}$ and map $x : B \to X$, there is given an extension $j_\epsilon(c, x)$,*

$$\begin{tikzcd} B \arrow[r, "x"] \arrow[d, tail, "c\otimes\delta_\epsilon"'] & X, \\ \mathrm{I}^{n+1} \arrow[ur, dotted, "j_\epsilon(c,x)"'] & \end{tikzcd} \tag{27}$$

*and the choice is* uniform in $\mathrm{I}^n$ *in the sense: given any cubical map $u : \mathrm{I}^m \to \mathrm{I}^n$, the pullback $u^*c \otimes \delta_\epsilon : B' \rightarrowtail \mathrm{I}^m \times \mathrm{I}$ fits into a commutative diagram of the form*

$$\begin{tikzcd} B' \arrow[r, "(u\times\mathrm{I})'"] \arrow[d, tail, "u^*c\otimes\delta_\epsilon"'] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[r, "x"] \arrow[d, tail, "c\otimes\delta_\epsilon"'] & X. \\ \mathrm{I}^m \times \mathrm{I} \arrow[r, "u\times\mathrm{I}"'] & \mathrm{I}^n \times \mathrm{I} \arrow[ur, dotted, "j(c,x)"'] & \end{tikzcd} \tag{28}$$

*Then for the pair $(u^*c \otimes \delta_\epsilon,\ x(u \times \mathrm{I})')$ in* (28) *the chosen extension*

$$j(u^*c \otimes \delta_\epsilon, x(u \times \mathrm{I})') : \mathrm{I}^m \times \mathrm{I} \longrightarrow X$$

*is equal to $j(c, x) \circ (u \times \mathrm{I})$,*

$$j(u^*c \otimes \delta_\epsilon, x(u \times \mathrm{I})') = j(c, x)(u \times \mathrm{I}). \tag{29}$$

## 5 Unbiased partial path-lifting

Rather than building a weak factorization system based on the foregoing notion of (biased) fibration (as is done in [**?**, **?**]), we shall first eliminate the "bias" on a choice of endpoint $\delta_\epsilon : 1 \longrightarrow \mathrm{I}$, expressed by the indexing $\epsilon = 0, 1$. This will have the effect of adding more trivial cofibrations, and thus more weak equivalences, to our model structure. Consider first the simple path-lifting condition, which is a special case of (19) with $c = ! : 0 \rightarrowtail 1$, since $! \otimes \delta_\epsilon = \delta_\epsilon$:

$$\begin{tikzcd} 1 \arrow[r] \arrow[d, tail, "\delta_\epsilon"'] & Y \arrow[d, "f"] \\ \mathrm{I} \arrow[r] \arrow[ur, dotted, "j_\epsilon"'] & X . \end{tikzcd}$$

(Note that $0 \rightarrowtail 1$ is a cofibration by axioms C4 and C5).

In toplogical spaces, rather than requiring lifts $j_\epsilon$ for each of the endpoints $\epsilon = 0, 1$, we could instead require that there be a lift $j_i$ *for each point* $i : 1 \longrightarrow \mathrm{I}$ *in the real interval* $\mathrm{I} = [0, 1]$. Such "unbiased path-lifting" can be formulated in $\mathsf{cSet}$ by introducing a "generic point" $\delta : 1 \longrightarrow \mathrm{I}$, by passing to $\mathsf{cSet}/\mathrm{I}$, and then requiring path-lifting with respect to $\delta$. The following specification implements that idea, while also adding partiality in the sense of the foregoing section. We need the following strengthening of axiom C5.

(C5') The diagonal map $\delta : \mathrm{I} \longrightarrow \mathrm{I} \times \mathrm{I}$ is a cofibration.

**Definition 12** (Fibration)**.** Let $\delta : \mathrm{I} \longrightarrow \mathrm{I} \times \mathrm{I}$ be the diagonal map.

1. An object $X$ is *(unbiased) fibrant* if the map

$$(\delta \Rightarrow X) = \langle \mathsf{eval}, p_2 \rangle : X^{\mathrm{I}} \times \mathrm{I} \longrightarrow X \times \mathrm{I}$$

   is a $+$-algebra.

2. A map $f : Y \to X$ is an *(unbiased) fibration* if the map

$$(\delta \Rightarrow f) = \langle f^{\mathrm{I}} \times \mathrm{I}, \langle \mathsf{eval}, p_2 \rangle \rangle : Y^{\mathrm{I}} \times \mathrm{I} \longrightarrow (X^{\mathrm{I}} \times \mathrm{I}) \times_{(X \times \mathrm{I})} (Y \times \mathrm{I})$$

   is a $+$-algebra.

Now we can run the proof of Proposition 8 backwards in order to determine a class of generating trivial cofibrations for the unbiased case. We consider pairs of maps $c : C \rightarrowtail Z$ and $z : Z \longrightarrow \mathrm{I}$, where the former is a cofibration and the latter is regarded as an "I-indexing", so that

$$\begin{tikzcd}
C \arrow[r, tail, "c"] \arrow[dr] & Z \arrow[d, "z"] \\
& \mathrm{I}
\end{tikzcd}$$

can be regarded as an I-indexed family of cofibrations. Let

$$\mathsf{Gph}(z) : Z \longrightarrow Z \times \mathrm{I}\,,$$

be the graph of $z : Z \longrightarrow \mathrm{I}$, i.e. $\mathsf{Gph}(z) = \langle 1_Z, z \rangle$, and then let

$$c \otimes_z \delta := [\mathsf{Gph}(z), c \times \mathrm{I}] : Z +_C (C \times \mathrm{I}) \longrightarrow Z \times \mathrm{I}\,,$$

which is easily seen to be well-defined on the indicated pushout.

$$\begin{tikzcd}
C \arrow[r, "\mathsf{Gph}(zc)"] \arrow[d, tail, "c"'] & C \times \mathrm{I} \arrow[d] \arrow[ddr, bend left, "c\times\mathrm{I}"] & \\
Z \arrow[r] \arrow[drr, bend right, "\mathsf{Gph}(z)"'] & Z +_C (C \times \mathrm{I}) \arrow[dr, dotted, "c\otimes_z\delta"'] & \\
& & Z \times \mathrm{I}\,.
\end{tikzcd} \tag{30}$$

This specification differs from the similar (18) by using $\mathsf{Gph}(z)$ for the inclusion $Z \rightarrowtail Z \times \mathrm{I}$, rather than one of the "face maps" associated to the endpoint inclusions $\delta_\epsilon : 1 \longrightarrow \mathrm{I}$. (Note that a graph is always a cofibration by pulling back a diagonal.) The subobject $c \otimes_z \delta \rightarrowtail Z \times \mathrm{I}$ is the join of the subobjects $\mathsf{Gph}(z) \rightarrowtail Z \times \mathrm{I}$ and the cylinder $C \times \mathrm{I} \rightarrowtail Z \times \mathrm{I}$.

The maps of the form $c \otimes_z \delta : Z +_C (C \times \mathrm{I}) \rightarrowtail Z$ now form a *class of generating trivial cofibrations* in the expected sense. Let

$$\mathcal{C} \otimes \delta \;=\; \{c \otimes_z \delta : D \rightarrowtail Z \times \mathrm{I} \mid c : C \rightarrowtail Z, z : Z \longrightarrow \mathrm{I}\}\,, \tag{31}$$

then the fibrations are exactly the right class of these,

$$\mathcal{F} = (\mathcal{C} \otimes \delta)^{\pitchfork}.$$

**Proposition 13.** *A map $f : Y \to X$ is an (unbiased) fibration iff for every pair of maps $c : C \rightarrowtail Z$ and $z : Z \to \mathrm{I}$, where the former is a cofibration,* *every commutative square of the following form has a diagonal filler, as indicated.*

$$\begin{tikzcd} Z +_C (C \times \mathrm{I}) \arrow[r] \arrow[d, tail, "c \otimes_z \delta"'] & Y \arrow[d, "f"] \\ Z \times \mathrm{I} \arrow[r] \arrow[ur, dotted, "j"] & X. \end{tikzcd} \tag{32}$$

*Proof.* Suppose that for all $c : C \rightarrowtail Z$ and $z : Z \to \mathrm{I}$, we have $(c \otimes_z \delta) \pitchfork f$ in $\mathsf{cSet}$. Pulling $f$ back over I, this is equivalent to the condition $c \otimes \delta \pitchfork \mathrm{I}^* f$ in $\mathsf{cSet}/\mathrm{I}$, for all cofibrations $c : C \rightarrowtail Z$ over I, which is equivalent to $c \pitchfork (\delta \Rightarrow \mathrm{I}^* f)$ in $\mathsf{cSet}/\mathrm{I}$ for all cofibrations $c : C \rightarrowtail Z$. But this in turn means that $\delta \Rightarrow \mathrm{I}^* f$ is a $+$-algebra, which by definition means that $f$ is a fibration. □

**Unbiased fibration structure.** As in the biased case, the fibrations can also be determined by *uniform* right-lifting against a generating *set* of trivial cofibrations, now consisting of all those $c \otimes_z \delta$ in $\mathcal{C} \otimes \delta$ for which $c : C \rightarrowtail Z$ has a representable codomain $Z = \mathrm{I}^n$. Call these maps the *basic (unbiased) trivial cofibrations*, and let

$$\mathcal{B} \otimes \delta = \{c \otimes_z \delta : B \rightarrowtail \mathrm{I}^{n+1} \mid c : C \rightarrowtail \mathrm{I}^n,\, z : \mathrm{I}^n \to \mathrm{I},\, n \geq 0\}\,, \tag{33}$$

where the pushout-product $c \otimes_z \delta$ now has the form

$$\begin{tikzcd} C \arrow[r, "\mathsf{Gph}(zc)"] \arrow[d, tail, "c"'] & C \times \mathrm{I} \arrow[d] \arrow[ddr, bend left, "c \times \mathrm{I}"] & \\ \mathrm{I}^n \arrow[r] \arrow[drr, bend right, "\mathsf{Gph}(z)"'] & \mathrm{I}^n +_C (C \times \mathrm{I}) \arrow[dr, dotted, "c \otimes_z \delta"'] & \\ & & \mathrm{I}^n \times \mathrm{I}\,. \end{tikzcd} \tag{34}$$

for a cofibration $c : C \rightarrowtail \mathrm{I}^n$, an indexing map $z : \mathrm{I}^n \to \mathrm{I}$, and with domain $B = \big(\mathrm{I}^n +_C (C \times \mathrm{I})\big)$. These subobjects $B \rightarrowtail \mathrm{I}^{n+1}$ can again be seen geometrically as "generalized open box" inclusions, but now the floor or lid of the open box may be replaced by a "cross-section" given by the graph of a map $z : \mathrm{I}^n \longrightarrow \mathrm{I}$.

For any map $f : Y \to X$ a (uniform, unbiased) fibration structure on $f$

is a choice of diagonal fillers $j(c,z,x,y)$,

$$\begin{tikzcd} B \arrow[r, "x"] \arrow[d, tail, "c\otimes_z\delta"'] & X \arrow[d, "f"] \\ \mathrm{I}^n\times\mathrm{I} \arrow[r, "y"'] \arrow[ur, dotted, "j(c,z,x,y)"] & Y, \end{tikzcd} \tag{35}$$

for each basic trivial cofibration $c\otimes_z\delta : B \rightarrowtail \mathrm{I}^{n+1}$, which is *uniform* in $\mathrm{I}^n$ in the following sense: given any cubical map $u:\mathrm{I}^m \longrightarrow \mathrm{I}^n$, the pullback $u^*c : u^*C \rightarrowtail \mathrm{I}^m$ and the reindexing $zu:\mathrm{I}^m \longrightarrow \mathrm{I}^n \longrightarrow \mathrm{I}$ determine another basic trivial cofibration $u^*c\otimes_{zu}\delta : B' = (\mathrm{I}^m +_{u^*C} (u^*C\times\mathrm{I})) \rightarrowtail \mathrm{I}^{m+1}$ which fits into a commutative diagram of the form

$$\begin{tikzcd} B' \arrow[r, "(u\times\mathrm{I})'"] \arrow[d, tail, "u^*c\otimes_{zu}\delta"'] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[r, "x"] \arrow[d, tail, "c\otimes_z\delta"'] & X \arrow[d, "f"] \\ \mathrm{I}^m\times\mathrm{I} \arrow[r, "u\times\mathrm{I}"'] & \mathrm{I}^n\times\mathrm{I} \arrow[r, "y"'] \arrow[ur, dotted, "j(c,z,x,y)"] & Y. \end{tikzcd} \tag{36}$$

For the outer rectangle in (36) there is a chosen diagonal filler

$$j(u^*c, zu, x(u\times\mathrm{I})', y(u\times\mathrm{I})) : \mathrm{I}^m\times\mathrm{I}\longrightarrow X,$$

and for this map we require that

$$j(u^*c, zu, x(u\times\mathrm{I})', y(u\times\mathrm{I})) = j(c,z,x,y)\circ(u\times\mathrm{I}). \tag{37}$$

**Definition 14.** A *(uniform, unbiased) fibration structure* on a map

$$f : Y \longrightarrow X$$

is a choice of fillers $j(c,z,x,y)$ as in (35) satisfying (37) for all $u:\mathrm{I}^m\to\mathrm{I}^n$.

In these terms, we have following analogue of corollary 11.

**Proposition 15.** *For any object $X$ in* $\mathsf{cSet}$ *the following are equivalent:*

1. *the canonical map $X^{\mathrm{I}}\times\mathrm{I}\longrightarrow X\times\mathrm{I}$ is a* +-algebra.
2. *$X$ has the right lifting property with respect to all generating trivial cofibrations,*
$$(\mathcal{C}\otimes_z\delta) \pitchfork X.$$
3. *$X$ has a* uniform fibration structure *in the sense of Definition 14.*

*Proof.* The equivalence between (1) and (2) is proposition 13. Suppose (1), i.e. that the map

$$(\delta \Rightarrow X) : X^{\mathrm{I}} \times \mathrm{I} \longrightarrow X \times \mathrm{I}$$

is a relative +-algebra over $X \times \mathrm{I}$. By proposition 6, this means that $(\delta \Rightarrow X)$, as an object of $\mathsf{cSet}/(X \times \mathrm{I})$, has a uniform filling structure with respect to all cofibrations $c : C \rightarrowtail \mathrm{I}^n$ over $(X \times \mathrm{I})$. Transposing by the $\otimes \dashv \Rightarrow$ adjunction and unwinding gives, equivalently, a uniform fibration structure on $X$. $\square$

A statement analogous to the foregoing also holds for maps $f : Y \to X$ in place of objects $X$. Indeed, as before, we have the following sharper formulation.

**Corollary 16.** *Fibration structures on a map $f : Y \to X$ correspond uniquely to +-algebra structures on the map $(\delta \Rightarrow f)$ (cf. definition 12),*

$$(\delta \Rightarrow f) : Y^{I} \times \mathrm{I} \longrightarrow (X^{I} \times \mathrm{I}) \times_{(X \times \mathrm{I})} (Y \times \mathrm{I})$$

## 6 The fibration weak factorization system

**Definition 17.** Summarizing the foregoing definitions and results, we have the following classes of maps:

- The *generating trivial cofibrations* were determined in (31) to be

$$\mathcal{C} \otimes \delta = \{ c \otimes_z \delta : D \rightarrowtail Z \times \mathrm{I} \mid c : C \rightarrowtail Z,\ z : Z \longrightarrow \mathrm{I} \}, \tag{38}$$

  where the pushout-product $c \otimes_z \delta$ has the form

$$\begin{tikzcd} C \arrow[r, "\mathsf{Gph}(zc)"] \arrow[d, tail, "c"'] & C \times \mathrm{I} \arrow[d] \arrow[ddr, bend left, "c \times \mathrm{I}"] & \\ Z \arrow[r] \arrow[drr, bend right, "\mathsf{Gph}(z)"'] & Z +_C (C \times \mathrm{I}) \arrow[dr, dotted, tail, "c \otimes_z \delta"] & \\ & & Z \times \mathrm{I}\,. \end{tikzcd} \tag{39}$$

  for any cofibration $c : C \rightarrowtail Z$ and indexing map $z : Z \to \mathrm{I}$, with domain $D = \big(Z +_C (C \times \mathrm{I})\big)$.

- The class $\mathcal{F}$ of *fibrations*, written $f : Y \twoheadrightarrow X$, may be characterized as the right-lifting class of the generating trivial cofibrations,

$$(\mathcal{C} \otimes \delta)^{\pitchfork} = \mathcal{F}.$$

- The class of *trivial cofibrations* is defined to be left class of the fibrations,

$$\mathsf{TrivCof} \;=\; {}^{\pitchfork}\mathcal{F}.$$

It follows from the specification that the classes $\mathsf{TrivCof}$ and $\mathcal{F}$ are mutually weakly orthogonal,

$$\mathsf{TrivCof} \pitchfork \mathcal{F},$$

and are both closed under retracts, so in order to have a weak factorization system $(\mathsf{TrivCof}, \mathcal{F})$ it just remains to show that every map $f : X \to Y$ can be factored as $f = g \circ h$ with $g \in \mathcal{F}$ and $h \in \mathsf{TrivCof}$.

**Proposition 18.** *Every map $f : X \longrightarrow Y$ in $\mathsf{cSet}$ can be factored as $f = g \circ h$,*

$$\begin{tikzcd} X \arrow[r, tail, "h"] \arrow[dr, "f"'] & X' \arrow[d, two heads, "g"] \\ & Y \end{tikzcd} \tag{40}$$

*with $h : X \rightarrowtail X'$ a trivial cofibration and $g : X' \twoheadrightarrow Y$ a fibration.*

*Proof.* This is a standard argument (cf. [**?**, **?**]), which can be simplified a bit in this particular case. We sketch the proof for the case $Y = 1$; the general case is not essentially different.

Thus let $X$ be any object, and we wish to find a fibrant object $X'$ and a trivial cofibration $h : X \rightarrowtail X'$. For each basic trivial cofibration $\beta : B \rightarrowtail \mathrm{I}^k$, we need to solve all extension problems of the form

$$\begin{tikzcd} B \arrow[r, "x"] \arrow[d, tail, "\beta"'] & X. \\ \mathrm{I}^k \arrow[ur, dotted] & \end{tikzcd} \tag{41}$$

We first combine these into a single problem by taking a coproduct over all maps $x : B \to X$,

$$\begin{tikzcd} \coprod_x B \arrow[r, "{[x]}"] \arrow[d, tail, "\coprod_x \beta"'] & X. \\ \coprod_x \mathrm{I}^k \arrow[ur, dotted] & \end{tikzcd}$$

We then take the coproduct over all basic trivial cofibrations $\beta : B \rightarrowtail \mathrm{I}^k$,

$$
\begin{tikzcd}
\coprod_\beta \coprod_x B \arrow[r, "{[[x]_\beta]}"] \arrow[d, tail, "\coprod_\beta \coprod_x \beta"'] & X. \\
\coprod_\beta \coprod_x \mathrm{I}^k \arrow[ur, dotted]
\end{tikzcd}
$$

Note that a coproduct of trivial cofibrations is clearly a trivial cofibration.

Taking a pushout, the indicated map $h_1$ is then also a trivial cofibration, because it is a pushout of one

$$
\begin{tikzcd}
\coprod_\beta \coprod_x B \arrow[r, "{[[x]_\beta]}"] \arrow[d, tail, "\coprod_\beta \coprod_x \beta"'] & X \arrow[d, tail, "h_1"] \\
\coprod_\beta \coprod_x \mathrm{I}^k \arrow[r] & X_1 \, .
\end{tikzcd}
$$

Now iterate the construction to get a sequence of trivial cofibrations, of which we take $X'$ to be the colimit and $h : X \longrightarrow X'$ the canonical map,

$$h : X \underset{h_1}{\rightarrowtail} X_1 \underset{h_2}{\rightarrowtail} X_2 \underset{h_3}{\rightarrowtail} \ldots \rightarrowtail \varinjlim X_n = X'. \tag{42}$$

To show that $X'$ is fibrant, consider an extension problem of the form (41) with $X'$ in place of $X$,

$$
\begin{tikzcd}
B \arrow[r, "x"] \arrow[d, tail, "\beta"'] & \varinjlim X_n \\
\mathrm{I}^k \arrow[ur, dotted]
\end{tikzcd}
$$

The subobject $B \rightarrowtail \mathrm{I}^k$ has as domain an object $B$ that is a *finite* colimit of maps $\mathrm{I}^m \to \mathrm{I}^n$ of representables (as can be seen by considering sieves in the category of cubes), and is therefore finitely presented, in the sense that mapping out of it preserves filtered colimits. Thus the map $x : B \longrightarrow \varinjlim X_n$

must factor through some $x_k : B \longrightarrow X_k$, giving rise to the problem

$$\begin{tikzcd} B \arrow[r, "x_k"] \arrow[d, tail, "\beta"'] & X_k \arrow[d] \\ \mathrm{I}^k \arrow[r, dotted] & \varinjlim X_n. \end{tikzcd}$$

But this has a solution in the next step, by the construction of $X_{k+1}$,

$$\begin{tikzcd} B \arrow[r, "x_k"] \arrow[d, tail, "\beta"'] & X_k \arrow[d, "h_{k+1}"] \\ \mathrm{I}^k \arrow[r] \arrow[dr, dotted, "j"'] & X_{k+1} \arrow[d] \\ & \varinjlim X_n. \end{tikzcd}$$

Finally, we need to show the uniformity condition on the resulting fillers $j = j(\beta, x)$. For this to work, we must modify the colimit construction (42) by interleaving certain coequalizers, in order to identify fillers added at different stages. For details, see [**?**, **?**]. $\square$

**Proposition 19.** *There is a weak factorization system on the category* $\mathsf{cSet}$ *in which the right maps are the fibrations and the left maps are the trivial cofibrations, both as specified in definition 17.*

This will be called the *fibration weak factorization system.* The following observation will be of use later on, the proof can be found in [**?**, **?**].

**Corollary 20.** *The construction given in* (42) *of the* fibrant replacement*,*

$$X' = \varinjlim_{n} X_n$$

*is functorial in $X$, and the canonical trivial cofibrations $h : X \rightarrowtail X'$ are natural in $X$. There is also a natural monad multiplication $\mu : X'' \to X'$.*

# 7 Weak equivalences

**Definition 21** (Weak equivalence)**.** A map $f : X \longrightarrow Y$ in $\mathsf{cSet}$ will be called a *weak equivalence* if can be factored as $f = g \circ h$,

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[dr, "f"'] & W \arrow[d, "g"] \\ & Y \end{tikzcd}$$

with $h : X \to W$ a trivial cofibration and $g : W \to Y$ a trivial fibration, i.e. a right map in the cofibration weak factorization system. Let

$$\mathcal{W} = \{f : X \longrightarrow Y \mid f = g \circ h \text{ for } g \in \mathsf{TrivFib} \text{ and } h \in \mathsf{TrivCof}\}$$

be the class of weak equivalences.

Now observe that every trivial fibration $f \in \mathcal{C}^{\pitchfork}$ is indeed a fibration, because the basic trivial cofibrations are cofibrations; moreover, every trivial fibration is also a weak equivalence, since the identity maps are trivial cofibrations. Thus we have

$$\mathsf{TrivFib} \subseteq (\mathcal{F} \cap \mathcal{W}).$$

Dually, every trivial cofibration $g \in {}^{\pitchfork}\mathcal{F}$ is a cofibration, because the trivial fibrations are fibrations; moreover, every trivial cofibration is also a weak equivalence, since the identity maps are trivial fibrations. Thus we also have

$$\mathsf{TrivCof} \subseteq (\mathcal{C} \cap \mathcal{W}).$$

**Lemma 22.** $(\mathcal{C} \cap \mathcal{W}) \subseteq \mathsf{TrivCof}$.

*Proof.* Let $c : A \rightarrowtail B$ be a cofibration with a factorization $c = tf \circ tc : A \longrightarrow W \longrightarrow B$ where $tc \in \mathsf{TrivCof}$ and $tf \in \mathsf{TrivFib}$. Let $f : Y \twoheadrightarrow X$ be a fibration and consider a commutative diagram,

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, tail, "c"'] & Y \arrow[d, two heads, "f"] \\ B \arrow[r, "b"'] & X. \end{tikzcd}$$

Inserting the factorization of $c$, we have $j : W \longrightarrow Y$ as indicated, with $j \circ tc = a$ and $f \circ j = b \circ tf$, since $tc \pitchfork f$.

$$
\begin{tikzcd}
A \arrow[rr, "a"] \arrow[dd, tail, "c"'] \arrow[dr, "tc"] & & Y \arrow[dd, two heads, "f"] \\
& W \arrow[ur, dotted, "j"'] \arrow[dl, "tf"'] & \\
B \arrow[rr, "b"'] & & X.
\end{tikzcd}
$$

Moreover, since $c \pitchfork tf$ there is an $i : B \to W$ as indicated, with $i \circ c = tc$ and $tf \circ i = 1_B$.

$$
\begin{tikzcd}
A \arrow[rr, "a"] \arrow[dd, tail, "c"'] \arrow[dr, "tc"] & & Y \arrow[dd, two heads, "f"] \\
& W \arrow[ur, dotted, "j"'] \arrow[dl, "tf"'] & \\
B \arrow[ur, dotted, bend right, "i"'] \arrow[rr, "b"'] & & X.
\end{tikzcd}
$$

Let $k = j \circ i$. Then $k \circ c = j \circ i \circ c = j \circ tc = a$, and $f \circ k = f \circ j \circ i = b \circ tf \circ i = b$. $\square$

The proof of the following is dual:

**Lemma 23.** $(\mathcal{F} \cap \mathcal{W}) \subseteq \mathsf{TrivFib}$.

**Proposition 24.** *For the three classes of maps $\mathcal{C}, \mathcal{W}, \mathcal{F}$ in $\mathsf{cSet}$, we have*

$$
\begin{aligned}
\mathcal{F} \cap \mathcal{W} &= \mathsf{TrivFib}, \\
\mathcal{C} \cap \mathcal{W} &= \mathsf{TrivCof},
\end{aligned}
$$

*and therefore two weak factorization systems:*

$$(\mathcal{C},\, \mathcal{W} \cap \mathcal{F}) \quad , \quad (\mathcal{C} \cap \mathcal{W},\, \mathcal{F}).$$

It thus just remains to prove that the weak equivalences satisfy the 3-for-2 property.

**Weak homotopy equivalence** Recall that a *homotopy* between parallel maps $f, g : X \rightrightarrows Y$, written $\vartheta : f \sim g$, is a map from the cylinder of $X$,

$$\vartheta : \mathrm{I} \times X \longrightarrow Y,$$

such that $\vartheta \circ \mathsf{Gph}(\delta_0!) = f$ and $\vartheta \circ \mathsf{Gph}(\delta_1!) = g$.

**Proposition 25.** *If $K$ is fibrant, then the relation of homotopy $f \sim g$ between maps $f, g : X \rightrightarrows K$ is an equivalence relation. Moreover, it is compatible with composition.*

*Proof.* add the proof ... □

**Definition 26** (Connected components)**.** The functor

$$\pi_0 : \mathsf{cSet} \longrightarrow \mathsf{Set}$$

is the left adjoint of the constant presheaf functor $\Delta : \mathsf{Set} \to \mathsf{cSet}$. For any cubical set $X$ we have a coequalizer $X_1 \rightrightarrows X_0 \to \pi_0 X$, where the two parallel arrows are the endpoints of the 1-cubes $X_1$ of $X$. Thus for any Kan complex $K$ we have $\pi_0 K = \mathrm{Hom}(1, K)/\sim$, i.e. $\pi_0 K$ is the set of points $1 \to K$, modulo the homotopy equivalence relation on them.

Recall that a map $f : X \to Y$ in $\mathsf{cSet}$ is called a *homotopy equivalence* if there is a *quasi-inverse* $g : Y \to X$ and homotopies $\vartheta : 1_X \sim g \circ f$ and $\varphi : 1_Y \sim f \circ g$.

**Definition 27** (Weak homotopy equivalence)**.** A map $f : X \to Y$ will be called a *weak homotopy equivalence* if for every fibrant object $K$, the "internal precomposition" map $K^f : K^Y \to K^X$ is bijective on connected components, i.e.

$$\pi_0 K^f : \pi_0 K^Y \longrightarrow \pi_0 K^X$$

is a bijection of sets.

The following is immediate.

**Lemma 28.** *A homotopy equivalence is weak homotopy equivalence.*

**Lemma 29.** *The weak homotopy equivalences $f : X \to Y$ satisfy the 3-for-2 condition.*

*Proof.* Follows from the corresponding fact about bijections of sets. □

**Lemma 30.** *A map $f : X \to Y$ is a* weak homotopy equivalence *iff it satisfies the following two conditions.*

1. *For every fibrant object $K$, every map $x : X \to K$ extends up to homotopy along $f$ to a map $y : Y \to K$ such that $y \circ f \sim x$,*

$$\begin{tikzcd} X \arrow[r, "x"] \arrow[d, "f"'] \arrow[dr, phantom, "\sim", near start] & K \\ Y. \arrow[ur, dotted, "y"'] & \end{tikzcd}$$

2. *For every fibrant object $K$, any maps $y, y' : Y \to K$ such that $yf \sim y'f$, there is a homotopy $y \sim y'$,*

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & K^{\mathrm{I}} \arrow[d] \\ Y \arrow[r] \arrow[ur, dotted] & K \times K. \end{tikzcd}$$

*Proof.* Unwind the definition. □

Thus our goal of showing that the weak equivalences satisfy 3-for-2 is reduced to showing that a map is a weak equivalence (WE) if and only if it is a weak homotopy equivalence (WHE). This is proved via separate lemmas showing that a (co)fibration is a WE if and only if it is a WHE.

**Lemma 31.** *A cofibration $c : A \rightarrowtail B$ that is a WE is a WHE.*

*Proof.* A cofibration $c : A \rightarrowtail B$ that is a WE is a trivial cofibration. So the result follows from Lemma 30. □

**Lemma 32.** *A fibration $p : Y \twoheadrightarrow X$ that is a WE is a WHE.*

*Proof.* A fibration weak equivalence $f : Y \twoheadrightarrow X$ is a trivial fibration, and therefore has a section $s : X \rightarrowtail Y$, by the lifting problem:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, tail] & Y \arrow[d, two heads, "f"] \\ X \arrow[r, "="'] & X. \end{tikzcd}$$

Moreover, there is a homotopy $\vartheta : sf \sim 1_Y$, resulting from the lifting problem

$$\begin{tikzcd} Y + Y \arrow[r, "{[sf,1]}"] \arrow[d, tail] & Y \arrow[d, two heads, "f"] \\ \mathrm{I} \times Y \arrow[r, "f\pi_2"'] & X. \end{tikzcd}$$

Thus $f$ is a homotopy equivalence, and so a WHE by lemma 28. □

**Lemma 33.** *If $K$ is fibrant, then any fibration $f : Y \twoheadrightarrow K$ that is a HE is a WE.*

*Proof.* add the proof ... $\square$

**Lemma 34.** *If $K$ is fibrant, then any fibration $f : Y \twoheadrightarrow K$ that is a WHE is a WE.*

*Proof.* Since $K$ is fibrant, so is $Y$, and since $f$ is a WHE, there are maps $s, t : K \to Y$ such that $fs \sim 1_K$ and $tf \sim 1_Y$. ... turn this into a HE ... apply lemma 33. $\square$

**Lemma 35.** *If $K$ is fibrant, then any cofibration $c : A \rightarrowtail K$ that is a WHE is a WE.*

*Proof.* Let $c : A \rightarrowtail K$ be a cofibration WHE and factor it into a trivial cofibration $i : A \rightarrowtail Z$ followed by a fibration $p : Z \twoheadrightarrow K$. Then both $c$ and $i$ are WHE, whence so is $p$. Since $K$ is fibrant, $p$ is a trivial fibration by lemma 34, a thus $c$ is a WE. $\square$

**Lemma 36** ([**?**], x.n.m)**.** *A cofibration $c : A \rightarrowtail B$ is a WHE if and only if it lifts against all fibrations $f : Y \twoheadrightarrow K$ with fibrant codomain.*

*Proof.* Let $c : A \rightarrowtail B$ be a cofibration and a WHE, and $f : Y \twoheadrightarrow K$ a fibration with fibrant codomain, and consider a lifting problem

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, tail, "c"'] & Y \arrow[d, two heads, "f"] \\ B \arrow[r, "b"'] & K. \end{tikzcd}$$

Let $\eta : B \rightarrowtail B'$ be a fibrant replacement of $B$, since $K$ is fibrant, $b$ extends along $\eta$ to give $b' : B' \to K$ as in

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, tail, "c"'] & Y \arrow[d, two heads, "f"] \\ B \arrow[r, "b"] \arrow[d, "\eta"'] & K \\ B' \arrow[ur, "b'"'] & \end{tikzcd}$$

Since $\eta$ is a trivial cofibration, it is a WHE, and so, therefore, is the composite $\eta c$. But since $B'$ is fibrant, $\eta c$ is then a trivial cofibration by lemma 35. Thus there is a lift $j : B' \to Y$, and therefore one $k = j\eta : B \to Y$.

Conversely, suppose the cofibration $c : A \rightarrowtail B$ lifts against every fibration with a fibrant codomain. Let $\eta : B \rightarrowtail B'$ be a fibrant replacement of $B$ as before, and factor $\eta c : A \longrightarrow B'$ into a trivial fibration $i : A \rightarrowtail Y$ followed by a fibration $f : Y \twoheadrightarrow B'$. Then in the diagram

$$\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, tail, "c"'] & Y \arrow[d, two heads, "p"] \\ B \arrow[r, "\eta"'] \arrow[ur, dashed, "j"] & B' \end{tikzcd}$$

there is a diagonal filler $j : B \to Y$. Moreover, there is a section $s$ of $p$ with $s\eta = j$, since $\eta$ is a trivial cofibration. This exhibits $\eta c$ as a retract of $i$, and therefore also a trivial cofibration, and so a WHE. Thus $c$ is also a WHE by 3 for 2. $\square$

To complete the proof that a cofibration WHE is a WE we require following important fact, the proof of which is deferred to the next section.

**Proposition 37** (Fibration extension)**.** *Given a fibration $f : Y \twoheadrightarrow X$ and a trivial cofibration $\eta : X \to X'$, there is a fibration $f' : Y' \twoheadrightarrow X'$ such that $f$ is a pullback of $f'$ along $\eta$.*

**Lemma 38.** *A cofibration that lifts against every fibration $f : Y \twoheadrightarrow K$ with fibrant codomain is a WE.*

*Proof.* Let $c : A \rightarrowtail B$ be a cofibration and consider a lifting problem against an arbitrary fibration $f : Y \twoheadrightarrow X$,

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, tail, "c"'] & Y \arrow[d, two heads, "f"] \\ B \arrow[r, "b"'] & X. \end{tikzcd} \tag{43}$$

Let $\eta : X \to X'$ be the fibrant replacement of lemma **??**, so $\eta$ is a trivial cofibration and $X'$ is fibrant. By lemma 37, there is a fibration $f' : Y' \twoheadrightarrow X'$ such that $f$ is a pullback of $f'$ along $\eta$. So we can extend diagram (43) to obtain the following, in which the righthand square is a pullback.

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, tail, "c"'] & Y \arrow[r, "y"] \arrow[d, two heads, "f"] \arrow[dr, phantom, "\lrcorner", very near start] & Y' \arrow[d, two heads, "f'"] \\ B \arrow[r, "b"'] & X \arrow[r, "\eta"'] & X'. \end{tikzcd} \tag{44}$$

By assumption, there is a lift $j' : B \to Y'$ with $f'j' = \eta b$ and $j'c = yb$. Therefore, by the property of the pullback, there is a map $j : B \to Y$ with $fj = b$ and $yj = j'$.

$$
\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, tail, "c"'] & Y \arrow[r, "y"] \arrow[d, two heads, "f"] \arrow[dr, phantom, "\lrcorner", very near start] & Y' \arrow[d, two heads, "f'"] \\
B \arrow[r, "b"'] \arrow[ur, dashed, "j"] \arrow[urr, dashed, "j'"] & X \arrow[r, "\eta"'] & X'.
\end{tikzcd}
\tag{45}
$$

Thus $yjc = j'c = ya$. But as a trivial cofibration, $\eta$ is monic, and as a pullback of $\eta$, $y$ is also monic. So $jc = a$. □

Combining the previous two lemmas 36 and 38 we now have.

**Corollary 39.** *A cofibration $c : A \rightarrowtail B$ that is a WHE is a WE.*

Finally, we can show:

**Lemma 40.** *A fibration $f : Y \twoheadrightarrow X$ that is a WHE is a WE.*

*Proof.* Factor $f : Y \twoheadrightarrow X$ into a cofibration $i : Y \rightarrowtail Z$ followed by a trivial fibration $p : Z \twoheadrightarrow X$. Then $f$ is a trivial fibration if $i \pitchfork f$, for then $f$ is a retract of $p$. Since $p$ is a trivial fibration, it is a WHE by lemma 32. Since $f$ is also a WHE, so is $i$ by 3-for-2. Thus $i$ is a trivial cofibration by corollary 39. Since $f$ is a fibration, $i \pitchfork f$ as required. □

**Proposition 41.** *A map $f : X \rightarrowtail Y$ is a WHE if and only if it is a WE. Thus the weak equivalences $\mathcal{W}$ satisfy the 3-for-2 condition.*

*Proof.* Let $f : X \twoheadrightarrow Y$ be a WE and factor it into a trivial cofibration $i : X \rightarrowtail Z$ followed by a trivial fibration $p : Z \twoheadrightarrow Y$. Then both $i$ and $p$ are WHE, whence so is $f$. Conversely, let $f$ be a WHE and factor it into a trivial cofibration $i : X \rightarrowtail Z$ followed by a fibration $p : Z \twoheadrightarrow Y$. Since $i$ is then a WHE, as is $f$, it follows that $p$ is as well. Thus $p$ is also a WE, hence a trivial fibration. So $f$ is a WE. □

## 8 Frobenius

In this section, we show that the fibration WFS from section 6 has the *Frobenius property*: the left maps are stable under pullback along the right maps (see [**?**]). This will eventually imply the *right properness* of our model structure: the weak equivalences are preserved by pullback along fibrations. But it is also needed in the proof of the equivalence extension property in the next section. A proof of Frobenius in a setting with connections can be found in [**?**]; however the type theoretic approach of [**?**, **?**] provides an alternate route that is perhaps more direct, and in any event can also be applied without connections. This approach proves the "dual" fact that the right adjoint to pullback along any *fibration* $f : Y \to X$ (which always exists in the ambient presheaf topos) preserves fibrations. This corresponds to the type-theoretic $\Pi$-formation rule. One aspect of this approach that deserves a closer analysis is the methodology (due to Coquand and first introduced in [**?**]) of defining an explicit type-theoretic notion of *fibration structure* with which one can efficiently calculate in type theory (and even in a proof assistant), making it possible to prove e.g. that the fibrations are closed under the type-forming operations. An algebraic/functorial analogue of this technique should be equally useful.

In order to simplify the exposition in this section we will first consider biased fibration structures in the sense of section 4. Recall that a (biased) fibration structure on a map $f : Y \to X$ is essentially the same thing as a pair of +-algebra structures on the maps

$$(\delta_\epsilon \Rightarrow f) : Y^{\mathrm{I}} \longrightarrow X^{\mathrm{I}} \times_X Y$$

Let us drop the index $\epsilon$ and consider only the case of $\epsilon = 0$, since later on we shall use the generic $\delta$ anyway. The construction of $\delta \Rightarrow f$ is recalled from (20) in the pullback diagram below.

$$\begin{tikzcd} Y^{\mathrm{I}} \arrow[dr, "\delta\Rightarrow f"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & X^{\mathrm{I}} \times_X Y \arrow[r] \arrow[d] & Y \arrow[d] \\ & X^{\mathrm{I}} \arrow[r] & X \end{tikzcd} \tag{46}$$

In order to compare with [**?**], let us switch to informal type-theoretic notation and write $Y = \sum_{x:X} A(x)$ for a type-valued map $A : X \to \mathcal{U}$. Let $\mathsf{eval}_0 = X^\delta : X^{\mathrm{I}} \to X$, so that the above pullback becomes

$$\begin{tikzcd} \sum_{p:X^{\mathrm{I}}} A(p0) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \sum_{x:X} A(x) \arrow[d] \\ X^{\mathrm{I}} \arrow[r, "\mathsf{eval}_0"'] & X \end{tikzcd}$$

The map $f^{\mathrm{I}} : Y^{\mathrm{I}} \to X^{\mathrm{I}}$ may be rewritten as a family over $X^{\mathrm{I}}$ in the form:

$$(\sum_{x:X} A(x))^{\mathrm{I}} \cong \prod_{i:\mathrm{I}} \sum_{x:X} A(x) \cong \sum_{p:X^{\mathrm{I}}} \prod_{i:\mathrm{I}} A(pi)$$

so that up to isomorphism our previous diagram (46) becomes:

$$\begin{tikzcd}
\sum_{p:X^{\mathrm{I}}} \prod_{i:\mathrm{I}} A(pi) \arrow[dr, "\delta\Rightarrow f"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& \sum_{p:X^{\mathrm{I}}} A(p0) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \sum_{x:X} A(x) \arrow[d] \\
& X^{\mathrm{I}} \arrow[r, "\mathsf{eval}_0"'] & X
\end{tikzcd}$$

Finally, pulling back along an arbitrary point $p : 1 \to X^{\mathrm{I}}$ we obtain the left-most map in the diagram below:

$$\begin{tikzcd}
\prod_{i:\mathrm{I}} A(pi) \arrow[r] \arrow[dr] \arrow[ddr, bend right] & \sum_{p:X^{\mathrm{I}}} \prod_{i:\mathrm{I}} A(pi) \arrow[dr, "\delta\Rightarrow f"] \arrow[drr, bend left] & & \\
& A(p0) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \sum_{p:X^{\mathrm{I}}} A(p0) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \sum_{x:X} A(x) \arrow[d] \\
& 1 \arrow[r, "p"'] & X^{\mathrm{I}} \arrow[r, "\mathsf{eval}_0"'] & X.
\end{tikzcd}$$

Since being a +-algebra (a trivial fibration) is a local property, we have shown:

**Proposition 42.** *The map $f : Y \to X$ is a fibration iff for all $p : X^{\mathrm{I}}$, the map $\prod_{i:\mathrm{I}} A(pi) \to A(p0)$ just constructed (and its counterpart for $\epsilon = 1$) is a +-algebra.*

The condition that $\prod_{i:\mathrm{I}} A(pi) \to A(p0)$ is a +-algebra for all $p : X^{\mathrm{I}}$ is shown in Appendix 1 to be equivalent to the type-theoretic definition of *fibration structure* on a type family $A : X \to \mathcal{U}$ given in [**?**].

# 9 The fibration extension property

# Appendix 1: Logical specification

To make the connection to the logical style of presentation used in [**?**, **?**], suppose we want to describe a (uniform) filling structure on an arbitrary

$f : X \longrightarrow Y$ with respect to all generating trivial cofibrations $m \otimes \delta_\epsilon : \mathrm{I}^n +_M (M \times \mathrm{I}) \longrightarrow \mathrm{I}^{n+1}$,

$$\begin{tikzcd}
\mathrm{I}^n +_M (M \times \mathrm{I}) \arrow[r] \arrow[d, "m\otimes\delta_\epsilon"'] & X \arrow[d, "f"] \\
\mathrm{I}^n \times \mathrm{I} \arrow[r, "c"'] \arrow[ur, dotted] & Y.
\end{tikzcd} \tag{47}$$

By pulling back along $c$, it suffices to consider the case $Y = \mathrm{I}^n \times \mathrm{I}$ and $c$ the identity map. Moreover, since we shall internalize the quantification over all cofibrations $m : M \rightarrowtail \mathrm{I}^n$ using the classifier $\Phi$, it suffices to consider just the following case internally,

$$\begin{tikzcd}
1 +_{[\phi]} ([\phi] \times \mathrm{I}) \arrow[r, "{[a_0,s]}"] \arrow[d, "\phi\otimes\delta_\epsilon"'] & X \arrow[d] \\
1 \times \mathrm{I} \arrow[r, "\cong"'] \arrow[ur, dotted, "a"] & \mathrm{I}
\end{tikzcd} \tag{48}$$

where the cofibration $[\phi] \rightarrowtail 1$ is classified by $\phi : 1 \to \Phi$.

Using a universe $\mathsf{Set}$ in the internal language of $\widehat{\mathbb{C}}$, we can regard the family $X \longrightarrow \mathrm{I}$ internally as a map $P : \mathrm{I} \to \mathsf{Set}$ (switching notation from $X$ to $P$ to agree with [**?**]). Thus we arrive at the following local specification, expressed logically in the internal language of $\widehat{\mathbb{C}}$, of the object of "(0-directed) lifting structures" $L^0(P)$ on a family $P : \mathrm{I} \to \mathsf{Set}$ :

$$L^0(P) \;=\; \prod_{\phi:\Phi} \; \prod_{s:\prod_{i:I}(Pi)^\phi} \; \prod_{a_0:P0} a_0|_\phi = s0 \;\longrightarrow\; \sum_{a:\prod_{i:I} Pi} (a0 = a_0) \times (a|_\phi = s)\,. \tag{49}$$

Here the variables $s : \prod_{i:I}(Pi)^\phi$ and $a_0 : P0$, and the condition $a_0|_\phi = s0$, give the domain $1 +_{[\phi]} ([\phi] \times \mathrm{I})$ of the arrow $[a_0, s]$ in (48), and $a : \prod_{i:I} Pi$ is the diagonal filler, with $(a0 = a_0) \times (a|_\phi = s)$ expressing the commutitivity of the top triangle.

There is an analogous condition $L^1(P)$ in which 1 replaces 0 everywhere, describing ("directed") filling from the other end of the interval. Note that [**?**, **?**] derive the "filling" conclusion

$$\sum_{a:\prod_{i:I} Pi} (a0 = a_0) \times (a|_\phi = s)$$

from (connections on I and) a weaker "composition operation"

$$\sum_{a_1:P1} a_1|_\phi = s_1\,,$$

but we will not take this approach.

The specification of the type $L^0(P)$ of (49) can also be represented diagrammatically as follows:

$$
\begin{tikzcd}
& P0 \arrow[rr] \arrow[dd] & & P \arrow[dd] \\
{[\phi]} \arrow[ur, "s0"] \arrow[rr] \arrow[dd, tail] & & {[\phi]\times\mathrm{I}} \arrow[ur, "s"] \arrow[dd, tail] & \\
& 1 \arrow[rr, "\delta_0"'] & & \mathrm{I} \\
1 \arrow[ur] \arrow[rr] \arrow[uuur, "a_0"'] & & 1\times\mathrm{I} \arrow[ur, "\pi_2"'] \arrow[uuur, dotted, "a"'] &
\end{tikzcd}
\tag{50}
$$

Here the left-hand vertical square is determined as a pullback of the right-hand one along the endpoint $\delta_0 : 1 \longrightarrow \mathrm{I}$.

Now write

$$\widetilde{P} \;=\; \prod_{i:I} Pi$$

for the type of sections of the projection $P = \sum_{i:I} Pi \longrightarrow \mathrm{I}$, and write

$$\pi_0 : \widetilde{P} \longrightarrow P0$$

for the $0^{th}$-projection (i.e. the evaluation of $P : \mathrm{I} \longrightarrow \mathsf{Set}$ at $0 : \mathrm{I}$).

Then the (0-directed) lifting structures on $P$ correspond to $+$-algebra structures on the projection $\pi_0 : \widetilde{P} \longrightarrow P0$, as follows.

**Proposition 43.** *For any $P : \mathsf{Set}^{\mathrm{I}}$, there is an isomorphism*

$$L^0(P) \;\cong\; {}^{+}\mathsf{Alg}(\pi_0 : \widetilde{P} \longrightarrow P0)\,.$$

*Proof.* Consider the following diagram,

$$
\begin{tikzcd}
& \widetilde{P} \arrow[rr] \arrow[d, "\pi_0"] & & \widetilde{P} \times \mathrm{I} \arrow[d, "\varepsilon"] \\
& P0 \arrow[rr] \arrow[dd] & & P \arrow[dd] \\
{[\phi]} \arrow[uur, "\overline{s}_0"] \arrow[ur, "s0"] \arrow[rr] \arrow[dd, tail] & & {[\phi]} \times \mathrm{I} \arrow[uur, "\overline{s}"] \arrow[ur, "s"'] \arrow[dd, tail] & \\
& 1 \arrow[rr, "\delta_0"'] & & \mathrm{I} \\
1 \arrow[uuur, "a_0"'] \arrow[ur] \arrow[rr] & & 1 \times \mathrm{I} \arrow[ur, "\pi_2"'] \arrow[uuur, dotted, "a"'] &
\end{tikzcd}
\tag{51}
$$

which is (50), extended by the counit (evaluation) $\varepsilon : \widetilde{P} \times \mathrm{I} \longrightarrow P$ over I on the right, and with 1 still representing the domain of a variable to reason internally. The pullback of $\varepsilon$ over I along $\delta_0$ is then the map $\pi_0 : \widetilde{P} \longrightarrow P0$ that we are interested in.

Given an $L^0(P)$-structure, reasoning internally we construct a ${}^{+}\mathsf{Alg}$-structure on $\pi_0 : \widetilde{P} \longrightarrow P0$ as follows: for any cofibration $i_\phi : [\phi] \rightarrowtail 1$ and any commutative square,

$$
\begin{tikzcd}
{[\phi]} \arrow[r, "s"] \arrow[d, tail, "i_\phi"'] & \widetilde{P} \arrow[d, "\pi_0"] \\
1 \arrow[r, "a_0"'] & P0\,,
\end{tikzcd}
\tag{52}
$$

we require a diagonal filler,

$$
\begin{tikzcd}
{[\phi]} \arrow[r, "s"] \arrow[d, tail, "i_\phi"'] & \widetilde{P} \arrow[d, "\pi_0"] \\
1 \arrow[r, "a_0"'] \arrow[ur, dotted, "j"'] & P0\,.
\end{tikzcd}
$$

Transposing the top left span in (52) formed by $i_\phi$ and $s$ along the adjunction $\mathrm{I}^* \dashv \prod_{\mathrm{I}}$ gives the right-hand square in (51), and the commutative square in (52) formed by $a_0$ and $\pi_0$ gives the rest of the data in (51). Thus the assumed $L^0(P)$-structure gives an $a : 1 \times \mathrm{I} \longrightarrow P$ as indicated in (51). But then $a$ lifts uniquely across $\varepsilon$ to a map $\overline{a} : 1 \times \mathrm{I} \longrightarrow \widetilde{P} \times \mathrm{I}$ over I, by the universal property of $\varepsilon : \widetilde{P} \times \mathrm{I} \longrightarrow P$. We can therefore set

$$j = \delta_0^*(\overline{a}) : 1 \longrightarrow \widetilde{P}\,.$$

Suppose conversely that we have a $^+\mathsf{Alg}$-structure on $\pi_0 : \widetilde{P} \longrightarrow P0$, and we want to build a (0-directed) lifting structure on $P$. Take any $\phi, s, a_0$ as indicated in (51), and we require an $a : 1 \times \mathrm{I} \longrightarrow P$ over I. From $s$ we get $\overline{s}$ by the universal property of $\varepsilon$, and we therefore have $\overline{s}_0$ by pullback. From $\overline{s}_0$ and $a_0$ and the $^+\mathsf{Alg}$ structure on $\pi_0$ we obtain a map $j : 1 \longrightarrow \widetilde{P}$ over $P0$ which is a diagonal filler of the indicated square formed by $i_\phi, \overline{s}_0, a_0$ and $\pi_0$. Finally, we obtain the required map $a : 1 \times \mathrm{I} \longrightarrow P$ over I as the $(\mathrm{I}^* \dashv \prod_{\mathrm{I}})$-transpose of $j$,

$$a = \varepsilon \circ (j \times \mathrm{I})\,.$$

We leave to the reader the verification that these assignments are mutually inverse. $\square$

$$\begin{tikzcd}
 & B_1 \arrow[dd, "q"] & & \\
\cdot \arrow[ur] \arrow[rr] \arrow[dd, "q_X"'] & & G \arrow[ul] \arrow[dd, "q'"] & \\
 & B_0 & & \\
X \times_{A_0} A_1 \arrow[ur, "b"] \arrow[rr, "\overline{b} \times_{A_0} A_1"'] \arrow[d, "p_X"'] & & B_0^{A_1} \times_{A_0} A_1 \arrow[ul, "\mathrm{ev}"'] \arrow[r] \arrow[d, "p'"] & A_1 \arrow[d, "p"] \\
X \arrow[rr, "\overline{b}"] \arrow[rrr, bend right, "a"'] \arrow[uuu, bend left, "{(a',b')}"] & & B_0^{A_1} \arrow[r] & A_0
\end{tikzcd}$$

$$
\begin{tikzcd}
B_1 \arrow[d, "q"] & & \\
B_0 & G \arrow[ul] \arrow[d, "q'"] \arrow[dd, bend right=60] & \\
& B_0^{A_1} \times_{A_0} A_1 \arrow[ul] \arrow[r] \arrow[d, "p'"] & A_1 \arrow[d, "p"] \\
& B_0^{A_1} \arrow[r] & A_0.
\end{tikzcd}
$$

# Appendix 2: A left-induced model structure on the Cartesian cubical sets

We make use of the Sattler model structure [**?**] on the *Dedekind cubical sets* $\widehat{\mathbb{D}} = \mathsf{Set}^{\mathbb{D}^{\mathrm{op}}}$, where $\mathbb{D}$ is the category of *Dedekind cubes*, defined as the Lawvere theory of distributive lattices. The unique product-preserving functor

$$i : \mathbb{C} \longrightarrow \mathbb{D}$$

classifying the Dedekind interval $\mathrm{I}_{\mathbb{D}} \in \mathbb{D}$ induces an adjunction,

$$i_! \dashv i^* \dashv i_* \ : \ \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{D}}\,,$$

where $i^*(Q) = Q \circ i$, for $Q \in \mathbb{D}$.

**Lemma 44.** *Observe that $i_!$ is left exact since the Dedekind interval $\mathrm{I}_{\mathbb{D}}$ is* strict, $0 \neq 1 : 1 \rightrightarrows \mathrm{I}_{\mathbb{D}}$. *Thus we have geometric morphisms:*

$$(i_! \dashv i^*) : \ \widehat{\mathbb{D}} \longrightarrow \widehat{\mathbb{C}}\,,$$

*classifying the bipointed object $i_!(\mathrm{I}_{\mathbb{C}}) = \mathrm{I}_{\mathbb{D}}$,*

$$(i^* \dashv i_*) : \ \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{D}}\,,$$

*classifying the dLat $i^*(\mathrm{I}_{\mathbb{D}}) := \mathbb{I}$, where $\eta : \mathrm{I}_{\mathbb{C}} \longrightarrow \mathbb{I}$ can be described pointwise as the distributive lattice completion of the corresponding bipointed set.*

*Also, since $i$ is faithful so is $i_!$, and since $i$ is surjective on objects $i^*$ is also faithful.*

*It follows that:*

- $\widehat{\mathbb{C}}$ *is* $(i_! \circ i^*)$*-coalgebras on* $\widehat{\mathbb{D}}$*,*
- $\widehat{\mathbb{D}}$ *is* $(i^* \circ i_*)$*-coalgebras on* $\widehat{\mathbb{C}}$*,*
- $\widehat{\mathbb{D}}$ *is* $(i^* \circ i_!)$*-algebras on* $\widehat{\mathbb{C}}$*.*

We will use the following transfer theorem for QMSs from [**?**, **?**]:

**Theorem** ([**?**, **?**])**.** *Suppose* $\widehat{\mathbb{D}}$ *has a (cofibrantly generated) model structure* $(\mathcal{C}_{\mathbb{D}}, \mathcal{W}_{\mathbb{D}}, \mathcal{F}_{\mathbb{D}})$*. Given an adjunction*

$$i_! \dashv i^* \;:\; \widehat{\mathbb{D}} \longrightarrow \widehat{\mathbb{C}}\,,$$

*there is a* left-induced model structure *on* $\widehat{\mathbb{C}}$ *if the following* acyclicity condition *holds:*

$$(i_!^{-1}\mathcal{C}_{\mathbb{D}})^{\pitchfork} \;\subset\; i_!^{-1}\mathcal{W}_{\mathbb{D}}\,.$$

*For the left-induced model structure* $(\mathcal{C}_{\mathbb{C}}, \mathcal{W}_{\mathbb{C}}, \mathcal{F}_{\mathbb{C}})$ *on* $\widehat{\mathbb{C}}$ *we then have:*

$$\begin{aligned} \mathcal{C}_{\mathbb{C}} &= i_!^{-1}\mathcal{C}_{\mathbb{D}}\,, \\ \mathcal{W}_{\mathbb{C}} &= i_!^{-1}\mathcal{W}_{\mathbb{D}}\,. \end{aligned}$$

The Sattler model structure on $\widehat{\mathbb{D}}$ is given as follows (for a constructive treatment a smaller class of "pointwise decidable cofibrations" is used, but we consider the classical case first):

$$\begin{aligned} \mathcal{C} &= \text{monomorphisms}\,, \\ \mathcal{W} &= \{f \mid f = p \circ i,\ p \in \mathcal{F} \cap \mathcal{W},\ i \in \mathcal{C} \cap \mathcal{W}\}\,, \\ \mathcal{F} &= (\mathcal{C} \otimes \delta)^{\pitchfork}\,. \end{aligned}$$

where $\delta : 1 \longrightarrow \mathrm{I}$ is either endpoint inclusion.

For the left-induced model structure on $\widehat{\mathbb{C}}$ we therefore have the following specification:

$$\begin{aligned} \mathcal{C} &= \text{monomorphisms,} \\ \mathcal{W} &= \{f \mid i_! f = p \circ i,\ p \in \mathcal{F}_{\mathbb{D}} \cap \mathcal{W}_{\mathbb{D}},\ i \in \mathcal{C}_{\mathbb{D}} \cap \mathcal{W}_{\mathbb{D}}\}\,, \\ \mathcal{F} &= (\mathcal{C} \cap \mathcal{W})^{\pitchfork}\,. \end{aligned}$$

The determination of $\mathcal{C}$ follows from the fact that $i_! : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{D}}$ is conservative.

To check the acyclicity condition,

$$(i_!^{-1}\mathcal{C}_{\mathbb{D}})^{\pitchfork} \;\subset\; i_!^{-1}\mathcal{W}_{\mathbb{D}}\,,$$

we know that $i_!^{-1}\mathcal{C}_{\mathbb{D}}$ consists of the monos in $\mathbb{C}$, so take $f : Y \longrightarrow X$ in $(i_!^{-1}\mathcal{C}_{\mathbb{D}})^{\pitchfork}$, apply $i_!$, and factor the result as $i_! f = p \circ m : i_! Y \longrightarrow Z \longrightarrow i_! X$ with $p \in \mathcal{F}_{\mathbb{D}} \cap \mathcal{W}_{\mathbb{D}}$ and $m : i_! Y \longrightarrow Z$ monic. We then need to show that $m$ is in $\mathcal{W}_{\mathbb{D}}$.

We can apply Theorem 2.2.1 of [?], with $\mathsf{K} = \widehat{\mathbb{C}}$, $\mathsf{M} = \widehat{\mathbb{D}}$, $V = i_!$, $k = i^*$, and:

1. $QX = X$ and $\epsilon = 1_X : X \longrightarrow X$, so that $i_! 1_X = 1_{i_! X}$ and therefore in $\mathcal{W}_{\mathbb{D}}$, while all objects are cofibrant,
2. $Qf = f$ for any $f : X \longrightarrow Y$ in $\widehat{\mathbb{C}}$, so that the naturality condition is similarly trivial,
3. factor the codiagonal $X + X \longrightarrow X$ as $\pi_2 \circ j : X + X \longrightarrow \mathrm{I} \times X \longrightarrow X$ with $j = (\partial\mathrm{I} \times X) : X + X \longrightarrow \mathrm{I} \times X$.

It remains only to show that $i_! p : \mathsf{i}_!(\mathrm{I} \times X) \longrightarrow i_! X$ is in $\mathcal{W}_{\mathbb{D}}$ and $i_! j : i_!(X + X) \longrightarrow i_!(\mathrm{I} \times X)$ is in $\mathcal{C}_{\mathbb{D}}$. The latter is clear, since $j$ is monic. To show the former, observe that for any $D \in \widehat{\mathbb{D}}$, the projection $\pi_2 : \mathrm{I}_{\mathbb{D}} \times D \longrightarrow D$ is in $\mathcal{W}_{\mathbb{D}}$ by 3-for-2, since the "cylinder end" inclusion $D \longrightarrow \mathrm{I}_{\mathbb{D}} \times D$, as a pullback of an endpoint inclusion, is a cofibration, and a strong deformation retract (using the connection on I), and hence is in $\mathcal{W}_{\mathbb{D}}$ by [?].

Thus we have shown:

**Theorem 45.** *There is a Quillen model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ on the category $\widehat{\mathbb{C}}$ of cartesian cubical sets, in which*

$$\begin{aligned} \mathcal{C} &= \textit{monomorphisms}, \\ \mathcal{W} &= \{ f \mid i_! f = p \circ i,\ p \in \mathcal{F}_{\mathbb{D}} \cap \mathcal{W}_{\mathbb{D}},\ i \in \mathcal{C}_{\mathbb{D}} \cap \mathcal{W}_{\mathbb{D}} \}, \\ \mathcal{F} &= (\mathcal{C} \cap \mathcal{W})^{\pitchfork}. \end{aligned}$$

*where $i_! : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{D}}$ is the left adjoint of precomposition along the canonical map $i : \mathbb{C} \longrightarrow \mathbb{D}$ from Cartesian cubes to Dedekind cubes, and $(\mathcal{C}_{\mathbb{D}}, \mathcal{W}_{\mathbb{D}}, \mathcal{F}_{\mathbb{D}})$ is the Sattler model structure on $\widehat{\mathbb{D}}$.*