Cartesian cubical model categories

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Abstract

Add an abstract.

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Introduction

novelties: univalence, universes, classifying types, uniformity, constructive, ...

1 Cartesian cubical sets

There are now many treatments of cubical sets $\mathsf{Set}^{\square^{\mathrm{op}}}$ in the literature, including [?, ?, ?, ?, ?, ?, ?, ?]. Our model structure is intended to work in any of these, insofar as they are *cartesian*, in the sense that the indexing cubes $[n] \in \square$ are closed under finite products $[m] \times [n] = [m + n]$. Rather than working axiomatically, however, we shall work in the initial such model, which we call simply the *Cartesian cube category* $\square$, defined as the free finite product category on an interval $\delta_0, \delta_1 : 1 \rightrightarrows \mathrm{I}$.

Definition 1. The objects $[n]$ of the *Cartesian cube category* $\square$, called n-cubes, are finite sets of the form

$$[n] = \{0, x_1, ..., x_n, 1\},$$

where the $x_1, ..., x_n$, are arbitrary but distinct elements, and $0, 1$ are further distinct, distinguished elements. The arrows,

$$f : [m] \to [n],$$

are arbitrary bipointed maps $f' : [n] \to [m]$ (note the variance!). Thus $\mathbb{B} = \square^{\mathrm{op}}$ is the category if finite, strictly bipointed sets.

As a Lawvere theory, the arrows $f : [m] \to [n]$ in $\square$ may also be regarded as n-tuples of elements from the set $\{0, x_1, ..., x_m, 1\}$. These can be generated under composition by faces, degeneracies, permutations, and diagonals (see [?] for further details).

Definition 2. The category cSet of *(Cartesian) cubical sets* is the category of presheaves on the (Cartesian) cube category $\square$,

$$\mathsf{cSet} \;=\; \mathsf{Set}^{\square^{\mathrm{op}}}.$$

It is of course generated by the representable presheaves $\mathsf{y}[n]$, to be written

$$\mathrm{I}^n = \mathsf{y}[n]$$

and called the *geometric n-cubes*.

Note that the representables I^n are also closed under finite products, $\mathrm{I}^m \times \mathrm{I}^n = \mathrm{I}^{m+n}$. We of course write I for I^1 and 1 for I^0, which is terminal. We will use the following basic fact about the cubes I^n in cSet.

Proposition 3. *The n-cubes I^n are* tiny*, in the sense that the endofunctor $X \mapsto X^{\mathrm{I}^n}$ is a left adjoint.*

(See [**?**] on such "amazing right adjoints".)

Proof. It clearly suffices to prove the claim for $n = 1$. For any cubical set X, the exponential X^{I} is a "shift by one dimension",

$$X^{\mathrm{I}}(n) \cong \mathrm{Hom}(\mathrm{I}^n, X^{\mathrm{I}}) \;\cong\; \mathrm{Hom}(\mathrm{I}^{n+1}, X) \cong\; X(n+1).$$

Thus X^{I} is given by precomposition with the "successor" functor $\Box \to \Box$ with $[n] \mapsto [n+1]$. Precomposition always has a right adjoint, which in this case we shall write as

$$(-)^{\mathrm{I}} \dashv (-)_{\mathrm{I}}$$

and call X_{I} the I^{th}-*root* of X. See Appendix B below for a calculation of the root X_{I}. $\Box$

The exponential X^{I} will be called the *pathobject* of X, and plays a special role. As we have just seen, it classifies "paths" in X; so the 0-cubes $p \in (X^I)_0$ in the pathobject correspond to 1-cubes $p \in X_1$, the "endpoints" of which $p_0, p_1 \in X_0$ are given by composing with the evaluation maps

$$\epsilon_0, \epsilon_1 : X^{\mathrm{I}} \rightrightarrows X$$

at the points $\delta_0, \delta_1 : 1 \rightrightarrows \mathrm{I}$. More generally, higher cubes $c \in X_{n+1}$ correspond to maps $c : \mathrm{I}^{n+1} \to X$, which are thus paths between the n-cubes $c_0, c_1 : \mathrm{I}^n \to X$, corresponding to $c_0, c_1 \in X_n$.

We mention two facts that will be needed below, concerning the base change functors

$$f_! \dashv f^* \dashv f_* : \mathsf{cSet}/_X \longrightarrow \mathsf{cSet}/_Y$$

associated to a map $f : X \to Y$ in cSet.

Lemma 4. *The pulled-back interval $\mathrm{I}^*\mathrm{I} = \mathrm{I} \times \mathrm{I} \to \mathrm{I}$ in $\mathsf{cSet}/_{\mathrm{I}}$ is also tiny.*

Proof. Since the interval $\mathrm{I} = \mathsf{y}[1]$ is representable, the slice category $\mathsf{cSet}/_{\mathrm{I}}$ is also a category of presheaves, namely over the sliced cube category $\square/_{[1]}$,

$$\mathsf{cSet}/_{\mathrm{I}} \;=\; \mathsf{Set}^{\square^{\mathrm{op}}}/_{\mathsf{y}[1]} \;\cong\; \mathsf{Set}^{(\square/_{[1]})^{\mathrm{op}}}\,.$$

However, since $\square$ does not have all finite limits, the sliced index category does not have all finite products, and so we cannot simply repeat the proof from Proposition 3. But as in that proof, we do have a "successor" functor

$$s_{[1]} : \square/_{[1]} \to \square/_{[1]}\,,$$

resulting from the "predecessor" natural transformation $s \Rightarrow 1_{\square}$ given by the projection $\mathrm{I} \times X \to X$. Evaluating s at each object $f : [n] \to [1]$ in $\square/_{[1]}$, we obtain a commutative diagram:

$$\begin{tikzcd}
s[n] \arrow[r, "\cong"] \arrow[d, "sf"'] & {[1]\times[n]} \arrow[r, "p_n"] & {[n]} \arrow[d, "f"] \\
s[1] \arrow[r, "\cong"'] & {[1]\times[1]} \arrow[r, "p_1"'] & {[1]}
\end{tikzcd} \tag{1}$$

We can then set $s_{[1]}(f) = p_1 \circ sf = f \circ p_n$. As in the foregoing proof, we can then calculate the values of the adjoints on presheaves, associated to $s_{[1]}$,

$$s_{[1]!} \dashv s_{[1]}{}^{*} : \widehat{\square/_{[1]}} \longrightarrow \widehat{\square/_{[1]}}$$

to be, successively,

$$\begin{aligned} s_{[1]!}(X) &= \mathrm{I}^{*}\mathrm{I} \times X\,, \\ s_{[1]}{}^{*}(X) &= X^{\mathrm{I}^{*}\mathrm{I}}\,. \end{aligned}$$

The first equation follows from the observation that the diagram (1) is a pullback, and so the object $s_{[1]}(f) : s[n] \to [1]$ of $\widehat{\square/_{[1]}}$ given by the evident composite is just $\mathrm{I}^{*}\mathrm{I} \times f$, and the diagram itself represents the counit map $(\mathrm{I}^{*}\mathrm{I} \times f) \to f$ over I. The second line then follows by adjointness, as does the fact that we have a further right adjoint, namely, the $\mathrm{I}^{*}\mathrm{I}^{th}$-*root*:

$$s_{[1]*}(X) =: X_{\mathrm{I}^{*}\mathrm{I}}\,.$$

$\square$

Lemma 5. *The pushforward functor along any map $f : X \to Y$ preserves pathspaces; for any object $A \to X$ over X, the pathobject of the pushforward f_*A is (canonically isomorphic to) the pushforward of the pathobject,*

$$(f_*A)^{\mathrm{I}} \cong f_*(A^{\mathrm{I}})$$

over Y.

Proof. This is true for any constant family $X^*C = X \times C \to X$ with C in place of I, as the reader can easily verify using the Beck-Chevalley condition. □

2 The cofibration weak factorization system

One can simply take as the cofibrations all monomorphisms in cSet, but for some purposes (*e.g.* []) it is convenient to know what is actually required of them. Thus to begin, the following axioms will be assumed.

Definition 6 (Cofibrations)**.** The *cofibrations* are a class $\mathcal{C}$ of monomorphisms satisfying the following conditions:

(C0) The map $0 \to C$ is always a cofibration.

(C1) All isomorphisms are cofibrations.

(C2) The composite of two cofibrations is a cofibration.

(C3) Any pullback of a cofibration is a cofibration.

We also require the cofibrations to be classified by a subobject $\Phi \hookrightarrow \Omega$ of the standard subobject classifier $\top : 1 \to \Omega$ of cSet:

(C4) There is a terminal object $t : 1 \to \Phi$ in the category of cofibrations and cartesian squares.

Further axioms for cofibrations will be added later as needed: two in Section 3.1, one in Section 3.2, and a final one in Section 6.4 (see Appendix 8 for a summary). Cofibrations will be written

$$c : A \rightarrowtail B\,.$$

The cofibrant partial map classifier. Consider the polynomial endofunctor $P_t : \mathsf{cSet} \to \mathsf{cSet}$ determined by the cofibration classifier $t : 1 \rightarrowtail \Phi$ (see [?]). We will write the value of this functor at an object X as

$$X^+ \ := \ \Phi_! \, t_*(X) \ = \ \sum_{\varphi:\Phi} X^{[\varphi]} \,. \tag{2}$$

The reader familiar with type theory will recognize the similarity to the "partiality" or "lifting" monad [?]. When all monos are cofibrations, so that $\Phi = \Omega$, this agrees with the partial map classifier $X^+ = \widetilde{X}$ from topos theory [?]. We may therefore regard X^+ as the object of *cofibrant partial elements* of X, as we now explain.

Since $t : 1 \rightarrowtail \Phi$ is monic, $t^* t_* \cong 1$, so X^+ fits into the pullback square

$$\begin{tikzcd} X \arrow[r, tail] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & X^+ \arrow[d, "t_* X"] \\ 1 \arrow[r, tail, "t"'] & \Phi . \end{tikzcd} \tag{3}$$

Let $\eta : X \rightarrowtail X^+$ be the indicated top horizontal map; we call this the *cofibrant partial map classifier* of X. By a *cofibrant partial map* (from an object Z) into X we mean a span $(c, x) : Z \leftarrowtail C \to X$ with a cofibration on the left. The object X^+ is a *classifying type* for such cofibrant partial maps, in that it has the following universal property.

Proposition 7. *Let $\eta : X \rightarrowtail X^+$ be as defined in (3).*

1. *The map $\eta : X \rightarrowtail X^+$ is a cofibration.*
2. *For any object Z and any partial map $(c, x) : Z \leftarrowtail C \to X$, with $c : C \rightarrowtail Z$ a cofibration, there is a unique $\chi : Z \to X^+$ fitting into a pullback square as follows.*

$$\begin{tikzcd} C \arrow[r, "x"] \arrow[d, tail, "c"'] \arrow[dr, phantom, very near start, "\lrcorner"] & X \arrow[d, tail, "\eta"] \\ Z \arrow[r, "\chi"'] & X^+ \end{tikzcd}$$

The map $\chi : Z \to X^+$ is said to classify *the partial map*

$$(c, x) : Z \leftarrowtail C \to X \,.$$

Proof. The map $\eta : X \rightarrowtail X^+$ is a cofibration since it is a pullback of the universal cofibration $t : 1 \rightarrowtail \Phi$. Observe that $(\eta, 1_X) : X^+ \leftarrowtail X \to X$ is therefore a cofibrant partial map into X. The second statement is just the universal property of X^+ as a polynomial (see [**?**], prop. 7). $\square$

Proposition 8. *The pointed endofunctor $\eta_X : X \rightarrowtail X^+$ has a natural multiplication $\mu_X : X^{++} \to X^+$ making it a monad.*

Proof. Since the cofibrations are closed under composition, the monad structure on X^+ follows as in [**?**], proposition XY. Explicitly, μ_X is determined by proposition 7 as the unique map making the following a pullback diagram.

$$\begin{tikzcd}
X \arrow[r, "="] \arrow[d, tail, "\eta_X"'] & X \arrow[dd, tail, "\eta"] \\
X^+ \arrow[d, tail, "\eta_{X^+}"'] & \\
X^{++} \arrow[r, dotted, "\mu"'] & X^+
\end{tikzcd}$$

$\square$

Relative partial map classifier. For any object $X \in \mathsf{cSet}$ the pullback functor

$$X^* : \mathsf{cSet} \to \mathsf{cSet}/_X\,,$$

taking any A to (say) the first projection $X \times A \to X$, not only preserves the subobject classifier Ω, but also the cofibration classifier $\Phi \hookrightarrow \Omega$, where a map in $\mathsf{cSet}/_X$ is defined to be a cofibration if it is one in cSet (under the forgetful functor $\mathsf{cSet}/_X \to \mathsf{cSet}$). Thus in $\mathsf{cSet}/_X$ we can define the *(relative) cofibration classifier* to be the map

$$X^*t : X^*1 \longrightarrow X^*\Phi \quad \text{over } X\,,$$

which we may also write $t_X : 1_X \to \Phi_X$. Like $t : 1 \to \Phi$, this map determines a polynomial endofunctor

$$+_X : \mathsf{cSet}/_X \to \mathsf{cSet}/_X\,,$$

which commutes (up to natural isomorphism) with $+ : \mathsf{cSet} \to \mathsf{cSet}$ and $X^* : \mathsf{cSet} \to \mathsf{cSet}/_X$ in the expected way, namely:

$$
\begin{tikzcd}
\mathsf{cSet}/_X \arrow[r, "+_X"] & \mathsf{cSet}/_X \\
\mathsf{cSet} \arrow[u, "X^*"] \arrow[r, "+"'] & \mathsf{cSet} \arrow[u, "X^*"']
\end{tikzcd}
\tag{4}
$$

The endofunctor $+_X$ is also pointed $\eta_Y : Y \to Y^+$ and has a natural monad multiplication $\mu_Y : Y^{++} \to Y^+$, for any $Y \to X$, for the same reason that $+$ has this structure. Summarizing, we may say:

Proposition 9. *The polynomial monad* $+ : \mathsf{cSet} \to \mathsf{cSet}$ *of* cofibrant partial elements *is indexed (or fibered) over* cSet*.*

Definition 10. A $+$-*algebra* in cSet is an algebra for the pointed endofunctor $+ : \mathsf{cSet} \to \mathsf{cSet}$. Explicitly, a $+$-algebra is a cubical set A together with a retraction $\alpha : A^+ \to A$ of the unit $\eta_A : A \to A^+$. Algebras for the monad $(+, \eta, \mu)$ will be referred to specifically as $(+, \eta, \mu)$-*algebras*, or $+$-*monad algebras.*

A *relative* $+$-*algebra* in cSet is a map $A \to X$, together with an algebra structure over the codomain X for the pointed endofunctor

$$
+_X : \mathsf{cSet}/_X \longrightarrow \mathsf{cSet}/_X \,.
$$

The cofibration weak factorization system. The following proposition generalizes one in [**?**].

Proposition 11. *There is an (algebraic) weak factoriation system on* cSet *with the cofibrations as the left class, and as the right class, the maps underlying relative* $+$*-algebras. Thus a right map is one* $p : A \to X$ *for which there is a retract* $\alpha : A' \to A$ *over* X *of the canonical map* $\eta : A \to A'$*,*

$$
\begin{tikzcd}
A \arrow[r, "\eta"] \arrow[rr, bend left, "="] \arrow[dr, "p"'] & A' \arrow[r, "\alpha"] \arrow[d, "p^+"] & A \arrow[dl, "p"] \\
& X. &
\end{tikzcd}
$$

(Note that the domain of $p^+ : A' \to X$ is not A^+, unless of course $X = 1$.)

Proof. The factorization of a map $f : Y \to X$ is given by applying the relative +-functor over the codomain,

$$\begin{tikzcd} Y \arrow[r, tail, "\eta_f"] \arrow[dr, "f"'] & Y' \arrow[d, "f^+"] \\ & X. \end{tikzcd}$$

We know by proposition 7 that the unit η_f is always a cofibration, and since f^+ is the free algebra for the relative +-monad, it is in particular a +-algebra.

For the lifting condition, consider a cofibration $c : B \rightarrowtail C$, and a right map $p : A \to X$ with +-algebra structure map $\alpha : A' \to A$ over X, and a commutative square as indicated below.

$$\begin{tikzcd} B \arrow[r, "a"] \arrow[dd, tail, "c"'] & A \arrow[dd, "p"'] \arrow[dr, "\eta"] & \\ & & A' \arrow[ul, bend right, "\alpha"'] \arrow[dl, "p^+"] \\ C \arrow[r, "x"'] & X & \end{tikzcd}$$

Thus in the slice category over X, we have

$$\begin{tikzcd} B \arrow[r, "a"] \arrow[d, tail, "c"'] & A \arrow[d, "\eta"] \\ C \arrow[ur, dotted, "d"'] & A^+ \arrow[u, bend right=60, "\alpha"'] \end{tikzcd}$$

and we seek a diagonal filler d as indicated. (Note that we are writing A^+ for the map $p^+ : A' \to A$ regarded as an object over X, and similarly C for $x : C \to X$ and B for $xc : B \to X$ and A for $p : A \to X$.) Since $(c, a) : B \leftarrowtail C \to A$ is a cofibrant partial map into A, by the universal property of $\eta : A \rightarrowtail A^+$ (Proposition 7) there is a unique classifying map $\varphi : C \to A^+$ (over X) making a pullback square,

$$\begin{tikzcd} B \arrow[r, "a"] \arrow[d, tail, "c"'] & A \arrow[d, "\eta"] \\ C \arrow[r, dotted, "\varphi"'] & A^+. \arrow[u, bend right=60, "\alpha"'] \end{tikzcd}$$

We can set $d := \alpha \circ \varphi : C \to A$ to obtain the required diagonal filler, since $dc = \alpha\varphi c = \alpha\eta a = a$, because α is a retract of η.

The closure of the cofibrations under retracts follows from their classification by a universal object $t : 1 \rightarrowtail \Phi$, and the closure of the right maps under retracts follows from their being the algebras for a pointed endofunctor underlying a monad (cf. [**?**]). Algebraicity of this weak factorization system is immediate, since $+$ is a monad. $\square$

Summarizing, we have an algebraic weak factorization system $(\mathcal{C}, \mathcal{C}^{\pitchfork})$ on the category cSet of cubical sets, where:

$$\begin{aligned} \mathcal{C} &= \text{the cofibrations} \\ \mathcal{C}^{\pitchfork} &= \text{the maps underlying relative +-algebras} \end{aligned}$$

We shall call this the *cofibration weak factorization system*. The right maps will be denoted

$$\mathsf{TFib} = \mathcal{C}^{\pitchfork}$$

and called *trivial fibrations*.

The cofibration algebraic weak factorization system is a generalization of the one defined in [**?**] and mentioned in [**?**].

Uniform filling structure. It will be useful to relate relative $+$-algebra structure with the more familiar diagonal filling condition of cofibrantly generated weak factorization systems, and specifically the special ones occurring in [CCHM16] under the name *uniform filling structure* (this notion is also closely related to that of an *algebraic weak factorization system*, *cf.* [GKR18, Rie14]).

Consider a generating sub*set* of cofibrations consisting of those with representable codomain $c : C \rightarrowtail \mathrm{I}^n$, and call these the *basic cofibrations*.

$$\mathsf{BCof} = \{c : C \rightarrowtail \mathrm{I}^n \mid c \in \mathcal{C}, n \geq 0\}. \tag{5}$$

Proposition 12. *For any object X in cSet the following are equivalent:*

1. *X admits a* $+$-algebra structure: *a retraction $\alpha : X^+ \to X$ of the unit $\eta : X \to X^+$.*

2. *$X \to 1$ is a* trivial fibration: *it has the right lifting property with respect to all cofibrations,*
$$\mathcal{C} \pitchfork X.$$

3. *X admits a* uniform filling structure: *for each basic cofibration $c : C \rightarrowtail \mathrm{I}^n$ and map $x : C \to X$ there is given an extension $j(c,x)$,*

$$\begin{tikzcd} C \arrow[r, "x"] \arrow[d, tail, "c"'] & X, \\ \mathrm{I}^n \arrow[ur, dotted, "j(c,x)"'] & \end{tikzcd} \tag{6}$$

and the choice is uniform in I^n *in the following sense.*

Given any cubical map $u : \mathrm{I}^m \to \mathrm{I}^n$, the pullback $u^*c : u^*C \rightarrowtail \mathrm{I}^m$, which is again a basic cofibration, fits into a commutative diagram of the form

$$\begin{tikzcd} u^*C \arrow[r, "c^*u"] \arrow[d, tail, "u^*c"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[r, "x"] \arrow[d, tail, "c"'] & X. \\ \mathrm{I}^m \arrow[r, "u"'] & \mathrm{I}^n \arrow[ur, dotted, "j(c,x)"'] & \end{tikzcd} \tag{7}$$

For the pair $(u^*c,\ x \circ c^*u)$ in (7), the chosen extension $j(u^*c, x \circ c^*u) : \mathrm{I}^m \to X$, is required to be equal to $j(c,x) \circ u$,

$$j(u^*c, x \circ c^*u) = j(c,x) \circ u. \tag{8}$$

Proof. Let (X, α) be a $+$-*algebra* and suppose given the span (c, x) as below, with c a cofibration.

$$\begin{tikzcd} C \arrow[r, "x"] \arrow[d, tail, "c"'] & X \\ Z & \end{tikzcd}$$

Let $\chi(c,x) : Z \to X^+$ be the classifying map of the cofibrant partial map $(c,x) : Z \leftarrowtail C \to X$, so that we have a pullback square as follows.

$$\begin{tikzcd} C \arrow[r, "x"] \arrow[d, tail, "c"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\eta"] \\ Z \arrow[r, "\chi(c,x)"'] & X^+ \end{tikzcd} \tag{9}$$

Then set

$$j = \alpha \circ \chi(c,x) : Z \to X \tag{10}$$

to get a filler,

$$
\begin{tikzcd}
C \arrow[r, "x"] \arrow[d, tail, "c"'] & X \arrow[d, "\eta"] \\
Z \arrow[r, "\chi(c,x)"'] \arrow[ur, dotted, "j"] & X^+ \arrow[u, bend right=60, "\alpha"']
\end{tikzcd}
\tag{11}
$$

since

$$
j \circ c = \alpha \circ \chi(c,x) \circ c = \alpha \circ \eta \circ x = x.
$$

Thus (1) implies (2). To see that it also implies (3), observe that in the case where $Z = I^n$ and we specify, in (10), that

$$
j(c,x) = \alpha \circ \chi(c,x) : \mathrm{I}^n \to X, \tag{12}
$$

then the assignment is natural in I^n. Indeed, given any $u : \mathrm{I}^m \to \mathrm{I}^n$, we have

$$
j(c', xu') = \alpha \circ \chi(c', xu') = \alpha \circ \chi(c,x) \circ u = j(c,x)u, \tag{13}
$$

by the uniqueness of the classifying maps.

It is clear that (2) implies (1), since if $\mathcal{C} \pitchfork X$ then we can take as an algebra structure $\alpha : X^+ \to X$ any filler for the universal span

$$
\begin{tikzcd}
X \arrow[r, "="] \arrow[d, tail, "\eta"'] & X. \\
X^+ \arrow[ur, dotted, "\alpha"'] &
\end{tikzcd}
$$

To see that (3) implies (1), suppose that X has a uniform filling structure j and we want to define an algebra structure $\alpha : X^+ \to X$. By Yoneda, for every $y : \mathrm{I}^n \to X^+$ we need a map $\alpha(y) : \mathrm{I}^n \to X$, naturally in I^n, in the sense that for any $u : \mathrm{I}^m \to \mathrm{I}^n$, we have

$$
\alpha(yu) = \alpha(y)u. \tag{14}
$$

Moreover, to ensure that $\alpha\eta = 1_X$, for any $x : \mathrm{I}^n \to X$ we must have $\alpha(\eta \circ x) = x$. So take $y : \mathrm{I}^n \to X^+$ and let

$$
\alpha(y) = j(y^*\eta, y'),
$$

as indicated on the right below.

$$
\begin{tikzcd}
u^*C \arrow[r, "u'"] \arrow[d, tail, "u^*y^*\eta"'] \arrow[dr, phantom, very near start, "\lrcorner"] & C \arrow[r, "y'"] \arrow[d, tail, "y^*\eta"'] \arrow[dr, phantom, very near start, "\lrcorner"] & X. \arrow[d, "\eta"] \\
\mathrm{I}^m \arrow[r, "u"'] & \mathrm{I}^n \arrow[r, "y"'] \arrow[ur, dotted, "j(y^*\eta,y')"'] & X^+
\end{tikzcd}
\tag{15}
$$

Then for any $u : \mathrm{I}^m \to \mathrm{I}^n$, we indeed have

$$\alpha(yu) = j\big((yu)^*\eta, y'u'\big) = j(y^*\eta, y') \circ u = \alpha(y)u,$$

by the uniformity of j. Finally, if $y = \eta \circ x$ for some $x : \mathrm{I}^n \to X$ then

$$\alpha(\eta x) = j\big((\eta x)^*\eta, (\eta x)'\big) = j(1_X, x) = x,$$

because the defining diagram for $\alpha(\eta x)$, i.e. the one on the right in (15), then factors as

$$\begin{tikzcd} \mathrm{I}^n \arrow[r, "x"] \arrow[d, tail, "="'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[r, "="] \arrow[d, tail, "="'] \arrow[dr, phantom, "\lrcorner", very near start] & X, \arrow[d, "\eta"] \\ \mathrm{I}^n \arrow[r, "x"'] & X \arrow[r, "\eta"'] & X^+ \end{tikzcd} \tag{16}$$

and the only possible extension $j(1_X, x)$ for the span $(1_{\mathrm{I}^n}, x)$ is x itself. □

Remark 13. Observe that the uniformilty condition (3) can be extended to the *class of all* cofibrations, in the form:

4. X admits a *(large) uniform filling structure:* for each cofibration $c : C \rightarrowtail Z$ and map $x : C \to X$ there is given an extension $j(c, x)$,

$$\begin{tikzcd} C \arrow[r, "x"] \arrow[d, tail, "c"'] & X, \\ Z \arrow[ur, dotted, "j(c,x)"'] & \end{tikzcd} \tag{17}$$

   and the choice is *uniform in Z* in the following sense: Given any map $u : Y \to Z$, the pullback $u^*c : u^*C \rightarrowtail Y$, which is again a cofibration, fits into a commutative diagram of the form

$$\begin{tikzcd} u^*C \arrow[r, "c^*u"] \arrow[d, tail, "u^*c"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[r, "x"] \arrow[d, tail, "c"'] & X. \\ Y \arrow[r, "u"'] & z \arrow[ur, dotted, "j(c,x)"'] & \end{tikzcd} \tag{18}$$

   For the pair $(u^*c,\ x \circ c^*u)$ in (18), the chosen extension $j(u^*c, x \circ c^*u) : \mathrm{I}^m \to X$, is required to be equal to $j(c, x) \circ u$,

$$j(u^*c, x \circ c^*u) = j(c, x) \circ u. \tag{19}$$

Indeed, the proof that (1) implies (2) and (3) works just as well to infer (4), which in turn implies (2) and (3) as special cases.

The relative version of the foregoing is entirely analogous, since the $+$-functor is fibered over cSet in the sense of diagram (4). We can therefore omit the entirely analogous proof of the following.

Proposition 14. *For any map $f : Y \to X$ in cSet the following are equivalent:*

1. $f : Y \to X$ *admits a* relative $+$-algebra structure over X, *i.e. there is a retraction $\alpha : Y' \to Y$ over X of the unit $\eta : Y \to Y'$, where $f^+ : Y' \to X$ is the result of the relative $+$-functor applied to f, as in definition 10.*

2. $f : Y \to X$ *is a* trivial fibration,

$$\mathcal{C} \pitchfork f.$$

3. $f : Y \to X$ *admits a (small)* uniform filling structure*: for each basic cofibration $c : C \rightarrowtail \mathrm{I}^n$ and maps $x : C \to X$ and $y : \mathrm{I}^n \to Y$ making the square below commute, there is given a diagonal filler $j(c, x, y)$,*

$$\begin{tikzcd} C \arrow[r, "x"] \arrow[d, tail, "c"'] & X \arrow[d, "f"] \\ \mathrm{I}^n \arrow[r, "y"'] \arrow[ur, dotted, "{j(c,x,y)}"'] & Y, \end{tikzcd} \tag{20}$$

and the choice is uniform in I^n *in the following sense: given any cubical map $u : \mathrm{I}^m \to \mathrm{I}^n$, the pullback $u^*c : u^*C \rightarrowtail \mathrm{I}^m$ is again a basic cofibration and fits into a commutative diagram of the form*

$$\begin{tikzcd} u^*C \arrow[r, "c^*u"] \arrow[d, tail, "u^*c"'] \arrow[dr, phantom, very near start, "\lrcorner"] & C \arrow[r, "x"] \arrow[d, tail, "c"'] & X \arrow[d, "f"] \\ \mathrm{I}^m \arrow[r, "u"'] & \mathrm{I}^n \arrow[r, "y"'] \arrow[ur, dotted, "{j(c,x,y)}"'] & Y. \end{tikzcd} \tag{21}$$

For the evident triple $(u^*c,\, x \circ c^*u, y \circ u)$ in (21) the chosen diagonal filler

$$j(u^*c, x \circ c^*u, y \circ u) : \mathrm{I}^m \to X$$

is equal to $j(c, x, y) \circ u$,

$$j(u^*c, x \circ c^*u, y \circ u) = j(c, x, y) \circ u. \tag{22}$$

And again, a large version of (3) with arbitrary cofibrations $c : C \rightarrowtail Z$ is again equivalent to (1)-(3).

We next collect some basic facts about trivial fibrations that will be needed later: they have sections, they are closed under composition and retracts, and they are closed under pullback and pushforward along all maps.

Corollary 15. *1. Every trivial fibration $A \to X$ has a section $s : X \to A$.*

2. If $a : A \to X$ is a trivial fibration and $b : B \to A$ is a trivial fibration, then $a \circ b : B \to X$ is a trivial fibration.

3. If $a : A \to X$ is a trivial fibration and $a' : A' \to X'$ is a retract of a in the arrow category, then a' is a trivial fibration.

4. For any map $f : X \to Y$ and any trivial fibration $B \to Y$, the pullback $f^*B \to X$ is a trivial fibration.

5. For any map $f : X \to Y$ and any trivial fibration $A \to X$, the pushforward $f_*A \to Y$ is a trivial fibration.

Proof. (1) holds because all objects are cofibrant by (C0). (5) is a consequence of (C3), stability of cofibrations under pullback, by a standard argument using the adjunction $f^* \dashv f_*$. The rest hold for the right maps in any weak factorization system. □

Remark 16. The structured notion of trivial fibration, *vis.* relative +-algebra, can also be shown algebraically (i.e. not using Proposition 14) to be preserved by pullback, pushforward, composition, and to be closed under retracts. We do just the case of pushforward as an example. Thus consider the following situation with $A \to X$ a +-algebra with structure α, as indicated.

$$\begin{tikzcd}
A \arrow[dd] \arrow[dr, "\eta_A"] & & f_*A \arrow[dd] \arrow[dr, "\eta_{f_*A}"] & \\
& A^+ \arrow[ul, bend right, "\alpha"'] \arrow[dl] & & (f_*A)^+ \arrow[ul, dotted, bend right, "\beta"'] \arrow[dl] \\
X \arrow[rr, "f"'] & & Y &
\end{tikzcd} , \tag{23}$$

A $+$-algebra structure for $f_*A \to Y$ would be a retract $\beta : (f_*A)^+ \to f_*A$ of $\eta_{f_*A} : f_*A \to (f_*A)^+$ over Y, which corresponds under $f^* \dashv f_*$ to a map $\tilde{\beta} : f^*((f_*A)^+) \to A$ over X with

$$\tilde{\beta} \circ f^*\eta_{f_*A} = \epsilon_A$$

as indicated below.

$$\begin{tikzcd}
f^*f_*A \arrow[r, "f^*\eta_{f_*A}"'] \arrow[dr, "\epsilon_A"'] & f^*((f_*A)^+) \arrow[d, dotted, "\tilde{\beta}"'] & \\
& A \arrow[r, "\eta_A"'] & A^+. \arrow[l, bend right, "\alpha"']
\end{tikzcd} \tag{24}$$

But since pullback f^* commutes with $+$, there is a canonical iso $c : f^*((f_*A)^+) \cong (f^*f_*A)^+$ with $c \circ f^*\eta_{f_*A} = \eta_{f^*f_*A}$. So we can set $\tilde{\beta} := \alpha \circ (\epsilon_A)^+ \circ c$.

$$\begin{tikzcd}
f^*f_*A \arrow[r, "f^*\eta_{f_*A}"'] \arrow[rr, bend left, "\eta_{f^*f_*A}"] \arrow[dr, "\epsilon_A"'] & f^*((f_*A)^+) \arrow[r, "\sim", "c"'] \arrow[d, dotted, "\tilde{\beta}"'] & (f^*f_*A)^+ \arrow[d, "(\epsilon_A)^+"] \\
& A \arrow[r, "\eta_A"'] & A^+ \arrow[l, bend right, "\alpha"']
\end{tikzcd} \tag{25}$$

3 The fibration weak factorization system

We now specify a second weak factorization system, with a restricted class of "trivial" cofibrations on the left, and an expanded class of right maps, the *fibrations*. As explained in the introduction, we first recall from [GS17] what we shall call the "biased" notion of fibration, before giving the "unbiased" one appropriate to the more general setting. The biased version makes use of *connections*,

$$\vee, \wedge : \mathrm{I} \times \mathrm{I} \longrightarrow \mathrm{I},$$

on the cubes, which are also used in [**?**] to determine a model structure with such fibrations. In [AGH21] it is shown that the fibrations of *op.cit.* agree with those specified in the "logical style" of [CCHM16, OP17]. Note that we do *not* assume connections in the category $\square$ of Cartesian cubical sets.

3.1 Partial box filling (biased version)

The *generating biased trivial cofibrations* are all maps of the form

$$c \otimes \delta_\epsilon : D \rightarrowtail Z \times \mathrm{I}\,, \tag{26}$$

where:

1. $c : C \rightarrowtail Z$ is an arbitrary cofibration,

2. $\delta_\epsilon : 1 \to \mathrm{I}$ is one of the two *endpoint inclusions*, for $\epsilon = 0, 1$.

3. $c \otimes \delta_\epsilon$ is the *pushout-product* indicated in the following diagram.

$$\begin{tikzcd}
C \times 1 \arrow[r, "C\times\delta_\epsilon"] \arrow[d, "C\times 1"'] & C \times \mathrm{I} \arrow[d] \arrow[ddr, bend left, "c\times\mathrm{I}"] & \\
Z \times 1 \arrow[r] \arrow[drr, bend right, "Z\times\delta_\epsilon"'] & Z +_C (C \times \mathrm{I}) \arrow[dr, dotted, "c\otimes\delta_\epsilon"'] & \\
 & & Z \times \mathrm{I}
\end{tikzcd} \tag{27}$$

4. $D = Z +_C (C \times \mathrm{I})$ is the indicated domain of the map $c \otimes \delta_\epsilon$.

In order to ensure that such maps are indeed cofibrations, we assume two further axioms in addition to (C1)–(C4) from Definition 6:

(C5) The endpoint inclusions $\delta_\epsilon : 1 \to \mathrm{I}$ are cofibrations, for $\epsilon = 0, 1$.

(C6) The cofibrations are closed under joins $A \vee B \rightarrowtail C$ of subobjects $A, B \rightarrowtail C$ of any object C.

Remark 17. Note that since $\delta_0 : 1 \to \mathrm{I}$ and $\delta_1 : 1 \to \mathrm{I}$ are disjoint, by (C5) and stability under pullbacks we have that $0 \to 1$ is a cofibration, so by stability again $0 \to A$ is always a cofibration. Thus (C0) is no longer required. From (C6) it follows that cofibrations are closed under pushout-products $a \otimes b$ in the arrow category. It also then follows from (C5) that the boundary $\partial : 1 + 1 \to \mathrm{I}$ is a cofibration.

Fibrations (biased version). Now let

$$\mathcal{C} \otimes \delta_\epsilon \;=\; \{c \otimes \delta_\epsilon : D \rightarrowtail Z \times \mathrm{I} \mid c \in \mathcal{C},\ \epsilon = 0, 1\}$$

be the class of all generating biased trivial cofibrations. The *biased fibrations* are defined to be the right class of these maps,

$$(\mathcal{C} \otimes \delta_\epsilon)^{\pitchfork} \;=\; \mathcal{F}\,.$$

Thus a map $f : Y \to X$ is a biased fibration just if for every commutative square of the form

$$\begin{tikzcd}
Z +_C (C \times \mathrm{I}) \arrow[r] \arrow[d, tail, "c \otimes \delta_\epsilon"'] & Y \arrow[d, "f"] \\
Z \times \mathrm{I} \arrow[r] \arrow[ur, dotted, "j"'] & X
\end{tikzcd} \tag{28}$$

with a generating biased trivial cofibration on the left, there is a diagonal filler j as indicated.

To relate this notion of fibration to the cofibration weak factorization system, fix any map $u : A \to B$, and recall (*e.g.* from [**?**, **?**]) that the pushout-product with u is a functor on the arrow category

$$(-)\otimes u : \mathsf{cSet}^{\mathbb{2}} \to \mathsf{cSet}^{\mathbb{2}}\,.$$

This functor has a right adjoint, the *pullback-hom*, which for a map $f : X \to Y$ we shall write as

$$(u \Rightarrow f) : Y^B \longrightarrow (X^B \times_{X^A} Y^A)\,.$$

The pullback-hom is determined as indicated in the following diagram.

$$\begin{tikzcd}
Y^B \arrow[dr, dotted, "u \Rightarrow f"] \arrow[drr, bend left, "Y^u"] \arrow[ddr, bend right, "f^B"'] & & \\
& X^B \times_{X^A} Y^A \arrow[r] \arrow[d] & Y^A \arrow[d, "f^A"] \\
& X^B \arrow[r, "X^u"'] & X^A
\end{tikzcd} \tag{29}$$

The $\otimes \dashv \Rightarrow$ adjunction on the arrow category has the following useful relation to weak factorization systems (cf. [GS17, Rie14, **?**]), where, as usual, for any maps $a : A \to B$ and $f : X \to Y$ we write

$$a \pitchfork f$$

to mean that for every solid square of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, "a"'] & X \arrow[d, "f"] \\ B \arrow[r] \arrow[ur, dotted, "j"'] & Y \end{tikzcd} \tag{30}$$

there exists a diagonal filler j as indicated.

Lemma 18. *For any maps $a : A_0 \to A_1, b : B_0 \to B_1, c : C_0 \to C_1$ in cSet,*

$$(a \otimes b) \pitchfork c \quad \textit{iff} \quad a \pitchfork (b \Rightarrow c)\,.$$

The following is now a direct corollary.

Proposition 19. *An object X is fibrant if and only if both of the endpoint projections $X^{\mathrm{I}} \to X$ from the pathspace are trivial fibrations. More generally, a map $f : Y \to X$ is a fibration iff both of the maps*

$$(\delta_\epsilon \Rightarrow f) : Y^I \to X^I \times_X Y$$

are trivial fibrations (for $\epsilon = 0, 1$).

Fibration structure (biased version). The $\otimes \dashv \Rightarrow$ adjunction determines the fibrations in terms of the trivial fibrations, which in turn can be determined by *uniform* lifting against a *small category* consisting of basic cofibrations and pullback squares between them, by proposition 14. The fibrations are similarly determined by *uniform* lifting against the *small category* of basic, biased trivial cofibrations, consisting of all those $c \otimes \delta_\epsilon$ in $\mathcal{C} \otimes \delta_\epsilon$ where $c : C \rightarrowtail \mathrm{I}^n$ is a *basic* cofibration, *i.e.* one with representable codomain. Thus the set of *basic biased trivial cofibrations* is

$$\mathsf{BCof} \otimes \delta_\epsilon = \{c \otimes \delta_\epsilon : B \rightarrowtail \mathrm{I}^{n+1} \mid c : C \rightarrowtail \mathrm{I}^n,\ \epsilon = 0, 1,\ n \geq 0\}, \tag{31}$$

where the pushout-product $c \otimes \delta_\epsilon$ now takes the simpler form

$$\begin{tikzcd} C \arrow[r] \arrow[d, tail] & C \times \mathrm{I} \arrow[d] \arrow[ddr, bend left] & \\ \mathrm{I}^n \arrow[r] \arrow[drr, bend right] & \mathrm{I}^n +_C (C \times \mathrm{I}) \arrow[dr, tail, "c \otimes \delta_\epsilon"'] & \\ & & \mathrm{I}^n \times \mathrm{I} \end{tikzcd} \tag{32}$$

for a basic cofibration $c : C \rightarrowtail \mathrm{I}^n$, an endpoint $\delta_\epsilon : 1 \to \mathrm{I}$, and with domain $B = \big(\mathrm{I}^n +_C (C \times \mathrm{I})\big)$. These subobjects $B \rightarrowtail \mathrm{I}^{n+1}$ can be seen geometrically as generalized open box inclusions.

For any map $f : Y \to X$ a *uniform, biased fibration structure* on f is a choice of diagonal fillers $j_\epsilon(c, x, y)$,

$$\begin{tikzcd} \mathrm{I}^n +_C (C \times \mathrm{I}) \arrow[r, "x"] \arrow[d, "c \otimes \delta_\epsilon"'] & X \arrow[d, "f"] \\ \mathrm{I}^n \times \mathrm{I} \arrow[r, "y"'] \arrow[ur, dotted, "j_\epsilon(c,x,y)"'] & Y, \end{tikzcd} \tag{33}$$

for each basic biased trivial cofibration $c \otimes \delta_\epsilon : B = (\mathrm{I}^n +_C (C \times \mathrm{I})) \rightarrowtail \mathrm{I}^{n+1}$ and maps $x : B \to X$ and $y : \mathrm{I}^{n+1} \to Y$, which is *uniform in* I^n in the following sense: Given any cubical map $u : \mathrm{I}^m \to \mathrm{I}^n$, the pullback $u^*c : u^*C \rightarrowtail \mathrm{I}^m$ of $c : C \rightarrowtail \mathrm{I}^n$ along u determines another basic biased trivial cofibration

$$u^*c \otimes \delta_\epsilon : B' = (\mathrm{I}^m +_{u^*C} (u^*C \times \mathrm{I})) \rightarrowtail \mathrm{I}^{m+1},$$

which fits into a commutative diagram of the form

$$\begin{tikzcd} \mathrm{I}^m +_{u^*C} (u^*C \times \mathrm{I}) \arrow[r, "(u\times \mathrm{I})'"] \arrow[d, "u^*c \otimes \delta_\epsilon"'] & \mathrm{I}^n +_C (C \times \mathrm{I}) \arrow[r, "x"] \arrow[d, "c \otimes \delta_\epsilon"'] & X \arrow[d, "f"] \\ \mathrm{I}^m \times \mathrm{I} \arrow[r, "u \times \mathrm{I}"'] & \mathrm{I}^n \times \mathrm{I} \arrow[r, "y"'] \arrow[ur, dotted, "j_\epsilon(c,x,y)"'] & Y, \end{tikzcd} \tag{34}$$

by applying the functor $(-) \otimes \delta_\epsilon$ to the pullback square relating u^*c to c. For the outer rectangle in (34) there is then a chosen diagonal filler

$$j_\epsilon(u^*c, x \circ (u \times \mathrm{I})', y \circ (u \times \mathrm{I})) : \mathrm{I}^m \times \mathrm{I} \to X\,,$$

and for this map we require that

$$j_\epsilon(u^*c, x \circ (u \times \mathrm{I})', y \circ (u \times \mathrm{I})) = j_\epsilon(c, x, y) \circ (u \times \mathrm{I}). \tag{35}$$

This can be seen to be a reformulation of the logical specification given in [CCHM16] (see [AGH21]).

Definition 20. A *uniform, biased fibration structure* on a map $f : Y \to X$ is a choice of fillers $j_\epsilon(c, x, y)$ as in (33) satisfying (35) for all maps $u : \mathrm{I}^m \to \mathrm{I}^n$.

Finally, we have the analogue of proposition 12 for fibrant objects. The analogous statement of proposition 14 for fibrations is omitted, as is the entirely analogous proof.

Corollary 21. *For any object X in* cSet *the following are equivalent:*

1. *X is* biased fibrant, *in the sense that every map $D \to X$ from the domain of a generating biased trivial cofibration $D \rightarrowtail Z \times \mathrm{I}$ extends to a total map $Z \times \mathrm{I} \to X$,*

$$\mathcal{C} \otimes \delta_\epsilon \ \pitchfork \ X\,.$$

2. *The canonical maps $(\delta_\epsilon \Rightarrow X) : X^I \to X$ are trivial fibrations.*

3. *$X \to 1$ admits a* uniform biased fibration structure*. Explicitly, for each basic biased trivial cofibration $c \otimes \delta_\epsilon : B \rightarrowtail \mathrm{I}^{n+1}$ and map $x : B \to X$, there is given an extension $j_\epsilon(c, x)$,*

$$\begin{tikzcd} B \arrow[r, "x"] \arrow[d, tail, "c\otimes\delta_\epsilon"'] & X, \\ \mathrm{I}^{n+1} \arrow[ur, dotted, "j_\epsilon(c,x)"'] & \end{tikzcd} \tag{36}$$

and, moreover, the choice is uniform in I^n *in the following sense: Given any cubical map $u : \mathrm{I}^m \to \mathrm{I}^n$, the pullback $u^*c \otimes \delta_\epsilon : B' \rightarrowtail \mathrm{I}^m \times \mathrm{I}$ fits into a commutative diagram of the form*

$$\begin{tikzcd} B' \arrow[r, "(u\times\mathrm{I})'"] \arrow[d, tail, "u^*c\otimes\delta_\epsilon"'] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[r, "x"] \arrow[d, tail, "c\otimes\delta_\epsilon"'] & X. \\ \mathrm{I}^m \times \mathrm{I} \arrow[r, "u\times\mathrm{I}"'] & \mathrm{I}^n \times \mathrm{I} \arrow[ur, dotted, "j(c,x)"'] & \end{tikzcd} \tag{37}$$

For the pair $(u^*c \otimes \delta_\epsilon,\ x \circ (u \times \mathrm{I})')$ in (37) the chosen extension

$$j(u^*c \otimes \delta_\epsilon, x \circ (u \times \mathrm{I})') : \mathrm{I}^m \times \mathrm{I} \to X$$

is equal to $j(c, x) \circ (u \times \mathrm{I})$,

$$j(u^*c \otimes \delta_\epsilon, x \circ (u \times \mathrm{I})') = j(c, x)(u \times \mathrm{I}). \tag{38}$$

3.2 Partial box filling (unbiased version)

Rather than building a weak factorization system based on the foregoing notion of biased fibration (as is done in [GS17]), we shall first eliminate the "bias" with respect to the endpoints $\delta_\epsilon : 1 \to \mathrm{I}$, for $\epsilon = 0, 1$. This will have the effect of adding more trivial cofibrations, and thus more weak equivalences, to our model structure. Consider first the simple path-lifting condition for a map $f : Y \to X$, which is a special case of (28) with $c = ! : 0 \rightarrowtail 1$, so that $! \otimes \delta_\epsilon = \delta_\epsilon$.

$$\begin{tikzcd} 1 \arrow[r] \arrow[d, tail, "\delta_\epsilon"'] & Y \arrow[d, "f"] \\ \mathrm{I} \arrow[r] \arrow[ur, dotted, "j_\epsilon"'] & X \end{tikzcd}$$

In topological spaces, for instance, rather than requiring lifts j_ϵ for each of the endpoints $\epsilon = 0, 1$ of the real interval $\mathrm{I} = [0, 1]$, one could equivalently require there to be a lift j_i *for each point* $i : 1 \to \mathrm{I}$. Such "unbiased path-lifting" can be formulated in cSet by introducing a "generic point" $\delta : 1 \to \mathrm{I}$ by passing to $\mathsf{cSet}/_\mathrm{I}$ via the pullback functor $\mathrm{I}^* : \mathsf{cSet} \to \mathsf{cSet}/_\mathrm{I}$, and then requiring path-lifting for I^*f with respect to $\delta : \mathrm{I} \to \mathrm{I} \times \mathrm{I}$, regarded as a map $\delta : 1 \to \mathrm{I}^*\mathrm{I}$ in $\mathsf{cSet}/_\mathrm{I}$. We shall therefore define f to be an unbiased fibration just if I^*f is a δ-biased fibration for the generic point δ. The following specification implements that idea, while also adding cofibrant partiality, as in the biased case.

We first replace axiom (C5) with the following stronger assumption.

(C7) The diagonal map $\delta : \mathrm{I} \to \mathrm{I} \times \mathrm{I}$ of the interval I is a cofibration.

The unbiased notion of a fibration for cSet is now as follows.

Definition 22 (unbiased fibration)**.** Let $\delta : \mathrm{I} \to \mathrm{I} \times \mathrm{I}$ be the diagonal map.

1. An object X is *unbiased fibrant* if the map

$$(\delta \Rightarrow X) = \langle \mathsf{eval}, p_2 \rangle : X^\mathrm{I} \times \mathrm{I} \to X \times \mathrm{I}$$

   is a trivial fibration.

2. A map $f : Y \to X$ is an *unbiased fibration* if the map

$$(\delta \Rightarrow f) = \langle f^\mathrm{I} \times \mathrm{I}, \langle \mathsf{eval}, p_2 \rangle \rangle : Y^\mathrm{I} \times \mathrm{I} \to (X^\mathrm{I} \times \mathrm{I}) \times_{(X \times \mathrm{I})} (Y \times \mathrm{I})$$

   is a trivial fibration.

Let us (temporarily) write $\mathbb{I} = \mathrm{I}^*\mathrm{I}$ for the pulled-back interval in the slice category $\mathsf{cSet}/_{\mathrm{I}}$, so that the generic point is written $\delta : 1 \to \mathbb{I}$. Condition (1) above (which of course is a special case of (2)) then says that evaluation at the generic point $\delta : 1 \to \mathbb{I}$, the map $(\mathrm{I}^*X)^\delta : (\mathrm{I}^*X)^{\mathbb{I}} \to \mathrm{I}^*X$, constructed in the slice category $\mathsf{cSet}/_{\mathrm{I}}$, is a trivial fibration. Condition (2) says that the pullback-hom of the generic point $\delta : 1 \to \mathbb{I}$ with I^*f, constructed in the slice category $\mathsf{cSet}/_{\mathrm{I}}$, is a trivial fibration. Thus a map $f : Y \to X$ is an *unbiased* fibration just if its base change I^*f is a δ-*biased* fibration in the slice category $\mathsf{cSet}/_{\mathrm{I}}$. The latter condition can also be reformulated as follows.

Proposition 23. *A map $f : Y \to X$ is an unbiased fibration if and only if the canonical map u to the pullback, in the following diagram in* cSet*, is a trivial fibration.*

$$\begin{tikzcd} Y^{\mathrm{I}} \times \mathrm{I} \arrow[dr, dotted, "u"] \arrow[drr, bend left, "\mathsf{eval}"] \arrow[ddr, bend right, "f^{\mathrm{I}}\times\mathrm{I}"'] & & \\ & Y_{\mathsf{eval}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "f"] \\ & X^{\mathrm{I}} \times \mathrm{I} \arrow[r, "\mathsf{eval}"'] & X. \end{tikzcd} \tag{39}$$

Proof. We interpolate another pullback into the rectangle in (41) to obtain

$$\begin{tikzcd} Y_{\mathsf{eval}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \times \mathrm{I} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "f"] \\ X^{\mathrm{I}} \times \mathrm{I} \arrow[r] & X \times \mathrm{I} \arrow[r] & X \end{tikzcd} \tag{40}$$

with the evident maps. The left hand square is therefore a pullback, so we indeed have that

$$Y_{\mathsf{eval}} \;\cong\; (X^{\mathrm{I}} \times \mathrm{I}) \times_{(X\times\mathrm{I})} (Y \times \mathrm{I}) \cong\; (X^{\mathrm{I}} \times \mathrm{I}) \times_X Y$$

and $u = (\delta \Rightarrow f)$. $\square$

As a special case, we have:

Corollary 24. *An object X is unbiased fibrant if and only if the canonical map u to the pullback, in the following diagram in* cSet*, is a trivial fibration.*

$$
\begin{tikzcd}
X^{\mathrm{I}} \times \mathrm{I} \arrow[dr, dotted, "u"] \arrow[drr, bend left, "\mathsf{eval}"] \arrow[ddr, bend right, "p_2"'] & & \\
& \mathrm{I} \times X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\
& \mathrm{I} \arrow[r] & 1.
\end{tikzcd}
\tag{41}
$$

Now we can run the proof of Proposition 19 backwards in order to determine a class of generating trivial cofibrations for the unbiased case. Consider pairs of maps $c : C \rightarrowtail Z$ and $i : Z \to \mathrm{I}$, where the former is a cofibration and the latter is regarded as an "I-indexing", so that

$$
\begin{tikzcd}
C \arrow[r, tail, "c"] \arrow[dr] & Z \arrow[d, "i"] \\
& \mathrm{I}
\end{tikzcd}
$$

is regarded as an "I-indexed family of cofibrations $c_i : C_i \rightarrowtail Z_i$". We shall use the notation

$$\langle i \rangle := \langle 1_Z, i \rangle : Z \longrightarrow Z \times \mathrm{I}\,, \tag{42}$$

for the graph of the indexing map $i : Z \to \mathrm{I}$. Then write

$$c \otimes_i \delta := [\langle i \rangle, c \times \mathrm{I}] : Z +_C (C \times \mathrm{I}) \longrightarrow Z \times \mathrm{I}\,,$$

which is easily seen to be well-defined on the indicated pushout below.

$$
\begin{tikzcd}
C \arrow[r, "\langle ic \rangle"] \arrow[d, tail, "c"'] & C \times \mathrm{I} \arrow[d] \arrow[ddr, bend left, "c \times \mathrm{I}"] & \\
Z \arrow[r] \arrow[drr, bend right, "\langle i \rangle"'] & Z +_C (C \times \mathrm{I}) \arrow[dr, dotted, "c \otimes_i \delta"] & \\
& & Z \times \mathrm{I}\,.
\end{tikzcd}
\tag{43}
$$

Remark 25. The specification (43) differs from the similar (27) by using the graph $\langle i\rangle : Z \rightarrowtail Z \times \mathrm{I}$ for the inclusion of Z into the cylinder over Z, rather than one of the two "ends",

$$\langle 1_Z, \delta_\epsilon ! \rangle : Z \cong Z \times 1 \xrightarrow{Z\times\delta_\epsilon} Z \times \mathrm{I} \tag{44}$$

arising from the endpoint inclusions $\delta_\epsilon : 1 \to \mathrm{I}$, for $\epsilon = 0, 1$. Note that as an arrow over I, the graph $\langle i\rangle : Z \to Z \times \mathrm{I}$ also takes the form (44), namely

$$\langle i\rangle = \langle 1_Z, \delta ! \rangle : Z \to Z \times \mathrm{I}\,.$$

If we also regard $c : C \to Z$ as an arrow over I via $i : Z \to \mathrm{I}$, and use the generic point $\delta : 1 \to \mathbb{I}$ over I in place of $\delta_\epsilon : 1 \to \mathrm{I}$, then (43) agrees with (27), up to those changes. Thus the indicated map $c \otimes_i \delta$ in (43) *is the pushout-product constructed over* I of the generic point δ with the map c regarded as an I-indexed family of cofibrations via the indexing $i : Z \to \mathrm{I}$.

Observe that for any map $f : X \to Y$, the graph $\langle f\rangle = \langle 1_X, f\rangle : X \to X \times Y$ is a cofibration, as a pullback of the diagonal of Y along $f \times Y$. The subobject

$$c \otimes_i \delta \rightarrowtail Z \times \mathrm{I}$$

constructed in (43) is then the join, in the lattice $\mathsf{Sub}(Z\times\mathrm{I})$, of the (cofibrant) subobjects $\langle i\rangle \rightarrowtail Z \times \mathrm{I}$ and $C \times \mathrm{I} \rightarrowtail Z \times \mathrm{I}$, where the latter is the "cylinder over $C \rightarrowtail Z$".

Definition 26. The maps of the form $c \otimes_i \delta : Z +_C (C \times \mathrm{I}) \rightarrowtail Z \times \mathrm{I}$ now form the class of *generating unbiased trivial cofibrations*,

$$\mathcal{C} \otimes \delta \;=\; \{c \otimes_i \delta : D \rightarrowtail Z \times \mathrm{I} \mid c : C \rightarrowtail Z, i : Z \to \mathrm{I}\}\,. \tag{45}$$

The unbiased fibrations are exactly the right class of these maps,

$$(\mathcal{C} \otimes \delta)^{\pitchfork} = \mathcal{F}.$$

Proposition 27. *A map $f : Y \to X$ is an unbiased fibration iff for every pair of maps $c : C \rightarrowtail Z$ and $i : Z \to \mathrm{I}$, where the former is a cofibration, every commutative square of the following form has a diagonal filler, as indicated in the following.*

$$\begin{tikzcd} Z +_C (C \times \mathrm{I}) \arrow[r] \arrow[d, tail, "c\otimes_i \delta"'] & Y \arrow[d, "f"] \\ Z \times \mathrm{I} \arrow[r] \arrow[ur, dotted, "j"] & X. \end{tikzcd} \tag{46}$$

Proof. Suppose that for all $c : C \rightarrowtail Z$ and $i : Z \to \mathrm{I}$, we have $(c \otimes_i \delta) \pitchfork f$ in cSet. Pulling f back over I, this is equivalent to the condition $c \otimes \delta \pitchfork \mathrm{I}^* f$ in $\mathsf{cSet}/_{\mathrm{I}}$, for all cofibrations $c : C \rightarrowtail Z$ over I, which is equivalent to $c \pitchfork (\delta \Rightarrow \mathrm{I}^* f)$ in $\mathsf{cSet}/_{\mathrm{I}}$ for all cofibrations $c : C \rightarrowtail Z$. But this in turn means that $\delta \Rightarrow \mathrm{I}^* f$ is a trivial fibration, which by definition means that f is an unbiased fibration. □

Remark 28. Note that the endpoints $\delta_\epsilon : 1 \to \mathrm{I}$, in particular, are of the form $c \otimes_i \delta$ by taking $Z = 1$ and $i = \delta_\epsilon$ and $c = \, ! : 0 \to 1$, so that the case of biased filling is subsumed. Moreover, for any $i : Z \to \mathrm{I}$ the graph $\langle i \rangle \rightarrowtail Z \times \mathrm{I}$ is itself of the form $0 \otimes_i \delta$ for the cofibration $0 \rightarrowtail Z$, so the graph of any "I-indexing" map $i : Z \to \mathrm{I}$ is also a trivial cofibration.

The following sanity check will be needed later.

Proposition 29. *Let $f : F \twoheadrightarrow X$ be an unbiased fibration in* cSet*. Then for the endpoints $\delta_0, \delta_1 : 1 \to \mathrm{I}$, the associated pullback-homs,*

$$\delta_\epsilon \Rightarrow f : F^{\mathrm{I}} \to X^{\mathrm{I}} \times_X F \qquad (\epsilon = 0, 1) \tag{47}$$

are also trivial fibrations. Thus unbiased fibrations are also δ_ϵ-biased fibrations, for $\epsilon = 0, 1$.

Proof. This follows from Remark 28 and the $\otimes \dashv \Rightarrow$ adjunction, but we give a different proof. Consider the case $X = 1$, the general one $f : F \to X$ being analogous. Thus let F be an unbiased fibrant object in cSet. So by definition $(\mathrm{I}^*F)^\delta : (\mathrm{I}^*F)^{\mathbb{I}} \longrightarrow \mathrm{I}^*F$ in $\mathsf{cSet}/_{\mathrm{I}}$ is a trivial fibration. Pulling back $\delta : 1 \to \mathbb{I}$ in $\mathsf{cSet}/_{\mathrm{I}}$ along the base change $\delta_\epsilon : 1 \to \mathrm{I}$ takes it to $\delta_\epsilon : 1 \to \mathrm{I}$ in cSet, by the universal property of the generic point $\delta : 1 \to \mathbb{I}$; that is $\delta_\epsilon^*(\delta) = \delta_\epsilon : 1 \to \mathrm{I}$. So $(\mathrm{I}^*F)^\delta : (\mathrm{I}^*F)^{\mathbb{I}} \longrightarrow \mathrm{I}^*F$ is taken by δ_ϵ^* to

$$\delta_\epsilon^*\big((\mathrm{I}^*F)^\delta\big) = (\delta_\epsilon^*\mathrm{I}^*F)^{\delta_\epsilon^*\delta} = F^{\delta_\epsilon} : F^{\mathrm{I}} \longrightarrow F\,,$$

as shown in the following.

$$\begin{tikzcd}
F^{\mathrm{I}} \arrow[r] \arrow[d, "F^{\delta_\epsilon}"'] \arrow[dr, phantom, "\lrcorner", very near start] & (\mathrm{I}^*F)^{\mathbb{I}} \arrow[d, "(\mathrm{I}^*F)^\delta"] & \\
F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{I}^*F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F \arrow[d] \\
1 \arrow[r, "\delta_\epsilon"'] & \mathrm{I} \arrow[r] & 1
\end{tikzcd} \tag{48}$$

And pullback preserves trivial fibrations. □

Unbiased fibration structure. As in the biased case, the fibrations can be determined by *uniform* right-lifting against a *small category* of unbiased trivial cofibrations, now consisting of all those $c \otimes_i \delta$ in $\mathcal{C} \otimes \delta$ for which $c : C \rightarrowtail \mathrm{I}^n$ is basic, *i.e.* has representable codomain. Call these maps the *basic unbiased trivial cofibrations*, and let

$$\mathsf{BCof} \otimes \delta = \{c \otimes_i \delta : B \rightarrowtail \mathrm{I}^{n+1} \mid c : C \rightarrowtail \mathrm{I}^n,\, i : \mathrm{I}^n \to \mathrm{I},\, n \geq 0\}\,, \tag{49}$$

where the pushout-product $c \otimes_i \delta$ now has the form

$$\begin{tikzcd}
C \arrow[r, "\langle ic \rangle"] \arrow[d, tail, "c"'] & C \times \mathrm{I} \arrow[d] \arrow[ddr, bend left, "c \times \mathrm{I}"] & \\
\mathrm{I}^n \arrow[r] \arrow[drr, bend right, "\langle i \rangle"'] & \mathrm{I}^n +_C (C \times \mathrm{I}) \arrow[dr, dotted, tail, "c \otimes_i \delta"] & \\
& & \mathrm{I}^n \times \mathrm{I}
\end{tikzcd} \tag{50}$$

for a basic cofibration $c : C \rightarrowtail \mathrm{I}^n$ and an indexing map $i : \mathrm{I}^n \to \mathrm{I}$, and with domain $B = \big(\mathrm{I}^n +_C (C \times \mathrm{I})\big)$. These subobjects $B \rightarrowtail \mathrm{I}^{n+1}$ can again be seen geometrically as "generalized open box inclusions", but now the floor and lid of the open box are generalized to the graph of an arbitrary map $i : \mathrm{I}^n \to \mathrm{I}$.

For any map $f : Y \to X$ a uniform, unbiased fibration structure on f is then a choice of diagonal fillers $j(c, i, x, y)$,

$$\begin{tikzcd}
B \arrow[r, "x"] \arrow[d, tail, "c \otimes_i \delta"'] & X \arrow[d, "f"] \\
\mathrm{I}^n \times \mathrm{I} \arrow[r, "y"'] \arrow[ur, dotted, "j(c,i,x,y)"'] & Y,
\end{tikzcd} \tag{51}$$

for each basic trivial cofibration $c \otimes_i \delta : B \rightarrowtail \mathrm{I}^{n+1}$, which is *uniform in* I^n in the following sense: Given any cubical map $u : \mathrm{I}^m \to \mathrm{I}^n$, the pullback $u^*c : u^*C \rightarrowtail \mathrm{I}^m$ and the reindexing $iu : \mathrm{I}^m \to \mathrm{I}^n \to \mathrm{I}$ determine another basic trivial cofibration $u^*c \otimes_{iu} \delta : B' = (\mathrm{I}^m +_{u^*C} (u^*C \times \mathrm{I})) \rightarrowtail \mathrm{I}^{m+1}$, which fits into a commutative diagram of the form

$$\begin{tikzcd}
B' \arrow[r, "(u \times \mathrm{I})'"] \arrow[d, tail, "u^*c \otimes_{iu} \delta"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[r, "x"] \arrow[d, tail, "c \otimes_i \delta"'] & X \arrow[d, "f"] \\
\mathrm{I}^m \times \mathrm{I} \arrow[r, "u \times \mathrm{I}"'] & \mathrm{I}^n \times \mathrm{I} \arrow[r, "y"'] \arrow[ur, dotted, "j(c,i,x,y)"'] & Y.
\end{tikzcd} \tag{52}$$

For the outer rectangle in (52) there is a chosen diagonal filler

$$j(u^*c, iu, x(u \times \mathrm{I})', y(u \times \mathrm{I})) : \mathrm{I}^m \times \mathrm{I} \to X,$$

and for this map we require that

$$j(u^*c, iu, x(u \times \mathrm{I})', y(u \times \mathrm{I})) = j(c, i, x, y) \circ (u \times \mathrm{I}). \tag{53}$$

Definition 30. A *uniform, unbiased fibration structure* on a map

$$f : Y \to X$$

is a choice of fillers $j(c, i, x, y)$ as in (51) satisfying (53) for all $u : \mathrm{I}^m \to \mathrm{I}^n$.

In these terms, we have the following analogue of corollary 21.

Proposition 31. *For any object X in cSet the following are equivalent:*

1. *X is an unbiased fibrant object in the sense of Definition 22: the canonical map $\delta \Rightarrow X : X^{\mathrm{I}} \times \mathrm{I} \to X \times \mathrm{I}$ is a trivial fibration.*
2. *X has the right lifting property with respect to all generating unbiased trivial cofibrations,*
$$(\mathcal{C} \otimes \delta) \pitchfork X.$$
3. *X has a uniform, unbiased fibration structure in the sense of Definition 30.*

Proof. The equivalence between (1) and (2) is proposition 27. So assume (1). Then in $\mathsf{cSet}/_{\mathrm{I}}$, the evaluation at $\delta : 1 \to \mathbb{I}$,

$$(\mathrm{I}^*X)^{\delta} : (\mathrm{I}^*X)^{\mathbb{I}} \longrightarrow X$$

is a trivial fibration. By Proposition 14 it therefore has a uniform filling structure with respect to all basic cofibrations $c : C \rightarrowtail \mathrm{I}^n$ over I. Transposing by the $\otimes \dashv \Rightarrow$ adjunction and unwinding then gives exactly a uniform fibration structure on X. □

A statement analogous to the foregoing also holds for maps $f : Y \to X$ in place of objects X. Indeed, as before, we have the following sharper formulation.

Corollary 32. *Fibration structures on a map $f : Y \to X$ correspond uniquely to relative $+$-algebra structures on the map $(\delta \Rightarrow f)$ (cf. definition 22),*

$$(\delta \Rightarrow f) : Y^I \times \mathrm{I} \longrightarrow (X^I \times \mathrm{I}) \times_{(X \times \mathrm{I})} (Y \times \mathrm{I})\,.$$

3.3 Factorization

Definition 33. Summarizing the foregoing definitions, we have the following classes of maps:

- The *generating unbiased trivial cofibrations* were determined in (45) as

$$\mathcal{C} \otimes \delta = \{c \otimes_i \delta : D \rightarrowtail Z \times \mathrm{I} \mid c : C \rightarrowtail Z,\ i : Z \to \mathrm{I}\}\,, \tag{54}$$

  where $D = \big(Z +_C (C \times \mathrm{I})\big)$ and the pushout-product $c \otimes_i \delta$ has the form

$$\begin{tikzcd}
C \arrow[r, "\langle ic \rangle"] \arrow[d, tail, "c"'] & C \times \mathrm{I} \arrow[d] \arrow[ddr, bend left, "c\times\mathrm{I}"] & \\
Z \arrow[r] \arrow[drr, bend right, "\langle ic \rangle"'] & Z +_C (C \times \mathrm{I}) \arrow[dr, dotted, "c \otimes_i \delta"'] & \\
 & & Z \times \mathrm{I}
\end{tikzcd} \tag{55}$$

  for any cofibration $c : C \rightarrowtail Z$ and indexing map $i : Z \to \mathrm{I}$.

- The class $\mathcal{F}$ of *unbiased fibrations*, which can be characterized as the right-lifting class of the generating unbiased trivial cofibrations,

$$(\mathcal{C} \otimes \delta)^{\pitchfork} \ = \ \mathcal{F}.$$

- The class of *unbiased trivial cofibrations* is then defined to be left-lifting class of the fibrations,

$$\mathsf{TCof} \ = \ {}^{\pitchfork}\mathcal{F}.$$

It follows that the classes TCof and $\mathcal{F}$ are closed under retracts and are mutually weakly orthogonal, $\mathsf{TCof} \pitchfork \mathcal{F}$. Thus in order to have a weak factorization system $(\mathsf{TCof}, \mathcal{F})$ it just remains to show the following.

Lemma 34. *Every map $f : X \to Y$ in cSet can be factored as $f = p \circ i$,*

$$\begin{tikzcd}
X \arrow[r, tail, "i"] \arrow[dr, "f"'] & X' \arrow[d, two heads, "p"] \\
 & Y
\end{tikzcd} \tag{56}$$

with $i : X \rightarrowtail X'$ an unbiased trivial cofibration and $p : X' \twoheadrightarrow Y$ an unbiased fibration.

Proof. We can use a standard argument (the "algebraic small objects argument", cf. [GKR18]), which can be further simplified using the fact that the codomains of the basic trivial cofibrations $c \otimes_i \delta : B \rightarrowtail \mathrm{I}^{n+1}$ are not just representable, but *tiny* in the sense of Proposition 3, and the domains are not merely "small", but *finitely presented*. The reader is referred to [**?**] for details in a similar case. $\square$

Remark 35. The proof in *ibid.* actually produces a stronger result than we need, namely an *algebraic* weak factorization system. This follows from the small generating *category* $\mathcal{C} \otimes \delta$ of basic unbiased trivial cofibrations (and pullback squares of the form on the left in (52)). The relationship between this stronger condition and the *classifying types* used in Section 6 is studied in [**?**], which also gives an even more "constructive" proof of the factorization Lemma 34, not requiring quotients, exactness, or impredicativity [**?**]. With this modification, the present approach can also be used in a *quasi*topos, as occurs in *e.g.* realizability and sheaves.

Proposition 36. *There is a weak factorization system on the category* cSet *in which the right maps are the unbiased fibrations and the left maps are the unbiased trivial cofibrations, both as specified in definition 33. This will be called the* (unbiased) fibration weak factorization system*.*

Hereafter, unless otherwise stated, all fibrations in cSet are assumed to be unbiased.

4 The weak equivalences

Our approach to proving that the classes $\mathcal{C}$ and $\mathcal{F}$ of cofibrations and fibrations, from Sections 2 and 3, determine a model structure will be to first identify a *premodel structure* in the sense of [**?**], and then turn to the question of the 3-for-2 property for the resulting weak equivalences.

Definition 37 (Weak equivalence)**.** A map $f : X \to Y$ in cSet is a *weak equivalence* if it can be factored as $f = g \circ h$,

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[dr, "f"'] & W \arrow[d, "g"] \\ & Y \end{tikzcd}$$

with $h \pitchfork \mathcal{F}$ and $\mathcal{C} \pitchfork g$. Accordingly, let

$$\begin{aligned}\mathcal{W} &= \mathsf{TFib} \circ \mathsf{TCof} \\ &= \{f : X \to Y \mid f = g \circ h \text{ for some } g \in \mathsf{TFib} \text{ and } h \in \mathsf{TCof}\}\end{aligned}$$

be the class of weak equivalences.

Observe first that every trivial fibration $f \in \mathsf{TFib} = \mathcal{C}^{\pitchfork}$ is indeed a fibration, because the generating trivial cofibrations $c \otimes_i \delta$ are cofibrations. Moreover, every trivial fibration $f : X \to Y$ is also a weak equivalence $f = f \circ 1_X$, since the identity map 1_X is (trivially) a trivial cofibration $\mathsf{TCof} = {}^{\pitchfork}\mathcal{F}$. Thus we have

$$\mathsf{TFib} \subseteq (\mathcal{F} \cap \mathcal{W}).$$

Similarly, because $\mathsf{TFib} \subseteq \mathcal{F}$, we have $\mathsf{TCof} \subseteq \mathcal{C}$. Moreover, since identity maps are also trivial fibrations we have $\mathsf{TCof} \subseteq \mathsf{TFib} \circ \mathsf{TCof} = \mathcal{W}$. Thus we also have

$$\mathsf{TCof} \subseteq (\mathcal{C} \cap \mathcal{W}).$$

Lemma 38. $(\mathcal{C} \cap \mathcal{W}) \subseteq \mathsf{TCof}$.

Proof. Let $c : A \rightarrowtail B$ be a cofibration with a factorization

$$c = tf \circ tc : A \to W \to B$$

where $tc \in \mathsf{TCof}$ and $tf \in \mathsf{TFib}$. Let $f : X \twoheadrightarrow Y$ be a fibration and consider a commutative diagram,

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, tail, "c"'] & X \arrow[d, two heads, "f"] \\ B \arrow[r, "y"'] & Y. \end{tikzcd}$$

Inserting the factorization of c, from $tc \pitchfork f$ we obtain $j : W \to X$ as indicated, with $j \circ tc = x$ and $f \circ j = y \circ tf$.

$$\begin{tikzcd} A \arrow[rr, "x"] \arrow[dd, tail, "c"'] \arrow[dr, "tc"] & & X \arrow[dd, two heads, "f"] \\ & W \arrow[ur, dotted, "j"'] \arrow[dl, "tf"] & \\ B \arrow[rr, "y"'] & & Y. \end{tikzcd}$$

Moreover, since $c \pitchfork tf$ there is an $i : B \to W$ as indicated, with $i \circ c = tc$ and $tf \circ i = 1_B$.

$$\begin{tikzcd}
A \arrow[rr, "x"] \arrow[dd, tail, "c"'] \arrow[dr, "tc"] & & X \arrow[dd, two heads, "f"] \\
& W \arrow[ur, dotted, "j"'] \arrow[dl, "tf"'] & \\
B \arrow[rr, "y"'] \arrow[ur, dotted, bend right, "i"'] & & Y.
\end{tikzcd}$$

Let $k = j \circ i$. Then $k \circ c = j \circ i \circ c = j \circ tc = x$, and $f \circ k = f \circ j \circ i = y \circ tf \circ i = y$. $\square$

The proof of the following is exactly dual.

Lemma 39. $(\mathcal{F} \cap \mathcal{W}) \subseteq \mathsf{TFib}$.

Proposition 40. *The three classes of maps $\mathcal{C}, \mathcal{W}, \mathcal{F}$ in cSet constitute a* premodel structure *in the sense of [**?**]. In particular, we have*

$$\mathcal{F} \cap \mathcal{W} = \mathsf{TFib},$$
$$\mathcal{C} \cap \mathcal{W} = \mathsf{TCof},$$

and therefore two interlocking weak factorization systems:

$$(\mathcal{C},\, \mathcal{W} \cap \mathcal{F}) \quad , \quad (\mathcal{C} \cap \mathcal{W},\, \mathcal{F}).$$

It now "only" remains to show that the weak equivalences $\mathcal{W}$ satisfy the 3-for-2 axiom, in order to verify that $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ is a model structure. Perhaps surprisingly, this will occupy the remainder of these lectures! We shall follow roughly the approach of [**?**]: the weak equivalences between fibrant objects are expected to be the usual *homotopy equivalences*, which evidently satisfy 3-for-2. So we reduce to this case using the fact that K^X is fibrant whenever K is. It should suffice, namely, to show that the weak equivalences $w : X \to Y$ are those maps which induce homotopy equivalences $K^w : K^Y \simeq K^X$ for fibrant K. Such maps are termed *weak homotopy equivalences* (Definition 45), and our task will be therefore to show that a map is a weak equivalence if and only if it is a weak homotopy equivalence.

Weak homotopy equivalence.

Definition 41. A *homotopy* $\vartheta : f \sim g$ between maps $f, g : X \rightrightarrows Y$ is a map,

$$\vartheta : \mathrm{I} \times X \longrightarrow Y,$$

such that $\vartheta \circ \iota_0 = f$ and $\vartheta \circ \iota_1 = g$,

$$\begin{tikzcd} X \arrow[r, "\iota_0"] \arrow[dr, "f"'] & \mathrm{I} \times X \arrow[d, "\vartheta"] & X, \arrow[l, "\iota_1"'] \arrow[dl, "g"] \\ & Y & \end{tikzcd} \tag{57}$$

where ι_0, ι_1 are the canonical inclusions into the ends of the cylinder,

$$\iota_\epsilon : X \cong 1 \times X \xrightarrow{\delta_\epsilon \times X} \mathrm{I} \times X\,, \qquad \epsilon = 0, 1.$$

Note that each of the inclusions $\iota_\epsilon : X \rightarrowtail \mathrm{I} \times X$ is a cofibration, as is their join $X + X \rightarrowtail \mathrm{I} \times X$, by Remark 17.

Proposition 42. *The relation of homotopy $f \sim g$ between maps $f, g : X \rightrightarrows Y$ is preserved by pre- and post-composition. If Y is fibrant, then $f \sim g$ is an equivalence relation.*

Proof. Inspecting (57), preservation of $f \sim g$ under post-composing with any $h : Y \to Z$ is obvious: we have $h \circ \vartheta : h \circ f \sim h \circ g$. Now observe that a homotopy $f \overset{\vartheta}{\sim} g : X \times \mathrm{I} \to Y$ determines a (unique) path $\tilde{\vartheta} : \mathrm{I} \to Y^X$ in the function space, with endpoints $\vartheta_0 = \vartheta \circ \delta_0 = \tilde{f} : 1 \to Y^X$ and $\vartheta_1 = \vartheta \circ \delta_1 = \tilde{g}$. Precomposing maps $f, g : X \rightrightarrows Y$ with any $e : W \to X$ is induced by post-composing $\tilde{f}, \tilde{g} : 1 \to Y^X$ with the map $Y^e : Y^X \to Y^W$, which then also takes the path $\tilde{\vartheta} : \mathrm{I} \to Y^X$ to a path $\tilde{\varphi} = Y^e \circ \tilde{\vartheta} : \mathrm{I} \to Y^W$ corresponding to a (unique) homotopy $\varphi : f \circ e \sim g \circ e$.

Now note that Y^X is fibrant if Y is fibrant, since the generating trivial cofibrations $c \times_i \delta$ are preserved by the functor $X \times (-)$. So we can use "box-filling" in Y^X to verify the claimed equivalence relation.

- Reflexivity $f \sim f$ is witnessed by the homotopy $\rho : \mathrm{I} \to 1 \xrightarrow{f} Y^X$.

- For symmetry $f \sim g \Rightarrow g \sim f$ take $\vartheta : \mathrm{I} \to Y^X$ with $\vartheta_0 = f$ and $\vartheta_1 = g$ and we want to build $\vartheta' : \mathrm{I} \to Y^X$ with $\vartheta'_0 = g$ and $\vartheta'_1 = f$. Take an open 2-box in Y^X of the following form.

$$\begin{tikzcd}
g & f \\
f \arrow[u, "\vartheta"] \arrow[r, "\rho"'] & f \arrow[u, "\rho"']
\end{tikzcd}$$

This box is a map $b : \mathrm{I} +_1 \mathrm{I} +_1 \mathrm{I} \to Y^X$ with the indicated components, and it has a filler $c : \mathrm{I} \times \mathrm{I} \to Y^X$, i.e. an extension along the canonical map $\mathrm{I}+_1\mathrm{I}+_1\mathrm{I} \rightarrowtail \mathrm{I}\times\mathrm{I}$, which is a trivial cofibration of the form $\partial\mathrm{I}\otimes\delta_0$. Let $t : \mathrm{I} \to \mathrm{I} \times \mathrm{I}$ be the top face of the 2-cube (the bipointed map $\{0, x_1, x_2, 1\} \to \{0, x, 1\}$ that is constantly 1). We can set $\vartheta' = c \circ t : \mathrm{I} \to Y^X$ to get a homotopy $\vartheta' : \mathrm{I} \to Y^X$ with $\vartheta'_0 = g$ and $\vartheta'_1 = f$ as required.

- For transitivity, $f \overset{\vartheta}{\sim} g$, $g \overset{\varphi}{\sim} h \Rightarrow f \sim h$, an analogous construction will fill the open box:

$$\begin{tikzcd}
f & h \\
f \arrow[u, "\rho"] \arrow[r, "\vartheta"'] & g \arrow[u, "\varphi"']
\end{tikzcd}$$

$\square$

Definition 43 (Connected components)**.** The functor

$$\pi_0 : \mathsf{cSet} \to \mathsf{Set}$$

is defined on a cubical set X as the coequalizer

$$X_1 \rightrightarrows X_0 \to \pi_0 X,$$

where the two parallel arrows are the maps $X_{\delta_0}, X_{\delta_1} : X_1 \rightrightarrows X_0$ for the endpoints $\delta_0, \delta_1 : 1 \rightrightarrows \mathrm{I}$. If K is fibrant, then by the foregoing Proposition 42, for any X we have

$$\pi_0(K^X) = \mathrm{Hom}(X, K)/\!\sim\ .$$

That is, $\pi_0(K^X)$ is the set $[X, K]$ of homotopy equivalence classes of maps $X \to K$.

Remark 44. One can show that in fact $\pi_0 X = \varinjlim X_n$ where the colimit is taken over *all* objects $[n]$ in the index category $\square^{\mathrm{op}} = \mathbb{B}$, rather than just the "last" two $[1] \rightrightarrows [0]$. Since the category $\mathbb{B}$ of finite strictly bipointed sets is sifted, the functor $\pi_0 : \mathsf{cSet} \to \mathsf{Set}$ preserves finite products.

As usual, a *homotopy equivalence* is a map $f : X \to Y$ together with a map $g : Y \to X$ and homotopies $\vartheta : 1_X \sim g \circ f$ and $\varphi : 1_Y \sim f \circ g$. We call g a *quasi-inverse* of f.

Definition 45 (Weak homotopy equivalence)**.** A map $f : X \to Y$ is called a *weak homotopy equivalence* if for every fibrant object K, the canonical map $K^f : K^Y \to K^X$ is bijective on connected components,

$$\pi_0 K^f : \pi_0 K^Y \cong \pi_0 K^X .$$

Lemma 46. *Every homotopy equivalence is a weak homotopy equivalence.*

Proof. Let $f : X \to Y$ be a homotopy equivalence. Then $K^f : K^Y \to K^X$ is also a homotopy equivalence for any K, since homotopy respects (post-) composition by all maps. If K is fibrant, then so is K^X and π_0 is well defined on homotopy classes of maps, by Proposition 42. It clearly takes homotopy equivalences to isomorphisms of sets, since it identifies homotopic maps. $\square$

Lemma 47. *The weak homotopy equivalences satisfy the 3-for-2 condition.*

Proof. This follows by applying the Set-valued functors $\pi_0 K^{(-)}$, for all fibrant objects K, and the corresponding fact about bijections of sets. $\square$

In virtue of Lemma 47 it will suffice to show that a map is a weak equivalence if and only if it is a weak homotopy equivalence. The following will be useful thereby.

Lemma 48. *A map $f : X \to Y$ is a* weak homotopy equivalence *if and only if it satisfies the following two conditions.*

1. *For every fibrant object K and every map $x : X \to K$ there is a map $y : Y \to K$ such that $y \circ f \sim x$,*

$$\begin{tikzcd} X \arrow[r, "x"] \arrow[d, "f"'] \arrow[dr, phantom, "\sim"] & K. \\ Y \arrow[ur, dotted, "y"'] & \end{tikzcd}$$

   We say that x "extends along f up to homotopy".

2. *For every fibrant object K and maps $y, y' : Y \to K$ such that $yf \sim y'f$, there is a homotopy $y \sim y'$,*

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & K^{\mathrm{I}} \arrow[d] \\
Y \arrow[r, "\langle y, y' \rangle"'] \arrow[ur, dotted] & K \times K.
\end{tikzcd}$$

Proof. Condition (1) says exactly that the internal precomposition map $K^f : K^Y \to K^X$ is surjective under connected components π_0, while (2) says just that it is injective under π_0. $\square$

Lemma 49. 1. *Any trivial fibration is a homotopy equivalence.*

2. *Any weak equivalence is a weak homotopy equivalence.*

Proof. For (1), any trivial fibration $f : X \twoheadrightarrow Y$ has a section $s : Y \to X$ by Corollary 15. Consider the following lifting problem:

$$\begin{tikzcd}
X + X \arrow[r, "{[sf, 1]}"] \arrow[d, tail, "{[\iota_0, \iota_1]}"'] & X \arrow[d, two heads, "f"] \\
\mathrm{I} \times X \arrow[r, "f\pi_2"'] & Y
\end{tikzcd}$$

Since the map on the left is a cofibration, a diagonal filler provides a homotopy $\vartheta : sf \sim 1_X$. Thus f is a homotopy equivalence.

For (2), by (1) and Lemma 46, a trivial fibration is also a weak homotopy equivalence. So it suffices to consider the trivial cofibrations, since weak homotopy equivalences are closed under composition, by Lemma 47. Thus let $f : X \rightarrowtail Y$ be a trivial cofibration, and apply Lemma 48: condition (1) is immediate, and (2) follows because $K^{\mathrm{I}} \twoheadrightarrow K \times K$ is a fibration when K is fibrant, since $\partial : 1 + 1 \rightarrowtail \mathrm{I}$ is a cofibration (by Remark 17). $\square$

To show the converse of Lemma 49(2), that a weak homotopy equivalence is a weak equivalence, we first consider maps $f : X \to K$ with a fibrant codomain K, and we split into the cases of a fibration and a cofibration.

Lemma 50. *If K is fibrant, then any fibration $f : X \twoheadrightarrow K$ that is a homotopy equivalence is a weak equivalence.*

Proof. This is a standard argument, which we just sketch. It suffices to show that any diagram of the form

$$\begin{tikzcd} C \arrow[r, "x"] \arrow[d, tail, "c"'] & X \arrow[d, two heads, "f"] \\ K \arrow[r, "="'] & K, \end{tikzcd} \tag{58}$$

with $c : C \rightarrowtail X$ a cofibration, has a diagonal filler, for then f is a trivial fibration. Since f is a homotopy equivalence, it has a quasi-inverse $s : K \to X$ with $\vartheta : fs \sim 1_K$, which we claim can be corrected to a section $s' : K \to X$. Indeed, consider

$$\begin{tikzcd} K \arrow[r, "s"] \arrow[d, tail, "\iota_0"'] & X \arrow[d, two heads, "f"] \\ K \times \mathrm{I} \arrow[r, "\vartheta"'] \arrow[ur, dotted, "\vartheta'"] & K \\ K \arrow[u, tail, "\iota_1"] \arrow[ur, "="'] & \end{tikzcd}$$

where ϑ' results from $\iota_0 \pitchfork f$. Let $s' = \vartheta' \iota_1$, so that $\vartheta' : s \sim s'$ and $fs' = 1_K$.

Thus we can assume that $s = s' : K \to X$ is a section, which fills the diagram (58) up to a homotopy in the upper triangle.

$$\begin{tikzcd} C \arrow[r, "x"] \arrow[d, tail, "c"'] \arrow[dr, phantom, "\sim" near start] & X \arrow[d, two heads, "f"] \\ K \arrow[r, "="'] \arrow[ur, "s"'] & K \end{tikzcd}$$

Now we can correct $s : K \to X$ to a homotopic $t : K \to X$ over f by using the homotopy $\varphi : sc \sim x$ to get a map $\varphi : C \to X^{\mathrm{I}}$ over f. Since f is a fibration, the projections $p_0, p_1 : X^{\mathrm{I}} \to X$ over f are trivial fibrations, and so there is a lift $\varphi' : K \to X^{\mathrm{I}}$ for which $t := p_1\varphi'$ has $tc = x$ and $ft = 1_K$, and so is a filler for (58). $\square$

Lemma 51. *If K is fibrant, then any fibration $f : X \twoheadrightarrow K$ that is a weak homotopy equivalence is a weak equivalence.*

Proof. Since K is fibrant, so is X, and since f is a weak homotopy equivalence, by lemma 48(1) there is then a map $s : K \to X$ and a homotopy

$\theta : sf \sim 1_X$. Postcomposing with f gives a homotopy $f\vartheta : fsf \sim f$, forming the outer commutative square in

$$\begin{tikzcd} X \arrow[r, "f\vartheta"] \arrow[d, "f"'] & K^{\mathrm{I}} \arrow[d] \\ K \arrow[r, "\langle fs, 1_K \rangle"'] \arrow[ur, dotted, "\varphi"'] & K \times K. \end{tikzcd}$$

By lemma 48(2) there is a diagonal filler $\varphi : fs \sim 1_K$, and so f is a homotopy equivalence. Now apply lemma 50. $\square$

We record the following intermediate step toward our goal.

Corollary 52. *For maps between fibrant objects, all three concepts coincide: weak equivalences, weak homotopy equivalences, and homotopy equivalences.*

Lemma 53. *If K is fibrant, then any cofibration $c : A \rightarrowtail K$ that is a weak homotopy equivalence is a weak equivalence.*

Proof. Let $c : A \rightarrowtail K$ be a cofibration weak homotopy equivalence and factor it into a trivial cofibration $i : A \rightarrowtail Z$ followed by a fibration $p : Z \twoheadrightarrow K$. By lemma 48, any trivial cofibration is clearly a weak homotopy equivalence. So both c and i are weak homotopy equivalence, and therefore so is p by 3-for-2 for weak homotopy equivalences. Since K is fibrant, p is a trivial fibration by lemma 51, and thus c is a weak equivalence. $\square$

It now follows that a weak homotopy equivalence $f : X \to K$ with a fibrant codomain is a weak equivalence. To eliminate the condition on the codomain we shall use the following lemma due to [**?**].

Lemma 54. *A cofibration $c : A \rightarrowtail B$ weak homotopy equivalence lifts against any fibration $f : Y \twoheadrightarrow K$ with fibrant codomain.*

Proof. Let $c : A \rightarrowtail B$ be a cofibration weak homotopy equivalence and $f : Y \twoheadrightarrow K$ a fibration with fibrant codomain K, and consider a lifting problem

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, tail, "c"'] & Y \arrow[d, two heads, "f"] \\ B \arrow[r, "b"'] & K. \end{tikzcd}$$

Let $\eta : B \rightarrowtail B'$ be a fibrant replacement of B, since K is fibrant, b extends along η to give $b' : B' \to K$ as shown below.

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, tail, "c"'] & Y \arrow[d, two heads, "f"] \\
B \arrow[r, "b"'] \arrow[d, "\eta"'] & K \\
B' \arrow[ur, "b'"']
\end{tikzcd}$$

Since η is a trivial cofibration, it is a weak homotopy equivalence. So the composite ηc is also a weak homotopy equivalence. But since B' is fibrant, ηc is then a trivial cofibration by lemma 53. Thus there is a lift $j : B' \to Y$, and therefore also one $k = j\eta : B \to Y$. $\square$

To complete the proof that a weak homotopy equivalence is a weak equivalence, we make use of the following *fibration extension property*, the proof of which is deferred to section 8.

Definition 55 (Fibration extension property)**.** For any fibration $f : Y \twoheadrightarrow X$ and trivial cofibration $\eta : X \to X'$, there is a fibration $f' : Y' \twoheadrightarrow X'$ that pulls back to f along η, as shown below.

$$\begin{tikzcd}
Y \arrow[r] \arrow[d, two heads, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y' \arrow[d, two heads, "f'"] \\
X \arrow[r, tail, "\eta"'] & X'
\end{tikzcd} \tag{59}$$

Lemma 56. *Assuming the fibration extension property, a cofibration that lifts against every fibration $f : Y \twoheadrightarrow K$ with fibrant codomain is a weak equivalence.*

Proof. Let $c : A \rightarrowtail B$ be a cofibration and consider a lifting problem against an arbitrary fibration $f : Y \twoheadrightarrow X$,

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, tail, "c"'] & Y \arrow[d, two heads, "f"] \\
B \arrow[r, "b"'] & X.
\end{tikzcd} \tag{60}$$

Let $\eta : X \to X'$ be a fibrant replacement, so η is a trivial cofibration and X' is fibrant. By the fibration extension property of definition 55, there is a fibration $f' : Y' \twoheadrightarrow X'$ such that f is a pullback of f' along η. So we can extend diagram (60) to obtain the following, in which the righthand square is a pullback.

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, tail, "c"'] & Y \arrow[r, "y"] \arrow[d, two heads, "f"] \arrow[dr, phantom, very near start, "\lrcorner"] & Y' \arrow[d, two heads, "f'"] \\ B \arrow[r, "b"'] & X \arrow[r, "\eta"'] & X'. \end{tikzcd} \tag{61}$$

By assumption, there is a lift $j' : B \to Y'$ with $f'j' = \eta b$ and $j'c = yb$. Therefore, since f is a pullback, there is a map $j : B \to Y$ with $fj = b$ and $yj = j'$.

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, tail, "c"'] & Y \arrow[r, "y"] \arrow[d, two heads, "f"] \arrow[dr, phantom, very near start, "\lrcorner"] & Y' \arrow[d, two heads, "f'"] \\ B \arrow[r, "b"'] \arrow[ur, dotted, "j"] \arrow[urr, dotted, "j'"] & X \arrow[r, "\eta"'] & X'. \end{tikzcd} \tag{62}$$

Thus $yjc = j'c = ya$. But as a trivial cofibration, η is monic, and as a pullback of η, y is also monic. So $jc = a$. $\square$

Corollary 57. *Assuming the fibration extension property,*

1. *a cofibration $c : A \rightarrowtail B$ weak homotopy equivalence is a weak equivalence,*
2. *a fibration $f : Y \twoheadrightarrow X$ weak homotopy equivalence is a weak equivalence.*

Proof. (1) follows immediately by combining the previous lemmas 54 and 56.

For (2), factor $f : Y \twoheadrightarrow X$ into a cofibration $i : Y \rightarrowtail Z$ followed by a trivial fibration $p : Z \twoheadrightarrow X$. Then f is itself a trivial fibration if $i \pitchfork f$, for then it is a retract of p. Since p is a trivial fibration, it is a weak homotopy equivalence by Lemma 49. Since f is also a weak homotopy equivalence, so is i by Lemma 47. Thus i is a trivial cofibration by (1). Since f is a fibration, $i \pitchfork f$ as required. $\square$

We have now shown:

Proposition 58. *Assuming the fibration extension property, a map $f : X \to Y$ is a weak homotopy equivalence if and only if it is a weak equivalence. The weak equivalences $\mathcal{W}$ therefore satisfy the 3-for-2 condition.*

Our results can be summarized as follows.

Theorem 59. *Assume the fibration weak factorization system of Definition 33 satisfies the fibration extension property of Definition 55 (as will be shown in Corollary 97). Then the weak equivalences $\mathcal{W}$ have the 3-for-2 property, and so by Proposition 40, the classes $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ form a Quillen model structure. The weak equivalences $\mathcal{W}$ are the* weak homotopy equivalences*: those maps $f : X \to Y$ for which $K^f : K^Y \to K^X$ is bijective on connected components whenever K is fibrant.*

The proof of the fibration extension property will be given in Section 8. It uses the equivalence extension property (Section 7), a universal fibration (Section 6), and the Frobenius condition (Section 5), to which we now turn.

5 The Frobenius condition

In this section, we show that the (unbiased) fibration weak factorization system from Section 3 satisfies what has been called the *Frobenius condition*: the left maps are stable under pullback along the right maps (see [**?**]). This will imply the *right properness* of our model structure: the weak equivalences are preserved by pullback along fibrations. In the present setting, it then follows that the entire model structure is stable under such a base change. The Frobenius condition will be used in the proof of the equivalence extension property in Section 7.

An proof of Frobenius in the related setting of cubical sets *with connections* was given in [GS17] using conventional, functorial methods. By contrast, the type theoretic approach of [CCHM16] provides a proof that is much more direct, and can also be modified to work without connections (as in [**?**]). That approach proves the dual fact that the *pushforward* operation, which is right adjoint to pullback and always exists in a topos, preserves fibrations when applied along a fibration. This corresponds to the type-theoretic Π-formation rule, and the proof given in *op.cit.* is entirely in type theory. It also employs a reduction of box filling (in all dimensions) to an apparently weaker condition of *Kan composition* (in all dimensions), which merely "puts a lid on" the open box, rather than filling it. This aspect of the type theoretic proof can also be described functorially, but is not used in the proof given here, and will therefore not be discussed further (see [**?**] for a description of Kan composition with connections, and [**?**, **?**, **?**, **?**] for the same without connections).

Our proof takes the approach that was used to determine the unbiased fibrations, namely we first establish the result in the *biased but generic* setting, and then transfer it to the unbiased setting by pulling back along the base change $\mathsf{cSet} \to \mathsf{cSet}/_{\mathrm{I}}$. We first give the second step as a conditional statement.

Proposition 60. *Suppose the δ-biased fibrations in $\mathsf{cSet}/_{\mathrm{I}}$ satisfy the Frobenius condition. Then the unbiased fibrations in cSet also satisfy the Frobenius condition.*

Proof. This follows almost immeditely from the fact that the pullback functor $\mathrm{I}^* : \mathsf{cSet} \to \mathsf{cSet}/_{\mathrm{I}}$ preserves the locally cartesian closed structure, takes unbiased fibrations to δ-biased ones, and reflects δ-biased fibrations to unbiased ones. In detail, let unbiased fibrations $B \twoheadrightarrow A$ and $A \twoheadrightarrow X$ in cSet be given, and we wish to find $C \twoheadrightarrow X$ and $e : A \times_X C \to B$ over A, universal in the way recalled in the diagram below.

$$\begin{tikzcd}
A \times_X C \arrow[r, dotted] \arrow[d, dotted, "e"] & C \arrow[dd, dotted, two heads] \\
B \arrow[d, two heads] & \\
A \arrow[r, two heads] & X
\end{tikzcd} \tag{63}$$

Take the pushforward $C := A_* B \to X$, and its associated map $e : A \times_X C \to B$, in the locally cartesian closed category cSet. Since fibrations are stable under (all) pullbacks, it then suffices to show that $C \to X$ is a fibration.

By definition, $C \to X$ is an unbiased fibration in cSet just in case the base change $\mathrm{I}^*C \to \mathrm{I}^*X$ is a δ-biased fibration in the slice category $\mathsf{cSet}/_{\mathrm{I}}$. Since the pullback functor $\mathrm{I}^* : \mathsf{cSet} \to \mathsf{cSet}/_{\mathrm{I}}$ preserves all lcc structure, over I^*X we have an iso,

$$\mathrm{I}^*C = \mathrm{I}^*(A_*B) \cong (\mathrm{I}^*A)_*\mathrm{I}^*B\,,$$

where the pushforward $(\mathrm{I}^*A)_*\mathrm{I}^*B$ is taken in the topos $\mathsf{cSet}/_{\mathrm{I}}$. But $\mathrm{I}^*B \to \mathrm{I}^*A$ and $\mathrm{I}^*A \to \mathrm{I}^*X$ are δ-biased fibrations in $\mathsf{cSet}/_{\mathrm{I}}$ because $B \to A$ and $A \to X$ were assumed to be unbiased fibrations in cSet. Since we are assuming the Frobenius condition for δ-biased fibrations in $\mathsf{cSet}/_{\mathrm{I}}$, the pushforward $\mathrm{I}^*C \cong (\mathrm{I}^*A)_*\mathrm{I}^*B \to \mathrm{I}^*X$ is also a δ-biased fibration, as required. □

Frobenius for biased fibrations. The results proved in this section will be applied to the slice category $\mathsf{cSet}/_{\mathrm{I}}$ and the generic point $\delta : 1 \to \mathrm{I} = \mathrm{I}^*\mathrm{I}$, but nothing depends on this particular case, and so we shall write simply $\delta : 1 \to \mathrm{I}$ for a chosen pointed object in an arbitrary topos $\mathcal{E}$. (Indeed, in this section $\mathcal{E}$ may even be just a locally cartesian closed category with a class of cofibrations in the sense of Appendix A.)

Recall from Definition 22 that a map $f : A \to X$ is a δ-biased fibration just if the map $\delta \Rightarrow f$ admits a relative +-algebra structure, and is therefore a trivial fibration. The definition of the pullback-hom $\delta \Rightarrow f$ is recalled below.

$$\begin{tikzcd}
A^{\mathrm{I}} \arrow[dr, dotted, "\delta\Rightarrow f"] \arrow[drr, bend left, "A^{\delta}"] \arrow[ddr, bend right, "f^{\mathrm{I}}"'] & & \\
& X^{\mathrm{I}} \times_X A \arrow[r] \arrow[d] & A \arrow[d, "f"] \\
& X^{\mathrm{I}} \arrow[r, "X^{\delta}"'] & X
\end{tikzcd} \tag{64}$$

Let us write this condition schematically as follows:

$$\begin{tikzcd}
A^{\mathrm{I}} \arrow[r, "|" marking] & A_{\epsilon} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
& X^{\mathrm{I}} \arrow[r, "\epsilon"'] & X
\end{tikzcd} \tag{65}$$

where $\epsilon = X^{\delta}$ and $A_{\epsilon} = X^{\mathrm{I}} \times_X A$, and the struck-through arrow indicates that it admits a +-algebra structure.

Lemma 61. *Let $A \to X$ be a δ-biased fibration and $t : Y \to X$ any map, then the pullback $t^*A \to Y$ is also a δ-biased fibration.*

Proof. This is clear from the fact that the δ-biased fibrations can be made into the right class of a weak factorization system (by reasoning analogous to that for Proposition 36), but it will be useful to see how the structure indicated in (65) is itself stable under pullback. Indeed, consider the following commutative diagram, in which the front face of the cube is the pullback in question, and the right and left sides are the respective versions of the construction in (65).

$$
\begin{tikzcd}
(t^*A)^{\mathrm{I}} \arrow[rr] \arrow[d] & & A^{\mathrm{I}} \arrow[d] & \\
(t^*A)_\epsilon \arrow[rr, dotted] \arrow[dr] \arrow[dd] & & A_\epsilon \arrow[dr] \arrow[dd] & \\
& t^*A \arrow[rr] \arrow[dd] & & A \arrow[dd] \\
Y^{\mathrm{I}} \arrow[rr] \arrow[dr, "\epsilon"'] & & X^{\mathrm{I}} \arrow[dr, "\epsilon"'] & \\
& Y \arrow[rr, "t"'] & & X
\end{tikzcd}
\tag{66}
$$

The rear square of solid arrows is the image of the front face under the pathobject functor and is therefore also a pullback. The base commutes by the naturality of the maps ϵ, as does a corresponding top square involving further such ϵ's not shown. Note that these naturality squares need not be pullbacks, but the vertical squares on the sides are, by construction. It follows that there is a dotted arrow as shown, making the resulting lower rear square commute. That lower square is then also a pullback, since the other vertical faces of the resulting cube are pullbacks, and thus finally, the upper rear square is also a pullback.

Now if $A \to X$ is a δ-biased fibration, then $A^{\mathrm{I}} \to A_\epsilon$ is a trivial fibration, and then so is its pullback $(t^*A)^{\mathrm{I}} \to (t^*A)_\epsilon$ since relative $+$-algebras are stable under pullback. Therefore the pullback $t^*A \to Y$ is also a δ-biased fibration. □

Remark 62. In this way we can show algebraically that the pullback of a δ-biased fibration is again one by pulling back the structure that makes it so. In Section 6.3, the pullback stability of the fibration structure will be used in the construction of a universal fibration via a closely related argument.

Lemma 63. *Let $\alpha : A \to X$ and $\beta : B \to A$ be δ-biased fibrations, then the composite $\alpha \circ \beta : B \to X$ is also a δ-biased fibration.*

Proof. Again for maps in the right class of a weak factorization system this is immediate. But let us see how the fibration structures also compose. We have the following diagram for the fibration structures on $B \to A$ and $A \to X$ (with obvious notation).

$$
\begin{tikzcd}
B^{\mathrm{I}} \arrow[r, "\shortmid" marking] & B_{\epsilon_A} \arrow[rr] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & & B \arrow[d] \\
& A^{\mathrm{I}} \arrow[r, "\shortmid" marking] & A_{\epsilon_X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d] \\
& & X^{\mathrm{I}} \arrow[r, "\epsilon_X"'] & X,
\end{tikzcd}
\tag{67}
$$

Pulling back $B \to A$ in two steps we therefore obtain the intermediate map $B_{\epsilon_X} \to A_{\epsilon_X}$ indicated in the following diagram.

$$
\begin{tikzcd}
B^{\mathrm{I}} \arrow[r, "\shortmid" marking] & B_{\epsilon_A} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B_{\epsilon_X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d] \\
& A^{\mathrm{I}} \arrow[r, "\shortmid" marking] & A_{\epsilon_X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d] \\
& & X^{\mathrm{I}} \arrow[r, "\epsilon_X"'] & X
\end{tikzcd}
\tag{68}
$$

Now use the fact that a trivial fibration structure (*i.e.* a +-algebra structure) has a canonical pullback along any map, and that two such structures have a canonical composition (cf. Remark 16), to obtain a trivial fibration structure for the indicated composite map $B^{\mathrm{I}} \to B_{\epsilon_X}$, which is then a fibration structure for the composite $B \to A \to X$. $\square$

Proposition 64 (Biased Frobenius)**.** *If $\alpha : A \to X$ and $\beta : B \to A$ are δ-biased fibrations, then the pushforward $\alpha_*\beta : \Pi_A B \to X$ is also a δ-biased fibration.*

Proof. Given δ-biased fibrations $\alpha : A \to X$ and $\beta : B \to A$, let $a : A^{\mathrm{I}} \to A_\epsilon$ and $b : B^{\mathrm{I}} \to a^* B_\epsilon$ be the associated trivial fibrations, so that we have the situation of diagram (68), with all three squares pullbacks.

$$\begin{tikzcd}
B^{\mathrm{I}} \arrow[r, "b", "|" marking] \arrow[dr, "\beta^{\mathrm{I}}"'] & a^*B_\epsilon \arrow[r] \arrow[d] & B_\epsilon \arrow[r] \arrow[d] & B \arrow[d, "\beta"] \\
& A^{\mathrm{I}} \arrow[r, "a", "|" marking] \arrow[dr, "\alpha^{\mathrm{I}}"'] & A_\epsilon \arrow[r] \arrow[d] & A \arrow[d, "\alpha"] \\
& & X^{\mathrm{I}} \arrow[r, "\epsilon"'] & X.
\end{tikzcd} \tag{69}$$

Taking the pushforward of the righthand vertical column give a map,

$$\gamma := \alpha_*\beta : \Pi_A B \to X\,,$$

and placing it underneath, along with the corresponding construction from (65), we then have the following commutative diagram.

$$\begin{tikzcd}
B^{\mathrm{I}} \arrow[r, "b", "|" marking] \arrow[dr, "\beta^{\mathrm{I}}"'] & a^*B_\epsilon \arrow[r] \arrow[d] & B_\epsilon \arrow[r] \arrow[d] & B \arrow[d, "\beta"] \\
& A^{\mathrm{I}} \arrow[r, "a", "|" marking] \arrow[dr, "\alpha^{\mathrm{I}}"'] & A_\epsilon \arrow[r] \arrow[d] & A \arrow[d, "\alpha"] \\
& & X^{\mathrm{I}} \arrow[r, "\epsilon"'] & X \\
& (\Pi_A B)^{\mathrm{I}} \arrow[r, "c"'] \arrow[ur, "\gamma^{\mathrm{I}}"] & (\Pi_A B)_\epsilon \arrow[r] \arrow[u] & \Pi_A B \arrow[u, "\gamma"']
\end{tikzcd} \tag{70}$$

We wish to show that the indicated map $c : (\Pi_A B)^{\mathrm{I}} \to (\Pi_A B)_\epsilon$ admits a +-algebra structure. This we will do by showing that it is a retract of a known +-algebra. Namely, we can apply the pushforward along the map $\alpha^{\mathrm{I}} : A^{\mathrm{I}} \to X^{\mathrm{I}}$ to the +-algebra $b : B^{\mathrm{I}} \to a^*B_\epsilon$ regarded as an arrow over A^{I}. We obtain an arrow over X^{I} of the form

$$\Pi_{A^{\mathrm{I}}}\, b : \Pi_{A^{\mathrm{I}}}\, B^{\mathrm{I}} \longrightarrow \Pi_{A^{\mathrm{I}}}\, a^*B_\epsilon \tag{71}$$

which is indeed a +-algebra, since these are preserved under pushing forward, by Remark 16.

Next, observe that by the Beck-Chevalley condition for the central pullback, for the codomain of c we have an isomorphism

$$(\Pi_A B)_\epsilon \;\cong\; \Pi_{A_\epsilon} B_\epsilon \qquad \text{over } X^{\mathrm{I}}.$$

And since $\Pi_{A^{\mathrm{I}}} \cong \Pi_{A_\epsilon} \circ a_*$, for the codomain of our $+$-algebra $\Pi_{A^{\mathrm{I}}}\, b$ from (71) we also have

$$\Pi_{A^{\mathrm{I}}}\, a^* B_\epsilon \;\cong\; \Pi_{A_\epsilon} a_* a^* B_\epsilon\,.$$

Thus the image of the unit $\eta : B_\epsilon \to a_* a^* B_\epsilon$ under Π_{A_ϵ} provides a map $\sigma := \Pi_{A_\epsilon}\eta$ over X^{I} of the form:

$$\begin{tikzcd}
& X^{\mathrm{I}} \\
(\Pi_A B)^{\mathrm{I}} \arrow[ur] \arrow[r, "c"'] & \Pi_{A_\epsilon} B_\epsilon \arrow[u] \arrow[d, "\sigma"] \\
\Pi_{A^{\mathrm{I}}} B^{\mathrm{I}} \arrow[r, "\Pi_{A^{\mathrm{I}}}\, b"'] & \Pi_{A_\epsilon} a_* a^* B_\epsilon
\end{tikzcd} \tag{72}$$

Our goal is now to determine further arrows φ, ψ, τ as indicated below, exhibiting c as a retract of $\Pi_{A^{\mathrm{I}}}\, b$ in the arrow category over X^{I}.

$$\begin{tikzcd}
& X^{\mathrm{I}} \\
(\Pi_A B)^{\mathrm{I}} \arrow[ur] \arrow[r, "c"'] \arrow[d, dotted, "\varphi"'] & \Pi_{A_\epsilon} B_\epsilon \arrow[u] \arrow[d, "\sigma"] \\
\Pi_{A^{\mathrm{I}}} B^{\mathrm{I}} \arrow[r, "\Pi_{A^{\mathrm{I}}}\, b"'] \arrow[d, dotted, "\psi"'] & \Pi_{A_\epsilon} a_* a^* B_\epsilon \arrow[d, dotted, "\tau"] \\
(\Pi_A B)^{\mathrm{I}} \arrow[r, "c"'] & \Pi_{A_\epsilon} B_\epsilon
\end{tikzcd} \tag{73}$$

- For φ, we require a map

$$\varphi : (\Pi_A B)^{\mathrm{I}} \to \Pi_{A^{\mathrm{I}}} B^{\mathrm{I}} \qquad \text{over } X^{\mathrm{I}}.$$

Consider the following diagram, which is based on (70).

$$
\begin{tikzcd}
B^{\mathrm{I}} \arrow[r, "\shortmid"{marking}, "b"'] \arrow[dr, "\beta^{\mathrm{I}}"'] & a^*B_\epsilon \arrow[r] \arrow[d] & B_\epsilon \arrow[r] \arrow[d] & B \arrow[d, "\beta"] & \\
 & A^{\mathrm{I}} \arrow[r, "\shortmid"{marking}, "a"'] \arrow[dr, "\alpha^{\mathrm{I}}"'] & A_\epsilon \arrow[r] \arrow[d] & A \arrow[d, "\alpha"] & \Pi_A B \times_X A \arrow[l] \arrow[ul, dotted, "e"'] \arrow[d] \\
(\Pi_A B \times_X A)^{\mathrm{I}} \arrow[uu, dotted, "e^{\mathrm{I}}"] \arrow[ur] \arrow[dr] & & X^{\mathrm{I}} \arrow[r, "\epsilon"'] & X & \Pi_A B \arrow[l] \\
 & (\Pi_A B)^{\mathrm{I}} \arrow[ur] \arrow[r, dotted, "\varphi"'] & \Pi_{A^{\mathrm{I}}} B^{\mathrm{I}} \arrow[u] & &
\end{tikzcd}
\tag{74}
$$

The map e is the counit at $\beta : B \to A$ of the pullback-pushforward adjunction along $\alpha : A \to X$. The right-hand side of the diagram, including e and the associated pullback square, reappears (mirrored) on the left under the functor $(-)^{\mathrm{I}}$, which preserves the pullback. Thus we can take φ to be the transpose of e^{I} under the pullback-pushforward adjunction along $\alpha^{\mathrm{I}} : A^I \to X^{\mathrm{I}}$,

$$\varphi := \widetilde{e^{\mathrm{I}}}\,.$$

An easy diagram chase involving the pullback-pushforward adjunction along $A_\epsilon \to X^{\mathrm{I}}$ shows that the upper square in (73) then commutes.

• For τ: referring to the diagram (70), since $a : A^{\mathrm{I}} \to A_\epsilon$ is a trivial fibration, it has a section $o : A_\epsilon \to A^{\mathrm{I}}$ by lemma 15. Pulling $a^*B_\epsilon \to A^{\mathrm{I}}$ back along o results in an iso,

$$o^*a^*B_\epsilon \cong B_\epsilon \quad \text{over } A_\epsilon$$

and so by the adjunction $o^* \dashv o_*$ there is an associated map,

$$a^*B_\epsilon \to o_*B_\epsilon \quad \text{over } A^{\mathrm{I}}$$

to which we can apply a_* to obtain a map,

$$t : a_*a^*B_\epsilon \to a_*o_*B_\epsilon \cong B_\epsilon \quad \text{over } A_\epsilon\,.$$

This map t is evidently a retraction of the unit $\eta : B_\epsilon \to a_*a^*B_\epsilon$ over A_ϵ. Applying the functor Π_{A_ϵ} therefore gives the desired retraction of σ,

$$\tau := \Pi_{A_\epsilon} t : \Pi_{A_\epsilon} a_*a^*B_\epsilon \to \Pi_{A_\epsilon} B_\epsilon\,.$$

- For ψ, we require a map

$$\psi : \Pi_{A^{\mathrm{I}}} B^{\mathrm{I}} \to (\Pi_A B)^{\mathrm{I}} \qquad \text{over } X^{\mathrm{I}}.$$

Consider the following diagram resulting from combining (70) and (73), in which all solid arrows are those already introduced. The dotted arrow labelled p is the evident composite.

$$\begin{tikzcd}
 & X^{\mathrm{I}} \arrow[r, "\epsilon"] & X \\
(\Pi_A B)^{\mathrm{I}} \arrow[ur] \arrow[r] \arrow[d] & \Pi_{A_\epsilon} B_\epsilon \arrow[u] \arrow[r] \arrow[d] & \Pi_A B \arrow[u] \arrow[dd, "="] \\
\Pi_{A^{\mathrm{I}}} B^{\mathrm{I}} \arrow[r] \arrow[drr, dotted, "p"] & \Pi_{A_\epsilon} a_* a^* B_\epsilon \arrow[d] & \\
(\Pi_A B)^{\mathrm{I}} \arrow[r] & \Pi_{A_\epsilon} B_\epsilon \arrow[r] & \Pi_A B
\end{tikzcd} \tag{75}$$

The lower horizontal composite is the evaluation of the pathobject $(\Pi_A B)^{\mathrm{I}}$ at the point $\delta : 1 \to \mathrm{I}$,

$$\epsilon_{\Pi_A B} = (\Pi_A B)^{\delta} : (\Pi_A B)^{\mathrm{I}} \longrightarrow (\Pi_A B)^1 \cong \Pi_A B\,.$$

This is constructed from the (cartesian closed) evaluation,

$$\mathsf{eval} : \mathrm{I} \times (\Pi_A B)^{\mathrm{I}} \longrightarrow \Pi_A B$$

which is the counit of $\mathrm{I} \times (-) \dashv (-)^{\mathrm{I}}$, as the composite shown below.

$$\begin{tikzcd}
(\Pi_A B)^{\mathrm{I}} \arrow[r, "\epsilon_{\Pi_A B}"] \arrow[d, "\cong"'] & \Pi_A B \\
1 \times (\Pi_A B)^{\mathrm{I}} \arrow[r, "\delta \times (\Pi_A B)^{\mathrm{I}}"'] & \mathrm{I} \times (\Pi_A B)^{\mathrm{I}} \arrow[u, "\mathsf{eval}"']
\end{tikzcd} \tag{76}$$

Let us analyse this evaluation at δ further, in terms of the *locally* cartesian closed structure associated to the base changes along the section $\delta : 1 \to \mathrm{I}$ and retraction $\mathrm{I} \to 1$ in $\mathcal{E}$. Since $\mathsf{id} \cong \delta^* \mathrm{I}^* : \mathcal{E} \to \mathcal{E}/_{\mathrm{I}} \to \mathcal{E}$, the map $\epsilon_{\Pi_A B}$ can be rewritten as follows.

$$
\begin{tikzcd}
(\Pi_A B)^{\mathrm{I}} \arrow[r, "\epsilon_{\Pi_A B}"] \arrow[d, "\cong"'] & \Pi_A B \arrow[d, "\cong"] \\
\delta^*\mathrm{I}^*((\Pi_A B)^{\mathrm{I}}) \arrow[r, "\delta^*\mathrm{I}^*\epsilon_{\Pi_A B}"] \arrow[d, "\cong"'] & \delta^*\mathrm{I}^*\Pi_A B \arrow[d, "="] \\
\delta^*\mathrm{I}^*\mathrm{I}_*\mathrm{I}^*\Pi_A B \arrow[r, "\delta^*\varepsilon"'] & \delta^*\mathrm{I}^*\Pi_A B
\end{tikzcd}
\tag{77}
$$

where the map $\delta^*\varepsilon$ across the bottom is the counit of the adjunction $\mathrm{I}^* \dashv \mathrm{I}_*$, taken at $\mathrm{I}^*\Pi_A B$, and then pulled back along $\delta : 1 \to \mathrm{I}$. Before taking the pullback, we therefore have the following iso over I between that counit $\varepsilon_{\mathrm{I}^*}$ and the image under I^* of the previously considered evaluation $\epsilon : (\Pi_A B)^{\mathrm{I}} \to \Pi_A B$ from (76).

$$
\begin{tikzcd}
\mathrm{I}^*((\Pi_A B)^{\mathrm{I}}) \arrow[r, "I^*\epsilon"] \arrow[d, "\cong"'] & \mathrm{I}^*\Pi_A B \arrow[d, "="] \\
\mathrm{I}^*\mathrm{I}_*\mathrm{I}^*\Pi_A B \arrow[r, "\varepsilon_{\mathrm{I}^*}"'] & \mathrm{I}^*\Pi_A B \,.
\end{tikzcd}
\tag{78}
$$

Now let us apply I^* to (75) to get the map I^*p in the diagram below, which therefore factors (up to (78)) through the counit $\varepsilon_{\mathrm{I}^*}$ as $\varepsilon_{\mathrm{I}^*} \circ \mathrm{I}^*(\widetilde{\mathrm{I}^*p})$, where $\widetilde{\mathrm{I}^*p}$ is the adjoint transpose of I^*p, as shown.

$$
\begin{tikzcd}
\mathrm{I}^*\Pi_{A^{\mathrm{I}}} B^{\mathrm{I}} \arrow[r] \arrow[d, "\mathrm{I}^*(\widetilde{\mathrm{I}^*p})"'] \arrow[drr, dotted, "\mathrm{I}^*p"] & \mathrm{I}^*\Pi_{A_\epsilon} a_* a^* B_\epsilon \arrow[d] & \\
\mathrm{I}^*\mathrm{I}_*\mathrm{I}^*\Pi_A B \arrow[r] \arrow[rr, bend right, "\varepsilon_{\mathrm{I}^*}"'] & \mathrm{I}^*\Pi_{A_\epsilon} B_\epsilon \arrow[r] & \mathrm{I}^*\Pi_A B
\end{tikzcd}
\tag{79}
$$

We can therefore set

$$\psi := \widetilde{\mathrm{I}^*p}\,,$$

and we obtain $\epsilon \circ \psi = p$, from which it follows that the square in (79) commutes by the definition of $\Pi_{A_\epsilon} B_\epsilon$ as a pullback. The same square without I^* then also commutes by applying the retraction δ^*.

We have now defined all the maps indicated below, the squares involving φ and ψ commute, and the composite of σ and τ is the identity.

$$\begin{tikzcd}
 & X^{\mathrm{I}} \arrow[r] & X \\
(\Pi_A B)^{\mathrm{I}} \arrow[ur] \arrow[r] \arrow[d, "\varphi"'] & \Pi_{A_\epsilon} B_\epsilon \arrow[u] \arrow[r] \arrow[d, "\sigma"] & \Pi_A B \arrow[u] \arrow[dd, "="] \\
\Pi_{A^{\mathrm{I}}} B^{\mathrm{I}} \arrow[r] \arrow[d, "\psi"'] \arrow[drr, dotted, "p"] & \Pi_{A_\epsilon} p_* p^* B_\epsilon \arrow[d, "\tau"] & \\
(\Pi_A B)^{\mathrm{I}} \arrow[r] \arrow[rr, bend right, "\epsilon"'] & \Pi_{A_\epsilon} B_\epsilon \arrow[r] & \Pi_A B
\end{tikzcd} \tag{80}$$

To see that $\psi \circ \varphi = 1$, an easy chase through the diagram (80) shows that

$$\epsilon \circ \psi \circ \varphi = p \circ \varphi = \epsilon \, .$$

Thus by applying I^* and using (78) we have $\varepsilon_{\mathrm{I}^*} \circ \mathrm{I}^*(\psi \circ \varphi) = \varepsilon_{\mathrm{I}^*}$, and so $\psi \circ \varphi = \widetilde{\varepsilon_{\mathrm{I}^*}} = 1$.

$\square$

From Proposition 60 we then have:

Corollary 65 (Unbiased Frobenius)**.** *The unbiased fibration weak factorization system on* cSet *satisfies the Frobenius condition.*

Remark 66. We note in passing that the proof just given for the δ-biased case of Frobenius, Proposition 64, made no use of the fact that $\delta : 1 \to \mathrm{I}$ is generic, nor even that we were working in the slice category over I. Indeed the same algebraic argument works for p-biased fibrations for any point $p : 1 \to \mathrm{I}$ of any object I in any (quasi-)topos $\mathcal{E}$.

6 A universal fibration

We shall construct a *universal small fibration* $\dot{\mathcal{U}} \to \mathcal{U}$, which is a classifier for small fibrations. It will be shown in Section 8 that the base object $\mathcal{U}$ is fibrant, using the fact to be proved in Section 7 that the map $\dot{\mathcal{U}} \to \mathcal{U}$ itself is *univalent*, in a sense to be made precise.[1]

Our construction of $\dot{\mathcal{U}} \to \mathcal{U}$ makes use, first of all, of a new description of the well-known Hofmann-Streicher universe in a category $\widehat{\mathbb{C}} = [\mathbb{C}^{\mathrm{op}}, \mathsf{Set}]$ of presheaves on a small category $\mathbb{C}$, which was used in [HS97] to interpret dependent type theory. See [**?**] for further details.

6.1 Classifying families

Definition 67 ([HS97])**.** Let $\mathbb{C}$ be a small category. A (type-theoretic) *universe* $(U, \mathsf{E}l)$ consists of $U \in \widehat{\mathbb{C}}$ and $\mathsf{E}l \in \widehat{\int_{\mathbb{C}} U}$ with:

$$U(c) \;=\; \mathsf{Cat}\big(\mathbb{C}/_c{}^{\mathrm{op}}, \mathsf{Set}\big) \tag{81}$$

$$\mathsf{E}l(c, A) \;=\; A(id_c) \tag{82}$$

with the evident associated action on morphisms.

A few comments are required:

- In contrast to [HS97], in (81) we take the underlying set of objects of the functor category $\widehat{\mathbb{C}/_c} = [\mathbb{C}/_c{}^{\mathrm{op}}, \mathsf{Set}]$.
- As in [HS97], (82) adopts the "categories with families" point of view in describing an arrow $E \to U$ in $\widehat{\mathbb{C}}$ equivalently as a presheaf on the category of elements $\int_{\mathbb{C}} U$, using

  $$\widehat{\mathbb{C}}/_U \;\simeq\; \widehat{\int_{\mathbb{C}} U} \tag{83}$$

  where

  $$E(c) \;=\; \textstyle\coprod_{A \in U(c)} \mathsf{E}l(c, A).$$

  The argument $(c, A) \in \int_{\mathbb{C}} U$ in (82) thus consists of an object $c \in \mathbb{C}$ and an element $A \in U(c)$.
- To account for size issues, the authors of [HS97] assume a Grothendieck universe u in Set, the elements of which are called *small*. The category $\mathbb{C}$ is assumed to be small, as are the values of the presheaves, unless otherwise stated.

[1] With only minor changes the position of this section and Section 7 could be exchanged.

The presheaf U, which is not small, is then regarded as the Grothendieck universe u "lifted" from Set to $[\mathbb{C}^{\mathrm{op}}, \mathsf{Set}]$. We first analyse this specification of $(U, \mathsf{E}l)$ from a different perspective, in order to establish its basic property as a classifier for small families in $\widehat{\mathbb{C}}$.

A realization-nerve adjunction. For a presheaf X on $\mathbb{C}$, recall that the category of elements is the comma category,

$$\textstyle\int_{\mathbb{C}} X \;=\; \mathsf{y}_{\mathbb{C}}/_X\,,$$

where $\mathsf{y}_{\mathbb{C}} : \mathbb{C} \to \mathsf{Set}^{\mathbb{C}^{\mathrm{op}}}$ is the Yoneda embedding, which we sometimes supress and write simply $\mathbb{C}/_X$ for $\mathsf{y}_{\mathbb{C}}/_X$.

Proposition 68 ([Gro83],§28)**.** *The category of elements functor*

$$\textstyle\int_{\mathbb{C}} : \widehat{\mathbb{C}} \longrightarrow \mathsf{Cat}$$

has a right adjoint,

$$\nu_{\mathbb{C}} : \mathsf{Cat} \longrightarrow \widehat{\mathbb{C}}\,.$$

For a small category $\mathbb{A}$, we shall call the presheaf $\nu_{\mathbb{C}}(\mathbb{A})$ the ($\mathbb{C}$-)nerve *of $\mathbb{A}$.*

Proof. The adjunction $\int_{\mathbb{C}} \dashv \nu_{\mathbb{C}}$ is an instance of the usual "realization/nerve" adjunction, here with respect to the covariant slice category functor $\mathbb{C}/- : \mathbb{C} \to \mathsf{Cat}$, as indicated below.

$$\begin{tikzcd}
\widehat{\mathbb{C}} \arrow[r, "\int_{\mathbb{C}}"'] & \mathsf{Cat} \arrow[l, bend right, "\nu_{\mathbb{C}}"'] \\
\mathbb{C} \arrow[u, hook, "\mathsf{y}"] \arrow[ur, "\mathbb{C}/_-"'] &
\end{tikzcd} \tag{84}$$

In detail, for $\mathbb{A} \in \mathsf{Cat}$ and $c \in \mathbb{C}$, let $\nu_{\mathbb{C}}(\mathbb{A})(c)$ be the Hom-set of functors,

$$\nu_{\mathbb{C}}(\mathbb{A})(c) = \mathsf{Cat}\big(\mathbb{C}/_c\,,\,\mathbb{A}\big)\,,$$

with contravariant action on $h : d \to c$ given by pre-composing a functor $P : \mathbb{C}/_c \to \mathbb{A}$ with the post-composition functor

$$\mathbb{C}/_h : \mathbb{C}/_d \longrightarrow \mathbb{C}/_c\,.$$

For the adjunction, observe that the slice category $\mathbb{C}/_c$ is the category of elements of the representable functor $\mathsf{y}c$,

$$\textstyle\int_{\mathbb{C}} \mathsf{y}c \;\cong\; \mathbb{C}/_c\,.$$

Thus for representables $\mathsf{y}c$, we have the required natural isomorphism

$$\widehat{\mathbb{C}}\big(\mathsf{y}c\,,\,\nu_{\mathbb{C}}(\mathbb{A})\big) \;\cong\; \nu_{\mathbb{C}}(\mathbb{A})(c) \;=\; \mathsf{Cat}\big(\mathbb{C}/_c\,,\,\mathbb{A}\big) \;\cong\; \mathsf{Cat}\big(\textstyle\int_{\mathbb{C}} \mathsf{y}c\,,\,\mathbb{A}\big)\,.$$

For arbitrary presheaves X, one uses the presentation of X as a colimit of representables over the index category $\int_{\mathbb{C}} X$, and the easy to prove fact that $\int_{\mathbb{C}}$ itself preserves colimits. Indeed, for any category $\mathbb{D}$, we have an isomorphism in Cat,

$$\varinjlim_{d\in\mathbb{D}} \mathbb{D}/_d \;\cong\; \mathbb{D}\,.$$

$\square$

When $\mathbb{C}$ is fixed, we may omit the subscript in the notation $\mathsf{y}_{\mathbb{C}}$ and $\int_{\mathbb{C}}$ and $\nu_{\mathbb{C}}$. The unit and counit maps of the adjunction $\int \dashv \nu$,

$$\begin{aligned} \eta &: X \longrightarrow \nu\textstyle\int X\,, \\ \epsilon &: \textstyle\int \nu\mathbb{A} \longrightarrow \mathbb{A}\,, \end{aligned}$$

are then as follows. At $c \in \mathbb{C}$, for $x : \mathsf{y}c \to X$, the functor $(\eta_X)_c(x) : \mathbb{C}/_c \to \mathbb{C}/_X$ is just composition with x,

$$(\eta_X)_c(x) = \mathbb{C}/_x : \mathbb{C}/_c \longrightarrow \mathbb{C}/_X\,. \tag{85}$$

For $\mathbb{A} \in \mathsf{Cat}$, the functor $\epsilon : \int \nu\mathbb{A} \to \mathbb{A}$ takes a pair $(c \in \mathbb{C}, f : \mathbb{C}/_c \to \mathbb{A})$ to the object $f(1_c) \in \mathbb{A}$,

$$\epsilon(c, f) = f(1_c).$$

Lemma 69. *For any $f : Y \to X$, the naturality square below is a pullback.*

$$\begin{tikzcd} Y \arrow[r, "\eta_Y"] \arrow[d, "f"'] & \nu\int Y \arrow[d, "\nu\int f"] \\ X \arrow[r, "\eta_X"'] & \nu\int X. \end{tikzcd} \tag{86}$$

Proof. It suffices to prove this for the case $f : X \to 1$. Thus consider the square

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d] & \nu\int X \arrow[d] \\ 1 \arrow[r, "\eta_1"'] & \nu\int 1. \end{tikzcd} \tag{87}$$

Evaluating at $c \in \mathbb{C}$ and applying (85) gives the following square in Set.

$$\begin{tikzcd} Xc \arrow[r, "\mathbb{C}/_{-}"] \arrow[d] & \mathsf{Cat}\big(\mathbb{C}/_c\,,\, \mathbb{C}/_X\big) \arrow[d] \\ 1c \arrow[r, "\mathbb{C}/_{-}"'] & \mathsf{Cat}\big(\mathbb{C}/_c\,,\, \mathbb{C}/_1\big) \end{tikzcd} \tag{88}$$

The image of $* \in 1c$ along the bottom is the forgetful functor $U_c : \mathbb{C}/_c \to \mathbb{C}$, and its fiber under the map on the right is the set of functors $F : \mathbb{C}/_c \to \mathbb{C}/_X$ such that $U_X \circ F = U_c$, where $U_X : \mathbb{C}/_X \to \mathbb{C}$ is also a forgetful functor. But any such F is uniquely of the form $\mathbb{C}/_x$ for $x = F(1_c) : \mathsf{y}c \to X$. $\square$

A universal family. For the terminal presheaf $1 \in \widehat{\mathbb{C}}$ we have an iso $\int 1 \cong \mathbb{C}$, so for every $X \in \widehat{\mathbb{C}}$ there is a canonical projection $\int X \to \mathbb{C}$, which is a discrete fibration. It follows that for any map $Y \to X$ of presheaves, the associated map $\int Y \to \int X$ is also a discrete fibration. Ignoring size issues temporarily, recall that discrete fibrations in Cat are classified by the forgetful functor $\dot{\mathsf{Set}}^{\mathrm{op}} \to \mathsf{Set}^{\mathrm{op}}$ from (the opposites of) the category of pointed sets to that of sets (cf. [Web07]). For every presheaf $X \in \widehat{\mathbb{C}}$, we therefore have a pullback diagram in Cat,

$$\begin{tikzcd} \int X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathsf{Set}}^{\mathrm{op}} \arrow[d] \\ \mathbb{C} \arrow[r, "X"'] & \mathsf{Set}^{\mathrm{op}}. \end{tikzcd} \tag{89}$$

Using $\mathbb{C} \cong \int 1$ and transposing by the adjunction $\int \dashv \nu$ then gives a commutative square in $\widehat{\mathbb{C}}$ of the form:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \nu\dot{\mathsf{Set}}^{\mathrm{op}} \arrow[d] \\ 1 \arrow[r, "\tilde{X}"'] & \nu\mathsf{Set}^{\mathrm{op}}. \end{tikzcd} \tag{90}$$

Lemma 70. *The square* (90) *is a pullback in* $\widehat{\mathbb{C}}$. *More generally, for any map* $Y \to X$ *in* $\widehat{\mathbb{C}}$, *there is a canonical pullback square*

$$\begin{tikzcd} Y \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \nu\dot{\mathsf{Set}}^{\mathrm{op}} \arrow[d] \\ X \arrow[r] & \nu\mathsf{Set}^{\mathrm{op}}. \end{tikzcd} \tag{91}$$

Proof. Apply the right adjoint ν to the pullback square (89) and paste the naturality square (86) from Lemma 69 on the left, to obtain the transposed square (91) as a pasting of two pullbacks. $\square$

Let us write $\dot{\mathcal{V}} \to \mathcal{V}$ for the vertical map on the right in (91), setting

$$\begin{aligned} \dot{\mathcal{V}} &:= \nu\dot{\mathsf{Set}}^{\mathrm{op}} \\ \mathcal{V} &:= \nu\mathsf{Set}^{\mathrm{op}}. \end{aligned} \tag{92}$$

We summarize our results so far as follows.

Proposition 71. *The nerve* $\dot{\mathcal{V}} \to \mathcal{V}$ *of the classifier for discrete fibrations* $\dot{\mathsf{Set}}^{\mathrm{op}} \to \mathsf{Set}^{\mathrm{op}}$, *as defined in* (92), *classifies natural transformations* $Y \to X$ *in* $\widehat{\mathbb{C}}$, *in the sense that there is always a pullback square,*

$$\begin{tikzcd} Y \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \dot{\mathcal{V}} \arrow[d] \\ X \arrow[r, "\tilde{Y}"'] & \mathcal{V}. \end{tikzcd} \tag{93}$$

The classifying map $\tilde{Y} : X \to \mathcal{V}$ *is determined by the adjunction* $\int \dashv \nu$ *as the transpose of the classifying map of the discrete fibration* $\int Y \to \int X$.

Given a natural transformation $Y \to X$, the classifying map $\tilde{Y} : X \to \mathcal{V}$ is of course not in general unique. Nonetheless, we can use the construction of $\dot{\mathcal{V}} \to \mathcal{V}$ as the nerve of the discrete fibration classifier $\dot{\mathsf{Set}}^{\mathrm{op}} \to \mathsf{Set}^{\mathrm{op}}$, for which classifying functors $\mathbb{C} \to \mathsf{Set}^{\mathrm{op}}$ are unique up to natural isomorphism, to infer the following proposition, which will be required below (cf. [Shu15, GSS22]).

Proposition 72 (Realignment for families)**.** *Given a monomorphism $c : C \rightarrowtail X$ and a family $Y \to X$, let $y_c : C \to \mathcal{V}$ classify the pullback $c^*Y \to C$. Then there is a classifying map $y : X \to \mathcal{V}$ for $Y \to X$ with $y \circ c = y_c$.*

$$
\begin{tikzcd}
c^*Y \arrow[rr] \arrow[dd] \arrow[dr] & & \dot{\mathcal{V}} \arrow[dd] \\
& Y \arrow[ur, dotted] \arrow[dd] & \\
C \arrow[rr, "y_c"] \arrow[dr, tail, "c"'] & & \mathcal{V} \\
& X \arrow[ur, dotted, "y"'] &
\end{tikzcd}
\tag{94}
$$

Proof. Transposing the realignment problem (94) for presheaves across the adjunction $\int \dashv \nu$ results in the following realignment problem for discrete fibrations.

$$
\begin{tikzcd}
\int c^*Y \arrow[rr] \arrow[dd] \arrow[dr] & & \dot{\mathsf{Set}}^{\mathrm{op}} \arrow[dd] \\
& \int Y \arrow[ur, dotted] \arrow[dd] & \\
\int C \arrow[rr, "\widetilde{y_c}"] \arrow[dr, tail, "\int c"'] & & \mathsf{Set}^{\mathrm{op}} \\
& \int X \arrow[ur, dotted, "\tilde{y}"'] &
\end{tikzcd}
\tag{95}
$$

The category of elements functor $\int$ is easily seen to preserve pullbacks, hence monos; thus let us consider the general case of a functor $C : \mathbb{C} \rightarrowtail \mathbb{D}$ which is monic in Cat, a pullback of discrete fibrations as on the left below, and a presheaf $E : \mathbb{C} \to \mathsf{Set}^{\mathrm{op}}$ with $\int E \cong \mathbb{E}$ over $\mathbb{C}$.

$$
\begin{tikzcd}
\mathbb{E} \arrow[rr] \arrow[dr] \arrow[dd] & & \dot{\mathsf{Set}}^{\mathrm{op}} \arrow[dd] \\
& \mathbb{F} \arrow[ur, dotted] \arrow[dd] & \\
\mathbb{C} \arrow[rr, "E"] \arrow[dr, tail, "C"'] & & \mathsf{Set}^{\mathrm{op}} \\
& \mathbb{D} \arrow[ur, dotted, "F"'] &
\end{tikzcd}
\tag{96}
$$

We seek $F : \mathbb{D} \to \mathsf{Set}^{\mathrm{op}}$ with $\int F \cong \mathbb{F}$ over $\mathbb{D}$ and $F \circ C = E$. Let $F_0 : \mathbb{D} \to \mathsf{Set}^{\mathrm{op}}$ with $\int F_0 \cong \mathbb{F}$ over $\mathbb{D}$, which exists since $\mathbb{F} \to \mathbb{D}$ is a discrete fibration. Since $F_0 \circ C$ and E both classify $\mathbb{E}$, there is a natural iso $e : F_0 \circ C \cong E$. Consider the following diagram

$$
\begin{tikzcd}
\mathbb{C} \arrow[r, "e"] \arrow[d, tail, "C"'] & (\mathsf{Set}^{\cong})^{\mathrm{op}} \arrow[r, "p_2"'] \arrow[d, "p_1"] & \mathsf{Set}^{\mathrm{op}} \\
\mathbb{D} \arrow[r, "F_0"'] \arrow[ur, dotted, "f"] & \mathsf{Set}^{\mathrm{op}} &
\end{tikzcd}
\tag{97}
$$

where $\mathsf{Set}^{\cong}$ is the category of isos in Set, with p_1, p_2 the (opposites of the) domain and codomain projections. There is a well-known weak factorization system on Cat (part of the "canonical model structure") with injective-on-objects functors on the left and isofibrations on the right. Thus there is a diagonal filler f as indicated. The functor $F := p_2 \circ f : \mathbb{D} \to \mathsf{Set}^{\mathrm{op}}$ is then the one we seek. $\square$

Small maps. Of course, as defined in (92), the classifier $\dot{\mathcal{V}} \to \mathcal{V}$ cannot be a map in $\widehat{\mathbb{C}}$, for reasons of size; we now address this. Let α be a cardinal number, and call the sets strictly smaller than it *α-small*. Let $\mathsf{Set}_\alpha \hookrightarrow \mathsf{Set}$ be the full subcategory of α-small sets. Call a presheaf $X : \mathbb{C}^{\mathrm{op}} \to \mathsf{Set}$ α-small if all of its values are α-small sets, and thus if, and only if, it factors through $\mathsf{Set}_\alpha \hookrightarrow \mathsf{Set}$. Call a map $f : Y \to X$ of presheaves α-small if all of the fibers $f_c^{-1}\{x\} \subseteq Yc$ are α-small sets (for all $c \in \mathbb{C}$ and $x \in Xc$). The latter condition is of course equivalent to saying that, in the pullback square over the element $x : \mathsf{y}c \to X$,

$$\begin{tikzcd} Y_x \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & Y \arrow[d, "f"] \\ \mathsf{y}c \arrow[r, "x"'] & X, \end{tikzcd} \tag{98}$$

the presheaf Y_x is α-small.

Now let us restrict the specification (92) of $\dot{\mathcal{V}} \to \mathcal{V}$ to the α-small sets:

$$\begin{aligned} \dot{\mathcal{V}}_\alpha &:= \nu\dot{\mathsf{Set}}_\alpha^{\mathsf{op}} \\ \mathcal{V}_\alpha &:= \nu\mathsf{Set}_\alpha^{\mathsf{op}}. \end{aligned} \tag{99}$$

Then the evident forgetful map $\dot{\mathcal{V}}_\alpha \to \mathcal{V}_\alpha$ *is* a map in the category $\widehat{\mathbb{C}}$ of presheaves, and it is in fact α-small. Moreover, it has the following basic property, which is just a restriction of the basic property of $\dot{\mathcal{V}} \to \mathcal{V}$ stated in Proposition 71.

Proposition 73. *The map $\dot{\mathcal{V}}_\alpha \to \mathcal{V}_\alpha$ classifies α-small maps $f : Y \to X$ in $\widehat{\mathbb{C}}$, in the sense that there is always a pullback square,*

$$\begin{tikzcd} Y \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \dot{\mathcal{V}}_\alpha \arrow[d] \\ X \arrow[r, "\tilde{Y}"'] & \mathcal{V}_\alpha. \end{tikzcd} \tag{100}$$

The classifying map $\tilde{Y} : X \to \mathcal{V}_\alpha$ is determined by the adjunction $\int \dashv \nu$ as (the factorization of) the transpose of the classifiyng map of the discrete fibration $\int X \to \int Y$.

Proof. If $Y \to X$ is α-small, its classifying map $\tilde{Y} : X \to \mathcal{V}$ factors through $\mathcal{V}_\alpha \hookrightarrow \mathcal{V}$, as indicated below,

$$\begin{tikzcd} Y \arrow[r] \arrow[d] \arrow[rr, bend left] & \nu\dot{\mathsf{Set}_\alpha}^{\mathrm{op}} \arrow[r, hook] \arrow[d] & \nu\dot{\mathsf{Set}}^{\mathrm{op}} \arrow[d] \\ X \arrow[r] \arrow[rr, bend right, "\tilde{Y}"'] & \nu\mathsf{Set}_\alpha{}^{\mathrm{op}} \arrow[r, hook] & \nu\mathsf{Set}^{\mathrm{op}}, \end{tikzcd} \tag{101}$$

in virtue of the following adjoint transposition,

$$\begin{tikzcd}
\int Y \arrow[r] \arrow[d] \arrow[rr, bend left] & \dot{\mathsf{Set}_\alpha}^{\mathrm{op}} \arrow[r, hook] \arrow[d] & \dot{\mathsf{Set}}^{\mathrm{op}} \arrow[d] \\
\int X \arrow[r] \arrow[rr, bend right] & \mathsf{Set}_\alpha{}^{\mathrm{op}} \arrow[r, hook] & \mathsf{Set}^{\mathrm{op}}.
\end{tikzcd} \tag{102}$$

Note that the square on the right is evidently a pullback, and so the one on the left is, too, because the outer rectangle is the classifying pulback of the discrete fibration $\int Y \to \int X$, as stated. Thus the left square in (101) is also a pullback. $\square$

Examples of universal families $\dot{\mathcal{V}}_\alpha \longrightarrow \mathcal{V}_\alpha$.

1. Let $\alpha = \kappa$ a strongly inaccessible cadinal, so that $\mathsf{ob}(\mathsf{Set}_\kappa)$ is a Grothendieck universe. Then the Hofmann-Streicher universe of Definition 67 is recovered as the κ-small map classifier

$$E \cong \dot{\mathcal{V}}_\kappa \longrightarrow \mathcal{V}_\kappa \cong U$$

in the sense of Proposition 73. Indeed, for $c \in \mathbb{C}$, we have

$$\mathcal{V}_\kappa c \;=\; \nu(\mathsf{Set}_\kappa^{\mathsf{op}})(c) = \mathsf{Cat}\big(\mathbb{C}/_c\,,\ \mathsf{Set}_\kappa^{\mathsf{op}}\big) \;=\; \mathsf{ob}(\widehat{\mathbb{C}/_c}) \;=\; Uc\,. \tag{103}$$

For $\dot{\mathcal{V}}_\kappa$ we then have,

$$\begin{aligned}\dot{\mathcal{V}}_\kappa c \;=\; \nu(\dot{\mathsf{Set}}_\kappa^{\mathsf{op}})(c) \;&=\; \mathsf{Cat}\big(\mathbb{C}/_c\,,\ \dot{\mathsf{Set}}_\kappa^{\mathsf{op}}\big) \\ &\cong\; \textstyle\coprod_{A\in\mathcal{V}_\kappa c} \mathsf{Cat}_{\mathbb{C}/_c}\big(\mathbb{C}/_c\,,\ A^*\mathsf{Set}_\kappa^{\mathsf{op}}\big)\end{aligned} \tag{104}$$

where the A-summand in (104) is defined by taking sections of the pullback indicated below.

$$\begin{tikzcd}
A^*\mathsf{Set}_\kappa^{\mathsf{op}} \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \dot{\mathsf{Set}}_\kappa^{\mathsf{op}} \arrow[d] \\
\mathbb{C}/_c \arrow[r, "A"'] \arrow[u, dotted, bend left] \arrow[ur, dotted] & \mathsf{Set}_\kappa^{\mathsf{op}}
\end{tikzcd} \tag{105}$$

But $A^*\mathsf{Set}_\kappa^{\mathsf{op}} \;\cong\; \int_{\mathbb{C}/_c} A$ over $\mathbb{C}/_c$, and sections of this discrete fibration in Cat correspond uniquely to natural maps $1 \to A$ in $\widehat{\mathbb{C}/_c}$. Since 1 is representable in $\widehat{\mathbb{C}/_c}$ we can continue (104) by

$$\begin{aligned}
\dot{\mathcal{V}}_\kappa c \;&\cong\; \textstyle\coprod_{A\in\mathcal{V}_\kappa c} \mathsf{Cat}_{\mathbb{C}/_c}\big(\mathbb{C}/_c\,,\, A^*\mathsf{Set}_\kappa^{\mathsf{op}}\big)\\
&\cong\; \textstyle\coprod_{A\in\mathcal{V}_\kappa c} \widehat{\mathbb{C}/_c}(1, A)\\
&\cong\; \textstyle\coprod_{A\in\mathcal{V}_\kappa c} A(1_c)\\
&=\; \textstyle\coprod_{A\in\mathcal{V}_\kappa c} \mathsf{E}l(\langle c, A\rangle)\\
&=\; Ec\,.
\end{aligned}$$

2. By functoriality of the nerve $\nu : \mathsf{Cat} \to \widehat{\mathbb{C}}$, a sequence of Grothendieck universes

$$\mathsf{Set}_\alpha \subseteq \mathsf{Set}_\beta \subseteq \ldots$$

in Set gives rise to a (cumulative) sequence of type-theoretic universes

$$\mathcal{V}_\alpha \rightarrowtail \mathcal{V}_\beta \rightarrowtail \ldots$$

in $\widehat{\mathbb{C}}$. More precisely, there is a sequence of cartesian squares,

$$\begin{tikzcd}
\dot{\mathcal{V}}_\alpha \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathcal{V}}_\beta \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \cdots \\
\mathcal{V}_\alpha \arrow[r, tail] & \mathcal{V}_\beta \arrow[r, tail] & \cdots\,,
\end{tikzcd} \tag{106}$$

in the image of $\nu : \mathsf{Cat} \longrightarrow \widehat{\mathbb{C}}$, classifying small maps in $\widehat{\mathbb{C}}$ of increasing size, in the sense of Proposition 73.

3. Let $\alpha = 2$ so that $1 \to 2$ is the subobject classifier of Set, and

$$\mathbb{1} = \dot{\mathsf{Set}}_2^{\mathsf{op}} \longrightarrow \mathsf{Set}_2^{\mathsf{op}} = \mathbb{2}$$

is then a classifier in Cat for *sieves*, i.e. full subcategories $\mathbb{S} \hookrightarrow \mathbb{A}$ closed under the domains of arrows $a \to s$ for $s \in \mathbb{S}$. The nerve $\dot{\mathcal{V}}_2 \to \mathcal{V}_2$ is then the usual subobject classifier $1 \to \Omega$ of $\widehat{\mathbb{C}}$,

$$1 = \nu\mathbb{1} = \dot{\mathcal{V}}_2 \longrightarrow \mathcal{V}_2 = \nu\mathbb{2} = \Omega\,.$$

4. For any $X \in \widehat{\mathbb{C}}$, we have an equivalence

$$\widehat{\mathbb{C}}/_X \ \simeq\ \widehat{\textstyle\int_{\mathbb{C}} X}\ \simeq\ \mathsf{dFib}/_{\int_{\mathbb{C}} X}$$

where, generally, $\mathsf{dFib}/_{\mathbb{D}}$ is the category of discrete fibrations over a category $\mathbb{D}$. This equivalence commutes with composition along discrete fibrations, in the sense that the forgetful functor

$$X_! : \widehat{\mathbb{C}}/_X \to \widehat{\mathbb{C}}$$

given by composition along $X \to 1$ agrees (up to canonical isomorphism) with the base change $(p_X)_! \dashv (p_X)^*$ of presheaves along the projection $p_X : \int_{\mathbb{C}} X \to \mathbb{C}$, and with composition along the discrete fibration p_X, as indicated in:

$$\begin{tikzcd} \widehat{\mathbb{C}}/_X \arrow[r, "\sim"] \arrow[d, "X_!"'] & \widehat{\int_{\mathbb{C}} X} \arrow[r, "\sim"] \arrow[d, "(p_X)_!"'] & \mathsf{dFib}/_{\int_{\mathbb{C}} X} \arrow[d, "p_X \circ (-)"] \\ \widehat{\mathbb{C}} \arrow[r, "\sim"'] & \widehat{\mathbb{C}} \arrow[r, "\sim"'] & \mathsf{dFib}/_{\mathbb{C}}. \end{tikzcd} \tag{107}$$

It follows that the pullback functor $X^* : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}/_X$ commutes with the corresponding right adjoints (one of which is the nerve), and therefore preserves the respective universes,

$$X^*\mathcal{V}_{\mathbb{C}} \ \cong\ (p_X)^*\nu_{\mathbb{C}}(\mathsf{Set}^{\mathsf{op}})\ \cong\ \nu_{\int_{\mathbb{C}} X}(\mathsf{Set}^{\mathsf{op}})\ \cong\ \mathcal{V}_{\int_{\mathbb{C}} X}\,.$$

Corollary 74. *Let $\dot{\mathcal{V}}_\alpha \to \mathcal{V}_\alpha$ classify α-small maps in $\widehat{\mathbb{C}}$, as in Proposition 73. Then for any $X \in \mathbb{C}$, the pullback $X^*\dot{\mathcal{V}}_\alpha \to X^*\mathcal{V}_\alpha$ classifies α-small maps in $\widehat{\mathbb{C}}/_X$*

6.2 Classifying trivial fibrations

Returning now to the presheaf category $\mathsf{cSet} = \mathsf{Set}^{\square^{\mathrm{op}}}$ of cubical sets, recall from section 2 that (uniform) trivial fibration structures on a map $A \to X$ correspond bijectively to relative $+$-algebra structures over X (definition 10). A relative $+$-algebra structure on $A \to X$ is an algebra structure for the pointed endofunctor $+_X : \mathsf{cSet}/X \to \mathsf{cSet}/X$, where recall from (2),

$$A^+ = \sum_{\varphi:\Phi} A^{[\varphi]} \quad \text{over } X.$$

A $+$-algebra structure is then a retract $\alpha : A^+ \to A$ over X of the canonical map $\eta_A : A \to A^+$,

$$\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[rr, bend left, "="] \arrow[dr] & A^+ \arrow[r, "\alpha"] \arrow[d] & A \arrow[dl] \\
& X. &
\end{tikzcd} \tag{108}$$

In more detail, let us write $A \to X$ as a family $(A_x)_{x \in X}$, so that $A = \sum_{x:X} A_x \to X$. Since the $+$-functor acts fiberwise, the object A^+ in (108) is then the indexing projection

$$\sum_{x:X} A_x^+ \to X.$$

Working in the slice cSet/X, the (relative) exponentials (internal Hom's) $[A^+, A]$ and $[A, A]$ and the "precomposition by η_A" map $[\eta_A, A]$, fit into the following pullback diagram

$$\begin{tikzcd}
+\mathsf{Alg}(A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & {[A^+, A]} \arrow[d, "{[\eta_A, A]}"] \\
1 \arrow[r, "{'id_A'}"'] & {[A, A]}.
\end{tikzcd} \tag{109}$$

The constructed object $+\mathsf{Alg}(A) \to X$ over X is then the *object of $+$-algebra structures on* $A \to X$, in the sense that sections $X \to +\mathsf{Alg}(A)$ correspond uniquely to $+$-algebra structures on $A \to X$. Moreover, $+\mathsf{Alg}(A) \to X$ is stable under pullback, in the sense that for any $f : Y \to X$, we have two pullback squares,

$$\begin{tikzcd}
f^*A \arrow[r] \arrow[d] & A \arrow[d] \\
Y \arrow[r, "f"'] & X \\
+\mathsf{Alg}(f^*A) \arrow[r] \arrow[u] & +\mathsf{Alg}(A) \arrow[u]
\end{tikzcd} \tag{110}$$

because the $+$-functor, exponentials and pullbacks occurring in the construction of $+\mathsf{Alg}(A) \to X$ are themselves all stable.

It then follows from Proposition 73 that, if $A \to X$ is small, then $+\mathsf{Alg}(A) \to X$ is itself a pullback of the analogous object $+\mathsf{Alg}(\dot{\mathcal{V}}) \to \mathcal{V}$ constructed from the universal small family $\dot{\mathcal{V}} \to \mathcal{V}$ of Proposition 73, so there are two pullback squares:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \dot{\mathcal{V}} \arrow[d] \\ X \arrow[r, "\chi_A"'] & \mathcal{V} \\ +\mathsf{Alg}(A) \arrow[r] \arrow[u] & +\mathsf{Alg}(\dot{\mathcal{V}}) \arrow[u] \end{tikzcd} \tag{111}$$

Proposition 75. *There is a* universal small trivial fibration

$$\dot{\mathsf{TFib}} \to \mathsf{TFib}.$$

Every small trivial fibration $A \to X$ is a pullback of $\dot{\mathsf{TFib}} \to \mathsf{TFib}$ along a canonically determined classifying map $X \to \mathsf{TFib}$.

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathsf{TFib}} \arrow[d] \\ X \arrow[r] & \mathsf{TFib} \end{tikzcd} \tag{112}$$

Proof. We can take

$$\mathsf{TFib} := +\mathsf{Alg}(\dot{\mathcal{V}}),$$

which comes with its projection $+\mathsf{Alg}(\dot{\mathcal{V}}) \to \mathcal{V}$ as in diagram (111). Now define $p_t : \dot{\mathsf{TFib}} \to \mathsf{TFib}$ by pulling back the universal small family,

$$\begin{tikzcd} \dot{\mathsf{TFib}} \arrow[r] \arrow[d, "p_t"'] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathcal{V}} \arrow[d, "p"] \\ \mathsf{TFib} \arrow[r] & \mathcal{V}. \end{tikzcd}$$

Consider the following diagram, in which all the squares (including the distorted ones) are pullbacks, with the outer one coming from proposition 73 and the lower one from (111).

$$
\begin{tikzcd}
& A \arrow[rr, "q_A"] \arrow[ddd] \arrow[dr, dotted] & & \dot{\mathcal{V}} \arrow[ddd, "p"] \\
& & \mathsf{T\dot{F}ib} \arrow[ur] \arrow[d, "p_t"'] & \\
\mathsf{TFib}(A) \arrow[rr] \arrow[dr] & & \mathsf{TFib} \arrow[dr] & \\
& X \arrow[rr, "\chi_A"'] \arrow[ur, dotted, "\alpha'"'] \arrow[ul, dotted, bend left, "\alpha"] & & \mathcal{V}.
\end{tikzcd}
\tag{113}
$$

A trivial fibration structure α on $A \to X$ is a section the object of $+$-algebra structures on A, occurring in the diagram as

$$\mathsf{TFib}(A) := +\mathsf{Alg}(A),$$

the pullback of $\mathsf{TFib} = +\mathsf{Alg}(\dot{\mathcal{V}})$ along the classifying map $\chi_A : X \to \mathcal{V}$ for the small family $A \to X$. Such sections correspond uniquely to factorizations α' of χ_A as indicated, which in turn induce pullback squares of the required kind (112).

Note that the map $p_t : \mathsf{T\dot{F}ib} \to \mathsf{TFib}$ has a canonical trivial fibration structure. Indeed, consider the following diagram, in which both squares are pullbacks.

$$
\begin{tikzcd}
\mathsf{T\dot{F}ib} \arrow[r] \arrow[d, "p_t"'] & \dot{\mathcal{V}} \arrow[d] \\
\mathsf{TFib} \arrow[r] & \mathcal{V} \\
\mathsf{TFib}(\mathsf{T\dot{F}ib}) \arrow[r] \arrow[u] & \mathsf{TFib}(\dot{\mathcal{V}}) \arrow[u]
\end{tikzcd}
\tag{114}
$$

$\mathsf{TFib}(\dot{\mathcal{V}})$ is the object of trivial fibration structures on $\dot{\mathcal{V}} \to \mathcal{V}$, and its pullback $\mathsf{TFib}(\mathsf{T\dot{F}ib})$ is therefore the object of trivial fibration structures on $p_t : \mathsf{T\dot{F}ib} \to \mathsf{TFib}$. Thus we seek a section of $\mathsf{TFib}(\mathsf{T\dot{F}ib}) \to \mathsf{TFib}$. But recall that $\mathsf{TFib} = \mathsf{TFib}(\dot{\mathcal{V}})$ by definition, so the lower pullback square is the pullback of $\mathsf{TFib}(\dot{\mathcal{V}}) \to \mathcal{V}$ against itself, which does indeed have a distinguished section, namely the diagonal

$$\Delta : \mathsf{TFib}(\dot{\mathcal{V}}) \to \mathsf{TFib}(\dot{\mathcal{V}}) \times_{\mathcal{V}} \mathsf{TFib}(\dot{\mathcal{V}}).$$

□

We record the following notation and corresponding fact from the foregoing proof for future reference:

Lemma 76. *The classifying type* $\mathsf{TFib}(A) := +\mathsf{Alg}(A) \to X$ *for trivial fibration structures on a map* $A \to X$ *is stable under pullback, in the sense that for any* $f : Y \to X$*, we have two pullback squares,*

$$\begin{tikzcd} f^*A \arrow[r] \arrow[d] & A \arrow[d] \\ Y \arrow[r, "f"'] & X \\ \mathsf{TFib}(f^*A) \arrow[r] \arrow[u] & \mathsf{TFib}(A) \arrow[u] \end{tikzcd} \tag{115}$$

Since the universal small trivial fibration $\dot{\mathsf{TFib}} \to \mathsf{TFib}$ in cSet from Proposition 75 was constructed as $\mathsf{TFib} = \mathsf{TFib}(\dot{\mathcal{V}})$ for the universal small family $\dot{\mathcal{V}} \to \mathcal{V}$, which in turn is stable under pullback by Corollary 74, we also have:

Corollary 77. *The base change of the universal small trivial fibration*

$$\dot{\mathsf{TFib}} \to \mathsf{TFib}$$

in cSet *along* $\mathrm{I}^* : \mathsf{cSet} \to \mathsf{cSet}/_{\mathrm{I}}$ *is a universal small trivial fibration in* $\mathsf{cSet}/_{\mathrm{I}}$*.*

6.3 Classifying fibrations

In order to classify fibrations $A \twoheadrightarrow X$, we shall proceed as for trivial fibrations by constructing, for any map $A \to X$, an object $\mathsf{Fib}(A) \to X$ of fibration structures which, moreover, is stable under pullback. We then apply the construction to the universal small family $\dot{\mathcal{V}} \to \mathcal{V}$ of Proposition 73 to obtain a universal small fibration. Here we will of course need to distinguish between biased and unbiased fibrations. In Proposition **??**, we first construct a stable classifying type $\mathsf{Fib}(A) \to X$ for δ-biased fibration structures on any map $A \to X$ in $\mathsf{cSet}/_{\mathrm{I}}$ where δ is the generic point. In Proposition **??** we then transfer the construction along the base change $\mathrm{I}^* : \mathsf{cSet} \to \mathsf{cSet}/_{\mathrm{I}}$ to obtain a classifier $\mathsf{Fib}(A) \to X$ for unbiased fibration structures on any $A \to X$ in cSet.

The construction of $\mathsf{Fib}(A) \to X$ for biased fibration structures with respect to a point $\delta : 1 \to \mathrm{I}$ is already a bit more involved than was that of $\mathsf{TFib}(A) \to X$. In particular, it requires the codomain I of δ to be *tiny*, which is indeed the case for the generic point $\delta : 1 \to \mathrm{I}^*\mathrm{I}$ in $\mathsf{cSet}/_{\mathrm{I}}$ by Lemma 4.

The classifying type of biased fibration structures. A classifying type $\mathsf{Fib}(A) \to X$ of (uniform, δ-biased) fibration structures on a map $p : A \to X$, as defined in Section 3.1, can be constructed as follows.

1. First form the pullback-hom $\delta \Rightarrow p : A^{\mathrm{I}} \to X^{\mathrm{I}} \times_X A$ with the point $\delta : 1 \to \mathrm{I}$, as indicated in the following diagram.

$$\begin{tikzcd}
A^{\mathrm{I}} \arrow[dr, dotted, "\delta\Rightarrow p"] \arrow[drr, bend left, "A^{\delta}"] \arrow[ddr, bend right, "(p_A)^{\mathrm{I}}"'] & & \\
& X^{\mathrm{I}} \times_X A \arrow[r] \arrow[d] & A \arrow[d, "p"] \\
& X^{\mathrm{I}} \arrow[r, "X^{\delta}"'] & X
\end{tikzcd} \tag{116}$$

2. A fibration structure on $p : A \to X$ is then a relative $+$-algebra structure on $\delta \Rightarrow p$ in the slice category over its codomain $X^{\mathrm{I}} \times_X A$. To construct a classifier for such structures, let us first relabel the objects and arrows in diagram (116) as follows:

$$\begin{aligned}
\epsilon &:= X^{\delta} : X^{\mathrm{I}} \to X \\
A_\epsilon &:= X^{\mathrm{I}} \times_X A \\
\epsilon_A &:= \delta \Rightarrow p
\end{aligned}$$

so that the working part of (116) becomes:

$$\begin{tikzcd}
A^{\mathrm{I}} \arrow[dr, "\epsilon_A"'] & & \\
& A_\epsilon \arrow[r] \arrow[d, "p_\epsilon"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "p"] \\
& X^{\mathrm{I}} \arrow[r, "\epsilon"'] & X
\end{tikzcd} \tag{117}$$

3. Now a relative +-algebra structure on ϵ_A (Definition 10) is a retract α over A_ϵ of the unit η, as indicated below, where D is simply the domain of the map $(\epsilon_A)^+$ resulting from applying the relative +-functor in the slice category over A_ϵ to the object ϵ_A.

$$\begin{tikzcd}
A^{\mathrm{I}} \arrow[r, "\eta"'] \arrow[dr, "\epsilon_A"'] & D \arrow[l, dotted, bend right, "\alpha"'] \arrow[d, "(\epsilon_A)^+"] & \\
& A_\epsilon \arrow[r] \arrow[d, "p_\epsilon"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "p"] \\
& X^{\mathrm{I}} \arrow[r, "\epsilon"'] & X
\end{tikzcd} \tag{118}$$

4. As in the construction (109), there is an object $\mathsf{TFib}(\epsilon_A) = +\mathsf{Alg}(\epsilon_A)$ over A_ϵ of relative +-algebra structures on ϵ_A, the sections of which correspond uniquely to relative +-algebra structures on ϵ_A (and thus to fibration structures on A).

$$\begin{tikzcd}
& A^{\mathrm{I}} \arrow[r, "\eta"'] \arrow[d, "\epsilon_A"'] & D \arrow[l, dotted, bend right, "\alpha"'] \arrow[dl, "(\epsilon_A)^+"] \\
\mathsf{TFib}(\epsilon_A) \arrow[r] & A_\epsilon \arrow[r] \arrow[d, "p_\epsilon"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "p"] \\
& X^{\mathrm{I}} \arrow[r, "\epsilon"'] & X
\end{tikzcd} \tag{119}$$

5. Sections of $\mathsf{TFib}(\epsilon_A) \longrightarrow A_\epsilon$ then correspond to sections of its push-forward along p_ϵ, which we shall call F_A:

$$F_A := (p_\epsilon)_* \mathsf{TFib}(\epsilon_A)\,.$$

$$\begin{tikzcd}
& A^{\mathrm{I}} \arrow[r, "\eta"'] \arrow[d, "\epsilon_A"'] & D \arrow[l, dotted, bend right, "\alpha"'] \arrow[dl, "(\epsilon_A)^+"] \\
\mathsf{TFib}(\epsilon_A) \arrow[r] & A_\epsilon \arrow[r] \arrow[d, "p_\epsilon"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "p"] \\
F_A \arrow[r] & X^{\mathrm{I}} \arrow[r, "\epsilon"'] & X
\end{tikzcd} \tag{120}$$

6. One might now try taking another pushforward of $F_A \to X^{\mathrm{I}}$ along $\epsilon : X^{\mathrm{I}} \to X$ to get the object $\mathsf{Fib}(A) \to X$ that we seek, but unfortunately, this would not be stable under pullback along arbitrary maps $Y \to X$, because the evaluation $\epsilon = X^\delta : X^{\mathrm{I}} \to X$ is not stable in that way. Instead we use the *root* functor, i.e. the right adjoint of the pathspace, $(-)^{\mathrm{I}} \dashv (-)_{\mathrm{I}}$ (Proposition 3).

   Let $f : F_A \to X^{\mathrm{I}}$ be the map $(p_\epsilon)_* \mathsf{TFib}(\epsilon_A)$ indicated in (120), and let $\eta : X \to (X^{\mathrm{I}})_{\mathrm{I}}$ be the unit of the root adjunction at X. Then define $\mathsf{Fib}(A) \to X$ by

$$\mathsf{Fib}(A) := \eta^* f_{\mathrm{I}}$$

   as indicated in the following pullback diagram.

$$\begin{tikzcd} \mathsf{Fib}(A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (F_A)_{\mathrm{I}} \arrow[d, "f_{\mathrm{I}}"] \\ X \arrow[r, "\eta"'] & (X^{\mathrm{I}})_{\mathrm{I}} \end{tikzcd} \tag{121}$$

   By adjointness, sections of $\mathsf{Fib}(A) \to X$ then correspond bijectively to sections of $f : F_A \to X^{\mathrm{I}}$.

Lemma 78. *For any map $A \to X$ in $\mathsf{cSet}/_{\mathrm{I}}$, the map $\mathsf{Fib}(A) \to X$ in (121) is a* classifying type for δ-biased fibration structures*: sections of $\mathsf{Fib}(A) \to X$ correspond bijectively to δ-biased fibration structures on $A \to X$, and the construction is stable under pullback in the sense that for any $f : Y \to X$, we have two pullback squares,*

$$\begin{tikzcd} f^*A \arrow[r] \arrow[d] & A \arrow[d] \\ Y \arrow[r, "f"'] & X \\ \mathsf{Fib}(f^*A) \arrow[r] \arrow[u] & \mathsf{Fib}(A) \arrow[u] \end{tikzcd} \tag{122}$$

Proof. It is clear from the construction that fibration structures on $A \to X$ correspond bijectively to sections of $\mathsf{Fib}(A) \to X$. We show that $\mathsf{Fib}(A) \to X$ is also stable under pullback. To that end, the relevant steps of the construction are recalled schematically below.

$$
\begin{tikzcd}
 & A^{\mathrm{I}} \arrow[d, "\epsilon_A"'] & & \\
\mathsf{TFib}(\epsilon_A) \arrow[r] & A_\epsilon \arrow[r] \arrow[d, "p_\epsilon"'] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, "p"] & \\
F_A \arrow[r] & X^{\mathrm{I}} \arrow[r, "\epsilon"'] & X & \mathsf{Fib}(A) \arrow[l]
\end{tikzcd}
\tag{123}
$$

Now consider the following diagram, in which the right hand side consists of the data from (123), and the front, central square is a pullback.

$$
\begin{tikzcd}
 & B^{\mathrm{I}} \arrow[rrr] \arrow[d, "\epsilon_B"'] & & & A^{\mathrm{I}} \arrow[d, "\epsilon_A"] & \\
\mathsf{TFib}(\epsilon_B) \arrow[r] & B_\epsilon \arrow[rrr, dotted] \arrow[d] \arrow[dr] & & & A_\epsilon \arrow[d] \arrow[dl] & \mathsf{TFib}(\epsilon_A) \arrow[l] \\
F_B \arrow[r] & Y^{\mathrm{I}} \arrow[rrr, dotted, bend right] \arrow[dr] & B \arrow[r] \arrow[d] & A \arrow[d] & X^{\mathrm{I}} \arrow[dl] & F_A \arrow[l] \\
 & \mathsf{Fib}(B) \arrow[r] & Y \arrow[r, "f"'] & X & \mathsf{Fib}(A) \arrow[l] &
\end{tikzcd}
\tag{124}
$$

As in the proof of Lemma 61, on the left side we repeat the construction with $B \to Y$ in place of $A \to X$. The left face of the indicated (distorted) cube is then also a pullback, whence the back (dotted) face is a pullback, since the two-story square in back is the image of the front pullback square under the right adjoint $(-)^{\mathrm{I}}$. Finally, the top rectangle in the back is therefore also a pullback.

It follows that $\mathsf{TFib}(\epsilon_B)$ is a pullback of $\mathsf{TFib}(\epsilon_A)$ along the upper dotted arrow, as in Lemma 76, and so the pushforward F_B is a pullback of the corresponding F_A, along the lower dotted arrow (which is f^{I}), by the Beck-Chevalley condition for the dotted pullback square. Let us record this for later reference:

$$F_B \cong (f^{\mathrm{I}})^* F_A. \tag{125}$$

It remains to show that $\mathsf{Fib}(B)$ is a pullback of $\mathsf{Fib}(A)$ along $f : Y \to X$, and now it is good that we did not take these to be pushforwards of F_B

and F_A, because the floor of the cube need not be a pullback, and so the Beck-Chavalley condition would not apply. Instead, consider the following diagram.

$$\begin{tikzcd}
& \mathsf{Fib}(B) \arrow[r] \arrow[dl] \arrow[dd] & \mathsf{Fib}(A) \arrow[dr] \arrow[dd] & \\
(F_B)_{\mathrm{I}} \arrow[rrr] \arrow[dd] & & & (F_A)_{\mathrm{I}} \arrow[dd] \\
& Y \arrow[r, "f"'] \arrow[dl, "\eta"'] & X \arrow[dr, "\eta"] & \\
(Y^{\mathrm{I}})_{\mathrm{I}} \arrow[rrr, "(f^{\mathrm{I}})_{\mathrm{I}}"'] & & & (X^{\mathrm{I}})_{\mathrm{I}}
\end{tikzcd} \tag{126}$$

The sides of the cube are pullbacks by the construction of $\mathsf{Fib}(A)$ and $\mathsf{Fib}(B)$. The front face is the root of the pullback (125) and is thus also a pullback, since the root is a right adjoint. The base commutes by naturality of the unit of the adjunction, and so the back face is also a pullback, as required. $\square$

Now let us apply the foregoing construction of $\mathsf{Fib}(A)$ to the universal family $\dot{\mathcal{V}} \to \mathcal{V}$ to get $\mathsf{Fib}(\dot{\mathcal{V}}) \to \mathcal{V}$, and define the universal small (δ-biased) fibration in $\mathsf{cSet}/_{\mathrm{I}}$ by setting $\mathsf{Fib} := \mathsf{Fib}(\dot{\mathcal{V}})$ and $\dot{\mathsf{Fib}} \twoheadrightarrow \mathsf{Fib}$ by pulling back the universal family,

$$\begin{tikzcd}
\dot{\mathsf{Fib}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathcal{V}} \arrow[d, "p"] \\
\mathsf{Fib} \arrow[r] & \mathcal{V}.
\end{tikzcd} \tag{127}$$

The proof of the following then proceeds just as that given for $\mathsf{T}\dot{\mathsf{Fib}} \to \mathsf{TFib}$ in Proposition 75.

Proposition 79. *The map* $\dot{\mathsf{Fib}} \to \mathsf{Fib}$ *constructed in* (127) *is a* universal small δ-biased fibration *in* $\mathsf{cSet}/_{\mathrm{I}}$*: every small δ-biased fibration $A \twoheadrightarrow X$ in* $\mathsf{cSet}/_{\mathrm{I}}$ *is a pullback of* $\dot{\mathsf{Fib}} \twoheadrightarrow \mathsf{Fib}$ *along a canonically determined classifying map* $X \to \mathsf{Fib}$*.*

$$\begin{tikzcd}
A \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathsf{Fib}} \arrow[d, two heads] \\
X \arrow[r] & \mathsf{Fib}
\end{tikzcd} \tag{128}$$

Remark 80. Proposition 79 made no use of the fact that we were working in the slice category $\mathsf{cSet}/_{\mathrm{I}}$ with $\delta : 1 \to \mathrm{I}$ the generic point. It holds equally for δ-biased fibrations with respect to any point $\delta : 1 \to \mathrm{I}$ of a tiny object I. Thus *e.g.* it could be used (with obvious adjustment) to construct a classifier for the $\{\delta_0, \delta_1\}$-biased fibrations of Section 3.1 in (Cartesian, Dedekind, or other varieties of) cubical sets cSet.

The classifying type of unbiased fibration structures. In order to classify *unbiased* fibration structures on maps $A \to X$ in cSet, we first apply the pullback $\mathrm{I}^* : \mathsf{cSet} \to \mathsf{cSet}/_{\mathrm{I}}$ and take the classifier $\mathsf{Fib}(\mathrm{I}^*A) \to \mathrm{I}^*X$ for δ-biased fibration structures, then apply the pushforward $\mathrm{I}_* : \mathsf{cSet}/_{\mathrm{I}} \to \mathsf{cSet}$ and pull the result $\mathrm{I}_*\mathsf{Fib}(\mathrm{I}^*A) \to \mathrm{I}_*\mathrm{I}^*X$ back along the unit $X \to \mathrm{I}_*\mathrm{I}^*X$.

To show that this indeed classifies unbiased fibration structures on $A \to X$, let us first rename the classifying type from Lemma 78, which was constructed over I, to $\mathsf{Fib}_i(\mathrm{I}^*A) \to \mathrm{I}^*X$, and then apply I_* to get the map,

$$\Pi_{i:\mathrm{I}}\mathsf{Fib}_i(\mathrm{I}^*A) := \mathrm{I}_*(\mathsf{Fib}_i(\mathrm{I}^*A)) \longrightarrow X^{\mathrm{I}}$$

in cSet. Then, as just said, we define the desired map $\mathsf{Fib}(A) \to X$ as the pullback along the unit $\rho : X \to X^{\mathrm{I}}$ of $\mathrm{I}^* \dashv \mathrm{I}_*$ as indicated below.

$$\begin{tikzcd}
\mathsf{Fib}(A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Pi_{i:\mathrm{I}}\mathsf{Fib}_i(\mathrm{I}^*A) \arrow[d] \\
X \arrow[r, "\rho"'] & X^{\mathrm{I}}
\end{tikzcd} \tag{129}$$

It now immediately from the adjunction $\mathrm{I}^* \dashv \mathrm{I}_*$ that sections of $\mathsf{Fib}(A) \to X$ correspond bijectively to sections of $\mathsf{Fib}_i(\mathrm{I}^*A) \to \mathrm{I}^*X$ over I, and thus to *unbiased* fibration structures on $A \to X$.

Lemma 81. *For any map $A \to X$ in cSet, the map $\mathsf{Fib}(A) \to X$ in (129) is a* classifying type for unbiased fibration structures*: sections of $\mathsf{Fib}(A) \to X$ correspond bijectively to unbiased fibration structures on $A \to X$, and the construction is stable under pullback in the expected sense (as in Lemma 78).*

Proof. It remains only to check the stability, but since both of the adjoints in $\mathrm{I}^* \dashv \mathrm{I}_* : \mathsf{cSet}/_{\mathrm{I}} \to \mathsf{cSet}$ preserve pullbacks, this follows easily from the fact that the classifying types Fib_i are stable under pullback by Lemma 78. $\square$

Finally, we can again take $\mathsf{Fib} := \mathsf{Fib}(\dot{\mathcal{V}})$ to now obtain a universal small *unbiased* fibration $\dot{\mathsf{Fib}} \to \mathsf{Fib}$ in cSet, as in (127), and the proof can conclude just as in that for Proposition 75.

Proposition 82. *The map* $\dot{\mathsf{Fib}} \to \mathsf{Fib}$ *just constructed is a* universal small unbiased fibration *in* cSet*: every small unbiased fibration* $A \twoheadrightarrow X$ *is a pullback of* $\dot{\mathsf{Fib}} \twoheadrightarrow \mathsf{Fib}$ *along a canonically determined classifying map* $X \to \mathsf{Fib}$*.*

$$\begin{tikzcd} A \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathsf{Fib}} \arrow[d, two heads] \\ X \arrow[r] & \mathsf{Fib} \end{tikzcd} \tag{130}$$

Remark 83. Recall from Proposition 74 that the universe in the slice category $\mathsf{cSet}/_{\mathrm{I}}$ is the pullback of the universe $\mathcal{V}$ from cSet along the base change $\mathrm{I}^* : \mathsf{cSet} \to \mathsf{cSet}/_{\mathrm{I}}$. Thus in the construction just given of the classifier $\dot{\mathsf{Fib}} \to \mathsf{Fib}$ for unbiased fibrations in cSet we are first building the classifying type

$$\mathsf{Fib}_i(\mathrm{I}^*\dot{\mathcal{V}}) \to \mathrm{I}^*\mathcal{V}$$

for δ-biased fibration structures on the universal family in $\mathsf{cSet}/_{\mathrm{I}}$, and then taking a pushforward $\mathrm{I}_* : \mathsf{cSet}/_{\mathrm{I}} \to \mathsf{cSet}$ to obtain the (base of the) classifier for unbiased fibrations as the pullback along the unit:

$$\begin{tikzcd} \mathsf{Fib}(\dot{\mathcal{V}}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Pi_{i:\mathrm{I}}\mathsf{Fib}_i(\mathrm{I}^*\dot{\mathcal{V}}) \arrow[d] \\ \mathcal{V} \arrow[r, "\rho"'] & \mathcal{V}^{\mathrm{I}} \end{tikzcd} \tag{131}$$

We remark for later reference that this classifying type $\mathsf{Fib} = \mathsf{Fib}(\dot{\mathcal{V}}) \to \mathcal{V}$ for unbiased fibration structures can therefore be constructed as the pushforward of the classifier $\mathsf{Fib}_i(\mathrm{I}^*\dot{\mathcal{V}}) \to \mathrm{I}^*\mathcal{V}$ for δ-biased fibration structures along the projection $q : \mathrm{I}^*\mathcal{V} = \mathrm{I} \times \mathcal{V} \to \mathcal{V}$ indicated below.

$$\begin{tikzcd} \mathsf{Fib}_i(\mathrm{I}^*\dot{\mathcal{V}}) \arrow[d] & \mathsf{Fib}(\dot{\mathcal{V}}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Pi_{i:\mathrm{I}}\mathsf{Fib}_i(\dot{\mathcal{V}}) \arrow[d] \\ \mathrm{I}^*\mathcal{V} \arrow[r, "q"'] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{V} \arrow[r, "\rho"'] \arrow[d] & \mathcal{V}^{\mathrm{I}} \\ \mathrm{I} \arrow[r] & 1 & \end{tikzcd} \tag{132}$$

We record this fact as:

Corollary 84. $\mathsf{Fib} = \Sigma_{\mathcal{V}}\, q_* \mathsf{Fib}_i(\mathrm{I}^* \dot{\mathcal{V}})$.

The reader may also find it illuminating to reconsider the construction of the universal small unbiased fibration in more type theoretic terms. It was defined to be $\dot{\mathsf{Fib}} \twoheadrightarrow \mathsf{Fib} = \mathsf{Fib}(\dot{\mathcal{V}})$, for the universal family $\dot{\mathcal{V}} \to \mathcal{V}$, with $\dot{\mathsf{Fib}}$ the pullback of $\dot{\mathcal{V}} \to \mathcal{V}$ along the canonical projection $\mathsf{Fib}(\dot{\mathcal{V}}) \to \mathcal{V}$. Since, type theoretically, we have $\dot{\mathcal{V}} = \Sigma_{A:\mathcal{V}} A$, by the stability of the classifying type $\mathsf{Fib}(-)$ we can write $\mathsf{Fib} = \Sigma_{A:\mathcal{V}} \mathsf{Fib}(A)$ so that:

$$\dot{\mathsf{Fib}} = \Sigma_{A:\mathcal{V}} \mathsf{Fib}(A) \times A \ \longrightarrow\ \Sigma_{A:\mathcal{V}} \mathsf{Fib}(A) \ = \ \mathsf{Fib}\,.$$

6.4 Realignment for fibration structure

The realignment for families of Proposition 72 will need to be extended to (structured) fibrations. Our approach makes use of the notion of a *weak proposition*. Informally, a map $P \to X$ may be said to be a weak proposition if it is "conditionally contractible", in the sense that it is contractible if it has a section (recall that a proposition may be defined as a fibration that is "contractible if inhabited"). More formally, we have the following.

Definition 85. A map $P \to X$ is said to be a *weak proposition* if the projection $P \times_X P \to P$ is a trivial fibration.

$$\begin{tikzcd} P^2 \arrow[r] \arrow[d, two heads, "\sim"'] \arrow[dr, phantom, "\lrcorner", very near start] & P \arrow[d] \\ P \arrow[r] & X. \end{tikzcd} \tag{133}$$

Note that if either projection is a trivial fibration, then both are.

As an object over the base, a weak proposition is thus one that "thinks it is contractible". The key fact needed for realignment is the following.

Lemma 86. *For any $A \to X$, the classifying type $\mathsf{TFib}(A) \to X$ is a weak proposition. Moreover, the same is true for $\mathsf{Fib}(A) \to X$ (both the biased and unbiased versions) if the cofibrations are closed under exponentiation by the interval* I.

Proof. Let $A \to X$ and consider the following diagram, in which we have written $A' = \mathsf{TFib}(A) \times_X A$ and $\mathsf{TFib}(A)^2 = \mathsf{TFib}(A) \times_X \mathsf{TFib}(A)$.

$$\begin{tikzcd}
A' \arrow[rr] \arrow[dd] \arrow[dr, phantom, very near start, "\lrcorner"] & & A \arrow[dd] & \\
& \mathsf{TFib}(A)^2 \arrow[rr] \arrow[dl] \arrow[dr, phantom, very near start, "\lrcorner"] & & \mathsf{TFib}(A) \arrow[dl] \\
\mathsf{TFib}(A) \arrow[rr] & & X &
\end{tikzcd} \tag{134}$$

Since TFib is stable under pullback (by Lemma 76), we have $\mathsf{TFib}(A)^2 \cong \mathsf{TFib}(A')$, and since $\mathsf{TFib}(A)^2$ has a canonical section, $A' \to \mathsf{TFib}(A)$ is therefore a trivial fibration. Inspecting the definition of $\mathsf{TFib}(A) = +\mathsf{Alg}(A)$ in (109), we see that if a map $A \to X$ is a trivial fibration, then so is $\mathsf{TFib}(A) \to X$ (since $\eta : A \to A^+$ is always a cofibration). Thus $\mathsf{TFib}(A)^2 \cong \mathsf{TFib}(A') \to \mathsf{TFib}(A)$ is also a trivial fibration.

For $\mathsf{Fib}(A) \to X$, with reference to the construction (123) we use the foregoing to infer that $\mathsf{TFib}(\epsilon_A) \to A_\epsilon$ is a weak proposition, and so therefore is its pushforward $F_A = (p_\epsilon)_* \mathsf{TFib}(\epsilon_A) \to X^{\mathrm{I}}$ along the projection $p_\epsilon : A_\epsilon = X^I \times_X A \to X^{\mathrm{I}}$, since pushforward clearly preserves weak propositions. Applying the root $(-)_{\mathrm{I}}$ preserves trivial fibrations, by the assumption that its left adjoint $(-)^{\mathrm{I}}$ preserves cofibrations, and so, as a right adjoint, it also preserves weak propositions. Therefore $(F_A)_{\mathrm{I}} \to (X^{\mathrm{I}})_{\mathrm{I}}$ is a weak proposition, but then so is its pullback along the unit $X \to (X^{\mathrm{I}})_{\mathrm{I}}$, which is $\mathsf{Fib}_i(A) \to X$, the classifier for δ-biased fibration structures. The same reasoning shows that $\mathsf{Fib}(A) = \rho^* \Pi_{i:\mathrm{I}} \mathsf{Fib}_i(\mathrm{I}^* A)$ (as in (129)) is also a weak proposition. $\square$

In light of Lemma 86 we shall assume as a final axiom on cofibrations:

(C8) The pathobject functor preserves cofibrations: thus $c : A \rightarrowtail B$ implies $c^{\mathrm{I}} : A^{\mathrm{I}} \rightarrowtail B^{\mathrm{I}}$.

Now, by Propositions 79 and 82 we have universal small δ-biased and unbiased fibrations, the former in $\mathsf{cSet}/_{\mathrm{I}}$, the latter in cSet. The following remarks apply to both, which we refer to neutrally as $\dot{\mathcal{U}} \twoheadrightarrow \mathcal{U}$. The base object $\mathcal{U}$ is (the domain of) the classifying type $\mathsf{Fib}(\dot{\mathcal{V}}) \to \mathcal{V}$, where $\dot{\mathcal{V}} \to \mathcal{V}$

is the universal small family. Type theoretically, this object can be written as

$$\mathcal{U} = \Sigma_{E:\mathcal{V}} \mathsf{Fib}(E)\,,$$

which comes with the canonical projection

$$\mathcal{U} = \Sigma_{E:\mathcal{V}} \mathsf{Fib}(E) \longrightarrow \mathcal{V}\,.$$

In these terms, a fibration $E \twoheadrightarrow X$ is a pair $\langle E, e\rangle$, consisting of the underlying family $E \to X$, equipped with a fibration structure $e : \mathsf{Fib}(E)$. Lemma 86 then allows us to establish the following, which was first isolated in [**?**] (as condition (2'), also see [Shu15]). It holds for both biased and unbiased fibrations, and will be used in the sequel to "correct" the fibration structure on certain maps.

Lemma 87 (Realignment for fibrations)**.** *Given a fibration $F \twoheadrightarrow X$ and a cofibration $c : C \rightarrowtail X$, let $f_c : C \to \mathcal{U}$ classify the pullback $c^*F \twoheadrightarrow C$. Then there is a classifying map $f : X \to \mathcal{U}$ for F with $f \circ c = f_c$.*

$$\begin{tikzcd}
c^*F \arrow[rr] \arrow[dr] \arrow[dd, two heads] & & \dot{\mathcal{U}} \arrow[dd, two heads] \\
& F \arrow[ur, dotted] \arrow[dd, two heads] & \\
C \arrow[rr, "f_c"] \arrow[dr, tail, "c"'] & & \mathcal{U} \\
& X \arrow[ur, dotted, "f"'] &
\end{tikzcd} \tag{135}$$

Proof. First, let $|f_c| : C \to \mathcal{V}$ be the composite of $f_c : C \to \mathcal{U}$ with the canonical projection $\mathcal{U} \to \mathcal{V}$, thus classifying the underlying family $c^*F \to C$. Next, let $f_0 : X \to \mathcal{V}$ classify the underlying family $F \to X$. We may assume that $f_0 \circ c = |f_c|$ by realignment for families, Proposition 72.

$$\begin{tikzcd}
c^*F \arrow[rr] \arrow[dr] \arrow[dd, two heads] & & \dot{\mathcal{U}} \arrow[r] \arrow[dd, two heads] & \dot{\mathcal{V}} \arrow[dd] \\
& F \arrow[urr, dotted] \arrow[dd, two heads] & & \\
C \arrow[rr, "f_c"] \arrow[dr, tail, "c"'] & & \mathcal{U} \arrow[r] & \mathcal{V} \\
& X \arrow[urr, dotted, "f_0"'] & &
\end{tikzcd} \tag{136}$$

Since $F \twoheadrightarrow X$ is a fibration, there is a lift $f_1 : X \to \mathcal{U}$ of f_0 classifying the fibration structure. We thus have the following commutative diagram in the base of (136).

$$\begin{tikzcd}
C \arrow[r, "f_c"'] \arrow[rr, bend left, "|f_c|"] \arrow[d, tail, "c"'] & \mathcal{U} \arrow[r] & \mathcal{V} \arrow[d, equal] \\
X \arrow[r, "f_1"] \arrow[rr, bend right, "f_0"'] & \mathcal{U} \arrow[r] & \mathcal{V}
\end{tikzcd} \tag{137}$$

Now pull $\mathcal{U} \to \mathcal{V}$ back against itself and rearrange the previous data to give (the solid part of) the following, which also commutes.

$$\begin{tikzcd}
C \arrow[r, "\langle f_1 c, f_c \rangle"'] \arrow[rr, bend left, "f_c"] \arrow[d, tail, "c"'] & \mathcal{U} \times_{\mathcal{V}} \mathcal{U} \arrow[r, "\pi_2"'] \arrow[d, two heads, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{U} \arrow[d] \\
X \arrow[ur, dotted, "f_2"] \arrow[r, "f_1"'] \arrow[rr, bend right, "f_0"'] & \mathcal{U} \arrow[r] & \mathcal{V}
\end{tikzcd} \tag{138}$$

Since $\mathcal{U} = \mathsf{Fib}(\dot{\mathcal{V}}) \to \mathcal{V}$ is a weak proposition by Lemma 86 and (C8), the projection $\pi_1 : \mathcal{U} \times_{\mathcal{V}} \mathcal{U} \twoheadrightarrow \mathcal{U}$ is a trivial fibration, so there is a diagonal filler $f_2 : X \to \mathcal{U} \times_{\mathcal{V}} \mathcal{U}$ as indicated. Taking $f := \pi_2 \circ f_2 : X \to \mathcal{U} \times_{\mathcal{V}} \mathcal{U} \to \mathcal{U}$ gives another classifying map for the fibration structure on $F \to X$, for which $f \circ c = f_c$ as required. $\square$

7 The equivalence extension property

The equivalence extension property is closely related to the *univalence* of the universal fibration $\dot{\mathcal{U}} \twoheadrightarrow \mathcal{U}$ constructed in section 6.3 (see [?]). It will be used in section 8 to show that the base object $\mathcal{U}$ is fibrant. The proof of the equivalence extension property given here is a reformulation of a type-theoretic argument due to Coquand [CCHM16], which in turn is a modification of the original argument of Voevodsky [?]. See [Sat17] for another reformulation.

In this section we are interested mainly in *unbiased* fibrations in cSet. Thus for the remainder of this section "fibration" unqualified will always mean "unbiased fibration in cSet".

We first recall some basic facts and make some simple observations. For any object $Z \in \mathsf{cSet}$, the slice category $\mathsf{cSet}/_Z$ inherits the premodel structure of Proposition 40 from cSet via the forgetful functor $\Sigma_Z : \mathsf{cSet}/_Z \to \mathsf{cSet}$. Thus a map $f : X \to Y$ over Z is a (trivial) cofibration, weak equivalence, or (trivial) fibration, over Z just if it is one in cSet after forgetting the Z-indexing. We shall call this the *sliced premodel structure* on $\mathsf{cSet}/_Z$. The reader is warned that when $Z = \mathrm{I}$ there a possibility of confusion with the δ-biased fibrations in $\mathsf{cSet}/_{\mathrm{I}}$, which do not in general agree with the sliced fibrations.

For any map $f : X \to Y$ in cSet, recall the *pathspace factorization* $f = t \circ s$ indicated below.

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "s"] \arrow[dd, bend right, "="'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "r"'] \arrow[dr, "="] & \\
P_f \arrow[r, "p_0^* f"] \arrow[d, "p_f"] \arrow[rr, dotted, bend right, "t"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y^{\mathrm{I}} \arrow[r, "p_1"] \arrow[d, "p_0"'] & Y \\
X \arrow[r, "f"'] & Y &
\end{tikzcd}
\tag{139}
$$

Here p_0, p_1 are the evaluations $Y^{\delta_0}, Y^{\delta_1}$ at the endpoints $\delta_0, \delta_1 : 1 \to \mathrm{I}$, and let $r := Y^!$ for $! : \mathrm{I} \to 1$, so that $p_0 r = p_1 r = 1_Y$. Then let $p_f := f^* p_0 : P_f \to Y$, the pullback of p_0 along f, and $s := f^* r : X \to P_f$ (over X). Finally, let $t := p_1 \circ p_0^* f : P_f \to Y$ be the indicated horizontal composite. The following facts are well-known in general model categories, but need to be checked again in this case, where we do not yet have a full model structure.

1. If $f : X \to Y$ is a map over a base Z in cSet, we can instead use the relative pathobject $Y^{\mathrm{I}} \to Z$, where the exponential is taken in the slice over Z, and the interval object occurring in the exponent is the result of pulling I from cSet back along $Z \to 1$. The pathspace factorization of (139) can then be constructed in just the same way as before, using the pulled back interval $1 \rightrightarrows \mathrm{I}$ in $\mathsf{cSet}/_Z$, and the factorization $t \circ s : X \to P_f \to Y$ is then stable under pullback along any map $g : Z' \to Z$, in the sense that $g^*(Y^{\mathrm{I}}) \cong g^*(Y)^{\mathrm{I}}$ and so $g^* P_f = P_{g^* f}$, where $g^* f : g^* X \to g^* Y$, and similarly for the factors $g^* s$ and $g^* t$.

2. The retraction $p_0 \circ r = 1_Y$ pulls back along f to a retraction $p_f \circ s = 1_X$.

3. If Y is a fibrant object, then $p_0, p_1 : Y^{\mathrm{I}} \to Y$ are both trivial fibrations by Proposition 29. If $f : X \to Y$ is over a base Z and Y is fibrant over Z, then the maps $p_0, p_1 : Y^{\mathrm{I}} \to Y$ over Z are also trivial fibrations, again by Proposition 29, but now applied to the fibration $p_Y : Y \twoheadrightarrow Z$. (Indeed, as the reader can easily check: the exponential of $p_Y : Y \twoheadrightarrow Z$, taken in $\mathsf{cSet}/_Z$, by a *constant* map $Z^*\delta_\epsilon : Z^*1 \to Z^*\mathrm{I}$, which we continue to write as $\delta_\epsilon : 1 \to \mathrm{I}$, is a map $Y^{\delta_\epsilon} : Y^{\mathrm{I}} \to Y$ over Z; when tested against an arbitrary cofibration $c : A \to C$ over Z via $z : C \to Z$ the map Y^{δ_ϵ} transposes to a filling problem over Z of the form $(c \otimes_z \delta_\epsilon) \pitchfork Y$, where the pushout-product $c \otimes_z \delta_\epsilon : D \to C \times_Z \mathrm{I}$ over Z is easily seen to be equal to the (non-relative) pushout-product $c \otimes \delta_\epsilon : D \to C \times \mathrm{I}$, *because* $\delta_\epsilon : 1 \to \mathrm{I}$ *is constant over* Z. Since $c \otimes \delta_\epsilon \pitchfork p_Y$, it follows that $(c \otimes_z \delta_\epsilon) \pitchfork Y$ over Z, and so $Y^{\delta_\epsilon} : Y^{\mathrm{I}} \to Y$ over Z is indeed a trivial fibration.)

4. If X and Y are both fibrant (over any base $Z \in \mathsf{cSet}$), then $t = p_1 \circ p_0^* f : P_f \to Y$ is a fibration. This can be seen by factoring the maps $p_0, p_1 : Y^{\mathrm{I}} \rightrightarrows Y$ through the product projections as
$$\pi_0 \circ p,\ \pi_1 \circ p : Y^{\mathrm{I}} \to Y \times Y \rightrightarrows Y$$
where $p = (p_0, p_1)$, and then interpolating the pullback $(f, 1_Y) : X \times$

$Y \to Y \times Y$ into (139) as indicated below.

$$
\begin{tikzcd}
 & X \arrow[r, "f"] \arrow[d, "s"] \arrow[dr, phantom, very near start, "\lrcorner"] & Y \arrow[d, "r"'] & \\
 & P_f \arrow[r, "p_0^* f"] \arrow[d, "f^* p"] \arrow[dl, "t"'] \arrow[dr, phantom, very near start, "\lrcorner"] & Y^{\mathrm{I}} \arrow[d, "p"'] \arrow[dr, "p_1"] & \\
Y & X \times Y \arrow[l, "\pi_1'"] \arrow[r, "{(f,1_Y)}"'] \arrow[d, "\pi_0'"'] \arrow[dr, phantom, very near start, "\lrcorner"] & Y \times Y \arrow[r, "\pi_1"'] \arrow[d, "\pi_0"] & Y \\
 & X \arrow[r, "f"'] & Y &
\end{tikzcd}
\tag{140}
$$

The second factor $t = p_1 \circ p_0^* f : P_f \to Y$ now appears also as $\pi_1 \circ (f, 1_Y) \circ f^* p$, which is equal to the pullback $f^* p : P_f \to X \times Y$ followed by the second projection $\pi_1' : X \times Y \to Y$ (which is not a pullback). But if Y is fibrant, then $p : Y^{\mathrm{I}} \to Y \times Y$ is a fibration, again by Proposition 29 applied to the fibration $p_Y : Y \twoheadrightarrow Z$ (by the same argument as for (3), but with the cofibration $\partial : 1 + 1 \rightarrowtail \mathrm{I}$ in place of the trivial cofibration $\delta_\epsilon : 1 \to \mathrm{I}$). Therefore the pullback $f^* p$ is also a fibration. And if X is fibrant, then the second projection $\pi_1' : X \times Y \to Y$ is a fibration. Thus in this case, $t = \pi_1' \circ f^* p : P_f \to Y$ is a fibration, as claimed.

Summarizing (1)-(4), we have shown the following.

Lemma 88. *For any map $f : X \to Y$ (possibly over a base $Z \in \mathsf{cSet}$), there is a stable factorization $f = t \circ s : X \to P_f \to Y$ (over Z), in which s is a section of a trivial fibration p_f if Y is fibrant, and t is a fibration if both X and Y are fibrant.*

$$
\begin{tikzcd}
X \arrow[r, "s"] \arrow[dr, "f"'] & P_f \arrow[d, "t"] \arrow[l, dotted, bend right, "p_f"'] \\
 & Y
\end{tikzcd}
\tag{141}
$$

Note that the retraction $p_f : P_f \to X$ is (over Z but) not over Y.

The following simple fact concerning just the cofibration weak factorization system will also be needed.

Lemma 89. *Let $p : E \twoheadrightarrow B$ be a trivial fibration and $c : C \rightarrowtail B$ a cofibration. Then the unit $\eta : E \to c_* c^* E$ of the base change $c^* \dashv c_*$ along c is a trivial fibration.*

Proof. The unit map $\eta : E \to c_* c^* E$ is the pullback-hom $c \Rightarrow p$, as is easily checked. By lemma 18, for any map $a : A \to Z$ we have the equivalence of diagonal filling conditions,

$$a \pitchfork c \Rightarrow p \quad \text{iff} \quad a \otimes c \pitchfork p.$$

But since $c : C \rightarrowtail B$ is a cofibration, $a \otimes c$ is also a cofibration if $a : A \rightarrowtail Z$ is one, by axiom (C6), which says that cofibrations are closed under pushout-products. So $a \otimes c \pitchfork p$ indeed holds, since p is a trivial fibration. $\square$

Proposition 90 (Equivalence extension property)**.** *Weak equivalences extended along cofibrations in the following sense: given a cofibration $c : C' \rightarrowtail C$ and fibrations $A' \twoheadrightarrow C'$ and $B \twoheadrightarrow C$, and a weak equivalence $w' : A' \simeq c^* B$ over C',*

$$\begin{tikzcd}
A' \arrow[rr, dotted] \arrow[dr, "w'", "\sim"'] \arrow[dd, two heads] & & A \arrow[dr, dotted, "w", "\sim"'] \arrow[dd, dotted, two heads] & \\
& c^*B \arrow[rr] \arrow[dl, two heads] & & B \arrow[dl, two heads] \\
C' \arrow[rr, tail, "c"'] & & C &
\end{tikzcd} \tag{142}$$

there is a fibration $A \twoheadrightarrow C$ and a weak equivalence $w : A \simeq B$ over C that pulls back along $c : C' \rightarrowtail C$ to w', so $c^* w = w'$.

Proof. Call the given fibration $q : B \to C$ and let $b := q^* c : c^* B \to B$ be the indicated pullback, which is thus also a cofibration. Let $w := b_* w' : A \to B$ be the pushforward of w' along b. Composing with q gives the map $p := q \circ w : A \to C$. Since b is monic, we indeed have $b^* w = w'$, thus filling in all the dotted arrows in (142). Note moreover that $c^* w = b^* w = w'$, as required. It remains to show that $p : A \to C$ is a fibration and $w : A \to B$ is a weak equivalence.

$$
\begin{tikzcd}
A' \arrow[rr, tail] \arrow[ddd, two heads] \arrow[dr, "w'", "\sim"'] & & A \arrow[r, "s"] \arrow[ddd, "p"] \arrow[dr, "w"] & P_w \arrow[l, bend right, dotted, two heads, "p_w"'] \arrow[d, "t"] \\
 & c^*B \arrow[rr, tail, "b"] \arrow[ddl, two heads] & & B \arrow[ddl, two heads, "q"] \\
 & & & \\
C' \arrow[rr, tail, "c"] & & C &
\end{tikzcd}
\tag{143}
$$

Let us rename $p' := c^*p : A' \to C'$ and $B' := c^*B$ and $q' := c^*q$. Now let $w = t \circ s$ be the pathspace factorization (139) of w, as a map over C. Since $q : B \to C$ is a fibration, by the foregoing remarks on pathspace factorizations, we know that $s : A \to P_w$ has a retraction $p_w : P_w \to A$ which is a trivial fibration. The retraction p_w is a map over C.

The pathspace factorization $w = t \circ s : A \to P_w \to B$ is stable under pullback along c, providing a pathspace factorization $w' = t' \circ s' : A' \to P_{w'} \to B'$ over C'. Since both p' and q' are fibrations, the retraction $p_{w'} : P_{w'} \to A'$ is a trivial fibration, and now $t' : P_{w'} \to B'$ is a fibration.

$$
\begin{tikzcd}
A' \arrow[r, "s'"] \arrow[dr, "w'", "\sim"'] \arrow[dd, two heads, "p'"'] & P_{w'} \arrow[l, bend right, dotted, two heads, "p_{w'}"'] \arrow[d, two heads, "t'"] \\
 & B' \arrow[dl, two heads, "q'"] \\
C' &
\end{tikzcd}
\tag{144}
$$

Thus the composite $q' \circ t' : P_{w'} \to B' \to C'$ is a fibration and therefore, by the retraction over C' with the trivial fibration $p_{w'}$, we have that $s' : A' \to P_{w'}$ is a weak equivalence, by 3-for-2 for weak equivalences between fibrations. For the same reason, t' is then a weak equivalence, and therefore a trivial fibration.

Since $t' = c^*t = b^*t$ is a trivial fibration, its pushforward b_*b^*t along b is also one by by Corollary 15. Moreover, $b_*b^*t : b_*b^*P_w \to B$ admits a unit $\eta : P_w \to b_*b^*P_w$ (over B).

$$\begin{tikzcd}
A' \arrow[rr, tail] \arrow[dr, "w'", "\sim"'] \arrow[dd, two heads] & & A \arrow[r, "s"] \arrow[drr, "w"] \arrow[dd, "p"] & P_w \arrow[l, dotted, bend right, "p_w"'] \arrow[r, "\eta"] \arrow[d, "t"] & b_*b^*P_w \arrow[dl, "b_*b^*t"] \\
& B' \arrow[rr, tail, "b"] \arrow[dl, two heads] & & B \arrow[dl, two heads, "q"] & \\
C' \arrow[rr, tail, "c"] & & C & &
\end{tikzcd} \tag{145}$$

We now *claim* that $\eta : P_w \to b_*b^*P_w$ is a trivial fibration. Given that, the composite $t = b_*b^*t \circ \eta$ is also a trivial fibration, whence $q \circ t : P_w \to C$ is a fibration, and so its retract $p : A \to C$ is a fibration. Moreover, since s is a section of the trivial fibration $p_w : P_w \to A$ between fibrations, as before it is also a weak equivalence. Thus $w = t \circ s$ is a weak equivalence, and we are finished.

To prove the remaining claim that $\eta : P_w \to b_*b^*P_w$ is a trivial fibration, we shall use lemma 89. It does not apply directly, however, since $t : P_w \to B$ is not yet known to be a trivial fibration. Instead, we show that η is a pullback of the corresponding unit at the trivial fibration $p_1 : B^{\mathrm{I}} \to B$.

Consider the following cube (viewed with $b : B' \to B$ at the front).

$$\begin{tikzcd}
P_{w'} \arrow[rr, tail, "\overline{a}"] \arrow[dr, "(p_0')^*w'"] \arrow[dd, two heads, "p_{w'}"'] & & P_w \arrow[dr, "p_0^*w"] \arrow[dd, "p_w" near start] & \\
& B'^{\mathrm{I}} \arrow[rr, tail, "\overline{b}" near start] \arrow[dd, two heads, "p_0'" near end] & & B^{\mathrm{I}} \arrow[dd, two heads, "p_0"] \\
A' \arrow[rr, tail, "a" near end] \arrow[dr, "w'"'] & & A \arrow[dr, "w"] & \\
& B' \arrow[rr, tail, "b"] & & B
\end{tikzcd} \tag{146}$$

The right hand face is a pullback by definition, and the remainder results from pulling the right face back along b, by the stability of the pathspace factorization (141). Thus all faces are pullbacks. The base is also a pushforward, $b_*w' = w$, again by definition. Thus the top face is also a pushforward, $\overline{b}_*((p_0')^*w') = p_0^*w$. Indeed, since the front face is a pullback, the Beck-Chevalley condition applies, and so we have $\overline{b}_*(p_0')^*(w') = p_0^*\, b_*(w') = p_0^*w$.

Now consider the following, in which the top square remains the same as in (146), but p_0 has been relaced by $p_1 : B^{\mathrm{I}} \to B$, so the composite at right is by definition $t = p_1 \circ p_0^*w$.

$$\begin{tikzcd}
P_{w'} \arrow[rr, tail, "\overline{a}"] \arrow[dr, "(p_0')^*w'"] \arrow[ddr, two heads, "t'"'] & & P_w \arrow[dr, "p_0^*w"] \arrow[ddr, two heads, "t"'] & \\
& B'^{\mathrm{I}} \arrow[rr, tail, "\overline{b}"] \arrow[d, two heads, "p_1'"] & & B^{\mathrm{I}} \arrow[d, two heads, "p_1"] \\
& B' \arrow[rr, tail, "b"'] & & B
\end{tikzcd} \tag{147}$$

The horizontal direction is still pullback along b; let us rename $p_0^*w =: u$ so that $(p_0')^*w' = b^*u$ and $t' = b^*t$ and $p_1' = b^*p_1$ to make this clear. We then add the pushforward along b on the right, in order to obtain the two units η.

$$\begin{tikzcd}
b^*P_w \arrow[rr, tail, "\overline{a}"] \arrow[dr, "b^*u"] \arrow[ddr, two heads, "b^*t"'] & & P_w \arrow[rr, "\eta_t"] \arrow[dr, "u"] \arrow[ddr, two heads, "t"'] & & b_*b^*P_w \arrow[dr, "b_*b^*u"] \arrow[ddr, "b_*b^*t"'] & \\
& b^*B^{\mathrm{I}} \arrow[rr, tail, "\overline{b}"] \arrow[d, two heads, "b^*p_1"] & & B^{\mathrm{I}} \arrow[rr, "\eta_{p_1}"] \arrow[d, two heads, "p_1"] & & b_*b^*B^{\mathrm{I}} \arrow[d, "b_*b^*p_1"] \\
& B' \arrow[rr, tail, "b"'] & & B \arrow[rr, "="'] & & B
\end{tikzcd} \tag{148}$$

By the usual calculation of pushforwards in slice categories, $\overline{b}_* \cong \eta_{p_1}^* \circ b_*$, and so for b^*u we have $\overline{b}_*b^*u = \eta_{p_1}^* b_*b^*u$. But as we just determined in (146) the top left square is already a pushforward, and therefore $u = \eta_{p_1}^* b_*b^*u$, so the top right naturality square is a pullback.

To finish the proof as planned, $p_1 : B^{\mathrm{I}} \to B$ is a trivial fibration because $q : B \to C$ is a fibration, and $b : B' \rightarrowtail B$ is a cofibration because it is a pullback of $c : C' \rightarrowtail C$. Thus by lemma 89, we have that $\eta_{p_1} : B^{\mathrm{I}} \to b_* b^* B^{\mathrm{I}}$ is a trivial fibration, and so its pullback $\eta_t : P_w \to b_* b^* P_w$ is a trivial fibration, as claimed. □

Remark 91. Note that $p : A \to C$ is small if $q : B \to C$ is small.

8 The fibration extension property

Given a universal fibration $\dot{\mathcal{U}} \twoheadrightarrow \mathcal{U}$, such as $\dot{\mathsf{Fib}} \twoheadrightarrow \mathsf{Fib}$ of Proposition 82, the fibration extension property (Definition 55) is closely related to the statement that the base object $\mathcal{U}$ is fibrant. For Kan simplicial sets, Voevodsky proved the latter directly using minimal fibrations [**?**]. In a more general setting, Shulman [**?**] gives a proof using univalence (in the form of the equivalence extension property as stated in section 7), but that proof also uses the 3-for-2 property for weak equivalences, which we do not yet have. For cubical sets, Coquand [CCHM16] uses the equivalence extension property to prove that $\mathcal{U}$ is fibrant, without assuming 3-for-2 for weak equivalences, via a neat argument reducing box-filling to an operation of *Kan-composition*. We shall prove that $\mathcal{U}$ is fibrant using the equivalence extension property, also without assuming 3-for-2 for weak equivalences, via a different argument than that in [CCHM16], one not using Kan composition.

Returning to the relation between the fibration extension property and the fibrancy of the base object of the universal fibration $\dot{\mathcal{U}} \twoheadrightarrow \mathcal{U}$, it is easy to see that the latter implies the former. Indeed, let $t : X \rightarrowtail X'$ be a trivial cofibration and $Y \twoheadrightarrow X$ a fibration. To extend Y along t, take a classifying map $y : X \to \mathcal{U}$, so that $Y \cong y^* \dot{\mathcal{U}}$ over X. If $\mathcal{U}$ is fibrant then we can extend y along $t : X \rightarrowtail X'$ to get $y' : X' \to \mathcal{U}$ with $y = y' \circ t$. The pullback $Y' = (y')^* \dot{\mathcal{U}} \twoheadrightarrow X'$ is then a (small) fibration such that $t^* Y' \cong t^* (y')^* \dot{\mathcal{U}} \cong$

$y^* \dot{\mathcal{U}} \cong Y$ over X.

$$\begin{tikzcd}
Y \arrow[rr] \arrow[dr] \arrow[dd, two heads] & & \dot{\mathcal{U}} \arrow[dd, two heads] \\
& Y' \arrow[ur, dotted] \arrow[dd, two heads] & \\
X \arrow[rr, "y"'] \arrow[dr, tail, "t"'] & & \mathcal{U} \\
& X' \arrow[ur, dotted, "y'"'] &
\end{tikzcd}$$

Thus, for the record, we have:

Proposition 92. *If the base object $\mathcal{U}$ of the universal fibration $\dot{\mathcal{U}} \twoheadrightarrow \mathcal{U}$ is fibrant, then the fibration weak factorization system has the fibration extension property.*

Conversely, given the Realignment Lemma 87, the fibration extension property also implies the fibrancy of $\mathcal{U}$:

Corollary 93. *The fibration extension property implies that the base $\mathcal{U}$ of the universal fibration $\dot{\mathcal{U}} \twoheadrightarrow \mathcal{U}$ is fibrant: given any $y : X \to \mathcal{U}$ and trivial cofibration $t : X \rightarrowtail X'$, there is a map $y' : X' \to \mathcal{U}$ with $y' \circ t = y$.*

Proof. Take the pullback of $\dot{\mathcal{U}} \twoheadrightarrow \mathcal{U}$ along $y : X \to \mathcal{U}$ to get a (small) a fibration $Y \twoheadrightarrow X$, which extends along the (trivial) cofibration $t : X \rightarrowtail X'$ by the fibration extension property, to a (small) fibration $Y' \twoheadrightarrow X'$ with $Y \cong t^* Y'$ over X. By realignment there is a classifying map $y' : X' \to \mathcal{U}$ for Y' with $y' \circ t = y$. $\square$

Now let us show the following.

Proposition 94. *The base $\mathcal{U}$ of the universal fibration $\dot{\mathcal{U}} \twoheadrightarrow \mathcal{U}$ in* cSet*, as constructed in section 6.3, is a fibrant object.*

Proof. By Corollary 24, $\mathcal{U}$ is an unbiased fibrant object if the canonical map

$u = \langle \mathsf{p}_2, \mathsf{eval} \rangle$ in the following diagram in cSet, is a trivial fibration.

$$\begin{tikzcd}
\mathcal{U}^{\mathrm{I}} \times \mathrm{I} \arrow[drr, bend left, "\mathsf{eval}"] \arrow[dr, dotted, "u"'] \arrow[ddr, bend right, "\mathsf{p}_2"'] & & \\
& \mathrm{I} \times \mathcal{U} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{U} \arrow[d] \\
& \mathrm{I} \arrow[r] & 1
\end{tikzcd} \tag{149}$$

Thus consider a filling problem of the following form, with an arbitrary cofibration $c : C \rightarrowtail Z$.

$$\begin{tikzcd}
C \arrow[r] \arrow[d, tail, "c"'] & \mathcal{U}^{\mathrm{I}} \times \mathrm{I} \arrow[d, "\langle \mathsf{p}_2, \mathsf{eval} \rangle"] \\
Z \arrow[r] \arrow[ur, dotted] & \mathrm{I} \times \mathcal{U}
\end{tikzcd} \tag{150}$$

The horizontal maps may be written in the form $\langle z, b \rangle : Z \to \mathrm{I} \times \mathcal{U}$ and $\langle \tilde{a}, zc \rangle : C \to \mathcal{U}^{\mathrm{I}} \times \mathrm{I}$, regarding $z : Z \to \mathrm{I}$ as an I-indexing.

Transposing $\tilde{a}$ to $a : C \times \mathrm{I} \to \mathcal{U}$ we obtain the new problem

$$\begin{tikzcd}
C \arrow[r, "\langle zc \rangle"] \arrow[d, tail, "c"'] & C \times \mathrm{I} \arrow[d, "c \times \mathrm{I}"] \arrow[ddr, bend left, "a"] & \\
Z \arrow[r, "\langle z \rangle"'] \arrow[drr, bend right, "b"'] & Z \times \mathrm{I} \arrow[dr, dotted, "d"] & \\
& & \mathcal{U}
\end{tikzcd} \tag{151}$$

in which we recall from (42) the graph notation $\langle z \rangle = \langle 1_Z, z \rangle : Z \to Z \times \mathrm{I}$. Given a map d as shown in (151), we can obtain the indicated diagonal filler in (150) as $\langle \tilde{d}, z \rangle : Z \to \mathcal{U}^{\mathrm{I}} \times \mathrm{I}$.

As a sanity check, note that $b \circ c = a \circ \langle zc \rangle$ turns the problem (151) into that of extending the copair $[b, a]$ along the unique map

$$Z +_C (C \times \mathrm{I}) \longrightarrow Z \times \mathrm{I}\,,$$

which is exactly the (trivial cofibration) pushout-product $c \otimes_z \delta$ from (43), recalled below for the reader's convenience.

$$
\begin{tikzcd}
C \arrow[r, "\langle zc \rangle"] \arrow[d, tail, "c"'] & C \times \mathrm{I} \arrow[d] \arrow[ddr, bend left, "c \times \mathrm{I}"] & \\
Z \arrow[r] \arrow[drr, bend right, "\langle z \rangle"'] & Z +_C (C \times \mathrm{I}) \arrow[dr, dotted, "c \otimes_z \delta"'] & \\
 & & Z \times \mathrm{I}
\end{tikzcd}
\tag{152}
$$

Returning to (151), take pullbacks of $\dot{\mathcal{U}} \twoheadrightarrow \mathcal{U}$ along a and b to get fibrations $p_a : A \twoheadrightarrow C \times \mathrm{I}$ and $p_b : B \twoheadrightarrow Z$ respectively, and let

$$
p_c := \langle zc \rangle^* p_a : A_c \longrightarrow C
$$

be the corresponding "fiber of A over the graph of zc". We then have $c^*B \cong A_c$ over C by the commutativity of the outer square of (150).

$$
\begin{tikzcd}
 & A_c \arrow[rrr] \arrow[d, two heads, "p_c"] \arrow[ddl] \arrow[dr, phantom, "\lrcorner", very near start] & & A \arrow[d, two heads, "p_a"] \\
 & C \arrow[rrr, "\langle zc \rangle"'] \arrow[ddl, "c"] & & C \times \mathrm{I} \arrow[ddl, "c \times \mathrm{I}"] \\
B \arrow[d, two heads, "p_b"'] & & & \\
Z \arrow[rr, "\langle z \rangle"'] & & Z \times \mathrm{I} &
\end{tikzcd}
$$

The diagonal filler sought in (150) corresponds, again by transposition and pullback of $\dot{\mathcal{U}} \twoheadrightarrow \mathcal{U}$, to a fibration $p_d : D \twoheadrightarrow Z \times \mathrm{I}$ with $\langle z \rangle^* D \cong B$ over Z and $(c \times \mathrm{I})^* D \cong A$ over $C \times \mathrm{I}$, as indicated below.

$$
\begin{tikzcd}
 & A_c \arrow[rrr] \arrow[d, two heads, "p_c"] \arrow[ddl] & & A \arrow[d, two heads, "p_a"] \arrow[ddl, dotted] \\
 & C \arrow[rrr, "\langle zc \rangle"'] \arrow[ddl, "c"] & & C \times \mathrm{I} \arrow[ddl, "c \times \mathrm{I}"] \\
B \arrow[d, two heads, "p_b"'] \arrow[rr, dotted] & & D \arrow[d, dotted, two heads, "p_d"'] & \\
Z \arrow[rr, "\langle z \rangle"'] & & Z \times \mathrm{I} &
\end{tikzcd}
\tag{153}
$$

We shall construct $p_d : D \twoheadrightarrow Z \times \mathrm{I}$ using the equivalence extension property (Proposition 90) as follows. First apply the functor $(-)\times\mathrm{I}$ to the left vertical (pullback) face of the cube in (153) to get the following, with a new pullback square on the right with the indicated fibrations.

$$\begin{tikzcd}
 & A_c \arrow[rr] \arrow[d, two heads, "p_c"] \arrow[ddl] & & A \arrow[d, two heads, "p_a"] \arrow[ddl, dotted] & & A_c \times \mathrm{I} \arrow[dll, two heads, "p_c\times\mathrm{I}"'] \arrow[ddl] \\
 & C \arrow[rr, "\langle zc\rangle"'] \arrow[ddl, "c"] & & C \times \mathrm{I} \arrow[ddl, "c\times\mathrm{I}"] & & \\
B \arrow[rr, dotted] \arrow[d, two heads, "p_b"'] & & D \arrow[d, dotted, two heads, "p_d"'] & & B \times \mathrm{I} \arrow[dll, two heads, "p_b\times\mathrm{I}"] & \\
Z \arrow[rr, "\langle z\rangle"'] & & Z \times \mathrm{I} & & &
\end{tikzcd} \tag{154}$$

We now *claim* that there is a weak equivalence $e : A \simeq A_c \times \mathrm{I}$ over $C \times \mathrm{I}$. From this it follows by the equivalence extension property (Proposition 90) that there are:

(i) a fibration $p_d : D \twoheadrightarrow Z \times \mathrm{I}$ with $(c \times \mathrm{I})^* D \cong A$ over $C \times \mathrm{I}$, and

(ii) a weak equivalence $f : D \simeq B \times \mathrm{I}$ over $Z \times \mathrm{I}$ with $(c \times \mathrm{I})^* f \cong e$ over $C \times \mathrm{I}$.

It then remains only to show that $B \cong \langle z\rangle^* D$ over Z to complete the proof.

To obtain the claimed weak equivalence e, consider the following square,

$$\begin{tikzcd}
A_c \arrow[r, "\langle zcp_c\rangle"] \arrow[d] & A_c \times \mathrm{I} \arrow[d, two heads, "p_c\times\mathrm{I}"] \\
A \arrow[r, two heads, "p_a"'] & C \times \mathrm{I}\,,
\end{tikzcd} \tag{155}$$

in which the top horizontal map is the graph of the composite,

$$\mathbb{A}_c \overset{p_c}{\twoheadrightarrow} C \overset{c}{\rightarrowtail} Z \overset{z}{\to} \mathrm{I}\,,$$

and the others are the evident ones from (154). The square is easily seen to commute, and the top map is a trivial cofibration (by Remark 28), because it is the graph of a map into I. The left map is also a trivial cofibration by Frobenius (Proposition 64), because by its definition in (153) it is the pullback of another such graph $\langle zc \rangle$ along the fibration p_a. A simple lemma (Lemma 95 below) provides the claimed weak equivalence $e : A \simeq A_c \times \mathrm{I}$ over $C \times \mathrm{I}$.

To see that $B \cong \langle z \rangle^* D$ over Z, recall from the proof of the equivalence extension property that the map $f : D \cong B \times \mathrm{I}$ is the pushforward of $e : A \simeq A_c \times \mathrm{I}$ along the cofibration $b_c \times \mathrm{I} : A_c \times \mathrm{I} \rightarrowtail B \times \mathrm{I}$, where we have christened the evident map in (154) $b_c : A_c \rightarrowtail B$. Thus by construction $f = (b_c \times \mathrm{I})_* \, e$. We can then apply the Beck-Chevalley condition for the pushforward using the pullback square on the left below.

$$\begin{tikzcd} A_c \arrow[r, tail, "\langle zcp_c \rangle"] \arrow[d, tail, "b_c"'] \arrow[dr, phantom, "\lrcorner", very near start] & A_c \times \mathrm{I} \arrow[d, tail, "b_c \times \mathrm{I}"] & A \arrow[l, "e"'] \\ B \arrow[r, tail, "\langle zp_b \rangle"'] & B \times \mathrm{I} & D \arrow[l, "f"] \end{tikzcd} \tag{156}$$

The pullback of e along the top of the square is the identity on A_c, as can be seen by pulling back e as a map over $C \times \mathrm{I}$ along $\langle zc \rangle : C \to C \times \mathrm{I}$. Thus the same is true up to isomorophism for the pullback of f along the bottom.

An application of the Realignment Lemma 87 along the trivial cofibration $c \otimes_z \delta$ completes the proof. $\square$

Lemma 95. *Suppose the following square commutes and the indicated cofibrations are trivial.*

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, tail] & C \arrow[d, two heads] \\ B \arrow[r, two heads] & D \end{tikzcd} \tag{157}$$

Then there is a weak equivalence $e : B \simeq C$ *over* D *(and under* A*).*

Proof. Use the fact that any two diagonal fillers are homotopic to get a homotopy equivalence $e : B \simeq C$ filling the square. $\square$

Remark 96. The foregoing proof also works, *mutatis mutandis*, for the case of *biased* fibrations, as in the setting of [CCHM16].

Applying proposition 92 now yields the following.

Corollary 97. *The fibration weak factorization system has the fibration extension property (definition 55).*

By Theorem 59, finally, we have the following.

Theorem 98. *There is a Quillen model structure* $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ *on the category* cSet *of cubical sets for which:*

1. *the cofibrations* $\mathcal{C}$ *are any class of maps satisfying (C0)-(C8) (equivalently, the simplified axioms in the Appendix),*

2. *the fibrations* $\mathcal{F}$ *are the maps* $f : Y \to X$ *for which the canonical map*

$$(f^{\mathrm{I}} \times \mathrm{I}, \mathsf{eval}_Y) : Y^{\mathrm{I}} \times \mathrm{I} \longrightarrow (X^{\mathrm{I}} \times \mathrm{I}) \times_X Y$$

lifts on the right against $\mathcal{C}$*.*

3. *the weak equivalences* $\mathcal{W}$ *are the maps* $w : X \to Y$ *for which the internal precomposition* $K^w : K^Y \to K^X$ *is bijective on connected components for every fibrant object* K*.*

Remark 99. We note that in terms of the universal fibration $\dot{\mathcal{U}} \twoheadrightarrow \mathcal{U}$ constructed in Section 6 the equivalence extension property Proposition 90 says that the second projection from the classifying type of equivalences $A \simeq B$ between small families,

$$\pi_2 : \Sigma_{A,B}\mathsf{Eq}(A, B) \longrightarrow \mathcal{U}\,,$$

is a trivial fibration. From this, it follows that the canonical transport map

$$* : \mathcal{U}^{\mathrm{I}} \longrightarrow \Sigma_{A,B}\mathsf{Eq}(A, B) \tag{158}$$

is an equivalence over the base $\mathcal{U}$ via $p_2 : \mathcal{U}^{\mathrm{I}} \to \mathcal{U}$, which is also a trivial fibration, because $\mathcal{U}$ is fibrant (by Proposition 94). In type theory, the equivalence (158) can be expressed as

$$(A = B) \simeq (A \simeq B)\,.$$

Appendix A: Axioms for cartesian cofibrations

A system of maps satisfying the axioms (C0)-(C8) in the main text, for the cofibrations in a cartesian cubical model category, will be called *cartesian cofibrations*. The axioms can be restated equivalently as follows.

(A0) All cofibrations are monomorphisms.

(A1) All isomorphisms are cofibrations.

(A2) The composite of two cofibrations is a cofibration.

(A3) Any pullback of a cofibration is a cofibration.

(A4) The category of cofibrations and cartesian squares has a terminal object.

(A5) The join of two cofibrant subobjects is a cofibration.

(A6) The diagonal map $\delta : \mathrm{I} \to \mathrm{I} \times \mathrm{I}$ is a cofibration.

(A7) If $c : A \to B$ is a cofibration, then so is $c^{\mathrm{I}} : A^{\mathrm{I}} \to B^{\mathrm{I}}$.

Proposition 100. *In any topos, the locally decidable subobjects satisfy the axioms for cartesian cofibrations.*

Proof. [*fill in ...*] □

Appendix B: Cubical sets as a classifying topos

Recall that the objects of the Cartesian cube category $\square$ may be taken concretely to be finite, strictly bipointed sets, written

$$[n] = \{\bot, x_1, ..., x_n, \top\},$$

and the arrows $f : [n] \to [m]$ to be all bipointed maps $[m] \to [n]$ (note the direction). The category of *(Cartesian) cubical sets* is the presheaf topos

$$\mathsf{cSet} = \mathsf{Set}^{\square^{\mathrm{op}}}.$$

It is generated by the representable presheaves $\mathrm{I}^n := \mathsf{y}[n]$, called the *n-cubes*. The 0-cube is $\mathrm{I}^0 = \mathsf{y}[0] = 1$; the 1-cube is $\mathrm{I} = \mathsf{y}[1]$; and $\mathrm{I}^n \times \mathrm{I}^m \cong \mathrm{I}^{n+m}$ in virtue of preservation of products by the Yoneda embedding $\mathsf{y} : \square^{\mathrm{op}} \hookrightarrow \mathsf{cSet}$. For a cubical set $X : \square^{\mathrm{op}} \to \mathsf{Set}$ we write $X_n = X[n]$ and call this the *set of n-cubes in X*, for which we have the usual Yoneda correspondence,

$$\{c \in X_n\} \;\cong\; \{c : \mathrm{I}^n \to X\}.$$

In particular, $\mathrm{I}^n_m = \mathsf{cSet}([m], [n])$ is the set of m-cubes in the n-cube.[2]

[2] Note that the cardinality of I^n_m is therefore just $(m+2)^n$, in comparison to the *Dedekind* cubes, for which *e.g.* $\mathsf{cSet}([1], [n])$ the n^{th} *Dedekind number*, the number of elements in the free distributive lattice on n generators, which in general is a number so large that it is not known, even for values of $n > 7$.

Proposition 101. *The category* cSet *of Cartesian cubical sets is the classifying topos for* intervals*: objects* $\mathcal{I}$ *with points* $i, j : 1 \rightrightarrows \mathcal{I}$ *the pullback of which is* 0*:*

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "j"] \\ 1 \arrow[r, "i"'] & \mathcal{I} \end{tikzcd}$$

Proof. Consider the covariant presentation $\mathsf{cSet} = \mathsf{Set}^{\mathbb{B}}$ where $\mathbb{B}$ is the category of finite, strictly bipointed sets and bipointed maps. We can extend $\mathbb{B} \hookrightarrow \mathbb{B}_{=}$ by freely adjoining coequalizers, making $\mathbb{B}_{=}$ the free finite *colimit* category on a co-bipointed object. An concrete presentation of $\mathbb{B}_{=}$ is the finite bipointed sets, including those with $\bot = \top$. Let us write (n) for the bipointed set $\{x_1, ..., x_n, \bot = \top\}$, with n (non-constant) elements and a further element $\bot = \top$. There is an evident coequalizer $[1] \rightrightarrows [n] \to (n)$, which just identifies the distinguished points, and every coqualizer has either the form $[m] \rightrightarrows [n] \to [k]$ or $[m] \rightrightarrows [n] \to (k)$, for a suitable choice of k. Note that there are no maps of the form $(m) \to [n]$, and that every map $[m] \to (n)$ factors uniquely as $[m] \to (m) \to (n)$ with $[m] \to (m)$ the canonical coequalizer of $\bot$ and $\top$. The category $\mathbb{B}_{=}$ can therefore be decomposed into two "levels", the upper one of which is essentially $\mathbb{B}$, the lower one consisting of just the objects (n) and thus essentially the finite *pointed* sets, and for each n, there is the canonical coequalizer $[n] \to (n)$ going from the upper level to the lower one.

$$\begin{tikzcd} \dots \arrow[r] & {[m]} \arrow[r] \arrow[d, two heads] & {[n]} \arrow[r] \arrow[d, two heads] & \dots \\ \dots \arrow[r] & (m) \arrow[r] & (n) \arrow[r] & \dots \end{tikzcd}$$

Write $u : \mathbb{B} \to \mathbb{B}_{=}$ for the upper inclusion, which is the classifying functor of generic co-bipointed object in $\mathbb{B}_{=}$ (which is strict).

Now consider the induced geometric morphism:

$$\begin{tikzcd} \mathsf{Set}^{\mathbb{B}} \arrow[r, shift left=3, "u_*"] \arrow[r, shift right=3, "u_!"'] & \mathsf{Set}^{\mathbb{B}_{=}} \arrow[l, "u^*" description] \end{tikzcd} \qquad u_! \dashv u^* \dashv u_*$$

Since u^* is the restriction along u, the right adjoint u_* must be "prolongation by 1”,

$$\begin{aligned} u_*(P)[n] &= P[n], \\ u_*(P)(n) &= \{*\}, \end{aligned}$$

with the obvious maps,

$$\begin{tikzcd} \ldots \arrow[r] & P[m] \arrow[r] \arrow[d, two heads] & P[n] \arrow[r] \arrow[d, two heads] & \ldots \\ \ldots \arrow[r] & \{*\} \arrow[r] & \{*\} \arrow[r] & \ldots \end{tikzcd}$$

as is easily seen by considering maps in $\mathsf{Set}^{\mathbb{B}_=}$ of the form

$$\begin{tikzcd} Q[n] \arrow[r] \arrow[d, "\cdot"'] & P[n] \arrow[d, two heads, "\cdot"] \\ Q(n) \arrow[r] & \{*\}. \end{tikzcd}$$

Since $u_* : \mathsf{Set}^{\mathbb{B}} \to \mathsf{Set}^{\mathbb{B}_=}$ is evidently full and faithful, it is the inclusion part of a sheaf subtopos $\mathsf{sh}(\mathbb{B}_=^{\mathsf{op}}, j) \hookrightarrow \mathsf{Set}^{\mathbb{B}_=}$ for a suitable Grothendieck topology j on $\mathbb{B}_=^{\mathsf{op}}$. We claim that j is the closed complement topology of the subobject $[\bot = \top] \rightarrowtail 1$ represented by the coequalizer $[0] \to (0)$. Indeed, in $\mathsf{Set}^{\mathbb{B}_=}$ we have the representable functors:

$$\begin{aligned} \mathrm{I} &= y[1], \\ 1 &= y[0], \\ [\bot = \top] &= y(0) \end{aligned}$$

fitting into an equalizer $[\bot = \top] \to 1 \rightrightarrows \mathrm{I}$, which is the image under Yoneda of the canonical coequalizer $[1] \rightrightarrows [0] \to (0)$ in $\mathbb{B}_=$. The closed complement topology for $[\bot = \top] \rightarrowtail 1$ is generated by the single cover $0 \to [\bot = \top]$, which can be described logically as forcing the sequent $(\bot = \top \vdash \bot)$ to hold. Recall from [**?**], Proposition 3.53, the following simple characterization of the sheaves for a closed topology generated by an object $U \rightarrowtail 1$: an object X is a sheaf iff $X \times U \cong U$. In the present case, it therefore suffices to show that for any $P : \mathbb{B}_= \to \mathsf{Set}$ we have:

$$P \times [\bot = \top] \cong [\bot = \top] \quad \text{iff} \quad P(n) = 1 \text{ for all } n.$$

For any object $b \in \mathbb{B}_=$, consider the map

$$\mathrm{Hom}(yb, P \times [0=1]) \cong \mathrm{Hom}(yb, P) \times \mathrm{Hom}(yb, [\bot = \top]) \to \mathrm{Hom}(yb, [\bot = \top]).$$

If $b = [k]$, then $\mathrm{Hom}(yb, [\bot = \top]) \cong \mathrm{Hom}_{\mathbb{B}_=}((0), [k]) \cong 0$, and so we always have an iso

$$\begin{aligned} \mathrm{Hom}(yb, P \times [\bot = \top]) &\cong \mathrm{Hom}(yb, P) \times \mathrm{Hom}(yb, [\bot = \top]) \\ &\cong \mathrm{Hom}(yb, P) \times 0 \cong 0. \end{aligned}$$

If $b = (k)$, then $\mathrm{Hom}(y(k), [\bot = \top]) \cong \mathrm{Hom}_{\mathbb{B}_=}((0), (k)) \cong 1$, and we have an iso

$$\begin{aligned} \mathrm{Hom}(y(k), P \times [\bot = \top]) &\cong \mathrm{Hom}(y(k), P) \times \mathrm{Hom}(y(k), [\bot = \top]) \\ &\cong \mathrm{Hom}(y(k), P) \times 1 \cong \mathrm{Hom}(y(k), P) \cong P(k). \end{aligned}$$

Thus we will have an iso $P \times [\bot = \top] \cong [\bot = \top]$ iff $P(k) \cong 1$.

Thus the presheaf topos $\mathsf{Set}^{\mathbb{B}}$ is the closed complement of the open subtopos

$$\mathsf{Set}^{\mathbb{B}_=}/_{[\bot=\top]} \hookrightarrow \mathsf{Set}^{\mathbb{B}_=}$$

given by forcing the proposition $\bot \neq \top$. Since $\mathsf{Set}^{\mathbb{B}_=}$ is clearly the classifying topos for *arbitrary* bipointed objects $\bot, \top : 1 \to B$, the subtopos $\mathsf{Set}^{\mathbb{B}}$ indeed classifies *strictly* bipointed objects, as claimed. □

Corollary 102. *The geometric realization functor to topological spaces*

$$R : \mathsf{cSet} \to \mathsf{Top}$$

preserves cartesian products, $R(X \times Y) \cong R(X) \times R(Y)$.[3]

Proof. This can of course be shown directly, but it follows immediately by composing the inverse image of the classifying geometric morphism $\mathsf{sSets} \to \mathsf{cSet}$ of the 1-simplex Δ^1 with the standard geometric realization $\mathsf{sSets} \to \mathsf{Top}$, each of which preserves finite products. □

We conclude with a few general remarks about the category cSet.

[3] This fact and the next one are to be contrasted with the case of monoidal cubical sets, *e.g.* as studied by [**?**, **?**]

Definition 103. Let $\square \to \mathsf{Cat}$ be the unique product-preserving functor taking the interval [1] to the one arrow category $\mathbb{2} = (0 \leq 1)$. This functor then takes $[n]$ to $\mathbb{2}^n$, the n-fold product in Cat, and maps $[m] \to [n]$ to the corresponding monotone functions of the posets $\mathbb{2}^n$. The *cubical nerve* functor

$$N : \mathsf{Cat} \to \mathsf{cSet}$$

is then defined by:

$$N(\mathbb{C})_n = \mathsf{Cat}(\mathbb{2}^n, \mathbb{C}).$$

Thus $N(\mathbb{C})_0$ is the set of objects of $\mathbb{C}$; $N(\mathbb{C})_1$ is the set of arrows; $N(\mathbb{C})_2$ consists of all commutative squares; $N(\mathbb{C})_3$ all commutative cubes, etc.

Proposition 104. *The nerve functor* $N : \mathsf{Cat} \to \mathsf{cSet}$ *is full and faithful.*

Proof. Given categories $\mathbb{C}$ and $\mathbb{D}$ and functors $F, G : \mathbb{C} \to \mathbb{D}$, suppose $F(f) \neq G(f)$ for some $f : A \to B$ in $\mathbb{C}$. Take $f^\sharp : \mathbb{2} \to \mathbb{C}$ with image f. Then $N(F)_1(f^\sharp) = F(f) \neq G(f) = N(G)_1(f^\sharp)$, and so $N(F) \neq N(G) : N(\mathbb{C}) \to N(\mathbb{D})$. So N is faithful.

For fullness, let $\varphi : N(\mathbb{C}) \to N(\mathbb{D})$ be a natural transformation, and define a proposed functor $F : \mathbb{C} \to \mathbb{D}$ by

$$\begin{aligned} F_0 = \varphi_0 : \mathbb{C}_0 = N(\mathbb{C})_0 \to N(\mathbb{D})_0 = \mathbb{D}_0 \\ F_1 = \varphi_1 : \mathbb{C}_1 = N(\mathbb{C})_1 \to N(\mathbb{D})_1 = \mathbb{D}_1. \end{aligned}$$

We just need to show that F preserves identity arrows and composition. Consider the following diagram.

$$\begin{tikzcd} \mathsf{Cat}(\mathbb{2}^1, \mathbb{C}) = N(\mathbb{C})_1 \arrow[r, "F_1"] & N(\mathbb{D})_1 = \mathsf{Cat}(\mathbb{2}^1, \mathbb{D}) \\ \mathsf{Cat}(\mathbb{2}^0, \mathbb{C}) = N(\mathbb{C})_0 \arrow[u, "!^*"] \arrow[r, "F_0"'] & N(\mathbb{D})_0 = \mathsf{Cat}(\mathbb{2}^0, \mathbb{D}). \arrow[u, "!^*"'] \end{tikzcd}$$

Here $!^* : \mathsf{Cat}(\mathbb{2}^0, \mathbb{C}) \to \mathsf{Cat}(\mathbb{2}, \mathbb{C})$ is precomposition with $! : \mathbb{2} = \mathbb{2}^1 \to \mathbb{2}^0 = \mathbb{1}$, so the diagram commutes. But since $! : \mathbb{2} \to \mathbb{1}$ is a functor,

$$\mathbb{C}_0 = \mathsf{Cat}(\mathbb{1}, \mathbb{C}) \xrightarrow{!^*} \mathsf{Cat}(\mathbb{2}, \mathbb{C}) = \mathbb{C}_1$$

takes objects in $\mathbb{C}$ to their identity arrows. Thus F preserves identity arrows. Similarly, for composition, consider

$$\begin{tikzcd} \mathsf{Cat}(\mathbb{2}^2, \mathbb{C}) = N(\mathbb{C})_2 \arrow[r, "\varphi_2"] \arrow[d, "d^*"'] & N(\mathbb{D})_2 = \mathsf{Cat}(\mathbb{2}^2, \mathbb{D}) \arrow[d, "d^*"] \\ \mathsf{Cat}(\mathbb{2}, \mathbb{C}) = N(\mathbb{C})_1 \arrow[r, "F_1"'] & N(\mathbb{D})_1 = \mathsf{Cat}(\mathbb{2}, \mathbb{D}). \end{tikzcd}$$

where $\varphi_2 : N(\mathbb{C})_2 \to N(\mathbb{D})_2$ is the action of φ on commutative squares of arrows, and $d^* : \mathsf{Cat}(\mathbb{2}^2, \mathbb{C}) \to \mathsf{Cat}(\mathbb{2}, \mathbb{C})$ is precomposition with the diagonal map $d : \mathbb{2} \to \mathbb{2}^2 = \mathbb{2} \times \mathbb{2}$, so the diagram commutes. For any composable pair of arrows $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathbb{C}$ there is a commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ B \arrow[r, "g"'] & C, \end{tikzcd}$$

and the effect of $d^* : \mathsf{Cat}(\mathbb{2}^2, \mathbb{C}) \to \mathsf{Cat}(\mathbb{2}, \mathbb{C})$ on this square is exactly $g \circ f : A \to C$, and similarly for $d^* : \mathsf{Cat}(\mathbb{2}^2, \mathbb{D}) \to \mathsf{Cat}(\mathbb{2}, \mathbb{D})$. Thus the commutativity of the above diagram implies that F preserves composition. Since clearly $N(F) = \varphi$, we indeed have that N is also full. □

Proposition 105. *For any cubical set X, the exponential X^{I} is calculated as the "shift by one dimension",*

$$X^{\mathrm{I}}(n) \cong X(n+1)\,.$$

Proof.

$$\begin{aligned} X^{\mathrm{I}}(n) &\cong \mathrm{Hom}(y[n], X^{\mathrm{I}}) \cong \mathrm{Hom}(\mathrm{I}^n, X^{\mathrm{I}}) \;\cong\; \mathrm{Hom}(\mathrm{I}^n \times \mathrm{I}, X) \\ &\cong\; \mathrm{Hom}(\mathrm{I}^{n+1}, X) \cong\; \mathrm{Hom}(y[n+1], X) \;\cong\; X(n+1). \end{aligned}$$

□

Corollary 106. *The functor $X \mapsto X^{\mathrm{I}}$ has a* right *adjoint.*

Proof. The functor $X \mapsto X^{\mathrm{I}}$ is given by precomposition with the "successor" functor $S : \square \to \square$ with $S[n] = [n+1]$. Thus $X^{\mathrm{I}}([n]) = X(S[n]) =$

$(S^*(X))([n])$. Precomposition always has a right adjoint $S^* \dashv S_*$, which can be calculated as:

$$S_*(X)(n) \cong \mathrm{Hom}(y[n], S_*X) \cong \mathrm{Hom}(S^*(y[n]), X) \ \cong \mathrm{Hom}(\square(S(-), [n]), X).$$

□

We need the following fact in order to calculate the right adjoint further.

Lemma 107. *In* cSet, *we have* $\mathrm{I}^{\mathrm{I}} \ \cong \ \mathrm{I} + 1$.

Proof. For any $[n] \in \square$ we have:

$$(\mathrm{I}^{\mathrm{I}})(n) \cong \mathrm{I}(n+1) \cong \mathrm{Hom}(\mathrm{I}^{(n+1)}, \mathrm{I}) \cong \square([n+1], [1]) \cong \mathbb{B}([1], [n+1]) \cong n+3.$$

On the other hand,

$$(\mathrm{I}+1)(n) \cong \mathrm{I}(n) + 1(n) \cong \mathrm{Hom}(\mathrm{I}^n, \mathrm{I}) + 1 \cong \mathbb{B}([1], [n]) + 1 \cong (n+2) + 1.$$

The isomorphism is natural in n. □

We mention that a similar fact holds for the generic object in the object classifier topos, and in the Schanuel topos, and is used in the theory of "abstract higher-order syntax" [**?**, **?**].

Definition 108. Let us write

$$X_{\mathrm{I}} = S_*(X)$$

for the right adjoint of the path object functor $X^{\mathrm{I}} = S^*X$.

Corollary 109. *We have the following calculation for the right adjoint* X_{I}*:*

$$\begin{aligned} X_{\mathrm{I}}(n) &\cong \mathrm{Hom}(\mathrm{I}^n, X_{\mathrm{I}}) \\ &\cong \mathrm{Hom}((\mathrm{I}^n)^{\mathrm{I}}, X) \\ &\cong \mathrm{Hom}((\mathrm{I}^{\mathrm{I}})^n, X) \\ &\cong \mathrm{Hom}((\mathrm{I}+1)^n, X) \\ &\cong \mathrm{Hom}(\mathrm{I}^n + C^n_{n-1}\mathrm{I}^{n-1} + \cdots + C^n_1 \mathrm{I} + 1, X) \\ &\cong X(n) \times X(n-1)^{C^n_{n-1}} \times \cdots \times X(1)^{C^n_1} \times X(0), \end{aligned}$$

where $C^n_k = \binom{n}{k}$ *is the usual binomial coefficient.*

Corollary 110. *There is a natural transformation $X_{\mathrm{I}} \to X$, given by the first projection from $X_{\mathrm{I}}(n) \cong X(n) \times X(n-1)^{C^n_{n-1}} \times \cdots \times X(1)^{C^n_1} \times X(0)$.*

Finally, we observe that that the path object functor X^{I} itself, as a left adjoint, preserves all *co*limits. This does not hold in general in type theory, but will be a special property of the cubical model. (Cf. Lawvere [**?**] on the notion of "tiny" objects and the "amazing right adjoint".)

Example. (P. Aczel) The cubical set P of polynomials (over the integers, say), is defined by:

$$P_n = \{p(x_1, ..., x_n) \mid \text{polynomials in at most } x_1, ..., x_n\}$$

with the evident maps $P_m \to P_n$ for each function $[m] \to [n]$.

This is a ring object in the category of cubical sets, and the interval $\mathrm{I} = y[1]$ embeds into P. The same is true for any algebraic theory $\mathbb{T}$ with two constants, such as boolean algebras: there is a cubical $\mathbb{T}$-algebra A and a monic $\mathrm{I} \rightarrowtail A$.

Let $\Box[\mathrm{I}] = \Box$ be the cube category, classifying intervals, and $\Box[\mathbb{T}]$ the classifying category for $\mathbb{T}$-algebras. There is an interval J in $\Box_{\mathbb{T}}$ consisting of the generic $\mathbb{T}$-algebra and its two constants. This J has a classifying functor $J : \Box_{\mathrm{I}} \to \Box_{\mathbb{T}}$, inducing functors on presheaves

$$J_! \dashv J^* \dashv J_* : \mathsf{Set}^{\Box_{\mathrm{I}}^{\mathrm{op}}} \to \mathsf{Set}^{\Box_{\mathbb{T}}^{\mathrm{op}}}$$

as usual, where $J_! \circ \mathsf{y}_{\Box_{\mathrm{I}}} = \mathsf{y}_{\Box_{\mathbb{T}}} \circ J$, with y the respective Yoneda embeddings.

We can calculate:

$$\begin{aligned} J^* J_!(\mathrm{I})([n]) &= J^* J_!(Y[1])([n]) \\ &= J^* Y(J[1])([n]) = Y(J[1])(J[n]) \\ &= \Box_{\mathbb{T}}(J[n], J[1]) = \mathbb{T}{-}\mathrm{Alg}(J[1], J[n]) \\ &= \mathbb{T}{-}\mathrm{Alg}(F(1), F(n)) = |F(n)|, \end{aligned} \tag{159}$$

where $F(n)$ is the free $\mathbb{T}$-algebra on n generators. So in the case of polynomials we indeed have

$$P = J^* J_!(\mathrm{I}).$$

The unit of the adjunction $\mathrm{I} \to J^* J_!(\mathrm{I})$ is faithful, since J itself is faithful and therefore the left adjoint $J_!$ is faithful. P is a ring in $\mathsf{Set}^{\Box_{\mathrm{I}}^{\mathrm{op}}}$ since $J_!(\mathrm{I})$ is a ring in $\mathsf{Set}^{\Box_{\mathbb{T}}^{\mathrm{op}}}$ and J^* is left exact.

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