Cartesian cubical model categories

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Abstract

Add an abstract.

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Introduction

1 Cartesian cubical sets

There are now many treatments of cubical sets in the literature, including [?, ?, ?, ?, ?, ?, ?, ?]. Our construction is intended to work in *all* of these, insofar as the axioms in Definition **??** below are satisfied. For the sake of concreteness, however, we shall consider what may be called the *cartesian* cube category $\square$, defined as the free finite product category on an interval $\delta_0, \delta_1 : 1 \rightrightarrows \mathrm{I}$.

Definition 1. The objects of the cartesian cube category $\square$, called n-cubes, are finite sets of the form

$$[n] = \{0, x_1, ..., x_n, 1\}\,,$$

where $x_1, ..., x_n$, are formal *generators*. The arrows,

$$f : [n] \to [m]\,,$$

may be taken to be m-tuples of elements drawn from the set $\{0, x_1, ..., x_n, 1\}$, regarded as formal terms representing composites of faces, degeneracies, permutations, and diagonals. Equivalently, the arrows $[n] \to [m]$ are arbitrary bipointed maps $[m] \to [n]$.

See [**?**] for further details.

Definition 2. The category cSet of *cubical sets* is the category of presheaves on the cartesian cube category $\square$,

$$\mathsf{cSet} \;=\; \mathsf{Set}^{\square^{\mathrm{op}}}.$$

It is of course generated by the representable presheaves $\mathsf{y}[n]$, to be written

$$\mathrm{I}^n = \mathsf{y}[n]$$

and called the *geometric n-cubes.*

Note that the representables I^n are closed under finite products, $\mathrm{I}^n \times \mathrm{I}^m = \mathrm{I}^{n+m}$. We of course write I for I^1 and 1 for I^0. We will need the following basic fact about the cubes I^n in cSet.

Proposition 3. *For each n, the n-cube I^n is* tiny*, in the sense that the exponential (or "internal* Hom*") functor $(-)^{\mathrm{I}^n} : \mathsf{cSet} \longrightarrow \mathsf{cSet}$ has a right adjoint.*

(See [**?**] for more on such "amazing right adjoints".)

Proof. It clearly suffices to prove the claim for $n = 1$. For any cubical set X, the exponential X^{I} is a "shift by one dimension",

$$X^{\mathrm{I}}(n) \cong \mathrm{Hom}(\mathrm{I}^n, X^{\mathrm{I}}) \;\cong\; \mathrm{Hom}(\mathrm{I}^{n+1}, X) \cong\; X(n+1).$$

Thus X^{I} is given by precomposition with the "successor" functor $\square \to \square$ with $[n] \mapsto [n+1]$. Precomposition always has a right adjoint, which in this case we write as

$$X^{\mathrm{I}} \dashv X_{\mathrm{I}}$$

and calculate to be:

$$\begin{aligned} X_{\mathrm{I}}(n) &\cong \mathrm{Hom}(\mathrm{I}^n, X_{\mathrm{I}}) \\ &\cong \mathrm{Hom}((\mathrm{I}^n)^{\mathrm{I}}, X) \\ &\cong \mathrm{Hom}((\mathrm{I}^{\mathrm{I}})^n, X) \\ &\cong \mathrm{Hom}((\mathrm{I}+1)^n, X) \\ &\cong \mathrm{Hom}\Big(\sum_{k=0}^{n} \binom{n}{k} \mathrm{I}^k, X\Big) \\ &\cong \prod_{k=0}^{n} X(k)^{\binom{n}{k}}, \end{aligned}$$

using the fact that $\mathrm{I}^{\mathrm{I}} \cong (\mathrm{I}+1)$ as in [**?**]. $\square$

Remark 4. size of the homsets in cartesian vs. other cubes. fullness of the cubical nerve functor from Cat. other stuff from old notes.

2 The cofibration weak factorization system

Definition 5 (Cofibration)**.** The *cofibrations*, written

$$c : A \rightarrowtail B\,,$$

are any class $\mathcal{C}$ of monomorphisms in cSet satisfying the following axioms:

(C0) The map $0 \to C$ is always a cofibration.

(C1) All isomorphisms are cofibrations.

(C2) The composite of two cofibrations is a cofibration.

(C3) Any pullback of a cofibration is a cofibration.

We also require the cofibrations to be classified by a subobject $\Phi \hookrightarrow \Omega$ of the standard subobject classifier $\top : 1 \to \Omega$ of cSet:

(C4) There is a terminal object $t : 1 \rightarrowtail \Phi$ in the category of cofibrations and cartesian squares.

Two further axioms for cofibrations will be added in Section 3.1, one in Section 3.2, and a final one in Section 6.4 (see Appendix 8 for a summary). Note that we also permit the case $\Phi = \Omega$, so that all monos are cofibrations, in which case no axioms are required.

The cofibrant partial map classifier. Write

$$X^+ := \sum_{\varphi : \Phi} X^{[\varphi]} \;=\; \Phi_! \, t_*(X)\,, \tag{1}$$

for the polynomial endofunctor $\mathsf{cSet} \longrightarrow \mathsf{cSet}$ determined by the cofibration classifier $t : 1 \rightarrowtail \Phi$ (see [?]). The reader familiar with type theory will recognize the similarity to the "partiality" or "lifting" monad.

Observe that since t is monic there is a pullback square,

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X^+ \arrow[d, "t_* X"] \\ 1 \arrow[r, "t"'] & \Phi. \end{tikzcd}$$

Let $\eta : X \rightarrowtail X^+$ be the indicated top horizontal map; we call this map the *cofibrant partial map classifier* of X.

Proposition 6. *The map $\eta : X \rightarrowtail X^+$ classifies partial maps into X with cofibrant domain, in the following sense.*

1. *The map $\eta : X \rightarrowtail X^+$ is a cofibration.*

2. *For any object Z and any partial map $(s, g) : Z \leftarrowtail S \to X$, with $s : S \rightarrowtail Z$ a cofibration, there is a unique $f : Z \to X^+$ making a*

pullback square as follows.

$$\begin{tikzcd} S \arrow[r, "g"] \arrow[d, tail, "s"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, tail, "\eta"] \\ Z \arrow[r, "f"'] & X^+ \end{tikzcd}$$

Proof. The map $\eta : X \rightarrowtail X^+$ is a cofibration since it is a pullback of $t : 1 \to \Phi$. Observe that $(\eta, 1_X) : X^+ \leftarrowtail X \to X$ is therefore a partial map into X with cofibrant domain. The second statement is the universal property of X^+ as a polynomial (see [?], prop. 7). □

Proposition 7. *The pointed endofunctor determined by* $\eta_X : X \rightarrowtail X^+$ *has a natural multiplication* $\mu_X : X^{++} \to X^+$ *making it a monad.*

Proof. Since the cofibrations are closed under composition, the monad structure on X^+ follows as in [?], proposition XY. Explicitly, μ_X is determined by proposition 6 as the unique map making the following a pullback diagram.

$$\begin{tikzcd} X \arrow[r, "="] \arrow[d, tail, "\eta_X"'] & X \arrow[dd, tail, "\eta"] \\ X^+ \arrow[d, tail, "\eta_{X^+}"'] & \\ X^{++} \arrow[r, dotted, "\mu"'] & X^+ \end{tikzcd}$$

□

Relative partial map classifier. For any object $X \in \mathsf{cSet}$ the usual pullback functor

$$X^* : \mathsf{cSet} \to \mathsf{cSet}/_X\,,$$

taking any A to the second projection $A \times X \to X$, not only preserves the subobject classifier Ω, but also the cofibration classifier $\Phi \hookrightarrow \Omega$, where a map in $\mathsf{cSet}/_X$ is defined to be a cofibration if it is one in cSet. Thus in $\mathsf{cSet}/_X$ the *(relative) cofibration classifier* is the map

$$t \times X : 1 \times X \to \Phi \times X \quad \text{over } X$$

which we may also write $t_X : 1_X \to \Phi_X$. Like $t : 1 \to \Phi$, this map determines a polynomial endofunctor

$$+_X : \mathsf{cSet}/_X \to \mathsf{cSet}/_X\,,$$

which commutes (up to natural isomorphism) with $+ : \mathsf{cSet} \to \mathsf{cSet}$ and $X^* : \mathsf{cSet} \to \mathsf{cSet}/_X$ in the evident way:

$$\begin{tikzcd} \mathsf{cSet}/_X \arrow[r, "+_X"] & \mathsf{cSet}/_X \\ \mathsf{cSet} \arrow[u, "X^*"] \arrow[r, "+"'] & \mathsf{cSet} \arrow[u, "X^*"'] \end{tikzcd} \tag{2}$$

The endofunctor $+_X$ is also pointed $\eta_Y : Y \to Y^+$ and has a natural monad multiplication $\mu_Y : Y^{++} \to Y^+$, for any $Y \to X$, for the same reason that $+$ has this structure. Summarizing, we may say that *the polynomial monad* $+ : \mathsf{cSet} \to \mathsf{cSet}$ *is indexed (or fibered) over* cSet.

Definition 8. A *+-algebra* in cSet is a cubical set A together with a retraction $\alpha : A^+ \to A$ of $\eta_A : A \to A^+$, i.e. an algebra for the pointed endofunctor ($+ : \mathsf{cSet} \to \mathsf{cSet}$, $\eta : 1 \to +$). Algebras for the monad $(+, \eta, \mu)$ will be referred to specifically as $(+, \eta, \mu)$*-algebras*, or *+-monad algebras.*

A *relative +-algebra* in cSet is a map $A \to X$ together with an algebra structure over the codomain X for the pointed endofunctor $+_X : \mathsf{cSet}/_X \to \mathsf{cSet}/_X$.

The cofibration weak factorization system.

Proposition 9. *There is an (algebraic) weak factoriation system on* cSet *with the cofibrations as the left class and as the right class, the maps underlying the relative +-algebras. Thus a right map is one* $f : A \to X$ *for which there is a retract* $\alpha : A' \to A$ *over* X *of the canonical map* $\eta_f : A \to A'$*,*

$$\begin{tikzcd} A \arrow[r, "\eta_f"] \arrow[rr, bend left, "="] \arrow[dr, "f"'] & A' \arrow[r, "\alpha"] \arrow[d, "f^+"] & A \arrow[dl, "f"] \\ & X. & \end{tikzcd}$$

Proof. The factorization of a map $f : Y \to X$ is given by applying the relative $+$-functor over the codomain,

$$\begin{tikzcd} Y \arrow[r, tail, "\eta_f"] \arrow[dr, "f"'] & Y' \arrow[d, "f^{+X}"] \\ & X. \end{tikzcd}$$

We know by proposition 6 that the unit η_f is always a cofibration, and since f^{+X} is the free algebra for the $+_X$-monad, it is in particular a $+_X$-algebra.

For the lifting condition, consider a cofibration $c : B \rightarrowtail C$, a right map $A \to X$, with $+_X$-algebra structure map $\alpha : A^{+X} \to A$ over X, and a commutative square as indicated in the following.

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, tail, "c"'] & A \arrow[d] \arrow[dr, "\eta"] & \\ & & A^{+X} \arrow[ul, bend right, "\alpha"'] \arrow[dl] \\ C \arrow[r, "f"'] & X & \end{tikzcd}$$

Thus over X, we have the situation

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, tail, "c"'] & A \arrow[d, "\eta"] \\ C \arrow[ur, dotted, "d"'] & A^+ \arrow[u, bend right=60, "\alpha"'] \end{tikzcd}$$

and we seek a diagonal filler d as indicated. Since $(c, g) : B \leftarrowtail C \to A$ is a cofibrant partial map into A, there is a map $\varphi : C \to A^+$ (over X) making a (pullback) square,

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, tail, "c"'] & A \arrow[d, "\eta"] \\ C \arrow[r, dotted, "\varphi"'] & A^+ \arrow[u, bend right=60, "\alpha"'] \end{tikzcd}$$

We thus have $d := \alpha \circ \varphi : C \to A$ as the required diagonal filler.

The closure of the cofibrations under retracts follows from their classification by a universal object $t : 1 \to \Phi$, and the closure of the right maps under retracts follows from their being the algebras for a pointed endofunctor underlying a monad (cf. [**?**]). Algebraicity of this weak factorization system is immediate, since $+$ is a monad. $\square$

Summarizing, we have an algebraic weak factorization system $(\mathcal{C}, \mathcal{C}^{\pitchfork})$ on the category cSet of cubical sets, where:

$$\begin{aligned} \mathcal{C} &= \text{the cofibrations} \\ \mathcal{C}^{\pitchfork} &= \text{the maps underlying relative } +\text{-algebras} \end{aligned}$$

We shall call this the *cofibration weak factorization system.* The right maps will be denoted

$$\mathsf{TFib} = \mathcal{C}^{\pitchfork}$$

and called *trivial fibrations.*

The cofibration algebraic weak factorization system is a refinement of the one defined in [**?**] and mentioned in [**?**].

Uniform filling structure. It is convenient to relate relative $+$-algebra structure with the more familiar diagonal filling condition of cofibrantly generated weak factorization systems, and specifically the special form occuring in [CCHM16] under the name *uniform filling structure.*

Consider a generating sub*set* of cofibrations consisting of those with representable codomain $c : C \rightarrowtail \mathrm{I}^n$, and call these the *basic cofibrations.*

$$\mathsf{BCof} = \{c : C \rightarrowtail \mathrm{I}^n \mid c \in \mathcal{C}, n \geq 0\}. \tag{3}$$

Proposition 10. *For any object X in cSet the following are equivalent:*

1. *X admits a* $+$-algebra structure: *a retraction $\alpha : X^+ \to X$ of the unit $\eta : X \to X^+$.*

2. *$X \to 1$ is a* trivial fibration: *it has the right lifting property with respect to all cofibrations,*

$$\mathcal{C} \pitchfork X.$$

3. *X admits a* uniform filling structure: *for each basic cofibration $c : C \rightarrowtail \mathrm{I}^n$ and map $x : C \to X$ there is given an extension $j(c, x)$,*

$$\begin{tikzcd} C \arrow[r, "x"] \arrow[d, tail, "c"'] & X, \\ \mathrm{I}^n \arrow[ur, dotted, "j(c,x)"'] & \end{tikzcd} \tag{4}$$

and the choice is uniform in I^n *in the following sense.*

Given any cubical map $u : \mathrm{I}^m \to \mathrm{I}^n$*, the pullback* $u^*c : u^*C \rightarrowtail \mathrm{I}^m$*, which is again a basic cofibration, fits into a commutative diagram of the form*

$$\begin{tikzcd} u^*C \arrow[r, "c^*u"] \arrow[d, tail, "u^*c"'] \arrow[dr, phantom, very near start, "\lrcorner"] & C \arrow[r, "x"] \arrow[d, tail, "c"] & X. \\ \mathrm{I}^m \arrow[r, "u"'] & \mathrm{I}^n \arrow[ur, dotted, "j(c,x)"'] & \end{tikzcd} \tag{5}$$

For the pair $(u^*c,\ x \circ c^*u)$ *in* (5)*, the chosen extension* $j(u^*c, x \circ c^*u) : \mathrm{I}^m \to X$*, is required to be equal to* $j(c,x) \circ u$*,*

$$j(u^*c, x \circ c^*u) = j(c,x) \circ u. \tag{6}$$

Proof. Let (X, α) be a $+$-*algebra* and suppose given the span (c,x) as below, with c a cofibration.

$$\begin{tikzcd} C \arrow[r, "x"] \arrow[d, tail, "c"'] & X \\ Z & \end{tikzcd}$$

Let $\chi(c,x) : Z \to X^+$ be the classifying map of the evident partial map $(c,x) : Z \rightharpoonup X$, so that we have a pullback square as follows.

$$\begin{tikzcd} C \arrow[r, "x"] \arrow[d, tail, "c"'] \arrow[dr, phantom, very near start, "\lrcorner"] & X \arrow[d, "\eta"] \\ Z \arrow[r, "\chi(c,x)"'] & X^+ \end{tikzcd} \tag{7}$$

Then set

$$j = \alpha \circ \chi(c,x) : Z \to X \tag{8}$$

to get a filler,

$$\begin{tikzcd} C \arrow[r, "x"] \arrow[d, tail, "c"'] & X \arrow[d, "\eta"] \\ Z \arrow[r, "\chi(c,x)"'] \arrow[ur, dotted, "j"] & X^+ \arrow[u, bend right=60, "\alpha"'] \end{tikzcd} \tag{9}$$

since

$$j \circ c = \alpha \circ \chi(c,x) \circ c = \alpha \circ \eta \circ x = x.$$

Thus (1) implies (2). To see that it also implies (3), observe that in the case where $Z = I^n$ and we specify, in (8), that

$$j(c, x) = \alpha \circ \chi(c, x) : \mathrm{I}^n \to X, \tag{10}$$

then the assignment is natural in I^n. Indeed, given any $u : \mathrm{I}^m \to \mathrm{I}^n$, we have

$$j(c', xu') = \alpha \circ \chi(c', xu') = \alpha \circ \chi(c, x) \circ u = j(c, x)u, \tag{11}$$

by the uniqueness of the classifying maps.

It is clear that (2) implies (1), since if $\mathcal{C} \pitchfork X$ then we can take as an algebra structure $\alpha : X^+ \to X$ any filler for the span

$$\begin{tikzcd}
X \arrow[r, "="] \arrow[d, tail, "\eta"'] & X. \\
X^+ \arrow[ur, dotted, "\alpha"'] &
\end{tikzcd}$$

To see that (3) implies (1), suppose that X has a uniform filling structure j and we want to define an algebra structure $\alpha : X^+ \to X$. By Yoneda, for every $y : \mathrm{I}^n \to X^+$ we need a map $\alpha(y) : \mathrm{I}^n \to X$, naturally in I^n, in the sense that for any $u : \mathrm{I}^m \to \mathrm{I}^n$, we have

$$\alpha(yu) = \alpha(y)u. \tag{12}$$

Moreover, to ensure that $\alpha\eta = 1_X$, for any $x : \mathrm{I}^n \to X$ we must have $\alpha(\eta \circ x) = x$. So take $y : \mathrm{I}^n \to X^+$ and let

$$\alpha(y) = j(y^*\eta, y'),$$

as indicated on the right below.

$$\begin{tikzcd}
u^*C \arrow[r, "u'"] \arrow[d, tail, "u^*y^*\eta"'] \arrow[dr, phantom, very near start, "\lrcorner"] & C \arrow[rr, "y'"] \arrow[d, tail, "y^*\eta"'] \arrow[dr, phantom, very near start, "\lrcorner"] & & X. \arrow[d, "\eta"] \\
\mathrm{I}^m \arrow[r, "u"'] & \mathrm{I}^n \arrow[rr, "y"'] \arrow[urr, dotted, "j(y^*\eta, y')"'] & & X^+
\end{tikzcd} \tag{13}$$

Then for any $u : \mathrm{I}^m \to \mathrm{I}^n$, we indeed have

$$\alpha(yu) = j\big((yu)^*\eta, y'u'\big) = j(y^*\eta, y') \circ u = \alpha(y)u,$$

by the uniformity of j. Finally, if $y = \eta \circ x$ for some $x : \mathrm{I}^n \to X$ then

$$\alpha(\eta x) = j\big((\eta x)^*\eta, (\eta x)'\big) = j(1_X, x) = x,$$

because the defining diagram for $\alpha(\eta x)$, i.e. the one on the right in (13), then factors as

$$\begin{tikzcd}
\mathrm{I}^n \arrow[r, "x"] \arrow[d, tail, "="'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[r, "="] \arrow[d, tail, "="'] \arrow[dr, phantom, "\lrcorner", very near start] & X, \arrow[d, "\eta"] \\
\mathrm{I}^n \arrow[r, "x"'] & X \arrow[r, "\eta"'] & X^+
\end{tikzcd} \tag{14}$$

and the only possible extension $j(1_X, x)$ for the span $(1_{\mathrm{I}^n}, x)$ is x itself. $\square$

The relative version of the foregoing is entirely analogous, since the +-functor is fibered over cSet in the sense of diagram (2). We can therefore omit the entirely analogous proof of the following.

Proposition 11. *For any map $f : Y \to X$ in cSet the following are equivalent:*

1. *$f : Y \to X$ admits a* relative +-algebra structure over X, *i.e. there is a retraction $\alpha : Y' \to Y$ over X of the unit $\eta : Y \to Y'$, where $f^+ : Y' \to X$ is the result of the relative +-functor applied to f, as in definition 8.*

2. *$f : Y \to X$ is a* trivial fibration,

$$\mathcal{C} \pitchfork f.$$

3. *$f : Y \to X$ admits a* uniform filling structure*: for each basic cofibration $c : C \rightarrowtail \mathrm{I}^n$ and maps $x : C \to X$ and $y : \mathrm{I}^n \to Y$ making the square below commute, there is given a diagonal filler $j(c, x, y)$,*

$$\begin{tikzcd}
C \arrow[r, "x"] \arrow[d, tail, "c"'] & X \arrow[d, "f"] \\
\mathrm{I}^n \arrow[r, "y"'] \arrow[ur, dotted, "{j(c,x,y)}"'] & Y,
\end{tikzcd} \tag{15}$$

and the choice is uniform in I^n *in the following sense: given any cubical map $u : \mathrm{I}^m \to \mathrm{I}^n$, the pullback $u^*c : u^*C \rightarrowtail \mathrm{I}^m$ is again a basic* *cofibration and fits into a commutative diagram of the form*

$$\begin{tikzcd}
u^*C \arrow[r, "c^*u"] \arrow[d, rightarrowtail, "u^*c"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[r, "x"] \arrow[d, rightarrowtail, "c"'] & X \arrow[d, "f"] \\
\mathrm{I}^m \arrow[r, "u"'] & \mathrm{I}^n \arrow[r, "y"'] \arrow[ur, dotted, "j(c,x,y)"'] & Y.
\end{tikzcd} \tag{16}$$

For the evident triple $(u^*c,\ x \circ c^*u, y \circ u)$ *in* (16) *the chosen diagonal filler*

$$j(u^*c, x \circ c^*u, y \circ u) : \mathrm{I}^m \to X$$

is equal to $j(c, x, y) \circ u$,

$$j(u^*c, x \circ c^*u, y \circ u) = j(c, x, y) \circ u. \tag{17}$$

We next collect some basic facts about trivial fibrations: they have sections, they are closed under composition and retracts, and they are closed under pullback and pushforward along all maps.

Corollary 12. *1. Every trivial fibration* $A \to X$ *has a section* $s : X \to A$.

2. If $f : Y \to X$ *is a trivial fibration and* $g : Z \to Y$ *is a trivial fibration, then* $f \circ g : Z \to X$ *is a trivial fibration.*

3. If $f : Y \to X$ *is a trivial fibration and* $f' : Y' \to X'$ *is a retract of* f *in the arrow category, then* f' *is a trivial fibration.*

4. For any map $f : Y \to X$ *and any trivial fibration* $A \to X$*, the pullback* $f^*A \to Y$ *is a trivial fibration.*

5. For any map $f : Y \to X$ *and any trivial fibration* $A \to Y$*, the pushforward* $f_*A \to X$ *is a trivial fibration.*

Proof. (1) holds because all objects are cofibrant by (C0). (5) is a consequence of (C3), stability of cofibrations under pullback, by a standard argument using the adjunction $f^* \dashv f_*$. The rest hold for the right maps in any weak factorization system. □

3 The fibration weak factorization system

We now specify a second weak factorization system, with a restricted class of "trivial" cofibrations on the left, and an expanded class of right maps, the *fibrations*. For comparison, we first recall the "biased" trivial-cofibration/fibration weak factorization system from [GS17], which makes use of *connections*,

$$\vee, \wedge : \mathrm{I} \times \mathrm{I} \longrightarrow \mathrm{I},$$

on the cubes, which we do not assume (in [AGH21] it is shown that the fibrations of [GS17] agree with those specified in the "logical style" of [CCHM16, OP17]).

3.1 Partial box filling (biased version)

A *generating class of biased trivial cofibrations* are all maps of the form

$$c \otimes \delta_\epsilon : D \rightarrowtail Z \times \mathrm{I}\,, \tag{18}$$

where:

1. $c : C \rightarrowtail Z$ is an arbitrary cofibration,
2. $\delta_\epsilon : 1 \to \mathrm{I}$ is one of the two *endpoint inclusions*, for $\epsilon = 0, 1$.
3. $c \otimes \delta_\epsilon$ is the *pushout-product* indicated in the following diagram.

$$\begin{tikzcd}
C \times 1 \arrow[r, "C \times \delta_\epsilon"] \arrow[d, "C \times 1"'] & C \times \mathrm{I} \arrow[d] \arrow[ddr, bend left, "c \times \mathrm{I}"] & \\
Z \times 1 \arrow[r] \arrow[drr, bend right, "Z \times \delta_\epsilon"'] & Z +_C (C \times \mathrm{I}) \arrow[dr, dotted, "c \otimes \delta_\epsilon"] & \\
 & & Z \times \mathrm{I}
\end{tikzcd} \tag{19}$$

4. $D = Z +_C (C \times \mathrm{I})$ is the indicated domain of $c \otimes \delta_\epsilon$.

In order to ensure that such maps are indeed cofibrations, we assume two further axioms:

(C5) The endpoint inclusions $\delta_\epsilon : 1 \to \mathrm{I}$ are cofibrations.

(C6) The cofibrations are closed under pushout-products.

Note that if we assume δ_0 and δ_1 are disjoint (as they are in most categories of cubical sets), then by (C5) we have that $0 \to 1$ is a cofibration, and hence that $0 \to A$ is a cofibration, for all objects A, so that (C0) is no longer required. In place of (C6), we could require the cofibrations to be closed under the join operation $A \vee B$ in the lattice of subobjects of an object (as is done in [CCHM16, OP17]).

Fibrations (biased version). Let

$$\mathcal{C} \otimes \delta_\epsilon \;=\; \{c \otimes \delta_\epsilon : D \rightarrowtail Z \times \mathrm{I} \mid c \in \mathcal{C},\ \epsilon = 0, 1\}$$

be the class of all such generating biased trivial cofibrations. The *biased fibrations* are defined to be the right class of these maps,

$$(\mathcal{C} \otimes \delta_\epsilon)^{\pitchfork} \;=\; \mathcal{F}\,.$$

Thus a map $f : Y \to X$ is a biased fibration if for every commutative square of the form

$$\begin{tikzcd} Z +_C (C \times \mathrm{I}) \arrow[r] \arrow[d, tail, "c \otimes \delta_\epsilon"'] & Y \arrow[d, "f"] \\ Z \times \mathrm{I} \arrow[r] \arrow[ur, dotted, "j"] & X \end{tikzcd} \tag{20}$$

with a generating biased trivial cofibration on the left, there is a diagonal filler j as indicated.

To relate this notion of fibration to the cofibration weak factorization system, fix any map $u : A \to B$, and recall (e.g. from [**?**]) that the pushout-product with u is a functor on the arrow category

$$(-) \otimes u : \mathsf{cSet}^{\mathbb{2}} \to \mathsf{cSet}^{\mathbb{2}}\,.$$

This functor has a right adjoint, the *pullback-hom*, which for a map $f : X \to Y$ we shall write as

$$(u \Rightarrow f) : Y^B \longrightarrow (X^B \times_{X^A} Y^A)\,.$$

The pullback-hom is determined as indicated in the following diagram.

$$
\begin{tikzcd}
Y^B \arrow[dr, dotted, "u \Rightarrow f"] \arrow[drr, bend left, "Y^u"] \arrow[ddr, bend right, "f^B"'] & & \\
& X^B \times_{X^A} Y^A \arrow[r] \arrow[d] & Y^A \arrow[d, "f^A"] \\
& X^B \arrow[r, "X^u"'] & X^A
\end{tikzcd}
\tag{21}
$$

The $\otimes \dashv \Rightarrow$ adjunction on the arrow category has the following useful relation to weak factorization systems (cf. [GS17, Rie14, **?**]), where, as usual, for any maps $a : A \to B$ and $f : X \to Y$ we write

$$a \pitchfork f$$

to mean that for every solid square of the form

$$
\begin{tikzcd}
A \arrow[r] \arrow[d, "a"'] & X \arrow[d, "f"] \\
B \arrow[r] \arrow[ur, dotted, "j"'] & Y
\end{tikzcd}
\tag{22}
$$

there exists a diagonal filler j as indicated.

Lemma 13. *For any maps $a : A_0 \to A_1, b : B_0 \to B_1, c : C_0 \to C_1$ in cSet,*

$$(a \otimes b) \pitchfork c \quad \textit{iff} \quad a \pitchfork (b \Rightarrow c)\,.$$

The following is now a direct corollary.

Proposition 14. *An object X is fibrant if and only if both of the endpoint projections $X^{\mathrm{I}} \to X$ from the pathspace are trivial fibrations. More generally, a map $f : Y \to X$ is a fibration iff both of the maps*

$$(\delta_\epsilon \Rightarrow f) : Y^I \to X^I \times_X Y$$

are trivial fibrations (for $\epsilon = 0, 1$).

Fibration structure (biased version). The $\otimes \dashv \Rightarrow$ adjunction determines the fibrations in terms of the trivial fibrations, which in turn can be determined by *uniform* lifting against a *set* of basic cofibrations, by proposition 11. The fibrations are similarly determined by *uniform* lifting against the *set* of biased trivial cofibrations consisting of all those $c \otimes \delta_\epsilon$ in $\mathcal{C} \otimes \delta_\epsilon$ where $c : C \rightarrowtail \mathrm{I}^n$ is a basic cofibration. Call these maps the *basic biased trivial cofibrations*, and let

$$\mathsf{BCof} \otimes \delta_\epsilon = \{c \otimes \delta_\epsilon : B \rightarrowtail \mathrm{I}^{n+1} \mid c : C \rightarrowtail \mathrm{I}^n,\ \epsilon = 0, 1,\ n \geq 0\}, \tag{23}$$

where the pushout-product $c \otimes \delta_\epsilon$ now takes the simpler form

$$\begin{tikzcd}
C \arrow[r] \arrow[d, tail] & C \times \mathrm{I} \arrow[d] \arrow[ddr, bend left] & \\
\mathrm{I}^n \arrow[r] \arrow[drr, bend right] & \mathrm{I}^n +_C (C \times \mathrm{I}) \arrow[dr, tail, "c \otimes \delta_\epsilon"'] & \\
& & \mathrm{I}^n \times \mathrm{I}
\end{tikzcd} \tag{24}$$

for a basic cofibration $c : C \rightarrowtail \mathrm{I}^n$, an endpoint $\delta_\epsilon : 1 \to \mathrm{I}$, and with domain $B = \big(\mathrm{I}^n +_C (C \times \mathrm{I})\big)$. These subobjects $B \rightarrowtail \mathrm{I}^{n+1}$ can be seen geometrically as generalized open box inclusions.

For any map $f : Y \to X$ a *uniform, biased fibration structure* on f is a choice of diagonal fillers $j_\epsilon(c, x, y)$,

$$\begin{tikzcd}
\mathrm{I}^n +_C (C \times \mathrm{I}) \arrow[r, "x"] \arrow[d, "c \otimes \delta_\epsilon"'] & X \arrow[d, "f"] \\
\mathrm{I}^n \times \mathrm{I} \arrow[r, "y"'] \arrow[ur, dotted, "j_\epsilon(c,x,y)"] & Y,
\end{tikzcd} \tag{25}$$

for each basic biased trivial cofibration $c \otimes \delta_\epsilon : B = (\mathrm{I}^n +_C (C \times \mathrm{I})) \rightarrowtail \mathrm{I}^{n+1}$ and maps $x : B \to X$ and $y : \mathrm{I}^{n+1} \to Y$, which is *uniform in* I^n in the following sense: Given any cubical map $u : \mathrm{I}^m \to \mathrm{I}^n$, the pullback $u^* c : u^* C \rightarrowtail \mathrm{I}^m$ of $c : C \rightarrowtail \mathrm{I}^n$ along u determines another basic biased trivial cofibration

$$u^* c \otimes \delta_\epsilon : B' = (\mathrm{I}^m +_{u^* C} (u^* C \times \mathrm{I})) \rightarrowtail \mathrm{I}^{m+1},$$

which fits into a commutative diagram of the form

$$\begin{tikzcd}
\mathrm{I}^m +_{u^*C} (u^*C \times \mathrm{I}) \arrow[r, "(u\times \mathrm{I})'"] \arrow[d, "u^*c\otimes\delta_\epsilon"'] & \mathrm{I}^n +_C (C \times \mathrm{I}) \arrow[r, "x"] \arrow[d, "c\otimes\delta_\epsilon"'] & X \arrow[d, "f"] \\
\mathrm{I}^m \times \mathrm{I} \arrow[r, "u\times\mathrm{I}"'] & \mathrm{I}^n \times \mathrm{I} \arrow[r, "y"'] \arrow[ur, dotted, "j_\epsilon(c,x,y)"'] & Y,
\end{tikzcd} \tag{26}$$

by applying the functor $(-) \otimes \delta_\epsilon$ to the pullback square relating u^*c to c. For the outer rectangle in (28) there is then a chosen diagonal filler

$$j_\epsilon(u^*c, x \circ (u \times \mathrm{I})', y \circ (u \times \mathrm{I})) : \mathrm{I}^m \times \mathrm{I} \to X$$

and for this map we require that

$$j_\epsilon(u^*c, x \circ (u \times \mathrm{I})', y \circ (u \times \mathrm{I})) = j_\epsilon(c, x, y) \circ (u \times \mathrm{I}). \tag{27}$$

This can be shown to be a reformulation of the logical specification given in [CCHM16] (see [AGH21]).

Definition 15. A *uniform, biased fibration structure* on a map $f : Y \to X$ is a choice of fillers $j_\epsilon(c, x, y)$ as in (25) satisfying (27) for all maps $u : \mathrm{I}^m \to \mathrm{I}^n$.

Finally, we have the analogue of proposition 10 for fibrant objects; we omit the analogous statement of proposition 11 for fibrations, as well as the entirely analogous proof.

Corollary 16. *For any object X in cSet the following are equivalent:*

1. *X is* biased fibrant, *i.e. every partial map to X with a generating biased trivial cofibration $D \rightarrowtail Z \times \mathrm{I}$ as domain of definition extends to a total map $Z \times \mathrm{I} \to X$,*
$$\mathcal{C} \otimes \delta_\epsilon \pitchfork X\,.$$
2. *The canonical maps $(\delta_\epsilon \Rightarrow X) : X^I \to X$ are trivial fibrations.*
3. *$X \to 1$ admits a* uniform biased fibration structure. *Explicitly, for each basic biased trivial cofibration $c \otimes \delta_\epsilon : B \rightarrowtail \mathrm{I}^{n+1}$ and map $x : B \to X$, there is given an extension $j_\epsilon(c, x)$,*

$$\begin{tikzcd}
B \arrow[r, "x"] \arrow[d, tail, "c\otimes\delta_\epsilon"'] & X, \\
\mathrm{I}^{n+1} \arrow[ur, dotted, "j_\epsilon(c,x)"'] &
\end{tikzcd} \tag{28}$$

and the choice is uniform in I^n *in the following sense: Given any cubical map* $u : \mathrm{I}^m \to \mathrm{I}^n$, *the pullback* $u^*c \otimes \delta_\epsilon : B' \rightarrowtail \mathrm{I}^m \times \mathrm{I}$ *fits into a commutative diagram of the form*

$$\begin{tikzcd} B' \arrow[r, "(u\times\mathrm{I})'"] \arrow[d, tail, "u^*c\otimes\delta_\epsilon"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[r, "x"] \arrow[d, tail, "c\otimes\delta_\epsilon"'] & X. \\ \mathrm{I}^m \times \mathrm{I} \arrow[r, "u\times\mathrm{I}"'] & \mathrm{I}^n \times \mathrm{I} \arrow[ur, dotted, "j(c,x)"'] & \end{tikzcd} \tag{29}$$

For the pair $(u^*c \otimes \delta_\epsilon,\ x \circ (u \times \mathrm{I})')$ *in* (29) *the chosen extension*

$$j(u^*c \otimes \delta_\epsilon, x \circ (u \times \mathrm{I})') : \mathrm{I}^m \times \mathrm{I} \to X$$

is equal to $j(c, x) \circ (u \times \mathrm{I})$,

$$j(u^*c \otimes \delta_\epsilon, x \circ (u \times \mathrm{I})') = j(c, x)(u \times \mathrm{I}). \tag{30}$$

3.2 Partial box filling (unbiased version)

Rather than building a weak factorization system based on the foregoing notion of biased fibration (as is done in [GS17]), we shall first eliminate the "bias" on a choice of endpoint $\delta_\epsilon : 1 \to \mathrm{I}$, expressed by the indexing $\epsilon = 0, 1$. This will have the effect of adding more trivial cofibrations, and thus more weak equivalences, to our model structure. Consider first the simple path-lifting condition for a map $f : Y \to X$, which is a special case of (20) with $c = ! : 0 \rightarrowtail 1$, since $! \otimes \delta_\epsilon = \delta_\epsilon$:

$$\begin{tikzcd} 1 \arrow[r] \arrow[d, tail, "\delta_\epsilon"'] & Y \arrow[d, "f"] \\ \mathrm{I} \arrow[r] \arrow[ur, dotted, "j_\epsilon"] & X \end{tikzcd}$$

In topological spaces, for instance, rather than requiring lifts j_ϵ for each of the endpoints $\epsilon = 0, 1$ of the real interval $\mathrm{I} = [0, 1]$, one could instead require there to be a lift j_i *for each point* $i : 1 \to \mathrm{I}$. Such "unbiased path-lifting" can be formulated in cSet by introducing a "generic point" $\delta : 1 \to \mathrm{I}$ by passing to cSet/I via the pullback functor $\mathrm{I}^* : \mathsf{cSet} \to \mathsf{cSet}/\mathrm{I}$, and then requiring path-lifting for I^*f with respect to δ. The following specification implements that idea, while also adding cofibrant partiality, as in the biased case. We first replace axiom (C5) with the following stronger assumption.

(C7) The diagonal map $\delta : \mathrm{I} \to \mathrm{I} \times \mathrm{I}$ is a cofibration.

The unbiased notion of a fibration is now as follows.

Definition 17 (Fibration)**.** Let $\delta : \mathrm{I} \to \mathrm{I} \times \mathrm{I}$ be the diagonal map.

1. An object X is *fibrant* if the map

$$(\delta \Rightarrow X) = \langle \mathsf{eval}, p_2 \rangle : X^{\mathrm{I}} \times \mathrm{I} \to X \times \mathrm{I}$$

   is a trivial fibration.

2. A map $f : Y \to X$ is an *fibration* if the map

$$(\delta \Rightarrow f) = \langle f^{\mathrm{I}} \times \mathrm{I}, \langle \mathsf{eval}, p_2 \rangle \rangle : Y^{\mathrm{I}} \times \mathrm{I} \to (X^{\mathrm{I}} \times \mathrm{I}) \times_{(X \times \mathrm{I})} (Y \times \mathrm{I})$$

   is a trivial fibration.

Condition (1) above, which is of course a special case of (2), says that evaluation at the generic point $\delta : 1 \to \mathrm{I}$, i.e. the map $X^{\delta} : X^{\mathrm{I}} \to X$ constructed in the slice category cSet/I, is a trivial fibration. Condition (2) says that the pullback-hom of the generic point $\delta : 1 \to \mathrm{I}$ with $\mathrm{I}^* f$, constructed in the slice category cSet/I, is a trivial fibration. The latter can be reformulated as follows.

Proposition 18. *A map $f : Y \to X$ is a fibration if and only if the canonical map c to the pullback, in the following diagram in* cSet*, is a trivial fibration.*

$$\begin{tikzcd}
Y^{\mathrm{I}} \times \mathrm{I} \arrow[dr, dotted, "c"] \arrow[drr, bend left, "\mathsf{eval}"] \arrow[ddr, bend right, "f^{\mathrm{I}} \times \mathrm{I}"'] & & \\
& Y_{\mathsf{eval}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "f"] \\
& X^{\mathrm{I}} \times \mathrm{I} \arrow[r, "\mathsf{eval}"'] & X.
\end{tikzcd} \tag{31}$$

Proof. We interpolate another pullback into the rectangle in (31) to obtain

$$\begin{tikzcd}
Y_{\mathsf{eval}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \times \mathrm{I} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "f"] \\
X^{\mathrm{I}} \times \mathrm{I} \arrow[r] & X \times \mathrm{I} \arrow[r] & X
\end{tikzcd} \tag{32}$$

with the evident maps. The left hand square is therefore a pullback, so we indeed have that

$$Y_{\mathsf{eval}} \;=\; (X^{\mathrm{I}} \times \mathrm{I}) \times_{(X\times\mathrm{I})} (Y \times \mathrm{I})$$

and $c = (\delta \Rightarrow f)$. $\square$

Now we can run the proof of Proposition 14 backwards in order to determine a class of generating trivial cofibrations for the unbiased case. Consider pairs of maps $c : C \rightarrowtail Z$ and $z : Z \to \mathrm{I}$, where the former is a cofibration and the latter is regarded as an "I-indexing", so that

$$\begin{tikzcd} C \arrow[r, tail, "c"] \arrow[dr] & Z \arrow[d, "z"] \\ & \mathrm{I} \end{tikzcd}$$

is regarded as an I-indexed family of cofibrations. Let

$$\mathsf{Gph}(z) = \langle 1_Z, z\rangle : Z \longrightarrow Z \times \mathrm{I}\,,$$

be the graph of $z : Z \to \mathrm{I}$, and define

$$c \otimes_z \delta := [\mathsf{Gph}(z), c \times \mathrm{I}] : Z +_C (C \times \mathrm{I}) \to Z \times \mathrm{I}\,,$$

which is easily seen to be well-defined on the indicated pushout.

$$\begin{tikzcd} C \arrow[r, "\mathsf{Gph}(zc)"] \arrow[d, tail, "c"'] & C \times \mathrm{I} \arrow[d] \arrow[ddr, bend left, "c\times\mathrm{I}"] & \\ Z \arrow[r] \arrow[drr, bend right, "\mathsf{Gph}(z)"'] & Z +_C (C \times \mathrm{I}) \arrow[dr, dotted, "c\otimes_z\delta"] & \\ & & Z \times \mathrm{I}\,. \end{tikzcd} \tag{33}$$

This specification differs from the similar (19) by using $\mathsf{Gph}(z)$ for the inclusion $Z \rightarrowtail Z \times \mathrm{I}$, rather than one of the "face maps" associated to the endpoint inclusions $\delta_\epsilon : 1 \to \mathrm{I}$. (Note that a graph is always a cofibration by pulling back a diagonal.) The subobject $c \otimes_z \delta \rightarrowtail Z \times \mathrm{I}$ is the join of the subobjects $\mathsf{Gph}(z) \rightarrowtail Z \times \mathrm{I}$ and the cylinder $C \times \mathrm{I} \rightarrowtail Z \times \mathrm{I}$.

Note that the endpoints $\delta_\epsilon : 1 \to \mathrm{I}$ are of the form $c \otimes_z \delta$ by taking $Z = 1$ and $z = \delta_\epsilon$ and $c = \,! : 0 \to 1$, so that biased filling is subsumed.

The maps of the form $c \otimes_z \delta : Z +_C (C \times \mathrm{I}) \rightarrowtail Z$ now form a *class of generating trivial cofibrations* in the expected sense. Let

$$\mathcal{C} \otimes \delta \;=\; \{c \otimes_z \delta : D \rightarrowtail Z \times \mathrm{I} \mid c : C \rightarrowtail Z, z : Z \to \mathrm{I}\}\,. \tag{34}$$

The fibrations are exactly the right class of these,

$$(\mathcal{C} \otimes \delta)^{\pitchfork} = \mathcal{F}.$$

Proposition 19. *A map $f : Y \to X$ is a fibration iff for every pair of maps $c : C \rightarrowtail Z$ and $z : Z \to \mathrm{I}$, where the former is a cofibration, every commutative square of the following form has a diagonal filler, as indicated.*

$$\begin{tikzcd} Z +_C (C \times \mathrm{I}) \arrow[r] \arrow[d, tail, "c \otimes_z \delta"'] & Y \arrow[d, "f"] \\ Z \times \mathrm{I} \arrow[r] \arrow[ur, dotted, "j"] & X. \end{tikzcd} \tag{35}$$

Proof. Suppose that for all $c : C \rightarrowtail Z$ and $z : Z \to \mathrm{I}$, we have $(c \otimes_z \delta) \pitchfork f$ in cSet. Pulling f back over I, this is equivalent to the condition $c \otimes \delta \pitchfork \mathrm{I}^* f$ in cSet/I, for all cofibrations $c : C \rightarrowtail Z$ over I, which is equivalent to $c \pitchfork (\delta \Rightarrow \mathrm{I}^* f)$ in cSet/I for all cofibrations $c : C \rightarrowtail Z$. But this in turn means that $\delta \Rightarrow \mathrm{I}^* f$ is a trivial fibration, which by definition means that f is a fibration. $\square$

Unbiased fibration structure. As in the biased case, the fibrations can be determined by *uniform* right-lifting against a generating *set* of trivial cofibrations, now consisting of all those $c \otimes_z \delta$ in $\mathcal{C} \otimes \delta$ for which $c : C \rightarrowtail \mathrm{I}^n$ is basic. Call these maps the *basic (unbiased) trivial cofibrations*, and let

$$\mathsf{BCof} \otimes \delta = \{c \otimes_z \delta : B \rightarrowtail \mathrm{I}^{n+1} \mid c : C \rightarrowtail \mathrm{I}^n,\ z : \mathrm{I}^n \to \mathrm{I},\ n \geq 0\}\,, \tag{36}$$

where the pushout-product $c \otimes_z \delta$ now has the form

$$\begin{tikzcd} C \arrow[r, "\mathsf{Gph}(zc)"] \arrow[d, tail, "c"'] & C \times \mathrm{I} \arrow[d] \arrow[ddr, bend left, "c \times \mathrm{I}"] & \\ \mathrm{I}^n \arrow[r] \arrow[drr, bend right, "\mathsf{Gph}(z)"'] & \mathrm{I}^n +_C (C \times \mathrm{I}) \arrow[dr, dotted, "c \otimes_z \delta"] & \\ & & \mathrm{I}^n \times \mathrm{I} \end{tikzcd} \tag{37}$$

for a basic cofibration $c : C \rightarrowtail \mathrm{I}^n$, an indexing map $z : \mathrm{I}^n \to \mathrm{I}$, and with domain $B = \big(\mathrm{I}^n +_C (C \times \mathrm{I})\big)$. These subobjects $B \rightarrowtail \mathrm{I}^{n+1}$ can again be seen geometrically as "generalized open box inclusions", but now the floor or lid of the open box may be replaced by a "cross-section" given by the graph of a map $z : \mathrm{I}^n \to \mathrm{I}$.

For any map $f : Y \to X$ a (uniform, unbiased) fibration structure on f is a choice of diagonal fillers $j(c, z, x, y)$,

$$\begin{tikzcd} B \arrow[r, "x"] \arrow[d, tail, "c \otimes_z \delta"'] & X \arrow[d, "f"] \\ \mathrm{I}^n \times \mathrm{I} \arrow[r, "y"'] \arrow[ur, dotted, "j(c,z,x,y)"'] & Y, \end{tikzcd} \tag{38}$$

for each basic trivial cofibration $c \otimes_z \delta : B \rightarrowtail \mathrm{I}^{n+1}$, which is *uniform* in I^n in the following sense: Given any cubical map $u : \mathrm{I}^m \to \mathrm{I}^n$, the pullback $u^*c : u^*C \rightarrowtail \mathrm{I}^m$ and the reindexing $zu : \mathrm{I}^m \to \mathrm{I}^n \to \mathrm{I}$ determine another basic trivial cofibration $u^*c \otimes_{zu} \delta : B' = (\mathrm{I}^m +_{u^*C} (u^*C \times \mathrm{I})) \rightarrowtail \mathrm{I}^{m+1}$ which fits into a commutative diagram of the form

$$\begin{tikzcd} B' \arrow[r, "(u \times \mathrm{I})'"] \arrow[d, "u^*c \otimes_{zu} \delta"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[r, "x"] \arrow[d, "c \otimes_z \delta"'] & X \arrow[d, "f"] \\ \mathrm{I}^m \times \mathrm{I} \arrow[r, "u \times \mathrm{I}"'] & \mathrm{I}^n \times \mathrm{I} \arrow[r, "y"'] \arrow[ur, dotted, "j(c,z,x,y)"'] & Y. \end{tikzcd} \tag{39}$$

For the outer rectangle in (39) there is a chosen diagonal filler

$$j(u^*c, zu, x(u \times \mathrm{I})', y(u \times \mathrm{I})) : \mathrm{I}^m \times \mathrm{I} \to X,$$

and for this map we require that

$$j(u^*c, zu, x(u \times \mathrm{I})', y(u \times \mathrm{I})) = j(c, z, x, y) \circ (u \times \mathrm{I}). \tag{40}$$

Definition 20. A *(uniform, unbiased) fibration structure* on a map

$$f : Y \to X$$

is a choice of fillers $j(c, z, x, y)$ as in (38) satisfying (40) for all $u : \mathrm{I}^m \to \mathrm{I}^n$.

In these terms, we have the following analogue of corollary 16.

Proposition 21. *For any object X in* cSet *the following are equivalent:*

1. *the canonical map* $X^{\mathrm{I}} \times \mathrm{I} \to X \times \mathrm{I}$ *is a trivial fibration.*

2. X *has the right lifting property with respect to all generating trivial cofibrations,*
$$(\mathcal{C} \otimes_z \delta) \pitchfork X.$$

3. X *has a uniform fibration structure in the sense of Definition 20.*

Proof. The equivalence between (1) and (2) is proposition 19. Suppose (1), i.e. that the map
$$(\delta \Rightarrow X) : X^{\mathrm{I}} \times \mathrm{I} \to X \times \mathrm{I}$$
is a relative $+$-algebra over $X \times \mathrm{I}$. By proposition 10, this means that $(\delta \Rightarrow X)$, as an object of $\mathsf{cSet}/(X \times \mathrm{I})$, has a uniform filling structure with respect to all cofibrations $c : C \rightarrowtail \mathrm{I}^n$ over $(X \times \mathrm{I})$. Transposing by the $\otimes \dashv \Rightarrow$ adjunction and unwinding gives, equivalently, a uniform fibration structure on X. □

A statement analogous to the foregoing also holds for maps $f : Y \to X$ in place of objects X. Indeed, as before, we have the following sharper formulation.

Corollary 22. *Fibration structures on a map* $f : Y \to X$ *correspond uniquely to relative* $+$*-algebra structures on the map* $(\delta \Rightarrow f)$ *(cf. definition 17),*
$$(\delta \Rightarrow f) : Y^I \times \mathrm{I} \longrightarrow (X^I \times \mathrm{I}) \times_{(X \times \mathrm{I})} (Y \times \mathrm{I})$$

3.3 Factorization

Definition 23. Summarizing the foregoing definitions and results, we have the following classes of maps:

- The *generating trivial cofibrations* were determined in (34) to be
$$\mathcal{C} \otimes \delta = \{c \otimes_z \delta : D \rightarrowtail Z \times \mathrm{I} \mid c : C \rightarrowtail Z,\ z : Z \to \mathrm{I}\}, \tag{41}$$
where the pushout-product $c \otimes_z \delta$ has the form
$$\begin{tikzcd}
C \arrow[r, "\mathsf{Gph}(zc)"] \arrow[d, tail, "c"'] & C \times \mathrm{I} \arrow[d] \arrow[ddr, bend left, "c \times \mathrm{I}"] & \\
Z \arrow[r] \arrow[drr, bend right, "\mathsf{Gph}(z)"'] & Z +_C (C \times \mathrm{I}) \arrow[dr, dotted, tail, "c \otimes_z \delta"'] & \\
& & Z \times \mathrm{I}\,.
\end{tikzcd} \tag{42}$$

for any cofibration $c : C \rightarrowtail Z$ and indexing map $z : Z \to \mathrm{I}$, with domain $D = \big(Z +_C (C \times \mathrm{I})\big)$.

- The class $\mathcal{F}$ of *fibrations*, written $f : Y \twoheadrightarrow X$, may be characterized as the right-lifting class of the generating trivial cofibrations,

$$(\mathcal{C} \otimes \delta)^{\pitchfork} \;=\; \mathcal{F}.$$

- The class of *trivial cofibrations* is defined to be left class of the fibrations,

$$\mathsf{TCof} \;=\; {}^{\pitchfork}\mathcal{F}.$$

It follows that the classes TCof and $\mathcal{F}$ are mutually weakly orthogonal,

$$\mathsf{TCof} \pitchfork \mathcal{F},$$

and are closed under retracts. Thus to have a weak factorization system $(\mathsf{TCof}, \mathcal{F})$ it just remains to show that every map $f : X \to Y$ can be factored as $f = g \circ h$ with $g \in \mathcal{F}$ and $h \in \mathsf{TCof}$.

Proposition 24. *Every map $f : X \to Y$ in cSet can be factored as $f = p \circ i$,*

$$\begin{tikzcd} X \arrow[r, tail, "i"] \arrow[dr, "f"'] & X' \arrow[d, two heads, "p"] \\ & Y \end{tikzcd} \tag{43}$$

with $i : X \rightarrowtail X'$ a trivial cofibration and $p : X' \twoheadrightarrow Y$ a fibration.

Proof. We can use a standard argument (the "algebraic small objects argument", cf. [GKR18]), further simpified by the fact that the codomains of the basic trivial cofibrations $c \otimes_z \delta : B \rightarrowtail \mathrm{I}^{n+1}$ are not just representable, but *tiny* in the sense of Proposition 3, while the domains are not merely "small", but finitely presented. The reader is referred to [**?**] for details (in a similar case). □

Proposition 25. *There is a weak factorization system on the category* cSet *in which the right maps are the fibrations and the left maps are the trivial cofibrations, both as specified in definition 23.*

This will be called the *fibration weak factorization system*. The following observation will be of use later on; a proof can be found in [GKR18, **?**].

Corollary 26. *The* fibrant replacement *of a map* $f : X \to Y$

$$\begin{tikzcd} X \arrow[r, tail, "i_f"] \arrow[dr, "f"'] & X' \arrow[d, two heads, "f'"] \\ & Y, \end{tikzcd} \tag{44}$$

can be given as an ω*-colimit in the slice category over* Y*,*

$$f' = \varinjlim_n f_n$$

so that it is functorial, and the canonical trivial cofibrations $i_f : X \rightarrowtail X'$ *over* Y *are natural, in* $f : X \to Y$*.*

4 The weak equivalences

Definition 27 (Weak equivalence)**.** A map $f : X \to Y$ in cSet is a *weak equivalence* if it can be factored as $f = g \circ h$,

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[dr, "f"'] & W \arrow[d, "g"] \\ & Y \end{tikzcd}$$

with $h : X \to W$ a trivial cofibration and $g : W \to Y$ a trivial fibration. Let

$$\mathcal{W} = \{f : X \to Y |\ f = g \circ h \text{ for } g \in \mathsf{TFib} \text{ and } h \in \mathsf{TCof}\}$$

be the class of weak equivalences.

Observe that every trivial fibration $f \in \mathcal{C}^{\pitchfork}$ is indeed a fibration, because the generating trivial cofibrations are cofibrations; moreover, every trivial fibration is also a weak equivalence, since the identity maps are trivial cofibrations. Thus we have

$$\mathsf{TFib} \subseteq (\mathcal{F} \cap \mathcal{W}).$$

Thus, because the trivial fibrations are fibrations, every trivial cofibration $g \in {}^{\pitchfork}\mathcal{F}$ is a cofibration; moreover, every trivial cofibration is also a weak equivalence, since the identity maps are also trivial fibrations. Thus we also have

$$\mathsf{TCof} \subseteq (\mathcal{C} \cap \mathcal{W}).$$

Lemma 28. $(\mathcal{C} \cap \mathcal{W}) \subseteq \mathsf{TCof}$.

Proof. Let $c : A \rightarrowtail B$ be a cofibration with a factorization

$$c = tf \circ tc : A \to W \to B$$

where $tc \in \mathsf{TCof}$ and $tf \in \mathsf{TFib}$. Let $f : Y \twoheadrightarrow X$ be a fibration and consider a commutative diagram,

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, tail, "c"'] & Y \arrow[d, two heads, "f"] \\ B \arrow[r, "b"'] & X. \end{tikzcd}$$

Inserting the factorization of c, we have $j : W \to Y$ as indicated, with $j \circ tc = a$ and $f \circ j = b \circ tf$, since $tc \pitchfork f$.

$$\begin{tikzcd} A \arrow[rr, "a"] \arrow[dr, "tc"] \arrow[dd, tail, "c"'] & & Y \arrow[dd, two heads, "f"] \\ & W \arrow[ur, dotted, "j"'] \arrow[dl, "tf"'] & \\ B \arrow[rr, "b"'] & & X. \end{tikzcd}$$

Moreover, since $c \pitchfork tf$ there is an $i : B \to W$ as indicated, with $i \circ c = tc$ and $tf \circ i = 1_B$.

$$\begin{tikzcd} A \arrow[rr, "a"] \arrow[dr, "tc"] \arrow[dd, tail, "c"'] & & Y \arrow[dd, two heads, "f"] \\ & W \arrow[ur, dotted, "j"'] \arrow[dl, "tf"'] & \\ B \arrow[ur, dotted, bend right, "i"'] \arrow[rr, "b"'] & & X. \end{tikzcd}$$

Let $k = j \circ i$. Then $k \circ c = j \circ i \circ c = j \circ tc = a$, and $f \circ k = f \circ j \circ i = b \circ tf \circ i = b$. $\square$

The proof of the following is dual:

Lemma 29. $(\mathcal{F} \cap \mathcal{W}) \subseteq \mathsf{TFib}$.

Proposition 30. *For the three classes of maps $\mathcal{C}, \mathcal{W}, \mathcal{F}$ in* cSet*, we have*

$$\mathcal{F} \cap \mathcal{W} = \mathsf{TFib},$$
$$\mathcal{C} \cap \mathcal{W} = \mathsf{TCof},$$

and therefore two weak factorization systems:

$$(\mathcal{C}, \mathcal{W} \cap \mathcal{F}) \quad , \quad (\mathcal{C} \cap \mathcal{W}, \mathcal{F}).$$

Corollary 31. *The following are equivalent for a map $f : X \to Y$.*

1. *$f : X \to Y$ is a weak equivalence*
2. *the first factor $\eta : X \to X'$ of the cofibration factorization of f is a trivial cofibration.*
3. *the second factor $p : Y' \to Y$ of the fibration factorization of f is a trivial fibration.*

Weak homotopy equivalence. To show that the weak equivalences satisfy the 3-for-2 condition, we shall follow the approach of [**?**], verifying that many of the same arguments go through in the current setting – up to a certain point.

Definition 32. By a *homotopy* between parallel maps $f, g : X \rightrightarrows Y$, written $\vartheta : f \sim g$, we mean a map from the *cylinder of* X built using the (representable) interval I,

$$\vartheta : \mathrm{I} \times X \to Y,$$

and such that $\vartheta \circ \iota_0 = f$ and $\vartheta \circ \iota_1 = g$,

$$\begin{tikzcd} X \arrow[r, "\iota_0"] \arrow[dr, "f"'] & \mathrm{I} \times X \arrow[d, "\vartheta"] & X, \arrow[l, "\iota_1"'] \arrow[dl, "g"] \\ & Y & \end{tikzcd}$$

where we write the canonical inclusions into the ends of the cylinder as

$$\iota_\epsilon = \mathsf{Gph}(\delta_\epsilon !) : X \to \mathrm{I} \times X\,, \qquad \epsilon = 0, 1.$$

Proposition 33. *If K is fibrant, then the relation of homotopy $f \sim g$ between maps $f, g : X \rightrightarrows K$ is an equivalence relation. Moreover, it is compatible with pre- and post-composition.*

Proof. For $f, g : X \rightrightarrows Y$, a homotopy $f \overset{\vartheta}{\sim} g : X \times \mathrm{I} \to Y$ is equivalent, under exponential transposition, to a path in the function space $\vartheta : \mathrm{I} \to Y^X$ with endpoints $\vartheta_0 = \vartheta \circ \delta_0 = f : 1 \to Y^X$ and $\vartheta_1 = g$. Note that Y^X is fibrant if Y is fibrant, since the generating trivial cofibrations are closed under taking the product with a fixed object. So we can use box-filling in Y^X.

The reflexivity of homotopy $f \sim f$ is witnessed by $\rho : \mathrm{I} \to 1 \xrightarrow{f} Y^X$.

For symmetry $f \sim g \Rightarrow g \sim f$ take $\vartheta : \mathrm{I} \to Y^X$ with $\vartheta_0 = f$ and $\vartheta_1 = g$ and we want to build $\vartheta' : \mathrm{I} \to Y^X$ with $\vartheta'_0 = g$ and $\vartheta'_1 = f$. Take an open 2-box in Y^X of the form

$$\begin{tikzcd} g & f \\ f \arrow[u, "\vartheta"] \arrow[r, "\rho"'] & f \arrow[u, "\rho"'] \end{tikzcd}$$

This box is a map $b : \mathrm{I} +_1 \mathrm{I} +_1 \mathrm{I} \to Y^X$ with the indicated components, and it has a filler $c : \mathrm{I} \times \mathrm{I} \to Y^X$, i.e. an extension along the canonical map $\mathrm{I} +_1 \mathrm{I} +_1 \mathrm{I} \rightarrowtail \mathrm{I} \times \mathrm{I}$, which is a trivial cofibration. Let $t : \mathrm{I} \to \mathrm{I} \times \mathrm{I}$ be the evident missing top face of the 2-cube. We can set $\vartheta' = c \circ t : \mathrm{I} \to Y^X$ to get a homotopy $\vartheta' : \mathrm{I} \to Y^X$ with required endpoints.

For transitivity, $f \overset{\vartheta}{\sim} g \;\&\; g \overset{\varphi}{\sim} h \Rightarrow f \sim h$, an analogous filling construction can be used with the open box:

$$\begin{tikzcd} f & h \\ f \arrow[u, "\rho"] \arrow[r, "\vartheta"'] & g \arrow[u, "\varphi"'] \end{tikzcd}$$

Compatibility under pre- and post-composition is shown by representing homotopies by mapping into the pathspace, for precomposition, and out of the cylinder, for post-composition. $\square$

Definition 34 (Connected components)**.** The functor

$$\pi_0 : \mathsf{cSet} \to \mathsf{Set}$$

is defined on a cubical set X as the coequalizer

$$X_1 \rightrightarrows X_0 \to \pi_0 X,$$

where the two parallel arrows are the maps $X_{\delta_0}, X_{\delta_1} : X_1 \rightrightarrows X_0$ induced by the endpoints $\delta_0, \delta_1 : 1 \rightrightarrows \mathrm{I}$. For any fibrant object K we therefore have

$\pi_0 K = \mathrm{Hom}(1, K)/\sim$, that is, $\pi_0 K$ is the set of points $1 \to K$, modulo the homotopy equivalence relation on them.

One can show that in fact $\pi_0 X = \varinjlim_n X_n$, the colimit being left adjoint to the constant presheaf functor $\Delta : \mathsf{Set} \to \mathsf{cSet}$. Since the category $\mathbb{B}$ of finite strictly bipointed sets is sifted, we have:

Corollary 35. *The functor* $\pi_0 : \mathsf{cSet} \to \mathsf{Set}$ *preserves finite products.*

As usual, a map $f : X \to Y$ in cSet will be called a *homotopy equivalence* if there is a *quasi-inverse* $g : Y \to X$ and homotopies $\vartheta : 1_X \sim g \circ f$ and $\varphi : 1_Y \sim f \circ g$.

Definition 36 (Weak homotopy equivalence)**.** A map $f : X \to Y$ is called a *weak homotopy equivalence* if for every fibrant object K, the “internal precomposition” map $K^f : K^Y \to K^X$ is bijective on connected components,

$$\pi_0 K^f : \pi_0 K^Y \cong \pi_0 K^X .$$

Lemma 37. *A homotopy equivalence is a weak homotopy equivalence.*

Proof. If $f : X \to Y$ is a homotopy equivalence, then so is $K^f : K^Y \to K^X$ for any K, since homotopy respects composition. Since K^X is always fibrant when K is, π_0 is well defined, and it clearly takes homotopy equivalences to isomorphisms of sets. □

Lemma 38. *The weak homotopy equivalences* $f : X \to Y$ *satisfy the 3-for-2 condition.*

Proof. This follows from the corresponding fact about bijections of sets. □

Our goal of showing that the weak equivalences satisfy 3-for-2 is now reduced to showing that a map is a weak equivalence (WE) if and only if it is a weak homotopy equivalence (WHE). This will be proved in four cases, showing that a (co)fibration is a WE if and only if it is a WHE.

Lemma 39. *A map* $f : X \to Y$ *is a* weak homotopy equivalence *iff it satisfies the following two conditions.*

1. *For every fibrant object K and every map $x : X \to K$ there is a map $y : Y \to K$ such that $y \circ f \sim x$,*

$$\begin{tikzcd} X \arrow[r, "x"] \arrow[d, "f"'] & K. \\ Y \arrow[ur, dotted, "y"'] \arrow[ur, phantom, "\sim", near end, shift left=3] & \end{tikzcd}$$

   We say that x "extends along f up to homotopy".

2. *For every fibrant object K and maps $y, y' : Y \to K$ such that $yf \sim y'f$, there is a homotopy $y \sim y'$,*

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & K^{\mathrm{I}} \arrow[d] \\ Y \arrow[r, "\langle y,y' \rangle"'] \arrow[ur, dotted] & K \times K. \end{tikzcd}$$

Proof. Condition (1) says exactly that the internal precomposition map $K^f : K^Y \to K^X$ is surjective on connected components, while (2) says just that it is injective. $\square$

Lemma 40. *A cofibration $c : A \rightarrowtail B$ that is a WE is a WHE.*

Proof. A cofibration $c : A \rightarrowtail B$ that is a WE is a trivial cofibration by proposition 30. So the result follows from Lemma 39, together with the fact that $K^{\partial} : K^{\mathrm{I}} \to K^{1+1} \cong K \times K$ is a fibration whenever K is fibrant, since $\partial : 1 + 1 \rightarrowtail \mathrm{I}$ is a cofibration, $\square$

Lemma 41. *A fibration $p : Y \twoheadrightarrow X$ that is a WE is a WHE.*

Proof. A fibration weak equivalence $f : Y \twoheadrightarrow X$ is a trivial fibration by proposition 30, and therefore has a section $s : X \rightarrowtail Y$, by the lifting problem

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, tail] & Y \arrow[d, two heads, "f"] \\ X \arrow[r, "="'] & X, \end{tikzcd}$$

since $0 \to X$ is a cofibration. Moreover, there is a homotopy $\vartheta : sf \sim 1_Y$, resulting from the lifting problem

$$\begin{tikzcd}
Y + Y \arrow[r, "{[sf,1]}"] \arrow[d, tail, "{[\iota_0,\iota_1]}"'] & Y \arrow[d, two heads, "f"] \\
\mathrm{I} \times Y \arrow[r, "f\pi_2"'] & X.
\end{tikzcd}$$

Thus f is a homotopy equivalence, and so a WHE by lemma 37. $\square$

Corollary 42. *A WE $e : X \simeq Y$ is a WHE, since e can be factored into a trivial cofibration followed by a trivial fibration, each of which is a WHE, and these are closed under composition.*

Lemma 43. *If K is fibrant, then any fibration $f : Y \twoheadrightarrow K$ that is a HE is a WE.*

Proof. This is a standard argument, which we just sketch. It suffices to show that any diagram of the form

$$\begin{tikzcd}
C \arrow[r, "y"] \arrow[d, tail, "c"'] & Y \arrow[d, two heads, "f"] \\
K \arrow[r, "="'] & K,
\end{tikzcd} \tag{45}$$

with $c : C \rightarrowtail X$ a cofibration, has a diagonal filler. Since f is a HE it has a quasi-inverse $s : X \to Y$ with $\vartheta : fs \sim 1_K$, which we can correct to a section $s' : K \to Y$. Indeed, consider

$$\begin{tikzcd}
K \arrow[r, "s"] \arrow[d, tail, "\iota_0"'] & Y \arrow[d, two heads, "f"] \\
K \times \mathrm{I} \arrow[r, "\vartheta"'] \arrow[ur, dotted, "\vartheta'"] & K \\
K \arrow[u, tail, "\iota_1"] \arrow[ur, "="']
\end{tikzcd}$$

where ϑ' results from $\iota_0 \pitchfork f$. Let $s' = \vartheta'\iota_1$, so that $\vartheta' : s \sim s'$ and $fs' = 1_K$.

Thus we can assume that $s = s' : K \to Y$ is a section, which fills the diagram (45) up to a homotopy in the upper triangle.

$$\begin{tikzcd}
C \arrow[r, "y"] \arrow[d, tail, "c"'] \arrow[dr, phantom, "\sim", pos=0.3] & Y \arrow[d, two heads, "f"] \\
K \arrow[r, "="'] \arrow[ur, "s"'] & K,
\end{tikzcd}$$

Now we can correct $s : K \to Y$ to a homotopic $t : K \to Y$ over f by using the homotopy $\varphi : sc \sim y$ to get a map $\varphi : C \to Y^{\mathrm{I}}$ over f. Since f is a fibration, the projections $p_0, p_1 : Y^{\mathrm{I}} \to Y$ over f are trivial fibrations, and so there is a lift $\varphi' : K \to Y^{\mathrm{I}}$ for which $t := p_1\varphi'$ has $tc = y$ and $ft = 1_K$, and so is a filler for (45). $\square$

Lemma 44. *If K is fibrant, then any fibration $f : Y \twoheadrightarrow K$ that is a WHE is a WE.*

Proof. Since K is fibrant, so is Y, and since f is a WHE, there is a map $s : K \to Y$ and a homotopy $\theta : sf \sim 1_Y$ by lemma 39(1). Thus, applying f again, we have a homotopy $f\vartheta : fsf \sim f$, forming the outer commutative square in

$$\begin{tikzcd} Y \arrow[r, "f\vartheta"] \arrow[d, "f"'] & K^{\mathrm{I}} \arrow[d] \\ K \arrow[r, "\langle fs, 1_K \rangle"'] \arrow[ur, dotted, "\varphi"] & K \times K. \end{tikzcd}$$

By lemma 39(2) there is a diagonal filler $\varphi : fs \sim 1_K$, and so f is a HE. Now apply lemma 43. $\square$

Lemma 45. *If K is fibrant, then any cofibration $c : A \rightarrowtail K$ that is a WHE is a WE.*

Proof. Let $c : A \rightarrowtail K$ be a cofibration WHE and factor it into a trivial cofibration $i : A \rightarrowtail Z$ followed by a fibration $p : Z \twoheadrightarrow K$. By lemma 39, it is clear that a trivial cofibration is a WHE. So both c and i are WHE, and therefore so is p by 3-for-2 for WHEs. Since K is fibrant, p is a trivial fibration by lemma 44, and thus c is a WE. $\square$

Lemma 46 ([**?**], x.n.m)**.** *A cofibration $c : A \rightarrowtail B$ WHE lifts against all fibrations $f : Y \twoheadrightarrow K$ with fibrant codomain.*

Proof. Let $c : A \rightarrowtail B$ be a cofibration WHE and $f : Y \twoheadrightarrow K$ a fibration with fibrant codomain K, and consider a lifting problem

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, tail, "c"'] & Y \arrow[d, two heads, "f"] \\ B \arrow[r, "b"'] & K. \end{tikzcd}$$

Let $\eta : B \rightarrowtail B'$ be a fibrant replacement of B, since K is fibrant, b extends along η to give $b' : B' \to K$ as in:

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, tail, "c"'] & Y \arrow[d, two heads, "f"] \\
B \arrow[r, "b"'] \arrow[d, "\eta"'] & K \\
B' \arrow[ur, "b'"'] &
\end{tikzcd}$$

Since η is a trivial cofibration, it is a WHE. So the composite ηc is also a WHE. But since B' is fibrant, ηc is then a trivial cofibration by lemma 45. Thus there is a lift $j : B' \to Y$, and therefore also one $k = j\eta : B \to Y$. $\square$

To complete the proof that a cofibration WHE is a WE, we use the following *fibration extension property* (FEP), the proof of which is deferred to section 8.

Definition 47 (Fibration extension property)**.** For any fibration $f : Y \twoheadrightarrow X$ and trivial cofibration $\eta : X \to X'$, there is a fibration $f' : Y' \twoheadrightarrow X'$ of which f is a pullback along η,

$$\begin{tikzcd}
Y \arrow[r] \arrow[d, two heads, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y' \arrow[d, two heads, "f'"] \\
X \arrow[r, tail, "\eta"'] & X'.
\end{tikzcd} \tag{46}$$

Lemma 48. *Assuming the FEP, a cofibration that lifts against every fibration $f : Y \twoheadrightarrow K$ with fibrant codomain is a WE.*

Proof. Let $c : A \rightarrowtail B$ be a cofibration and consider a lifting problem against an arbitrary fibration $f : Y \twoheadrightarrow X$,

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, tail, "c"'] & Y \arrow[d, two heads, "f"] \\
B \arrow[r, "b"'] & X.
\end{tikzcd} \tag{47}$$

Let $\eta : X \to X'$ be a fibrant replacement, so η is a trivial cofibration and X' is fibrant. By the fibration extension property of definition 47, there is a fibration $f' : Y' \twoheadrightarrow X'$ such that f is a pullback of f' along η. So we can extend diagram (47) to obtain the following, in which the righthand square is a pullback.

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, tail, "c"'] & Y \arrow[r, "y"] \arrow[d, two heads, "f"] \arrow[dr, phantom, "\lrcorner" very near start] & Y' \arrow[d, two heads, "f'"] \\ B \arrow[r, "b"'] & X \arrow[r, "\eta"'] & X'. \end{tikzcd} \tag{48}$$

By assumption, there is a lift $j' : B \to Y'$ with $f'j' = \eta b$ and $j'c = yb$. Therefore, since f is a pullback, there is a map $j : B \to Y$ with $fj = b$ and $yj = j'$.

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, tail, "c"'] & Y \arrow[r, "y"] \arrow[d, two heads, "f"] \arrow[dr, phantom, "\lrcorner" very near start] & Y' \arrow[d, two heads, "f'"] \\ B \arrow[r, "b"'] \arrow[ur, dotted, "j"] \arrow[urr, dotted, "j'"'] & X \arrow[r, "\eta"'] & X'. \end{tikzcd} \tag{49}$$

Thus $yjc = j'c = ya$. But as a trivial cofibration, η is monic, and as a pullback of η, y is also monic. So $jc = a$. $\square$

Combining the previous two lemmas 46 and 48 we now have the following.

Corollary 49. *Assuming the FEP, a cofibration $c : A \rightarrowtail B$ that is a WHE is a WE.*

The following is not required, but we state it anyway for the record:

Lemma 50. *Assuming the FEP, a fibration $f : Y \twoheadrightarrow X$ that is a WHE is a WE.*

Proof. Factor $f : Y \twoheadrightarrow X$ into a cofibration $i : Y \rightarrowtail Z$ followed by a trivial fibration $p : Z \twoheadrightarrow X$. Then f is a trivial fibration if $i \pitchfork f$, for then f is a retract of p. Since p is a trivial fibration, it is a WHE by lemma 41. Since f is also a WHE, so is i by 3-for-2. Thus i is a trivial cofibration by corollary 49. Since f is a fibration, $i \pitchfork f$ as required. $\square$

Proposition 51. *Assuming the FEP, a map $f : X \rightarrowtail Y$ is a WHE if and only if it is a WE. Thus the weak equivalences $\mathcal{W}$ satisfy the 3-for-2 condition.*

Proof. Let $f : X \twoheadrightarrow Y$ be a WE and factor it into a trivial cofibration $i : X \rightarrowtail Z$ followed by a trivial fibration $p : Z \twoheadrightarrow Y$. Then both i and p are WHE, whence so is f. Conversely, let f be a WHE and factor it into a cofibration $i : X \rightarrowtail Z$ followed by a trivial fibration $p : Z \twoheadrightarrow Y$. Since p is then a WHE, as is f, it follows that i is as well. Thus i is also a WE, by lemma 49, hence a trivial cofibration. So f is a WE. $\square$

Our results thus far can now be summarized as follows.

Theorem 52. *Assume the fibration weak factorization system of Definition 23 satisfies the fibration extension property of Definition 47 (as will be shown in Corollary 78). Then the weak equivalences $\mathcal{W}$ have the 3-for-2 property, and so by Proposition 30, the three classes $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ determined by Definition 23 form a Quillen model structure on the category* cSet *of cubical sets.*

The weak equivalences are those maps $f : X \to Y$ for which $K^f : K^Y \to K^X$ is bijective on connected components whenever K is fibrant.

The proof of the fibration extension property will occupy the second half of these lectures, concluding in Section 8. It requires several intermediate results, namely the equivalence extension property (Section 7), a universal fibration (Section 6.1), and the Frobenius condition (Section 5), to which we now turn.

5 The Frobenius condition

In this section, we show that the fibration WFS from section 3 has the *Frobenius property*: the left maps are stable under pullback along the right maps (see [**?**]). This will imply the *right properness* of our model structure: the weak equivalences are preserved by pullback along fibrations. The Frobenius property is also needed in the proof of the equivalence extension property in the next section. A proof of Frobenius in a related setting of cubical sets with connections can be found in [GS17]; however the type theoretic approach of [OP17, CCHM16] can be applied without connections and is also more direct. This approach proves the dual fact that the *pushforward* operation, which is right adjoint to pullback, and which always exists in a topos, when applied along any *fibration* $f : Y \twoheadrightarrow X$ preserves fibrations. This fact corresponds to the type-theoretic Π-formation rule.

Recall from Definition 17 that a map $f : A \to X$ is a fibration just if, when pulled back to the slice category cSet/I where there is a generic point $\delta : 1 \to \mathrm{I}$, the map $\delta \Rightarrow f$ admits a $+$-algebra structure, and so is a trivial fibration. The definition of $\delta \Rightarrow f$ is recalled below.

$$\begin{tikzcd}
A^{\mathrm{I}} \arrow[dr, "\delta\Rightarrow f"] \arrow[drr, bend left, "A^{\delta}"] \arrow[ddr, bend right, "f^{\mathrm{I}}"'] & & \\
& X^{\mathrm{I}} \times_X A \arrow[r] \arrow[d] & A \arrow[d, "f"] \\
& X^{\mathrm{I}} \arrow[r, "X^{\delta}"'] & X.
\end{tikzcd} \tag{50}$$

Let us write this condition schematically as follows:

$$\begin{tikzcd}
A^{\mathrm{I}} \arrow[r, "+" marking] & A_{\epsilon} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
& X^{\mathrm{I}} \arrow[r, "\epsilon"'] & X,
\end{tikzcd} \tag{51}$$

where $\epsilon = X^{\delta}$, $A_{\epsilon} = X^{\mathrm{I}} \times_X A$, and the struck-through arrow indicates that it admits a $+$-algebra structure.

Lemma 53. *Let $g : Y \to X$ be any map and $f : A \to X$ a fibration, then the pullback $g^* f : g^* A \to Y$ is also a fibration.*

Proof. This is clear, since fibrations are the right class of a weak factorization system, but note that the algebraic structure indicated in (51) is also stable under pullback, since the $+$-algebras are pullback stable. $\square$

Lemma 54. *Let $\alpha : A \to X$ and $\beta : B \to A$ be fibrations, then the composite $\alpha \circ \beta : B \to X$ is also a fibration.*

Proof. Again for maps in the right class of a weak factorization system this is immediate. But let us see how the *fibration structures* also compose. We have the following diagram for the fibration structures on $B \to A$ and $A \to X$ (with obvious notation).

$$\begin{tikzcd}
B^{\mathrm{I}} \arrow[r, "+" marking] & B_{\epsilon_A} \arrow[rr] \arrow[d] \arrow[drr, phantom, "\lrcorner", very near start] & & B \arrow[d] \\
& A^{\mathrm{I}} \arrow[r, "+" marking] & A_{\epsilon_X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d] \\
& & X^{\mathrm{I}} \arrow[r, "\epsilon_X"'] & X,
\end{tikzcd} \tag{52}$$

Pulling back $B \to A$ in two steps we therefore obtain the interpolant B_{ϵ_X},

$$
\begin{tikzcd}
B^{\mathrm{I}} \arrow[r, "\shortmid" marking] & B_{\epsilon_A} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B_{\epsilon_X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d] \\
& A^{\mathrm{I}} \arrow[r, "\shortmid" marking] & A_{\epsilon_X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d] \\
& & X^{\mathrm{I}} \arrow[r, "\epsilon_X"'] & X,
\end{tikzcd}
\tag{53}
$$

Now use the fact that trivial fibrations are closed under pullback along all maps, and under composition, to infer that the indicated composite map $B^{\mathrm{I}} \to B_{\epsilon_X}$ is also a trivial fibration, as required. $\square$

Proposition 55 (Frobenius)**.** *Let $\alpha : A \to X$ and $\beta : B \to A$ be fibrations, then the pushforward $\alpha_*\beta : \Pi_A B \to X$ is also a fibration.*

Proof. Given the fibrations $\alpha : A \to X$ and $\beta : B \to A$, let $p : A^{\mathrm{I}} \to A_\epsilon$ and $q : B^{\mathrm{I}} \to p^*B_\epsilon$ be the associated +-algebras, so that we have the following situation, with all squares pullbacks.

$$
\begin{tikzcd}
B^{\mathrm{I}} \arrow[r, "\shortmid" marking, "q"'] & p^*B_\epsilon \arrow[r] \arrow[d] & B_\epsilon \arrow[r] \arrow[d] & B \arrow[d] \\
& A^{\mathrm{I}} \arrow[r, "\shortmid" marking, "p"'] & A_\epsilon \arrow[r] \arrow[d] & A \arrow[d] \\
& & X^{\mathrm{I}} \arrow[r, "\epsilon"'] & X,
\end{tikzcd}
\tag{54}
$$

Adding (some composites and) the relevant pushforward underneath, we have

$$
\begin{tikzcd}
B^{\mathrm{I}} \arrow[r, "\shortmid" marking, "q"'] \arrow[dr] & p^*B_\epsilon \arrow[r] \arrow[d] & B_\epsilon \arrow[r] \arrow[d] & B \arrow[d] \\
& A^{\mathrm{I}} \arrow[r, "\shortmid" marking, "p"'] \arrow[dr] & A_\epsilon \arrow[r] \arrow[d] & A \arrow[d] \\
& & X^{\mathrm{I}} \arrow[r, "\epsilon"'] & X \\
& (\Pi_A B)^{\mathrm{I}} \arrow[r, "r"'] \arrow[ur] & (\Pi_A B)_\epsilon \arrow[r] \arrow[u] & \Pi_A B \arrow[u]
\end{tikzcd}
\tag{55}
$$

and we wish to show that the indicated map $r : (\Pi_A B)^{\mathrm{I}} \to (\Pi_A B)_\epsilon$ admits a +-algebra structure. We will do so by showing that it is a retract of a known +-algebra.

Indeed, let us apply the pushforward, along the indicated canonical map $\alpha^{\mathrm{I}} : A^{\mathrm{I}} \to X^{\mathrm{I}}$, to the +-algebra $q : B^{\mathrm{I}} \to p^* B_\epsilon$, regarded as an arrow over A^{I}. We obtain an arrow over X^{I} of the form

$$\Pi_{A^{\mathrm{I}}}\, q : \Pi_{A^{\mathrm{I}}}\, B^{\mathrm{I}} \to \Pi_{A^{\mathrm{I}}}\, p^* B_\epsilon$$

which is a +-algebra, because these are preserved by pushforward, according to Lemma **??**.

Now observe that by the Beck-Chevalley condition, we have an isomorphism

$$(\Pi_A B)_\epsilon \cong \Pi_{A_\epsilon} B_\epsilon\,.$$

Moreover, since $\Pi_{A^{\mathrm{I}}} \cong \Pi_{A_\epsilon} \circ p_*$, we have

$$\Pi_{A^{\mathrm{I}}}\, p^* B_\epsilon \cong \Pi_{A_\epsilon} p_* p^* B_\epsilon\,.$$

Thus the image of the unit $\eta : B_\epsilon \to p_* p^* B_\epsilon$ under Π_{A_ϵ} is a map $s = \Pi_{A_\epsilon} \eta$ over X^{I} of the form:

$$\begin{tikzcd}
 & X^{\mathrm{I}} \\
(\Pi_A B)^{\mathrm{I}} \arrow[ur] \arrow[r, "r"'] & \Pi_{A_\epsilon} B_\epsilon \arrow[u] \arrow[d, "s"] \\
\Pi_{A^{\mathrm{I}}} B^{\mathrm{I}} \arrow[r, "\Pi_{A^{\mathrm{I}}}\, q"'] & \Pi_{A_\epsilon} p_* p^* B_\epsilon
\end{tikzcd} \tag{56}$$

Our objective is now to fill in the further arrows φ, ψ, ϑ indicated below in order to exhibit r as a retract of $\Pi_{A^{\mathrm{I}}}\, q$ in the arrow category over X^{I}.

$$\begin{tikzcd}
 & X^{\mathrm{I}} \\
(\Pi_A B)^{\mathrm{I}} \arrow[ur] \arrow[r, "r"'] \arrow[d, dotted, "\varphi"'] & \Pi_{A_\epsilon} B_\epsilon \arrow[u] \arrow[d, "s"] \\
\Pi_{A^{\mathrm{I}}} B^{\mathrm{I}} \arrow[r, "\Pi_{A^{\mathrm{I}}}\, q"'] \arrow[d, dotted, "\psi"'] & \Pi_{A_\epsilon} p_* p^* B_\epsilon \arrow[d, dotted, "\vartheta"] \\
(\Pi_A B)^{\mathrm{I}} \arrow[r, "r"'] & \Pi_{A_\epsilon} B_\epsilon
\end{tikzcd} \tag{57}$$

• For φ, we require a map

$$\varphi : (\Pi_A B)^{\mathrm{I}} \to \Pi_{A^{\mathrm{I}}} B^{\mathrm{I}} \qquad \text{over } X^{\mathrm{I}}.$$

Consider the following diagram, which is based on (55).

$$\begin{tikzcd}
B^{\mathrm{I}} \arrow[r, "q"', "\shortmid" marking] \arrow[dr] & p^*B_\epsilon \arrow[r] \arrow[d] & B_\epsilon \arrow[r] \arrow[d] & B \arrow[d] & \\
& A^{\mathrm{I}} \arrow[r, "p"', "\shortmid" marking] \arrow[dr] & A_\epsilon \arrow[r] \arrow[d] & A & \Pi_A B \times_X A \arrow[l] \arrow[d] \arrow[ul, dotted, "c"'] \\
(\Pi_A B \times_X A)^{\mathrm{I}} \arrow[uu, dotted, "c^{\mathrm{I}}"] \arrow[ur] \arrow[dr] & & X^{\mathrm{I}} \arrow[r, "\epsilon"'] & X & \Pi_A B \arrow[l] \\
& (\Pi_A B)^{\mathrm{I}} \arrow[ur] \arrow[r, dotted, "\varphi"'] & \Pi_{A^{\mathrm{I}}} B^{\mathrm{I}} \arrow[u] & &
\end{tikzcd} \tag{58}$$

The map c is the counit at $B \to A$ of the pullback-pushforward adjunction along $A \to X$. The right-hand side of the diagram, including c and the associated pullback square, reappear on the left under the functor $(-)^{\mathrm{I}}$, which preserves the pullback. Thus we can take φ to be the transpose of c^{I} under the pullback-pushforward adjunction along $A^I \to X^{\mathrm{I}}$,

$$\varphi = \widetilde{c^{\mathrm{I}}}.$$

A diagram chase involving the pullback-pushforward adjunction along $A_\epsilon \to X^{\mathrm{I}}$ shows that the upper square in (57) commutes.

• For ϑ: referring to the diagram (55), since $p : A^{\mathrm{I}} \to A_\epsilon$ is a trivial fibration, it has a section $o : A_\epsilon \to A^{\mathrm{I}}$ by lemma 12. Pulling $p^*B_\epsilon \to A^{\mathrm{I}}$ back along o results in an iso over A_ϵ,

$$o^*p^*B_\epsilon \cong B_\epsilon\,,$$

and so by the adjunction $o^* \dashv o_*$ there is a map over A^{I},

$$p^*B_\epsilon \to o_*B_\epsilon\,,$$

to which we can apply p_* to obtain a map,

$$\rho : p_*p^*B_\epsilon \to p_*o_*B_\epsilon \cong B_\epsilon \quad \text{over } A_\epsilon.$$

This is a retraction of the unit $\eta : B_\epsilon \to p_* p^* B_\epsilon$ over A_ϵ. Applying the functor Π_{A_ϵ} therefore gives the desired retraction

$$\vartheta = \Pi_{A_\epsilon}\rho : \Pi_{A_\epsilon} p_* p^* B_\epsilon \to \Pi_{A_\epsilon} B_\epsilon$$

of s.

- For ψ, we require a map

$$\psi : \Pi_{A^{\mathrm{I}}} B^{\mathrm{I}} \to (\Pi_A B)^{\mathrm{I}} \qquad \text{over } X^{\mathrm{I}}.$$

Consider the following diagram resulting from combining (55) and (57), and in which all solid arrows are those already introduced. The dotted arrow labelled α is the evident composite.

$$\begin{tikzcd}
 & X^{\mathrm{I}} \arrow[r, "\epsilon"] & X \\
(\Pi_A B)^{\mathrm{I}} \arrow[ur] \arrow[r] \arrow[d] & \Pi_{A_\epsilon} B_\epsilon \arrow[u] \arrow[r] \arrow[d] & \Pi_A B \arrow[u] \arrow[dd, "="] \\
\Pi_{A^{\mathrm{I}}} B^{\mathrm{I}} \arrow[r] \arrow[drr, dotted, "\alpha"] & \Pi_{A_\epsilon} p_* p^* B_\epsilon \arrow[d] & \\
(\Pi_A B)^{\mathrm{I}} \arrow[r] & \Pi_{A_\epsilon} B_\epsilon \arrow[r] & \Pi_A B
\end{tikzcd} \tag{59}$$

Now recall that we are working in the slice category over I, and the objects $\Pi_{A^{\mathrm{I}}} B^{\mathrm{I}}$, $\Pi_A B$, and $(\Pi_A B)^{\mathrm{I}}$ are in the image of the base change I^*, and so are actually of the form $\mathrm{I}^*\Pi_{A^{\mathrm{I}}} B^{\mathrm{I}}$, $\mathrm{I}^*\Pi_A B$, and $\mathrm{I}^*((\Pi_A B)^{\mathrm{I}})$. Indeed, the latter is

$$\mathrm{I}^*((\Pi_A B)^{\mathrm{I}}) = \mathrm{I}^*\mathrm{I}_*\mathrm{I}^*\Pi_A B\,.$$

Since the lower horizontal map is the counit ε of the base change $\mathrm{I}^* \dashv \mathrm{I}_*$, the map α factors as $\varepsilon \circ \mathrm{I}^*\tilde{\alpha}$, where $\tilde{\alpha}$ is the adjoint transpose of α, as shown in the following.

$$\begin{tikzcd}
\mathrm{I}^*\Pi_{A^{\mathrm{I}}} B^{\mathrm{I}} \arrow[r] \arrow[d, "\mathrm{I}^*\tilde{\alpha}"'] \arrow[drr, dotted, "\alpha"] & \Pi_{A_\epsilon} p_* p^* B_\epsilon \arrow[d] & \\
\mathrm{I}^*\mathrm{I}_*\mathrm{I}^*\Pi_A B \arrow[r] \arrow[rr, bend right, "\varepsilon"'] & \Pi_{A_\epsilon} B_\epsilon \arrow[r] & \mathrm{I}^*\Pi_A B
\end{tikzcd} \tag{60}$$

We set $\psi = \mathrm{I}^*\tilde{\alpha}$, making the square commute.

We have now defined all the maps below, the squares involving φ and ψ commute, and the composite of ϑ and s is the identity.

$$
\begin{tikzcd}
 & X^{\mathrm{I}} \arrow[r] & X \\
(\Pi_A B)^{\mathrm{I}} \arrow[ur] \arrow[r] \arrow[d, "\varphi"'] & \Pi_{A_\epsilon} B_\epsilon \arrow[u] \arrow[r] \arrow[d, "s"] & \Pi_A B \arrow[u] \arrow[dd, "="] \\
\Pi_{A^{\mathrm{I}}} B^{\mathrm{I}} \arrow[r] \arrow[d, "\psi"'] & \Pi_{A_\epsilon} p_* p^* B_\epsilon \arrow[d, "\vartheta"] & \\
(\Pi_A B)^{\mathrm{I}} \arrow[r] \arrow[rr, bend right, "\varepsilon"'] & \Pi_{A_\epsilon} B_\epsilon \arrow[r] & \Pi_A B
\end{tikzcd}
\tag{61}
$$

To see that $\psi \circ \varphi = 1$, observe that each map is in the image of I^*, say:

$$
\begin{aligned}
\varphi &= \mathrm{I}^* f \\
\psi &= \mathrm{I}^* g\,,
\end{aligned}
$$

where $g = \tilde{\alpha}$. Recall that in general the unit ε satisfies,

$$\varepsilon \circ \mathrm{I}^*(h) = \tilde{h}$$

for any map $h : X \to \mathrm{I}_* Y$. Thus

$$
\begin{aligned}
\varepsilon \circ \psi \circ \varphi &= \varepsilon \circ \mathrm{I}^* g \circ \mathrm{I}^* f \\
&= \varepsilon \circ \mathrm{I}^*(g \circ f) \\
&= \widetilde{(g \circ f)}.
\end{aligned}
$$

On the other hand, a diagram chase on (61) shows that

$$\varepsilon \circ \psi \circ \varphi = \varepsilon.$$

Therefore $g \circ f = \tilde{\varepsilon} = 1$, so $\psi \circ \varphi = \mathrm{I}^* g \circ \mathrm{I}^* f = \mathrm{I}^*(g \circ f) = \mathrm{I}^* 1 = 1$.

$\square$

6 A universal fibration

In this section we construct a *universal small fibration* $\dot{\mathcal{U}} \to \mathcal{U}$. It will then be shown in Section 8 that the base object $\mathcal{U}$ is fibrant, using the equivalence extension property to be proved in Section 7. Our construction of $\dot{\mathcal{U}} \to \mathcal{U}$ makes use of a new analysis of the well-known Hofmann-Streicher universe in a category $\widehat{\mathbb{C}} = [\mathbb{C}^{\mathrm{op}}, \mathsf{Set}]$ of presheaves on a small category $\mathbb{C}$, which was used in [HS97] to interpret dependent type theory (but occurs already in [?]). See [?] for further details.

6.1 Classifying families

Definition 56 ([HS97]). Let $\mathbb{C}$ be a small category. A (type-theoretic) *universe* $(U, \mathsf{E}l)$ consists of $U \in \widehat{\mathbb{C}}$ and $\mathsf{E}l \in \widehat{\int_{\mathbb{C}} U}$ with:

$$U(c) \;=\; \mathsf{Cat}\big(\mathbb{C}/_c{}^{\mathrm{op}}, \mathsf{Set}\big) \tag{62}$$

$$\mathsf{E}l(c, A) \;=\; A(id_c) \tag{63}$$

with the evident associated action on morphisms.

A few comments are required:

- In contrast to [HS97], in (62) we take the underlying set of objects of the functor category $\widehat{\mathbb{C}/_c} = [\mathbb{C}/_c{}^{\mathrm{op}}, \mathsf{Set}]$.
- As in [HS97], (63) adopts the "categories with families" point of view in describing an arrow $E \to U$ in $\widehat{\mathbb{C}}$ equivalently as a presheaf on the category of elements $\int_{\mathbb{C}} U$, using

$$\widehat{\mathbb{C}}/_U \;\simeq\; \widehat{\textstyle\int_{\mathbb{C}} U} \tag{64}$$

  where

$$E(c) \;=\; \textstyle\coprod_{A \in U(c)} \mathsf{E}l(c, A).$$

  The argument $(c, A) \in \int_{\mathbb{C}} U$ in (63) thus consists of an object $c \in \mathbb{C}$ and an element $A \in U(c)$.

- To account for size issues, the authors of [HS97] assume a Grothendieck universe u in Set, the elements of which are called *small*. The category $\mathbb{C}$ is assumed to be small, as are the values of the presheaves, unless otherwise stated.

The presheaf U, which is not small, is then regarded as the Grothendieck universe u "lifted" from Set to $[\mathbb{C}^{\mathrm{op}}, \mathsf{Set}]$. We first analyse this specification of $(U, \mathsf{E}l)$ from a different perspective, in order to establish its basic property as a classifier for small families in $\widehat{\mathbb{C}}$.

A realization-nerve adjunction. For a presheaf X on $\mathbb{C}$, recall that the category of elements is the comma category,

$$\textstyle\int_{\mathbb{C}} X \;=\; \mathsf{y}_{\mathbb{C}}/_X\,,$$

where $\mathsf{y}_{\mathbb{C}} : \mathbb{C} \to \mathsf{Set}^{\mathbb{C}^{\mathrm{op}}}$ is the Yoneda embedding, which we sometimes supress and write simply $\mathbb{C}/_X$ for $\mathsf{y}_{\mathbb{C}}/_X$.

Proposition 57 ([Gro83],§28)**.** *The category of elements functor*

$$\textstyle\int_{\mathbb{C}} : \widehat{\mathbb{C}} \longrightarrow \mathsf{Cat}$$

has a right adjoint,

$$\nu_{\mathbb{C}} : \mathsf{Cat} \longrightarrow \widehat{\mathbb{C}}\,.$$

For a small category $\mathbb{A}$, we shall call the presheaf $\nu_{\mathbb{C}}(\mathbb{A})$ the ($\mathbb{C}$-)nerve *of $\mathbb{A}$.*

Proof. The adjunction $\int_{\mathbb{C}} \dashv \nu_{\mathbb{C}}$ is an instance of the usual "realization/nerve" adjunction, here with respect to the covariant slice category functor $\mathbb{C}/- : \mathbb{C} \to \mathsf{Cat}$, as indicated below.

$$\begin{tikzcd}
\widehat{\mathbb{C}} \arrow[r, "\int_{\mathbb{C}}"'] & \mathsf{Cat} \arrow[l, bend right, "\nu_{\mathbb{C}}"'] \\
\mathbb{C} \arrow[u, hook, "\mathsf{y}"] \arrow[ur, "\mathbb{C}/_-"'] &
\end{tikzcd} \tag{65}$$

In detail, for $\mathbb{A} \in \mathsf{Cat}$ and $c \in \mathbb{C}$, let $\nu_{\mathbb{C}}(\mathbb{A})(c)$ be the Hom-set of functors,

$$\nu_{\mathbb{C}}(\mathbb{A})(c) = \mathsf{Cat}\big(\mathbb{C}/_c\,,\, \mathbb{A}\big)\,,$$

with contravariant action on $h : d \to c$ given by pre-composing a functor $P : \mathbb{C}/_c \to \mathbb{A}$ with the post-composition functor

$$\mathbb{C}/_h : \mathbb{C}/_d \longrightarrow \mathbb{C}/_c\,.$$

For the adjunction, observe that the slice category $\mathbb{C}/_c$ is the category of elements of the representable functor $\mathsf{y}c$,

$$\textstyle\int_{\mathbb{C}} \mathsf{y}c \;\cong\; \mathbb{C}/_c\,.$$

Thus for representables $\mathsf{y}c$, we have the required natural isomorphism

$$\widehat{\mathbb{C}}\big(\mathsf{y}c\,,\,\nu_{\mathbb{C}}(\mathbb{A})\big) \;\cong\; \nu_{\mathbb{C}}(\mathbb{A})(c) \;=\; \mathsf{Cat}\big(\mathbb{C}/_c\,,\,\mathbb{A}\big) \;\cong\; \mathsf{Cat}\big(\textstyle\int_{\mathbb{C}} \mathsf{y}c\,,\,\mathbb{A}\big)\,.$$

For arbitrary presheaves X, one uses the presentation of X as a colimit of representables over the index category $\int_{\mathbb{C}} X$, and the easy to prove fact that $\int_{\mathbb{C}}$ itself preserves colimits. Indeed, for any category $\mathbb{D}$, we have an isomorphism in Cat,

$$\varinjlim_{d\in\mathbb{D}} \mathbb{D}/_d \;\cong\; \mathbb{D}\,.$$

$\square$

When $\mathbb{C}$ is fixed, we may omit the subscript in the notation $\mathsf{y}_{\mathbb{C}}$ and $\int_{\mathbb{C}}$ and $\nu_{\mathbb{C}}$. The unit and counit maps of the adjunction $\int \dashv \nu$,

$$\begin{aligned} \eta &: X \longrightarrow \nu{\textstyle\int} X\,, \\ \epsilon &: {\textstyle\int}\nu\mathbb{A} \longrightarrow \mathbb{A}\,, \end{aligned}$$

are then as follows. At $c \in \mathbb{C}$, for $x : \mathsf{y}c \to X$, the functor $(\eta_X)_c(x) : \mathbb{C}/_c \to \mathbb{C}/_X$ is just composition with x,

$$(\eta_X)_c(x) = \mathbb{C}/_x : \mathbb{C}/_c \longrightarrow \mathbb{C}/_X\,. \tag{66}$$

For $\mathbb{A} \in \mathsf{Cat}$, the functor $\epsilon : \int \nu\mathbb{A} \to \mathbb{A}$ takes a pair $(c \in \mathbb{C}, f : \mathbb{C}/_c \to \mathbb{A})$ to the object $f(1_c) \in \mathbb{A}$,

$$\epsilon(c, f) = f(1_c).$$

Lemma 58. *For any $f : Y \to X$, the naturality square below is a pullback.*

$$\begin{tikzcd} Y \arrow[r, "\eta_Y"] \arrow[d, "f"'] & \nu\int Y \arrow[d, "\nu\int f"] \\ X \arrow[r, "\eta_X"'] & \nu\int X. \end{tikzcd} \tag{67}$$

Proof. It suffices to prove this for the case $f : X \to 1$. Thus consider the square

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d] & \nu\int X \arrow[d] \\ 1 \arrow[r, "\eta_1"'] & \nu\int 1. \end{tikzcd} \tag{68}$$

Evaluating at $c \in \mathbb{C}$ and applying (66) gives the following square in Set.

$$\begin{tikzcd} Xc \arrow[r, "\mathbb{C}/_{-}"] \arrow[d] & \mathsf{Cat}\big(\mathbb{C}/_c\,, \mathbb{C}/_X\big) \arrow[d] \\ 1c \arrow[r, "\mathbb{C}/_{-}"'] & \mathsf{Cat}\big(\mathbb{C}/_c\,, \mathbb{C}/_1\big) \end{tikzcd} \tag{69}$$

The image of $* \in 1c$ along the bottom is the forgetful functor $U_c : \mathbb{C}/_c \to \mathbb{C}$, and its fiber under the map on the right is the set of functors $F : \mathbb{C}/_c \to \mathbb{C}/_X$ such that $U_X \circ F = U_c$, where $U_X : \mathbb{C}/_X \to \mathbb{C}$ is also a forgetful functor. But any such F is uniquely of the form $\mathbb{C}/_x$ for $x = F(1_c) : \mathsf{y}c \to X$. $\square$

A universal family. For the terminal presheaf $1 \in \widehat{\mathbb{C}}$ we have an iso $\int 1 \cong \mathbb{C}$, so for every $X \in \widehat{\mathbb{C}}$ there is a canonical projection $\int X \to \mathbb{C}$, which is a discrete fibration. It follows that for any map $Y \to X$ of presheaves, the associated map $\int Y \to \int X$ is also a discrete fibration. Ignoring size issues temporarily, recall that discrete fibrations in Cat are classified by the forgetful functor $\dot{\mathsf{Set}}^{\mathrm{op}} \to \mathsf{Set}^{\mathrm{op}}$ from (the opposites of) the category of pointed sets to that of sets (cf. [Web07]). For every presheaf $X \in \widehat{\mathbb{C}}$, we therefore have a pullback diagram in Cat,

$$\begin{tikzcd} \int X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathsf{Set}}^{\mathrm{op}} \arrow[d] \\ \mathbb{C} \arrow[r, "X"'] & \mathsf{Set}^{\mathrm{op}}. \end{tikzcd} \tag{70}$$

Using $\mathbb{C} \cong \int 1$ and transposing by the adjunction $\int \dashv \nu$ then gives a commutative square in $\widehat{\mathbb{C}}$ of the form:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \nu \dot{\mathsf{Set}}^{\mathrm{op}} \arrow[d] \\ 1 \arrow[r, "\tilde{X}"'] & \nu \mathsf{Set}^{\mathrm{op}}. \end{tikzcd} \tag{71}$$

Lemma 59. *The square (71) is a pullback in $\widehat{\mathbb{C}}$. More generally, for any map $Y \to X$ in $\widehat{\mathbb{C}}$, there is a canonical pullback square*

$$\begin{tikzcd} Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \nu \dot{\mathsf{Set}}^{\mathrm{op}} \arrow[d] \\ X \arrow[r] & \nu \mathsf{Set}^{\mathrm{op}}. \end{tikzcd} \tag{72}$$

Proof. Apply the right adjoint ν to the pullback square (70) and paste the naturality square (67) from Lemma 58 on the left, to obtain the transposed square (72) as a pasting of two pullbacks. □

Let us write $\dot{\mathcal{V}} \to \mathcal{V}$ for the vertical map on the right in (72), setting

$$\begin{aligned} \dot{\mathcal{V}} &:= \nu \dot{\mathsf{Set}}^{\mathrm{op}} \\ \mathcal{V} &:= \nu \mathsf{Set}^{\mathrm{op}}. \end{aligned} \tag{73}$$

We summarize our results so far as follows.

Proposition 60. *The nerve $\dot{\mathcal{V}} \to \mathcal{V}$ of the classifier for discrete fibrations $\dot{\mathsf{Set}}^{\mathrm{op}} \to \mathsf{Set}^{\mathrm{op}}$, as defined in (73), classifies natural transformations $Y \to X$ in $\widehat{\mathbb{C}}$, in the sense that there is always a pullback square,*

$$\begin{tikzcd} Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathcal{V}} \arrow[d] \\ X \arrow[r, "\tilde{Y}"'] & \mathcal{V}. \end{tikzcd} \tag{74}$$

The classifying map $\tilde{Y} : X \to \mathcal{V}$ is determined by the adjunction $\int \dashv \nu$ as the transpose of the classifying map of the discrete fibration $\int Y \to \int X$.

Given a natural transformation $Y \to X$, the classifying map $\tilde{Y} : X \to \mathcal{V}$ is of course not in general unique. Nonetheless, we can use the construction of $\dot{\mathcal{V}} \to \mathcal{V}$ as the nerve of the discrete fibration classifier $\dot{\mathsf{Set}}^{\mathrm{op}} \to \mathsf{Set}^{\mathrm{op}}$, for which classifying functors $\mathbb{C} \to \mathsf{Set}^{\mathrm{op}}$ are unique up to natural isomorphism, to infer the following proposition, which will be required below (cf. [Shu15, GSS22]).

Proposition 61 (Realignment for families)**.** *Given a monomorphism $c : C \rightarrowtail X$ and a family $Y \to X$, let $y_c : C \to \mathcal{V}$ classify the pullback $c^*Y \to C$. Then there is a classifying map $y : X \to \mathcal{V}$ for $Y \to X$ with $y \circ c = y_c$.*

$$
\begin{tikzcd}
c^*Y \arrow[rr] \arrow[dr] \arrow[dd] & & \dot{\mathcal{V}} \arrow[dd] \\
& Y \arrow[ur, dotted] \arrow[dd] & \\
C \arrow[rr, "y_c"] \arrow[dr, tail, "c"'] & & \mathcal{V} \\
& X \arrow[ur, dotted, "y"'] &
\end{tikzcd}
\tag{75}
$$

Proof. Transposing the realignment problem (75) for presheaves across the adjunction $\int \dashv \nu$ results in the following realignment problem for discrete fibrations.

$$
\begin{tikzcd}
\int c^*Y \arrow[rr] \arrow[dr] \arrow[dd] & & \dot{\mathsf{Set}}^{\mathrm{op}} \arrow[dd] \\
& \int Y \arrow[ur, dotted] \arrow[dd] & \\
\int C \arrow[rr, "\widetilde{y_c}"] \arrow[dr, tail, "\int c"'] & & \mathsf{Set}^{\mathrm{op}} \\
& \int X \arrow[ur, dotted, "\tilde{y}"'] &
\end{tikzcd}
\tag{76}
$$

The category of elements functor $\int$ is easily seen to preserve pullbacks, hence monos; thus let us consider the general case of a functor $C : \mathbb{C} \rightarrowtail \mathbb{D}$ which is monic in Cat, a pullback of discrete fibrations as on the left below, and a presheaf $E : \mathbb{C} \to \mathsf{Set}^{\mathrm{op}}$ with $\int E \cong \mathbb{E}$ over $\mathbb{C}$.

$$\begin{tikzcd}
\mathbb{E} \arrow[rr] \arrow[dr] \arrow[dd] & & \dot{\mathsf{Set}}^{\mathrm{op}} \arrow[dd] \\
& \mathbb{F} \arrow[ur, dotted] \arrow[dd] & \\
\mathbb{C} \arrow[rr, "E"] \arrow[dr, tail, "C"'] & & \mathsf{Set}^{\mathrm{op}} \\
& \mathbb{D} \arrow[ur, dotted, "F"'] &
\end{tikzcd} \tag{77}$$

We seek $F : \mathbb{D} \to \mathsf{Set}^{\mathrm{op}}$ with $\int F \cong \mathbb{F}$ over $\mathbb{D}$ and $F \circ C = E$. Let $F_0 : \mathbb{D} \to \mathsf{Set}^{\mathrm{op}}$ with $\int F_0 \cong \mathbb{F}$ over $\mathbb{D}$, which exists since $\mathbb{F} \to \mathbb{D}$ is a discrete fibration. Since $F_0 \circ C$ and E both classify $\mathbb{E}$, there is a natural iso $e : F_0 \circ C \cong E$. Consider the following diagram

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "e"] \arrow[d, tail, "C"'] & (\mathsf{Set}^{\cong})^{\mathrm{op}} \arrow[r, "p_2"'] \arrow[d, "p_1"] & \mathsf{Set}^{\mathrm{op}} \\
\mathbb{D} \arrow[r, "F_0"'] \arrow[ur, dotted, "f"] & \mathsf{Set}^{\mathrm{op}} &
\end{tikzcd} \tag{78}$$

where $\mathsf{Set}^{\cong}$ is the category of isos in Set, with p_1, p_2 the (opposites of the) domain and codomain projections. There is a well-known weak factorization system on Cat (part of the "canonical model structure") with injective-on-objects functors on the left and isofibrations on the right. Thus there is a diagonal filler f as indicated. The functor $F := p_2 \circ f : \mathbb{D} \to \mathsf{Set}^{\mathrm{op}}$ is then the one we seek. $\square$

Small maps. Of course, as defined in (73), the classifier $\dot{\mathcal{V}} \to \mathcal{V}$ cannot be a map in $\widehat{\mathbb{C}}$, for reasons of size; we now address this. Let α be a cardinal number, and call the sets strictly smaller than it *α-small*. Let $\mathsf{Set}_\alpha \hookrightarrow \mathsf{Set}$ be the full subcategory of α-small sets. Call a presheaf $X : \mathbb{C}^{\mathrm{op}} \to \mathsf{Set}$ α-small if all of its values are α-small sets, and thus if, and only if, it factors through $\mathsf{Set}_\alpha \hookrightarrow \mathsf{Set}$. Call a map $f : Y \to X$ of presheaves α-small if all of the fibers $f_c^{-1}\{x\} \subseteq Yc$ are α-small sets (for all $c \in \mathbb{C}$ and $x \in Xc$). The latter condition is of course equivalent to saying that, in the pullback square over the element $x : \mathsf{y}c \to X$,

$$\begin{tikzcd} Y_x \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "f"] \\ \mathsf{y}c \arrow[r, "x"'] & X, \end{tikzcd} \tag{79}$$

the presheaf Y_x is α-small.

Now let us restrict the specification (73) of $\dot{\mathcal{V}} \to \mathcal{V}$ to the α-small sets:

$$\begin{aligned} \dot{\mathcal{V}}_\alpha &:= \nu\dot{\mathsf{Set}}_\alpha^{\mathsf{op}} \\ \mathcal{V}_\alpha &:= \nu\mathsf{Set}_\alpha^{\mathsf{op}}. \end{aligned} \tag{80}$$

Then the evident forgetful map $\dot{\mathcal{V}}_\alpha \to \mathcal{V}_\alpha$ *is* a map in the category $\widehat{\mathbb{C}}$ of presheaves, and it is in fact α-small. Moreover, it has the following basic property, which is just a restriction of the basic property of $\dot{\mathcal{V}} \to \mathcal{V}$ stated in Proposition 60.

Proposition 62. *The map $\dot{\mathcal{V}}_\alpha \to \mathcal{V}_\alpha$ classifies α-small maps $f : Y \to X$ in $\widehat{\mathbb{C}}$, in the sense that there is always a pullback square,*

$$\begin{tikzcd} Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathcal{V}}_\alpha \arrow[d] \\ X \arrow[r, "\tilde{Y}"'] & \mathcal{V}_\alpha. \end{tikzcd} \tag{81}$$

The classifying map $\tilde{Y} : X \to \mathcal{V}_\alpha$ is determined by the adjunction $\int \dashv \nu$ as (the factorization of) the transpose of the classifiyng map of the discrete fibration $\int X \to \int Y$.

Proof. If $Y \to X$ is α-small, its classifying map $\tilde{Y} : X \to \mathcal{V}$ factors through $\mathcal{V}_\alpha \hookrightarrow \mathcal{V}$, as indicated below,

$$\begin{tikzcd} Y \arrow[r] \arrow[d] \arrow[rr, bend left] & \nu\dot{\mathsf{Set}}_\alpha{}^{\mathrm{op}} \arrow[r, hook] \arrow[d] & \nu\dot{\mathsf{Set}}{}^{\mathrm{op}} \arrow[d] \\ X \arrow[r] \arrow[rr, bend right, "\tilde{Y}"'] & \nu\mathsf{Set}_\alpha{}^{\mathrm{op}} \arrow[r, hook] & \nu\mathsf{Set}^{\mathrm{op}}, \end{tikzcd} \tag{82}$$

in virtue of the following adjoint transposition,

$$
\begin{tikzcd}
\int Y \arrow[r] \arrow[d] \arrow[rr, bend left] & \dot{\mathsf{Set}}_{\alpha}{}^{\mathrm{op}} \arrow[r, hook] \arrow[d] & \dot{\mathsf{Set}}^{\mathrm{op}} \arrow[d] \\
\int X \arrow[r] \arrow[rr, bend right] & \mathsf{Set}_{\alpha}{}^{\mathrm{op}} \arrow[r, hook] & \mathsf{Set}^{\mathrm{op}}.
\end{tikzcd}
\tag{83}
$$

Note that the square on the right is evidently a pullback, and so the one on the left is, too, because the outer rectangle is the classifying pulback of the discrete fibration $\int Y \to \int X$, as stated. Thus the left square in (82) is also a pullback. $\square$

Examples of universal families $\dot{\mathcal{V}}_{\alpha} \longrightarrow \mathcal{V}_{\alpha}$.

1. Let $\alpha = \kappa$ a strongly inaccessible cadinal, so that $\mathsf{ob}(\mathsf{Set}_{\kappa})$ is a Grothendieck universe. Then the Hofmann-Streicher universe of Definition 56 is recovered as the κ-small map classifier

   $$E \cong \dot{\mathcal{V}}_{\kappa} \longrightarrow \mathcal{V}_{\kappa} \cong U$$

   in the sense of Proposition 62. Indeed, for $c \in \mathbb{C}$, we have

   $$\mathcal{V}_{\kappa} c \;=\; \nu(\mathsf{Set}_{\kappa}^{\mathsf{op}})(c) = \mathsf{Cat}\big(\mathbb{C}/_c\,,\, \mathsf{Set}_{\kappa}^{\mathsf{op}}\big) \;=\; \mathsf{ob}(\widehat{\mathbb{C}/_c}) \;=\; Uc\,. \tag{84}$$

   For $\dot{\mathcal{V}}_{\kappa}$ we then have,

   $$\begin{aligned} \dot{\mathcal{V}}_{\kappa} c \;=\; \nu(\dot{\mathsf{Set}}_{\kappa}^{\mathsf{op}})(c) \;&=\; \mathsf{Cat}\big(\mathbb{C}/_c\,,\, \dot{\mathsf{Set}}_{\kappa}^{\mathsf{op}}\big) \\ &\cong\; \textstyle\coprod_{A \in \mathcal{V}_{\kappa} c} \mathsf{Cat}_{\mathbb{C}/_c}\big(\mathbb{C}/_c\,,\, A^{*}\mathsf{Set}_{\kappa}^{\mathsf{op}}\big) \end{aligned} \tag{85}$$

   where the A-summand in (85) is defined by taking sections of the pullback indicated below.

   $$
   \begin{tikzcd}
   A^{*}\mathsf{Set}_{\kappa}^{\mathsf{op}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \dot{\mathsf{Set}}_{\kappa}^{\mathsf{op}} \arrow[d] \\
   \mathbb{C}/_c \arrow[r, "A"'] \arrow[u, dotted, bend left] \arrow[ur, dotted] & \mathsf{Set}_{\kappa}^{\mathsf{op}}
   \end{tikzcd}
   \tag{86}
   $$

But $A^*\mathsf{Set}_\kappa^{\mathsf{op}} \;\cong\; \int_{\mathbb{C}/_c} A$ over $\mathbb{C}/_c$, and sections of this discrete fibration in Cat correspond uniquely to natural maps $1 \to A$ in $\widehat{\mathbb{C}/_c}$. Since 1 is representable in $\widehat{\mathbb{C}/_c}$ we can continue (85) by

$$\begin{aligned}
\dot{\mathcal{V}}_\kappa c \;&\cong\; \textstyle\coprod_{A \in \mathcal{V}_\kappa c} \mathsf{Cat}_{\mathbb{C}/_c}\big(\mathbb{C}/_c\,,\, A^*\mathsf{Set}_\kappa^{\mathsf{op}}\big) \\
&\cong\; \textstyle\coprod_{A \in \mathcal{V}_\kappa c} \widehat{\mathbb{C}/_c}(1, A) \\
&\cong\; \textstyle\coprod_{A \in \mathcal{V}_\kappa c} A(1_c) \\
&=\; \textstyle\coprod_{A \in \mathcal{V}_\kappa c} \mathsf{E}l(\langle c, A\rangle) \\
&=\; Ec\,.
\end{aligned}$$

2. By functoriality of the nerve $\nu : \mathsf{Cat} \to \widehat{\mathbb{C}}$, a sequence of Grothendieck universes
$$\mathsf{Set}_\alpha \subseteq \mathsf{Set}_\beta \subseteq \ldots$$
in Set gives rise to a (cumulative) sequence of type-theoretic universes
$$\mathcal{V}_\alpha \rightarrowtail \mathcal{V}_\beta \rightarrowtail \ldots$$
in $\widehat{\mathbb{C}}$. More precisely, there is a sequence of cartesian squares,
$$\begin{tikzcd}
\dot{\mathcal{V}}_\alpha \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathcal{V}}_\beta \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \ldots \\
\mathcal{V}_\alpha \arrow[r, tail] & \mathcal{V}_\beta \arrow[r, tail] & \ldots\,,
\end{tikzcd} \tag{87}$$
in the image of $\nu : \mathsf{Cat} \longrightarrow \widehat{\mathbb{C}}$, classifying small maps in $\widehat{\mathbb{C}}$ of increasing size, in the sense of Proposition 62.

3. Let $\alpha = 2$ so that $1 \to 2$ is the subobject classifier of Set, and
$$\mathbb{1} = \dot{\mathsf{Set}}_2^{\mathsf{op}} \longrightarrow \mathsf{Set}_2^{\mathsf{op}} = \mathbb{2}$$
is then a classifier in Cat for *sieves*, i.e. full subcategories $\mathbb{S} \hookrightarrow \mathbb{A}$ closed under the domains of arrows $a \to s$ for $s \in \mathbb{S}$. The nerve $\dot{\mathcal{V}}_2 \to \mathcal{V}_2$ is then the usual subobject classifier $1 \to \Omega$ of $\widehat{\mathbb{C}}$,
$$1 = \nu\mathbb{1} = \dot{\mathcal{V}}_2 \longrightarrow \mathcal{V}_2 = \nu\mathbb{2} = \Omega\,.$$

6.2 Classifying trivial fibrations

Recall from section 2 that (uniform) trivial fibration structures on a map $A \to X$ correspond bijectively to relative $+$-algebra structures over X (definition 8). A relative $+$-algebra structure on $A \to X$ is an algebra structure for the pointed endofunctor $+_X : \mathsf{cSet}/X \to \mathsf{cSet}/X$, where recall from (1),

$$A^+ = \sum_{\varphi:\Phi} A^{[\varphi]} \quad \text{over } X.$$

A $+$-algebra structure is then a retract $\alpha : A^+ \to A$ over X of the canonical map $\eta_A : A \to A^+$,

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[rr, bend left, "="] \arrow[dr] & A^+ \arrow[r, "\alpha"] \arrow[d] & A \arrow[dl] \\ & X. & \end{tikzcd} \tag{88}$$

In more detail, let us write $A \to X$ as a family $(A_x)_{x \in X}$, so that $A = \sum_{x:X} A_x \to X$. Since the $+$-functor acts fiberwise, the object A^+ in (88) is then the indexing projection

$$\sum_{x:X} A_x^+ \to X.$$

Working in the slice cSet/X, the (relative) exponentials (internal Hom's) $[A^+, A]$ and $[A, A]$ and the “precomposition by η_A” map $[\eta_A, A]$, fit into the following pullback diagram

$$\begin{tikzcd} +\mathsf{Alg}(A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & {[A^+, A]} \arrow[d, "{[\eta_A, A]}"] \\ 1 \arrow[r, "'id_A'"'] & {[A, A].} \end{tikzcd} \tag{89}$$

The constructed object $+\mathsf{Alg}(A) \to X$ over X is then the *object of $+$-algebra structures on* $A \to X$, in the sense that sections $X \to +\mathsf{Alg}(A)$ correspond uniquely to $+$-algebra structures on $A \to X$. Moreover, $+\mathsf{Alg}(A) \to X$ is stable under pullback, in the sense that for any $f : Y \to X$, we have two pullback squares,

$$\begin{tikzcd}
f^*A \arrow[r] \arrow[d] & A \arrow[d] \\
Y \arrow[r, "f"'] & X \\
+\mathsf{Alg}(f^*A) \arrow[r] \arrow[u] & +\mathsf{Alg}(A) \arrow[u]
\end{tikzcd} \tag{90}$$

because the +-functor, exponentials and pullbacks occurring in the construction of $+\mathsf{Alg}(A) \to X$ are themselves all stable.

It then follows from Proposition 62 that, if $A \to X$ is small, then $+\mathsf{Alg}(A) \to X$ is itself a pullback of the analogous object $+\mathsf{Alg}(\dot{\mathcal{V}}) \to \mathcal{V}$ constructed from the universal small family $\dot{\mathcal{V}} \to \mathcal{V}$, so there are two pullback squares:

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & \dot{\mathcal{V}} \arrow[d] \\
X \arrow[r, "\chi_A"'] & \mathcal{V} \\
+\mathsf{Alg}(A) \arrow[r] \arrow[u] & +\mathsf{Alg}(\dot{\mathcal{V}}) \arrow[u]
\end{tikzcd} \tag{91}$$

Proposition 63. *There is a* universal small trivial fibration

$$\mathsf{T\dot{F}ib} \to \mathsf{TFib}.$$

Every small trivial fibration $A \to X$ is a pullback of $\mathsf{T\dot{F}ib} \to \mathsf{TFib}$ along a canonically determined classifying map $X \to \mathsf{TFib}$.

$$\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{T\dot{F}ib} \arrow[d] \\
X \arrow[r] & \mathsf{TFib}
\end{tikzcd} \tag{92}$$

Proof. We can take

$$\mathsf{TFib} := +\mathsf{Alg}(\dot{\mathcal{V}}),$$

which comes with its projection $+\mathsf{Alg}(\dot{\mathcal{V}}) \to \mathcal{V}$ as in diagram (91). Now define $p_t : \mathsf{T\dot{F}ib} \to \mathsf{TFib}$ by pulling back the universal small family,

$$\begin{tikzcd}
\mathsf{T\dot{F}ib} \arrow[r] \arrow[d, "p_t"'] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathcal{V}} \arrow[d, "p"] \\
\mathsf{TFib} \arrow[r] & \mathcal{V}.
\end{tikzcd}$$

Consider the following diagram, in which all the squares (including the distorted ones) are pullbacks, with the outer one coming from proposition 62 and the lower one from (91).

$$
\begin{tikzcd}
 & A \arrow[rr, "q_A"] \arrow[dddd] \arrow[dr, dotted] & & \dot{\mathcal{V}} \arrow[dddd, "p"] \\
 & & \dot{\mathsf{TFib}} \arrow[ur] \arrow[dd, "p_t"'] & \\
 & & & \\
\mathsf{TFib}(A) \arrow[rr] \arrow[dr] & & \mathsf{TFib} \arrow[dr] & \\
 & X \arrow[rr, "\chi_A"'] \arrow[ur, dotted, "\alpha'"'] \arrow[ul, dotted, bend left, "\alpha"] & & \mathcal{V}.
\end{tikzcd}
\tag{93}
$$

A trivial fibration structure α on $A \to X$ is a section the object of $+$-algebra structures on A, occurring in the diagram as

$$\mathsf{TFib}(A) := +\mathsf{Alg}(A),$$

the pullback of $\mathsf{TFib} = +\mathsf{Alg}(\dot{\mathcal{V}})$ along the classifying map $\chi_A : X \to \mathcal{V}$ for the small family $A \to X$. Such sections correspond uniquely to factorizations α' of χ_A as indicated, which in turn induce pullback squares of the required kind (92).

Note that the map $p_t : \dot{\mathsf{TFib}} \to \mathsf{TFib}$ has a canonical trivial fibration structure. Indeed, consider the following diagram, in which both squares are pullbacks.

$$
\begin{tikzcd}
\dot{\mathsf{TFib}} \arrow[r] \arrow[d, "p_t"'] & \dot{\mathcal{V}} \arrow[d] \\
\mathsf{TFib} \arrow[r] & \mathcal{V} \\
\mathsf{TFib}(\dot{\mathsf{TFib}}) \arrow[r] \arrow[u] & \mathsf{TFib}(\dot{\mathcal{V}}) \arrow[u]
\end{tikzcd}
\tag{94}
$$

$\mathsf{TFib}(\dot{\mathcal{V}})$ is the object of trivial fibration structures on $\dot{\mathcal{V}} \to \mathcal{V}$, and its pullback $\mathsf{TFib}(\dot{\mathsf{TFib}})$ is therefore the object of trivial fibration structures on $p_t : \dot{\mathsf{TFib}} \to \mathsf{TFib}$. Thus we seek a section of $\mathsf{TFib}(\dot{\mathsf{TFib}}) \to \mathsf{TFib}$. But recall that $\mathsf{TFib} = \mathsf{TFib}(\dot{\mathcal{V}})$ by definition, so the lower pullback square is the pullback of $\mathsf{TFib}(\dot{\mathcal{V}}) \to \mathcal{V}$ against itself, which does indeed have a distinguished section, namely the diagonal

$$\Delta : \mathsf{TFib}(\dot{\mathcal{V}}) \to \mathsf{TFib}(\dot{\mathcal{V}}) \times_{\mathcal{V}} \mathsf{TFib}(\dot{\mathcal{V}}).$$

□

We record the following notation and corresponding fact from the foregoing proof for future reference:

Lemma 64. *The classifying type* $\mathsf{TFib}(A) := +\mathsf{Alg}(A) \to X$ *for trivial fibration structures on a map* $A \to X$ *is stable under pullback, in the sense that for any* $f : Y \to X$*, we have two pullback squares,*

$$\begin{tikzcd} f^*A \arrow[r] \arrow[d] & A \arrow[d] \\ Y \arrow[r, "f"'] & X \\ \mathsf{TFib}(f^*A) \arrow[r] \arrow[u] & \mathsf{TFib}(A) \arrow[u] \end{tikzcd} \tag{95}$$

6.3 Classifying fibrations

In order to classify fibrations $A \twoheadrightarrow X$, we proceed as for trivial fibrations by constructing an object $\mathsf{Fib}(A) \to X$ of fibration structures on a map $A \to X$ which, moreover, is stable under pullback. We then apply the construction to the universal small family $\dot{\mathcal{V}} \to \mathcal{V}$ to obtain a universal small fibration.

The construction of $\mathsf{Fib}(A) \to X$ is a bit more involved than that of $\mathsf{TFib}(A) \to X$. Recall from section 3.2 the characterization of (uniform, unbiased) fibration structures on a map $p_A : A \to X$ in terms of +-algebra structures:

1. First, pull the map $p_A : A \to X$ back to $\mathsf{cSet}/_{\mathrm{I}}$ by applying the functor $\mathrm{I} \times (-) : \mathsf{cSet} \to \mathsf{cSet}/_{\mathrm{I}}$. We continue to write $p_A : A \to X$ the resulting map over I.

2. Form the pullback-hom $\delta \Rightarrow p_A : A^{\mathrm{I}} \to X^{\mathrm{I}} \times_X A$ of p_A with the generic

point $\delta : \mathrm{I} \to \mathrm{I} \times \mathrm{I}$ over I, as indicated in the following diagram.

$$\begin{tikzcd}
A^{\mathrm{I}} \arrow[dr, "\delta \Rightarrow p_A"] \arrow[drr, bend left, "A^{\delta}"] \arrow[ddr, bend right, "(p_A)^{\mathrm{I}}"'] & & \\
& X^{\mathrm{I}} \times_X A \arrow[r] \arrow[d] & A \arrow[d, "p_A"] \\
& X^{\mathrm{I}} \arrow[r, "X^{\delta}"'] & X
\end{tikzcd} \tag{96}$$

3. A fibration structure on $p_A : A \to X$ is then a relative +-algebra structure on the map $\delta \Rightarrow p_A$ in the slice category over the codomain $X^{\mathrm{I}} \times_X A$ (and that in the slice over I).

   In order to construct the object $\mathsf{Fib}(A) \to X$ classifying such structures, let us first relabel the objects and arrows in diagram (96) as follows:

$$\begin{aligned}
p &:= p_A \\
\epsilon &:= X^{\delta} : X^{\mathrm{I}} \to X \\
A_{\epsilon} &:= X^{\mathrm{I}} \times_X A \\
\epsilon^A &:= \delta \Rightarrow p_A
\end{aligned}$$

   so that (the working part of) our diagram becomes:

$$\begin{tikzcd}
A^{\mathrm{I}} \arrow[dr, "\epsilon^A"'] & & \\
& A_{\epsilon} \arrow[r] \arrow[d, "p_{\epsilon}"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "p"] \\
& X^{\mathrm{I}} \arrow[r, "\epsilon"'] & X
\end{tikzcd} \tag{97}$$

4. A +-algebra structure on ϵ^A is a retract α over A_{ϵ} of the unit η as

indicated below, where D is the domain of the map $(\epsilon^A)^+$,

$$\begin{tikzcd}
A^{\mathrm{I}} \arrow[r, "\eta"'] \arrow[dr, "\epsilon^A"'] & D \arrow[l, bend right, dotted, "\alpha"'] \arrow[d, "(\epsilon^A)^+"] & \\
& A_\epsilon \arrow[r] \arrow[d, "p_\epsilon"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "p"] \\
& X^{\mathrm{I}} \arrow[r, "\epsilon"'] & X
\end{tikzcd} \tag{98}$$

5. As in the construction (89), there is an object $+\mathsf{Alg}(\epsilon^A)$ over A_ϵ of +-algebra structures on ϵ^A, the sections of which correspond uniquely to +-algebra structures on ϵ^A (and thus fibration structures on A).

$$\begin{tikzcd}
& A^{\mathrm{I}} \arrow[r, "\eta"'] \arrow[d, "\epsilon^A"'] & D \arrow[l, bend right, dotted, "\alpha"'] \arrow[dl, "(\epsilon^A)^+"] \\
+\mathsf{Alg}(\epsilon^A) \arrow[r] & A_\epsilon \arrow[r] \arrow[d, "p_\epsilon"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "p"] \\
& X^{\mathrm{I}} \arrow[r, "\epsilon"'] & X
\end{tikzcd} \tag{99}$$

6. Sections of $+\mathsf{Alg}(\epsilon^A) \to A_\epsilon$ then correspond to sections of its push-forward along p_ϵ, which we shall call F_A:

$$F_A := (p_\epsilon)_*(+\mathsf{Alg}(\epsilon^A))\,.$$

$$\begin{tikzcd}
& A^{\mathrm{I}} \arrow[r, "\eta"'] \arrow[d, "\epsilon^A"'] & D \arrow[l, bend right, dotted, "\alpha"'] \arrow[dl, "(\epsilon^A)^+"] \\
+\mathsf{Alg}(\epsilon^A) \arrow[r] & A_\epsilon \arrow[r] \arrow[d, "p_\epsilon"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "p"] \\
F_A \arrow[r] & X^{\mathrm{I}} \arrow[r, "\epsilon"'] & X
\end{tikzcd} \tag{100}$$

7. One might now think of taking another pushforward of $F_A \to X^{\mathrm{I}}$ along $\epsilon : X^{\mathrm{I}} \to X$ to get the object $\mathsf{Fib}(A) \to X$ that we seek, but unfortunately, this would not be stable under pullback along arbitrary maps $Y \to X$, because $\epsilon : X^{\mathrm{I}} \to X$ is not stable in that way. Instead we use the *root* functor of Proposition (3), i.e. the right adjoint of the pathspace, $(-)^{\mathrm{I}} \dashv (-)_{\mathrm{I}}$.

   Let $f : F_A \to X^{\mathrm{I}}$ be the map indicated in (100), and $\eta_X : X \to (X^{\mathrm{I}})_{\mathrm{I}}$ the unit of the root adjunction. Then define $\mathsf{Fib}(A) \to X$ by

$$\mathsf{Fib}(A) := \eta^* f_{\mathrm{I}}$$

   as indicated in the following pullback diagram.

$$\begin{tikzcd} \mathsf{Fib}(A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (F_A)_{\mathrm{I}} \arrow[d, "f_{\mathrm{I}}"] \\ X \arrow[r, "\eta"'] & (X^{\mathrm{I}})_{\mathrm{I}} \end{tikzcd} \tag{101}$$

   By adjointness, sections of $\mathsf{Fib}(A) \to X$ then correspond bijectively to sections of $f : F_A \to X^{\mathrm{I}}$.

8. Finally, recall that we are still working in the slice cSet/I and need to get back to cSet by applying the functor $\mathrm{I}_* : \mathsf{cSet}/\mathrm{I} \to \mathsf{cSet}$. Call the map $\mathsf{Fib}(A) \to X$ constructed over I in the last step $\mathsf{Fib}(A)_i \to \mathrm{I}^* X$, and apply I_* to get,

$$\mathrm{I}_*(\mathsf{Fib}(A)_i) = \Pi_{i:\mathrm{I}} \mathsf{Fib}(A)_i \to X^{\mathrm{I}}$$

   in cSet. We then define the desired map $\mathsf{Fib}(A) \to X$ as the pullback along the unit $\rho : X \to X^{\mathrm{I}}$ of $\mathrm{I}^* \dashv \mathrm{I}_*$, as indicated below.

$$\begin{tikzcd} \mathsf{Fib}(A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Pi_{i:\mathrm{I}} \mathsf{Fib}(A)_i \arrow[d] \\ X \arrow[r, "\rho"'] & X^{\mathrm{I}} \end{tikzcd} \tag{102}$$

   It then follows directly from the adjunction $\mathrm{I}^* \dashv \mathrm{I}_*$ that sections of $\mathsf{Fib}(A) \to X$ correspond bijectively to sections of $\mathsf{Fib}(A)_i \to \mathrm{I}^* X$ over I.

Now for the universal family $\dot{\mathcal{V}} \to \mathcal{V}$, let $\mathsf{Fib} = \mathsf{Fib}(\dot{\mathcal{V}})$, which comes with its projection $\mathsf{Fib}(\dot{\mathcal{V}}) \to \mathcal{V}$, and define the universal small fibration $\dot{\mathsf{Fib}} \twoheadrightarrow \mathsf{Fib}$ by pulling back the universal small family,

$$\begin{tikzcd} \dot{\mathsf{Fib}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathcal{V}} \arrow[d, "p"] \\ \mathsf{Fib} \arrow[r] & \mathcal{V}. \end{tikzcd} \tag{103}$$

Proposition 65. *The map*

$$\dot{\mathsf{Fib}} \to \mathsf{Fib}$$

just constructed is a universal small fibration*: every small fibration $A \twoheadrightarrow X$ is a pullback of $\dot{\mathsf{Fib}} \twoheadrightarrow \mathsf{Fib}$ along a canonically determined classifying map $X \to \mathsf{Fib}$.*

$$\begin{tikzcd} A \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathsf{Fib}} \arrow[d, two heads] \\ X \arrow[r] & \mathsf{Fib} \end{tikzcd} \tag{104}$$

Proof. First, we need to show that the construction of $\mathsf{Fib}(A) \to X$ as the object of fibration structures on a map $A \to X$ is stable under pullback along all maps $f : Y \to X$. The relevant parts of the construction diagram (100) are repeated below,

$$\begin{tikzcd} & A^{\mathrm{I}} \arrow[d, "\epsilon^A"'] & \\ +\mathsf{Alg}(\epsilon^A) \arrow[r] & A_\epsilon \arrow[r] \arrow[d, "p_\epsilon"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "p"] \\ F_A \arrow[r] & X^{\mathrm{I}} \arrow[r, "\epsilon"'] & X \end{tikzcd} \tag{105}$$

Now consider the following in which the front face of the central cube is a pullback.

$$\begin{tikzcd}
 & B^{\mathrm{I}} \arrow[rrr] \arrow[d, "\epsilon^B"'] & & & A^{\mathrm{I}} \arrow[d, "\epsilon^A"] & \\
+\mathsf{Alg}(\epsilon^B) \arrow[r] & B_\epsilon \arrow[rrr, dotted] \arrow[d] \arrow[dr] & & & A_\epsilon \arrow[d] \arrow[dl] & +\mathsf{Alg}(\epsilon^A) \arrow[l] \\
F_B \arrow[r] & Y^{\mathrm{I}} \arrow[rrr, dotted, bend right=15] \arrow[dr] & B \arrow[r] \arrow[d] & A \arrow[d] & X^{\mathrm{I}} \arrow[dl] & F_A \arrow[l] \\
 & \mathsf{Fib}(B) \arrow[r] & Y \arrow[r, "f"'] & X & \mathsf{Fib}(A) \arrow[l] &
\end{tikzcd} \tag{106}$$

On the left side we repeat the construction with $B \to Y$ in place of $A \to X$. The left face is thus a pullback, whence the back (dotted) face is a pullback. The two-story square in back is the image of the front square under the right adjoint $(-)^{\mathrm{I}}$ and is therefore a pullback, therefore the top rectangle in the back is a pullback. It follows that $+\mathsf{Alg}(\epsilon^B)$ is a pullback of $+\mathsf{Alg}(\epsilon^A)$ along the upper dotted arrow, as in diagram (90), and so the pushforward F_B is a pullback of the corresponding F_A, along the lower dotted arrow (which is f^{I}), by the Beck-Chevalley condition. Thus we have shown

$$F_B \cong (f^{\mathrm{I}})^* F_A. \tag{107}$$

It remains to show that $\mathsf{Fib}(B)$ is a pullback of $\mathsf{Fib}(A)$ along $f : Y \to X$, and now it is good that we did not take these to be pushforwards of F_B and F_A, because the floor of the cube need not be a pullback, and so the Beck-Chavalley condition would not apply. Instead, consider the following diagram.

$$\begin{tikzcd}
 & \mathsf{Fib}(B) \arrow[r] \arrow[dl] \arrow[dd] & \mathsf{Fib}(A) \arrow[dr] \arrow[dd] & \\
(F_B)_{\mathrm{I}} \arrow[rrr] \arrow[dd] & & & (F_A)_{\mathrm{I}} \arrow[dd] \\
 & Y \arrow[r, "f"'] \arrow[dl, "\eta"] & X \arrow[dr, "\eta"'] & \\
(Y^{\mathrm{I}})_{\mathrm{I}} \arrow[rrr, "(f^{\mathrm{I}})_{\mathrm{I}}"'] & & & (X^{\mathrm{I}})_{\mathrm{I}}
\end{tikzcd} \tag{108}$$

The sides of the cube are pullbacks by the construction of $\mathsf{Fib}(A)$ and $\mathsf{Fib}(B)$. The front face is the root of the pullback (107) and is thus also a pullback, since the root is a right adjoint. The base commutes by naturality of the unit, and so the back face is also a pullback, as required. Finally, the base change along $\mathrm{I}_* : \mathsf{cSet}/\mathrm{I} \to \mathsf{cSet}$ in step 8 above clearly also preserves the pullback.

Thus we can indeed use $\mathsf{Fib} := \mathsf{Fib}(\dot{\mathcal{V}})$ to define the universal small fibration $\dot{\mathsf{Fib}} \to \mathsf{Fib}$ as in (103), and the proof can conclude just as that for proposition 63. $\square$

Definition 66. Write $\dot{\mathcal{U}} \twoheadrightarrow \mathcal{U}$ for the universal small fibration $\dot{\mathsf{Fib}} \twoheadrightarrow \mathsf{Fib}$ just constructed in proposition 65.

6.4 Realignment for fibration structure

The realignment for families of Proposition 61 will need to be extended to (structured) fibrations. Our approach makes use of the notion of a *weak proposition.* Informally, a map $P \to X$ may be said to be a weak proposition if it is "conditionally contractible", in the sense that it is contractible if it has a section (recall that a proposition may be defined as a fibration that is "contractible if inhabited"). More formally, we have the following.

Definition 67. A map $P \to X$ is said to be a *weak proposition* if the projection $P \times_X P \to P$ is a trivial fibration.

$$
\begin{tikzcd}
P^2 \arrow[r] \arrow[d, two heads, "\sim"'] \arrow[dr, phantom, "\lrcorner", very near start] & P \arrow[d] \\
P \arrow[r] & X.
\end{tikzcd}
\tag{109}
$$

Note that if either projection is a trivial fibration, then both are.

As an object over the base, a weak proposition is thus one that "thinks it is contractible". The key fact needed for realignment is the following.

Lemma 68. *For any $Y \to X$, the classifying type $\mathsf{TFib}(Y) \to X$ is a weak proposition. Moreover, the same is true for $\mathsf{Fib}(Y) \to X$ if the cofibrations are closed under exponentiation by the interval I.*

Proof. Let $Y \to X$ and consider the following diagram, in which we have written $Y' = \mathsf{TFib}(Y) \times_X Y$ and $\mathsf{TFib}(Y)^2 = \mathsf{TFib}(Y) \times_X \mathsf{TFib}(Y)$.

$$
\begin{tikzcd}
Y' \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & Y \arrow[dd] & \\
& \mathsf{TFib}(Y)^2 \arrow[rr] \arrow[dl] \arrow[dr, phantom, "\lrcorner", very near start] & & \mathsf{TFib}(Y) \arrow[dl] \\
\mathsf{TFib}(Y) \arrow[rr] & & X &
\end{tikzcd}
\tag{110}
$$

Since TFib is stable under pullback (by Lemma 64), we have $\mathsf{TFib}(Y)^2 \cong \mathsf{TFib}(Y')$, and since $\mathsf{TFib}(Y)^2$ has a canonical section, $Y' \to \mathsf{TFib}(Y)$ is therefore a trivial fibration. Inspecting the definition of $\mathsf{TFib}(A) = +\mathsf{Alg}(A)$ in (89), we see that if a map $A \to X$ is a trivial fibration, then so is $\mathsf{TFib}(A) \to X$ (since $A \to A^+$ is always a cofibration). Thus $\mathsf{TFib}(Y)^2 \cong \mathsf{TFib}(Y') \to \mathsf{TFib}(Y)$ is also a trivial fibration.

For $\mathsf{Fib}(Y) \to X$, we use the foregoing to infer that $\mathsf{TFib}(Y^I) \to X^I \times_X Y$ is a weak proposition, and so therefore is its pushforward $p_* \mathsf{TFib}(Y^{\mathrm{I}}) \to X^{\mathrm{I}}$ along the first projection $p : X^I \times_X Y \to X^{\mathrm{I}}$, since pushforward clearly preserve weak propositions. Applying the root $(-)_{\mathrm{I}}$ preserves trivial fibrations, by the assumption that its left adjoint $(-)^{\mathrm{I}}$ preserves cofibrations, so as a right adjoint, it also preserves weak propositions. Therefore $(p_* \mathsf{TFib}(Y^{\mathrm{I}}))_{\mathrm{I}} \to (X^{\mathrm{I}})_{\mathrm{I}}$ is a weak proposition, but then so is its pullback along the unit $X \to (X^{\mathrm{I}})_{\mathrm{I}}$, which is $\mathsf{Fib}(Y) \to X$. $\square$

In light of Lemma 68 we assume a final axiom on cofibrations:

(C8) The map $c^{\mathrm{I}} : A^{\mathrm{I}} \to B^{\mathrm{I}}$ is a cofibration if $c : A \to B$ is a cofibration.

Now by Proposition 65 we have a universal (small) fibration $\dot{\mathcal{U}} \twoheadrightarrow \mathcal{U}$, with $\mathcal{U} = \mathsf{Fib}(\dot{\mathcal{V}})$, where $\dot{\mathcal{V}} \to \mathcal{V}$ is the universal (small) family. Type theoretically, we therefore have

$$\mathcal{U} = \Sigma_{E:\mathcal{V}} \mathsf{Fib}(E)\,,$$

with canonical projection

$$\mathcal{U} = \Sigma_{E:\mathcal{V}} \mathsf{Fib}(E) \to \mathcal{V}\,,$$

so that a fibration $E \twoheadrightarrow X$ is a pair $\langle E, e \rangle$, consisting of the underlying family $E \to X$, equipped with a fibration structure $e : \mathsf{Fib}(E)$. Lemma 68 then allows us to establish the following, which was first isolated in [**?**] (as condition (2'), also see [Shu15]). It will be needed in the sequel to correct the fibration structure on certain maps.

Lemma 69 (Realignment for fibrations)**.** *Given a fibration $F \twoheadrightarrow X$ and a cofibration $c : C \rightarrowtail X$, let $f_c : C \to \mathcal{U}$ classify the pullback $c^*F \twoheadrightarrow C$. Then there is a classifying map $f : X \to \mathcal{U}$ for F with $f \circ c = f_c$.*

$$\begin{tikzcd}
c^*F \arrow[rr] \arrow[dr] \arrow[dd, two heads] & & \dot{\mathcal{U}} \arrow[dd, two heads] \\
& F \arrow[ur, dotted] \arrow[dd, two heads] & \\
C \arrow[rr, "f_c"] \arrow[dr, tail, "c"'] & & \mathcal{U} \\
& X \arrow[ur, dotted, "f"'] &
\end{tikzcd} \tag{111}$$

Proof. First, let $|f_c| : C \to \mathcal{V}$ be the composite of $f_c : C \to \mathcal{U}$ with the canonical projection $\mathcal{U} \to \mathcal{V}$, thus classifying the underlying family $c^*F \to C$. Next, let $f_0 : X \to \mathcal{V}$ classify the underlying family $F \to X$. We may assume that $f_0 \circ c = |f_c|$ by realignment for families, Proposition 61.

$$\begin{tikzcd}
c^*F \arrow[rr] \arrow[dr] \arrow[dd, two heads] & & \dot{\mathcal{U}} \arrow[r] \arrow[dd, two heads] & \dot{\mathcal{V}} \arrow[dd] \\
& F \arrow[urr, dotted] \arrow[dd, two heads] & & \\
C \arrow[rr, "f_c"] \arrow[dr, tail, "c"'] & & \mathcal{U} \arrow[r] & \mathcal{V} \\
& X \arrow[urr, dotted, "f_0"'] & &
\end{tikzcd} \tag{112}$$

Since $F \twoheadrightarrow X$ is a fibration, there is a lift $f_1 : X \to \mathcal{U}$ of f_0 classifying the fibration structure. We thus have the following commutative diagram in the base of (112).

$$\begin{tikzcd} C \arrow[r, "f_c"'] \arrow[d, tail, "c"'] \arrow[rr, bend left, "|f_c|"] & \mathcal{U} \arrow[r] & \mathcal{V} \arrow[d, equal] \\ X \arrow[r, "f_1"] \arrow[rr, bend right, "f_0"'] & \mathcal{U} \arrow[r] & \mathcal{V} \end{tikzcd} \tag{113}$$

Now pull $\mathcal{U} \to \mathcal{V}$ back against itself and rearrange the previous data to give (the solid part of) the following, which also commutes.

$$\begin{tikzcd} C \arrow[r, "\langle f_1 c, f_c \rangle"'] \arrow[d, tail, "c"'] \arrow[rr, bend left, "f_c"] & \mathcal{U} \times_{\mathcal{V}} \mathcal{U} \arrow[r, "\pi_2"'] \arrow[d, two heads, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{U} \arrow[d] \\ X \arrow[r, "f_1"'] \arrow[ur, dotted, "f_2"] \arrow[rr, bend right, "f_0"'] & \mathcal{U} \arrow[r] & \mathcal{V} \end{tikzcd} \tag{114}$$

Since $\mathcal{U} = \mathsf{Fib}(\dot{\mathcal{V}}) \to \mathcal{V}$ is a weak proposition by Lemma 68 and (C8), the projection $\pi_1 : \mathcal{U} \times_{\mathcal{V}} \mathcal{U} \twoheadrightarrow \mathcal{U}$ is a trivial fibration, so there is a diagonal filler $f_2 : X \to \mathcal{U} \times_{\mathcal{V}} \mathcal{U}$ as indicated. Taking $f := \pi_2 \circ f_2 : X \to \mathcal{U} \times_{\mathcal{V}} \mathcal{U} \to \mathcal{U}$ gives another classifying map for the fibration structure on $F \to X$, for which $f \circ c = f_c$ as required. $\square$

7 The equivalence extension property

The equivalence extension property (EEP) is closely related to the *univalence* of the universal fibration $\dot{\mathcal{U}} \to \mathcal{U}$ constructed in section 6.3 (see [**?**]). We shall use it in section 8 to show that the base object $\mathcal{U}$ is fibrant, which implies the fibration extension property. Our proof of the EEP is a reformulation of a type-theoretic argument due to Coquand [CCHM16], which in turn is a modification of the original argument of Voevodsky [**?**]. See [Sat17] for another reformulation.

We first recall some basic facts and make some simple observations. For any map $f : X \to Y$, recall the *pathspace factorization* $f = t \circ s$ indicated below.

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "s"] \arrow[dd, bend right=50, "="'] \arrow[dr, phantom, very near start, "\lrcorner"] & Y \arrow[d, "r"'] \arrow[dr, "="] & \\
P_f \arrow[r, "p_0^* f"] \arrow[d, "p_f"] \arrow[rr, dotted, bend right, "t"'] \arrow[dr, phantom, very near start, "\lrcorner"] & Y^{\mathrm{I}} \arrow[r, "p_1"] \arrow[d, "p_0"'] & Y \\
X \arrow[r, "f"'] & Y &
\end{tikzcd}
\tag{115}
$$

Here p_0, p_1 are the evaluations $Y^{\delta_0}, Y^{\delta_1}$ at the endpoints $\delta_0, \delta_1 : 1 \to \mathrm{I}$, and let $r := Y^!$ for $! : \mathrm{I} \to 1$, so that $p_0 r = p_1 r = 1_Y$. Then let $p_f := f^* p_0 : P_f \to Y$, the pullback of p_0 along f, and $s := f^* r : X \to P_f$ (over X). Finally, let $t := p_1 \circ p_0^* f : P_f \to Y$ be the indicated horizontal composite.

We make the following well-known observations.

1. If $f : X \to Y$ is over a base Z, then the factorization $t \circ s : X \to P_f \to Y$ is stable under pullback along any map $g : Z' \to Z$, in the sense that $g^* P_f = P_{g^* f} : g^* X \to g^* Y$, and similarly for $g^* s$ and $g^* t$. Note that in this case we form the pathspace Y^{I} as an exponential in the slice category over Z.

2. The retraction $p_0 \circ r = 1_Y$ pulls back along f to a retraction $p_f \circ s = 1_X$.

3. If Y is fibrant (either as an object, or over a base $Y \to Z$), then $p_0 : Y^{\mathrm{I}} \to Y$ is a trivial fibration (as is p_1). In that case, its pullback $p_f : P_f \to X$ is also a trivial fibration.

4. If X and Y are both fibrant, then $t = p_1 \circ p_0^* f : P_f \to Y$ is a fibration. This can be seen by factoring the maps $p_0, p_1 : Y^{\mathrm{I}} \rightrightarrows Y$ through the product projections as $\pi_0 \circ p, \pi_1 \circ p : Y^{\mathrm{I}} \to Y \times Y \rightrightarrows Y$, with $p = (p_0, p_1)$, and then interpolating the pullback along the map $(f, 1_X) : X \times Y \to$

$Y \times Y$ into (115) as indicated below.

$$
\begin{tikzcd}
 & X \arrow[r, "f"] \arrow[d, "s"] \arrow[dr, phantom, very near start, "\lrcorner"] & Y \arrow[d, "r"'] & \\
 & P_f \arrow[r, "p_0^* f"] \arrow[d, "f^* p"] \arrow[dl, "t"'] \arrow[dr, phantom, very near start, "\lrcorner"] & Y^{\mathrm{I}} \arrow[d, "p"'] \arrow[dr, "p_1"] & \\
Y & X \times Y \arrow[l, "\pi_1'"] \arrow[r, "{(f,1_Y)}"'] \arrow[d, "\pi_0'"'] \arrow[dr, phantom, very near start, "\lrcorner"] & Y \times Y \arrow[r, "\pi_1"'] \arrow[d, "\pi_0"] & Y \\
 & X \arrow[r, "f"'] & Y &
\end{tikzcd}
\tag{116}
$$

The second factor $t = p_1 \circ p_0^* f : P_f \to Y$ now appears also as $\pi_1 \circ (f, 1_Y) \circ f^* p$, which is the pullback $f^* p : P_f \to X \times Y$ followed by the second projection $\pi_1' : X \times Y \to Y$ (which is not a pullback). But if Y is fibrant, then $p : Y^{\mathrm{I}} \to Y \times Y$ is a fibration, and then so is $f^* p$. And if X is fibrant, then the projection $\pi_1' : X \times Y \to Y$ is a fibration. Thus in this case, $t = \pi_1' \circ f^* p : P_f \to Y$ is a fibration, as claimed.

5. Summarizing (1)-(4), for any map $f : X \to Y$, we have a stable factorization $f = t \circ s : X \to P_f \to Y$, in which s has a retraction p_f, which is a trivial fibration when Y is fibrant, and t is a fibration when both X and Y are fibrant.

$$
\begin{tikzcd}
X \arrow[r, "s"] \arrow[dr, "f"'] & P_f \arrow[l, bend right, dotted, "p_f"'] \arrow[d, "t"] \\
 & Y
\end{tikzcd}
\tag{117}
$$

Note that the retraction $p_f : P_f \to X$ is not over Y.

The following simple fact concerning just the cofibration weak factorization system will also be needed.

Lemma 70. *Let $p : E \twoheadrightarrow B$ be a trivial fibration and $c : C \rightarrowtail B$ a cofibration. Then the unit $\eta : E \to c_* c^* E$ of the base change $c^* \dashv c_*$ along c is a trivial fibration.*

Proof. The unit map $\eta : E \to c_* c^* E$ is the pullback-hom $c \Rightarrow p$, as is easily checked. By lemma 13, for any map $a : A \to Z$ we have the equivalence of diagonal filling conditions,

$$a \pitchfork c \Rightarrow p \quad \text{iff} \quad a \otimes c \pitchfork p.$$

But since $c : C \rightarrowtail B$ is a cofibration, $a \otimes c$ is also a cofibration if $a : A \rightarrowtail Z$ is one, by axiom (C6), which says that cofibrations are closed under pushout-products. So $a \otimes c \pitchfork p$ indeed holds, since p is a trivial fibration. $\square$

Proposition 71 (EEP)**.** *Weak equivalences extended along cofibrations in the following sense: given a cofibration $c : C' \rightarrowtail C$ and fibrations $A' \twoheadrightarrow C'$ and $B \twoheadrightarrow C$, and a weak equivalence $w' : A' \simeq c^* B$ over C',*

$$\begin{tikzcd}
A' \arrow[rr, dotted] \arrow[dr, "w'", "\sim"'] \arrow[dd, two heads] & & A \arrow[dr, dotted, "w", "\sim"'] \arrow[dd, dotted, two heads] & \\
& c^*B \arrow[rr] \arrow[dl] & & B \arrow[dl] \\
C' \arrow[rr, tail, "c"'] & & C &
\end{tikzcd} \tag{118}$$

there is a fibration $A \twoheadrightarrow C$ and a weak equivalence $w : A \simeq B$ over C that pulls back along $c : C' \rightarrowtail C$ to w', so $c^* w = w'$.

Proof. Call the given fibration $q : B \to C$ and let $b := q^* c : c^* B \to B$ be the indicated pullback, which is thus also a cofibration. Let $w := b_* w' : A \to B$ be the pushforward of w' along b. Composing with q gives the map $p := q \circ w : A \to C$. Since b is monic, we indeed have $b^* w = w'$, thus filling in all the dotted arrows in (118). Note moreover that $c^* w = b^* w = w'$, as required. It remains to show that $p : A \to C$ is a fibration and $w : A \to B$ is a weak equivalence.

$$\begin{tikzcd}
A' \arrow[rr, tail] \arrow[dr, "w'", "\sim"'] \arrow[dd, two heads] & & A \arrow[r, "s"] \arrow[dr, "w"] \arrow[dd, "p"] & P_w \arrow[l, dotted, bend right, "p_w"'] \arrow[d, "t"] \\
& c^*B \arrow[rr, tail, "b"'] \arrow[dl, two heads] & & B \arrow[dl, two heads, "q"] \\
C' \arrow[rr, tail, "c"'] & & C &
\end{tikzcd} \tag{119}$$

Let us rename $p' := c^*p : A' \to C'$ and $B' := c^*B$ and $q' := c^*q$. Now let $w = t \circ s$ be the pathspace factorization (115) of w, as a map over C. Since $q : B \to C$ is a fibration, by the foregoing remarks on pathspace factorizations, we know that $s : A \to P_w$ has a retraction $p_w : P_w \to A$ which is a trivial fibration. The retraction p_w is a map over C.

The pathspace factorization $w = t \circ s : A \to P_w \to B$ is stable under pullback along c, providing a pathspace factorization $w' = t' \circ s' : A' \to P_{w'} \to B'$ over C'. Since both p' and q' are fibrations, the retraction $p_{w'} : P_{w'} \to A'$ is a trivial fibration, and now $t' : P_{w'} \to B'$ is a fibration.

$$\begin{tikzcd}
A' \arrow[r, "s'"] \arrow[dr, "w'", "\sim"'] \arrow[dd, two heads, "p'"'] & P_{w'} \arrow[l, bend right, dotted, two heads, "p_{w'}"'] \arrow[d, two heads, "t'"] \\
 & B' \arrow[dl, two heads, "q'"] \\
C' &
\end{tikzcd} \tag{120}$$

Thus the composite $q' \circ t' : P_{w'} \to B' \to C'$ is a fibration and therefore, by the retraction over C' with the trivial fibration $p_{w'}$, we have that $s' : A' \to P_{w'}$ is a weak equivalence, by 3-for-2 for weak equivalences between fibrations. For the same reason, t' is then a weak equivalence, and therefore a trivial fibration.

Since $t' = c^*t = b^*t$ is a trivial fibration, its pushforward b_*b^*t along b is also one by by Corollary 12. Moreover, $b_*b^*t : b_*b^*P_w \to B$ admits a unit

$\eta : P_w \to b_* b^* P_w$ (over B).

$$
\begin{tikzcd}
A' \arrow[rr, tail] \arrow[dr, "w'", "\sim"'] \arrow[dd, two heads] & & A \arrow[r, "s"] \arrow[dr, "w"] \arrow[dd, "p"' near end] & P_w \arrow[l, bend right, dotted, "p_w"'] \arrow[r, "\eta"] \arrow[d, "t"] & b_* b^* P_w \arrow[dl, "b_* b^* t"] \\
& B' \arrow[rr, tail, "b"' near start] \arrow[dl] & & B \arrow[dl, "q"] & \\
C' \arrow[rr, tail, "c"'] & & C & &
\end{tikzcd}
\tag{121}
$$

We now *claim* that $\eta : P_w \to b_* b^* P_w$ is a trivial fibration. Given that, the composite $t = b_* b^* t \circ \eta$ is also a trivial fibration, whence $q \circ t : P_w \to C$ is a fibration, and so its retract $p : A \to C$ is a fibration. Moreover, since s is a section of the trivial fibration $p_w : P_w \to A$ between fibrations, as before it is also a weak equivalence. Thus $w = t \circ s$ is a weak equivalence, and we are finished.

To prove the remaining claim that $\eta : P_w \to b_* b^* P_w$ is a trivial fibration, we shall use lemma 70. But it does not apply directly since $t : P_w \to B$ is not yet known to be a trivial fibration. Instead, we show that η is a pullback of the corresponding unit at the trivial fibration $p_1 : B^{\mathrm{I}} \to B$.

Consider the following cube (viewed with $b : B' \to B$ at the front).

$$
\begin{tikzcd}
P_{w'} \arrow[rr, tail, "\overline{a}"] \arrow[dr, "(p_0')^* w'"] \arrow[dd, two heads, "p_{w'}"'] & & P_w \arrow[dr, "p_0^* w"] \arrow[dd, "p_w" near start] & \\
& B'^{\mathrm{I}} \arrow[rr, tail, "\overline{b}" near end] \arrow[dd, two heads, "p_0'" near end] & & B^{\mathrm{I}} \arrow[dd, two heads, "p_0"] \\
A' \arrow[rr, tail, "a" near end] \arrow[dr, "w'"'] & & A \arrow[dr, "w"] & \\
& B' \arrow[rr, tail, "b"'] & & B
\end{tikzcd}
\tag{122}
$$

The right hand face is a pullback by definition, and the remainder results from pulling the right face back along b, by the stability of the pathspace factorization (117). Thus all faces are pullbacks. The base is also a pushforward, $b_* w' = w$, again by definition. Thus the top face is also a pushforward, $\overline{b}_*((p_0')^* w') = p_0^* w$. Indeed, since the front face is a pullback, the Beck-Chevalley condition applies, and so we have $\overline{b}_*(p_0')^*(w') = p_0^*\, b_*(w') = p_0^* w$.

Now consider the following, in which the top square remains the same as in (122), but p_0 has been relaced by $p_1 : B^{\mathrm{I}} \to B$, so the composite at right is by definition $t = p_1 \circ p_0^* w$.

$$
\begin{tikzcd}
P_{w'} \arrow[rr, tail, "\overline{a}"] \arrow[dr, "(p_0')^* w'"] \arrow[ddr, two heads, "t'"'] & & P_w \arrow[dr, "p_0^* w"] \arrow[ddr, two heads, "t"' near start] & \\
& B'^{\mathrm{I}} \arrow[rr, tail, "\overline{b}"] \arrow[d, two heads, "p_1'"] & & B^{\mathrm{I}} \arrow[d, two heads, "p_1"] \\
& B' \arrow[rr, tail, "b"'] & & B
\end{tikzcd}
\tag{123}
$$

The horizontal direction is still pullback along b; let us rename $p_0^* w =: u$ so that $(p_0')^* w' = b^* u$ and $t' = b^* t$ and $p_1' = b^* p_1$ to make this clear. We then add the pushforward along b on the right, in order to obtain the two units η.

$$
\begin{tikzcd}
b^* P_w \arrow[rr, tail, "\overline{a}"] \arrow[dr, "b^* u"] \arrow[ddr, two heads, "b^* t"'] & & P_w \arrow[rr, "\eta_t"] \arrow[dr, "u"] \arrow[ddr, two heads, "t"' near start] & & b_* b^* P_w \arrow[dr, "b_* b^* u"] \arrow[ddr, "b_* b^* t"' near start] & \\
& b^* B^{\mathrm{I}} \arrow[rr, tail, "\overline{b}"] \arrow[d, two heads, "b^* p_1"] & & B^{\mathrm{I}} \arrow[rr, "\eta_{p_1}"] \arrow[d, two heads, "p_1"] & & b_* b^* B^{\mathrm{I}} \arrow[d, "b_* b^* p_1"] \\
& B' \arrow[rr, tail, "b"'] & & B \arrow[rr, "="'] & & B
\end{tikzcd}
\tag{124}
$$

By the usual calculation of pushforwards in slice categories, $\overline{b}_* \cong \eta_{p_1}^* \circ b_*$, and so for $b^* u$ we have $\overline{b}_* b^* u = \eta_{p_1}^* b_* b^* u$. But as we just determined in (122) the top left square is already a pushforward, and therefore $u = \eta_{p_1}^* b_* b^* u$, so the top right naturality square is a pullback.

To finish the proof as planned, $p_1 : B^{\mathrm{I}} \to B$ is a trivial fibration because $q : B \to C$ is a fibration, and $b : B' \rightarrowtail B$ is a cofibration because it is a pullback of $c : C' \rightarrowtail C$. Thus by lemma 70, we have that $\eta_{p_1} : B^{\mathrm{I}} \to b_* b^* B^{\mathrm{I}}$ is a trivial fibration, and so its pullback $\eta_t : P_w \to b_* b^* P_w$ is a trivial fibration, as claimed. □

Remark 72. Note that $p : A \to C$ is small if $q : B \to C$ is small.

8 The fibration extension property

In the presence of a universal fibration $\dot{\mathcal{U}} \twoheadrightarrow \mathcal{U}$, as given by Proposition 65, the fibration extension property (Definition 47) is closely related to the statement that the base object $\mathcal{U}$ is fibrant. For Kan simplicial sets, Voevodsky proved the latter directly using minimal fibrations [?]. Shulman [?] gives a proof from univalence (in the form of the equivalence extension property as stated in section 7) in a more general setting, but it uses the 3-for-2 property for weak equivalences, which is what we are trying to prove. In [CCHM16], Coquand uses the equivalence extension property to prove that $\mathcal{U}$ is fibrant, without assuming 3-for-2 for weak equivalences, by a neat argument using a reduction of general box-filling to a condition called "Kan-composition". We shall prove that $\mathcal{U}$ is fibrant using the equivalence extension property via a different argument than that in [CCHM16], avoiding the reduction of filling to composition, which we therefore do not require.

Returning to the relation between the fibration extension property and the condition that the base object $\mathcal{U}$ is fibrant, it is easily seen that the latter implies the former. Indeed, let $t : X \rightarrowtail X'$ be a trivial cofibration and $Y \twoheadrightarrow X$ a fibration. To extend Y along t, take a classifying map $y : X \to \mathcal{U}$, so that $Y \cong y^* \dot{\mathcal{U}}$ over X. If $\mathcal{U}$ is fibrant then we can extend y along $t : X \rightarrowtail X'$ to get $y' : X' \to \mathcal{U}$ with $y = y' \circ t$. The pullback $Y' = (y')^* \dot{\mathcal{U}} \twoheadrightarrow X'$ is then a (small) fibration such that $t^* Y' \cong t^*(y')^* \dot{\mathcal{U}} \cong y^* \dot{\mathcal{U}} \cong Y$ over X.

$$\begin{tikzcd}
Y \arrow[rr] \arrow[dr] \arrow[dd, two heads] & & \dot{\mathcal{U}} \arrow[dd, two heads] \\
& Y' \arrow[ur, dotted] \arrow[dd, two heads] & \\
X \arrow[rr, "y"'] \arrow[dr, tail, "t"'] & & \mathcal{U} \\
& X' \arrow[ur, dotted, "y'"'] &
\end{tikzcd}$$

Thus for the record, we have:

Proposition 73. *If the base object $\mathcal{U}$ of the universal fibration $\dot{\mathcal{U}} \twoheadrightarrow \mathcal{U}$ is fibrant, then the fibration weak factorization system has the fibration extension property.*

Conversely, given the Realignment Lemma 69, the FEP also implies the fibrancy of $\mathcal{U}$:

Corollary 74. *The fibration extension property implies that the base $\mathcal{U}$ of the universal fibration $\dot{\mathcal{U}} \twoheadrightarrow \mathcal{U}$ is fibrant: given any $y : X \to \mathcal{U}$ and trivial cofibration $t : X \rightarrowtail X'$, there is a map $y' : X' \to \mathcal{U}$ with $y' \circ t = y$.*

Proof. Take the pullback of $\dot{\mathcal{U}} \twoheadrightarrow \mathcal{U}$ along $y : X \to \mathcal{U}$ to get a (small) a fibration $Y \twoheadrightarrow X$, which extends along the (trivial) cofibration $t : X \rightarrowtail X'$ by the FEP, to a (small) fibration $Y' \twoheadrightarrow X'$ with $Y \cong t^*Y'$ over X. By realignment there is a classifying map $y' : X' \to \mathcal{U}$ for Y' with $y' \circ t = y$. $\square$

Now let us use the EEP, Proposition 71, to show that $\mathcal{U}$ is fibrant. We shall need the following two lemmas.

Lemma 75. *The base change $\mathrm{I}^* : \mathsf{cSet} \to \mathsf{cSet}/\mathrm{I}$ takes the universal fibration $\dot{\mathcal{U}} \to \mathcal{U}$ in cSet to a universal fibration in the slice category cSet/I.*

Proof. [*fill in*] $\square$

Proposition 76. *The base object $\mathcal{U}$ of the universal fibration $\dot{\mathcal{U}} \twoheadrightarrow \mathcal{U}$ in cSet constructed in section 6.1 is fibrant.*

Proof. Moving $\mathcal{U}$ to the slice category cSet/I by (silently) applying the base change $\mathrm{I}^* : \mathsf{cSet} \to \mathsf{cSet}/\mathrm{I}$, we need to solve the following filling problem there, for $\delta : 1 \to \mathrm{I}$ the generic point and $c : C \rightarrowtail Z$ an arbitrary cofibration (over I),

$$\begin{tikzcd} C \arrow[r, "\tilde{a}"] \arrow[d, tail, "c"'] & \mathcal{U}^{\mathrm{I}} \arrow[d, "\mathcal{U}^{\delta}"] \\ Z \arrow[r, "b"'] \arrow[ur, dotted] & \mathcal{U} \end{tikzcd} \tag{125}$$

This shows that the map $\delta \Rightarrow \mathcal{U} = \mathcal{U}^{\delta} : \mathcal{U}^{\mathrm{I}} \longrightarrow \mathcal{U}$ over I is a trivial fibration in cSet/I, and so $\mathcal{U}$ is fibrant in cSet, by Definition 17. The remainder of the proof occurs in the Cartesian cubical presheaf category cSet/I, for which we have all the same results of sections 1 through 7 as for cSet. In particular, by Lemma 75 there is a universal fibration $\dot{\mathcal{U}} \to \mathcal{U}$ resulting from the base change.

Transposing $\tilde{a}$ to $a : C \times \mathrm{I} \to \mathcal{U}$ and taking pullbacks of $\dot{\mathcal{U}} \twoheadrightarrow \mathcal{U}$ along a and b to get corresponding fibrations $A \twoheadrightarrow C \times \mathrm{I}$ and $B \twoheadrightarrow Z$, we have the following equivalent condition. Letting

$$C_\delta : C \cong C \times 1 \to C \times \mathrm{I}$$

be the evident inclusion of the "δ-section" of the cylinder, let $A_\delta = (C_\delta)^* A \twoheadrightarrow C$ be the "section of A over C_δ". We then have $c^* B \cong A_\delta$ over C by the outer square of (125).

$$\begin{tikzcd}
 & A_\delta \arrow[rr] \arrow[d, two heads] \arrow[ddl] & & A \arrow[d, two heads] \\
 & C \arrow[rr, "C_\delta"'] \arrow[ddl, "c"] & & C \times \mathrm{I} \arrow[ddl] \\
B \arrow[d, two heads] & & & \\
Z \arrow[rr, "Z_\delta"'] & & Z \times \mathrm{I} &
\end{tikzcd}$$

The diagonal filler sought in (125) corresponds, again by transposition and pullback of $\dot{\mathcal{U}} \twoheadrightarrow \mathcal{U}$, to a fibration $D \twoheadrightarrow Z \times \mathrm{I}$ with $(c \times \mathrm{I})^* D \cong A$ over $C \times \mathrm{I}$ and $(Z_\delta)^* D \cong B$ over Z, as indicated below.

$$\begin{tikzcd}
 & A_\delta \arrow[rr] \arrow[d, two heads] \arrow[ddl] & & A \arrow[d, two heads] \arrow[ddl, dotted] \\
 & C \arrow[rr, "C_\delta"'] \arrow[ddl, "c"] & & C \times \mathrm{I} \arrow[ddl, "c\times\mathrm{I}"] \\
B \arrow[d, two heads] \arrow[rr, dotted] & & D \arrow[d, dotted, two heads] & \\
Z \arrow[rr, "Z_\delta"'] & & Z \times \mathrm{I} &
\end{tikzcd} \tag{126}$$

We can construct such a $D \twoheadrightarrow Z \times \mathrm{I}$ using the equivalence extension property, as follows. First apply the functor $(-) \times \mathrm{I}$ to the left (pullback) face of the cube in (126) to get the following, with a new pullback square on the right, involving the indicated fibrations.

$$
\begin{tikzcd}
 & A_\delta \arrow[rr] \arrow[d, two heads] \arrow[ddl] & & A \arrow[d, two heads] \arrow[ddl, dotted] & & A_\delta \times \mathrm{I} \arrow[dll, two heads] \arrow[ddl] \\
 & C \arrow[rr, "C_\delta"'] \arrow[ddl, "c"] & & C \times \mathrm{I} \arrow[ddl, "c\times\mathrm{I}"] & & \\
B \arrow[rr, dotted] \arrow[d, two heads] & & D \arrow[d, dotted, two heads] & & B \times \mathrm{I} \arrow[dll, two heads] & \\
Z \arrow[rr, "Z_\delta"'] & & Z \times \mathrm{I} & & &
\end{tikzcd} \tag{127}
$$

We next *claim* that there is a weak equivalence $e : A \simeq A_\delta \times \mathrm{I}$ over $C \times \mathrm{I}$, from which follow by the EEP:

(i) a fibration $D \twoheadrightarrow Z \times \mathrm{I}$ with $(c \times \mathrm{I})^* D \cong A$ over $C \times \mathrm{I}$, and

(ii) a weak equivalence $f : D \simeq B \times \mathrm{I}$ over $Z \times \mathrm{I}$ with $(c \times \mathrm{I})^* f \cong e$ over $C \times \mathrm{I}$.

It then remains only to show that $B \cong (Z_\delta)^* D$ over Z to complete the proof.

To obtain e, consider the following square, in which the top map is $A_\delta \times \delta$ (after $A_\delta \cong A_\delta \times 1$) and the others are those from diagram (127).

$$
\begin{tikzcd}
A_\delta \arrow[r, tail] \arrow[d, tail] & A_\delta \times \mathrm{I} \arrow[d, two heads] \\
A \arrow[r, two heads] & C \times \mathrm{I}
\end{tikzcd} \tag{128}
$$

The square is easily seen to commute, and the maps with A_δ as domain are trivial cofibrations by Frobenius (Proposition 55), because each is the pullback of a trivial cofibration along a fibration. A simple lemma (given below as Lemma 77) provides the claimed weak equivalence $e : A \simeq A_\delta \times \mathrm{I}$ over $C \times \mathrm{I}$.

To see that $B \cong (Z_\delta)^* D$ over Z, recall from the proof of the EEP that the map $f : B \cong (Z_\delta)^* D$ is the pushforward of $e : A \simeq A_\delta \times \mathrm{I}$ along the cofibration $b_\delta \times \mathrm{I} : A_\delta \times \mathrm{I} \rightarrowtail B \times \mathrm{I}$, where we have named the evident map in (127) $b_\delta : A_\delta \rightarrowtail B$. That is, by construction $f = (b_\delta \times \mathrm{I})_* \, e$. We can then apply the Beck-Chevalley condition for the pushforward using the pullback square on the left below.

$$\begin{tikzcd} A_\delta \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & A_\delta \times \mathrm{I} \arrow[d, two heads] & A \arrow[l, tail, "e"'] \\ B \arrow[r, tail] & B \times \mathrm{I} & D \arrow[l, tail, "f"] \end{tikzcd} \tag{129}$$

The pullback of e along the top of the square is the identity on A_δ, as can be seen by pulling back e as a map over $C \times \mathrm{I}$ along $C_\delta : C \to C \times \mathrm{I}$. Thus the same is true (up to isomorophism) for the pullback of f along the bottom.

An application of the Realignment Lemma 69 along the trivial cofibration $c \otimes \delta$ completes the proof. □

Lemma 77. *Suppose the following square commutes and the indicated cofibrations are trivial.*

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, tail] & C \arrow[d, two heads] \\ B \arrow[r, two heads] & D \end{tikzcd} \tag{130}$$

Then there is a weak equivalence $e : B \simeq C$ over D (and under A).

Proof. Use the fact that any two diagonal fillers are homotopic to get a homotopy equivalence $e : B \simeq C$ filling the square. □

Applying proposition 73 now yields the following.

Corollary 78. *The fibration weak factorization system has the fibration extension property (definition 47).*

By Theorem 52, finally, we have the following.

Theorem 79. *There is a Quillen model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ on the category of cubical sets* cSet*, where:*

1. *the cofibrations $\mathcal{C}$ are any class of maps satisfying (C0)-(C8) (equivalently, the simplified axioms in the Appendix),*

2. *the fibrations $\mathcal{F}$ are the maps $f : Y \to X$ for which the canonical map*

$$(f^{\mathrm{I}} \times \mathrm{I}, \mathrm{eval}_Y) : Y^{\mathrm{I}} \times \mathrm{I} \to (X^{\mathrm{I}} \times \mathrm{I}) \times_X Y$$

lifts on the right against $\mathcal{C}$.

3. *the weak equivalences $\mathcal{W}$ are the maps $w : X \to Y$ for which the internal precomposition $K^w : K^Y \to K^X$ is bijective on connected components for every fibrant object K.*

Appendix: Axioms for cofibrations

The axioms (C0)–(C8) for the *cofibrations* $c : A \rightarrowtail B$ in cartesian cubical sets, may be restated equivalently as follows.

(C1) All isomorphisms are cofibrations.

(C2) The composite of two cofibrations is a cofibration.

(C3) Any pullback of a cofibration is a cofibration.

(C4) There is a terminal object of the form $t : 1 \rightarrowtail \Phi$ in the category of cofibrations and cartesian squares.

(C5) The diagonal map $\delta : \mathrm{I} \to \mathrm{I} \times \mathrm{I}$ is a cofibration.

(C6) The join of two cofibrant subobjects is a cofibration.

For example, we have the following.

Proposition 80. *The locally decidable subobjects satisfy the axioms.*

Proof. [*fill in*] □

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