On Hofmann-Streicher universes

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to the memory of Erik Palmgren

Abstract

We have another look at the construction by Hofmann and Streicher of a universe $(U, \mathsf{E}l)$ for the interpretation of Martin-Löf type theory in a presheaf category $[\mathbb{C}^{\mathrm{op}}, \mathsf{Set}]$. It turns out that $(U, \mathsf{E}l)$ can be described as the *categorical nerve* of the classifier $\dot{\mathsf{Set}}^{\mathsf{op}} \to \mathsf{Set}^{\mathsf{op}}$ for discrete fibrations in Cat, where the nerve functor is right adjoint to the so-called "Grothendieck construction" taking a presheaf $P : \mathbb{C}^{\mathrm{op}} \to \mathsf{Set}$ to its category of elements $\int_{\mathbb{C}} P$.

Let $\widehat{\mathbb{C}} = [\mathbb{C}^{\mathrm{op}}, \mathsf{Set}]$ be the category of presheaves on a small category $\mathbb{C}$.

1. The Hofmann-Streicher universe

In [HS97] the authors define a (type-theoretic) *universe* $(U, \mathsf{E}l)$ with $U \in \widehat{\mathbb{C}}$ and $\mathsf{E}l \in \widehat{\int_{\mathbb{C}} U}$ as follows. For $I \in \mathbb{C}$, set

$$U(I) \;=\; \mathsf{Cat}\big(\mathbb{C}/_{I}{}^{\mathrm{op}}, \mathsf{Set}\big) \tag{1}$$

$$\mathsf{E}l(I, A) \;=\; A(id_I) \tag{2}$$

with an evident associated action on morphisms, which need not concern us for the moment. A few comments are required:

1. In (1), we have taken the *underlying set of objects* of the category $\widehat{\mathbb{C}/_I} = [\mathbb{C}/_I{}^{\mathrm{op}}, \mathsf{Set}]$ (in contrast to the specification in [HS97]).
2. In (2), and throughout, the authors steadfastly adopt a "categories with families" point of view in describing a morphism $E \to U$ in $\widehat{\mathbb{C}}$ instead as an object in

$$\widehat{\int_{\mathbb{C}} U} \;\simeq\; \widehat{\mathbb{C}}/_U\,, \tag{3}$$

that is, as a presheaf on the *category of elements* $\int_{\mathbb{C}} U$, rather than specifying an arrow $E \to U$ in $\widehat{\mathbb{C}}$ with,

$$E(I) \;=\; \coprod_{A \in U(I)} \mathsf{E}l(I, A)$$

Thus the argument $(I, A) \in \int_{\mathbb{C}} U$ in (2) consists of an object $I \in \mathbb{C}$ and an element $A \in U(I)$.

3. In order to account for size issues, the authors assume a Grothendieck universe $\mathcal{U}$ in Set, the elements of which are called *small*. The category $\mathbb{C}$ is then assumed to be small, as are the values of the presheaves (unless otherwise stated).

The presheaf U, which is not small, is regarded as the Grothendieck universe $\mathcal{U}$ "lifted" from Set to $[\mathbb{C}^{\mathrm{op}}, \mathsf{Set}]$. We will analyse the construction of $(U, \mathsf{E}l)$ from a slightly different perspective in order to arrive at its basic property as a classifier for small families in $\widehat{\mathbb{C}}$.

2. An unused adjunction

For a presheaf X on $\mathbb{C}$, recall that the category of elements is the comma category,

$$\textstyle\int_{\mathbb{C}} X \;=\; \mathsf{y}_{\mathbb{C}}/_X\,,$$

where $\mathsf{y}_{\mathbb{C}} : \mathbb{C} \to [\mathbb{C}^{\mathrm{op}}, \mathsf{Set}]$ is the Yoneda embedding, which we may supress and write simply $\mathbb{C}/_X$. While the category of elements $\int_{\mathbb{C}} X$ is used in the specification of the Hofmann-Streicher universe $(U, \mathsf{E}l)$ at the point (3), the authors seem to have missed a trick which would have simplified things:

Proposition 1 ([Gro83],§28)**.** *The category of elements functor* $\int_{\mathbb{C}} : \widehat{\mathbb{C}} \longrightarrow \mathsf{Cat}$ *has a right adjoint, which we denote*

$$\nu_{\mathbb{C}} : \mathsf{Cat} \longrightarrow \widehat{\mathbb{C}}\,.$$

For a small category $\mathbb{A}$*, we call the presheaf* $\nu_{\mathbb{C}}(\mathbb{A})$ *the* $\mathbb{C}$-nerve *of* $\mathbb{A}$*.*

Proof. As suggested by the name, the adjunction $\int_{\mathbb{C}} \dashv \nu_{\mathbb{C}}$ can be seen as the familiar "realization $\dashv$ nerve" construction with respect to the covariant functor $\mathbb{C}/- : \mathbb{C} \to \mathsf{Cat}$, as indicated below.

$$\begin{tikzcd}
\widehat{\mathbb{C}} \arrow[r, "\int_{\mathbb{C}}"'] & \mathsf{Cat} \arrow[l, bend right, "\nu_{\mathbb{C}}"'] \\
\mathbb{C} \arrow[u, "\mathsf{y}"] \arrow[ur, "\mathbb{C}/_{-}"'] &
\end{tikzcd} \tag{4}$$

In detail, for $\mathbb{A} \in \mathsf{Cat}$ and $c \in \mathbb{C}$, let $\nu_{\mathbb{C}}(\mathbb{A})(c)$ be the Hom-set of functors,

$$\nu_{\mathbb{C}}(\mathbb{A})(c) = \mathsf{Cat}\big(\mathbb{C}/_c\,,\ \mathbb{A}\big)\,,$$

with contravariant action on $h : d \to c$ given by pre-composing a functor $P : \mathbb{C}/_c \to \mathbb{A}$ with the post-composition functor

$$\mathbb{C}/_h : \mathbb{C}/_d \longrightarrow \mathbb{C}/_c\,.$$

For the adjunction, observe that the slice category $\mathbb{C}/_c$ is the category of elements of the representable functor $\mathsf{y}c$,

$$\textstyle\int_{\mathbb{C}} \mathsf{y}c \;\cong\; \mathbb{C}/_c\,.$$

Thus for representables $\mathsf{y}c$, we have the required natural isomorphism

$$\widehat{\mathbb{C}}\big(\mathsf{y}c\,,\ \nu_{\mathbb{C}}(\mathbb{A})\big) \;\cong\; \nu_{\mathbb{C}}(\mathbb{A})(c) \;=\; \mathsf{Cat}\big(\mathbb{C}/_c\,,\ \mathbb{A}\big) \;\cong\; \mathsf{Cat}\big(\textstyle\int_{\mathbb{C}} \mathsf{y}c\,,\ \mathbb{A}\big)\,.$$

For arbitrary presheaves X, one uses the presentation of X as a colimit of representables over the index category $\int_{\mathbb{C}} X$, and the easy to prove fact that $\int_{\mathbb{C}}$ itself preserves colimits. Indeed, for any category $\mathbb{D}$, we have an isomorphism in Cat,

$$\varinjlim_{d \in \mathbb{D}} \mathbb{D}/_d \;\cong\; \mathbb{D}\,.$$

$\square$

When $\mathbb{C}$ is fixed, as here, we may omit the subscript from the notation $\mathsf{y}_{\mathbb{C}}$ and $\int_{\mathbb{C}}$ and $\nu_{\mathbb{C}}$. The unit and counit maps of the adjunction $\int \dashv \nu$, vis.

$$\begin{aligned} \eta &: X \longrightarrow \nu \textstyle\int X\,, \\ \epsilon &: \textstyle\int \nu\mathbb{A} \longrightarrow \mathbb{A}\,, \end{aligned}$$

are as follows. At $c \in \mathbb{C}$, for $x : \mathsf{y}c \to X$, the functor $(\eta_X)_c(x) : \mathbb{C}/_c \to \mathbb{C}/_X$ is just composition with x,

$$(\eta_X)_c(x) = \mathbb{C}/_x : \mathbb{C}/_c \longrightarrow \mathbb{C}/_X\,. \tag{5}$$

For $\mathbb{A} \in \mathsf{Cat}$, the functor $\epsilon : \int \nu\mathbb{A} \to \mathbb{A}$ takes a pair $(c \in \mathbb{C}, f : \mathbb{C}/_c \to \mathbb{A})$ to the object $f(1_c) \in \mathbb{A}$,

$$\epsilon(c, f) = f(1_c).$$

Lemma 2. *For any $f : Y \to X$, the naturality square below is a pullback.*

$$\begin{tikzcd} Y \arrow[r, "\eta_Y"] \arrow[d, "f"'] & \nu \int Y \arrow[d, "\nu \int f"] \\ X \arrow[r, "\eta_X"'] & \nu \int X. \end{tikzcd} \tag{6}$$

Proof. It suffices to prove it for the case $f : X \to 1$. Thus consider the square

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d] & \nu \int X \arrow[d] \\ 1 \arrow[r, "\eta_1"'] & \nu \int 1. \end{tikzcd} \tag{7}$$

Evaluating at $c \in \mathbb{C}$ and applying (5) then gives the following square in Set.

$$\begin{tikzcd} Xc \arrow[r, "\mathbb{C}/_-"] \arrow[d] & \mathsf{Cat}\big(\mathbb{C}/_c\,, \mathbb{C}/_X\big) \arrow[d] \\ 1c \arrow[r, "\mathbb{C}/_-"'] & \mathsf{Cat}\big(\mathbb{C}/_c\,, \mathbb{C}/_1\big) \end{tikzcd} \tag{8}$$

The image of $* \in 1c$ along the bottom is the forgetful functor $U_c : \mathbb{C}/_c \to \mathbb{C}$, and its fiber under the map on the right is therefore the set of functors $F : \mathbb{C}/_c \to \mathbb{C}/_X$ such that $U_X \circ F = U_c$, where $U_X : \mathbb{C}/_X \to \mathbb{C}$ is also a forgetful functor. But any such F is easily seen to be uniquely of the form $\mathbb{C}/_x$ for $x = F(1_c) : \mathsf{y}c \to X$. $\square$

3. Classifying families

For the terminal presheaf $1 \in \widehat{\mathbb{C}}$, we have $\int 1 \cong \mathbb{C}$, so for every $X \in \widehat{\mathbb{C}}$ there is a canonical projection $\int X \to \mathbb{C}$, which is easily seen to be a discrete fibration. It follows that for any map $Y \to X$ of presheaves, the associated map $\int Y \to \int X$ is also a discrete fibration. Ignoring size issues for the moment, recall that discrete fibrations in Cat are classified by the forgetful functor $\dot{\mathsf{Set}}^{\mathrm{op}} \to \mathsf{Set}^{\mathrm{op}}$ from (the opposites of) the category of pointed sets to that of sets (cf. [Web07]). For every presheaf $X \in \widehat{\mathbb{C}}$, we therefore have a pullback diagram in Cat,

$$\begin{tikzcd} \int X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathsf{Set}}^{\mathrm{op}} \arrow[d] \\ \mathbb{C} \arrow[r, "X"'] & \mathsf{Set}^{\mathrm{op}}. \end{tikzcd} \tag{9}$$

Using $\int 1 \cong \mathbb{C}$ and transposing by the adjunction $\int \dashv \nu$ then gives a commutative square in $\widehat{\mathbb{C}}$,

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \nu\dot{\mathsf{Set}}^{\mathrm{op}} \arrow[d] \\ 1 \arrow[r, "\tilde{X}"'] & \nu\mathsf{Set}^{\mathrm{op}}. \end{tikzcd} \tag{10}$$

Lemma 3. *The square (10) is a pullback in $\widehat{\mathbb{C}}$. More generally, for any map $Y \to X$ in $\widehat{\mathbb{C}}$, there is a pullback square*

$$\begin{tikzcd} Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \nu\dot{\mathsf{Set}}^{\mathrm{op}} \arrow[d] \\ X \arrow[r] & \nu\mathsf{Set}^{\mathrm{op}}. \end{tikzcd} \tag{11}$$

Proof. Apply the right adjoint ν to the pullback square (9) and paste the naturality square (6) from Lemma 2 on the left, to obtain the transposed square (11) as a pasting of two pullbacks. □

Let us write $\dot{\mathcal{V}} \to \mathcal{V}$ for the vertical map on the right in (11), that is, let

$$\begin{aligned} \dot{\mathcal{V}} &= \nu\dot{\mathsf{Set}}^{\mathrm{op}} \\ \mathcal{V} &= \nu\mathsf{Set}^{\mathrm{op}}. \end{aligned} \tag{12}$$

We can then summarize our results so far as follows.

Proposition 4. *The nerve $\dot{\mathcal{V}} \to \mathcal{V}$ of the classifier for discrete fibrations $\dot{\mathsf{Set}}^{\mathrm{op}} \to \mathsf{Set}^{\mathrm{op}}$, as defined in (12), classifies natural transformations $Y \to X$ in $\widehat{\mathbb{C}}$, in the sense that there is always a pullback square,*

$$\begin{tikzcd} Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathcal{V}} \arrow[d] \\ X \arrow[r, "\tilde{Y}"'] & \mathcal{V}. \end{tikzcd} \tag{13}$$

The classifying map $\tilde{Y} : X \to \mathcal{V}$ is determined by the adjunction $\int \dashv \nu$ as the transpose of the classifying map of the discrete fibration $\int Y \to \int X$.

For a given natural transformation $Y \to X$, the classifying map $\tilde{Y} : X \to \mathcal{V}$ is not in general unique. Nonetheless, we can make use of the construction of $\dot{\mathcal{V}} \to \mathcal{V}$ as the nerve of the discrete fibration classifier $\dot{\mathsf{Set}}^{\mathrm{op}} \to \mathsf{Set}^{\mathrm{op}}$, for which classifying functors $\mathbb{C} \to \mathsf{Set}^{\mathrm{op}}$ are unique up to natural isomorphism, to infer the following proposition, which plays a role in [**?**, **?**] and elsewhere.

Proposition 5 (Realignment)**.** *Given a monomorphism $c : C \rightarrowtail X$ and a family $Y \to X$, let $y_c : C \to \mathcal{V}$ classify the pullback $c^*Y \to C$. Then there is a classifying map $y : X \to \mathcal{V}$ for $Y \to X$ with $y \circ c = y_c$.*

$$\begin{tikzcd}
c^*Y \arrow[rr] \arrow[dr] \arrow[dd] & & \dot{\mathcal{V}} \arrow[dd] \\
& Y \arrow[ur, dotted] \arrow[dd] & \\
C \arrow[rr, "y_c"] \arrow[dr, tail, "c"'] & & \mathcal{V} \\
& X \arrow[ur, dotted, "y"'] &
\end{tikzcd} \tag{14}$$

Proof. Transposing the realignment problem (14) for presheaves across the adjunction $\int \dashv \nu$ results in the following realignment problem for discrete fibrations.

$$\begin{tikzcd}
\int c^*Y \arrow[rr] \arrow[dr] \arrow[dd] & & \dot{\mathsf{Set}}^{\mathrm{op}} \arrow[dd] \\
& \int Y \arrow[ur, dotted] \arrow[dd] & \\
\int C \arrow[rr, "\widetilde{y_c}"] \arrow[dr, tail, "\int c"'] & & \mathsf{Set}^{\mathrm{op}} \\
& \int X \arrow[ur, dotted, "\tilde{y}"'] &
\end{tikzcd} \tag{15}$$

The category of elements functor $\int$ is easily seen to preserve pullbacks, hence monos; thus let us consider the general case of a functor $C : \mathbb{C} \rightarrowtail \mathbb{D}$ which is monic in Cat, a pullback of discrete fibrations as on the left below, and a presheaf $E : \mathbb{C} \to \mathsf{Set}^{\mathrm{op}}$ with $\int E \cong \mathbb{E}$ over $\mathbb{C}$.

$$\begin{tikzcd}
\mathbb{E} \arrow[rr] \arrow[dr] \arrow[dd] & & \dot{\mathsf{Set}}^{\mathrm{op}} \arrow[dd] \\
& \mathbb{F} \arrow[ur, dotted] \arrow[dd] & \\
\mathbb{C} \arrow[rr, "E"] \arrow[dr, tail, "C"'] & & \mathsf{Set}^{\mathrm{op}} \\
& \mathbb{D} \arrow[ur, dotted, "F"'] &
\end{tikzcd} \tag{16}$$

We seek $F : \mathbb{D} \to \mathsf{Set}^{\mathrm{op}}$ with $\int F \cong \mathbb{F}$ over $\mathbb{D}$ and $F \circ C = E$. Let $F_0 : \mathbb{D} \to \mathsf{Set}^{\mathrm{op}}$ with $\int F_0 \cong \mathbb{F}$ over $\mathbb{D}$. Since $F_0 \circ C$ and E both classify $\mathbb{E}$, there is a natural iso $e : F_0 \circ C \cong E$. Consider the following diagram

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "e"] \arrow[d, tail, "C"'] & (\mathsf{Set}^{\cong})^{\mathrm{op}} \arrow[r, "p_2"'] \arrow[d, "p_1"] & \mathsf{Set}^{\mathrm{op}} \\
\mathbb{D} \arrow[r, "F_0"'] \arrow[ur, dotted, "f"] & \mathsf{Set}^{\mathrm{op}} &
\end{tikzcd} \tag{17}$$

where $\mathsf{Set}^{\cong}$ is the category of isos in Set, with p_1, p_2 the (opposites of the) domain and codomain projections. There is a well-known weak factorization system on Cat (part of the "canonical model structure") with injective-on-objects functors on the left and isofibrations on the right. Thus there is a diagonal filler f as indicated. The functor $F := p_2 f : \mathbb{D} \to \mathsf{Set}^{\mathrm{op}}$ is then the one we seek. $\square$

Of course, as defined in (12), the classifier $\dot{\mathcal{V}} \to \mathcal{V}$ cannot be a map in $\widehat{\mathbb{C}}$, for reasons of size; we now address this.

4. Small maps

Let α be a cardinal number, and call the sets that are strictly smaller than it *α-small*. Let $\mathsf{Set}_\alpha \hookrightarrow \mathsf{Set}$ be the full subcategory of α-small sets. Call a presheaf $X : \mathbb{C}^{\mathrm{op}} \to \mathsf{Set}$ α-small if all of its values are α-small sets, and thus if, and only if, it factors through $\mathsf{Set}_\alpha \hookrightarrow \mathsf{Set}$. Call a map $f : Y \to X$ of presheaves α-small if all of the fibers $f_c^{-1}\{x\} \subseteq Yc$ are α-small sets (for all $c \in \mathbb{C}$ and $x \in Xc$). The latter condition is of course equivalent to saying that, in the pullback square over the element $x : \mathsf{y}c \to X$,

$$\begin{tikzcd}
Y_x \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "f"] \\
\mathsf{y}c \arrow[r, "x"'] & X,
\end{tikzcd} \tag{18}$$

the presheaf Y_x is α-small.

Now let us restrict the specification (12) of $\dot{\mathcal{V}} \to \mathcal{V}$ to the α-small sets:

$$\begin{aligned} \dot{\mathcal{V}}_\alpha &= \nu \dot{\mathsf{Set}}_\alpha^{\mathsf{op}} \\ \mathcal{V}_\alpha &= \nu \mathsf{Set}_\alpha^{\mathsf{op}}. \end{aligned} \tag{19}$$

Then the evident forgetful map $\dot{\mathcal{V}}_\alpha \to \mathcal{V}_\alpha$ *is* a map in the category $\widehat{\mathbb{C}}$ of presheaves, and it is in fact α-small. Moreover, it has the following basic property, which is just a restriction of the basic property of $\dot{\mathcal{V}} \to \mathcal{V}$ stated in Proposition 4.

Proposition 6. *The map $\dot{\mathcal{V}}_\alpha \to \mathcal{V}_\alpha$ classifies α-small maps $f : Y \to X$ in $\widehat{\mathbb{C}}$, in the sense that there is always a pullback square,*

$$\begin{tikzcd} Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathcal{V}}_\alpha \arrow[d] \\ X \arrow[r, "\tilde{Y}"'] & \mathcal{V}_\alpha. \end{tikzcd} \tag{20}$$

The classifying map $\tilde{Y} : X \to \mathcal{V}_\alpha$ is determined by the adjunction $\int \dashv \nu$ as (the factorization of) the transpose of the classifiyng map of the discrete fibration $\int X \to \int Y$.

Proof. If $Y \to X$ is α-small, its classifying map $\tilde{Y} : X \to \mathcal{V}$ factors through $\mathcal{V}_\alpha \hookrightarrow \mathcal{V}$, as indicated below,

$$\begin{tikzcd} Y \arrow[r] \arrow[d] \arrow[rr, bend left] & \nu\dot{\mathsf{Set}_\alpha}^{\mathrm{op}} \arrow[r, hook] \arrow[d] & \nu\dot{\mathsf{Set}}^{\mathrm{op}} \arrow[d] \\ X \arrow[r] \arrow[rr, bend right, "\tilde{Y}"'] & \nu\mathsf{Set}_\alpha^{\mathrm{op}} \arrow[r, hook] & \nu\mathsf{Set}^{\mathrm{op}}, \end{tikzcd} \tag{21}$$

in virtue of the following adjoint transposition,

$$\begin{tikzcd} \int Y \arrow[r] \arrow[d] \arrow[rr, bend left] & \dot{\mathsf{Set}_\alpha}^{\mathrm{op}} \arrow[r, hook] \arrow[d] & \dot{\mathsf{Set}}^{\mathrm{op}} \arrow[d] \\ \int X \arrow[r] \arrow[rr, bend right] & \mathsf{Set}_\alpha^{\mathrm{op}} \arrow[r, hook] & \mathsf{Set}^{\mathrm{op}}. \end{tikzcd} \tag{22}$$

Note that the square on the right is evidently a pullback, and the one on the left therefore is, too, because the outer rectangle is the classifying pulback of the discrete fibration $\int Y \to \int X$, as stated. Thus the left square in (21) is a pullback. $\square$

5. Examples

1. Let $\alpha = \kappa$ a strongly inaccessible cadinal, so that $\mathsf{ob}(\mathsf{Set}_\kappa)$ is a Grothendieck universe. Then the Hofmann-Streicher universe of (**??**) is recovered in the present setting as the κ-small map classifier

   $$E \;\cong\; \dot{\mathcal{V}}_\kappa \longrightarrow \mathcal{V}_\kappa \;\cong\; U$$

   in the sense of Proposition 6. Indeed, for $c \in \mathbb{C}$, we have

   $$\mathcal{V}_\kappa c \;=\; \nu(\mathsf{Set}_\kappa^{\mathsf{op}})(c) = \mathsf{Cat}\big(\mathbb{C}/_c\,,\, \mathsf{Set}_\kappa^{\mathsf{op}}\big) \;=\; \mathsf{ob}(\widehat{\mathbb{C}/_c}) \;=\; U c\,. \tag{23}$$

   For $\dot{\mathcal{V}}_\kappa$ we then have,

   $$\begin{aligned} \dot{\mathcal{V}}_\kappa c \;=\; \nu(\,\dot{\mathsf{Set}}_\kappa^{\mathsf{op}})(c) \;&=\; \mathsf{Cat}\big(\mathbb{C}/_c\,,\; \dot{\mathsf{Set}}_\kappa^{\mathsf{op}}\big) \\ &\cong\; \textstyle\coprod_{A \in \mathcal{V}_\kappa c} \mathsf{Cat}_{\mathbb{C}/_c}\big(\mathbb{C}/_c\,,\, A^*\mathsf{Set}_\kappa^{\mathsf{op}}\big) \end{aligned} \tag{24}$$

   where the A-summand in (24) is defined by taking sections of the pullback indicated below.

   $$\begin{tikzcd} A^*\mathsf{Set}_\kappa^{\mathsf{op}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathsf{Set}}_\kappa^{\mathsf{op}} \arrow[d] \\ \mathbb{C}/_c \arrow[r, "A"'] \arrow[u, dotted, bend left] \arrow[ur, dotted] & \mathsf{Set}_\kappa^{\mathsf{op}} \end{tikzcd} \tag{25}$$

   But $A^*\mathsf{Set}_\kappa^{\mathsf{op}} \;\cong\; \int_{\mathbb{C}/_c} A$ over $\mathbb{C}/_c$, and sections of this discrete fibration in Cat correspond uniquely to natural maps $1 \to A$ in $\widehat{\mathbb{C}/_c}$. Since 1 is representable in $\widehat{\mathbb{C}/_c}$ we can continue (24) by

   $$\begin{aligned} \dot{\mathcal{V}}_\kappa c \;&\cong\; \textstyle\coprod_{A \in \mathcal{V}_\kappa c} \mathsf{Cat}_{\mathbb{C}/_c}\big(\mathbb{C}/_c\,,\, A^*\mathsf{Set}_\kappa^{\mathsf{op}}\big) \\ &\cong\; \textstyle\coprod_{A \in \mathcal{V}_\kappa c} \widehat{\mathbb{C}/_c}(1, A) \\ &\cong\; \textstyle\coprod_{A \in \mathcal{V}_\kappa c} A(1_c) \\ &=\; \textstyle\coprod_{A \in \mathcal{V}_\kappa c} \mathsf{E}l(\langle c, A\rangle) \\ &=\; E c\,. \end{aligned}$$

2. By functoriality of the nerve $\nu : \mathsf{Cat} \to \widehat{\mathbb{C}}$, a sequence of Grothendieck universes

   $$\mathcal{U} \subseteq \mathcal{U}' \subseteq \ldots$$

   in Set gives rise to a (cumulative) sequence of type-theoretic universes

   $$\mathcal{V} \rightarrowtail \mathcal{V}' \rightarrowtail \ldots$$

in $\widehat{\mathbb{C}}$. More precisely, there is a sequence of cartesian squares,

$$\begin{tikzcd}
\dot{\mathcal{V}} \arrow[r, tail] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \dot{\mathcal{V}}' \arrow[r, tail] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \dots \\
\mathcal{V} \arrow[r, tail] & \mathcal{V}' \arrow[r, tail] & \dots\,,
\end{tikzcd} \tag{26}$$

in the image of $\nu : \mathsf{Cat} \longrightarrow \widehat{\mathbb{C}}$, classifying small maps in $\widehat{\mathbb{C}}$ of increasing size, in the sense of Proposition 6.

3. Let $\alpha = 2$ so that $1 \to 2$ is the subobject classifier of Set, and

$$\mathbb{1} = \dot{\mathsf{Set}}_2^{\mathsf{op}} \longrightarrow \mathsf{Set}_2^{\mathsf{op}} = \mathbb{2}$$

is then a classifier in Cat for *sieves*, i.e. full subcategories $\mathbb{S} \hookrightarrow \mathbb{A}$ closed under the domains of arrows $a \to s$ for $s \in \mathbb{S}$. The nerve $\dot{\mathcal{V}}_2 \to \mathcal{V}_2$ is then exactly the subobject classifier $1 \to \Omega$ of $\widehat{\mathbb{C}}$,

$$1 = \nu\mathbb{1} = \dot{\mathcal{V}}_2 \longrightarrow \mathcal{V}_2 = \nu\mathbb{2} = \Omega\,.$$

4. Let $i : \mathbb{2} \hookrightarrow \mathsf{Set}_\kappa$ and $p : \mathsf{Set}_\kappa \to \mathbb{2}$ be the embedding-retraction pair with $i : \mathbb{2} \hookrightarrow \mathsf{Set}_\kappa$ the inclusion of the full subcategory on the sets $\{0, 1\}$ and $p : \mathsf{Set}_\kappa \to \mathbb{2}$ the retraction that takes $0 = \emptyset$ to itself, and everything else (i.e. the non-empty sets) to $1 = \{\emptyset\}$. There is a retraction (of arrows) in Cat,

$$\begin{tikzcd}
\mathbb{1} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \dot{\mathsf{Set}}_\kappa \arrow[r] \arrow[d] & \mathbb{1} \arrow[d] \\
\mathbb{2} \arrow[r, hook, "i"'] & \mathsf{Set}_\kappa \arrow[r, two heads, "p"'] & \mathbb{2}
\end{tikzcd} \tag{27}$$

where the left square is a pullback.

By the functoriality of $(\,-^{\mathrm{op}}$ and) $\nu : \mathsf{Cat} \to \widehat{\mathbb{C}}$ we then have a retract diagram in $\widehat{\mathbb{C}}$, again with a pullback on the left,

$$\begin{tikzcd}
1 \arrow[r, hook] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \dot{\mathcal{V}}_\kappa \arrow[r] \arrow[d] & 1 \arrow[d] \\
\Omega \arrow[r, hook, "\{-\}"'] & \mathcal{V}_\kappa \arrow[r, two heads, "{[-]}"'] & \Omega
\end{tikzcd} \tag{28}$$

where for any $\phi : X \to \Omega$ the subobject $\{\phi\} \rightarrowtail X$ is classified as a small map by the composite $\{\phi\} : X \to \mathcal{V}_\kappa$, and for any small map $A \to X$, the image $[A] \rightarrowtail X$ is classified as a subobject by the composite $[\alpha] : X \to \mathcal{V}_\kappa \to \Omega$, where $\alpha : X \to \mathcal{V}_\kappa$ classifies $A \to X$. The idempotent composite

$$\|-\| = \{[-]\} : \mathcal{V}_\kappa \longrightarrow \mathcal{V}_\kappa$$

is the *propositional truncation modality* in the natural model of type theory given by $\dot{\mathcal{V}}_\kappa \to \mathcal{V}_\kappa$ (see [AGH21]).

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References

[AGH21] S. Awodey, N. Gambino, and S. Hazratpour. Kripke-Joyal forcing for type theory and uniform fibrations, October 2021. Preprint available as https://arxiv.org/abs/2110.14576.

[Gro83] Alexander Grothendieck. Pursuing stacks. 1983. Unpublished.

[HS97] Martin Hofmann and Thomas Streicher. Lifting Grothendieck universes. Spring 1997. Unpublished.

[Web07] Mark Weber. Yoneda structures from 2-toposes. *Applied Categorical Structures*, 15(3):259–323, 2007.