A Quillen model structure on the category of cartesian cubical sets

Steve Awodey

January 29, 2019

1 The cartesian cube category

In contrast to some other treatments of cubical sets [?, ?, ?, ?, ?, ?, ?, ?], we consider what may be termed the *cartesian* cube category $\mathbb{C}$, defined as the free finite product category on an interval $\delta_0, \delta_1 : 1 \rightrightarrows \mathrm{I}$. As a classifying category for an algebraic theory with two constant symbols $\mathbb{T} = \{0, 1\}$, the category $\mathbb{C}$ is dual to the full subcategory of finitely-generated, free $\mathbb{T}$-algebras $\mathsf{Alg}(\mathbb{T})_{\mathrm{fg}}$ (by Lawvere duality). In this case, the algebras are thus simply *bipointed sets* (A, a_0, a_1), and the free ones are the *strictly* bipointed sets $a_0 \neq a_1$. Thus $\mathsf{Alg}(\mathbb{T})_{\mathrm{fg}}$ consists of the finite, strictly bipointed sets and all bipointed maps between them. We will use the following specific presentation.

Definition 1. The objects of the cartesian cube category $\mathbb{C}$, called n-cubes, will be written

$$[n] = \{0, x_1, ..., x_n, 1\}\,.$$

The arrows,

$$f : [n] \longrightarrow [m]\,,$$

maybe taken to be m-tuples of elements drawn from the set $\{0, x_1, ..., x_n, 1\}$ regarded as formal terms representing composites of faces, degeneracies, permutations, and diagonals. Equivalently, the arrows $[n] \longrightarrow [m]$ are arbitrary bipointed maps $[m] \longrightarrow [n]$.

See [?] for further details.

2 Cubical sets

The category cSet of *cubical sets* is the category of presheaves on the cartesian cube category $\mathbb{C}$,

$$\mathsf{cSet} \;=\; \mathsf{Set}^{\mathbb{C}^{\mathrm{op}}}\,.$$

It is thus generated by the representable presheaves $y([n])$, which will be written

$$\mathrm{I}^n = y([n])$$

and called the *standard n-cubes*.

3 The cofibration weak factorization system

Cofibrations. The *cofibrations* are a class $\mathcal{C}$ of maps in cSet, written

$$c : A \rightarrowtail B\,,$$

and are assumed to satisy the following axioms:

(C1) All isomorphisms are cofibrations.

(C2) The composite of two cofibrations is a cofibration.

(C3) Cofibrations are monomorphisms.

(C4) Any pullback of a cofibration is a cofibration.

By conditions (C3-4), the cofibrations are classified by a (pointed) subobject $\Phi \hookrightarrow \Omega$ (a not necessarily cofibrant mono) of the standard subobject classifier $\top : 1 \longrightarrow \Omega$ of cSet. We shall call the canonical factorization $t : 1 \longrightarrow \Phi$ the *cofibration classifier.* Note that we permit the case where $\Phi = \Omega$, i.e. all monos are cofibrant.

Cofibrant partial map classifier. The polynomial endofunctor [**?**] determined by the cofibration classifier $t : 1 \longrightarrow \Phi$ is defined on objects by

$$X \;\mapsto\; \Phi_!\, t_*(X) \;=\; \sum_{\phi:\Phi} X^{\phi}\,.$$

We shall write $X^+ := \sum_{\phi:\Phi} X^{\phi}$.

Observe that by the definition of X^+ there is a pullback square,

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X^+ \arrow[d, "t_*X"] \\ 1 \arrow[r, "t"'] & \Phi \end{tikzcd}$$

since t is monic. Let $\eta : X \rightarrowtail X^+$ be the indicated top horizontal map; we call this map the *cofibrant partial map classifier* of X.

Proposition 2. *The map $\eta : X \rightarrowtail X^+$ classifies partial maps with cofibrant domain, in the following sense.*

1. *The map $\eta : X \rightarrowtail X^+$ is a cofibration.*
2. *For any object Z and any partial map $(s, g) : Z \leftarrowtail S \longrightarrow X$, with $s : S \rightarrowtail Z$ cofibrant, there is a unique $f : Z \longrightarrow X^+$ making a pullback square,*

$$\begin{tikzcd} S \arrow[r, "g"] \arrow[d, tail, "s"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, tail, "\eta"] \\ Z \arrow[r, "f"'] & X^+ . \end{tikzcd}$$

Proof. $\eta : X \rightarrowtail X^+$ is a cofibration since it is a pullback of $t : 1 \longrightarrow \Phi$. The second statement follows directly from the definition of X^+ as a polynomial (see [**?**], prop. 7). □

The +-Monad.

Proposition 3. *The pointed endofunctor determined by $\eta_X : X \rightarrowtail X^+$ has a natural multiplication $\mu_X : X^{++} \longrightarrow X^+$ making it a monad.*

Proof. Since the cofibrations are closed under composition, the monad structure on X^+ follows as in [**?**], proposition nm. Explicitly, μ_X is determined as the unique map making the following a pullback diagram.

$$\begin{tikzcd} X \arrow[r, "="] \arrow[d, tail, "\eta_X"'] & X \arrow[dd, tail, "\eta"] \\ X^+ \arrow[d, tail, "\eta_{X^+}"'] & \\ X^{++} \arrow[r, dotted, "\mu"'] & X^+ \end{tikzcd}$$

□

Relative partial map classifier. For any object $X \in \mathsf{cSet}$ the usual pullback functor

$$X^* : \mathsf{cSet} \longrightarrow \mathsf{cSet}/X\,,$$

taking any A to the second projection $A \times X \longrightarrow X$, not only preserves the subobject classifier Ω, but also the cofibration classifier $\Phi \hookrightarrow \Omega$, where a map in cSet/X is defined to be a cofibration if it is one in cSet. Thus in cSet/X the *(relative) cofibration classifier* is the map

$$t \times X : 1 \times X \longrightarrow \Phi \times X \quad \text{over } X$$

which we may also write $t_X : 1_X \longrightarrow \Phi_X$. Like $t : 1 \longrightarrow \Phi$, this map determines a polynomial endofunctor

$$+_X : \mathsf{cSet}/X \longrightarrow \mathsf{cSet}/X\,,$$

which commutes (up to natural isomorphism) with $+ : \mathsf{cSet} \longrightarrow \mathsf{cSet}$ and $X^* : \mathsf{cSet} \longrightarrow \mathsf{cSet}/X$ in the evident way:

$$\begin{tikzcd}
\mathsf{cSet}/X \arrow[r, "+_X"] & \mathsf{cSet}/X \\
\mathsf{cSet} \arrow[u, "X^*"] \arrow[r, "+"'] & \mathsf{cSet} \arrow[u, "X^*"']
\end{tikzcd}$$

The endofunctor $+_X$ is also pointed $\eta : Y \longrightarrow Y^+$ and has a monad multiplication $\mu_Y : Y^{++} \longrightarrow Y^+$, for any $Y \longrightarrow X$, for the same reason that $+$ has this structure. Summarizing, we may say that *the polynomial monad* $+ : \mathsf{cSet} \longrightarrow \mathsf{cSet}$ *is fibered over* cSet.

Definition 4. A $+$-*algebra* in cSet is a cubical set A together with a retraction $\alpha : A^+ \longrightarrow A$ of $\eta_A : A \longrightarrow A^+$, i.e. an algebra for the pointed endofunctor $(+ : \mathsf{cSet} \longrightarrow \mathsf{cSet},\ \eta : 1 \longrightarrow +)$. Algebras for the monad $(+, \eta, \mu)$ will be referred to specifically as $(+, \eta, \mu)$-*algebras*, or $+$-*monad algebras*.

A *relative* $+$-*algebra* in cSet is a map $A \longrightarrow X$ together with an algebra structure for the pointed endofunctor $+_X : \mathsf{cSet}/X \longrightarrow \mathsf{cSet}/X$.

The factorization system.

Proposition 5. *There is an (algebraic) weak factoriation system on* cSet *given by taking as the left class the cofibrations and as the right class the (maps underlying) the relative* $+$-*algebras. Thus a right map is one* f :

$A \longrightarrow X$ for which there is a retract $\alpha : A' \longrightarrow A$ over X of the canonical map $\eta_f : A \longrightarrow A'$ over X,

$$\begin{tikzcd} A \arrow[r, "\eta_f"] \arrow[rr, bend left, "="] \arrow[dr, "f"'] & A' \arrow[r, "\alpha"] \arrow[d, "f^+"] & A \arrow[dl, "f"] \\ & X. & \end{tikzcd}$$

Proof. The factorization of any map $f : Y \longrightarrow X$ is given simply by applying the (relative) +-functor

$$\begin{tikzcd} Y \arrow[r, "\eta_f"] \arrow[dr, "f"'] & Y' \arrow[d, "f^+"] \\ & X. \end{tikzcd}$$

We know that the unit η_f is always a cofibration, and since f^+ is the free algebra for the +-monad, it is in particular a +-algebra.

For the lifting condition, consider a cofibration $c : B \rightarrowtail C$, a right map $A \longrightarrow X$, with a $+_X$-algebra structure map $\alpha : A^+ \longrightarrow A$ over X, and a commutative square as indicated in the following.

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, tail, "c"'] & A \arrow[d] \arrow[dr, "\eta"] & \\ & & A^+ \arrow[ul, bend right, "\alpha"'] \arrow[dl] \\ C \arrow[r, "f"'] & X & \end{tikzcd}$$

Thus over X, we have the situation

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, tail, "c"'] & A \arrow[d, "\eta"] \\ C \arrow[ur, dotted, "d"] & A^+ \arrow[u, bend right, "\alpha"'] \end{tikzcd}$$

and we seek a diagonal filler as indicated. Since $(c, g) : B \leftarrowtail C \longrightarrow A$ is a cofibrant partial map into A, there is a map $\varphi : C \longrightarrow A^+$ (over X) making a (pullback) square,

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, tail, "c"'] & A \arrow[d, "\eta"] \\ C \arrow[r, dotted, "\varphi"'] & A^+ \arrow[u, bend right, "\alpha"'] \end{tikzcd}$$

We thus have $d := \alpha \circ \varphi : C \longrightarrow A$ as the required diagonal filler.

The closure of the cofibrations under retracts follows from their classification by a universal object $t : 1 \longrightarrow \Phi$, and the closure of the right maps under retracts follows from their being the algebras for a pointed endofunctor underlying a monad (cf. [**?**]). Algebraicity of this weak factorization system also follows directly, since $+$ is a monad. $\square$

Summarizing, we have a weak factorization system $(\mathcal{L}, \mathcal{R})$ on the category cSet of cubical sets, in which:

$$\begin{aligned} \mathcal{L} &= \mathcal{C} \quad \text{(the cofibrations)} \\ \mathcal{R} &= +\mathsf{Alg} \quad \text{(the relative +-algebras)} \end{aligned}$$

We shall call this the *cofibration weak factorization system*. As here, we will sometimes say that an object (or map) is a (relative) +-algebra when it can be equipped with a (relative) +-algebra structure; such maps will also be called *trivial fibrations* and the class of all such will be denoted $\mathsf{TrivFib}$,

$$\mathcal{C}^{\pitchfork} = \mathsf{TrivFib}.$$

Uniform filling structure. It will be convenient later to relate +-algebra structure and the more familiar diagonal filling condition of weak factorization systems with a special form of the latter that occurs in [**?**] in the form of a *uniform filling structure*.

Consider a generating sub*set* of cofibrations, consisting of all those $c \in \mathcal{C}$ where $c : C \rightarrowtail Z$ has a representable codomain $Z = \mathrm{I}^n$. Call these maps the *basic cofibrations*, and let

$$\mathsf{BCof} = \{c : C \rightarrowtail \mathrm{I}^n \mid c \in \mathcal{C}\}. \tag{1}$$

Proposition 6. *For any object X in* cSet *the following are equivalent:*

1. *X is a* +-algebra*, i.e. there is a retraction $\alpha : X^+ \longrightarrow X$ of the unit $\eta : X \longrightarrow X^+$.*

2. *X is* contractible *in the sense that it has the right lifting property with respect to all cofibrations,*
$$\mathcal{C} \pitchfork X.$$

3. *X has a* uniform filling structure*: for each basic cofibration $c : C \rightarrowtail \mathrm{I}^n$ and map $x : C \longrightarrow X$ there is given an extension $j(c, x)$,*
$$\begin{tikzcd} C \arrow[r, "x"] \arrow[d, tail, "c"'] & X, \\ \mathrm{I}^n \arrow[ur, dotted, "j(c,x)"'] & \end{tikzcd} \tag{2}$$
that is uniform in I^n *in the following sense: given any cubical map $u : \mathrm{I}^m \longrightarrow \mathrm{I}^n$, the pullback $u^*c : u^*C \rightarrowtail \mathrm{I}^m$ is again a basic cofibration and fits into a commutative diagram of the form*
$$\begin{tikzcd} u^*C \arrow[r, "c^*u"] \arrow[d, tail, "u^*c"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[r, "x"] \arrow[d, tail, "c"'] & X. \\ \mathrm{I}^m \arrow[r, "u"'] & \mathrm{I}^n \arrow[ur, dotted, "j(c,x)"'] & \end{tikzcd} \tag{3}$$
For the pair $(u^*c,\ x' := x \circ c^*u)$ in (3) the chosen extension $j(u^*c, x') : \mathrm{I}^m \longrightarrow X$, then satisfies
$$j(u^*c, x') = j(c, x) \circ (u). \tag{4}$$

Proof. Yoneda. □

4 Partial path lifting (biased version)

Our next goal is the specification of a second weak factorization system (the *fibration weak factorization system*) with a restricted class of "trivial" cofibrations on the left, and an expanded class of right maps, the fibrations.

As a warm-up, we first recall the specification of the trivial-cofibration/fibration WFS from [**?**]. (In an appendix we show that these fibrations agree with those specified in the "logical style" of [**?**, **?**]). In the subsequent section we shall modify the specification of fibrations in order to arrive at an "unbiased" version more suitable for the cartesian setting.

A *generating class of (biased) trivial cofibrations* are all maps of the form
$$c \otimes \delta_\epsilon : D \longrightarrow Z \times \mathrm{I}\,, \tag{5}$$
where:

1. $c : C \rightarrowtail Z$ is an arbitrary cofibration,

2. $\delta_\epsilon : 1 \longrightarrow \mathrm{I}$ is one of the two "endpoint inclusions" where, recall, $1 = y[0]$, and $I = y[1]$, and for $\epsilon = 0, 1$, we have the maps $\delta_\epsilon : 1 \longrightarrow \mathrm{I}$ corresponding to the two bipointed maps $0, 1 : \{0, x, 1\} \longrightarrow \{0, 1\}$.

3. $c \otimes \delta_\epsilon$ is the pushout-product (resp. "Leibniz tensor") of the cofibration $c : C \rightarrowtail Z$ and an endpoint $\delta_\epsilon : 1 \longrightarrow \mathrm{I}$, as indicated in the following diagram (in which the unlabelled maps are the expected ones).

$$\begin{tikzcd}
C \times 1 \arrow[r] \arrow[d] & C \times \mathrm{I} \arrow[d] \arrow[ddr, bend left] & \\
Z \times 1 \arrow[r] \arrow[drr, bend right] & Z +_C (C \times \mathrm{I}) \arrow[dr, "c\otimes\delta_\epsilon"'] & \\
 & & Z \times \mathrm{I}
\end{tikzcd} \tag{6}$$

4. $D = Z +_C (C \times \mathrm{I})$ is the indicated pushout, the domain of $c \otimes \delta_\epsilon$.

In order to insure that such maps are indeed cofibrations, we assume two further axioms:

(C5) The endpoint inclusions $\delta_\epsilon : 1 \longrightarrow \mathrm{I}$ are cofibrations.

(C6) The cofibrations are closed under pushout-products.

Fibrations (biased version). Let

$$\mathcal{C} \otimes \delta_\epsilon \;=\; \{c \otimes \delta_\epsilon : D \rightarrowtail Z \times \mathrm{I} \mid c \in \mathcal{C},\ \epsilon = 0, 1\}$$

be the class of all such pushout-products of arbitrary cofibrations $c : C \rightarrowtail Z$ with endpoint inclusions $\delta_\epsilon : 1 \rightarrowtail \mathrm{I}$. The *(biased) fibrations* are defined to be the right class of these generating trivial cofibrations,

$$(\mathcal{C} \otimes \delta_\epsilon)^{\pitchfork} \;=\; \mathcal{F}\,.$$

Thus a map $f : Y \longrightarrow X$ is a (biased) fibration if for every commutative square of the form

$$\begin{tikzcd}
Z +_C (C \times \mathrm{I}) \arrow[r] \arrow[d, tail, "c\otimes\delta_\epsilon"'] & Y \arrow[d, "f"] \\
Z \times \mathrm{I} \arrow[r] \arrow[ur, dotted, "j"] & X\,,
\end{tikzcd} \tag{7}$$

with a generating trivial cofibration on the left, there is a diagonal filler j as indicated. This condition can be seen as a generalized homotopy lifting property.

To now relate this notion of fibration to the cofibration weak factorization system, fix any map $u : A \longrightarrow B$, and recall (e.g. from [**?**]) that the pushout-product with u is a functor on the arrow category

$$(-) \otimes u : \mathsf{cSet}^{\mathbb{2}} \longrightarrow \mathsf{cSet}^{\mathbb{2}} .$$

This functor has a right adjoint, the *pullback-hom* (or "Leibniz exponential"), which for a map $f : X \longrightarrow Y$ we will write as

$$(u \Rightarrow f) : Y^B \longrightarrow (X^B \times_{X^A} Y^A) .$$

The pullback-hom is determined as indicated in the following diagram (in which the unlabelled maps are the expected ones).

$$\begin{tikzcd}
Y^B \arrow[dr, "u \Rightarrow f"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X^B \times_{X^A} Y^A \arrow[r] \arrow[d] & Y^A \arrow[d] \\
& X^B \arrow[r] & X^A
\end{tikzcd} \tag{8}$$

Using the $\otimes \dashv \Rightarrow$ adjunction on the arrow category, we can now show the following (cf. [**?**], prop. n.m).

Proposition 7. *An object X is fibrant if and only if both of the endpoint projections $X^{\mathrm{I}} \longrightarrow X$ from the pathspace are $+$-algebras. More generally, a map $f : Y \longrightarrow X$ is a fibration iff both of the maps*

$$(\delta_\epsilon \Rightarrow f) : Y^I \longrightarrow X^I \times_X Y$$

are $+$-algebras (for $\epsilon = 0, 1$).

Proof. The first statement follows from the second, since the pathspace projections $X^{\mathrm{I}} \longrightarrow X$ are just the maps

$$(\delta_\epsilon \Rightarrow !_X) : X^I \longrightarrow (1^I \times_1 X) \cong X ,$$

for $!_X : X \longrightarrow 1$.

By definition, $f : X \longrightarrow Y$ is a fibration if every square of the form

$$\begin{tikzcd} Z +_C (C \times \mathrm{I}) \arrow[r] \arrow[d, tail, "c \otimes \delta_\epsilon"'] & Y \arrow[d, "f"] \\ Z \times \mathrm{I} \arrow[r] \arrow[ur, dotted, "j"'] & X , \end{tikzcd} \tag{9}$$

with a generating trivial cofibration $c \otimes \delta_\epsilon$ on the left, has is a diagonal filler j as indicated. In brief,

$$(c \otimes \delta_\epsilon) \pitchfork f \qquad (\text{for } c \in \mathcal{C},\ \epsilon = 0, 1).$$

By the $\otimes \dashv \Rightarrow$ adjunction, this is equivalent to the condition

$$c \pitchfork (\delta_\epsilon \Rightarrow f) \qquad (\text{for } c \in \mathcal{C},\ \epsilon = 0, 1).$$

That is, for every square

$$\begin{tikzcd} C \arrow[r] \arrow[d, tail, "c"'] & Y^I \arrow[d, "\delta_\epsilon \Rightarrow f"] \\ Z \arrow[r] \arrow[ur, dotted, "k"] & X^I \times_X Y , \end{tikzcd}$$

with an arbitrary cofibration $c : C \rightarrowtail Z$ on the left, there is a diagonal filler k as indicated, for $\epsilon = 0, 1$. But this is just to say that the maps $\delta_\epsilon \Rightarrow f$ are in the right class of the cofibrations, which is equivalent to their being $+$-algebras, as claimed. $\square$

Fibration structure. It follows from the $\otimes \dashv \Rightarrow$ adjunction that the pushout-product preserves colimits in the arrow category. Thus we see that the fibrations can equivalently be determined by right-lifting against a generating *set* of trivial cofibrations, consisting of all those $c \otimes \delta_\epsilon$ in $\mathcal{C} \otimes \delta_\epsilon$ where $c : C \rightarrowtail Z$ has a representable codomain $Z = \mathrm{I}^n$. Call these maps the *basic (biased) trivial cofibrations*, and let

$$\mathcal{B} \otimes \delta_\epsilon = \{ c \otimes \delta_\epsilon : B \rightarrowtail \mathrm{I}^{n+1} \mid n \geq 0\,, c : C \rightarrowtail \mathrm{I}^n,\ \epsilon = 0, 1 \}\,, \tag{10}$$

where the pushout-pullback $c \otimes \delta_\epsilon$ now takes the simpler form

$$\begin{tikzcd} C \arrow[r] \arrow[d, tail] & C \times \mathrm{I} \arrow[d] \arrow[ddr, bend left] & \\ \mathrm{I}^n \arrow[r] \arrow[drr, bend right] & \mathrm{I}^n +_C (C \times \mathrm{I}) \arrow[dr, tail, "m \otimes \delta_\epsilon"'] & \\ & & \mathrm{I}^n \times \mathrm{I} \end{tikzcd} \tag{11}$$

for a cofibration $c : C \rightarrowtail \mathrm{I}^n$, an endpoint $\delta_\epsilon : 1 \longrightarrow \mathrm{I}$, and with domain $B = \big(\mathrm{I}^n +_C (C \times \mathrm{I})\big)$. These subobjects $B \rightarrowtail \mathrm{I}^{n+1}$ can be seen geometrically as generalized open box inclusions.

For any fibration $f : Y \longrightarrow X$ a (uniform, biased) fibration structure on f is a choice of diagonal fillers $j(c, \epsilon)$,

$$\begin{tikzcd} \mathrm{I}^n +_C (C \times \mathrm{I}) \arrow[r] \arrow[d, "c \otimes \delta_\epsilon"'] & X \arrow[d, "f"] \\ \mathrm{I}^n \times \mathrm{I} \arrow[r] \arrow[ur, dotted, "j(c,\epsilon)"'] & Y, \end{tikzcd} \tag{12}$$

for each basic trivial cofibration $c \otimes \delta_\epsilon : B = (\mathrm{I}^n +_C (C \times \mathrm{I})) \rightarrowtail \mathrm{I}^{n+1}$, which is *uniform* in the following sense: given any cubical map $u : \mathrm{I}^m \longrightarrow \mathrm{I}^n$, the pullback $u^*c : u^*C \rightarrowtail \mathrm{I}^m$ of $c : C \rightarrowtail \mathrm{I}^n$ along u determines another basic trivial cofibration $u^*c \otimes \delta_\epsilon : B' = (\mathrm{I}^m +_{u^*C} (u^*C \times \mathrm{I})) \rightarrowtail \mathrm{I}^{m+1}$ which fits into a commutative diagram of the form

$$\begin{tikzcd} \mathrm{I}^m +_{u^*C} (u^*C \times \mathrm{I}) \arrow[r] \arrow[d, "u^*c \otimes \delta_\epsilon"'] & \mathrm{I}^n +_C (C \times \mathrm{I}) \arrow[r] \arrow[d, "c \otimes \delta_\epsilon"'] & X \arrow[d, "f"] \\ \mathrm{I}^m \times \mathrm{I} \arrow[r, "u \times \mathrm{I}"'] & \mathrm{I}^n \times \mathrm{I} \arrow[r] \arrow[ur, dotted, "j(c,\epsilon)"'] & Y, \end{tikzcd} \tag{13}$$

by applying the functor $(-) \otimes \delta_\epsilon$ to the pullback square relating u^*c to c. Now for the outer rectangle in (13) there is a chosen diagonal filler $j(u^*c, \epsilon) : \mathrm{I}^m \times \mathrm{I} \longrightarrow X$, and for this map we require that

$$j(u^*c, \epsilon) = j(c, \epsilon) \circ (u \times \mathrm{I}). \tag{14}$$

This is a reformulation of the logical specification given in [**?**] (see the appendix).

Definition 8. A *(uniform, biased) fibration structure* on a map $f : Y \longrightarrow X$ is a choice of fillers $j(c, \epsilon)$ as in (12) satisfying (14) for all $u : \mathrm{I}^m \longrightarrow \mathrm{I}^n$.

Essentially the same argument as that given for Proposition 7 also yields the following sharper formulation in terms of fibration structure.

Corollary 9. *Fibration structure on a map $f : Y \longrightarrow X$ is equivalent to a pair of $+$-algebra structures on the maps*

$$(\delta_\epsilon \Rightarrow f) : Y^I \longrightarrow X^I \times_X Y$$

for $\epsilon = 0, 1$.

Finally, it is clear that any map $f : Y \longrightarrow X$ that can be equipped with a fibration structure is a fibration in the sense of (7). Conversely, it can be shown that every fibration in the latter sense can be equipped with a fibration structure, as in Proposition 6.

5 Unbiased partial path-lifting

Rather than building a weak factorization system based on the foregoing notion of (biased) fibration (as is done in [**?**, **?**]), we shall first eliminate the "bias" on a choice of endpoint $\delta_\epsilon : 1 \longrightarrow \mathrm{I}$, expressed by the indexing $\epsilon = 0, 1$. Consider first the simple path-lifting condition, which is a special case of (7) with $c = ! : 0 \rightarrowtail 1$, since $! \otimes \delta_\epsilon = \delta_\epsilon$:

$$\begin{tikzcd} 1 \arrow[r] \arrow[d, tail, "\delta_\epsilon"'] & Y \arrow[d, "f"] \\ \mathrm{I} \arrow[r] \arrow[ur, dotted, "j_\epsilon"'] & X . \end{tikzcd}$$

(Note that $0 \rightarrowtail 1$ is a cofibration by axioms C4 and C5).

In toplogical spaces, rather than requiring lifts j_ϵ for each of the endpoints $\epsilon = 0, 1$, we could instead require that there be a lift j_i *for each point* $i : 1 \longrightarrow \mathrm{I}$ *in the real interval* $\mathrm{I} = [0, 1]$. Such "unbiased path-lifting" can be formulated in cSet by introducing a "generic point" $\delta : 1 \longrightarrow \mathrm{I}$, by passing to cSet/I, and then requiring path-lifting with respect to δ. The following specification implements that idea, while also adding partiality in the sense of the foregoing section. We need the following strengthening of axiom C5.

(C5') The diagonal map $\delta : \mathrm{I} \longrightarrow \mathrm{I} \times \mathrm{I}$ is a cofibration.

Definition 10 (Fibration)**.** Let $\delta : \mathrm{I} \longrightarrow \mathrm{I} \times \mathrm{I}$ be the diagonal map.

1. An object X is *(unbiased) fibrant* if the map

$$(\delta \Rightarrow X) = \langle \mathsf{eval}, p_2 \rangle : X^{\mathrm{I}} \times \mathrm{I} \longrightarrow X \times \mathrm{I}$$

   is a $+$-algebra.

2. A map $f : Y \longrightarrow X$ is an *(unbiased) fibration* if the map

$$(\delta \Rightarrow f) = \langle f^{\mathrm{I}} \times \mathrm{I}, \langle \mathsf{eval}, p_2 \rangle \rangle : Y^{I} \times \mathrm{I} \longrightarrow (X^{I} \times \mathrm{I}) \times_{(X \times \mathrm{I})} (Y \times \mathrm{I})$$

   is a $+$-algebra.

Now we can play the proof of Proposition 7 backwards in order to determine a class of generating trivial cofibrations for the unbiased case. We consider pairs of maps $c : C \rightarrowtail Z$ and $z : Z \longrightarrow \mathrm{I}$, where the former is a cofibration and the latter is regarded as an "I-indexing", so that

$$\begin{tikzcd} C \arrow[r, tail, "c"] \arrow[dr] & Z \arrow[d, "z"] \\ & \mathrm{I} \end{tikzcd}$$

can be regarded as an I-indexed family of cofibrations. Let

$$\mathsf{Gph}(z) : Z \longrightarrow Z \times \mathrm{I}\,,$$

be the graph of $z : Z \longrightarrow \mathrm{I}$, i.e. $\mathsf{Gph}(z) = \langle 1_Z, z \rangle$, and then let

$$c \otimes_z \delta := [\mathsf{Gph}(z), c \times \mathrm{I}] : Z +_C (C \times \mathrm{I}) \longrightarrow Z \times \mathrm{I}\,,$$

which is easily seen to be well-defined on the indicated pushout.

$$\begin{tikzcd} C \arrow[r, "\mathsf{Gph}(zc)"] \arrow[d, tail, "c"'] & C \times \mathrm{I} \arrow[d] \arrow[ddr, bend left, "c\times\mathrm{I}"] & \\ Z \arrow[r] \arrow[drr, bend right, "\mathsf{Gph}(z)"'] & Z +_C (C \times \mathrm{I}) \arrow[dr, dotted, "c\otimes_z\delta"'] & \\ & & Z \times \mathrm{I}\,. \end{tikzcd} \tag{15}$$

This specification differs from the similar (6) by using $\mathsf{Gph}(z)$ for the inclusion $Z \rightarrowtail Z \times \mathrm{I}$, rather than one of the "face maps" associated to the endpoint inclusions $\delta_\epsilon : 1 \longrightarrow \mathrm{I}$. (Note that a graph is always a cofibration by pulling back a diagonal.) The subobject $c \otimes_z \delta \rightarrowtail Z \times \mathrm{I}$ is the join of the subobjects $\mathsf{Gph}(z) \rightarrowtail Z \times \mathrm{I}$ and the cylinder $C \times \mathrm{I} \rightarrowtail Z \times \mathrm{I}$.

The maps of the form $c \otimes_z \delta : Z +_C (C \times \mathrm{I}) \rightarrowtail Z$ now form a *class of generating trivial cofibrations* in the expected sense: Let

$$\mathcal{C} \otimes \delta \;=\; \{ c \otimes_z \delta : D \rightarrowtail Z \times \mathrm{I} \mid c : C \rightarrowtail Z, z : Z \longrightarrow \mathrm{I} \}\,,$$

then the fibrations are exactly the right class $(\mathcal{C} \otimes \delta)^{\pitchfork}$ of these.

Proposition 11. *A map $f : Y \longrightarrow X$ is an unbiased fibration iff for every pair of maps $c : C \rightarrowtail Z$ and $z : Z \longrightarrow \mathrm{I}$, where the former is a cofibration,* *every commutative square of the following form has a diagonal filler, as indicated.*

$$\begin{tikzcd} Z +_C (C \times \mathrm{I}) \arrow[r] \arrow[d, tail, "c \otimes_z \delta"'] & Y \arrow[d, "f"] \\ Z \times \mathrm{I} \arrow[r] \arrow[ur, dotted, "j"'] & X. \end{tikzcd} \tag{16}$$

Proof. ☐

Unbiased fibration structure. As in the biased case, the fibrations can also be determined by right-lifting against a generating *set* of trivial cofibrations, now consisting of all those $c \otimes_z \delta$ in $\mathcal{C} \otimes \delta$ for which $c : C \rightarrowtail Z$ has a representable codomain $Z = \mathrm{I}^n$. Call these maps the *basic (unbiased) trivial cofibrations*, and let

$$\mathcal{B} \otimes \delta = \{c \otimes_z \delta : B \rightarrowtail \mathrm{I}^{n+1} \mid n \geq 0\,, c : C \rightarrowtail \mathrm{I}^n,\, z : \mathrm{I}^n \longrightarrow \mathrm{I}\}\,, \tag{17}$$

where the pushout-product $c \otimes_z \delta$ now has the form

$$\begin{tikzcd} C \arrow[r, "\mathsf{Gph}(zc)"] \arrow[d, tail, "c"'] & C \times \mathrm{I} \arrow[d] \arrow[ddr, bend left, "c \times \mathrm{I}"] & \\ \mathrm{I}^n \arrow[r] \arrow[drr, bend right, "\mathsf{Gph}(z)"'] & \mathrm{I}^n +_C (C \times \mathrm{I}) \arrow[dr, dotted, tail, "c \otimes_z \delta"'] & \\ & & \mathrm{I}^n \times \mathrm{I}\,. \end{tikzcd} \tag{18}$$

for a cofibration $c : C \rightarrowtail \mathrm{I}^n$, a indexing map $z : \mathrm{I}^n \longrightarrow \mathrm{I}$, and with domain $B = \big(\mathrm{I}^n +_C (C \times \mathrm{I})\big)$. These subobjects $B \rightarrowtail \mathrm{I}^{n+1}$ can again be seen geometrically as "generalized open box" inclusions, but now the floor or lid of the open box may be replaced by a cross-section, given by the graph of a map $z : \mathrm{I}^n \longrightarrow \mathrm{I}$.

For any map $f : Y \longrightarrow X$ a (uniform, unbiased) fibration structure on f is a choice of diagonal fillers $j(c, \epsilon)$,

$$\begin{tikzcd} \mathrm{I}^n +_C (C \times \mathrm{I}) \arrow[r] \arrow[d, "c \otimes_z \delta"'] & X \arrow[d, "f"] \\ \mathrm{I}^n \times \mathrm{I} \arrow[r] \arrow[ur, dotted, "j(c,z)"'] & Y, \end{tikzcd} \tag{19}$$

for each basic trivial cofibration $c \otimes_z \delta : B \rightarrowtail \mathrm{I}^{n+1}$, which is *uniform* in I^n in the following sense: given any cubical map $u : \mathrm{I}^m \longrightarrow \mathrm{I}^n$, the pullback

$u^*c : u^*C \rightarrowtail \mathrm{I}^m$ and the reindexing $zu : \mathrm{I}^m \longrightarrow \mathrm{I}^n \longrightarrow \mathrm{I}$ determine another basic trivial cofibration $u^*c \otimes_{zu} \delta : B' = (\mathrm{I}^m +_{u^*C} (u^*C \times \mathrm{I})) \rightarrowtail \mathrm{I}^{m+1}$ which fits into a commutative diagram of the form

$$\begin{tikzcd}
\mathrm{I}^m +_{u^*C} (u^*C \times \mathrm{I}) \arrow[r] \arrow[d, "u^*c \otimes_{zu} \delta"'] & \mathrm{I}^n +_C (C \times \mathrm{I}) \arrow[r] \arrow[d, "c \otimes_z \delta"'] & X \arrow[d, "f"] \\
\mathrm{I}^m \times \mathrm{I} \arrow[r, "u \times \mathrm{I}"'] & \mathrm{I}^n \times \mathrm{I} \arrow[r] \arrow[ur, dotted, "j(c,z)"'] & Y.
\end{tikzcd} \tag{20}$$

For the outer rectangle in (20) there is a chosen diagonal filler $j(u^*c, zu) : \mathrm{I}^m \times \mathrm{I} \longrightarrow X$, and for this map we require that

$$j(u^*c, zu) = j(c, z) \circ (u \times \mathrm{I}). \tag{21}$$

Definition 12. A *(uniform, unbiased) fibration structure* on a map

$$f : Y \longrightarrow X$$

is a choice of fillers $j(c, z)$ as in (19) satisfying (21) for all $u : \mathrm{I}^m \longrightarrow \mathrm{I}^n$.

In these terms, we have following analogue of Proposition 6.

Proposition 13. *For any object X in* cSet *the following are equivalent:*

1. *the canonical map $X^{\mathrm{I}} \times \mathrm{I} \longrightarrow X \times \mathrm{I}$ is a* +-algebra*.*
2. *X has the right lifting property with respect to all generating trivial cofibrations,*
$$(\mathcal{C} \otimes \delta) \pitchfork X.$$
3. *X has a* uniform fibration structure *in the sense of Definition 12.*

Proof. Yoneda. $\square$

A statement analogous to Proposition 13 also holds for maps $f : Y \longrightarrow X$ in place of objects X. Indeed, as before, we have the following sharper formulation.

Corollary 14. *Fibration structures on a map $f : Y \longrightarrow X$ correspond uniquely to +-algebra structures on the map $(\delta \Rightarrow f)$.*

6 The fibration weak factorization system

Summarizing the foregoing definitions and results, we have the following classes of maps:

- The set of *basic trivial cofibrations* was determined to be

$$\mathcal{B} \otimes \delta = \{ c \otimes_z \delta : B \rightarrowtail \mathrm{I}^{n+1} \mid n \geq 0 \,, c : C \rightarrowtail \mathrm{I}^n,\ z : \mathrm{I}^n \longrightarrow \mathrm{I} \} \,, \quad (22)$$

  where the pushout-product $c \otimes_z \delta$ has the form

$$\begin{tikzcd} C \arrow[r, "\mathsf{Gph}(zc)"] \arrow[d, tail, "c"'] & C \times \mathrm{I} \arrow[d] \arrow[ddr, bend left, "c\times \mathrm{I}"] & \\ \mathrm{I}^n \arrow[r] \arrow[drr, bend right, "\mathsf{Gph}(z)"'] & \mathrm{I}^n +_C (C \times \mathrm{I}) \arrow[dr, dotted, "c \otimes_z \delta"'] & \\ & & \mathrm{I}^n \times \mathrm{I} \,. \end{tikzcd} \quad (23)$$

  for any basic cofibration $c : C \rightarrowtail \mathrm{I}^n$ and indexing map $z : \mathrm{I}^n \longrightarrow \mathrm{I}$, with domain $B = \big(\mathrm{I}^n +_C (C \times \mathrm{I})\big)$.

- The class $\mathcal{F}$ of *fibrations*, written $f : Y \longrightarrow\!\!\!\!\!\rightarrow X$, may be characterized as the right class of these,

$$(\mathcal{B} \otimes \delta)^{\pitchfork} = \mathcal{F}.$$

- The class of *trivial cofibrations* is then defined to be left class of the fibrations,

$$\mathsf{TrivCof} = {}^{\pitchfork}\mathcal{F}.$$

It follows that the classes $\mathsf{TrivCof}$ and $\mathcal{F}$ are mutually weakly orthogonal,

$$\mathsf{TrivCof} \pitchfork \mathcal{F},$$

and are both closed under retracts, so it just remains to show that every map $f : X \longrightarrow Y$ can be factored as $f = g \circ h$ with $g \in \mathcal{F}$ and $h \in \mathsf{TrivCof}$.

Proposition 15. *Every map $f : X \longrightarrow Y$ in cSet can be factored as $f = g \circ h$,*

$$\begin{tikzcd} X \arrow[r, tail, "h"] \arrow[dr, "f"'] & X' \arrow[d, two heads, "g"] \\ & Y \end{tikzcd} \quad (24)$$

with $h : X \rightarrowtail X'$ a trivial cofibration and $g : X' \longrightarrow\!\!\!\!\!\rightarrow Y$ a fibration.

Proof. This is a standard argument (cf. [**?**, **?**]), which can be simplified a bit in this particular case. We sketch the proof for the case $Y = 1$; the general case is not essentially different.

Thus let X be any object, and we wish to find a fibrant object X' and a trivial cofibration $h : X \rightarrowtail X'$. For each basic trivial cofibration $\beta : B \rightarrowtail \mathrm{I}^k$, we need to solve all extension problems of the form

$$\begin{tikzcd} B \arrow[r, "x"] \arrow[d, tail, "\beta"'] & X. \\ \mathrm{I}^k \arrow[ur, dotted] & \end{tikzcd} \tag{25}$$

We first combine these into a single problem by taking a coproduct over all maps $x : B \to X$,

$$\begin{tikzcd} \coprod_{x:B\to X} B \arrow[r, "{[x]}"] \arrow[d, tail, "\coprod_{x:B\to X} \beta"'] & X. \\ \coprod_{x:B\to X} \mathrm{I}^k \arrow[ur, dotted] & \end{tikzcd}$$

We then take the coproduct over all basic trivial cofibrations $\beta : B \rightarrowtail \mathrm{I}^k$,

$$\begin{tikzcd} \coprod_{\beta:B\rightarrowtail\mathrm{I}^k} \coprod_{x:B\to X} B \arrow[r, "{[[x]_\beta]}"] \arrow[d, tail, "\coprod_{\beta:B\rightarrowtail\mathrm{I}^k} \coprod_{x:B\to X} \beta"'] & X. \\ \coprod_{\beta:B\rightarrowtail\mathrm{I}^k} \coprod_{x:B\to X} \mathrm{I}^k \arrow[ur, dotted] & \end{tikzcd}$$

Note that a coproduct of trivial cofibrations is clearly a trivial cofibration.

Taking a pushout, the indicated map h_1 is then also a trivial cofibration because it is a pushout of one

$$\begin{tikzcd}
\coprod\limits_{\beta:B\rightarrowtail \mathrm{I}^k}\coprod\limits_{x:B\to X} B \arrow[rr, "{[[x]_\beta]}"] \arrow[d, tail, "\coprod\limits_{\beta:B\rightarrowtail \mathrm{I}^k}\coprod\limits_{x:B\to X}\beta"'] & & X. \arrow[d, tail, "h_1"] \\
\coprod\limits_{\beta:B\rightarrowtail \mathrm{I}^k}\coprod\limits_{x:B\to X} \mathrm{I}^k \arrow[rr] & & X_1 \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

Now iterate the construction to get a sequence of trivial cofibrations, of which we take X' to be the colimit and $h : X \longrightarrow X'$ the canonical map,

$$h : X \underset{h_1}{\rightarrowtail} X_1 \underset{h_2}{\rightarrowtail} X_2 \underset{h_3}{\rightarrowtail} \ldots \rightarrowtail \varinjlim X_n = X'. \tag{26}$$

To show that X' is fibrant, consider an extension problem of the form (25) with X' in place of X,

$$\begin{tikzcd}
B \arrow[r, "x"] \arrow[d, tail, "\beta"'] & \varinjlim X_n \\
\mathrm{I}^k \arrow[ur, dotted]
\end{tikzcd}$$

The subobject $B \rightarrowtail \mathrm{I}^k$ has as domain an object B that is a *finite* colimit of maps $\mathrm{I}^m \to \mathrm{I}^n$ of representables (as can be seen by considering sieves in the category of cubes), and is therefore finitely presented, in the sense that mapping out of it preserves filtered colimits. Thus the map $x : B \longrightarrow \varinjlim X_n$ must factor through some $x_k : B \longrightarrow X_k$, giving rise to the problem

$$\begin{tikzcd}
B \arrow[r, "x_k"] \arrow[d, tail, "\beta"'] & X_k \arrow[d] \\
\mathrm{I}^k \arrow[r, dotted] & \varinjlim X_n.
\end{tikzcd}$$

But this has a solution in the next step, by the construction of X_{k+1},

$$\begin{tikzcd}
B \arrow[r, "x_k"] \arrow[d, tail, "\beta"'] & X_k \arrow[d, "h_{k+1}"] \\
\mathrm{I}^k \arrow[r] \arrow[dr, dotted] & X_{k+1} \arrow[d] \\
& \varinjlim X_n .
\end{tikzcd}$$

Thus X' is fibrant. Finally, $h : X \rightarrowtail X'$ is clearly a trivial cofibration, as a sequential colimit of such. $\square$

We therefore have shown the following.

Proposition 16. *There is a weak factorization system on the category* cSet *in which the right maps are the fibrations and the left maps are the trivial cofibrations.*

This will be called the *fibration weak factorization system.* The following observation, which is easily proved, will be of use later on.

Corollary 17. *The construction given in* (26) *of the* fibrant replacement*,*

$$X' = \varinjlim_n X_n$$

is functorial in X*, and the canonical trivial cofibrations* $h : X \rightarrowtail X'$ *are natural in* X*.*

7 Weak equivalences

Definition 18 (Weak equivalence)**.** A map $f : X \longrightarrow Y$ in cSet will be called a *weak equivalence* if can be factored as $f = g \circ h$,

$$\begin{tikzcd}
X \arrow[r, "h"] \arrow[dr, "f"'] & W \arrow[d, "g"] \\
& Y
\end{tikzcd}$$

with $h : X \to W$ a trivial cofibration and $g : W \to Y$ a trivial fibration, i.e. a right map in the cofibration weak factorization system. Let

$$\mathcal{W} = \{f : X \longrightarrow Y \mid\ f = g \circ h \text{ for } g \in \mathsf{TrivFib} \text{ and } h \in \mathsf{TrivCof}\}$$

be the class of weak equivalences.

Now observe that every trivial fibration $f \in \mathcal{C}^{\pitchfork}$ is indeed a fibration, because the basic trivial cofibrations are cofibrations; moreover, every trivial fibration is also a weak equivalence, since the identity maps are trivial cofibrations. Thus we have

$$\mathsf{TrivFib} \subseteq (\mathcal{F} \cap \mathcal{W}).$$

Dually, every trivial cofibration $g \in {}^{\pitchfork}\mathcal{F}$ is a cofibration, because the trivial fibrations are fibrations; moreover, every trivial cofibration is also a weak equivalence, since the identity maps are trivial fibrations. Thus we also have

$$\mathsf{TrivCof} \subseteq (\mathcal{C} \cap \mathcal{W}).$$

Lemma 19. $(\mathcal{C} \cap \mathcal{W}) \subseteq \mathsf{TrivCof}$.

Proof. Let $c : A \rightarrowtail B$ be a cofibration with a factorization $c = tf \circ tc : A \longrightarrow W \longrightarrow B$ where $tc \in \mathsf{TrivCof}$ and $tf \in \mathsf{TrivFib}$. Let $f : Y \longrightarrow\!\!\!\!\rightarrow X$ be a fibration and consider a commutative diagram,

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, tail, "c"'] & Y \arrow[d, two heads, "f"] \\
B \arrow[r, "b"'] & X.
\end{tikzcd}$$

Inserting the factorization of c, we have $j : W \longrightarrow Y$ as indicated, with $j \circ tc = a$ and $f \circ j = b \circ tf$, since $tc \pitchfork f$.

$$\begin{tikzcd}
A \arrow[rr, "a"] \arrow[dd, tail, "c"'] \arrow[dr, "tc"'] & & Y \arrow[dd, two heads, "f"] \\
& W \arrow[ur, dotted, "j"'] \arrow[dl, "tf"'] & \\
B \arrow[rr, "b"'] & & X.
\end{tikzcd}$$

Moreover, since $c \pitchfork tf$ there is an $i : B \to W$ as indicated, with $i \circ c = tc$ and $tf \circ i = 1_B$.

$$\begin{tikzcd}
A \arrow[rr, "a"] \arrow[dd, tail, "c"'] \arrow[dr, "tc"'] & & Y \arrow[dd, two heads, "f"] \\
& W \arrow[ur, dotted, "j"'] \arrow[dl, "tf"'] & \\
B \arrow[rr, "b"'] \arrow[ur, dotted, bend right, "i"'] & & X.
\end{tikzcd}$$

Let $k = j \circ i$. Then $k \circ c = j \circ i \circ c = j \circ tc = a$, and $f \circ k = f \circ j \circ i = b \circ tf \circ i = b$. $\square$

The proof of the following is dual:

Lemma 20. $(\mathcal{F} \cap \mathcal{W}) \subseteq \mathsf{TrivFib}$.

Proposition 21. *For the three classes of maps $\mathcal{C}, \mathcal{W}, \mathcal{F}$ in cSet, we have*

$$\mathcal{F} \cap \mathcal{W} = \mathsf{TrivFib},$$
$$\mathcal{C} \cap \mathcal{W} = \mathsf{TrivCof},$$

and therefore two weak factorization systems:

$$(\mathcal{C},\, \mathcal{W} \cap \mathcal{F}) \quad , \quad (\mathcal{C} \cap \mathcal{W},\, \mathcal{F}).$$

It thus remains "only" to prove that the weak equivalences satisfy the 3-for-2 property.

Weak homotopy equivalence Recall that a *homotopy* between parallel maps $f, g : X \rightrightarrows Y$, written $\vartheta : f \sim g$, is a map from the cylinder,

$$\vartheta : \mathrm{I} \times X \longrightarrow Y,$$

such that $\vartheta \circ \mathsf{Gph}(\delta_0!) = f$ and $\vartheta \circ \mathsf{Gph}(\delta_1!) = g$.

Proposition 22. *If K is fibrant, then the relation of homotopy $f \sim g$ between maps $f, g : X \rightrightarrows K$ is an equivalence relation. Moreover, it is compatible with composition.*

Definition 23 (Connected components)**.** The functor

$$\pi_0 : \mathsf{cSet} \longrightarrow \mathsf{Set}$$

is the left adjoint of the constant presheaf functor $\Delta : \mathsf{Set} \to \mathsf{cSet}$. For any cubical set X we have a coequalizer $X_1 \rightrightarrows X_0 \to \pi_0 X$, where the two parallel arrows are the endpoints of the 1-cubes X_1 of X. Thus for any Kan complex K we have $\pi_0 K = \mathrm{Hom}(1, K)/\!\sim$, i.e. $\pi_0 K$ is the set of points $1 \to K$, modulo the homotopy equivalence relation on them.

Recall that a map $f : X \to Y$ in cSet is called a *homotopy equivalence* if there is a *quasi-inverse* $g : Y \to X$ and homotopies $\vartheta : 1_X \sim g \circ f$ and $\varphi : 1_Y \sim f \circ g$.

Definition 24 (Weak homotopy equivalence)**.** A map $f : X \to Y$ will be called a *weak homotopy equivalence* if for every fibrant object K, the "internal precomposition" map $K^f : K^Y \to K^X$ is bijective on connected components, i.e.

$$\pi_0 K^f : \pi_0 K^Y \longrightarrow \pi_0 K^X$$

is a bijection of sets.

Corollary 25. *The weak homotopy equivalences $f : X \to Y$ satisfy the 3-for-2 condition.*

Thus our goal of showing that the weak equivalences satisfy 3-for-2 is reduced to the following:

Proposition 26. *A map $f : X \to Y$ is a weak equivalence if and only if it is a weak homotopy equivalence.*

For the proof, in addition to several lemmas, we will require the following proposition, the proof of which is deferred to the next section.

Proposition 27. *Every fibration is a pullback of one over a fibrant object. More precisely, for every fibration $f : Y \twoheadrightarrow X$ there exists a fibrant object K, a map $X \longrightarrow K$, and a fibration $p : Z \twoheadrightarrow K$ fitting into a pullback square,*

$$\begin{tikzcd} Y \arrow[r] \arrow[d, two heads, "f"'] & Z \arrow[d, two heads, "p"] \\ X \arrow[r] & K. \end{tikzcd}$$

This proposition is a consequence of several stronger ones, e.g. the equivalence extension property of [**?**], but we will prefer to prove it directly.