A model structure on the cartesian cubical sets

### 1 The cartesian cube category

We consider the cartesian cube category  $\mathbb{C}$ , defined as the free finite product category on an interval  $\delta_0, \delta_1 : 1 \rightrightarrows I$ . As a classifying category for an algebraic theory  $\mathbb{T} = \{0,1\}$ ,  $\mathbb{C}$  has a covariant presentation by Lawvere duality, namely as the dual of the full subcategory of finitely-generated, free  $\mathbb{T}$ -algebras  $\mathsf{Alg}(\mathbb{T})_{\mathrm{fg}}$ . In this case, the algebras are simply bipointed sets  $(A, a_0, a_1)$ , and the free ones are the strictly bipointed sets  $a_0 \neq a_1$ . Thus  $\mathsf{Alg}(\mathbb{T})_{\mathrm{fg}}$  consists of the finite, strictly bipointed sets and all bipointed maps between them.

**Definition 1.** The objects of the cartesian cube category  $\mathbb C$  are themselves called cubes, and will be written

$$[n] = \{x_1, ..., x_n\},\$$

where the  $x_i$  may be regarded as coordinate axes. The arrows,

$$f: [n] \longrightarrow [m],$$

are then taken to be m-tuples of elements drawn from the set

$$[n]^+ = \{0, x_1, ..., x_n, 1\},\$$

regarded as formal terms representing composites of faces, degeneracies, permutations, and diagonals (alternately, the arrows are arbitrary bipointed maps  $[m]^+ \longrightarrow [n]^+$ ).

#### 2 Cubical sets

The category cSet of *cubical sets* is the category of presheaves on the cartesian cube category  $\mathbb{C}$ . It is generated by the representable presheaves y([n]), which will be written  $I^n = y([n])$  and called the *standard n-cubes*.

# 3 Partial map classification and the +-algebra weak factorization system

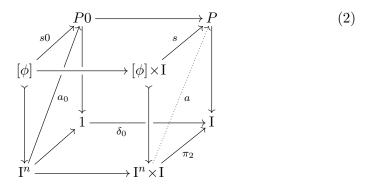
Cofibrations, partial map classification, the functor  $X^+$ , the awfs of +-algebras.

### 4 Partial path lifting (biased version)

Given a type  $P: I \longrightarrow Set$ , the type of (0-biased) partial path-lifting structures  $L^0(P)$  may be defined in the "logical style" of [?] as:

$$L^{0}(P) = \prod_{\phi:\Phi} \prod_{s:\prod_{i:I}(Pi)^{\phi}} \prod_{a_{0}:P0} a_{0}|\phi = s0 \longrightarrow \sum_{a:\prod_{i:I}Pi} (a0 = a_{0}) \times (a|\phi = s).$$
(1)

The data involved in this type can be represented as follows:



Here the left-hand vertical square is understood to be a pullback of the right-hand one along the chosen endpoint  $\delta_0: 1 \longrightarrow I$  (the "bias").

Now write

$$\widetilde{P} = \prod_{i:I} Pi$$

for the type of sections of the projection  $P = \sum_{i:I} Pi \longrightarrow I$ , and write

$$\pi_0: \widetilde{P} \longrightarrow P0$$

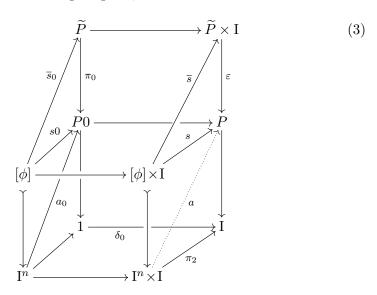
for the  $0^{th}$ -projection (i.e. the evaluation of  $P: I \longrightarrow \mathsf{Set}$  at 0: I).

Then the (0-biased) partial path-lifting structures on P correspond to +-algebra structures on the projection  $\pi_0: \widetilde{P} \longrightarrow P0$ , as follows.

**Proposition 2.** For any  $P : Set^{I}$ , there is an isomorphism

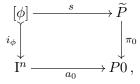
$$L^0(P) \cong {}^+ Alg(\pi_0 : \widetilde{P} \longrightarrow P0)$$
.

*Proof.* Consider the following diagram,

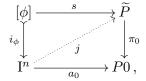


which is (2), extended by the counit (evaluation)  $\varepsilon: \widetilde{P} \times I \longrightarrow P$  over I on the right. The pullback of  $\varepsilon$  over I along  $\delta_0$  is just  $\pi_0: \widetilde{P} \longrightarrow P0$ .

Given an  $L^0(P)$ -structure we construct a +Alg-structure on  $\pi_0: \widetilde{P} \longrightarrow P0$  as follows: for any  $I^n$  and cofibration  $i_{\phi}: [\phi] \longrightarrow I^n$  and any commutative square,



we require a diagonal filler,



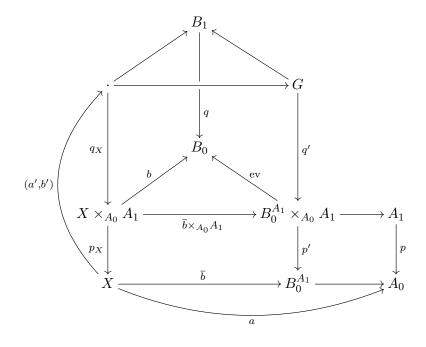
uniformly in  $I^n$  and  $\phi$ . Transposing the span formed by  $i_{\phi}$  and s along the adjunction  $I^* \dashv \prod_I$  gives the right-hand square in (??), and the commutative square formed by  $a_0$  and  $\pi_0$  gives the rest of the data in that diagram. Thus the  $L^0(P)$ -structure gives an  $a: I^n \times I \longrightarrow P$  as indicated. Looking at (3), we see that a lifts across  $\varepsilon$  to a unique map  $\overline{a}: I^n \times I \longrightarrow \widetilde{P} \times I$  over I, by the universal property of  $\varepsilon: \widetilde{P} \times I \longrightarrow P$ . We can therefore set

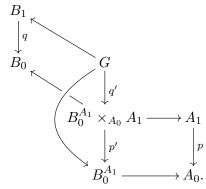
$$j = \delta_0^*(\overline{a}) : \mathbf{I}^n \longrightarrow \widetilde{P}$$
.

Suppose conversely that we have a  ${}^{+}$ Alg-structure on  $\pi_0: \widetilde{P} \longrightarrow P0$ , and we want to build a (0-biased) partial path-lifting structure on P. Take any  $I^n, \phi, s, a_0$  as indicated and we require an  $a: I^n \times I \longrightarrow P$  over I. From s we get  $\overline{s}$  by the universal property of  $\varepsilon$ , and therefore we get  $\overline{s}_0$  by pullback. From  $\overline{s}_0$  and  $a_0$  and the  ${}^{+}$ Alg structure on  $\pi_0$  we get a map  $j: I^n \longrightarrow \widetilde{P}$  over P0 which is a diagonal filler of the indicated square formed by  $i_{\phi}, \overline{s}_0, a_0$  and  $\pi_0$ . We then get the required map  $a: I^n \times I \longrightarrow P$  over I as the  $(I^* \dashv \prod_I)$ -transpose of j,

$$a = \varepsilon \circ (i \times I)$$
.

We leave to the reader the verification that these assignments are mutually inverse.  $\Box$ 





## 5 Unbiased partial path lifting

# 6 A left-induced model structure on the Cartesian cubical sets

We make use of the Sattler model structure [?] on the *Dedekind cubical* sets  $\widehat{\mathbb{D}} = \mathsf{Set}^{\mathbb{D}^{\mathrm{op}}}$ , where  $\mathbb{D}$  is the category of *Dedekind cubes*, defined as

the Lawvere theory of distributive lattices. The unique product-preserving functor

$$i: \mathbb{C} \longrightarrow \mathbb{D}$$

classifying the Dedekind interval  $I_{\mathbb{D}} \in \mathbb{D}$  induces an adjunction,

$$i_! \dashv i^* \dashv i_* : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{D}},$$

where  $i^*(Q) = Q \circ i$ , for  $Q \in \mathbb{D}$ .

**Lemma 3.** Observe that  $i_!$  is left exact since the Dedekind interval  $I_{\mathbb{D}}$  is strict,  $0 \neq 1 : 1 \rightrightarrows I_{\mathbb{D}}$ . Thus we have geometric morphisms:

$$(i_! \dashv i^*): \widehat{\mathbb{D}} \longrightarrow \widehat{\mathbb{C}},$$

classifying the bipointed object  $i_!(I_{\mathbb{C}}) = I_{\mathbb{D}}$ ,

$$(i^* \dashv i_*): \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{D}},$$

classifying the dLat  $i^*(I_{\mathbb{D}}) := \mathbb{I}$ , where  $\eta : I_{\mathbb{C}} \longrightarrow \mathbb{I}$  can be described pointwise as the distributive lattice completion of the corresponding bipointed set.

Also, since i is faithful so is  $i_!$ , and since i is surjective on objects  $i^*$  is also faithful.

It follows that:

- $\widehat{\mathbb{C}}$  is  $(i_! \circ i^*)$ -coalgebras on  $\widehat{\mathbb{D}}$ ,
- $\widehat{\mathbb{D}}$  is  $(i^* \circ i_*)$ -coalgebras on  $\widehat{\mathbb{C}}$ ,
- $\widehat{\mathbb{D}}$  is  $(i^* \circ i_!)$ -algebras on  $\widehat{\mathbb{C}}$ .

We will use the following transfer theorem for QMSs from  $\cite{MSs}$  from  $\cite{M$ 

**Theorem** ([?, ?]). Suppose  $\widehat{\mathbb{D}}$  has a (cofibrantly generated) model structure  $(\mathcal{C}_{\mathbb{D}}, \mathcal{W}_{\mathbb{D}}, \mathcal{F}_{\mathbb{D}})$ . Given an adjunction

$$i_! \dashv i^* : \widehat{\mathbb{D}} \longrightarrow \widehat{\mathbb{C}},$$

there is a left-induced model structure on  $\widehat{\mathbb{C}}$  if the following acyclicity condition holds:

$$(i_!^{-1}\mathcal{C}_{\mathbb{D}})^{\pitchfork} \subset i_!^{-1}\mathcal{W}_{\mathbb{D}}.$$

For the left-induced model structure  $(\mathcal{C}_{\mathbb{C}}, \mathcal{W}_{\mathbb{C}}, \mathcal{F}_{\mathbb{C}})$  on  $\widehat{\mathbb{C}}$  we then have:

$$\mathcal{C}_{\mathbb{C}} = i_{!}^{-1} \mathcal{C}_{\mathbb{D}},$$
 $\mathcal{W}_{\mathbb{C}} = i_{!}^{-1} \mathcal{W}_{\mathbb{D}}.$ 

The Sattler model structure on  $\widehat{\mathbb{D}}$  is given as follows (for a constructive treatment a smaller class of "pointwise decidable cofibrations" is used, but we consider the classical case first):

where  $\delta: 1 \longrightarrow I$  is either endpoint inclusion.

For the left-induced model structure on  $\widehat{\mathbb{C}}$  we therefore have the following specification:

$$\mathcal{C} = \text{monomorphisms},$$

$$\mathcal{W} = \left\{ f \mid i_! f = p \circ i, \ p \in \mathcal{F}_{\mathbb{D}} \cap \mathcal{W}_{\mathbb{D}}, \ i \in \mathcal{C}_{\mathbb{D}} \cap \mathcal{W}_{\mathbb{D}} \right\},$$

$$\mathcal{F} = (\mathcal{C} \cap \mathcal{W})^{\pitchfork}.$$

The determination of  $\mathcal{C}$  follows from the fact that  $i_!: \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{D}}$  is conservative. To check the acyclicity condition,

$$(i_!^{-1}\mathcal{C}_{\mathbb{D}})^{\pitchfork} \subset i_!^{-1}\mathcal{W}_{\mathbb{D}},$$

we know that  $i_!^{-1}\mathcal{C}_{\mathbb{D}}$  consists of the monos in  $\mathbb{C}$ , so take  $f: Y \longrightarrow X$  in  $(i_!^{-1}\mathcal{C}_{\mathbb{D}})^{\pitchfork}$ , apply  $i_!$ , and factor the result as  $i_!f = p \circ m: i_!Y \longrightarrow Z \longrightarrow i_!X$  with  $p \in \mathcal{F}_{\mathbb{D}} \cap \mathcal{W}_{\mathbb{D}}$  and  $m: i_!Y \longrightarrow Z$  monic. We then need to show that m is in  $\mathcal{W}_{\mathbb{D}}$ .

We can apply Theorem 2.2.1 of [?], with  $K = \widehat{\mathbb{C}}$ ,  $M = \widehat{\mathbb{D}}$ ,  $V = i_!$ ,  $k = i^*$ , and:

- 1. QX = X and  $\epsilon = 1_X : X \longrightarrow X$ , so that  $i_! 1_X = 1_{i_!X}$  and therefore in  $\mathcal{W}_{\mathbb{D}}$ , while all objects are cofibrant,
- 2. Qf = f for any  $f: X \longrightarrow Y$  in  $\widehat{\mathbb{C}}$ , so that the naturality condition is similarly trivial,
- 3. factor the codiagonal  $X + X \longrightarrow X$  as  $\pi_2 \circ j : X + X \longrightarrow I \times X \longrightarrow X$  with  $j = (\partial I \times X) : X + X \longrightarrow I \times X$ .

It remains only to show that  $i_!p: i_!(I \times X) \longrightarrow i_!X$  is in  $\mathcal{W}_{\mathbb{D}}$  and  $i_!j: i_!(X+X) \longrightarrow i_!(I \times X)$  is in  $\mathcal{C}_{\mathbb{D}}$ . The latter is clear, since j is monic. To show the former, observe that for any  $D \in \widehat{\mathbb{D}}$ , the projection  $\pi_2: I_{\mathbb{D}} \times D \longrightarrow D$  is in  $\mathcal{W}_{\mathbb{D}}$  by 3-for-2, since an endpoint inclusion  $D \longrightarrow I_{\mathbb{D}} \times D$  is a cofibration and a strong deformation retract, hence in  $\mathcal{W}_{\mathbb{D}}$ .

Thus we have shown:

**Theorem 4.** There is a Quillen model structure  $(C, W, \mathcal{F})$  on the category  $\widehat{\mathbb{C}}$  of cartesian cubical sets, in which

$$\mathcal{C} = monomorphisms,$$

$$\mathcal{W} = \{ f \mid i_! f = p \circ i, \ p \in \mathcal{F}_{\mathbb{D}} \cap \mathcal{W}_{\mathbb{D}}, \ i \in \mathcal{C}_{\mathbb{D}} \cap \mathcal{W}_{\mathbb{D}} \},$$

$$\mathcal{F} = (\mathcal{C} \cap \mathcal{W})^{\pitchfork}.$$

where  $i_!: \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{D}}$  is the left adjoint of precomposition along the canonical map  $i: \mathbb{C} \longrightarrow \mathbb{D}$  from Cartesian cubes to Dedekind cubes, and  $(\mathcal{C}_{\mathbb{D}}, \mathcal{W}_{\mathbb{D}}, \mathcal{F}_{\mathbb{D}})$  is the Sattler model structure on  $\widehat{\mathbb{D}}$ .

#### References:

- Gambino-Sattler
- Sattler
- Hess, Kedziorek, Riehl, Shipley
- Garner, Kedziorek, Riehl