ALGEBRAIC TYPE THEORY
PART 1: MARTIN-LÖF ALGEBRAS

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In memory of Phil Scott, mentor and friend.

ABSTRACT. A new algebraic treatment of dependent type theory is proposed using ideas derived from topos theory and algebraic set theory.

One of the most beautiful aspects of the book *Introduction to Higher-Order Categorical Logic* by Lambek and Scott is the almost entirely algebraic treatment of higher-order logic, with operations and equations in place of the traditional presentation of logic by rules of inference. This is made possible in a general way by F.W. Lawvere's profound analysis of all of the logical primitives as adjoints [**?**], but more specifically by the presence in a topos $\mathcal{E}$ of a subobject classifier Ω that represents the presheaf of subobjects via a natural isomorphism of sets,

$$\mathsf{Sub}(X) \cong \mathsf{Hom}_{\mathcal{E}}(X, \Omega)\,.$$

The logical operations on subobjects $\{x \mid \varphi(x)\} \rightarrowtail X$, represented by formulas $\varphi(x) : X \to \Omega$, are themselves represented by operations on Ω, such as conjunction $\wedge : \Omega \times \Omega \to \Omega$. Thus given two "propositional functions" $\varphi(x), \psi(x) : X \to \Omega$ we obtain the meet of their subobjects $\{x \mid \varphi(x)\} \cap \{x \mid \psi(x)\}$ from the conjunction $\varphi(x) \wedge \psi(x)$ as

$$\{x \mid \varphi(x)\} \cap \{x \mid \psi(x)\} = \{x \mid \varphi(x) \wedge \psi(x)\}\,.$$

The conjunction arises simply by (pairing and) composing:

$$\begin{tikzcd}
X \arrow[r, "\langle\varphi(x),\psi(x)\rangle"] \arrow[dr, "\varphi(x)\wedge\psi(x)"'] & \Omega \times \Omega \arrow[d, "\wedge"] \\
& \Omega
\end{tikzcd}$$

It follows immediately that the operation $\varphi(x)[t(y)/x] = \varphi(t(y))$ of substitution of a term $t(y) : Y \to X$ for the variable x necessarily

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respects conjunction, just by the associativity of composition:

$$\begin{aligned}
(\varphi(x) \wedge \psi(x))[t(y)/x] &= (\wedge \circ \langle \varphi(x), \psi(x) \rangle) \circ t(y) \\
&= \wedge \circ (\langle \varphi(x), \psi(x) \rangle \circ t(y)) \\
&= \wedge \circ \langle \varphi(x) \circ t(y), \psi(x) \circ t(y) \rangle \\
&= \wedge \circ \langle \varphi(t(y)), \psi(t(y)) \rangle \\
&= \varphi(t(y)) \wedge \psi(t(y))
\end{aligned}$$

It therefore also follows that the corresponding meet operation $\cap$ on subobjects also respects substitution, which is interpreted by pullback of subobjects.

The same thing holds for the other propositional operations $\top$, $\bot$, $\neg$, $\vee$, $\Rightarrow$, $\Leftrightarrow$, which are also representable. Moreover, representability also implies the Beck-Chevalley condition for the quantifiers $\forall$ and $\exists$; e.g.

$$\begin{aligned}
(\forall z.\vartheta(x, z))[t(y)/x] &= (\forall z.\vartheta(x, z)) \circ t(y) \\
&= (\forall z \circ \vartheta(x, \hat{z})) \circ t(y) \\
&= \forall z \circ (\vartheta(x, \hat{z}) \circ t(y)) \\
&= \forall z \circ \vartheta(t(y), \hat{z}) \\
&= \forall z.\vartheta(t(y), z)\,,
\end{aligned}$$

in virtue of the universal quantifier $\forall z$ also being represented by a map on Ω, namely $\forall_z : \Omega^Z \to \Omega$, with which we simply compose (this time after an exponential transposition).

$$\begin{tikzcd}
X \arrow[r, "\vartheta(x,\hat{z})"] \arrow[dr, swap, "\forall z.\vartheta(x,z)"] & \Omega^Z \arrow[d, "\forall z"] \\
& \Omega
\end{tikzcd} \tag{1}$$

Again, the corresponding equations then also hold for the classified subobjects.

The general idea is that, because they are natural in the context of variables X, the logical operations on subobjects are represented by "homming in" to an algebra of propositions Ω (by Yoneda, of course). And since they are then just pointwise operations on propositional functions $\varphi(x) : X \to \Omega$, they automatically respect substitutions of terms $t(y) : Y \to X$ into the context of variables. In this way, the internal logic of a topos arises almost entirely from homming into the internal *complete Heyting algebra* Ω — combined with some basic λ-calculus, enabling higher types constructed from Ω. This, in a nut shell, is what permits the lovely algebraic formulation of even higher-order logic in [**?**].

Martin-Löf algebras. One of the motivations for the present work was to apply this same approach to *dependent type theory* in place of predicate logic, by determining a suitable algebraic gadget $\mathcal{U}$ in place of Ω, representing the *presheaf of types*, rather than the presheaf of subobjects. In fibrational terms, over the category $\mathcal{C}$ of contexts of variables, we would like a representing object $\dot{\mathcal{U}} \to \mathcal{U}$ for the *codomain fibration* $\mathcal{C}^{\downarrow} \twoheadrightarrow \mathcal{C}$, rather than the object $1 \rightarrowtail \Omega$ representing the fibration of subobjects. Unlike the discrete fibration, or presheaf $\mathsf{Sub} : \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}$, of subobjects, however, which is (at best) poset valued, the pseudofunctor of slice categories $\mathcal{C}/_{(-)} : \mathcal{C}^{\mathsf{op}} \to \mathsf{Cat}$ cannot be representable, even in the weaker sense of a natural equivalence of indexed categories,

$$\mathcal{C}/_X \simeq \mathsf{Hom}_{\mathcal{C}}(X,\mathcal{U})\,.$$

There are really two different problems here: size and coherence of structure. We solve both simultaneously by taking $\dot{\mathcal{U}} \to \mathcal{U}$ to determine a full internal subcategory (with suitable additional structure) in the category $\mathcal{E} = \widehat{\mathcal{C}}$ of presheaves over $\mathcal{C}$, splitting the codomain fibration as in [**?**]. Whatever one may think of this solution, it is of obvious interest to determine what additional algebraic structure on $\dot{\mathcal{U}} \to \mathcal{U}$ will serve to model dependent type theory. And we know how to find the answer: by Yoneda! We call the resulting gadget a *Martin-Löf algebra* ("ML-algebra" for short), and it has a remarkably simple description as an algebra for a polynomial monad, giving a complete answer to our question (Theorem **??**).

The polynomial endofunctor in question $\mathsf{P_u} : \mathcal{E} \to \mathcal{E}$ is that of the algebra $\mathsf{u} : \dot{\mathcal{U}} \to \mathcal{U}$, which therefore has the form

$$\mathsf{P_u}(X) = \Sigma_{A:\mathcal{U}} X^A\,,$$

allowing e.g. the formation rule for the Π-type to be expressed by a composition:

$$\begin{tikzcd} \Gamma \arrow[r, "{(A,B)}"] \arrow[dr, "\Pi_A B"'] & \mathsf{P_u}(\mathcal{U}) \arrow[d, "\Pi"] \\ & \mathcal{U} \end{tikzcd}$$

This ensures not only the strict Beck-Chevalley rules for the type formers, as in (**??**), but also a solution (due originally to Voevodsky) to the old bugbear of coherence in dependent type theory [**?**], since substitution $\sigma : \Delta \to \Gamma$ into types and terms $\Gamma \vdash a : A$ is now interpreted simply by composition, which unlike pullback, is strictly associative.

$$\begin{tikzcd}
 & & \dot{\mathcal{U}} \arrow[d] \\
\Delta \arrow[r, "\sigma"] \arrow[urr, bend left, "{a[\sigma]}"] \arrow[rr, bend right, "{A[\sigma]}"'] & \Gamma \arrow[r, "A"'] \arrow[ur, "a"] \arrow[r, phantom, shift left=3, "a{:}A"] & \mathcal{U}
\end{tikzcd}$$

The full rules for Π-types turn out to state exactly that $\mathsf{u} : \dot{\mathcal{U}} \to \mathcal{U}$ is an algebra for its own polynomial endofunctor $\mathsf{P}_\mathsf{u} : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}}$ lifted to the (cartasian) arrow category $\mathsf{P}^{\downarrow}_\mathsf{u} : \widehat{\mathcal{C}}^{\downarrow}_{\mathsf{cart}} \longrightarrow \widehat{\mathcal{C}}^{\downarrow}_{\mathsf{cart}}$. Even more strikingly, the type formers $1, \Sigma$ endow $\mathsf{P}_\mathsf{u} : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}}$ with the underlying structure of a polynomial monad. The monad and algebra laws are then seen to express fundamental type isomorphisms, the analysis of which requires the important Id-types (as recently reformulated in terms of polynomials by R. Garner), which forms one of the main new advances of the current work.

Algebraic type theory. There turns out to be an intriguing analogy between ML-algebras and the Zermelo-Fraenkel algebras from the *Algebraic Set Theory* of Joyal and Moerdijk [**?**]. A second motivation for the present work was to explore this analogy, which is discernible in the similarity between the universal small map $\pi : E \to B$ and an ML-algebra $\mathsf{u} : \dot{\mathcal{U}} \to \mathcal{U}$, and between a ZF-algebra structure map $P_s V \to V$ (especially in the formulation of [**?**]) and the Σ-type former $\mathsf{P}_\mathsf{u}\mathcal{U} \to \mathcal{U}$, as well as between the respective monads P_s and P_u.

Indeed, an ML-algebra is in some sense a proof-relevant version of the ZF-algebras from *op. cit.* To be sure, only the most basic aspects of this connection have been developed here. Apart from the obviously missing successor operation $s : V \to V$, one still needs to consider morphisms of ML-algebras, free and initial algebras, as well as the functor induced by a change of context $\mathcal{C} \to \mathcal{C}'$. Some of this is underway in the work in progress [**?**].

We begin in Section 1 below by recalling from [**?**] the notion of a *natural model* of dependent type theory and its relation to the *categories with families* of [**?**]. In section **??** we abstract the main features of a natural model with the type formers $1, \Sigma, \Pi$ to form the notion of a *Martin-Löf algebra*, the basic theory of which is also begun in this section with the addition of identity types. After briefly indicating the relation to the *tribes* of Joyal [**?**] we provide some examples of ML-algebras in Section **??**, including the subobject classifier Ω of a topos, and the Hofmann-Streicher universe $\dot{\mathcal{V}} \to \mathcal{V}$ in presheaves. The rest of the paper is devoted to the relationship between ML-algebras and polynomial monads, which was already considered from a somewhat different point of view in [**?**] and [**?**]. We conclude with the main new result, Theorem **??**, in Section **??**.

1. Natural models of type theory

We write $\widehat{\mathcal{C}} = [\mathcal{C}^{\mathsf{op}}, \mathsf{Set}]$ for the category of presheaves on a small category $\mathcal{C}$. In [**?**], a *natural model of type theory* is defined to be a representable natural transformation $\mathsf{u} : \dot{\mathsf{U}} \to \mathsf{U}$ of presheaves in $\widehat{\mathcal{C}}$.

Definition 1. A natural transformation $p : Y \to X$ of presheaves on a category $\mathcal{C}$ is *representable* if the pullback of p along any element $x : \mathsf{y}C \to X$ is representable.

$$\begin{tikzcd} \mathsf{y}D \arrow[r, "y"] \arrow[d, "\mathsf{y}c"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "p"] \\ \mathsf{y}C \arrow[r, "x"'] & X \end{tikzcd}$$

We may assume a choice of pullback data $c : D \to C$ in $\mathcal{C}$ and $y \in Y(D)$ for all $x \in X(C)$ (but no coherence conditions).

Proposition 2 of *op. cit.* shows that such a map is essentially the same thing as a *category with families* (CwF) in the sense of [**?**] when $\mathcal{C}$ is regarded as the category of contexts of a type theory, $\mathsf{U} : \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}$ is regarded as the presheaf of types in context, $\dot{\mathsf{U}} : \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}$ as the presheaf of terms in context, and $\mathsf{u} : \dot{\mathsf{U}} \to \mathsf{U}$ as the typing of the terms.

Proposition 2 ([**?**, **?**])**.** *A representable natural transformation is the same thing as a* category with families (CwF) *in the sense of Dybjer* [**?**].

We sketch the correspondence from [**?**]. Let us write the objects and arrows of $\mathcal{C}$ as $\sigma : \Delta \to \Gamma$, giving the *category of contexts and substitutions*. A CwF is usually defined as a presheaf of *types in context*,

$$\mathsf{Ty} : \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}\,,$$

together with a presheaf of *typed terms in context*,

$$\mathsf{Tm} : (\textstyle\int_{\mathcal{C}} \mathsf{Ty})^{\mathsf{op}} \to \mathsf{Set}\,.$$

We reformulate this using the familiar equivalence

$$\mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}/_{\mathsf{Ty}} \;\simeq\; \mathsf{Set}^{(\int_{\mathcal{C}} \mathsf{Ty})^{\mathsf{op}}}$$

in order to obtain a map $\mathsf{tp} : \mathsf{Tm} \to \mathsf{Ty}$.

Formally, we then interpret:

$$\mathsf{Ty}(\Gamma) = \{A \mid \Gamma \vdash A\}$$
$$\mathsf{Tm}(\Gamma) = \{a \mid \Gamma \vdash a : A\}$$

Under the Yoneda lemma we therefore have a bijective correspondence:

$$\begin{array}{rcll} \Gamma \vdash A & \approx & A : \mathsf{y}\Gamma \to \mathsf{Ty} & \\ \Gamma \vdash a : A & \approx & a : \mathsf{y}\Gamma \to \mathsf{Tm} & \quad (\mathsf{tp} \circ a = A) \end{array}$$

as indicated in the following.

$$\begin{tikzcd} & \mathsf{Tm} \arrow[d, "\mathsf{tp}"] \\ \mathsf{y}\Gamma \arrow[ur, "a"] \arrow[r, "A"'] & \mathsf{Ty} \end{tikzcd}$$

The action of a substitution of contexts $\sigma : \Delta \to \Gamma$ on types and terms,

$$\frac{\sigma : \Delta \to \Gamma, \quad \Gamma \vdash a : A}{\Delta \vdash a[\sigma] : A[\sigma]}$$

is then interpreted simply as composition:

$$\begin{tikzcd} & & \mathsf{Tm} \arrow[d, "\mathsf{tp}"] \\ \mathsf{y}\Delta \arrow[r, "\mathsf{y}\sigma"] \arrow[urr, bend left, "{a[\sigma]}"] \arrow[rr, bend right, "{A[\sigma]}"'] & \mathsf{y}\Gamma \arrow[ur, "a"] \arrow[r, "A"'] & \mathsf{Ty} \end{tikzcd}$$

We may hereafter omit the y for the Yoneda embedding, letting the Greek letters serve to distinguish representable presheaves and their maps.

The CwF operation of *context extension*

$$\frac{\Gamma \vdash A}{\Gamma, A \vdash}$$

is modeled by the representability of $\mathsf{tp} : \mathsf{Tm} \to \mathsf{Ty}$ as follows. Given $\Gamma \vdash A$ we need a new context Γ, A together with a substitution $\pi_A : \Gamma, A \to \Gamma$ and a term

$$\Gamma, A \vdash q_A : A[\pi_A]\,.$$

This models the usual rule of "weakening the context" from $\Gamma \vdash A :$ type to $\Gamma, x : A \vdash x : A$. The substitution $A[\pi_A]$ is given by composition $A[\pi_A] = A \circ \pi_A$, where $\pi_A : \Gamma, A \to \Gamma$ is the pullback of tp along A, which exists as an arrow in $\mathcal{C}$ since tp is representable.

$$\begin{tikzcd} \Gamma, A \arrow[r, "q_A"] \arrow[d, "\pi_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Tm} \arrow[d, "\mathsf{tp}"] \\ \Gamma \arrow[r, "A"'] & \mathsf{Ty} \end{tikzcd} \tag{2}$$

The map $q_A : \Gamma, A \to \mathsf{Tm}$ arising from the pullback then gives the required term $\Gamma, A \vdash q_A : A[\pi_A]$ since $\mathsf{tp} \circ q_A = A \circ \pi_A = A[\pi_A]$. The remaining laws of a CwF follow from the pullback condition on (**??**); see [**?**].

1.1. **Modeling the type formers.** Given a natural model $\mathsf{u} : \dot{\mathsf{U}} \to \mathsf{U}$, we will make extensive use of the associated *polynomial endofunctor* $\mathsf{P_u} : \widehat{\mathcal{C}} \longrightarrow \widehat{\mathcal{C}}$ (cf. [**?**]), defined by

$$\mathsf{P_u} = \mathsf{U}_! \circ \mathsf{u}_* \circ \dot{\mathsf{U}}^* : \widehat{\mathcal{C}} \longrightarrow \widehat{\mathcal{C}},$$

as indicated below.

$$\begin{tikzcd} \mathsf{Set}^{\mathcal{C}^{\mathsf{op}}} \arrow[r, "\mathsf{P_u}"] \arrow[d, "\dot{\mathsf{U}}^*"'] & \mathsf{Set}^{\mathcal{C}^{\mathsf{op}}} \\ \mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}/_{\dot{\mathsf{U}}} \arrow[r, "\mathsf{u}_*"'] & \mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}/_{\mathsf{U}} \arrow[u, "\mathsf{U}_!"'] \end{tikzcd}$$

The action of $\mathsf{P_u}$ on an object X may be depicted:

$$\begin{tikzcd} X & X \times \dot{\mathsf{U}} \arrow[l] \arrow[d] & & \mathsf{P_u}(X) \arrow[d] \\ & \dot{\mathsf{U}} \arrow[rr, "\mathsf{u}"'] & & \mathsf{U} \end{tikzcd}$$

We call $\mathsf{u} : \dot{\mathsf{U}} \to \mathsf{U}$ the *signature* of $\mathsf{P_u}$ and briefly recall the following *universal mapping property* from [**?**].

Lemma 3. *For $p : E \to B$ in a locally cartesian closed category $\mathcal{E}$ we have the following universal property of the polynomial functor P_p. For any objects $X, Y \in \mathcal{E}$, maps $f : Y \to \mathsf{P}_p(X)$ correspond bijectively to pairs* *of maps* $f_1 : Y \to B$ *and* $f_2 : Y \times_B E \to X$*, as indicated below.*

$$
\begin{array}{c}
Y \xrightarrow{\quad f \quad} \mathsf{P}_p(X) \\
\hline\hline
\begin{tikzcd}
X & Y \times_B E \arrow[l, "f_2"'] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\
& Y \arrow[r, "f_1"'] & B
\end{tikzcd}
\end{array}
\tag{3}
$$

The correspondence is natural in both X *and* Y*, in the expected sense.*

This universal property is also suggested by the conventional type theoretic notation, namely:

$$\mathsf{P}_p(X) = \Sigma_{b:B} X^{E_b}$$

The lemma can be used to determine the signature $p \cdot q$ for the composite $\mathsf{P}_p \circ \mathsf{P}_q$ of two polynomial functors, which is again polynomial, and for which we have

$$\mathsf{P}_{p \cdot q} \cong \mathsf{P}_p \circ \mathsf{P}_q \,. \tag{4}$$

Indeed, let $p : B \to A$ and $q : D \to C$, and consider the following diagram resulting from applying the correspondence (**??**) to the identity arrow,

$$\langle a, c \rangle = 1_{\mathsf{P}_p(C)} : \mathsf{P}_p(C) \to \mathsf{P}_p(C) \,,$$

and taking Q to be the indicated pullback.

$$
\begin{tikzcd}
D \arrow[d, "q"'] & Q \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \arrow[ddr, dotted, bend left, "p \cdot q"] & \\
C & \pi^* B \arrow[l, "c"] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "p"] \\
& \mathsf{P}_p(C) \arrow[r, "a"'] & A
\end{tikzcd}
\tag{5}
$$

The map $p \cdot q$ is then defined to be the indicated composite,

$$p \cdot q \;=\; a^* p \circ c^* q \,.$$

The condition (**??**) can then be checked using the correspondence (**??**) (also see [**?**]).

Definition 4. A natural model $\mathsf{u} : \dot{\mathsf{U}} \to \mathsf{U}$ over $\mathcal{C}$ will be said to *model* the type formers $1, \Sigma, \Pi$ if there are pullback squares in $\widehat{\mathcal{C}}$ of the following form,

$$
\begin{tikzcd}
1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathsf{U}} \arrow[d, "\mathsf{u}"] \\
1 \arrow[r] & \mathsf{U}
\end{tikzcd}
\qquad
\begin{tikzcd}
\dot{\mathsf{U}}_2 \arrow[r] \arrow[d, "\mathsf{u}\cdot\mathsf{u}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathsf{U}} \arrow[d, "\mathsf{u}"] \\
\mathsf{U}_2 \arrow[r] & \mathsf{U}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{P}_{\mathsf{u}}(\dot{\mathsf{U}}) \arrow[r] \arrow[d, "\mathsf{P}_{\mathsf{u}}(\mathsf{u})"'] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathsf{U}} \arrow[d, "\mathsf{u}"] \\
\mathsf{P}_{\mathsf{u}}(\mathsf{U}) \arrow[r] & \mathsf{U}
\end{tikzcd}
\tag{6}
$$

where $\mathsf{u}\cdot\mathsf{u} : \dot{\mathsf{U}}_2 \to \mathsf{U}_2$ is determined by $\mathsf{P}_{\mathsf{u}\cdot\mathsf{u}} \cong \mathsf{P}_{\mathsf{u}} \circ \mathsf{P}_{\mathsf{u}}$ as in (**??**).

The terminology is justified by the following result from [**?**, Theorem 16].

Theorem 5. *Let* $\mathsf{u} : \dot{\mathsf{U}} \to \mathsf{U}$ *be a natural model. The associated category with families satisfies the usual rules for the type-formers* $1, \Sigma, \Pi$ *just if* $\mathsf{u} : \dot{\mathsf{U}} \to \mathsf{U}$ *models the same in the sense of Definition* **??**.

We only sketch the case of Π-types; the other type formers are treated in detail in [**?**, **?**, **?**].

Proposition 6. *The natural model* $\mathsf{u} : \dot{\mathsf{U}} \to \mathsf{U}$ *models* Π*-types just if there are maps* λ *and* Π *making the following a pullback diagram.*

$$
\begin{tikzcd}
\mathsf{P}_{\mathsf{u}}(\dot{\mathsf{U}}) \arrow[r, "\lambda"] \arrow[d, "\mathsf{P}_{\mathsf{u}}(\mathsf{u})"'] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathsf{U}} \arrow[d, "\mathsf{u}"] \\
\mathsf{P}_{\mathsf{u}}(\mathsf{U}) \arrow[r, "\Pi"'] & \mathsf{U}
\end{tikzcd}
\tag{7}
$$

Proof. Unpacking the definitions, we have $\mathsf{P}_{\mathsf{u}}(\mathsf{U}) = \Sigma_{A:\mathsf{U}}\mathsf{U}^A$, etc., so diagram (**??**) becomes:

$$
\begin{tikzcd}
\Sigma_{A:\mathsf{U}}\dot{\mathsf{U}}^A \arrow[r, "\lambda"] \arrow[d, "\Sigma_{A:\mathsf{U}}\mathsf{u}^A"'] & \dot{\mathsf{U}} \arrow[d, "\mathsf{u}"] \\
\Sigma_{A:\mathsf{U}}\mathsf{U}^A \arrow[r, "\Pi"'] & \mathsf{U}
\end{tikzcd}
$$

For $\Gamma \in \mathcal{C}$, maps $\Gamma \to \Sigma_{A:\mathsf{U}}\mathsf{U}^A$ correspond to pairs (A, B) with $A : \Gamma \to \mathsf{U}$ and $B : \Gamma, A \to \mathsf{U}$, and thus to $\Gamma \vdash A$ and $\Gamma, A \vdash B$. Similarly, a map $\Gamma \to \Sigma_{A:\mathsf{U}}\dot{\mathsf{U}}^A$ corresponds to a pair (A, b) with $\Gamma \vdash A$ and $\Gamma, A \vdash b : B$, the typing of b resulting from composing with the map

$$\Sigma_{A:\mathsf{U}}\mathsf{u}^A : \Sigma_{A:\mathsf{U}}\dot{\mathsf{U}}^A \to \Sigma_{A:\mathsf{U}}\mathsf{U}^A\,.$$

$$\begin{tikzcd}
 & \Sigma_{A:\mathsf{U}}\dot{\mathsf{U}}^A \arrow[r, "\lambda"] \arrow[dd] & \dot{\mathsf{U}} \arrow[dd, "\mathsf{u}"] \\
\Gamma \arrow[ur, "{(A,b)}"] \arrow[urr, "\lambda_A b"'] \arrow[dr, "{(A,B)}"'] \arrow[drr, "\Pi_A B"] & & \\
 & \Sigma_{A:\mathsf{U}}\mathsf{U}^A \arrow[r, "\Pi"'] & \mathsf{U}
\end{tikzcd}$$

The composition across the top is then the term $\Gamma \vdash \lambda_{x:A}b$, the type of which is determined by composing with u and comparing with the composition across the bottom, namely $\Gamma \vdash \Pi_{x:A}B$. In this way, the lower horizontal arrow in the diagram models the Π-*formation rule*:

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash \Pi_{x:A}B}$$

and the upper horizontal arrow, along with the commutativity of the diagram, models the Π-*introduction rule*:

$$\frac{\Gamma, A \vdash b : B}{\Gamma \vdash \lambda_{x:A}b : \Pi_{x:A}B}$$

The square (**??**) is a pullback just if, for every $(A, B) : \Gamma \to \Sigma_{A:\mathsf{U}}\mathsf{U}^A$ and every $t : \Gamma \to \dot{\mathsf{U}}$ with $\mathsf{u} \circ t = \Pi_A B$, there is a unique $(A, b) : \Gamma \to \Sigma_{A:\mathsf{U}}\dot{\mathsf{U}}^A$ with $b : B$ and $\lambda_A b = t$. In terms of the interpretation, given $\Gamma, A \vdash B$ and $\Gamma \vdash t : \Pi_{x:A}B$, there is required to be a term $\Gamma, A \vdash t' : B$ with $\lambda_{x:A}t' = t$, and t' is unique with this property. This is just what is provided by the Π-*elimination rule*:

$$\frac{\Gamma, A \vdash B \qquad \Gamma \vdash t : \Pi_{x:A}B \qquad \Gamma \vdash x : A}{\Gamma, A \vdash t\,x : B}$$

in conjunction with the Π-*computation rules*:

$$\lambda_{x:A}(t\,x) = t : \Pi_A B$$
$$(\lambda_{x:A}b)\,x = b : B$$

□

2. Martin-Löf algebras

Now let $\mathcal{E}$ be a locally cartesian closed category (lccc) and $\mathsf{u} : \dot{\mathsf{U}} \to \mathsf{U}$ a map in $\mathcal{E}$. As in the foregoing case where $\mathcal{E}$ was a category of presheaves $\mathcal{E} = \widehat{\mathcal{C}}$, the map u gives rise to a polynomial endofunctor,

$$\mathsf{P_u} = \mathsf{U}_! \circ \mathsf{u}_* \circ \dot{\mathsf{U}}^* : \mathcal{E} \longrightarrow \mathcal{E}\,,$$

which we may use to define the following abstraction of the notion of a natural model.

Definition 7. A *Martin-Löf algebra* in an lccc $\mathcal{E}$ is a map $\mathsf{u} : \dot{\mathsf{U}} \to \mathsf{U}$ equipped with structure maps $(*, 1, \sigma, \Sigma, \lambda, \Pi)$ making pullback squares

$$\begin{tikzcd} 1 \arrow[r, "*"] \arrow[d, "!"'] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathsf{U}} \arrow[d, "\mathsf{u}"] \\ 1 \arrow[r, "1"'] & \mathsf{U} \end{tikzcd} \qquad \begin{tikzcd} \dot{\mathsf{U}}_2 \arrow[r, "\sigma"] \arrow[d, "\mathsf{u}\cdot\mathsf{u}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathsf{U}} \arrow[d, "\mathsf{u}"] \\ \mathsf{U}_2 \arrow[r, "\Sigma"'] & \mathsf{U} \end{tikzcd} \qquad \begin{tikzcd} \mathsf{P}_{\mathsf{u}}\dot{\mathsf{U}} \arrow[r, "\lambda"] \arrow[d, "\mathsf{P}_{\mathsf{u}}(\mathsf{u})"'] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathsf{U}} \arrow[d, "\mathsf{u}"] \\ \mathsf{P}_{\mathsf{u}}\mathsf{U} \arrow[r, "\Pi"'] & \mathsf{U} \end{tikzcd} \tag{8}$$

where the map $\mathsf{u} \cdot \mathsf{u}$ is defined in terms of P_{u} as in (**??**) via

$$\mathsf{P}_{\mathsf{u}\cdot\mathsf{u}} = \mathsf{P}_{\mathsf{u}} \circ \mathsf{P}_{\mathsf{u}} \,.$$

In place of representability in the elementary setting we may sometimes require the further condition that $\mathsf{u} : \dot{\mathsf{U}} \to \mathsf{U}$ be *tiny* in the following sense.

Definition 8. A map $p : Y \to X$ in a locally cartesian closed category $\mathcal{E}$ will be said to be *tiny* if it is so as an object in $\mathcal{E}/_X$ in the sense that exponentiation by p has a right adjoint $(-)^p \dashv (-)_p$.

Note that a map $p : Y \to X$ in an lccc is tiny just if the *pushforward* functor $p_* : \mathcal{E}/_Y \to \mathcal{E}/_X$ has a right adjoint,

$$f_* \dashv f^! : \mathcal{E}/_X \longrightarrow \mathcal{E}/_Y \,.$$

Proposition 9. *If $\mathcal{C}$ has finite limits, a map $p : Y \to X$ in $\widehat{\mathcal{C}}$ is representable just if it is tiny in the sense of Definition* **??**, *which is the case just if the* pushforward functor *(the right adjoint to pullback)*

$$p^* \dashv p_* : \widehat{\mathcal{C}}/_Y \longrightarrow \widehat{\mathcal{C}}/_X$$

itself has a right *adjoint:*

$$p_! \dashv p^* \dashv p_* \dashv p^!$$

Proof. The elementary definition **??** clearly states that, for the category of elements $\int X \simeq \mathsf{y}/_X$, the composition functor

$$\textstyle\int p : \int Y \to \int X$$

has a right adjoint, say $\int p \dashv (\int p)^\sharp$. Now recall that $\widehat{\mathcal{C}}/Y \simeq [(\int X)^{\mathsf{op}}, \mathsf{Set}]$, so that the precomposition functor $(\int p)^* = [(\int p)^{\mathsf{op}}, \mathsf{Set}]$ gives rise to a commutative diagram with left and right Kan extensions:

$$\textstyle(\int p)_! \dashv (\int p)^* \dashv (\int p)_*$$

$$\begin{tikzcd}
{[(\int Y)^{\mathsf{op}}, \mathsf{Set}]} \arrow[r, shift left=2] \arrow[r, shift right=2, "\int p_!"'] & {[(\int X)^{\mathsf{op}}, \mathsf{Set}]} \arrow[l] \\
\int Y \arrow[u, hook, "\mathsf{y}"] \arrow[r, "\int p"'] & \int X \arrow[u, hook, "\mathsf{y}"']
\end{tikzcd} \tag{9}$$

But since $\int p \dashv (\int p)^\sharp$, there is a further right adjoint $(\int p)_* \dashv ((\int p)^\sharp)_*$ to precomposition with $(\int p)^\sharp$. Moving back across the equivalence $[(\int X)^{\mathsf{op}}, \mathsf{Set}] \simeq \widehat{\mathcal{C}}/_X$ we obtain the claimed further right adjoint:

$$p_! \dashv p^* \dashv p_* \dashv p^! : \widehat{\mathcal{C}}/_X \longrightarrow \widehat{\mathcal{C}}/_Y$$

Conversely, a right adjoint

$$(\textstyle\int p)_* \dashv R : [(\int X)^{\mathsf{op}}, \mathsf{Set}] \to [(\int Y)^{\mathsf{op}}, \mathsf{Set}]$$

is easily shown to be induced by precomposing with a right adjoint $\int p \dashv r : \int X \to \int Y$ if $\mathcal{C}$ has all finite products and idempotents split in $\mathcal{C}$, which is the case if $\mathcal{C}$ has finite limits.

We leave the construction of the right adjoint $p^! : \widehat{\mathcal{C}}/_X \longrightarrow \widehat{\mathcal{C}}/_Y$ from the right adjoint $(-)^p \dashv (-)_p : \widehat{\mathcal{C}}/_X \to \widehat{\mathcal{C}}/_X$ to the reader. □

Remark 10. In [**?**] it is shown how a *clan* in the sense of Joyal [**?**], or *category with display maps* in the sense of [**?**], say $(\mathcal{C}, \mathcal{D})$, gives rise to a natural model $\mathsf{u} : \dot{\mathsf{U}} \to \mathsf{U}$ in $\widehat{\mathcal{C}}$, namely with

$$\mathsf{u} = \coprod_{d \in \mathcal{D}} \mathsf{y}d\,.$$

In particular, the fibrations in any right-proper Quillen model category (which determine a Π-*tribe* on the fibrant objects, in the language of [**?**]), or simply all maps in an lccc, are shown to give rise to a natural model with $1, \Sigma, \Pi$, and thus a Martin-Löf algebra, which moreover has *identity types*, in the sense of the next section.

2.1. **Identity Types.** A natural model $\mathsf{u} : \dot{\mathsf{U}} \to \mathsf{U}$ in a presheaf category $\widehat{\mathcal{C}}$ was determined in [**?**] to model either the extensional or intensional identity types of Martin-Löf type theory in terms of the existence of certain additional structures. We transfer these definitions to the elementary setting of Martin-Löf algebras. Condition (**??**) below is shown in [**?**] to capture the extensional identity types of Martin-Löf type theory. The condition given in *op.cit.* for the intensional case is replaced in (**??**) below by a simplification suggested by R. Garner.

Definition 11. Let $\mathsf{u} : \dot{\mathsf{U}} \to \mathsf{U}$ be a map in an lccc $\mathcal{E}$.

(1) $\mathsf{u} : \dot{\mathsf{U}} \to \mathsf{U}$ is said to model the (extensional) *equality type former* if there are structure maps $(\mathsf{refl}, \mathsf{Eq})$ making a pullback square:

$$\begin{tikzcd}
\dot{\mathsf{U}} \arrow[r, "\mathsf{refl}"] \arrow[d, "\delta"'] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathsf{U}} \arrow[d, "\mathsf{u}"] \\
\dot{\mathsf{U}} \times_{\mathsf{U}} \dot{\mathsf{U}} \arrow[r, "\mathsf{Eq}"'] & \mathsf{U}
\end{tikzcd}$$

(2) $\mathsf{u} : \dot{\mathsf{U}} \to \mathsf{U}$ is said to model the (intensional) *identity type former* if there are structure maps $(\mathsf{i}, \mathsf{Id})$ making a commutative square,

$$\begin{tikzcd}
\dot{\mathsf{U}} \arrow[r, "\mathsf{i}"] \arrow[d, "\delta"'] & \dot{\mathsf{U}} \arrow[d, "\mathsf{u}"] \\
\dot{\mathsf{U}} \times_{\mathsf{U}} \dot{\mathsf{U}} \arrow[r, "\mathsf{Id}"'] & \mathsf{U}
\end{tikzcd} \tag{10}$$

together with a weak pullback structure J for the resulting comparison square, in the sense of (**??**) below.

To describe the map J, let us see how (**??**) models identity types. Under the interpretation already described in Section **??** the maps Id and i in

$$\begin{tikzcd}
\dot{\mathsf{U}} \arrow[r, "\mathsf{i}"] \arrow[d, "\delta"'] & \dot{\mathsf{U}} \arrow[d, "\mathsf{u}"] \\
\dot{\mathsf{U}} \times_{\mathsf{U}} \dot{\mathsf{U}} \arrow[r, "\mathsf{Id}"'] & \mathsf{U}
\end{tikzcd}$$

respectively, directly model the formation and introduction rules.

$$x, y : A \vdash \mathsf{Id}_A(x, y)$$
$$x : A \vdash \mathsf{i}(x) : \mathsf{Id}_A(x, x)$$

Next, pull u back along Id to get an object I and a map $\rho : \dot{\mathsf{U}} \to \mathsf{I}$,

$$\begin{tikzcd}
\dot{\mathsf{U}} \arrow[r, "\rho"] \arrow[rr, bend left, "\mathsf{i}"] \arrow[dr, "\delta"'] & \mathsf{I} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathsf{U}} \arrow[d, "\mathsf{u}"] \\
& \dot{\mathsf{U}} \times_{\mathsf{U}} \dot{\mathsf{U}} \arrow[r, "\mathsf{Id}"'] & \mathsf{U}
\end{tikzcd}$$

which commutes with the compositions to U as indicated below.

$$\begin{tikzcd} \dot{\mathsf{U}} \arrow[r, "\rho"] \arrow[dr, "\mathsf{u}"'] & \mathsf{I} \arrow[d, "q"] \\ & \mathsf{U} \end{tikzcd}$$

The map $\rho : \dot{\mathsf{U}} \to \mathsf{I}$, which can be interpreted as the substitution $(x) \mapsto (x, x, \mathsf{i}x)$, gives rise to a "restriction" natural transformation of polynomial endofunctors ([**?**]),

$$\rho^* : P_q \to P_{\mathsf{u}} \,,$$

evaluation of which at $\mathsf{u} : \dot{\mathsf{U}} \to \mathsf{U}$ results in the following commutative naturality square.

$$\begin{tikzcd} P_q\dot{\mathsf{U}} \arrow[r, "\rho^*_{\dot{\mathsf{U}}}"] \arrow[d, "P_q\mathsf{u}"'] & P_{\mathsf{u}}\dot{\mathsf{U}} \arrow[d, "P_{\mathsf{u}}\mathsf{u}"] \\ P_q\mathsf{U} \arrow[r, "\rho^*_{\mathsf{U}}"'] & P_{\mathsf{u}}\mathsf{U} \end{tikzcd} \tag{11}$$

Note that (**??**) is a pullback in the extensional case; here we require it to be a *weak* pullback by taking a section of the resulting "gap map". Explicitly, *weak pullback structure* J is a section of the resulting comparison map.

$$\begin{tikzcd} P_q\dot{\mathsf{U}} \arrow[r] & P_q\mathsf{U} \times_{P_{\mathsf{u}}\mathsf{U}} P_{\mathsf{u}}\dot{\mathsf{U}} \arrow[l, bend right, dotted, "\mathsf{J}"'] \end{tikzcd} \tag{12}$$

To show that this models the standard elimination rule, namely

$$\frac{x : A \vdash c(x) : C(\rho x)}{x, y : A, z : \mathsf{Id}_A(x, y) \vdash \mathsf{J}_c(x, y, z) : C(x, y, z)}$$

take any object $\Gamma \in \mathcal{E}$ and maps $(A, A, \mathsf{Id}_A \vdash C) : \Gamma \to P_q\mathsf{U}$ and $(A \vdash c) : \Gamma \to P_{\mathsf{u}}\dot{\mathsf{U}}$ with equal composites to $P_{\mathsf{u}}\mathsf{U}$, meaning that $A \vdash c : C(\rho x)$. Composing the resulting map

$$(A \vdash c(x) : C(\rho x)) : \Gamma \to P_q\mathsf{U} \times_{P_{\mathsf{u}}\mathsf{U}} P_{\mathsf{u}}\dot{\mathsf{U}}$$

with $\mathsf{J} : P_q\mathsf{U} \times_{P_{\mathsf{u}}\mathsf{U}} P_{\mathsf{u}}\dot{\mathsf{U}} \to P_q\dot{\mathsf{U}}$ then indeed provides a term

$$x : A, y : A, z : \mathsf{Id}_A(x, y) \vdash \mathsf{J}_c(x, y, z) : C(x, y, z) \,.$$

The computation rule

$$x : A \vdash \mathsf{J}_c(\rho x) = c(x) : C(\rho x)$$

then says exactly that J is indeed a section of the comparison map (**??**).

Proposition 12 (R. Garner)**.** *A natural model* $\mathsf{u} : \dot{\mathsf{U}} \to \mathsf{U}$ *satisfies the rules for intensional identity types just if the map* $\mathsf{u} : \dot{\mathsf{U}} \to \mathsf{U}$ models *the same in the sense of Definition* **??***: there are maps* $(\mathsf{i}, \mathsf{Id})$ *making the diagram* (**??**) *commute, together with a weak pullback structure* J *for the resulting comparison square* (**??**)*.*

The proposition clearly generalizes to arbitrary ML-algebras.

2.2. **Identity Types via an Interval.**

Definition 13. By an *interval* in an lccc $\mathcal{E}$ we simply mean a bipointed object $d_0, d_1 : 1 \rightrightarrows \mathrm{I}$. In terms of an interval, we then define further:

(1) for every object A, a *path-object* factorization of the diagonal $\delta : A \to A \times A$, obtained by exponentiating A by $1 \rightrightarrows \mathrm{I} \to 1$,

$$\begin{tikzcd} A \arrow[r, "\rho"] \arrow[dr, "\delta"'] & A^{\mathrm{I}} \arrow[d, "\langle \varepsilon_0, \varepsilon_1 \rangle"] \\ & A \times A \end{tikzcd}$$

where we write $\rho = A^{!_{\mathrm{I}}}$, and $\varepsilon_0 = A^{d_0}$, and $\varepsilon_1 = A^{d_1}$.

(2) Similarly, and abusing notation slightly, for any $A \to X$ regarded as an object in the slice category $\mathcal{E}/_X$, we define the *relative pathobject factorization*

$$\begin{tikzcd} A \arrow[r, "\rho"] \arrow[dr, "\delta"'] & A^{\mathrm{I}} \arrow[d, "\langle \varepsilon_0, \varepsilon_1 \rangle"] \\ & A \times_X A \end{tikzcd}$$

to be the pathobject factorization in $\mathcal{E}/_X$ with respect to the pulled-back interval $1_X \rightrightarrows X^*\mathrm{I} \to 1_X$, where we are using the pullback functor $X^* = !_X^* : \mathcal{E} \to \mathcal{E}/_X$ along $!_X : X \to 1$.

Lemma 14. *For any object* $A \to X$ *over any base* X*, the relative pathobject factorization*

$$\begin{tikzcd} A \arrow[r, "\rho"] \arrow[dr, "\delta"'] & A^{\mathrm{I}} \arrow[d, "\langle \varepsilon_0, \varepsilon_1 \rangle"] \\ & A \times_X A\,, \end{tikzcd}$$

is stable under pullback along any map $f : Y \to X$*, in the sense that the factorization pulls back to the relative pathobject factorization over* Y *of the pullback* $f^*A \to Y$*, resulting in a canonical isomorphism over* Y*,*

$$(f^*A)^{\mathrm{I}} \cong f^*(A^{\mathrm{I}})\,.$$

Definition 15. A natural model $\mathsf{u} : \dot{\mathsf{U}} \to \mathsf{U}$ will be said to *have path types* if there are structure maps $(\mathsf{Id}, \mathsf{j})$ making a pullback square,

$$\begin{array}{ccc} \dot{\mathsf{U}}^{\mathrm{I}} & \xrightarrow{\mathsf{j}} & \dot{\mathsf{U}} \\ {\scriptstyle\varepsilon}\downarrow & & \downarrow{\scriptstyle\mathsf{u}} \\ \dot{\mathsf{U}} \times_{\mathsf{U}} \dot{\mathsf{U}} & \xrightarrow[\mathsf{Id}]{} & \mathsf{U} \end{array} \tag{13}$$

where $\varepsilon = \langle \varepsilon_0, \varepsilon_1 \rangle : \dot{\mathsf{U}}^{\mathrm{I}} \to \dot{\mathsf{U}} \times_{\mathsf{U}} \dot{\mathsf{U}}$ is the relative pathobject of $\mathsf{u} : \dot{\mathsf{U}} \to \mathsf{U}$ over U.

In order to show that a natural model with path types also has intensional Identity types in the sense of Definition **??**, we require an additional condition on the map $\mathsf{u} : \dot{\mathsf{U}} \to \mathsf{U}$, namely that it is a "Hurewicz fibration", in the following sense.

Definition 16. A map $f : Y \to X$ will be called a *Hurewicz fibration* (with respect to an interval $d_0, d_1 : 1 \rightrightarrows \mathrm{I}$), if it has the right lifting property with respect to every cylinder $Z \times d_0 : Z \times 1 \to Z \times \mathrm{I}$. In detail, given any object Z and maps y and h as indicated below, there exists a diagonal filler $\tilde{h}$ making the following diagram commute.

$$\begin{tikzcd} Z \times 1 \arrow[r, "y"] \arrow[d, "Z\times d_0"'] & Y \arrow[d, "f"] \\ Z \times \mathrm{I} \arrow[r, "h"'] \arrow[ur, dotted, "\tilde{h}"'] & X \end{tikzcd} \tag{14}$$

One regards $h : Z \times \mathrm{I} \to X$ as a homotopy between the maps $h_0, h_1 : Z \to X$ obtained by composing it with the two ends of the cylinder $Z \times 1 \rightrightarrows Z \times \mathrm{I}$, and $\tilde{h} : Z \times \mathrm{I} \to Y$ as a lift of h to the specified 0-end y.

Proposition 17. *A map $f : Y \to X$ is a Hurewicz fibration just if the following diagram is a weak pullback.*

$$\begin{array}{ccc} Y^{\mathrm{I}} & \xrightarrow{\varepsilon_0} & Y \\ {\scriptstyle f^{\mathrm{I}}}\downarrow & & \downarrow{\scriptstyle f} \\ X^{\mathrm{I}} & \xrightarrow[\varepsilon_0]{} & X \end{array} \tag{15}$$

Proof. The diagram (**??**) is a weak pullback just if the comparison map to the actual pullback has a section $\ell : X^{\mathrm{I}} \times_X Y \to Y^{\mathrm{I}}$, as indicated below.

$$\begin{tikzcd}
Y^{\mathrm{I}} \arrow[dr] \arrow[drr, bend left, "\varepsilon_0"] \arrow[ddr, bend right, "f^{\mathrm{I}}"'] & & \\
& X^{\mathrm{I}} \times_X Y \arrow[r] \arrow[d] \arrow[ul, dotted, "\ell"'] & Y \arrow[d, "f"] \\
& X^{\mathrm{I}} \arrow[r, "\varepsilon_0"'] & X
\end{tikzcd} \tag{16}$$

In terms of the so-called "Leibniz adjunction" $\otimes \dashv \Rightarrow$ (see [**?**]), the comparison map $\langle f^{\mathrm{I}}, \varepsilon_0 \rangle : Y^{\mathrm{I}} \to X^{\mathrm{I}} \times_X Y$ is the "pullback-hom", $\langle f^{\mathrm{I}}, \varepsilon_0 \rangle = d_0 \Rightarrow f$.

Now, an arbitrary map $g : A \to B$ has a section just if it lifts on the right against $0 \to Z$, for all objects Z (one can take $Z = B$). Thus $d_0 \Rightarrow f$ has a section just if $0_Z \pitchfork (d_0 \Rightarrow f)$ for all Z, which is equivalent by the adjunction to $(0_Z \otimes d_0) \pitchfork f$ for all Z. But we have $0_Z \otimes d_0 = Z \times d_0 : Z \to Z \times \mathrm{I}$. $\square$

A section $\ell : X^{\mathrm{I}} \times_X Y \to Y^{\mathrm{I}}$ may be regarded as a "lifting operation" that takes a path $p : x_0 \rightsquigarrow x_1$ in X, and a point $y_0 \in Y$ over x_0, to a lifted path $\tilde{p} : y_0 \rightsquigarrow y$ in Y lying over p. A Hurewicz fibration $f : Y \to X$ will be called *normal* if it comes with such a lifting $\ell : X^{\mathrm{I}} \times_X Y \to Y^{\mathrm{I}}$ that takes an identity path in X to an identity path in Y, as indicated below.

$$\begin{tikzcd}
Y \arrow[r, "\rho"] & Y^{\mathrm{I}} \\
X \times_X Y \arrow[u, "p_2"] \arrow[r, "\rho \times_X Y"'] & X^{\mathrm{I}} \times_X Y \arrow[u, "\ell"']
\end{tikzcd} \tag{17}$$

Now consider the following *axioms* for a natural model $\mathsf{u} : \dot{\mathsf{U}} \to \mathsf{U}$ with an interval $d_0, d_1 : 1 \rightrightarrows \mathrm{I}$.

(A1) $\mathsf{u} : \dot{\mathsf{U}} \to \mathsf{U}$ has path types, as in Definition **??**.
(A2) $\mathsf{u} : \dot{\mathsf{U}} \to \mathsf{U}$ is a Hurewicz fibration, as in Definition **??**.

We also assume that $\mathsf{u} : \dot{\mathsf{U}} \to \mathsf{U}$ is normal.

Lemma 18. *Assuming axioms (A1) and (A2), the map* $\mathsf{u} : \dot{\mathsf{U}} \to \mathsf{U}$ *has a* connection*: a map* $\chi : \dot{\mathsf{U}}^{\mathrm{I}} \to (\dot{\mathsf{U}}^{\mathrm{I}})^{\mathrm{I}} \cong \dot{\mathsf{U}}^{\mathrm{I} \times \mathrm{I}}$ *(over* U*) making the following* *commute.*

$$
\begin{tikzcd}
 & \dot{\mathsf{U}}^{\mathrm{I}} \\
\dot{\mathsf{U}}^{\mathrm{I}} \arrow[ur, "="] \arrow[r, "\chi"'] \arrow[d, "\varepsilon_0"'] & \dot{\mathsf{U}}^{\mathrm{I}\times\mathrm{I}} \arrow[u, "\varepsilon_1"'] \arrow[d, "\varepsilon_0"] \\
\dot{\mathsf{U}} \arrow[r, "\rho"'] & \dot{\mathsf{U}}^{\mathrm{I}}
\end{tikzcd}
\tag{18}
$$

Moreover, the connection χ *is* normal *in the sense that* $\chi\rho = \rho\rho$ *,*

$$
\begin{tikzcd}
\dot{\mathsf{U}}^{I} \arrow[r, "\chi"] & \dot{\mathsf{U}}^{\mathrm{I}\times\mathrm{I}} \\
\dot{\mathsf{U}} \arrow[u, "\rho"] \arrow[r, "\rho"'] & \dot{\mathsf{U}}^{\mathrm{I}}. \arrow[u, "\rho"']
\end{tikzcd}
\tag{19}
$$

Proof. We can use "box-filling" to construct the connection as the transpose of a certain diagonal filler

$$\tilde{\chi} : \dot{\mathsf{U}}^{\mathrm{I}} \times \mathrm{I} \to (\dot{\mathsf{U}}^{\mathrm{I}})$$

as follows. Given any pathspace $A^{\mathrm{I}} \to A \times A$ that is a Hurewicz fibration, consider a lifting problem of the form:

$$
\begin{tikzcd}
1 \arrow[r, "r"] \arrow[d, "\delta_0"'] & A^{\mathrm{I}} \arrow[d, "A^{\partial}"] \\
\mathrm{I} \arrow[r, "\langle q,p\rangle"'] \arrow[ur, dotted, "\tilde{b}"] & A \times A
\end{tikzcd}
\tag{20}
$$

This can be regarded as a filler for an "open box" (p, q, r) in A to give a 2-cube $b : \mathrm{I} \times \mathrm{I} \to A$, which can be depicted as follows:

$$
\begin{tikzcd}
q_0 \arrow[r, "r"] \arrow[d, "q"'] \arrow[dr, phantom, "b"] & p_0 \arrow[d, "p"] \\
q_1 \arrow[r, dotted] & p_1
\end{tikzcd}
\tag{21}
$$

Now consider the case $q = \rho_{p_0} = r$ and take $\chi_p : \mathrm{I} \to A^{\mathrm{I}}$ to be the resulting box:

$$
\begin{tikzcd}
p_0 \arrow[r, "="] \arrow[d, "="'] \arrow[dr, phantom, "\chi_p"] & p_0 \arrow[d, "p"] \\
p_0 \arrow[r, dotted] & p_1
\end{tikzcd}
\tag{22}
$$

We apply this to the case where $A = \dot{\mathsf{U}}$ (over U), and with object of parameters $\dot{\mathsf{U}}^{\mathrm{I}}$, to obtain the following:

$$\begin{tikzcd} \dot{\mathsf{U}}^{\mathrm{I}} \times 1 \arrow[r, "r"] \arrow[d, "\dot{\mathsf{U}}^{\mathrm{I}}\times\delta_0"'] & \dot{\mathsf{U}}^{\mathrm{I}} \arrow[d, "\dot{\mathsf{U}}^{\partial}"] \\ \dot{\mathsf{U}}^{\mathrm{I}} \times \mathrm{I} \arrow[r, "\langle q,p\rangle"'] \arrow[ur, dotted, "\tilde{\chi}"'] & \dot{\mathsf{U}} \times \dot{\mathsf{U}} \end{tikzcd} \tag{23}$$

The maps p, q, r are as follows:

$$\begin{aligned} p &= \varepsilon \\ q &= \varepsilon(\rho\varepsilon_0 \times \mathrm{I}) \\ r &= \rho\varepsilon_0 \end{aligned}$$

Transposing provides the desired map $\chi : \dot{\mathsf{U}}^{\mathrm{I}} \to \dot{\mathsf{U}}^{\mathrm{I}\times\mathrm{I}}$ with evaluations $\chi(p)_0 = \rho p_0$ and $\chi(p)_1 = p$ for all $p : \dot{\mathsf{U}}^{\mathrm{I}}$. □

The Id-Elim rule now follows for any $A \to X$ that is classified by $\mathsf{u} : \dot{\mathsf{U}} \to \mathsf{U}$, as follows. By (A2), $A \to X$ is Hurewicz (since it's a pullback of $\mathsf{u} : \dot{\mathsf{U}} \to \mathsf{U}$), and by (A1) the pathtype $A^{\mathrm{I}} \to A \times_X A$ is also Hurewicz (since it's therefore a pullback of $\dot{\mathsf{U}}^{\mathrm{I}} \to \dot{\mathsf{U}} \times_{\mathsf{U}} \dot{\mathsf{U}}$). By Lemma **??** there is also a (normal) connection on $A \to X$ (again since it's a pullback of a map with a connection). Consider an Id-elimination problem as follows:

$$\begin{tikzcd} A \arrow[r, "c"] \arrow[d, "\rho"'] & C \arrow[d, "\pi"] \\ A^{\mathrm{I}} \arrow[r, "="'] \arrow[ur, dotted, "j"'] & A^{\mathrm{I}} \end{tikzcd} \tag{24}$$

where $\pi : C \to A^{\mathrm{I}}$ is a pullback of $\mathsf{u} : \dot{\mathsf{U}} \to \mathsf{U}$, and therefor a Hurewicz fibration, with (normal) lifting operation $\ell : (A^{\mathrm{I}})^{\mathrm{I}} \to C^{\mathrm{I}}$.

We argue first with elements to give the idea of the proof, before giving a diagrammatic version of the same argument. Thus take any $p : A^{\mathrm{I}}$, which is a path $p : p_0 \rightsquigarrow p_1$ in A. Applying the connection on A we obtain a (higher) path $\chi_p : \rho p_0 \rightsquigarrow p$ in A^{I}. Since the outer square of (**??**) commutes, for every $p_0 : A$ the term $cp_0 : C$ lies over ρp_0, thus we can lift χ_p to a path $\ell\langle\chi_p, cp_0\rangle : cp_0 \rightsquigarrow \tilde{p}$ in C, with the endpoint $\tilde{p}$ over p. We then set

$$jp := \tilde{p} = \varepsilon_1\ell\langle\chi_p, cp_0\rangle\,,$$

which plainly makes the bottom triangle of (**??**) commute. Observe that in the case $p = \rho a$ for $a : A$, we have $\chi_{\rho a} = \rho_{\rho a}$ since χ is normal, and then for the lift we have $\ell\langle\chi_{\rho a}, ca\rangle = \ell\langle\rho_{\rho a}, ca\rangle = \rho ca$, since ℓ is normal. Whence $j\rho a = \varepsilon_1 \rho c a = ca$, as required for the top triangle of (**??**) to commute.

We summarize the result as follows.

Proposition 19. *Let* $\mathsf{u} : \dot{\mathsf{U}} \to \mathsf{U}$ *be a natural model in a category* $\mathcal{E}$ *with an interval* $1 \rightrightarrows \mathrm{I}$*, and assume axioms (A1) and (A2) above. Then* $\mathsf{u} : \dot{\mathsf{U}} \to \mathsf{U}$ *has Identity-types of the form* $A \to A^{\mathrm{I}} \to A \times A$ *validating the usual rules of intensional type theory.*

Proof. The foregoing "elementary" proof is depicted in the following diagram:

$$
\begin{tikzcd}
A \arrow[rr, "c"] \arrow[ddd, "\rho"'] & & C \arrow[dd, "="] \\
& C^{\mathrm{I}} \arrow[ur, "\varepsilon_1"] & \\
& * \arrow[u, "\ell"] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d] \\
A^{\mathrm{I}} \arrow[r, "\chi"'] \arrow[ur] \arrow[uuurr, dotted, "j"] \arrow[rr, bend right, "="'] & (A^{\mathrm{I}})^{\mathrm{I}} \arrow[r, "\varepsilon_0"'] & A^{\mathrm{I}}
\end{tikzcd}
\tag{25}
$$

Where $* = (A^{\mathrm{I}})^{\mathrm{I}} \times_{A^{\mathrm{I}}} C$ is the indicated pullback, and the unlabelled map into it is the pair

$$\langle \chi, c\varepsilon_0 \rangle : A^{\mathrm{I}} \longrightarrow * \,.$$

□

Since the map $j = \varepsilon_1 \circ \ell \circ \langle \chi, c\varepsilon_0 \rangle$ is defined from others that are stable under pullback (themselves being pullbacks of a universal instance, defined from structure on $\mathsf{u} : \dot{\mathsf{U}} \to \mathsf{U}$), the substitution condition with respect to a change of context $\sigma : X' \to X$ will obtain; that is we shall have:

$$(j^c)_\sigma = j^{c\sigma} : C_\sigma \,.$$

From this, it follows that we can determine a weak pullback structure map J as in (**??**), relating the polynomial functors associated with $\mathsf{u} : \dot{\mathsf{U}} \to \mathsf{U}$ and $\dot{\mathsf{U}}^{\mathrm{I}} \to \mathsf{U}$, evaluated at $\mathsf{u} : \dot{\mathsf{U}} \to \mathsf{U}$, as in (**??**).

A simpler formulation of this condition is available in the case when the interval I is a tiny object—as obtains in the presheaves over the (finite product) category $\mathcal{C}$ of contexts. In that case, let us reformulate the elimination diagram (**??**) in the equivalent form

$$\begin{tikzcd} A \arrow[r, "c"] \arrow[d, "\rho"'] & \dot{\mathsf{U}} \arrow[d, "\mathsf{u}"] \\ A^{\mathrm{I}} \arrow[r, "C"'] \arrow[ur, dotted, "j"'] & \mathsf{U} \end{tikzcd} \tag{26}$$

Then, writing $\rho = A^{!} : A \to A^{\mathrm{I}}$, for $! : \mathrm{I} \to 1$, we can transpose across the adjunction $(-)^{\mathrm{I}} \dashv (-)_{\mathrm{I}}$ to obtain the following equivalent problem:

$$\begin{tikzcd} A \arrow[drr, bend left, "c"] \arrow[dr, dotted, "\tilde{j}"'] \arrow[ddr, bend right, "\tilde{C}"'] & & \\ & \dot{\mathsf{U}}_{\mathrm{I}} \arrow[r, "\dot{\mathsf{U}}_{!}"] \arrow[d, "\mathsf{u}_{\mathrm{I}}"'] & \dot{\mathsf{U}} \arrow[d, "\mathsf{u}"] \\ & \mathsf{U}_{\mathrm{I}} \arrow[r, "\mathsf{U}_{!}"'] & \mathsf{U} \end{tikzcd} \tag{27}$$

Note that we are using the fact that the terminal object 1 is also tiny and $(-)_1 = \mathsf{id}$ is the identity functor, so $! : \mathrm{I} \to 1$ gives rise to a natural transformation $(-)_{!} : (-)_{\mathrm{I}} \to \mathsf{id}$.

The problem (**??**) can be reformulated without reference to $A, \tilde{C}, c$ as stating that the inner square is a weak pullback, or, again equivalently, that there exists a section $\tilde{\mathsf{J}}$ of the comparison map $\langle \mathsf{u}_{\mathrm{I}}, \dot{\mathsf{U}}_{!} \rangle$,

$$\begin{tikzcd} \dot{\mathsf{U}}_{\mathrm{I}} \arrow[r] & \mathsf{U}_{\mathrm{I}} \times_{\mathsf{U}} \dot{\mathsf{U}} \arrow[l, bend right, dotted, "\tilde{\mathsf{J}}"'] \end{tikzcd} \tag{28}$$

This condition is clearly independent of the domain of any maps $A : X \to \mathsf{U}$, etc., into $\mathsf{u} : \dot{\mathsf{U}} \to \mathsf{U}$, and therefore evidently satisfies the strict rule of coherence under substitution. Comparing the condition (**??**) with the similar (**??**), we have achieved a simplification in replacing the polynomial functors $P_{\mathsf{u}}, P_q : \hat{\mathcal{C}} \to \hat{\mathcal{C}}$ by the "root" functor $(-)_{\mathrm{I}} : \hat{\mathcal{C}} \to \hat{\mathcal{C}}$, which is available when an interval $1 \rightrightarrows \mathrm{I}$ is present in $\mathcal{C}$, and of course in replacing the *axioms* (**??**) and (**??**) by the axioms (A1) and (A2) above.