A Quillen model structure on the category of cartesian cubical sets

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1 The cartesian cube category

In contrast to some other treatments of cubical sets [?, ?, ?, ?, ?, ?, ?, ?], we consider what may be termed the *cartesian* cube category $\mathbb{C}$, defined as the free finite product category on an interval $\delta_0, \delta_1 : 1 \rightrightarrows \mathrm{I}$. As a classifying category for an algebraic theory with two constant symbols $\mathbb{T} = \{0, 1\}$, the category $\mathbb{C}$ is dual to the full subcategory of finitely-generated, free $\mathbb{T}$-algebras $\mathsf{Alg}(\mathbb{T})_{\mathrm{fg}}$ (by Lawvere duality). In this case, the algebras are thus simply *bipointed sets* (A, a_0, a_1), and the free ones are the *strictly* bipointed sets $a_0 \neq a_1$. Thus $\mathsf{Alg}(\mathbb{T})_{\mathrm{fg}}$ consists of the finite, strictly bipointed sets and all bipointed maps between them. We will use the following specific presentation.

Definition 1. The objects of the cartesian cube category $\mathbb{C}$, called n-cubes, will be written

$$[n] = \{0, x_1, ..., x_n, 1\}\,.$$

The arrows,

$$f : [n] \longrightarrow [m]\,,$$

maybe taken to be m-tuples of elements drawn from the set $\{0, x_1, ..., x_n, 1\}$ regarded as formal terms representing composites of faces, degeneracies, permutations, and diagonals. Equivalently, the arrows $[n] \to [m]$ are arbitrary bipointed maps $[m] \to [n]$.

See [?] for further details.

2 Cubical sets

The category cSet of *cubical sets* is the category of presheaves on the cartesian cube category $\mathbb{C}$,

$$\mathsf{cSet} \;=\; \mathsf{Set}^{\mathbb{C}^{\mathrm{op}}}\,.$$

It is thus generated by the representable presheaves $y([n])$, which will be written

$$\mathrm{I}^n = y([n])$$

and called the *standard n-cubes*.

3 The cofibration weak factorization system

Cofibrations. The *cofibrations* are a class $\mathcal{C}$ of maps in cSet, written

$$c : A \rightarrowtail B\,,$$

and are assumed to satisfy the following axioms:

(C1) All isomorphisms are cofibrations.

(C2) The composite of two cofibrations is a cofibration.

(C3) Cofibrations are monomorphisms.

(C4) Any pullback of a cofibration is a cofibration.

Moreover, we want the cofibrations to be classified by a subobject $\Phi \hookrightarrow \Omega$ of the standard subobject classifier $\top : 1 \to \Omega$ of cSet. One way to ensure this is to further assume that they are closed under all colimits in the category of cartesian squares. An equivalent way is to just assume it from the start:

(C0) There is a terminal object $\Phi' \rightarrowtail \Phi$ in the category of cofibrations and cartesian squares.

It follows from (C1) that $\Phi' = 1$ and the terminal cofibration $1 \rightarrowtail \Phi$ is a factorization of $\top : 1 \to \Omega$. We call this map $t : 1 \rightarrowtail \Phi$ the *cofibration classifier*. Note that we permit the case where $\Phi = \Omega$, i.e. all monos are cofibrations.

Cofibrant partial map classifier. The polynomial endofunctor [?] determined by the cofibration classifier $t : 1 \rightarrowtail \Phi$ is defined on objects by

$$X \;\mapsto\; \Phi_!\, t_*(X) \;=\; \sum_{\varphi:\Phi} X^{\varphi}\,.$$

We shall write $X^+ := \sum_{\varphi:\Phi} X^{\varphi}$.

Observe that by the definition of X^+ there is a pullback square,

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X^+ \arrow[d, "t_* X"] \\ 1 \arrow[r, "t"'] & \Phi \end{tikzcd}$$

since t is monic. Let $\eta : X \rightarrowtail X^+$ be the indicated top horizontal map; we call this map the *cofibrant partial map classifier* of X.

Proposition 2. *The map $\eta : X \rightarrowtail X^+$ classifies partial maps with cofibrant domain, in the following sense.*

1. *The map $\eta : X \rightarrowtail X^+$ is a cofibration.*
2. *For any object Z and any partial map $(s, g) : Z \leftarrowtail S \to X$, with $s : S \rightarrowtail Z$ a cofibration, there is a unique $f : Z \to X^+$ making a pullback square,*

$$\begin{tikzcd} S \arrow[r, "g"] \arrow[d, tail, "s"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, tail, "\eta"] \\ Z \arrow[r, "f"'] & X^+ . \end{tikzcd}$$

Proof. $\eta : X \rightarrowtail X^+$ is a cofibration since it is a pullback of $t : 1 \longrightarrow \Phi$. The second statement follows directly from the definition of X^+ as a polynomial (see [?], prop. 7). □

The +-Monad.

Proposition 3. *The pointed endofunctor determined by $\eta_X : X \rightarrowtail X^+$ has a natural multiplication $\mu_X : X^{++} \longrightarrow X^+$ making it a monad.*

Proof. Since the cofibrations are closed under composition, the monad structure on X^+ follows as in [?], proposition nm. Explicitly, μ_X is determined as the unique map making the following a pullback diagram.

$$\begin{tikzcd} X \arrow[r, "="] \arrow[d, tail, "\eta_X"'] & X \arrow[dd, tail, "\eta"] \\ X^+ \arrow[d, tail, "\eta_{X^+}"'] & \\ X^{++} \arrow[r, dotted, "\mu"'] & X^+ \end{tikzcd}$$

□

Relative partial map classifier. For any object $X \in \mathsf{cSet}$ the usual pullback functor

$$X^* : \mathsf{cSet} \longrightarrow \mathsf{cSet}/X ,$$

taking any A to the second projection $A \times X \longrightarrow X$, not only preserves the subobject classifier Ω, but also the cofibration classifier $\Phi \hookrightarrow \Omega$, where a map in cSet/X is defined to be a cofibration if it is one in cSet. Thus in cSet/X the *(relative) cofibration classifier* is the map

$$t \times X : 1 \times X \longrightarrow \Phi \times X \quad \text{over } X$$

which we may also write $t_X : 1_X \longrightarrow \Phi_X$. Like $t : 1 \longrightarrow \Phi$, this map determines a polynomial endofunctor

$$+_X : \mathsf{cSet}/X \longrightarrow \mathsf{cSet}/X\,,$$

which commutes (up to natural isomorphism) with $+ : \mathsf{cSet} \longrightarrow \mathsf{cSet}$ and $X^* : \mathsf{cSet} \longrightarrow \mathsf{cSet}/X$ in the evident way:

$$\begin{tikzcd}
\mathsf{cSet}/X \arrow[r, "+_X"] & \mathsf{cSet}/X \\
\mathsf{cSet} \arrow[u, "X^*"] \arrow[r, "+"'] & \mathsf{cSet} \arrow[u, "X^*"']
\end{tikzcd} \tag{1}$$

The endofunctor $+_X$ is also pointed $\eta : Y \longrightarrow Y^+$ and has a monad multiplication $\mu_Y : Y^{++} \longrightarrow Y^+$, for any $Y \longrightarrow X$, for the same reason that $+$ has this structure. Summarizing, we may say that *the polynomial monad* $+ : \mathsf{cSet} \longrightarrow \mathsf{cSet}$ *is fibered over* cSet.

Definition 4. A *+-algebra* in cSet is a cubical set A together with a retraction $\alpha : A^+ \longrightarrow A$ of $\eta_A : A \longrightarrow A^+$, i.e. an algebra for the pointed endofunctor $(+ : \mathsf{cSet} \longrightarrow \mathsf{cSet},\ \eta : 1 \longrightarrow +)$. Algebras for the monad $(+, \eta, \mu)$ will be referred to specifically as $(+, \eta, \mu)$*-algebras*, or *+-monad algebras.*

A *relative +-algebra* in cSet is a map $A \longrightarrow X$ together with an algebra structure for the pointed endofunctor $+_X : \mathsf{cSet}/X \longrightarrow \mathsf{cSet}/X$.

The factorization system.

Proposition 5. *There is an (algebraic) weak factoriation system on* cSet *given by taking as the left class the cofibrations and as the right class the (maps underlying) the relative +-algebras. Thus a right map is a map* $f : A \longrightarrow X$ *for which there is a retract* $\alpha : A' \longrightarrow A$ *over* X *of the canonical map*

$\eta_f : A \longrightarrow A'$ *over* X,

$$\begin{tikzcd} A \arrow[r, "\eta_f"] \arrow[rr, bend left, "="] \arrow[dr, "f"'] & A' \arrow[r, "\alpha"] \arrow[d, "f^+"] & A \arrow[dl, "f"] \\ & X. & \end{tikzcd}$$

Proof. The factorization of any map $f : Y \longrightarrow X$ is given simply by applying the (relative) +-functor

$$\begin{tikzcd} Y \arrow[r, "\eta_f"] \arrow[dr, "f"'] & Y' \arrow[d, "f^+"] \\ & X. \end{tikzcd}$$

We know that the unit η_f is always a cofibration, and since f^+ is the free algebra for the +-monad, it is in particular a +-algebra.

For the lifting condition, consider a cofibration $c : B \rightarrowtail C$, a right map $A \longrightarrow X$, with a $+_X$-algebra structure map $\alpha : A^+ \longrightarrow A$ over X, and a commutative square as indicated in the following.

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[dd, tail, "c"'] & A \arrow[dd] \arrow[dr, "\eta"'] & \\ & & A^+ \arrow[ul, bend right, "\alpha"'] \arrow[dl] \\ C \arrow[r, "f"'] & X & \end{tikzcd}$$

Thus over X, we have the situation

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, tail, "c"'] & A \arrow[d, "\eta"] \\ C \arrow[ur, dotted, "d"] & A^+ \arrow[u, bend right, "\alpha"'] \end{tikzcd}$$

and we seek a diagonal filler as indicated. Since $(c, g) : B \leftarrowtail C \longrightarrow A$ is a cofibrant partial map into A, there is a map $\varphi : C \longrightarrow A^+$ (over X) making a (pullback) square,

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, tail, "c"'] & A \arrow[d, "\eta"] \\ C \arrow[r, dotted, "\varphi"'] & A^+ \arrow[u, bend right, "\alpha"'] \end{tikzcd}$$

We thus have $d := \alpha \circ \varphi : C \longrightarrow A$ as the required diagonal filler.

The closure of the cofibrations under retracts follows from their classification by a universal object $t : 1 \longrightarrow \Phi$, and the closure of the right maps under retracts follows from their being the algebras for a pointed endofunctor underlying a monad (cf. [**?**]). Algebraicity of this weak factorization system also follows directly, since $+$ is a monad. □

Summarizing, we have a weak factorization system $(\mathcal{L}, \mathcal{R})$ on the category cSet of cubical sets, in which:

$$\begin{aligned} \mathcal{L} &= \mathcal{C} \quad \text{(the cofibrations)} \\ \mathcal{R} &= +\mathsf{Alg} \quad \text{(the relative +-algebras)} \end{aligned}$$

We shall call this the *cofibration weak factorization system.* As here, we will sometimes say that an object (or map) is a (relative) +-algebra when it can be equipped with a (relative) +-algebra structure; such maps will also be called *trivial fibrations* and the class of all such is denoted $\mathsf{TrivFib}$,

$$\mathsf{TrivFib} = \mathcal{C}^{\pitchfork}.$$

Uniform filling structure. It will be convenient to relate +-algebra structure with the more familiar diagonal filling condition of weak factorization systems, and specifically a special form of the latter that occurs in [CCHM16] under the name *uniform filling structure.*

Consider a generating sub*set* of cofibrations, consisting of all those cofibrations $c : C \rightarrowtail Z$ where Z is representable, $Z = \mathrm{I}^n$. Call these maps the *basic cofibrations*, and let

$$\mathsf{BCof} = \{c : C \rightarrowtail \mathrm{I}^n \mid c \in \mathcal{C}, n \geq 0\}. \tag{2}$$

Proposition 6. *For any object X in cSet the following are equivalent:*

1. *X is a* +-algebra*, i.e. there is a retraction $\alpha : X^+ \longrightarrow X$ of the unit $\eta : X \longrightarrow X^+$.*

2. *X is $\mathcal{C}$-*injective*, in the sense that it has the right lifting property with respect to all cofibrations,*

$$\mathcal{C} \pitchfork X.$$

3. *X has a* uniform filling structure*: for each basic cofibration $c : C \rightarrowtail \mathrm{I}^n$ and map $x : C \to X$ there is given an extension $j(c,x)$,*

$$\begin{tikzcd} C \arrow[r, "x"] \arrow[d, tail, "c"'] & X, \\ \mathrm{I}^n \arrow[ur, dotted, "j(c,x)"'] & \end{tikzcd} \tag{3}$$

and the choice is uniform in I^n *in the following sense: given any cubical map $u : \mathrm{I}^m \to \mathrm{I}^n$, the pullback $u^*c : u^*C \rightarrowtail \mathrm{I}^m$ is again a basic cofibration and fits into a commutative diagram of the form*

$$\begin{tikzcd} u^*C \arrow[r, "c^*u"] \arrow[d, tail, "u^*c"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[r, "x"] \arrow[d, tail, "c"'] & X. \\ \mathrm{I}^m \arrow[r, "u"'] & \mathrm{I}^n \arrow[ur, dotted, "j(c,x)"'] & \end{tikzcd} \tag{4}$$

For the pair $(u^*c,\ xc^*u)$ in (34) the chosen extension $j(u^*c, xc^*u) : \mathrm{I}^m \to X$, is equal to $j(c,x) \circ u$,

$$j(u^*c, xc^*u) = j(c,x)u. \tag{5}$$

Proof. Let (X,α) be a $+$-*algebra* and suppose given the span (c,x) as below, with c a cofibration.

$$\begin{tikzcd} C \arrow[r, "x"] \arrow[d, tail, "c"'] & X \\ Z & \end{tikzcd}$$

Let $\chi(c,x) : Z \longrightarrow X^+$ be the classifying map of the evident partial map $(c,x) : Z \rightharpoonup X$, so that we have a pullback square as follows.

$$\begin{tikzcd} C \arrow[r, "x"] \arrow[d, tail, "c"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\eta"] \\ Z \arrow[r, "\chi(c,x)"'] & X^+ \end{tikzcd} \tag{6}$$

Then set

$$j = \alpha \circ \chi(c, x) : Z \longrightarrow X \tag{7}$$

to get a filler,

$$\begin{tikzcd} C \arrow[r, "x"] \arrow[d, tail, "c"'] & X \arrow[d, "\eta"] \\ Z \arrow[r, "\chi(c,x)"'] \arrow[ur, dotted, "j"] & X^{+} \arrow[u, bend right=60, "\alpha"'] \end{tikzcd} \tag{8}$$

since $j \circ c = \alpha \circ \chi(c, x) \circ c = \alpha \circ \eta \circ x = x$. Thus (1) implies (2). To see that it also implies (3), observe that in the case where $Z = I^n$ and we specify, in (7), that

$$j(c, x) = \alpha \circ \chi(c, x) : \mathrm{I}^n \longrightarrow X, \tag{9}$$

then the assignment is natural in I^n. Indeed, given any $u : \mathrm{I}^m \longrightarrow \mathrm{I}^n$, we have

$$j(c', xu') = \alpha \circ \chi(c', xu') = \alpha \circ \chi(c, x) \circ u = j(c, x)u, \tag{10}$$

by the uniqueness of classifying maps.

It is clear that (2) implies (1), since if $\mathcal{C} \pitchfork X$ then we can take as an algebra structure $\alpha : X^{+} \longrightarrow X$ any filler for the span

$$\begin{tikzcd} X \arrow[r, "="] \arrow[d, tail, "\eta"'] & X. \\ X^{+} \arrow[ur, dotted, "\alpha"'] & \end{tikzcd}$$

To see that (3) implies (1), suppose that X has a uniform filling structure j and we want to define an algebra structure $\alpha : X^{+} \longrightarrow X$. By Yoneda, for every $y : \mathrm{I}^n \longrightarrow X^{+}$ we need a map $\alpha(y) : \mathrm{I}^n \longrightarrow X$, naturally in I^n, in the sense that for any $u : \mathrm{I}^m \longrightarrow \mathrm{I}^n$, we have

$$\alpha(yu) = \alpha(y)u. \tag{11}$$

Moreover, to ensure that $\alpha\eta = 1_X$, for any $x : \mathrm{I}^n \longrightarrow X$ we must have $\alpha(\eta \circ x) = x$. So take $y : \mathrm{I}^n \longrightarrow X^{+}$ and let

$$\alpha(y) = j(y^{*}\eta, y'),$$

as indicated on the right below.

$$\begin{tikzcd} u^{*}C \arrow[r, "u'"] \arrow[d, tail, "u^{*}y^{*}\eta"'] \arrow[dr, phantom, very near start, "\lrcorner"] & C \arrow[r, "y'"] \arrow[d, tail, "y^{*}\eta"'] \arrow[dr, phantom, very near start, "\lrcorner"] & X. \arrow[d, "\eta"] \\ \mathrm{I}^m \arrow[r, "u"'] & \mathrm{I}^n \arrow[r, "y"'] \arrow[ur, dotted, "j(y^{*}\eta, y')"'] & X^{+} \end{tikzcd} \tag{12}$$

Then for any $u : \mathrm{I}^m \longrightarrow \mathrm{I}^n$, we indeed have

$$\alpha(yu) = j\big((yu)^*\eta, y'u'\big) = j(y^*\eta, y') \circ u = \alpha(y)u,$$

by the uniformity of j. Finally, if $y = \eta \circ x$ for some $x : \mathrm{I}^n \longrightarrow X$ then

$$\alpha(\eta x) = j\big((\eta x)^*\eta, (\eta x)'\big) = j(1_X, x) = x,$$

because the defining diagram for $\alpha(\eta x)$, i.e. the one on the right in (12), then factors as

$$\begin{tikzcd} \mathrm{I}^n \arrow[r, "x"] \arrow[d, tail, "="'] \arrow[dr, phantom, very near start, "\lrcorner"] & X \arrow[r, "="] \arrow[d, tail, "="'] \arrow[dr, phantom, very near start, "\lrcorner"] & X, \arrow[d, "\eta"] \\ \mathrm{I}^n \arrow[r, "x"'] & X \arrow[r, "\eta"'] & X^+ \end{tikzcd} \tag{13}$$

and the only possible extension $j(1_X, x)$ for the span $(1_{\mathrm{I}^n}, x)$ is x itself. $\square$

The relative version of the foregoing is entirely analogous, since the $+$-functor is fibered over cSet in the sense of diagram (1). We can therefore omit the entirely analogous proof. The statement is as follows.

Proposition 7. *For any map $f : Y \longrightarrow X$ in* cSet *the following are equivalent:*

1. $f : Y \to X$ *is a* (relative) $+$-algebra (over X)*, i.e. there is a retraction $\alpha : Y' \to Y$ over X of the unit $\eta : Y \to Y'$ over X, where $f^+ : Y' \to X$ is the result of the relative $+_X$-functor applied to f, as in definition 4.*

2. $f : Y \to X$ *is* trivial fibration *in the sense that it has the right lifting property with respect to all cofibrations,*

$$\mathcal{C} \pitchfork f.$$

3. $f : Y \to X$ *has a* uniform filling structure*: for each basic cofibration $c : C \rightarrowtail \mathrm{I}^n$ and maps $x : C \to X$ and $y : \mathrm{I}^n \to Y$ making the square below commute, there is given a diagonal filler $j(c, x, y)$,*

$$\begin{tikzcd} C \arrow[r, "x"] \arrow[d, tail, "c"'] & X \arrow[d, "f"] \\ \mathrm{I}^n \arrow[r, "y"'] \arrow[ur, dotted, "{j(c,x,y)}"] & Y, \end{tikzcd} \tag{14}$$

and the choice is uniform in I^n *in the following sense: given any cubical map* $u : \mathrm{I}^m \to \mathrm{I}^n$*, the pullback* $u^*c : u^*C \rightarrowtail \mathrm{I}^m$ *is again a basic cofibration and fits into a commutative diagram of the form*

$$\begin{tikzcd} u^*C \arrow[r, "c^*u"] \arrow[d, tail, "u^*c"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[r, "x"] \arrow[d, tail, "c"] & X \arrow[d, "f"] \\ \mathrm{I}^m \arrow[r, "u"'] & \mathrm{I}^n \arrow[r, "y"'] \arrow[ur, dotted, "j(c,x,y)"'] & Y. \end{tikzcd} \tag{15}$$

For the evident triple $(u^*c,\, xc^*u, yu)$ *in* (15) *the chosen diagonal filler*

$$j(u^*c, xc^*u, yu) : \mathrm{I}^m \longrightarrow X$$

is equal to $j(c, x, y) \circ u$*,*

$$j(u^*c, xc^*u, yu) = j(c, x, y)u. \tag{16}$$

We next collect some basic facts about +-algebras/trivial fibrations: they have sections, they are closed under composition and retracts, and they are closed under pullback and pushforward along all maps.

Corollary 8. *A +-algebra* $A \to X$ *always has a section* $s : X \to A$*.*

Proof. It suffices to find a filler for the following diagram.

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, tail] & A \arrow[d] \\ X \arrow[r, "="'] \arrow[ur, dotted, "s"] & X \end{tikzcd} \tag{17}$$

But $0 \to X$ is always a cofibration by assumption. □

Corollary 9. *If* $f : Y \to X$ *is a +-algebra over* X *and* $g : Z \to Y$ *is a +-algebra over* Y*, then there is a canonical way of making* $f \circ g : Z \to X$ *into a +-algebra over* X*.*

Proof. One of several equivalent constructions is as follows. The square below has a caninical diagonal filler j as indicated, because η is a cofibration and f a +-algebra.

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, tail, "\eta"'] & Y \arrow[d, "f"] \\ (\Sigma_Y Z)^+ \arrow[r] \arrow[ur, dotted, "j"] & X, \end{tikzcd} \tag{18}$$

Now use the fact that g is a $+$-algebra to get a canonical retraction of η over Y. $\square$

Corollary 10. *If $f : Y \to X$ is a $+$-algebra over X and $f' : Y' \to X'$ is a retract of f in the arrow category, then f' is a $+$-algebra over X'.*

Proof. The right class of a weak factorization system is always closed under retracts. $\square$

Corollary 11. *For any map $f : Y \to X$ and any $+$-algebra $A \to B$ over X, the pullback $f^*A \to Y$ is a $+$-algebra over Y.*

$$\begin{tikzcd} f^*A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d] \\ Y \arrow[r, "f"'] & X \end{tikzcd} \tag{19}$$

Proof. The right class of a weak factorization system is always stable under pullback, but one can also see directly that a pullback of a $+$-algebra is again a $+$-algebra, using the fact that the $+$-endofunctor commutes with pullbacks. $\square$

The proof of the following is a straighforward application of the general fact that a left adjoint preserves the left class of a weak factorization system if and only if its right adjoint preserves the right class. We give a more detailed argument for this special case since we will need an analogous version later.

Corollary 12. *For any map $f : Y \to X$ and any $+$-algebra $A \to Y$ over Y, the pushforward $f_*A \to X$ is a $+$-algebra over X.*

$$\begin{tikzcd} A \arrow[d] & f_*A \arrow[d] \\ Y \arrow[r, "f"'] & X \end{tikzcd} \tag{20}$$

More generally, if $B \to Y$ is any map and $g : A \to B$ is a $+$-algebra over B, then the pushforward $f_*g : f_*A \to f_*B$ is a $+$-algebra over f_*B.

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d] & B \arrow[dl] & f_*A \arrow[r, "f_*g"] \arrow[d] & f_*B \arrow[dl] \\ Y \arrow[rr, "f"'] & & X & \end{tikzcd} \tag{21}$$

Proof. Note that the first statement follows from the second one by taking $B = Y$ and $B \to Y$ the identity map.

To prove the second statement, by Proposition 7 it suffices to show that $f_* g : f_* A \to f_* B$ has the right lifting property, in the slice category over X, against all cofibrations. So consider a lifting problem over X, with a cofibration on the left:

$$\begin{tikzcd} C \arrow[r] \arrow[d, tail, "c"'] & f_* A \arrow[d, "f_* g"] \\ Z \arrow[r] & f_* B \end{tikzcd} \tag{22}$$

Transposing across the adjunction $f^* \dashv f_*$ results in a lifting problem over Y of the form

$$\begin{tikzcd} f^* C \arrow[r] \arrow[d, tail, "f^* c"'] & A \arrow[d, "g"] \\ f^* Z \arrow[r] \arrow[ur, dotted, "j"] & B \end{tikzcd}$$

in which $f^* c$ is a cofibration, because these are preserved by pullbacks. Thus there is a filler j as indicated, since by assumption $g : A \to B$ is a $+$-algebra over B and therefore a trivial fibration. Transposing the filler j then provides a filler for the original problem (22). □

4 Partial box filling (biased)

Our next goal is the specification of a second weak factorization system (the *fibration weak factorization system*) with a restricted class of "trivial" cofibrations on the left, and an expanded class of right maps, the fibrations.

As a warm-up, we first recall the specification of the trivial-cofibration/fibration WFS from [GS17]. (In an appendix we show that these fibrations agree with those specified in the "logical style" of [CCHM16, OP17]). In the subsequent section we shall modify the specification of fibrations in order to arrive at an "unbiased" version that is more appropriate for the cartesian setting.

A *generating class of (biased) trivial cofibrations* are all maps of the form

$$c \otimes \delta_\epsilon : D \longrightarrow Z \times \mathrm{I}\,, \tag{23}$$

where:

1. $c : C \rightarrowtail Z$ is an arbitrary cofibration,

2. $\delta_\epsilon : 1 \longrightarrow \mathrm{I}$ is one of the two "endpoint inclusions" where, recall, $1 = y[0]$, and $I = y[1]$, and for $\epsilon = 0, 1$, we have the maps $\delta_\epsilon : 1 \longrightarrow \mathrm{I}$ corresponding to the two bipointed maps $0, 1 : \{0, x, 1\} \longrightarrow \{0, 1\}$.

3. $c \otimes \delta_\epsilon$ is the pushout-product (resp. "Leibniz tensor") of the cofibration $c : C \rightarrowtail Z$ and an endpoint $\delta_\epsilon : 1 \longrightarrow \mathrm{I}$, as indicated in the following diagram (in which the unlabelled maps are the expected ones).

$$\begin{tikzcd}
C \times 1 \arrow[r] \arrow[d] & C \times \mathrm{I} \arrow[d] \arrow[ddr, bend left] & \\
Z \times 1 \arrow[r] \arrow[drr, bend right] & Z +_C (C \times \mathrm{I}) \arrow[dr, "c\otimes\delta_\epsilon"] & \\
& & Z \times \mathrm{I}
\end{tikzcd} \tag{24}$$

4. $D = Z +_C (C \times \mathrm{I})$ is the indicated pushout, the domain of $c \otimes \delta_\epsilon$.

In order to insure that such maps are indeed cofibrations, we assume two further axioms:

(C5) The endpoint inclusions $\delta_\epsilon : 1 \longrightarrow \mathrm{I}$ are cofibrations.

(C6) The cofibrations are closed under pushout-products.

In place of (C6), we could require that cofibrations be closed under the join operation $A \vee B$ in the lattice of subobjects of an object.

Fibrations (biased version). Let

$$\mathcal{C} \otimes \delta_\epsilon \;=\; \{c \otimes \delta_\epsilon : D \rightarrowtail Z \times \mathrm{I} \mid c \in \mathcal{C},\ \epsilon = 0, 1\}$$

be the class of all such pushout-products of arbitrary cofibrations $c : C \rightarrowtail Z$ with endpoint inclusions $\delta_\epsilon : 1 \rightarrowtail \mathrm{I}$. The *(biased) fibrations* are defined to be the right class of these generating trivial cofibrations,

$$(\mathcal{C} \otimes \delta_\epsilon)^{\pitchfork} \;=\; \mathcal{F}\,.$$

Thus a map $f : Y \longrightarrow X$ is a (biased) fibration if for every commutative square of the form

$$\begin{tikzcd}
Z +_C (C \times \mathrm{I}) \arrow[r] \arrow[d, tail, "c\otimes\delta_\epsilon"'] & Y \arrow[d, "f"] \\
Z \times \mathrm{I} \arrow[r] \arrow[ur, dotted, "j"] & X
\end{tikzcd} \tag{25}$$

with a generating trivial cofibration on the left, there is a diagonal filler j as indicated. This condition can be seen as a generalized homotopy lifting property.

To relate this notion of fibration to the cofibration weak factorization system, fix any map $u : A \longrightarrow B$, and recall (e.g. from [?]) that the pushout-product with u is a functor on the arrow category

$$(-) \otimes u : \mathsf{cSet}^{\mathbb{2}} \longrightarrow \mathsf{cSet}^{\mathbb{2}} .$$

This functor has a right adjoint, the *pullback-hom* (or "Leibniz exponential"), which for a map $f : X \longrightarrow Y$ we will write as

$$(u \Rightarrow f) : Y^B \longrightarrow (X^B \times_{X^A} Y^A) .$$

The pullback-hom is determined as indicated in the following diagram (in which the unlabelled maps are the expected ones).

$$\begin{tikzcd}
Y^B \arrow[dr, "u \Rightarrow f"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X^B \times_{X^A} Y^A \arrow[r] \arrow[d] & Y^A \arrow[d] \\
& X^B \arrow[r] & X^A
\end{tikzcd} \tag{26}$$

Using the $\otimes \dashv \Rightarrow$ adjunction on the arrow category, we can now show the following (cf. [?], prop. n.m).

Proposition 13. *An object X is fibrant if and only if both of the endpoint projections $X^{\mathrm{I}} \longrightarrow X$ from the pathspace are (relative) $+$-algebras (over X). More generally, a map $f : Y \longrightarrow X$ is a fibration iff both of the maps*

$$(\delta_\epsilon \Rightarrow f) : Y^I \longrightarrow X^I \times_X Y$$

are $+$-algebras (for $\epsilon = 0, 1$).

Proof. The first statement follows from the second, since the pathspace projections $X^{\mathrm{I}} \longrightarrow X$ are just the maps

$$(\delta_\epsilon \Rightarrow !_X) : X^I \longrightarrow (1^I \times_1 X) \cong X ,$$

for $!_X : X \longrightarrow 1$.

By definition, $f : X \longrightarrow Y$ is a fibration iff every square of the form

$$
\begin{tikzcd}
Z +_C (C \times \mathrm{I}) \arrow[r] \arrow[d, tail, "c \otimes \delta_\epsilon"'] & Y \arrow[d, "f"] \\
Z \times \mathrm{I} \arrow[r] \arrow[ur, dotted, "j"] & X ,
\end{tikzcd}
\tag{27}
$$

with a generating trivial cofibration $c \otimes \delta_\epsilon$ on the left, has is a diagonal filler j as indicated. Briefly,

$$(c \otimes \delta_\epsilon) \pitchfork f \qquad (\text{for } c \in \mathcal{C},\ \epsilon = 0, 1).$$

By the $\otimes \dashv \Rightarrow$ adjunction, this is equivalent to the condition

$$c \pitchfork (\delta_\epsilon \Rightarrow f) \qquad (\text{for } c \in \mathcal{C},\ \epsilon = 0, 1).$$

That is, for every square

$$
\begin{tikzcd}
C \arrow[r] \arrow[d, tail, "c"'] & Y^I \arrow[d, "\delta_\epsilon \Rightarrow f"] \\
Z \arrow[r] \arrow[ur, dotted, "k"] & X^I \times_X Y ,
\end{tikzcd}
$$

with an arbitrary cofibration $c : C \rightarrowtail Z$ on the left, there is a diagonal filler k as indicated, for $\epsilon = 0, 1$. But this is just to say that the maps $\delta_\epsilon \Rightarrow f$ are in the right class of the cofibrations, which is equivalent to their being $+$-algebras, as claimed. $\square$

Fibration structure. The $\otimes \dashv \Rightarrow$ adjunction determines the fibrations in terms of the trivial fibrations, which in turn can be determined by *uniform* lifting against a *set* of basic cofibrations, by proposition 7. We can similarly determine the fibrations by uniform lifting against a *set* of trivial cofibrations, consisting of all those $c \otimes \delta_\epsilon$ in $\mathcal{C} \otimes \delta_\epsilon$ where $c : C \rightarrowtail Z$ has a representable codomain $Z = \mathrm{I}^n$. Call these maps the *basic (biased) trivial cofibrations*, and let

$$\mathcal{B} \otimes \delta_\epsilon = \{c \otimes \delta_\epsilon : B \rightarrowtail \mathrm{I}^{n+1} \mid c : C \rightarrowtail \mathrm{I}^n,\ \epsilon = 0, 1,\ n \geq 0\}, \tag{28}$$

where the pushout-product $c \otimes \delta_\epsilon$ now takes the simpler form

$$\begin{tikzcd}
C \arrow[r] \arrow[d, tail] & C \times \mathrm{I} \arrow[d] \arrow[ddr, bend left] & \\
\mathrm{I}^n \arrow[r] \arrow[drr, bend right] & \mathrm{I}^n +_C (C \times \mathrm{I}) \arrow[dr, tail, "c \otimes \delta_\epsilon"'] & \\
 & & \mathrm{I}^n \times \mathrm{I}
\end{tikzcd} \tag{29}$$

for a cofibration $c : C \rightarrowtail \mathrm{I}^n$, an endpoint $\delta_\epsilon : 1 \longrightarrow \mathrm{I}$, and with domain $B = \big(\mathrm{I}^n +_C (C \times \mathrm{I})\big)$. These subobjects $B \rightarrowtail \mathrm{I}^{n+1}$ can be seen geometrically as generalized open box inclusions.

For any map $f : Y \longrightarrow X$ a (uniform, biased) fibration structure on f is a choice of diagonal fillers $j_\epsilon(c, x, y)$,

$$\begin{tikzcd}
\mathrm{I}^n +_C (C \times \mathrm{I}) \arrow[r, "x"] \arrow[d, "c \otimes \delta_\epsilon"'] & X \arrow[d, "f"] \\
\mathrm{I}^n \times \mathrm{I} \arrow[r, "y"'] \arrow[ur, dotted, "j_\epsilon(c,x,y)"'] & Y,
\end{tikzcd} \tag{30}$$

for each basic trivial cofibration $c \otimes \delta_\epsilon : B = (\mathrm{I}^n +_C (C \times \mathrm{I})) \rightarrowtail \mathrm{I}^{n+1}$ and maps $x : B \longrightarrow X$ and $y : \mathrm{I}^{n+1} \longrightarrow Y$, which is *uniform in* I^n in the following sense: given any cubical map $u : \mathrm{I}^m \longrightarrow \mathrm{I}^n$, the pullback $u^*c : u^*C \rightarrowtail \mathrm{I}^m$ of $c : C \rightarrowtail \mathrm{I}^n$ along u determines another basic trivial cofibration

$$u^*c \otimes \delta_\epsilon : B' = (\mathrm{I}^m +_{u^*C} (u^*C \times \mathrm{I})) \rightarrowtail \mathrm{I}^{m+1},$$

which fits into a commutative diagram of the form

$$\begin{tikzcd}
\mathrm{I}^m +_{u^*C} (u^*C \times \mathrm{I}) \arrow[r, "(u \times \mathrm{I})'"] \arrow[d, "u^*c \otimes \delta_\epsilon"'] & \mathrm{I}^n +_C (C \times \mathrm{I}) \arrow[r, "x"] \arrow[d, "c \otimes \delta_\epsilon"'] & X \arrow[d, "f"] \\
\mathrm{I}^m \times \mathrm{I} \arrow[r, "u \times \mathrm{I}"'] & \mathrm{I}^n \times \mathrm{I} \arrow[r, "y"'] \arrow[ur, dotted, "j_\epsilon(c,x,y)"'] & Y,
\end{tikzcd} \tag{31}$$

by applying the functor $(-) \otimes \delta_\epsilon$ to the pullback square relating u^*c to c. Now for the outer rectangle in (31) there is a chosen diagonal filler

$$j_\epsilon(u^*c, x(u \times \mathrm{I})', y(u \times \mathrm{I})) : \mathrm{I}^m \times \mathrm{I} \longrightarrow X$$

and for this map we require that

$$j_\epsilon(u^*c, x(u \times \mathrm{I})', y(u \times \mathrm{I})) = j_\epsilon(c, x, y) \circ (u \times \mathrm{I}). \tag{32}$$

This is a reformulation of the logical specification given in [CCHM16] (see the appendix).

Definition 14. A *(uniform, biased) fibration structure* on a map $f : Y \to X$ is a choice of fillers $j_\epsilon(c, x, y)$ as in (33) satisfying (35) for all maps $u : \mathrm{I}^m \to \mathrm{I}^n$.

Essentially the same argument as that given for Proposition 13 also yields the following sharper formulation in terms of fibration structure.

Corollary 15. *Fibration structure on a map $f : Y \to X$ is equivalent to a pair of $+$-algebra structures on the maps*

$$(\delta_\epsilon \Rightarrow f) : Y^I \longrightarrow X^I \times_X Y$$

for $\epsilon = 0, 1$.

Finally, we have the analogue of proposition 6 for fibrant objects; we omit the analogous statement of proposition 7 for fibrations, as well as the entirely analogous proof.

Corollary 16. *For any object X in* cSet *the following are equivalent:*

1. *X is* fibrant, *i.e. every partial map to X with a generating trivial cofibration $D \rightarrowtail Z \times \mathrm{I}$ as domain of definition extends to a total map $Z \times \mathrm{I} \to X$,*

$$\mathcal{C} \otimes \delta_\epsilon \ \pitchfork \ f$$

2. *There are $+$-algebra structures on the canonical maps*

$$(\delta_\epsilon \Rightarrow X) : X^I \longrightarrow X,$$

   for $\epsilon = 0, 1$.

3. *$X \to 1$ has a* (uniform, biased) fibration structure. *Explicitly, for each basic trivial cofibration $c \otimes \delta_\epsilon : B \rightarrowtail \mathrm{I}^{n+1}$ and map $x : B \to X$, there is given an extension $j_\epsilon(c, x)$,*

$$\begin{tikzcd} B \arrow[r, "x"] \arrow[d, tail, "c\otimes\delta_\epsilon"'] & X, \\ \mathrm{I}^{n+1} \arrow[ur, dotted, "j_\epsilon(c,x)"'] & \end{tikzcd} \tag{33}$$

and the choice is uniform in I^n *in the sense: given any cubical map* $u : \mathrm{I}^m \to \mathrm{I}^n$*, the pullback* $u^*c \otimes \delta_\epsilon : B' \rightarrowtail \mathrm{I}^m \times \mathrm{I}$ *fits into a commutative diagram of the form*

$$\begin{tikzcd} B' \arrow[r, "(u\times\mathrm{I})'"] \arrow[d, tail, "u^*c\otimes\delta_\epsilon"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[r, "x"] \arrow[d, tail, "c\otimes\delta_\epsilon"] & X. \\ \mathrm{I}^m \times \mathrm{I} \arrow[r, "u\times\mathrm{I}"'] & \mathrm{I}^n \times \mathrm{I} \arrow[ur, dotted, "j(c,x)"'] & \end{tikzcd} \tag{34}$$

Then for the pair $(u^*c \otimes \delta_\epsilon,\ x(u \times \mathrm{I})')$ *in* (34) *the chosen extension*

$$j(u^*c \otimes \delta_\epsilon, x(u \times \mathrm{I})') : \mathrm{I}^m \times \mathrm{I} \longrightarrow X$$

is equal to $j(c, x) \circ (u \times \mathrm{I})$*,*

$$j(u^*c \otimes \delta_\epsilon, x(u \times \mathrm{I})') = j(c, x)(u \times \mathrm{I}). \tag{35}$$

5 Partial box filling (unbiased)

Rather than building a weak factorization system based on the foregoing notion of (biased) fibration (as is done in [**?**, OP17]), we shall first eliminate the "bias" on a choice of endpoint $\delta_\epsilon : 1 \longrightarrow \mathrm{I}$, expressed by the indexing $\epsilon = 0, 1$. This will have the effect of adding more trivial cofibrations, and thus more weak equivalences, to our model structure. Consider first the simple path-lifting condition, which is a special case of (25) with $c = ! : 0 \rightarrowtail 1$, since $! \otimes \delta_\epsilon = \delta_\epsilon$:

$$\begin{tikzcd} 1 \arrow[r] \arrow[d, tail, "\delta_\epsilon"'] & Y \arrow[d, "f"] \\ \mathrm{I} \arrow[r] \arrow[ur, dotted, "j_\epsilon"'] & X\,. \end{tikzcd}$$

(Note that $0 \rightarrowtail 1$ is a cofibration by axioms C4 and C5).

In toplogical spaces, rather than requiring lifts j_ϵ for each of the endpoints $\epsilon = 0, 1$, we could instead require that there be a lift j_i *for each point* $i : 1 \longrightarrow \mathrm{I}$ *in the real interval* $\mathrm{I} = [0, 1]$. Such "unbiased path-lifting" can be formulated in cSet by introducing a "generic point" $\delta : 1 \longrightarrow \mathrm{I}$, by passing to cSet/I, and then requiring path-lifting with respect to δ. The following specification implements that idea, while also adding partiality in the sense of the foregoing section. We need the following strengthening of axiom C5.

(C5') The diagonal map $\delta : \mathrm{I} \longrightarrow \mathrm{I} \times \mathrm{I}$ is a cofibration.

Definition 17 (Fibration)**.** Let $\delta : \mathrm{I} \longrightarrow \mathrm{I} \times \mathrm{I}$ be the diagonal map.

1. An object X is *(unbiased) fibrant* if the map

$$(\delta \Rightarrow X) = \langle \mathsf{eval}, p_2 \rangle : X^{\mathrm{I}} \times \mathrm{I} \longrightarrow X \times \mathrm{I}$$

is a $+$-algebra.

2. A map $f : Y \to X$ is an *(unbiased) fibration* if the map

$$(\delta \Rightarrow f) = \langle f^{\mathrm{I}} \times \mathrm{I}, \langle \mathsf{eval}, p_2 \rangle \rangle : Y^{\mathrm{I}} \times \mathrm{I} \longrightarrow (X^{\mathrm{I}} \times \mathrm{I}) \times_{(X \times \mathrm{I})} (Y \times \mathrm{I})$$

is a $+$-algebra.

Condition (2) above says that the pullback-hom of the generic point $\delta : 1 \to \mathrm{I}$ with $\mathrm{I}^* f$, constructed in the slice category cSet/I, is a $+$-algebra. This can be reformulated as follows.

Proposition 18. *A map $f : Y \to X$ is an (unbiased) fibration if and only if the canonical map c to the pullback, in the following diagram, is a $+$-algebra.*

$$\begin{tikzcd}
Y^{\mathrm{I}} \times \mathrm{I} \arrow[dr, dotted, "c"] \arrow[drr, bend left, "\mathsf{eval}"] \arrow[ddr, bend right, "f^{\mathrm{I}} \times \mathrm{I}"'] & & \\
& Y_{\mathsf{eval}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "f"] \\
& X^{\mathrm{I}} \times \mathrm{I} \arrow[r, "\mathsf{eval}"'] & X.
\end{tikzcd} \tag{36}$$

Proof. We interpolate another pullback into the rectangle in (36) to obtain

$$\begin{tikzcd}
Y_{\mathsf{eval}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \times \mathrm{I} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "f"] \\
X^{\mathrm{I}} \times \mathrm{I} \arrow[r] & X \times \mathrm{I} \arrow[r] & X
\end{tikzcd} \tag{37}$$

with the evident maps. The lefthand square is therefore a pullback, so we indeed have that

$$Y_{\mathsf{eval}} \;=\; (X^{\mathrm{I}} \times \mathrm{I}) \times_{(X \times \mathrm{I})} (Y \times \mathrm{I})$$

and $c = (\delta \Rightarrow f)$. $\square$

Now we can run the proof of Proposition 13 backwards in order to determine a class of generating trivial cofibrations for the unbiased case. We consider pairs of maps $c : C \rightarrowtail Z$ and $z : Z \longrightarrow \mathrm{I}$, where the former is a cofibration and the latter is regarded as an "I-indexing", so that

$$\begin{tikzcd} C \arrow[r, tail, "c"] \arrow[dr] & Z \arrow[d, "z"] \\ & \mathrm{I} \end{tikzcd}$$

can be regarded as an I-indexed family of cofibrations. Let

$$\mathsf{Gph}(z) : Z \longrightarrow Z \times \mathrm{I}\,,$$

be the graph of $z : Z \longrightarrow \mathrm{I}$, i.e. $\mathsf{Gph}(z) = \langle 1_Z, z \rangle$, and then let

$$c \otimes_z \delta := [\mathsf{Gph}(z), c \times \mathrm{I}] : Z +_C (C \times \mathrm{I}) \longrightarrow Z \times \mathrm{I}\,,$$

which is easily seen to be well-defined on the indicated pushout.

$$\begin{tikzcd} C \arrow[r, "\mathsf{Gph}(zc)"] \arrow[d, tail, "c"'] & C \times \mathrm{I} \arrow[d] \arrow[ddr, bend left, "c\times \mathrm{I}"] & \\ Z \arrow[r] \arrow[drr, bend right, "\mathsf{Gph}(z)"'] & Z +_C (C \times \mathrm{I}) \arrow[dr, dotted, "c \otimes_z \delta"] & \\ & & Z \times \mathrm{I}\,. \end{tikzcd} \tag{38}$$

This specification differs from the similar (24) by using $\mathsf{Gph}(z)$ for the inclusion $Z \rightarrowtail Z \times \mathrm{I}$, rather than one of the "face maps" associated to the endpoint inclusions $\delta_\epsilon : 1 \longrightarrow \mathrm{I}$. (Note that a graph is always a cofibration by pulling back a diagonal.) The subobject $c \otimes_z \delta \rightarrowtail Z \times \mathrm{I}$ is the join of the subobjects $\mathsf{Gph}(z) \rightarrowtail Z \times \mathrm{I}$ and the cylinder $C \times \mathrm{I} \rightarrowtail Z \times \mathrm{I}$.

The maps of the form $c \otimes_z \delta : Z +_C (C \times \mathrm{I}) \rightarrowtail Z$ now form a *class of generating trivial cofibrations* in the expected sense. Let

$$\mathcal{C} \otimes \delta \;=\; \{c \otimes_z \delta : D \rightarrowtail Z \times \mathrm{I} \mid c : C \rightarrowtail Z, z : Z \longrightarrow \mathrm{I}\}\,, \tag{39}$$

then the fibrations are exactly the right class of these,

$$\mathcal{F} = (\mathcal{C} \otimes \delta)^{\pitchfork}.$$

Proposition 19. *A map $f : Y \to X$ is an (unbiased) fibration iff for every pair of maps $c : C \rightarrowtail Z$ and $z : Z \to \mathrm{I}$, where the former is a cofibration, every commutative square of the following form has a diagonal filler, as indicated.*

$$\begin{tikzcd} Z +_C (C \times \mathrm{I}) \arrow[r] \arrow[d, tail, "c \otimes_z \delta"'] & Y \arrow[d, "f"] \\ Z \times \mathrm{I} \arrow[r] \arrow[ur, dotted, "j"] & X. \end{tikzcd} \tag{40}$$

Proof. Suppose that for all $c : C \rightarrowtail Z$ and $z : Z \to \mathrm{I}$, we have $(c \otimes_z \delta) \pitchfork f$ in cSet. Pulling f back over I, this is equivalent to the condition $c \otimes \delta \pitchfork \mathrm{I}^* f$ in cSet/I, for all cofibrations $c : C \rightarrowtail Z$ over I, which is equivalent to $c \pitchfork (\delta \Rightarrow \mathrm{I}^* f)$ in cSet/I for all cofibrations $c : C \rightarrowtail Z$. But this in turn means that $\delta \Rightarrow \mathrm{I}^* f$ is a +-algebra, which by definition means that f is a fibration. □

Unbiased fibration structure. As in the biased case, the fibrations can also be determined by *uniform* right-lifting against a generating *set* of trivial cofibrations, now consisting of all those $c \otimes_z \delta$ in $\mathcal{C} \otimes \delta$ for which $c : C \rightarrowtail Z$ has a representable codomain $Z = \mathrm{I}^n$. Call these maps the *basic (unbiased) trivial cofibrations*, and let

$$\mathcal{B} \otimes \delta = \{ c \otimes_z \delta : B \rightarrowtail \mathrm{I}^{n+1} \mid c : C \rightarrowtail \mathrm{I}^n,\ z : \mathrm{I}^n \to \mathrm{I},\ n \geq 0 \}\,, \tag{41}$$

where the pushout-product $c \otimes_z \delta$ now has the form

$$\begin{tikzcd} C \arrow[r, "\mathsf{Gph}(zc)"] \arrow[d, tail, "c"'] & C \times \mathrm{I} \arrow[d] \arrow[ddr, bend left, "c \times \mathrm{I}"] & \\ \mathrm{I}^n \arrow[r] \arrow[drr, bend right, "\mathsf{Gph}(z)"'] & \mathrm{I}^n +_C (C \times \mathrm{I}) \arrow[dr, dotted, "c \otimes_z \delta"'] & \\ & & \mathrm{I}^n \times \mathrm{I}\,. \end{tikzcd} \tag{42}$$

for a cofibration $c : C \rightarrowtail \mathrm{I}^n$, an indexing map $z : \mathrm{I}^n \to \mathrm{I}$, and with domain $B = \big(\mathrm{I}^n +_C (C \times \mathrm{I})\big)$. These subobjects $B \rightarrowtail \mathrm{I}^{n+1}$ can again be seen geometrically as "generalized open box" inclusions, but now the floor or lid of the open box may be replaced by a "cross-section" given by the graph of a map $z : \mathrm{I}^n \longrightarrow \mathrm{I}$.

For any map $f : Y \to X$ a (uniform, unbiased) fibration structure on f is a choice of diagonal fillers $j(c, z, x, y)$,

$$\begin{tikzcd}
B \arrow[r, "x"] \arrow[d, tail, "c \otimes_z \delta"'] & X \arrow[d, "f"] \\
\mathrm{I}^n \times \mathrm{I} \arrow[r, "y"'] \arrow[ur, dotted, "j(c,z,x,y)"'] & Y,
\end{tikzcd} \tag{43}$$

for each basic trivial cofibration $c \otimes_z \delta : B \rightarrowtail \mathrm{I}^{n+1}$, which is *uniform* in I^n in the following sense: given any cubical map $u : \mathrm{I}^m \longrightarrow \mathrm{I}^n$, the pullback $u^*c : u^*C \rightarrowtail \mathrm{I}^m$ and the reindexing $zu : \mathrm{I}^m \longrightarrow \mathrm{I}^n \longrightarrow \mathrm{I}$ determine another basic trivial cofibration $u^*c \otimes_{zu} \delta : B' = (\mathrm{I}^m +_{u^*C} (u^*C \times \mathrm{I})) \rightarrowtail \mathrm{I}^{m+1}$ which fits into a commutative diagram of the form

$$\begin{tikzcd}
B' \arrow[r, "(u\times \mathrm{I})'"] \arrow[d, tail, "u^*c \otimes_{zu} \delta"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[r, "x"] \arrow[d, tail, "c \otimes_z \delta"'] & X \arrow[d, "f"] \\
\mathrm{I}^m \times \mathrm{I} \arrow[r, "u \times \mathrm{I}"'] & \mathrm{I}^n \times \mathrm{I} \arrow[r, "y"'] \arrow[ur, dotted, "j(c,z,x,y)"'] & Y.
\end{tikzcd} \tag{44}$$

For the outer rectangle in (44) there is a chosen diagonal filler

$$j(u^*c, zu, x(u \times \mathrm{I})', y(u \times \mathrm{I})) : \mathrm{I}^m \times \mathrm{I} \longrightarrow X,$$

and for this map we require that

$$j(u^*c, zu, x(u \times \mathrm{I})', y(u \times \mathrm{I})) = j(c, z, x, y) \circ (u \times \mathrm{I}). \tag{45}$$

Definition 20. A *(uniform, unbiased) fibration structure* on a map

$$f : Y \longrightarrow X$$

is a choice of fillers $j(c, z, x, y)$ as in (43) satisfying (45) for all $u : \mathrm{I}^m \to \mathrm{I}^n$.

In these terms, we have following analogue of corollary 16.

Proposition 21. *For any object X in* cSet *the following are equivalent:*

1. *the canonical map $X^{\mathrm{I}} \times \mathrm{I} \longrightarrow X \times \mathrm{I}$ is a* $+$-algebra.
2. *X has the right lifting property with respect to all generating trivial cofibrations,*

$$(\mathcal{C} \otimes_z \delta) \pitchfork X.$$

3. X has a uniform fibration structure *in the sense of Definition 20.*

Proof. The equivalence between (1) and (2) is proposition 19. Suppose (1), i.e. that the map

$$(\delta \Rightarrow X) : X^{\mathrm{I}} \times \mathrm{I} \longrightarrow X \times \mathrm{I}$$

is a relative $+$-algebra over $X \times \mathrm{I}$. By proposition 6, this means that $(\delta \Rightarrow X)$, as an object of $\mathsf{cSet}/(X \times \mathrm{I})$, has a uniform filling structure with respect to all cofibrations $c : C \rightarrowtail \mathrm{I}^n$ over $(X \times \mathrm{I})$. Transposing by the $\otimes \dashv \Rightarrow$ adjunction and unwinding gives, equivalently, a uniform fibration structure on X. $\square$

A statement analogous to the foregoing also holds for maps $f : Y \to X$ in place of objects X. Indeed, as before, we have the following sharper formulation.

Corollary 22. *Fibration structures on a map $f : Y \to X$ correspond uniquely to $+$-algebra structures on the map $(\delta \Rightarrow f)$ (cf. definition 17),*

$$(\delta \Rightarrow f) : Y^I \times \mathrm{I} \longrightarrow (X^I \times \mathrm{I}) \times_{(X \times \mathrm{I})} (Y \times \mathrm{I})$$

6 The fibration weak factorization system

Definition 23. Summarizing the foregoing definitions and results, we have the following classes of maps:

- The *generating trivial cofibrations* were determined in (39) to be

$$\mathcal{C} \otimes \delta = \{c \otimes_z \delta : D \rightarrowtail Z \times \mathrm{I} \mid c : C \rightarrowtail Z,\, z : Z \longrightarrow \mathrm{I}\}\,, \tag{46}$$

where the pushout-product $c \otimes_z \delta$ has the form

$$\begin{tikzcd} C \arrow[r, "\mathsf{Gph}(zc)"] \arrow[d, tail, "c"'] & C \times \mathrm{I} \arrow[d] \arrow[ddr, bend left, "c \times \mathrm{I}"] & \\ Z \arrow[r] \arrow[drr, bend right, "\mathsf{Gph}(z)"'] & Z +_C (C \times \mathrm{I}) \arrow[dr, dotted, "c \otimes_z \delta"] & \\ & & Z \times \mathrm{I}\,. \end{tikzcd} \tag{47}$$

for any cofibration $c : C \rightarrowtail Z$ and indexing map $z : Z \to \mathrm{I}$, with domain $D = \big(Z +_C (C \times \mathrm{I})\big)$.

- The class $\mathcal{F}$ of *fibrations*, written $f : Y \twoheadrightarrow X$, may be characterized as the right-lifting class of the generating trivial cofibrations,

$$(\mathcal{C} \otimes \delta)^{\pitchfork} = \mathcal{F}.$$

- The class of *trivial cofibrations* is defined to be left class of the fibrations,

$$\mathsf{TrivCof} = {}^{\pitchfork}\mathcal{F}.$$

It follows from the specification that the classes $\mathsf{TrivCof}$ and $\mathcal{F}$ are mutually weakly orthogonal,

$$\mathsf{TrivCof} \pitchfork \mathcal{F},$$

and are both closed under retracts, so in order to have a weak factorization system $(\mathsf{TrivCof}, \mathcal{F})$ it just remains to show that every map $f : X \to Y$ can be factored as $f = g \circ h$ with $g \in \mathcal{F}$ and $h \in \mathsf{TrivCof}$.

Proposition 24. *Every map $f : X \longrightarrow Y$ in cSet can be factored as $f = g \circ h$,*

$$\begin{tikzcd} X \arrow[r, tail, "h"] \arrow[dr, "f"'] & X' \arrow[d, two heads, "g"] \\ & Y \end{tikzcd} \tag{48}$$

with $h : X \rightarrowtail X'$ a trivial cofibration and $g : X' \twoheadrightarrow Y$ a fibration.

Proof. This is a standard argument (cf. [**?**, GKR18]), which can be simplified a bit in this particular case. We sketch the proof for the case $Y = 1$; the general case is not essentially different.

Thus let X be any object, and we wish to find a fibrant object X' and a trivial cofibration $h : X \rightarrowtail X'$. For each basic trivial cofibration $\beta : B \rightarrowtail \mathrm{I}^k$, we need to solve all extension problems of the form

$$\begin{tikzcd} B \arrow[r, "x"] \arrow[d, tail, "\beta"'] & X. \\ \mathrm{I}^k \arrow[ur, dotted] & \end{tikzcd} \tag{49}$$

We first combine these into a single problem by taking a coproduct over all maps $x : B \to X$,

$$\begin{tikzcd}
\coprod_x B \arrow[r, "{[x]}"] \arrow[d, tail, "\coprod_x \beta"'] & X. \\
\coprod_x \mathrm{I}^k \arrow[ur, dotted] &
\end{tikzcd}$$

We then take the coproduct over all basic trivial cofibrations $\beta : B \rightarrowtail \mathrm{I}^k$,

$$\begin{tikzcd}
\coprod_\beta \coprod_x B \arrow[r, "{[[x]_\beta]}"] \arrow[d, tail, "\coprod_\beta \coprod_x \beta"'] & X. \\
\coprod_\beta \coprod_x \mathrm{I}^k \arrow[ur, dotted] &
\end{tikzcd}$$

Note that a coproduct of trivial cofibrations is clearly a trivial cofibration.

Taking a pushout, the indicated map h_1 is then also a trivial cofibration, because it is a pushout of one

$$\begin{tikzcd}
\coprod_\beta \coprod_x B \arrow[r, "{[[x]_\beta]}"] \arrow[d, tail, "\coprod_\beta \coprod_x \beta"'] & X \arrow[d, tail, "h_1"] \\
\coprod_\beta \coprod_x \mathrm{I}^k \arrow[r] & X_1 \, . \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

Now iterate the construction to get a sequence of trivial cofibrations, of which we take X' to be the colimit and $h : X \longrightarrow X'$ the canonical map,

$$h : X \underset{h_1}{\rightarrowtail} X_1 \underset{h_2}{\rightarrowtail} X_2 \underset{h_3}{\rightarrowtail} \dots \rightarrowtail \varinjlim X_n = X'. \tag{50}$$

To show that X' is fibrant, consider an extension problem of the form (49) with X' in place of X,

$$\begin{tikzcd}
B \arrow[r, "x"] \arrow[d, tail, "\beta"'] & \varinjlim X_n \\
\mathrm{I}^k \arrow[ur, dotted] &
\end{tikzcd}$$

The subobject $B \rightarrowtail \mathrm{I}^k$ has as domain an object B that is a *finite* colimit of maps $\mathrm{I}^m \to \mathrm{I}^n$ of representables (as can be seen by considering sieves in the category of cubes), and is therefore finitely presented, in the sense that mapping out of it preserves filtered colimits. Thus the map $x : B \longrightarrow \varinjlim X_n$ must factor through some $x_k : B \longrightarrow X_k$, giving rise to the problem

$$\begin{tikzcd}
B \arrow[r, "x_k"] \arrow[d, tail, "\beta"'] & X_k \arrow[d] \\
\mathrm{I}^k \arrow[r, dotted] & \varinjlim X_n.
\end{tikzcd}$$

But this has a solution in the next step, by the construction of X_{k+1},

$$\begin{tikzcd}
B \arrow[r, "x_k"] \arrow[d, tail, "\beta"'] & X_k \arrow[d, "h_{k+1}"] \\
\mathrm{I}^k \arrow[r] \arrow[dr, dotted, "j"'] & X_{k+1} \arrow[d] \\
 & \varinjlim X_n.
\end{tikzcd}$$

Finally, we need to show the uniformity condition on the resulting fillers $j = j(\beta, x)$. For this to work, we must modify the colimit construction (50) by interleaving certain coequalizers, in order to identify fillers added at different stages. For details, see [GKR18, **?**]. $\square$

Proposition 25. *There is a weak factorization system on the category* cSet *in which the right maps are the fibrations and the left maps are the trivial cofibrations, both as specified in definition 23.*

This will be called the *fibration weak factorization system.* The following observation will be of use later on, the proof can be found in [GKR18, **?**].

Corollary 26. *The construction given in* (50) *of the* fibrant replacement*,*

$$X' = \varinjlim_n X_n$$

is functorial in X, and the canonical trivial cofibrations $h : X \rightarrowtail X'$ are natural in X.

7 Weak equivalences

Definition 27 (Weak equivalence)**.** A map $f : X \longrightarrow Y$ in cSet will be called a *weak equivalence* if can be factored as $f = g \circ h$,

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[dr, "f"'] & W \arrow[d, "g"] \\ & Y \end{tikzcd}$$

with $h : X \to W$ a trivial cofibration and $g : W \to Y$ a trivial fibration, i.e. a right map in the cofibration weak factorization system. Let

$$\mathcal{W} = \{f : X \longrightarrow Y \mid f = g \circ h \text{ for } g \in \mathsf{TrivFib} \text{ and } h \in \mathsf{TrivCof}\}$$

be the class of weak equivalences.

Observe that every trivial fibration $f \in \mathcal{C}^{\pitchfork}$ is indeed a fibration, because the generating trivial cofibrations are indeed cofibrations; moreover, every trivial fibration is also a weak equivalence, since the identity maps are trivial cofibrations. Thus we have

$$\mathsf{TrivFib} \subseteq (\mathcal{F} \cap \mathcal{W}).$$

Thus, because the trivial fibrations are fibrations, every trivial cofibration $g \in {}^{\pitchfork}\mathcal{F}$ is a cofibration; moreover, every trivial cofibration is also a weak equivalence, since the identity maps are also trivial fibrations. Thus we also have

$$\mathsf{TrivCof} \subseteq (\mathcal{C} \cap \mathcal{W}).$$

Lemma 28. $(\mathcal{C} \cap \mathcal{W}) \subseteq \mathsf{TrivCof}$.

Proof. Let $c : A \rightarrowtail B$ be a cofibration with a factorization

$$c = tf \circ tc : A \to W \to B$$

where $tc \in \mathsf{TrivCof}$ and $tf \in \mathsf{TrivFib}$. Let $f : Y \twoheadrightarrow X$ be a fibration and consider a commutative diagram,

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, tail, "c"'] & Y \arrow[d, two heads, "f"] \\ B \arrow[r, "b"'] & X. \end{tikzcd}$$

Inserting the factorization of c, we have $j : W \longrightarrow Y$ as indicated, with $j \circ tc = a$ and $f \circ j = b \circ tf$, since $tc \pitchfork f$.

$$
\begin{tikzcd}
A \arrow[rr, "a"] \arrow[dd, tail, "c"'] \arrow[dr, "tc"] & & Y \arrow[dd, two heads, "f"] \\
& W \arrow[ur, dotted, "j"'] \arrow[dl, "tf"] & \\
B \arrow[rr, "b"'] & & X.
\end{tikzcd}
$$

Moreover, since $c \pitchfork tf$ there is an $i : B \to W$ as indicated, with $i \circ c = tc$ and $tf \circ i = 1_B$.

$$
\begin{tikzcd}
A \arrow[rr, "a"] \arrow[dd, tail, "c"'] \arrow[dr, "tc"] & & Y \arrow[dd, two heads, "f"] \\
& W \arrow[ur, dotted, "j"'] \arrow[dl, "tf"] & \\
B \arrow[ur, dotted, bend right, "i"'] \arrow[rr, "b"'] & & X.
\end{tikzcd}
$$

Let $k = j \circ i$. Then $k \circ c = j \circ i \circ c = j \circ tc = a$, and $f \circ k = f \circ j \circ i = b \circ tf \circ i = b$. $\square$

The proof of the following is dual:

Lemma 29. $(\mathcal{F} \cap \mathcal{W}) \subseteq \mathsf{TrivFib}$.

Proposition 30. *For the three classes of maps $\mathcal{C}, \mathcal{W}, \mathcal{F}$ in cSet, we have*

$$
\begin{aligned}
\mathcal{F} \cap \mathcal{W} &= \mathsf{TrivFib}, \\
\mathcal{C} \cap \mathcal{W} &= \mathsf{TrivCof},
\end{aligned}
$$

and therefore two weak factorization systems:

$$(\mathcal{C},\, \mathcal{W} \cap \mathcal{F}) \quad , \quad (\mathcal{C} \cap \mathcal{W},\, \mathcal{F}).$$

It thus remains only to prove that the weak equivalences satisfy the 3-for-2 property.

Weak homotopy equivalence

Definition 31. By a *homotopy* between parallel maps $f, g : X \rightrightarrows Y$, written $\vartheta : f \sim g$, we shall mean a map from the *cylinder of* X built using the (representable) interval I,

$$\vartheta : \mathrm{I} \times X \longrightarrow Y,$$

and such that $\vartheta \circ \iota_0 = f$ and $\vartheta \circ \iota_1 = g$,

$$\begin{tikzcd} X \arrow[r, "\iota_0"] \arrow[dr, "f"'] & \mathrm{I} \times X \arrow[d, "\vartheta"] & X, \arrow[l, "\iota_1"'] \arrow[dl, "g"] \\ & Y & \end{tikzcd}$$

where we write the canonical inclusions into the ends of the cylinder as

$$\iota_\epsilon = \mathsf{Gph}(\delta_\epsilon !) : X \longrightarrow \mathrm{I} \times X\,, \qquad \epsilon = 0, 1.$$

Proposition 32. *If K is fibrant, then the relation of homotopy $f \sim g$ between maps $f, g : X \rightrightarrows K$ is an equivalence relation. Moreover, it is compatible with pre- and post-composition.*

Proof. For $f, g : X \rightrightarrows Y$, a homotopy $f \overset{\vartheta}{\sim} g : X \times \mathrm{I} \longrightarrow Y$ is equivalent, under exponential transposition, to a path in the function space $\vartheta : \mathrm{I} \to Y^X$ with endpoints $\vartheta_0 = \vartheta \circ \delta_0 = f : 1 \to Y^X$ and $\vartheta_1 = g$. Note that Y^X is fibrant if Y is fibrant, so we can use box-filling in Y^X.

The reflexivity of homotopy $f \sim f$ is witnessed by $\rho : \mathrm{I} \to 1 \xrightarrow{f} Y^X$.

For symmetry $f \sim g \Rightarrow g \sim f$ take $\vartheta : \mathrm{I} \to Y^X$ with $\vartheta_0 = f$ and $\vartheta_1 = g$ and we want to build $\vartheta' : \mathrm{I} \to Y^X$ with $\vartheta'_0 = g$ and $\vartheta'_1 = f$. Take an open 2-box in Y^X of the form

$$\begin{tikzcd} g & f \\ f \arrow[u, "\vartheta"] \arrow[r, "\rho"'] & f \arrow[u, "\rho"'] \end{tikzcd}$$

This box is a map $b : \mathrm{I} +_1 \mathrm{I} +_1 \mathrm{I} \longrightarrow Y^X$ with the indicated components, and it has a filler $c : \mathrm{I} \times \mathrm{I} \longrightarrow Y^X$, i.e. an extension along the canonical map $\mathrm{I} +_1 \mathrm{I} +_1 \mathrm{I} \rightarrowtail \mathrm{I} \times \mathrm{I}$, which is a trivial cofibration. Let $t : \mathrm{I} \to \mathrm{I} \times \mathrm{I}$ be the evident missing top face of the 2-cube. Then we can set $\vartheta' = ct : \mathrm{I} \to Y^X$ to get a homotopy $\vartheta' : \mathrm{I} \to Y^X$ with required endpoints.

For transitivity, $f \overset{\vartheta}{\sim} g$ & $g \overset{\varphi}{\sim} h \Rightarrow f \sim h$, an analogous filling construction is used with the open box:

$$\begin{tikzcd} f & h \\ f \arrow[u, "\rho"] \arrow[r, "\vartheta"'] & g \arrow[u, "\varphi"'] \end{tikzcd}$$

Compatibility under pre- and post-composition is shown by representing homotopy by mapping into the pathspace, for precomposition, and out of the cylinder, for post-composition. $\square$

Definition 33 (Connected components)**.** The functor

$$\pi_0 : \mathsf{cSet} \longrightarrow \mathsf{Set}$$

is defined, for any cubical set X, to be the coequalizer

$$X_1 \rightrightarrows X_0 \to \pi_0 X,$$

where the two parallel arrows are the maps $X_{\delta_0}, X_{\delta_1} : X_1 \rightrightarrows X_0$ induced by the endpoints $\delta_0, \delta_1 : 1 \rightrightarrows \mathrm{I}$. For any Kan complex K we therefore have $\pi_0 K = \mathrm{Hom}(1, K)/{\sim}$, that is, $\pi_0 K$ is the set of points $1 \to K$, modulo the homotopy equivalence relation on them.

One can show that in fact $\pi_0 X = \varinjlim_n X_n$, the colimit being left adjoint to the constant presheaf functor $\Delta : \mathsf{Set} \longrightarrow \mathsf{cSet}$. Since the category $\mathbb{B}$ of finite strictly bipointed sets is sifted, we have:

Corollary 34. *The functor* $\pi_0 : \mathsf{cSet} \longrightarrow \mathsf{Set}$ *preserves finite products.*

As usual, a map $f : X \to Y$ in cSet will be called a *homotopy equivalence* if there is a *quasi-inverse* $g : Y \to X$ and homotopies $\vartheta : 1_X \sim g \circ f$ and $\varphi : 1_Y \sim f \circ g$.

Definition 35 (Weak homotopy equivalence)**.** A map $f : X \to Y$ will be called a *weak homotopy equivalence* if for every fibrant object K, the "internal precomposition" map $K^f : K^Y \to K^X$ is bijective on connected components, i.e.

$$\pi_0 K^f : \pi_0 K^Y \longrightarrow \pi_0 K^X$$

is a bijection of sets.

Lemma 36. *A homotopy equivalence is weak homotopy equivalence.*

Proof. If $f : X \to Y$ is a homotopy equivalence, then so is $K^f : K^Y \longrightarrow K^X$ for any K, since homotopy respects composition. Since K^X is always fibrant when K is, π_0 is well defined, and it clearly takes homotopy equivalences to isomorphisms of sets. $\square$

Lemma 37. *The weak homotopy equivalences $f : X \to Y$ satisfy the 3-for-2 condition.*

Proof. Follows from the corresponding fact about bijections of sets. $\square$

Our goal of showing that the weak equivalences satisfy 3-for-2 is now reduced to showing that a map is a weak equivalence (WE) if and only if it is a weak homotopy equivalence (WHE). This will be proved in four cases, showing that a (co)fibration is a WE if and only if it is a WHE.

Lemma 38. *A map $f : X \to Y$ is a* weak homotopy equivalence *iff it satisfies the following two conditions.*

1. *For every fibrant object K and every map $x : X \to K$ there is a map $y : Y \to K$ such that $y \circ f \sim x$,*

$$\begin{tikzcd} X \arrow[r, "x"] \arrow[d, "f"'] \arrow[dr, phantom, "\sim"] & K. \\ Y \arrow[ur, dotted, "y"'] & \end{tikzcd}$$

   We say that x "extends along f up to homotopy".

2. *For every fibrant object K and maps $y, y' : Y \to K$ such that $yf \sim y'f$, there is a homotopy $y \sim y'$,*

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & K^{\mathrm{I}} \arrow[d] \\ Y \arrow[r, "\langle y,y' \rangle"'] \arrow[ur, dotted] & K \times K. \end{tikzcd}$$

Proof. Unwind the definition. $\square$

Lemma 39. *A cofibration $c : A \rightarrowtail B$ that is a WE is a WHE.*

Proof. A cofibration $c : A \rightarrowtail B$ that is a WE is a trivial cofibration by proposition 30. So the result follows from Lemma 38, and the fact that $K^{\mathrm{I}} \to K \times K$ is always a fibration when K is fibrant. $\square$

Lemma 40. *A fibration $p : Y \twoheadrightarrow X$ that is a WE is a WHE.*

Proof. A fibration weak equivalence $f : Y \twoheadrightarrow X$ is a trivial fibration by proposition 30, and therefore has a section $s : X \rightarrowtail Y$, by the lifting problem

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, tail] & Y \arrow[d, two heads, "f"] \\ X \arrow[r, "="'] & X, \end{tikzcd}$$

since $0 \to X$ is always a cofibration. Moreover, there is a homotopy $\vartheta : sf \sim 1_Y$, resulting from the lifting problem

$$\begin{tikzcd} Y + Y \arrow[r, "{[sf,1]}"] \arrow[d, tail, "{[\iota_0,\iota_1]}"'] & Y \arrow[d, two heads, "f"] \\ \mathrm{I} \times Y \arrow[r, "f\pi_2"'] & X. \end{tikzcd}$$

Thus f is a homotopy equivalence, and so a WHE by lemma 36. $\square$

Lemma 41. *If K is fibrant, then any fibration $f : Y \twoheadrightarrow K$ that is a HE is a WE.*

Proof. This is a standard argument, which we just sketch. It suffices to show that any diagram of the form

$$\begin{tikzcd} C \arrow[r, "y"] \arrow[d, tail, "c"'] & Y \arrow[d, two heads, "f"] \\ K \arrow[r, "="'] & K, \end{tikzcd} \tag{51}$$

with $c : C \rightarrowtail X$ a cofibration, has a diagonal filler. Since f is a HE it has a quasi-inverse $s : X \to Y$ with $\vartheta : fs \sim 1_K$, which we can correct to a section $s' : K \to Y$. Indeed, consider

$$\begin{tikzcd} K \arrow[r, "s"] \arrow[d, tail, "\iota_0"'] & Y \arrow[d, two heads, "f"] \\ K \times \mathrm{I} \arrow[r, "\vartheta"'] \arrow[ur, dotted, "\vartheta'"] & K \\ K \arrow[u, tail, "\iota_1"'] \arrow[ur, "="'] & \end{tikzcd}$$

where ϑ' results from $\iota_0 \pitchfork f$. Let $s' = \vartheta'\iota_1$, so that $\vartheta' : s \sim s'$ and $fs' = 1_K$.

Thus we can assume that $s = s' : K \to Y$ is a section, which fills the diagram (51) up to a homotopy in the upper triangle.

$$\begin{tikzcd}
C \arrow[r, "y"] \arrow[d, tail, "c"'] \arrow[dr, phantom, "\sim", near start] & Y \arrow[d, two heads, "f"] \\
K \arrow[r, "="'] \arrow[ur, "s"'] & K,
\end{tikzcd}$$

Now we can correct $s : K \to Y$ to a homotopic $t : K \to Y$ over f by using the homotopy $\varphi : sc \sim y$ to get a map $\varphi : C \to Y^{\mathrm{I}}$ over f. Since f is a fibration, the projections $p_0, p_1 : Y^{\mathrm{I}} \to Y$ over f are trivial fibrations, and so there is a lift $\varphi' : K \to Y^{\mathrm{I}}$ for which $t := p_1\varphi'$ has $tc = y$ and $ft = 1_K$, and so is a filler for (51). □

Lemma 42. *If K is fibrant, then any fibration $f : Y \twoheadrightarrow K$ that is a WHE is a WE.*

Proof. Since K is fibrant, so is Y, and since f is a WHE, there is a map $s : K \to Y$ and a homotopy $\theta : sf \sim 1_Y$ by lemma 38(1). Thus, applying f again, we have a homotopy $f\vartheta : fsf \sim f$, forming the outer commutative square in

$$\begin{tikzcd}
Y \arrow[r, "f\vartheta"] \arrow[d, "f"'] & K^{\mathrm{I}} \arrow[d] \\
K \arrow[r, "\langle fs, 1_K \rangle"'] \arrow[ur, dotted, "\varphi"'] & K \times K.
\end{tikzcd}$$

By lemma 38(2) there is a diagonal filler $\varphi : fs \sim 1_K$, and so f is a HE. Now apply lemma 41. □

Lemma 43. *If K is fibrant, then any cofibration $c : A \rightarrowtail K$ that is a WHE is a WE.*

Proof. Let $c : A \rightarrowtail K$ be a cofibration WHE and factor it into a trivial cofibration $i : A \rightarrowtail Z$ followed by a fibration $p : Z \twoheadrightarrow K$. By lemma 38, it is clear that a trivial cofibration is a WHE. So both c and i are WHE, and therefore so is p by 3-for-2 for WHEs. Since K is fibrant, p is a trivial fibration by lemma 42, and thus c is a WE. □

Lemma 44 ([?], x.n.m)**.** *A cofibration $c : A \rightarrowtail B$ WHE lifts against all fibrations $f : Y \twoheadrightarrow K$ with fibrant codomain.*

Proof. Let $c : A \rightarrowtail B$ be a cofibration WHE and $f : Y \twoheadrightarrow K$ a fibration with fibrant codomain K, and consider a lifting problem

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, tail, "c"'] & Y \arrow[d, two heads, "f"] \\ B \arrow[r, "b"'] & K. \end{tikzcd}$$

Let $\eta : B \rightarrowtail B'$ be a fibrant replacement of B, since K is fibrant, b extends along η to give $b' : B' \to K$ as in:

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, tail, "c"'] & Y \arrow[d, two heads, "f"] \\ B \arrow[r, "b"] \arrow[d, "\eta"'] & K \\ B' \arrow[ur, "b'"'] & \end{tikzcd}$$

Since η is a trivial cofibration, it is a WHE. So the composite ηc is also a WHE. But since B' is fibrant, ηc is then a trivial cofibration by lemma 43. Thus there is a lift $j : B' \to Y$, and therefore also one $k = j\eta : B \to Y$. $\square$

To complete the proof that a cofibration WHE is a WE we will use the fact that the fibration weak factorization system satisfies the fibration extension property, the proof of which is deferred to section XX**??**.

Definition 45 (Fibration extension)**.** The fibration weak factorization system is said to satisfy the *fibration extension property* if the following holds: Given a fibration $f : Y \twoheadrightarrow X$ and a trivial cofibration $\eta : X \to X'$, there is a fibration $f' : Y' \twoheadrightarrow X'$ such that f is a pullback of f' along η.

$$\begin{tikzcd} Y \arrow[r] \arrow[d, two heads, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y' \arrow[d, two heads, "f'"] \\ X \arrow[r, tail, "\eta"'] & X'. \end{tikzcd} \tag{52}$$

Lemma 46. *A cofibration that lifts against every fibration $f : Y \twoheadrightarrow K$ with fibrant codomain is a WE.*

Proof. Let $c : A \rightarrowtail B$ be a cofibration and consider a lifting problem against an arbitrary fibration $f : Y \twoheadrightarrow X$,

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, tail, "c"'] & Y \arrow[d, two heads, "f"] \\ B \arrow[r, "b"'] & X. \end{tikzcd} \tag{53}$$

Let $\eta : X \to X'$ be a fibrant replacement, so η is a trivial cofibration and X' is fibrant. By the fibration extension property of definition 45, there is a fibration $f' : Y' \twoheadrightarrow X'$ such that f is a pullback of f' along η. So we can extend diagram (53) to obtain the following, in which the righthand square is a pullback.

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, tail, "c"'] & Y \arrow[r, "y"] \arrow[d, two heads, "f"] \arrow[dr, phantom, "\lrcorner", very near start] & Y' \arrow[d, two heads, "f'"] \\ B \arrow[r, "b"'] & X \arrow[r, "\eta"'] & X'. \end{tikzcd} \tag{54}$$

By assumption, there is a lift $j' : B \to Y'$ with $f'j' = \eta b$ and $j'c = yb$. Therefore, since f is a pullback, there is a map $j : B \to Y$ with $fj = b$ and $yj = j'$.

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, tail, "c"'] & Y \arrow[r, "y"] \arrow[d, two heads, "f"] \arrow[dr, phantom, "\lrcorner", very near start] & Y' \arrow[d, two heads, "f'"] \\ B \arrow[r, "b"'] \arrow[ur, dotted, "j"] \arrow[urr, dotted, "j'"'] & X \arrow[r, "\eta"'] & X'. \end{tikzcd} \tag{55}$$

Thus $yjc = j'c = ya$. But as a trivial cofibration, η is monic, and as a pullback of η, y is also monic. So $jc = a$. $\square$

Combining the previous two lemmas 44 and 46 we now have.

Corollary 47. *A cofibration $c : A \rightarrowtail B$ that is a WHE is a WE.*

The following is not required, but we state it anyway for the record:

Lemma 48. *A fibration $f : Y \twoheadrightarrow X$ that is a WHE is a WE.*

Proof. Factor $f : Y \twoheadrightarrow X$ into a cofibration $i : Y \rightarrowtail Z$ followed by a trivial fibration $p : Z \twoheadrightarrow X$. Then f is a trivial fibration if $i \pitchfork f$, for then f is a retract of p. Since p is a trivial fibration, it is a WHE by lemma 40. Since f is also a WHE, so is i by 3-for-2. Thus i is a trivial cofibration by corollary 47. Since f is a fibration, $i \pitchfork f$ as required. $\square$

Proposition 49. *A map $f : X \rightarrowtail Y$ is a WHE if and only if it is a WE. Thus the weak equivalences $\mathcal{W}$ satisfy the 3-for-2 condition.*

Proof. Let $f : X \twoheadrightarrow Y$ be a WE and factor it into a trivial cofibration $i : X \rightarrowtail Z$ followed by a trivial fibration $p : Z \twoheadrightarrow Y$. Then both i and p are WHE, whence so is f. Conversely, let f be a WHE and factor it into a cofibration $i : X \rightarrowtail Z$ followed by a trivial fibration $p : Z \twoheadrightarrow Y$. Since p is then a WHE, as is f, it follows that i is as well. Thus i is also a WE, by lemma 47, hence a trivial fibration. So f is a WE. □

We summarize the results to this point in the following.

Theorem 50. *Assume the fibration weak factorization system of Definition 23 satisfies the fibration extension property of Definition 45. Then the weak equivalences have the 3-for-2 property and we therefore have a Quillen model structure.*

The proof of the fibration extension property will require several intermediate results: the equivalence extension property (Section 10), the Frobenius condition (Section 9), and the reduction of filling to composition (Section 8), to which we now turn.

8 Composition

A novelty of the type-theoretic notion of fibration is the method (due to Coquand and first introduced in [CCHM16]) of reducing the (type-theoretically specified) notion of *fibration structure* to the apparently weaker notion of a *composition structure*. Composition structure is more easily shown to be preserved by the type-forming operations like Σ and Π, when these concepts are formulated in type theory, as is done in [CCHM16], or in the internal lanuage of the ambient presheaf topos, as in [OP17]. This is due to the fact that one can then efficiently calculate using the rules of type theory (and even in a proof assistant), making it possible to prove e.g. that the fibrations are closed under Π-types. The approach taken here is a reformulation into diagrammatic language of those type-theoretic calculations.

8.1 Composition for an object

Let $p : 1 \to \mathrm{I}$ be any point of the interval (e.g. δ_0), and $\epsilon_p := X^p : X^{\mathrm{I}} \longrightarrow X$ the corresponding "evaluation at p" map. Given another point $q : 1 \longrightarrow \mathrm{I}$, there is an evident factorization

$$\epsilon_p = \pi_1 \circ \langle \epsilon_p, \epsilon_q \rangle : X^{\mathrm{I}} \longrightarrow X \times X \longrightarrow X.$$

We will say that the object X *has composition (from p to q)* if for every object Z and cofibration $c : C \rightarrowtail Z$ and commutative square

$$\begin{tikzcd} C \arrow[r] \arrow[d, tail, "c"'] & X^{\mathrm{I}} \arrow[d, "\epsilon_p"] \\ Z \arrow[r] & X, \end{tikzcd} \tag{56}$$

there is an arrow $k : Z \longrightarrow X \times X$ as indicated below making both subdiagrams commute.

$$\begin{tikzcd} C \arrow[r] \arrow[dd, tail, "c"'] & X^{\mathrm{I}} \arrow[d, "\langle \epsilon_p, \epsilon_q \rangle"] \\ & X \times X \arrow[d, "\pi_1"] \\ Z \arrow[r] \arrow[ur, dotted, "k"] & X, \end{tikzcd} \tag{57}$$

If X is fibrant in the *biased* sense of section 4, then X clearly has composition from δ_0 to δ_1 (and back), since the outer rectangle then has a diagonal filler (as does the corresponding one with π_2 for π_1). In the case where the category of cubes is assumed to have connections, one can also show the converse, that having composition implies diagonal filling for all such squares (56); logical proofs of this fact can be found in [CCHM16, OP17], and a diagrammatic proof is given in Appendix 1.

If X is fibrant in the *unbiased* sense of section 5, with the generic point $\delta : 1 \to \mathrm{I}$ over I, then X will have composition over I from δ to either of δ_0, δ_1, for the same reason as before: the map $(\delta \Rightarrow X) = X^{\delta} : X^{\mathrm{I}} \longrightarrow X$ is a trivial fibration over I, by the definition of fibrancy. In order to arrive at a property equivalent to unbiased filling, we shall use a generic form of composition from δ to a second generic point δ'. This is obtained by pulling back along the (say, second) projection $\pi : \mathrm{I} \times \mathrm{I} \to \mathrm{I}$ to work in $\mathsf{cSet}/(\mathrm{I} \times \mathrm{I})$, where in addition to $\delta, \delta_0, \delta_1$ we now also have a point $\delta' : 1 \to \mathrm{I}$, given by the additional diagonal map over $\mathrm{I}\times\mathrm{I}$,

$$\begin{tikzcd} \mathrm{I}\times\mathrm{I} \arrow[r, "\langle\pi_1,\pi_2,\pi_1\rangle"] \arrow[dr, "\mathsf{id}"'] & \mathrm{I}\times\mathrm{I}\times\mathrm{I} \arrow[d, "\langle\pi_1,\pi_2\rangle"] \\ & \mathrm{I}\times\mathrm{I} \end{tikzcd} . \tag{58}$$

Observe that in $\mathsf{cSet}/\mathrm{I}\times\mathrm{I}$ the (binary) diagonal $\Delta : \mathrm{I} \rightarrowtail \mathrm{I}\times\mathrm{I}$ is a subobject of the terminal object $\Delta \rightarrowtail 1$, with associated base change

$$\mathsf{cSet}/\mathrm{I} \underset{\Delta_*}{\overset{\Delta^*}{\leftrightarrows}} \mathsf{cSet}/\mathrm{I}\times\mathrm{I}\,. \tag{59}$$

For any object X in $\mathsf{cSet}/\mathrm{I}\times\mathrm{I}$, let

$$\eta_X : X \to X^\Delta$$

be the unit of $\Delta^* \dashv \Delta_*$. Given objects and arrows $f, g : X \rightrightarrows Y$ in $\mathsf{cSet}/\mathrm{I}\times\mathrm{I}$, observe that $\Delta^* f = \Delta^* g$ in cSet/I if and only if the composites with η_Y are equal,

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "\eta_Y"] & Y^\Delta. \end{tikzcd} \tag{60}$$

Indeed, consider the double naturality square

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, "\eta_X"'] & Y \arrow[d, "\eta_Y"] \\ X^\Delta \arrow[r, shift left, "f^\Delta"] \arrow[r, shift right, "g^\Delta"'] & Y^\Delta. \end{tikzcd} \tag{61}$$

Then $\eta_Y f = \eta_Y g$ iff $f^\Delta\eta_X = g^\Delta\eta_X : X \to Y^\Delta$ iff $\Delta^* f = \Delta^* g : \Delta^* X \to \Delta^* Y$, by transposition, since $Y^\Delta = \Delta_*\Delta^* Y$.

We can now define what it means for an object X to have (unbiased) composition by saying that it has composition, in the sense of (57), from δ to δ' over $\mathrm{I}\times\mathrm{I}$, and that, moreover, the two resulting maps $k_1, k_2 : Z \rightrightarrows X$ are coequalized by the unit $\eta_X : X \to X^\Delta$. We reformulate this condition equivalently as follows (cf. [**?**, **?**]).

Definition 51. An object X will be said to have *(unbiased) composition* if it satisfies the following condition: for every cofibration $c : C \rightarrowtail Z$ in cSet/I and every commutative square in cSet/I of the form

$$\begin{tikzcd} C \arrow[r, "y"] \arrow[d, tail, "c"'] & X^{\mathrm{I}} \arrow[d, "\epsilon_\delta"] \\ Z \arrow[r, "x"'] & X \end{tikzcd}$$

with $\epsilon_\delta := X^\delta : X^{\mathrm{I}} \longrightarrow X$ the evaluation at the generic point $\delta : 1 \to \mathrm{I}$, upon pulling back along $\pi : \mathrm{I} \times \mathrm{I} \to \mathrm{I}$ to $\mathsf{cSet}/\mathrm{I} \times \mathrm{I}$, there is a map $x' : Z \to X$ making the following commute,

$$\begin{tikzcd} C \arrow[r, "y"] \arrow[d, tail, "c"'] & X^{\mathrm{I}} \arrow[r, "\epsilon_{\delta'}"] \arrow[d, "\epsilon_\delta"'] & X \arrow[d, "\eta_X"] \\ Z \arrow[r, "x"'] \arrow[urr, dotted, "x'"] & X \arrow[r, "\eta_X"'] & X^{\Delta} \end{tikzcd} \tag{62}$$

where $\epsilon_{\delta'} := X^{\delta'} : X^{\mathrm{I}} \longrightarrow X$ is the evaluation at the second generic point $\delta' : 1 \to \mathrm{I}$.

Proposition 52. *Composition implies filling; that is, every cubical set X with composition is fibrant.*

Proof. Let X be a cubical set with composition, and suppose given a filling problem in cSet/I of the form

$$\begin{tikzcd} C \arrow[r, "y"] \arrow[d, tail, "c"'] & X^{\mathrm{I}} \arrow[d, "\epsilon_\delta"] \\ Z \arrow[r, "x"'] & X. \end{tikzcd} \tag{63}$$

Pulling back to $\mathsf{cSet}/(\mathrm{I} \times \mathrm{I})$ (but omitting the π^* everywhere), since X has composition there is a map $x' : Z \to X$ making the following commute,

$$\begin{tikzcd} C \arrow[r, "y"] \arrow[d, tail, "c"'] & X^{\mathrm{I}} \arrow[r, "\epsilon_{\delta'}"] \arrow[d, "\epsilon_\delta"'] & X \arrow[d, "\eta_X"] \\ Z \arrow[r, "x"'] \arrow[urr, dotted, "x'"] & X \arrow[r, "\eta_X"'] & X^{\Delta}\,. \end{tikzcd} \tag{64}$$

Transposing by the adjunction $\pi^* \dashv \pi_*$ results in a commutative diagram in cSet/I of the form

$$
\begin{tikzcd}
C \arrow[r, "y"] \arrow[d, tail, "c"'] & X^{\mathrm{I}} \arrow[r, "\widetilde{\epsilon_{\delta'}}"] \arrow[d, "\epsilon_\delta"'] & \pi_*\pi^* X \arrow[d, "\pi_*\eta_X"] \\
Z \arrow[r, "x"'] \arrow[urr, dotted, "\widetilde{x'}"] & X \arrow[r, "\widetilde{\eta_X}"'] & \pi_* X^{\Delta},
\end{tikzcd}
\tag{65}
$$

where $\widetilde{\epsilon_{\delta'}} : X^{\mathrm{I}} \longrightarrow \pi_*\pi^* X$ is the adjoint transpose of $\epsilon_{\delta'}$, and similarly for $\widetilde{x'}$ and $\widetilde{\eta_X}$. To compute these transpositions, we factor them through the unit maps η^π of the adjunction $\pi^* \dashv \pi_*$,

$$
\begin{tikzcd}
C \arrow[r, "y"] \arrow[d, tail, "c"'] & X^{\mathrm{I}} \arrow[r, "\eta^\pi_{X^{\mathrm{I}}}"] \arrow[d, "\epsilon_\delta"] & (X^{\mathrm{I}})^{\mathrm{I}} \arrow[r, "\pi_*\epsilon_{\delta'}"] \arrow[d, "(\epsilon_\delta)^{\mathrm{I}}"] & \pi_*\pi^* X \arrow[d, "\pi_*\eta_X"] \\
Z \arrow[r, "x"'] \arrow[urrr, dotted, "\widetilde{x'}"] & X \arrow[r, "\eta^\pi_X"'] & X^{\mathrm{I}} \arrow[r, "\pi_*\eta_X"'] & \pi_* X^{\Delta}.
\end{tikzcd}
\tag{66}
$$

Next, observe that $\pi_*\pi^* X = X^{\mathrm{I}}$ and, up to the iso $(X^{\mathrm{I}})^{\mathrm{I}} \cong X^{\mathrm{I}\times\mathrm{I}}$, the map $\pi_*\epsilon_{\delta'}$ is

$$\pi_*\epsilon_{\delta'} = X^{(\Delta:\mathrm{I}\rightarrowtail\mathrm{I}\times\mathrm{I})} : X^{\mathrm{I}\times\mathrm{I}} \longrightarrow X^{\mathrm{I}}\,,$$

which we write as $\Delta^* : X^{\mathrm{I}\times\mathrm{I}} \longrightarrow X^{\mathrm{I}}$ to avoid confusion with the exponential object X^{Δ}. The map Δ^* is plainly a retraction of

$$\eta^\pi_{X^{\mathrm{I}}} = X^{(\pi:\mathrm{I}\times\mathrm{I}\to\mathrm{I})} : X^{\mathrm{I}} \longrightarrow (X^{\mathrm{I}})^{\mathrm{I}} \cong X^{\mathrm{I}\times\mathrm{I}}\,.$$

The last diagram (66) now becomes

$$
\begin{tikzcd}
C \arrow[r, "y"] \arrow[d, tail, "c"'] & X^{\mathrm{I}} \arrow[r, "\eta^\pi_{X^{\mathrm{I}}}"] \arrow[d, "\epsilon_\delta"] \arrow[rr, bend left, "="] & X^{\mathrm{I}\times\mathrm{I}} \arrow[r, "\Delta^*"] \arrow[d, "(\epsilon_\delta)^{\mathrm{I}}"] & X^{\mathrm{I}} \arrow[d, "\pi_*\eta_X"] \\
Z \arrow[r, "x"'] \arrow[urrr, dotted, "\widetilde{x'}"] & X \arrow[r, "\eta^\pi_X"'] & X^{\mathrm{I}} \arrow[r, "\pi_*\eta_X"'] & \pi_* X^{\Delta}.
\end{tikzcd}
\tag{67}
$$

Finally, we claim that $\pi_* X^{\Delta} \cong X$, and that, up to this iso,

$$\pi_*\eta_X = \epsilon_\delta : X^{\mathrm{I}} \to X,$$

which will finish the proof, since this is a retraction of $\eta^{\pi}_X : X \to X^{\mathrm{I}}$. Indeed, writing out the object $\pi_* X^{\Delta}$ explicitly, in terms of the two adjunctions $\pi^* \dashv \pi_*$ and $\Delta^* \dashv \Delta_*$, we have

$$\pi_* X^{\Delta} = \pi_* \Delta_* \Delta^* \pi^* X \cong (\pi \circ \Delta)_* (\pi \circ \Delta)^* X \cong X,$$

since $(\pi \circ \Delta) = 1$.

To see that $\pi_* \eta_X = \epsilon_\delta$, first let us make the base change

$$\mathrm{I}^* : \mathsf{cSet} \longrightarrow \mathsf{cSet}/\mathrm{I}$$

explicit, so that ϵ_δ is the counit of the adjunction $\mathrm{I}^* \dashv \mathrm{I}_*$ at $\mathrm{I}^* X$,

$$\epsilon_\delta = \epsilon_{(\mathrm{I}^* X)} : X^{\mathrm{I}} \times \mathrm{I} \longrightarrow X \times \mathrm{I} \quad \text{(over I)}.$$

By a triangle law, this map has the inverse $\mathrm{I}^*(\eta^{\mathrm{I}}_X) : X \times \mathrm{I} \longrightarrow X^{\mathrm{I}} \times \mathrm{I}$, where

$$\eta^{\mathrm{I}}_X : X \to X^{\mathrm{I}}$$

is the unit of $\mathrm{I}^* \dashv \mathrm{I}_*$ at X. It suffices to show that $\mathrm{I}^*(\eta^{\mathrm{I}}_X)$ is also an inverse for $\pi_* \eta_X$ which, more explicitly is:

$$\pi_* \eta_X = \pi_*(\eta^{\Delta}_{\pi^*(\mathrm{I}^* X)}),$$

where η^{Δ} is the unit of $\Delta^* \dashv \Delta_*$. Since I^* preserves exponentials, we have

$$\mathrm{I}^*(\eta^{\mathrm{I}}_X) = \eta^{\pi}_{(\mathrm{I}^* X)} : \mathrm{I}^* X \longrightarrow (\mathrm{I}^* X)^{\mathrm{I}^* \mathrm{I}} \cong \pi_* \pi^* (\mathrm{I}^* X) \quad \text{(over I)}.$$

Now, for the composition of the $\pi^* \dashv \pi_*$ and $\Delta^* \dashv \Delta_*$ adjunctions,

$$\begin{tikzcd} \mathsf{cSet}/\mathrm{I} \arrow[r, shift right, "\Delta_*"'] & \mathsf{cSet}/\mathrm{I} \times \mathrm{I} \arrow[l, shift right, "\Delta^*"'] \arrow[r, shift right, "\pi_*"'] & \mathsf{cSet}/\mathrm{I} \arrow[l, shift right, "\pi^*"'] \end{tikzcd} \tag{68}$$

the units η^{π} and η^{Δ} satisfy the well-known law

$$\eta^{\pi \circ \Delta} = \pi_*(\eta^{\Delta}_{\pi^*}) \circ (\eta^{\pi}).$$

But $\eta^{\pi \circ \Delta} = \mathsf{id}$, since $\pi \circ \Delta = 1$. We therefore have

$$\pi_*(\eta^{\Delta}_{\pi^*(\mathrm{I}^* X)}) \circ \eta^{\pi}_{(\mathrm{I}^* X)} = \eta^{(\pi \circ \Delta)}_{(\mathrm{I}^* X)} = \mathsf{id}_{(\mathrm{I}^* X)},$$

as required. $\square$

8.2 Composition for a map

We next generalize the notion of compostion for an object X to composition for a map $f : Y \longrightarrow X$. First consider biased fibrations in the sense of section 4; recall from Corollary 15 that a (biased) fibration structure on a map $f : Y \to X$ is the same thing as a pair of +-algebra structures on the maps

$$(\delta_\epsilon \Rightarrow f) : Y^{\mathrm{I}} \longrightarrow X^{\mathrm{I}} \times_X Y$$

for $\epsilon = 0, 1$. The construction of $\delta_0 \Rightarrow f$ is recalled from (26) in the pullback diagram below, in which $X^{\delta_0} : X^{\mathrm{I}} \to X$ is the evaluation map at $\delta_0 : 1 \to \mathrm{I}$.

$$\begin{tikzcd}
Y^{\mathrm{I}} \arrow[dr, "\delta_0 \Rightarrow f"] \arrow[drr, bend left, "Y^{\delta_0}"] \arrow[ddr, bend right, "f^{\mathrm{I}}"'] & & \\
& X^{\mathrm{I}} \times_X Y \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\
& X^{\mathrm{I}} \arrow[r, "X^{\delta_0}"'] & X
\end{tikzcd} \tag{69}$$

Given another point $\delta_1 : 1 \to \mathrm{I}$, we have an analogous construction

$$\begin{tikzcd}
Y^{\mathrm{I}} \arrow[dr, "\delta_1 \Rightarrow f"] \arrow[drr, bend left, "Y^{\delta_1}"] \arrow[ddr, bend right, "f^{\mathrm{I}}"'] & & \\
& X^{\mathrm{I}} \times_X Y \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\
& X^{\mathrm{I}} \arrow[r, "X^{\delta_1}"'] & X.
\end{tikzcd} \tag{70}$$

But note that now the pullback object $X^{\mathrm{I}} \times_X Y$ is a different one, with fiber over $p : \mathrm{I} \to X$ being the fiber of f over $p\delta_1$ rather than over $p\delta_0$. Let us call these two different pulled-back maps $f_0 : Y_0 \to X^{\mathrm{I}}$ and $f_1 : Y_1 \to X^{\mathrm{I}}$ and write $f^\epsilon := (\delta_\epsilon \Rightarrow f)$ for $\epsilon = 0, 1$. There is then a commutative diagram,

$$\begin{tikzcd}
Y^{\mathrm{I}} \arrow[r, "f^1"] \arrow[d, "f^0"'] \arrow[dr, "f^{\mathrm{I}}"] & Y_1 \arrow[d, "f_1"] \\
Y_0 \arrow[r, "f_0"'] & X^{\mathrm{I}},
\end{tikzcd} \tag{71}$$

We will say that $f : Y \to X$ *has composition from 0 to 1* if for every cofibration $c : C \rightarrowtail Z$ and maps $y_0 : Z \to Y_0$ and $y : C \to Y^{\mathrm{I}}$ making the square on the left below commute, there is a map $y_1 : Z \to Y_1$ making the following commute.

$$\begin{tikzcd} C \arrow[r, "y"] \arrow[d, tail, "c"'] & Y^{\mathrm{I}} \arrow[r, "f^1"] \arrow[d, "f^0"'] & Y_1 \arrow[d, "f_1"] \\ Z \arrow[r, "y_0"'] \arrow[urr, dotted, "y_1"] & Y_0 \arrow[r, "f_0"'] & X^{\mathrm{I}} \end{tikzcd} \tag{72}$$

To define *unbiased* composition, we begin with $f : Y \to X$ in cSet and then move to cSet/I, where we have the generic point $\delta : 1 \to \mathrm{I}$. Now we consider an arbitrary cofibration $c : C \rightarrowtail Z$ and maps $y_\delta : Z \to Y_\delta$ and $y : C \to Y^{\mathrm{I}}$ making the square below commute

$$\begin{tikzcd} C \arrow[r, "y"] \arrow[d, tail, "c"'] & Y^{\mathrm{I}} \arrow[d, "f^\delta"] \\ Z \arrow[r, "y_\delta"'] & Y_\delta \end{tikzcd} \tag{73}$$

where Y_δ and f^δ are defined in terms of $\delta : 1 \to \mathrm{I}$ just as were Y_0 and f^0 in terms of $\delta_0 : 1 \to \mathrm{I}$. Passing to $\mathsf{cSet}/(\mathrm{I} \times \mathrm{I})$ by a further pullback, as before we have another point $\delta' : 1 \to \mathrm{I}$, as well as a subobject $\Delta \rightarrowtail 1$, determined by the further diagonals.

Definition 53. The map $f : Y \to X$ *has (unbiased) composition* if, in cSet/I, for any cofibration $c : C \rightarrowtail Z$ and maps $y_\delta : Z \to Y_\delta$ and $y : C \to Y^{\mathrm{I}}$ as on the left below, there is in $\mathsf{cSet}/(\mathrm{I} \times \mathrm{I})$ a map $y_{\delta'} : Z \to Y_{\delta'}$ making the following commute

$$\begin{tikzcd} C \arrow[r, "y"] \arrow[d, tail, "c"'] & Y^{\mathrm{I}} \arrow[r, "f^{\delta'}"] \arrow[d, "f^\delta"'] & Y_{\delta'} \arrow[d, "\eta_{Y_{\delta'}}"] \\ Z \arrow[r, "y_\delta"'] \arrow[urr, dotted, "y_{\delta'}"] & Y_\delta \arrow[r, "\eta_{Y_\delta}"'] & Y^\Delta \end{tikzcd} , \tag{74}$$

where $Y_{\delta'}$ and $f^{\delta'}$ are defined in terms of $\delta' : 1 \to \mathrm{I}$, and Y^Δ is $Y_\delta^\Delta = Y_{\delta'}^\Delta$, since $\Delta^*\delta = \Delta^*\delta'$.

Proposition 54. *Composition implies filling for maps; that is, every $f : Y \to X$ with composition is a fibration.*

Proof. Analogous to the proof of Proposition 52. □

Remark 55. One can also promote the *property* of an object or map of "having composition" to the notion of a *composition structure.* This procedes via the notion of a *uniform composition structure*, which is defined with respect to cofibrations $c : C \rightarrowtail \mathrm{I}^n$ with representable codomains, and a requirement of naturality in I^n, and which can then be internalized as a suitable map representing the uniform structure, in a way that is analogous to the case for trivial fibrations formulated in proposition 7.

9 The Frobenius condition

In this section, we show that the fibration WFS from section 6 has the *Frobenius property*: the left maps are stable under pullback along the right maps (see [?]). This will imply the *right properness* of our model structure: the weak equivalences are preserved by pullback along fibrations. The Frobenius property is also needed in the proof of the equivalence extension property in the next section. A proof of Frobenius in a related setting of cubical sets with connections can be found in [GS17]; however the type theoretic approach of [OP17, CCHM16] can be applied without connections and is also more direct. This approach proves the "dual" fact that the *pushforward* operation, which is right adjoint to pullback, and which always exists in a topos, when applied along any *fibration* $f : Y \twoheadrightarrow X$ preserves fibrations. This corresponds to the type-theoretic Π-formation rule.

Recall that a map $f : A \to X$ is a fibration if (in the slice cSet/I, where there is a generic point $\delta : 1 \to \mathrm{I}$) the map $\delta \Rightarrow f$ admits a +-algebra structure (and so is a trivial fibration), where the definition of $\delta \Rightarrow f$ is recalled below.

$$\begin{tikzcd}
A^{\mathrm{I}} \arrow[dr, "\delta\Rightarrow f"] \arrow[drr, bend left, "A^{\delta}"] \arrow[ddr, bend right, "f^{\mathrm{I}}"'] & & \\
& X^{\mathrm{I}} \times_X A \arrow[r] \arrow[d] & A \arrow[d, "f"] \\
& X^{\mathrm{I}} \arrow[r, "X^{\delta}"'] & X.
\end{tikzcd} \tag{75}$$

Let us write this condition schematically as follows:

$$\begin{tikzcd}
A^{\mathrm{I}} \arrow[r, "+" marking] & A_{\epsilon} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
& X^{\mathrm{I}} \arrow[r, "\epsilon"'] & X,
\end{tikzcd} \tag{76}$$

where $\epsilon = X^{\delta}$, $A_{\epsilon} = X^{\mathrm{I}} \times_X A$, and the struck-through arrow indicates that it admits a $+$-algebra structure.

Lemma 56. *Let $g : Y \to X$ be any map and $f : A \to X$ a fibration, then the pullback $g^* f : g^* A \to Y$ is also a fibration.*

Proof. This is clear, since fibrations are the right class of a weak factorization system, but let us see how the "algebraic" specification (79) is also stable under pullback ... □

Lemma 57. *Let $\alpha : A \to X$ and $\beta : B \to A$ be fibrations, then the composite $\alpha \circ \beta : B \to X$ is also a fibration.*

Proof. We have the following for the fibration structures on $B \to A$ and $A \to X$ (with obvious notation).

$$\begin{tikzcd}
B^{\mathrm{I}} \arrow[r, "+" marking] & B_{\epsilon_A} \arrow[rr] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & & B \arrow[d] \\
& A^{\mathrm{I}} \arrow[r, "+" marking] & A_{\epsilon_X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d] \\
& & X^{\mathrm{I}} \arrow[r, "\epsilon_X"'] & X,
\end{tikzcd} \tag{77}$$

Pulling back $B \to A$ in two steps we therefore obtain,

$$\begin{tikzcd}
B^{\mathrm{I}} \arrow[r, "+" marking] & B_{\epsilon_A} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B_{\epsilon_X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d] \\
& A^{\mathrm{I}} \arrow[r, "+" marking] & A_{\epsilon_X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d] \\
& & X^{\mathrm{I}} \arrow[r, "\epsilon_X"'] & X,
\end{tikzcd} \tag{78}$$

Now use the fact that trivial fibrations are closed under pullback along all maps, and under composition, to infer that the indicated composite map $B^{\mathrm{I}} \longrightarrow B_{\epsilon X}$ is also a trivial fibration, as required. $\square$

Proposition 58 (Frobenius)**.** *Let $\alpha : A \to X$ and $\beta : B \to A$ be fibrations, then the pushforward $\alpha_* \beta : \Pi_A B \to X$ is also a fibration.*

Proof. Given the fibrations $\alpha : A \to X$ and $\beta : B \to A$, let $p : A^{\mathrm{I}} \to A_\epsilon$ and $q : B^{\mathrm{I}} \to p^* B_\epsilon$ be the associated +-algebras, so that we have the following situation, with all squares pullbacks.

$$\begin{tikzcd}
B^{\mathrm{I}} \arrow[r, "|" marking, "q"'] & p^*B_\epsilon \arrow[r] \arrow[d] & B_\epsilon \arrow[r] \arrow[d] & B \arrow[d] \\
& A^{\mathrm{I}} \arrow[r, "|" marking, "p"'] & A_\epsilon \arrow[r] \arrow[d] & A \arrow[d] \\
& & X^{\mathrm{I}} \arrow[r, "\epsilon"'] & X,
\end{tikzcd} \tag{79}$$

Adding (some composites and) the relevant pushforward underneath, we have

$$\begin{tikzcd}
B^{\mathrm{I}} \arrow[r, "|" marking, "q"'] \arrow[dr] & p^*B_\epsilon \arrow[r] \arrow[d] & B_\epsilon \arrow[r] \arrow[d] & B \arrow[d] \\
& A^{\mathrm{I}} \arrow[r, "|" marking, "p"'] \arrow[dr] & A_\epsilon \arrow[r] \arrow[d] & A \arrow[d] \\
& & X^{\mathrm{I}} \arrow[r, "\epsilon"'] & X \\
& (\Pi_A B)^{\mathrm{I}} \arrow[r, "r"'] \arrow[ur] & (\Pi_A B)_\epsilon \arrow[r] \arrow[u] & \Pi_A B \arrow[u]
\end{tikzcd} \tag{80}$$

and we wish to show that the indicated map $r : (\Pi_A B)^{\mathrm{I}} \longrightarrow (\Pi_A B)_\epsilon$ admits a +-algebra structure. We will do so by showing that it is a retract of a known +-algebra.

Indeed, let us apply the pushforward, along the indicated canonical map $\alpha^{\mathrm{I}} : A^{\mathrm{I}} \to X^{\mathrm{I}}$, to the +-algebra $q : B^{\mathrm{I}} \to p^* B_\epsilon$, regarded as an arrow over A^{I}. We obtain an arrow over X^{I} of the form

$$\Pi_{A^{\mathrm{I}}}\, q : \Pi_{A^{\mathrm{I}}}\, B^{\mathrm{I}} \longrightarrow \Pi_{A^{\mathrm{I}}}\, p^* B_\epsilon$$

which is a +-algebra, because these are preserved by pushforward, according to Lemma 12.

Now observe that by the Beck-Chevalley condition, we have an isomorphism

$$(\Pi_A B)_\epsilon \cong \Pi_{A_\epsilon} B_\epsilon \,.$$

Moreover, since $\Pi_{A^{\mathrm{I}}} \cong \Pi_{A_\epsilon} \circ p_*$, we have

$$\Pi_{A^{\mathrm{I}}}\, p^* B_\epsilon \cong \Pi_{A_\epsilon} p_* p^* B_\epsilon \,.$$

Thus the image of the unit $\eta : B_\epsilon \to p_* p^* B_\epsilon$ under Π_{A_ϵ} is a map $s = \Pi_{A_\epsilon}\eta$ over X^{I} of the form:

$$\begin{tikzcd}
& X^{\mathrm{I}} \\
(\Pi_A B)^{\mathrm{I}} \arrow[ur] \arrow[r, "r"'] & \Pi_{A_\epsilon} B_\epsilon \arrow[u] \arrow[d, "s"] \\
\Pi_{A^{\mathrm{I}}} B^{\mathrm{I}} \arrow[r, "\Pi_{A^{\mathrm{I}}}\, q"'] & \Pi_{A_\epsilon} p_* p^* B_\epsilon
\end{tikzcd} \tag{81}$$

Our objective is now to fill in the further arrows φ, ψ, ϑ indicated below in order to exhibit r as a retract of $\Pi_{A^{\mathrm{I}}}\, q$ in the arrow category over X^{I}.

$$\begin{tikzcd}
& X^{\mathrm{I}} \\
(\Pi_A B)^{\mathrm{I}} \arrow[ur] \arrow[r, "r"'] \arrow[d, dotted, "\varphi"'] & \Pi_{A_\epsilon} B_\epsilon \arrow[u] \arrow[d, "s"] \\
\Pi_{A^{\mathrm{I}}} B^{\mathrm{I}} \arrow[r, "\Pi_{A^{\mathrm{I}}}\, q"'] \arrow[d, dotted, "\psi"'] & \Pi_{A_\epsilon} p_* p^* B_\epsilon \arrow[d, dotted, "\vartheta"] \\
(\Pi_A B)^{\mathrm{I}} \arrow[r, "r"'] & \Pi_{A_\epsilon} B_\epsilon
\end{tikzcd} \tag{82}$$

- For φ, we require a map

$$\varphi : (\Pi_A B)^{\mathrm{I}} \longrightarrow \Pi_{A^{\mathrm{I}}} B^{\mathrm{I}} \qquad \text{over } X^{\mathrm{I}}.$$

Consider the following diagram, which is based on (80).

$$
\begin{tikzcd}
B^{\mathrm{I}} \arrow[r, "q"', "|" marking] \arrow[dr] & p^*B_\epsilon \arrow[r] \arrow[d] & B_\epsilon \arrow[r] \arrow[d] & B \arrow[d] & \\
 & A^{\mathrm{I}} \arrow[r, "p"', "|" marking] \arrow[dr] & A_\epsilon \arrow[r] \arrow[d] & A & \Pi_A B \times_X A \arrow[l] \arrow[d] \arrow[ul, dotted, "c"'] \\
(\Pi_A B \times_X A)^{\mathrm{I}} \arrow[uu, dotted, "c^{\mathrm{I}}"] \arrow[ur] \arrow[dr] & & X^{\mathrm{I}} \arrow[r, "\epsilon"'] & X & \Pi_A B \arrow[l] \\
 & (\Pi_A B)^{\mathrm{I}} \arrow[ur] \arrow[r, dotted, "\varphi"'] & \Pi_{A^{\mathrm{I}}} B^{\mathrm{I}} \arrow[u] & &
\end{tikzcd}
\tag{83}
$$

The map c is the counit at $B \to A$ of the pullback-pushforward adjunction along $A \to X$. The right-hand side of the diagram, including c and the associated pullback square, reappear on the left under the functor $(-)^{\mathrm{I}}$, which preserves the pullback. Thus we can take φ to be the transpose of c^{I} under the pullback-pushforward adjunction along $A^I \to X^{\mathrm{I}}$,

$$\varphi = \widetilde{c^{\mathrm{I}}}.$$

A diagram chase involving the pullback-pushforward adjunction along $A_\epsilon \to X^{\mathrm{I}}$ shows that the upper square in (82) commutes.

• For ϑ: referring to the diagram (80), since $p : A^{\mathrm{I}} \to A_\epsilon$ is a trivial fibration, it has a section $o : A_\epsilon \to A^{\mathrm{I}}$ by lemma 8. Pulling $p^*B_\epsilon \to A^{\mathrm{I}}$ back along o results in an iso over A_ϵ,

$$o^*p^*B_\epsilon \cong B_\epsilon\,,$$

and so by the adjunction $o^* \dashv o_*$ there is a map over A^{I},

$$p^*B_\epsilon \longrightarrow o_*B_\epsilon\,,$$

to which we can apply p_* to obtain a map,

$$\rho : p_*p^*B_\epsilon \longrightarrow p_*o_*B_\epsilon \cong B_\epsilon \quad \text{over } A_\epsilon.$$

This is a retraction of the unit $\eta : B_\epsilon \to p_*p^*B_\epsilon$ over A_ϵ. Applying the functor Π_{A_ϵ} therefore gives the desired retraction

$$\vartheta = \Pi_{A_\epsilon}\rho : \Pi_{A_\epsilon} p_*p^*B_\epsilon \to \Pi_{A_\epsilon} B_\epsilon$$

of s.

- For ψ, we require a map

$$\psi : \Pi_{A^{\mathrm{I}}} B^{\mathrm{I}} \longrightarrow (\Pi_A B)^{\mathrm{I}} \qquad \text{over } X^{\mathrm{I}}.$$

Consider the following diagram resulting from combining (80) and (82), and in which all arrows are those already introduced and the dotted one labelled α is the evident composite.

$$\begin{tikzcd}
 & X^{\mathrm{I}} \arrow[r, "\epsilon"] & X \\
(\Pi_A B)^{\mathrm{I}} \arrow[ur] \arrow[r] \arrow[d] & \Pi_{A_\epsilon} B \arrow[u] \arrow[r] \arrow[d] & \Pi_A B \arrow[u] \arrow[dd, "="] \\
\Pi_{A^{\mathrm{I}}} B^{\mathrm{I}} \arrow[r] \arrow[drr, dotted, "\alpha"] & \Pi_{A_\epsilon} p_* p^* B_\epsilon \arrow[d] & \\
(\Pi_A B)^{\mathrm{I}} \arrow[r] & \Pi_{A_\epsilon} B_\epsilon \arrow[r] & \Pi_A B
\end{tikzcd} \tag{84}$$

Now recall that we are working in the slice category over I, and the objects $\Pi_{A^{\mathrm{I}}} B^{\mathrm{I}}$, $\Pi_A B$, and $(\Pi_A B)^{\mathrm{I}}$ are in the image of the base change I^*, and so are actually of the form $\mathrm{I}^*\Pi_{A^{\mathrm{I}}} B^{\mathrm{I}}$, $\mathrm{I}^*\Pi_A B$, and $\mathrm{I}^*((\Pi_A B)^{\mathrm{I}})$. Indeed, the latter is

$$\mathrm{I}^*((\Pi_A B)^{\mathrm{I}}) = \mathrm{I}^*\mathrm{I}_*\mathrm{I}^*\Pi_A B\,.$$

Since the lower horizontal map is the counit ε of the base change $\mathrm{I}^* \dashv \mathrm{I}_*$, the map α factors as $\varepsilon \circ \mathrm{I}^*\tilde{\alpha}$, where $\tilde{\alpha}$ is the adjoint transpose of α, as shown in the following.

$$\begin{tikzcd}
\mathrm{I}^*\Pi_{A^{\mathrm{I}}} B^{\mathrm{I}} \arrow[r] \arrow[d, "\mathrm{I}^*\tilde{\alpha}"'] \arrow[drr, dotted, "\alpha"] & \Pi_{A_\epsilon} p_* p^* B_\epsilon \arrow[d] & \\
\mathrm{I}^*\mathrm{I}_*\mathrm{I}^*\Pi_A B \arrow[r] \arrow[rr, bend right, "\varepsilon"'] & \Pi_{A_\epsilon} B_\epsilon \arrow[r] & \mathrm{I}^*\Pi_A B
\end{tikzcd} \tag{85}$$

We set $\psi = \mathrm{I}^*\tilde{\alpha}$, making the square commute.

We have now defined all the maps below, the squares involving φ and ψ

commute, and the composite of ϑ and s is the identity.

$$
\begin{tikzcd}
& X^{\mathrm{I}} \arrow[r] & X \\
(\Pi_A B)^{\mathrm{I}} \arrow[ur] \arrow[r] \arrow[d, "\varphi"'] & \Pi_{A_\epsilon} B \arrow[u] \arrow[r] \arrow[d, "s"] & \Pi_A B \arrow[u] \arrow[dd, "="] \\
\Pi_{A^{\mathrm{I}}} B^{\mathrm{I}} \arrow[r] \arrow[d, "\psi"'] & \Pi_{A_\epsilon} p_* p^* B_\epsilon \arrow[d, "\vartheta"] & \\
(\Pi_A B)^{\mathrm{I}} \arrow[r] \arrow[rr, bend right, "\varepsilon"'] & \Pi_{A_\epsilon} B_\epsilon \arrow[r] & \Pi_A B
\end{tikzcd}
\tag{86}
$$

To see that $\psi \circ \varphi = 1$, observe that each map is in the image of I^*, say:

$$
\begin{aligned}
\varphi &= \mathrm{I}^* f \\
\psi &= \mathrm{I}^* g \,,
\end{aligned}
$$

where $g = \tilde{\alpha}$. Recall that in general the unit ε satisfies,

$$\varepsilon \circ \mathrm{I}^*(h) = \tilde{h}$$

for any map $h : X \to \mathrm{I}_* Y$. Thus

$$
\begin{aligned}
\varepsilon \circ \psi \circ \varphi &= \varepsilon \circ \mathrm{I}^* g \circ \mathrm{I}^* f \\
&= \varepsilon \circ \mathrm{I}^*(g \circ f) \\
&= \widetilde{(g \circ f)}.
\end{aligned}
$$

On the other hand, a diagram chase on (86) shows that

$$\varepsilon \circ \psi \circ \varphi = \varepsilon.$$

Therefore $g \circ f = \tilde{\varepsilon} = 1$, so $\psi \circ \varphi = \mathrm{I}^* g \circ \mathrm{I}^* f = \mathrm{I}^*(g \circ f) = \mathrm{I}^* 1 = 1$.

□

10 The equivalence extension property

[... as usual ...]

11 The universe

In this section, we define a universal small fibration $\dot{\mathcal{U}} \longrightarrow \mathcal{U}$. In the next section we shall use the EEP to show that $\mathcal{U}$ is a fibrant object.

11.1 Classifying families

Let κ be an inaccessible cardinal number, and call the sets of size strictly less than κ *small*. Write Set_κ for the category of small sets and $\mathsf{cSet}_\kappa = \mathsf{Set}_\kappa^{\mathbb{C}^{\mathrm{op}}}$ for the category of small set valued presheaves on the cube category $\mathbb{C}$. By a *small fibration* we mean a fibration in the category of small cubical sets, which we identify with the evident subcategory $\mathsf{cSet}_\kappa \subseteq \mathsf{cSet}$. Finally, let $\dot{\mathsf{Set}}_\kappa$ be the category of small pointed sets, i.e. the coslice category $1/\mathsf{Set}_\kappa$. There is an evident forgetful functor $U : \dot{\mathsf{Set}}_\kappa \longrightarrow \mathsf{Set}_\kappa$.

Definition 59. The (κ-)*universe* $p : \dot{\mathcal{V}} \longrightarrow \mathcal{V}$ in cSet is defined:

1. $\mathcal{V}_n = \{A : \mathbb{C}/[n] \longrightarrow \mathsf{Set}_\kappa^{op}\}$, the *set* of small presheaves on $\mathbb{C}/[n]$.

   The action of a map $h : [m] \to [n]$ in $\mathbb{C}$ is given by precomposition with postcomposition: from $h : [m] \to [n]$ we have $\mathbb{C}/h : \mathbb{C}/[m] \to \mathbb{C}/[n]$, which we precompose with any $A : \mathbb{C}/[n] \longrightarrow \mathsf{Set}_\kappa^{op}$ to get $A.h = A \circ \mathbb{C}/h$,

$$\begin{tikzcd} {[n]} & \mathbb{C}/[n] \arrow[r, "A"] & \mathsf{Set}_\kappa^{op} \\ {[m]} \arrow[u, "h"] & \mathbb{C}/[m] \arrow[u, "\mathbb{C}/h"] \arrow[ur, "A.h"'] & \end{tikzcd} \tag{87}$$

2. $\dot{\mathcal{V}}_n = \{a : \mathbb{C}/[n] \longrightarrow \dot{\mathsf{Set}}_\kappa^{op}\}$, the *set* of small pointed presheaves on $\mathbb{C}/[n]$, with the corresponding action.

3. For $a \in \dot{\mathcal{V}}_n$, let $p_n(a) = U(a) \in \mathcal{V}_n$, where $U : \dot{\mathsf{Set}}_\kappa \longrightarrow \mathsf{Set}_\kappa$.

Functoriality of $\mathcal{V}$ and $\dot{\mathcal{V}}$ and naturality of $p : \dot{\mathcal{V}} \to \mathcal{V}$ are immediate.

Lemma 60. *For each $A : \mathrm{I}^n \to \mathcal{V}$ there is a canonical choice of a small family $p_A : E_A \to \mathrm{I}^n$ and a map $q_A : E_A \to \dot{\mathcal{V}}$ making a pullback square as follows.*

$$\begin{tikzcd} E_A \arrow[r, "q_A"] \arrow[d, "p_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathcal{V}} \arrow[d, "p"] \\ \mathrm{I}^n \arrow[r, "A"'] & \mathcal{V} \end{tikzcd} \tag{88}$$

Proof. Since $\mathrm{I}^n \cong y[n]$ is representable, there is a distinguished associated presheaf $A : (\mathbb{C}/[n])^{op} \to \mathsf{Set}_\kappa$. Define $p_A : E_A \to \mathrm{I}^n$ by

$$(E_A)_k = \coprod_{h \in \mathbb{C}(k,n)} A(h) \qquad \ni (h, a)$$

with first projection $(p_A)_k(h, a) = h$. Note that $(E_A)_k$ is small. Then let $q_A : E_A \to \dot{\mathcal{V}}$ be defined on $(h, a) : \mathrm{I}^k \to E_A$ by

$$(q_A) \circ (h, a) = a \in Ah$$

as illustrated below.

$$\begin{tikzcd}
 & E_A \arrow[r, dotted, "q_A"] \arrow[d, "p_A"] & \dot{\mathcal{V}} \arrow[d, "p"] \\
\mathrm{I}^k \arrow[ur, "(h{,}a)"] \arrow[urr, "a"'] \arrow[r, "h"'] & \mathrm{I}^n \arrow[r, "A"'] & \mathcal{V}
\end{tikzcd} \tag{89}$$

The proof that the square is a pullback is left to the reader. □

Lemma 61. *For each small family $p_E : E \to \mathrm{I}^n$ there is a canonical map $\chi_E : \mathrm{I}^n \to \mathcal{V}$ and a map $q_E : E \to \dot{\mathcal{V}}$ making a pullback square as follows.*

$$\begin{tikzcd}
E \arrow[r, "q_E"] \arrow[d, "p_E"'] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathcal{V}} \arrow[d, "p"] \\
\mathrm{I}^n \arrow[r, "\chi_E"'] & \mathcal{V}
\end{tikzcd} \tag{90}$$

Proof. It suffices to give a small set $(\chi_E)_k(h)$ for each $h : [k] \to [n]$ in a way that is functorial in $h \in \mathbb{C}/[n]$ and natural in $[k]$. Thus let

$$(\chi_E)_k(h) := \Gamma(h, E) \;=\; \{e : \mathrm{I}^k \to E \mid p_E \circ e = h\}.$$

$$\begin{tikzcd}
 & E \arrow[d, "p_E"] \\
\mathrm{I}^k \arrow[ur, dotted, "e"] \arrow[r, "h"'] & \mathrm{I}^n
\end{tikzcd}$$

which is small if each E_k is.

To define $q_E : E \to \dot{\mathcal{V}}$, take any $e : \mathrm{I}^k \to E$ and first compose with p_E and observe that $e \in \Gamma(p_E e, E)$. Thus the assignment gives a map $\dot{e} : \mathrm{I}^k \to \dot{\mathcal{V}}$ making the solid arrows in the following commute.

$$\begin{tikzcd} & E \arrow[r, dotted, "q_E"] \arrow[d, "p_E"] & \dot{\mathcal{V}} \arrow[d, "p"] \\ \mathrm{I}^k \arrow[ur, "e"] \arrow[urr, "\dot{e}"'] \arrow[r, "p_E e"'] & \mathrm{I}^n \arrow[r, "E"'] & \mathcal{V} \end{tikzcd} \tag{91}$$

Since the assignment of $\dot{e}$ to e is natural in $[k]$, we get the required map $q_E : E \to \dot{\mathcal{V}}$. The proof that the square is a pullback is again left to the reader. $\square$

Corollary 62. *Given a small family $p_E : E \to \mathrm{I}^n$ there is a unique isomorphism $E \cong E_{\chi_E}$ over I^n making a commutative diagram as follows.*

$$\begin{tikzcd} E \arrow[r, "\cong"] \arrow[rr, bend left, "q_E"] \arrow[dr, "p_E"'] & E_{\chi_E} \arrow[r, "q_{\chi_E}"'] \arrow[d, "p_{\chi_E}"] & \dot{\mathcal{V}} \arrow[d, "p"] \\ & \mathrm{I}^n \arrow[r, "\chi_E"'] & \mathcal{V} \end{tikzcd} \tag{92}$$

Proposition 63. *For any cubical set X and any small family $p_E : E \to X$ there are canonical maps $\chi_E : X \to \mathcal{V}$ and $q_E : E \to \dot{\mathcal{V}}$ making a pullback square as follows.*

$$\begin{tikzcd} E \arrow[r, "q_E"] \arrow[d, "p_E"'] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathcal{V}} \arrow[d, "p"] \\ X \arrow[r, "\chi_E"'] & \mathcal{V} \end{tikzcd} \tag{93}$$

Moreover, χ_E and q_E are uniquely determined by the equations (95) *below.*

Proof. Write $X = \varinjlim_x \mathrm{I}^n$ as a colimit of a cocone of maps $x : \mathrm{I}^n \to X$ from representables, over the canonical index category $([n], x) \in \int_{\mathbb{C}} X$. Form the family of pullback squares below, where the arrows with a dot represent cocones, and the cocone consisting of the $q_x : E_x \to E$ is determined by taking pullbacks along p_E, and is therefore also a colimit.

$$\begin{tikzcd}
E_x \arrow[r, "q_x"', "\cdot"] \arrow[d, "p_{E_x}"'] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[rr, bend left, "q_{E_x}"] & E \arrow[r, dotted, "q_E"] \arrow[d, "p_E"'] & \dot{\mathcal{V}} \arrow[d, "p"] \\
\mathrm{I}^n \arrow[r, "\cdot", "x"'] \arrow[rr, bend right, "\chi_{E_x}"'] & X \arrow[r, dotted, "\chi_E"'] & \mathcal{V}
\end{tikzcd} \tag{94}$$

The maps χ_{E_x} and q_{E_x} are determined by lemma 61, since the families p_{E_x} are small if $p_E : E \to X$ is. Thus we can define the indicated maps χ_E and q_E from the colimits as those uniquely determined by the equations:

$$\chi_E \circ x = \chi_{E_x} \tag{95}$$

$$q_E \circ q_x = q_{E_x} \tag{96}$$

The square on the right is a pullback because the outer squares are all pullbacks, the family of left-hand squares are pullbacks, and the family of maps $x : \mathrm{I}^n \to X$ covers X. □

Remark 64. Note that the classification operation

$$\chi : \mathsf{cSet}_\kappa / X \longrightarrow \mathsf{cSet}(X, \mathcal{V})$$

again has the evident "pullback of $p : \dot{\mathcal{V}} \to \mathcal{V}$" operation

$$E : \mathsf{cSet}(X, \mathcal{V}) \longrightarrow \mathsf{cSet}_\kappa / X$$

as a left (quasi-)inverse $E \cong E_{\chi_E}$, which is (pseudo-)natural in X. But there is no corresponding uniqueness of classifying maps, relating $A : X \to \mathcal{V}$ and $\chi_{E_A} : X \to \mathcal{V}$. This is what is provided by the univalence of a universe

$$p : \dot{\mathcal{U}} \to \mathcal{U}$$

of *fibrations*, which we now procede to construct.

11.2 Classifying trivial fibrations

Recall from section 3 that (uniform) trivial fibration structures on a map $A \to X$ correspond bijectively to relative +-algebra structures over X (definition 4). A relative +-algebra structure on $A \longrightarrow X$ is an algebra structure for the pointed endofunctor $+_X : \mathsf{cSet}/X \longrightarrow \mathsf{cSet}/X$, where

$$A^+ = \sum_{\varphi:\Phi} A^\varphi \quad \text{over } X.$$

A +-algebra structure is then a retract $\alpha : A^+ \longrightarrow A$ over X of the canonical map $\eta_A : A \longrightarrow A^+$,

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[rr, bend left, "="] \arrow[dr] & A^+ \arrow[r, "\alpha"] \arrow[d] & A \arrow[dl] \\ & X. & \end{tikzcd}$$

In more detail, let us write $A \to X$ as a family $\sum_{x:X} A_x \to X$ over X. Since the +-functor acts fiberwise, the A^+ in the diagram above is then the indexing projection

$$\sum_{x:X} A_x^+ \longrightarrow X.$$

Working in the slice cSet/X, we make the (relative) exponentials (internal Hom's) $[A^+, A]$ and $[A, A]$ with the "precomposition by η_A" map $[\eta_A, A]$, which fit into the following pullback diagram

$$\begin{tikzcd} +\mathsf{Alg}(A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & {[A^+, A]} \arrow[d, "{[\eta_A, A]}"] \\ 1 \arrow[r, "{'id_A'}"'] & {[A, A].} \end{tikzcd}$$

The constructed object $+\mathsf{Alg}(A) \to X$ over X is the *object of +-algebra structures on* $A \to X$, in the sense that sections $X \to +\mathsf{Alg}(A)$ correspond isomorphically to +-algebra structures on $A \to X$. Moreover, $+\mathsf{Alg}(A) \to X$ is stable under pullback in the sense that for any $f : Y \to X$, we have two pullback squares,

$$\begin{tikzcd} f^*A \arrow[r] \arrow[d] & A \arrow[d] \\ Y \arrow[r, "f"'] & X \\ +\mathsf{Alg}(f^*A) \arrow[r] \arrow[u] & +\mathsf{Alg}(A) \arrow[u] \end{tikzcd} \tag{97}$$

because the +-functor, exponentials and pullbacks occurring in the construction of $+\mathsf{Alg}(A) \to X$ are themselves all stable.

It follows from proposition 63 that if $A \to X$ is small, then $+\mathsf{Alg}(A) \to X$ is itself a pullback of the analogous object $+\mathsf{Alg}(\dot{\mathcal{V}}) \to \mathcal{V}$ constructed from the universal small family $\dot{\mathcal{V}} \to \mathcal{V}$,

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \dot{\mathcal{V}} \arrow[d] \\ X \arrow[r, "\chi_A"'] & \mathcal{V} \\ +\mathsf{Alg}(A) \arrow[r] \arrow[u] & +\mathsf{Alg}(\dot{\mathcal{V}}) \arrow[u] \end{tikzcd} \tag{98}$$

Proposition 65. *There is a* universal small trivial fibration

$$\dot{\mathsf{TFib}} \to \mathsf{TFib}.$$

Every small trivial fibration $A \to X$ is a pullback of $\dot{\mathsf{TFib}} \to \mathsf{TFib}$ along a canonically determined classifying map $X \to \mathsf{TFib}$.

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & \dot{\mathsf{TFib}} \arrow[d] \\ X \arrow[r] & \mathsf{TFib} \end{tikzcd} \tag{99}$$

Proof. We can take $\mathsf{TFib} = +\mathsf{Alg}(\dot{\mathcal{V}})$, which comes with its projection $+\mathsf{Alg}(\dot{\mathcal{V}}) \to \mathcal{V}$ as in diagram (98). Now define $p_t : \dot{\mathsf{TFib}} \to \mathsf{TFib}$ by pulling back the universal small family,

$$\begin{tikzcd} \dot{\mathsf{TFib}} \arrow[r] \arrow[d, "p_t"'] \arrow[dr, phantom, "\lrcorner" very near start] & \dot{\mathcal{V}} \arrow[d, "p"] \\ \mathsf{TFib} \arrow[r] & \mathcal{V}. \end{tikzcd}$$

Consider the following diagram, in which all the squares (including the distorted ones) are pullbacks, with the outer one coming from proposition 63 and the lower one from (98).

$$
\begin{tikzcd}
 & A \arrow[rr, "q_A"] \arrow[ddd] \arrow[dr, dotted] & & \dot{\mathcal{V}} \arrow[ddd, "p"] \\
 & & \dot{\mathsf{TFib}} \arrow[ur] \arrow[d, "p_t"] & \\
\mathsf{TFib}(A) \arrow[rr, crossing over] \arrow[dr] & & \mathsf{TFib} \arrow[dr] & \\
 & X \arrow[rr, "\chi_A"'] \arrow[ur, dotted, "\alpha'"'] \arrow[ul, dotted, bend left, "\alpha"] & & \mathcal{V}.
\end{tikzcd}
\tag{100}
$$

A trivial fibration structure α on $A \to X$ is a section the object of $+$-algebra structures on A, occurring in the diagram as $\mathsf{TFib}(A)$, the pullback of TFib. Such sections correspond uniquely to factorizations α' of χ_A as indicated, which in turn induce pullback squares of the required kind (99). $\square$

11.3 Classifying fibrations

In order to classify fibrations $A \to X$, we shall proceed as for trivial fibrations by constructing an object $\mathsf{Fib}(A) \to X$ of fibration structures on $A \to X$ which, moreover, is stable under pullback. We then apply the construction to the universal small family $\dot{\mathcal{V}} \to \mathcal{V}$ to get a universal small fibration.

The construction of $\mathsf{Fib}(A) \to X$ is a bit more involved than that of $\mathsf{TFib}(A) \to X$. Recall from section 5 the characterization of (uniform, unbiased) fibration structures on a map $p_A : A \to X$ in terms of $+$-algebra structures:

1. First, pass to cSet/I where there is a generic point $\delta : 1 \to \mathrm{I}$,

2. Form the pullback-hom $\delta \Rightarrow p_A : A^{\mathrm{I}} \to X^{\mathrm{I}} \times_X A$ as indicated in the

following diagram.

$$
\begin{tikzcd}
A^{\mathrm{I}} \arrow[dr, "\delta \Rightarrow p_A"] \arrow[drr, bend left, "A^{\delta}"] \arrow[ddr, bend right, "(p_A)^{\mathrm{I}}"'] & & \\
& X^{\mathrm{I}} \times_X A \arrow[r] \arrow[d] & A \arrow[d, "p_A"] \\
& X^{\mathrm{I}} \arrow[r, "X^{\delta}"'] & X
\end{tikzcd}
\tag{101}
$$

3. A fibration structure on $p_A : A \to X$ is then a relative +-algebra structure on $\delta \Rightarrow p_A$ in the slice category over the codomain.

   We next construct an object $\mathsf{Fib}(A) \to X$ classifying such structures. For convenience, let us relabel the objects and arrows in the previous diagram as follows:

$$
\begin{aligned}
p &:= p_A \\
\epsilon &:= X^{\delta} : X^{\mathrm{I}} \to X \\
A_{\epsilon} &:= X^{\mathrm{I}} \times_X A \\
\epsilon^A &:= \delta \Rightarrow p_A
\end{aligned}
$$

   so that (the working part of) our diagram becomes:

$$
\begin{tikzcd}
A^{\mathrm{I}} \arrow[dr, "\epsilon^A"'] & & \\
& A_{\epsilon} \arrow[r] \arrow[d, "p_{\epsilon}"'] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, "p"] \\
& X^{\mathrm{I}} \arrow[r, "\epsilon"'] & X
\end{tikzcd}
\tag{102}
$$

4. A +-algebra structure on ϵ^A is a retract α over A_{ϵ} of the unit η as

indicated below, where D is the domain of the map $(\epsilon^A)^+$:

$$
\begin{tikzcd}
A^{\mathrm{I}} \arrow[r, "\eta"'] \arrow[dr, "\epsilon^A"'] & D \arrow[l, bend right, dotted, "\alpha"'] \arrow[d, "(\epsilon^A)^+"] & \\
& A_\epsilon \arrow[r] \arrow[d, "p_\epsilon"'] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, "p"] \\
& X^{\mathrm{I}} \arrow[r, "\epsilon"'] & X
\end{tikzcd}
\tag{103}
$$

5. Thus, as in the previous section, there is an object $+\mathsf{Alg}(\epsilon^A)$ over A_ϵ of $+$-algebra structures on ϵ^A, the sections of which correspond uniquely to $+$-algebra structures on ϵ^A (and thus fibration structures on A).

$$
\begin{tikzcd}
& A^{\mathrm{I}} \arrow[r, "\eta"'] \arrow[d, "\epsilon^A"'] & D \arrow[l, bend right, dotted, "\alpha"'] \arrow[dl, "(\epsilon^A)^+"] \\
+\mathsf{Alg}(\epsilon^A) \arrow[r] & A_\epsilon \arrow[r] \arrow[d, "p_\epsilon"'] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, "p"] \\
& X^{\mathrm{I}} \arrow[r, "\epsilon"'] & X
\end{tikzcd}
\tag{104}
$$

6. Sections of $+\mathsf{Alg}(\epsilon^A) \to A_\epsilon$ correspond to sections of its push-forward along p_ϵ, which we shall call F_A:

$$
F_A := (p_\epsilon)_*(+\mathsf{Alg}(\epsilon^A))\,.
$$

$$
\begin{tikzcd}
& A^{\mathrm{I}} \arrow[r, "\eta"'] \arrow[d, "\epsilon^A"'] & D \arrow[l, bend right, dotted, "\alpha"'] \arrow[dl, "(\epsilon^A)^+"] \\
+\mathsf{Alg}(\epsilon^A) \arrow[r] & A_\epsilon \arrow[r] \arrow[d, "p_\epsilon"'] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, "p"] \\
F_A \arrow[r] & X^{\mathrm{I}} \arrow[r, "\epsilon"'] & X
\end{tikzcd}
\tag{105}
$$

7. We might now think of taking another pushforward of $F_A \to X^{\mathrm{I}}$ along $\epsilon : X^{\mathrm{I}} \to X$ to get the object $\mathsf{Fib}(A) \to X$ that we seek, but unfortunately, this would not be stable under pullback along arbitrary maps $Y \to X$, because $\epsilon : X^{\mathrm{I}} \to X$ is not stable in this way. Instead we will use the *root* functor, i.e. the "amazing right adjoint" to the pathspace (see [**?**]).

$$(-)^{\mathrm{I}} \dashv (-)_{\mathrm{I}}$$

Let $f : F_A \to X^{\mathrm{I}}$ be the map indicated in (105), and let $\eta_X : X \to (X^{\mathrm{I}})_{\mathrm{I}}$ be the unit of the root adjunction. Then define $\mathsf{Fib}(A) \to X$ as the following pullback.

$$\begin{tikzcd} \mathsf{Fib}(A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (F_A)_{\mathrm{I}} \arrow[d, "f_{\mathrm{I}}"] \\ X \arrow[r, "\eta"'] & (X^{\mathrm{I}})_{\mathrm{I}} \end{tikzcd} \tag{106}$$

By adjointness, sections of $\mathsf{Fib}(A) \to X$ correspond uniquely to sections of $f : F_A \to X^{\mathrm{I}}$.

8. Finally, we are still working in the slice cSet/I and need to get back to cSet by applying the functor $\mathrm{I}_* : \mathsf{cSet}/\mathrm{I} \to \mathsf{cSet}$. Call the map $\mathsf{Fib}(A) \to X$ constructed over I in the last step $\mathsf{Fib}(A)_i \to \mathrm{I}^*X$ and apply I_* to get,

$$\mathrm{I}_*(\mathsf{Fib}(A)_i) = \Pi_{i:\mathrm{I}}\mathsf{Fib}(A)_i \longrightarrow X^{\mathrm{I}}$$

in cSet. We then define the desired map $\mathsf{Fib}(A) \to X$ as the pullback along the unit $\rho : X \to X^{\mathrm{I}}$ of $\mathrm{I}^* \dashv \mathrm{I}_*$, as indicated below.

$$\begin{tikzcd} \mathsf{Fib}(A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Pi_{i:\mathrm{I}}\mathsf{Fib}(A)_i \arrow[d] \\ X \arrow[r, "\rho"'] & X^{\mathrm{I}} \end{tikzcd} \tag{107}$$

It follows directly from the adjunction $\mathrm{I}^* \dashv \mathrm{I}_*$ that sections of $\mathsf{Fib}(A) \to X$ correspond bijectively to sections of $\mathsf{Fib}(A)_i \to \mathrm{I}^*X$ over I.

Proposition 66. *There is a* universal small fibration

$$\dot{\mathsf{Fib}} \to \mathsf{Fib}.$$

Every small fibration $A \to X$ is a pullback of $\dot{\mathsf{Fib}} \to \mathsf{Fib}$ along a canonically determined classifying map $X \to \mathsf{Fib}$.

$$
\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & \dot{\mathsf{Fib}} \arrow[d] \\
X \arrow[r] & \mathsf{Fib}
\end{tikzcd}
\tag{108}
$$

Proof. First, we need to show that the construction of $\mathsf{Fib}(A) \to X$ as the object of fibration structures on a map $A \to X$ is stable under pullback along all maps $f : Y \to X$. The relevant parts of the construction diagram (109) are repeated below,

$$
\begin{tikzcd}
 & A^{\mathrm{I}} \arrow[d, "\epsilon^A"'] & \\
+\mathsf{Alg}(\epsilon^A) \arrow[r] & A_\epsilon \arrow[r] \arrow[d, "p_\epsilon"'] \arrow[dr, phantom, "\lrcorner" very near start] & A \arrow[d, "p"] \\
F_A \arrow[r] & X^{\mathrm{I}} \arrow[r, "\epsilon"'] & X
\end{tikzcd}
\tag{109}
$$

Now consider the following in which the front face of the central cube is a pullback.

$$
\begin{tikzcd}
 & B^{\mathrm{I}} \arrow[rrr] \arrow[d, "\epsilon^B"'] & & & A^{\mathrm{I}} \arrow[d, "\epsilon^A"] & \\
+\mathsf{Alg}(\epsilon^B) \arrow[r] & B_\epsilon \arrow[rrr, dotted] \arrow[d] \arrow[dr] & & & A_\epsilon \arrow[d] \arrow[dl] & +\mathsf{Alg}(\epsilon^A) \arrow[l] \\
F_B \arrow[r] & Y^{\mathrm{I}} \arrow[rrr, dotted, bend right] \arrow[dr] & B \arrow[r] \arrow[d] & A \arrow[d] & X^{\mathrm{I}} \arrow[dl] & F_A \arrow[l] \\
 & \mathsf{Fib}(B) \arrow[r] & Y \arrow[r, "f"'] & X & \mathsf{Fib}(A) \arrow[l] &
\end{tikzcd}
\tag{110}
$$

On the left side we will repeat the construction with $B \to Y$ in place of $A \to X$. The left face is thus a pullback, whence the back (dotted) face is a pullback. The two-story square in back is the image of the front square under the right adjoint $(-)^{\mathrm{I}}$ and is therefore a pullback, therefore the top rectangle in the back is a pullback. It follows that $+\mathsf{Alg}(\epsilon^B)$ is a pullback of $+\mathsf{Alg}(\epsilon^A)$ along the upper dotted arrow, as in diagram (97), and so the pushforward F_B

is a pullback of the corresponding F_A, along the lower dotted arrow (which is f^{I}), by the Beck-Chevalley condition. Thus we have shown

$$F_B \cong (f^{\mathrm{I}})^* F_A. \tag{111}$$

It remains to show that $\mathsf{Fib}(B)$ is a pullback of $\mathsf{Fib}(A)$ along $f : Y \to X$, and now it is good that we did not take these to be pushforwards of F_B and F_A, because the floor of the cube is not a pullback, and so the Beck-Chavalley condition would not apply. Instead, consider the following diagram.

$$\begin{tikzcd}
 & \mathsf{Fib}(B) \arrow[r] \arrow[dl] \arrow[dd] & \mathsf{Fib}(A) \arrow[dr] \arrow[dd] & \\
(F_B)_{\mathrm{I}} \arrow[rrr] \arrow[dd] & & & (F_A)_{\mathrm{I}} \arrow[dd] \\
 & Y \arrow[r, "f"'] \arrow[dl, "\eta"'] & X \arrow[dr, "\eta"] & \\
(Y^{\mathrm{I}})_{\mathrm{I}} \arrow[rrr, "(f^{\mathrm{I}})_{\mathrm{I}}"'] & & & (X^{\mathrm{I}})_{\mathrm{I}}
\end{tikzcd} \tag{112}$$

The sides of the cube are pullbacks by the construction of $\mathsf{Fib}(A)$ and $\mathsf{Fib}(B)$. The front face is the root of the pullback (111) and is thus also a pullback, since the root is a right adjoint. The base commutes by naturality of the unit, and so the back face is also a pullback as required.

Now we can take $\mathsf{Fib} = \mathsf{Fib}(\dot{\mathcal{V}})$, which comes with its projection $\mathsf{Fib}(\dot{\mathcal{V}}) \to \mathcal{V}$, and define the universal small fibration $\dot{\mathsf{Fib}} \to \mathsf{Fib}$ by pulling back the universal small family,

$$\begin{tikzcd}
\dot{\mathsf{Fib}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathcal{V}} \arrow[d, "p"] \\
\mathsf{Fib} \arrow[r] & \mathcal{V}.
\end{tikzcd}$$

The remainder of the proof is just as for proposition 65. □

Proposition 67. *The map* $\dot{\mathsf{Fib}} \to \mathsf{Fib}$ *just constructed has a canonical fibration structure.*

Proof. Consider the following diagram, in which both squares are pullbacks.

$$\begin{tikzcd}
\dot{\mathsf{Fib}} \arrow[r] \arrow[d] & \dot{\mathcal{V}} \arrow[d] \\
\mathsf{Fib} \arrow[r] & \mathcal{V} \\
\mathsf{Fib}(\dot{\mathsf{Fib}}) \arrow[r] \arrow[u] & \mathsf{Fib}(\dot{\mathcal{V}}) \arrow[u]
\end{tikzcd} \tag{113}$$

$\mathsf{Fib}(\dot{\mathcal{V}})$ is the object of fibration structures on $\dot{\mathcal{V}} \to \mathcal{V}$, and its pullback $\mathsf{Fib}(\dot{\mathsf{Fib}})$ is therefore the object of fibration structures on $\dot{\mathsf{Fib}} \to \mathsf{Fib}$. Thus we seek a section of $\mathsf{Fib}(\dot{\mathsf{Fib}}) \to \mathsf{Fib}$. But recall that $\mathsf{Fib} = \mathsf{Fib}(\dot{\mathcal{V}})$ by definition, so the lower pullback square is the pullback of $\mathsf{Fib}(\dot{\mathcal{V}}) \to \mathcal{V}$ against itself, which does indeed have a distinguished section, namely the diagonal

$$\Delta : \mathsf{Fib}(\dot{\mathcal{V}}) \longrightarrow \mathsf{Fib}(\dot{\mathcal{V}}) \times_{\mathcal{V}} \mathsf{Fib}(\dot{\mathcal{V}}).$$

□

12 The fibration extension property

In the presence of a universal fibration $\dot{\mathcal{U}} \to \mathcal{U}$ as was constructed in section 11, the fibration extension property is closely related to the statement that the base object $\mathcal{U}$ is fibrant (see below). For Kan simplicial sets, Voevodsky proved this directly using minimal fibrations [**?**]. Shulman [**?**] gives a proof from univalence (in the form of the equivalence extension property as in section 10) in a more general setting, but it uses the 3-for-2 property for weak equivalences, which is what we are trying to prove. The following argument, also using the equivalence extension property, is due to Coquand and employs the reduction of filling to composition (section 8).

Proposition 68. *The universe $\mathcal{U}$ is fibrant.*

By the reduction of filling to composition (Proposition 52), it suffices to show:

Lemma 69. *The universe $\mathcal{U}$ has composition.*

Proof. Consider a composition problem

$$
\begin{tikzcd}
C \arrow[r] \arrow[dd, tail, "c"'] & \mathcal{U}^{\mathbb{I}} \arrow[d] \\
 & \mathcal{U} \times \mathcal{U} \arrow[d] \\
Z \arrow[r] \arrow[ur, dotted, "k"] & \mathcal{U}
\end{tikzcd}
$$

We claim that the canonical map $\mathcal{U}^{\mathbb{I}} \longrightarrow \mathcal{U} \times \mathcal{U}$ factors over $\mathcal{U} \times \mathcal{U}$ through the object Eq of equivalences, via a map i as indicated below.

$$
\begin{tikzcd}
C \arrow[r] \arrow[ddd, tail, "c"'] & \mathcal{U}^{\mathbb{I}} \arrow[d, "i"] \\
 & \mathsf{Eq} \arrow[d] \\
 & \mathcal{U} \times \mathcal{U} \arrow[d] \\
Z \arrow[r] \arrow[uur, dotted, "j"] \arrow[ur, dotted, "k"'] & \mathcal{U}
\end{tikzcd}
\tag{114}
$$

Since the projection $\mathsf{Eq} \longrightarrow \mathcal{U}$ is a trivial fibration by the equivalence extension property **??**, there is a diagonal filler j. Composing gives the required k.

The claimed map i is usually known as $\mathsf{IdtoEq} : \mathcal{U}^{\mathbb{I}} \to \mathsf{Eq}$, and is defined in type theory by transport.

[fill this in ...]

□

Finally, for the case of unbiased filling we need the more general form of composition going from one generic point $p : 1 \to \mathbb{I}$ to another $q : 1 \to \mathbb{I}$, rather than from one endpoint to another.

[fill this in ...]

Returning to the exact relation between the fibration extension property (Definition 45) and the condition that the base object $\mathcal{U}$ is fibrant, we can easily see that the latter implies the former. Indeed, let $t : X \rightarrowtail X'$ be a trivial cofibration and $Y \twoheadrightarrow X$ a fibration. To extend Y along t, take a classifying map $y : X \longrightarrow \mathcal{U}$, so that $Y \cong y^* \dot{\mathcal{U}}$ over X. If $\mathcal{U}$ is fibrant then we can extend y along $t : X \rightarrowtail X'$ to get $y' : X' \longrightarrow \mathcal{U}$ with $y = y' \circ t$. The pullback $Y' = (y')^*\dot{\mathcal{U}} \twoheadrightarrow X'$ is then a fibration such that $t^*Y' \cong t^*(y')^*\dot{\mathcal{U}} \cong y^*\dot{\mathcal{U}} \cong Y$ over X.

Conversely, we remark that the FEP implies the fibrancy of $\mathcal{U}$, given the following "alignment" lemma, which will be required below (cf.(2') of [**?**]).

Lemma 70. *Given a fibration $Y \twoheadrightarrow X$ with classifying map $y : X \longrightarrow \mathcal{U}$, a cofibration $t : X \rightarrowtail X'$, and a (small) fibration $Y' \twoheadrightarrow X'$ with $Y \cong t^*Y'$ over X, there is a classifying map $y' : X' \longrightarrow \mathcal{U}$ for Y' with $y' \circ t = y$.*

$$
\begin{tikzcd}
Y \arrow[rr] \arrow[dr] \arrow[dd, two heads] & & \dot{\mathcal{U}} \arrow[dd, two heads] \\
& Y' \arrow[ur, dotted] \arrow[dd, two heads] & \\
X \arrow[rr, "y"'] \arrow[dr, tail, "t"'] & & \mathcal{U} \\
& X' \arrow[ur, dotted, "y'"'] &
\end{tikzcd}
$$

Proof. This is routine using Yoneda and the fact that cofibrations have decidable monos as components. In more detail, since $Y' \twoheadrightarrow X'$ is small, there is a classifying map $z : X' \longrightarrow \mathcal{U}$, perhaps not commuting with t. Nonetheless we can use z to define the desired $y' : X' \longrightarrow \mathcal{U}$ objectwise as follows: Take any map from a representable $x' : \mathrm{I}^n \longrightarrow X'$ and consider whether it factors through t, say as $x' = t \circ x$ for some (necessarily unique) $x : \mathrm{I}^n \longrightarrow X$. If x' does factor, set $y' \circ x' = y \circ x$; if not, set $y' \circ x' = z \circ x'$. This specification is clearly natural in I^n, so it defines $y' : X' \longrightarrow \mathcal{U}$, and the specification ensures that $y' \circ t = y$, and that the pullback of $\dot{\mathcal{U}}$ along y' is the same as that along z, namely $Y' \twoheadrightarrow X'$. $\square$

We shall prove that $\mathcal{U}$ is fibrant using the equivalence extension property (section 10). This approach was first used in [CCHM16] via a different argument employing the reduction of filling to composition, which we do not require.

Proposition 71. *The base $\mathcal{U}$ of the universal fibration $\dot{\mathcal{U}} \to \mathcal{U}$ constructed in section 11 is fibrant.*

Proof. To show that $\mathcal{U}$ is fibrant, we need to solve the following filling problem for an arbitrary cofibration $c : C \rightarrowtail Z$, thus showing that $\mathcal{U}^{\delta_0} : \mathcal{U}^{\mathrm{I}} \longrightarrow \mathcal{U}$ is a trivial cofibration (for, say, the point $\delta_0 : 1 \longrightarrow \mathrm{I}$).

$$
\begin{tikzcd}
C \arrow[r, "\tilde{a}"] \arrow[d, tail, "c"'] & \mathcal{U}^{\mathrm{I}} \arrow[d, "\mathcal{U}^{\delta_0}"] \\
Z \arrow[r, "b"'] \arrow[ur, dotted] & \mathcal{U}
\end{tikzcd}
\tag{115}
$$

Transposing $\tilde{a}$ to $a : C \times \mathrm{I} \longrightarrow \mathcal{U}$ and taking pullbacks of $\dot{\mathcal{U}} \twoheadrightarrow \mathcal{U}$ along a and b to get corresponding fibrations $A \twoheadrightarrow C \times \mathrm{I}$ and $B \twoheadrightarrow Z$, we have the following equivalent condition. Letting

$$C_0 : C \cong C \times 1 \longrightarrow C \times \mathrm{I}$$

be the evident inclusion of the 0-end of the cylinder, let $A_0 = (C_0)^* A \twoheadrightarrow C$ be the "slice of A over C_0". We then have $c^* B \cong A_0$ over C by the outer square of (115).

$$
\begin{tikzcd}
 & A_0 \arrow[rr] \arrow[d, two heads] \arrow[ddl] & & A \arrow[d, two heads] \\
 & C \arrow[rr, "C_0"'] \arrow[ddl, "c"] & & C \times \mathrm{I} \arrow[ddl] \\
B \arrow[d, two heads] & & & \\
Z \arrow[rr, "Z_0"'] & & Z \times \mathrm{I} &
\end{tikzcd}
$$

The diagonal filler in (115) corresponds, again by transposition and pullback of $\dot{\mathcal{U}} \twoheadrightarrow \mathcal{U}$, to a fibration $D \twoheadrightarrow Z \times \mathrm{I}$ with $(c \times \mathrm{I})^* D \cong A$ over $C \times \mathrm{I}$ and

$(Z_0)^*D \cong B$ over Z, as indicated below.

$$
\begin{tikzcd}
 & A_0 \arrow[rr] \arrow[d, two heads] \arrow[ddl] & & A \arrow[d, two heads] \arrow[ddl, dotted] \\
 & C \arrow[rr, "C_0"'] \arrow[ddl, "c"] & & C \times \mathrm{I} \arrow[ddl, "c\times\mathrm{I}"] \\
B \arrow[rr, dotted] \arrow[d, two heads] & & D \arrow[d, dotted, two heads] & \\
Z \arrow[rr, "Z_0"'] & & Z \times \mathrm{I} &
\end{tikzcd}
$$

We construct $D \twoheadrightarrow Z \times \mathrm{I}$ by the equivalence extension property as follows. Apply the functor $(-) \times \mathrm{I}$ to the left (pullback) face of the above cube to get the following with a new pullback square on the right, with the indicated fibrations.

$$
\begin{tikzcd}
 & A_0 \arrow[rr] \arrow[d, two heads] \arrow[ddl] & & A \arrow[d, two heads] \arrow[ddl, dotted] & & A_0 \times \mathrm{I} \arrow[dll, two heads] \arrow[ddl, two heads] \\
 & C \arrow[rr, "C_0"'] \arrow[ddl, "c"] & & C \times \mathrm{I} \arrow[ddl, "c\times\mathrm{I}"] & & \\
B \arrow[rr, dotted] \arrow[d, two heads] & & D \arrow[d, dotted, two heads] & & B \times \mathrm{I} \arrow[dll, two heads] & \\
Z \arrow[rr, "Z_0"'] & & Z \times \mathrm{I} & & &
\end{tikzcd}
\tag{116}
$$

We claim there is a weak equivalence $e : A \simeq A_0 \times \mathrm{I}$ over $C \times \mathrm{I}$, from which follow by the EEP:

(i) a fibration $D \twoheadrightarrow Z \times \mathrm{I}$ with $(c \times \mathrm{I})^*D \cong A$ over $C \times \mathrm{I}$, and

(ii) a weak equivalence $f : D \simeq B \times \mathrm{I}$ over $Z \times \mathrm{I}$ with $(c \times \mathrm{I})^* f \cong e$ over $C \times \mathrm{I}$.

It then remains only to show that $B \cong (Z_0)^* D$ over Z.

To get e, consider the following square, in which the top map is $A_0 \times \delta_0$ (after $A_0 \cong A_0 \times 1$) and the others are those from the previous diagram.

$$\begin{tikzcd} A_0 \arrow[r, tail] \arrow[d, tail] & A_0 \times \mathrm{I} \arrow[d, two heads] \\ A \arrow[r, two heads] & C \times \mathrm{I} \end{tikzcd} \tag{117}$$

The square is easily seen to commute, and the maps with A_0 as domain are trivial cofibrations by Frobenius (proposition **??**), because each is the pullback of a trivial cofibration along a fibration. Applying a simple lemma (given below as 72) gives the required weak equivalence $e : A \simeq A_0 \times \mathrm{I}$ over $C \times \mathrm{I}$.

To see that $B \cong (Z_0)^* D$ over Z, recall from the proof of the EEP that the map $f : B \cong (Z_0)^* D$ is the pushforward of $e : A \simeq A_0 \times \mathrm{I}$ along the cofibration $d_0 \times \mathrm{I} : A_0 \times \mathrm{I} \rightarrowtail B \times \mathrm{I}$, calling the evident map $d_0 : A_0 \rightarrowtail B$ in (116). That is, by construction $f = (d_0 \times \mathrm{I})_* \, e$. We can apply the Beck-Chevalley condition for the pushforward using the pullback square on the left below.

$$\begin{tikzcd} A_0 \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & A_0 \times \mathrm{I} \arrow[d, two heads] & A \arrow[l, "e"'] \\ B \arrow[r, tail] & B \times \mathrm{I} & D \arrow[l, "f"] \end{tikzcd} \tag{118}$$

Since the pullback of e along the top of the square is the identity, the same is true (up to isomorophism) for the pullback of f along the bottom.

An application of the alignment lemma 70 completes the proof. $\square$

Lemma 72. *Suppose the following square commutes and the indicated cofi-*

brations are trivial.

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, tail] & C \arrow[d, two heads] \\ B \arrow[r, two heads] & D \end{tikzcd} \tag{119}$$

Then there is a weak equivalence $e : B \simeq C$ over D (and under A).

Proof. Use the fact that any two diagonal fillers are homotopic to get a homotopy equivalence $e : B \simeq C$ filling the square. □

Appendix 1: Filling from composition, with connections

For two points $p, q : 1 \to \mathrm{I}$, a cubical set X *has composition from p to q* if for every object Z and cofibration $c : C \rightarrowtail Z$ and commutative square

$$\begin{tikzcd} C \arrow[r] \arrow[d, tail, "c"'] & X^{\mathrm{I}} \arrow[d, "\epsilon_p"] \\ Z \arrow[r] & X, \end{tikzcd} \tag{120}$$

there is a diagonal arrow $k : Z \longrightarrow X \times X$ making both subdiagrams below commute,

$$\begin{tikzcd} C \arrow[r] \arrow[dd, tail, "c"'] & X^{\mathrm{I}} \arrow[d, "\langle \epsilon_p, \epsilon_q \rangle"] \\ & X \times X \arrow[d, "\pi_1"] \\ Z \arrow[r] \arrow[ur, dotted, "k"] & X, \end{tikzcd} \tag{121}$$

where $\epsilon_p : X^{\mathrm{I}} \longrightarrow X$ is the "evaluation at p" map X^p, and similarly for $\epsilon_q : X^{\mathrm{I}} \longrightarrow X$.

Proposition 73. *In cubical sets with connections, if an object X has composition from δ_0 to δ_1 and back, then X has filling for all trivial cofibrations $c \otimes \delta : B \rightarrowtail Z \times \mathrm{I}$, where $c : C \rightarrowtail Z$ is any cofibraiton and $\delta = \delta_0, \delta_1 : 1 \longrightarrow \mathrm{I}$.*

An object X has filling for all trivial cofibrations $c \otimes \delta : B \rightarrowtail Z \times \mathrm{I}$ iff for all cofibrations $c : C \rightarrowtail Z$ and squares as below there is a diagonal filler

$$\begin{tikzcd}
C \arrow[r] \arrow[d, tail, "c"'] & X^{\mathbb{I}} \arrow[d, "\delta \Rightarrow X"] \\
Z \arrow[r] \arrow[ur, dotted] & X
\end{tikzcd}$$

where the Leibniz exponential $\delta \Rightarrow X : X^{\mathbb{I}} \to X$ is "evaluation at the endpoint $\delta : 1 \to \mathbb{I}$" (and we require the condition for both endpoints $\delta = \delta_0, \delta_1$). Clearly if X has filling then it has composition, since there is then a diagonal filler k making both subdiagrams commute in

$$\begin{tikzcd}
C \arrow[r] \arrow[dd, tail, "c"'] & X^{\mathbb{I}} \arrow[d, "\partial \Rightarrow X"] \\
 & X \times X \arrow[d] \\
Z \arrow[r] \arrow[ur, dotted, "k"] & X
\end{tikzcd}$$

where $(\partial \Rightarrow X) : X^{\mathbb{I}} \to X \times X$ is the Leibniz exponential of X by the boundary map $\partial : 1{+}1 \to \mathbb{I}$, and we require the condition for both projections $X \times X \to X$.

Conversely, we can obtain filling from composition as follows: to fill the following open 2-box in X:

$$\begin{tikzcd}
A' & B' \\
A \arrow[u, "a"] \arrow[r, "p"'] & B \arrow[u, "b"']
\end{tikzcd}$$

First make a higher-dimensional composition problem using the connections on the right and left sides:

$$\begin{tikzcd}
A' & & & B' \\
& A \arrow[ul, "a"'] \arrow[r, "p"] & B \arrow[ur, "b"] & \\
& A \arrow[u, "="] \arrow[r, "p"'] \arrow[dl, "="'] & B \arrow[u, "="'] \arrow[dr, "="] & \\
A \arrow[uuu, "a"] \arrow[rrr, "p"'] & & & B \arrow[uuu, "b"']
\end{tikzcd}$$

Then since X has composition, the (partial) open 3-box has a top face, which is then a filler for the original open 2-box.

$$\begin{tikzcd}
A' & & & B' \\
& A \arrow[ul, "a"'] \arrow[r, "p"] & B \arrow[ur, "b"] & \\
& A \arrow[u, "="] \arrow[r, "p"'] \arrow[dl, "="'] & B \arrow[u, "="'] \arrow[dr, "="] & \\
A \arrow[uuu, "a"] \arrow[rrr, "p"'] & & & B \arrow[uuu, "b"']
\end{tikzcd}$$

For a general, algebraic proof, first use the connections to get maps in $\mathcal{E}^2$ of the form

$$\begin{tikzcd}
\delta & \delta \arrow[l, "="'] \arrow[d] \\
\delta \otimes \delta \arrow[u] & i \otimes \delta \arrow[l]
\end{tikzcd}$$

where $i : 1 \to 1 + 1$.

Applying the functor $(-) \Rightarrow X$ gives the top square in:

$$\begin{tikzcd}
\delta \Rightarrow X \arrow[r, "="] \arrow[d] & \delta \Rightarrow X \\
\delta \otimes \delta \Rightarrow X \arrow[r] \arrow[d, "\cong"'] & i \otimes \delta \Rightarrow X \arrow[u] \\
\delta \Rightarrow (\delta \Rightarrow X) \arrow[r] & \delta \Rightarrow (i \Rightarrow X) \arrow[u, "\cong"']
\end{tikzcd}$$

while the bottom one is by the $\otimes \dashv \Rightarrow$ adjunction.

So for any cofibration $c : C \rightarrowtail Z$ and filling problem

$$\begin{tikzcd}
C \arrow[r] \arrow[d, tail, "c"'] & X^{\mathbb{I}} \arrow[d, "\delta\Rightarrow X"] \\
Z \arrow[r] & X
\end{tikzcd}$$

we can extend on the right as follows.

$$\begin{tikzcd}
C \arrow[r] \arrow[d, tail, "c"'] & X^{\mathbb{I}} \arrow[r] \arrow[d, "\delta\Rightarrow X"] \arrow[rrr, bend left, "="] & \cdot \arrow[r] \arrow[d, "\delta\Rightarrow(\delta\Rightarrow X)"] & \cdot \arrow[r] \arrow[d, "\delta\Rightarrow(i\Rightarrow X)"] & X^{\mathbb{I}} \arrow[d, "\delta\Rightarrow X"] \\
Z \arrow[r] & X \arrow[r] \arrow[rrr, bend right, "="'] & \cdot \arrow[r] & \cdot \arrow[r] & X
\end{tikzcd}$$

Transposing the left three squares yields

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d, tail, "c\otimes\delta"'] & \cdot \arrow[r] \arrow[d, tail, "(\delta\Rightarrow X)\otimes\delta"'] & X^{\mathbb{I}} \arrow[r] \arrow[d, "\delta\Rightarrow X"] & X^{1+1} \arrow[d, "i\Rightarrow X"] \\
\cdot \arrow[r] & \cdot \arrow[r] & X \arrow[r, "="'] & X
\end{tikzcd}$$

which has a diagonal filler by composition, since $c \otimes \delta$ is also a cofibration.

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d, tail, "c\otimes\delta"'] & \cdot \arrow[r] \arrow[d, tail, "(\delta\Rightarrow X)\otimes\delta"'] & X^{\mathbb{I}} \arrow[r] \arrow[d, "\delta\Rightarrow X"] & X^{1+1} \arrow[d, "i\Rightarrow X"] \\
\cdot \arrow[r] \arrow[urrr, dotted] & \cdot \arrow[r] & X \arrow[r, "="'] & X
\end{tikzcd}$$

Transposing back thus gives a diagonal filler

$$\begin{tikzcd}
C \arrow[r] \arrow[d, tail, "c"'] & X^{\mathbb{I}} \arrow[r] \arrow[d, "\delta\Rightarrow X"] \arrow[rrr, bend left, "="] & \cdot \arrow[r] \arrow[d, "\delta\Rightarrow(\delta\Rightarrow X)"] & \cdot \arrow[r] \arrow[d, "\delta\Rightarrow(i\Rightarrow X)"] & X^{\mathbb{I}} \arrow[d, "\delta\Rightarrow X"] \\
Z \arrow[r] \arrow[urrr, dotted] & X \arrow[r] \arrow[rrr, bend right, "="'] & \cdot \arrow[r] & \cdot \arrow[r] & X
\end{tikzcd}$$

which provides a filler for the original problem

$$\begin{tikzcd}
C \arrow[r] \arrow[d, tail, "c"'] & X^{\mathbb{I}} \arrow[d, "\delta\Rightarrow X"] \\
Z \arrow[r] \arrow[ur, dotted] & X
\end{tikzcd}$$

$\square$

Appendix 2: A left-induced model structure on the Cartesian cubical sets

We make use of the Sattler model structure [Sat17] on the *Dedekind cubical sets* $\widehat{\mathbb{D}} = \mathsf{Set}^{\mathbb{D}^{\mathrm{op}}}$, where $\mathbb{D}$ is the category of *Dedekind cubes*, defined as the Lawvere theory of distributive lattices. The unique product-preserving functor

$$i : \mathbb{C} \longrightarrow \mathbb{D}$$

classifying the Dedekind interval $\mathrm{I}_{\mathbb{D}} \in \mathbb{D}$ induces an adjunction,

$$i_! \dashv i^* \dashv i_* \ : \ \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{D}}\,,$$

where $i^*(Q) = Q \circ i$, for $Q \in \mathbb{D}$.

Lemma 74. *Observe that $i_!$ is left exact since the Dedekind interval $\mathrm{I}_{\mathbb{D}}$ is* strict, *$0 \neq 1 : 1 \rightrightarrows \mathrm{I}_{\mathbb{D}}$. Thus we have geometric morphisms:*

$$(i_! \dashv i^*) : \ \widehat{\mathbb{D}} \longrightarrow \widehat{\mathbb{C}}\,,$$

classifying the bipointed object $i_!(\mathrm{I}_{\mathbb{C}}) = \mathrm{I}_{\mathbb{D}}$,

$$(i^* \dashv i_*) : \ \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{D}}\,,$$

classifying the dLat $i^*(\mathrm{I}_{\mathbb{D}}) := \mathbb{I}$, where $\eta : \mathrm{I}_{\mathbb{C}} \longrightarrow \mathbb{I}$ can be described pointwise as the distributive lattice completion of the corresponding bipointed set.

Also, since i is faithful so is $i_!$, and since i is surjective on objects i^* is also faithful.

It follows that:

- *$\widehat{\mathbb{C}}$ is $(i_! \circ i^*)$-coalgebras on $\widehat{\mathbb{D}}$,*
- *$\widehat{\mathbb{D}}$ is $(i^* \circ i_*)$-coalgebras on $\widehat{\mathbb{C}}$,*
- *$\widehat{\mathbb{D}}$ is $(i^* \circ i_!)$-algebras on $\widehat{\mathbb{C}}$.*

We will use the following transfer theorem for QMSs from [HKRS17, GKR18]:

Theorem ([HKRS17, GKR18]). *Suppose $\widehat{\mathbb{D}}$ has a (cofibrantly generated) model structure $(\mathcal{C}_{\mathbb{D}}, \mathcal{W}_{\mathbb{D}}, \mathcal{F}_{\mathbb{D}})$. Given an adjunction*

$$i_! \dashv i^* \;:\; \widehat{\mathbb{D}} \longrightarrow \widehat{\mathbb{C}}\,,$$

there is a left-induced model structure *on $\widehat{\mathbb{C}}$ if the following* acyclicity condition *holds:*

$$(i_!^{-1}\mathcal{C}_{\mathbb{D}})^{\pitchfork} \;\subset\; i_!^{-1}\mathcal{W}_{\mathbb{D}}\,.$$

For the left-induced model structure $(\mathcal{C}_{\mathbb{C}}, \mathcal{W}_{\mathbb{C}}, \mathcal{F}_{\mathbb{C}})$ on $\widehat{\mathbb{C}}$ we then have:

$$\begin{aligned} \mathcal{C}_{\mathbb{C}} &= i_!^{-1}\mathcal{C}_{\mathbb{D}}\,, \\ \mathcal{W}_{\mathbb{C}} &= i_!^{-1}\mathcal{W}_{\mathbb{D}}\,. \end{aligned}$$

The Sattler model structure on $\widehat{\mathbb{D}}$ is given as follows (for a constructive treatment a smaller class of "pointwise decidable cofibrations" is used, but we consider the classical case first):

$$\begin{aligned} \mathcal{C} &= \text{monomorphisms}\,, \\ \mathcal{W} &= \{f \mid f = p \circ i,\ p \in \mathcal{F} \cap \mathcal{W},\ i \in \mathcal{C} \cap \mathcal{W}\}\,, \\ \mathcal{F} &= (\mathcal{C} \otimes \delta)^{\pitchfork}\,. \end{aligned}$$

where $\delta : 1 \longrightarrow \mathrm{I}$ is either endpoint inclusion.

For the left-induced model structure on $\widehat{\mathbb{C}}$ we therefore have the following specification:

$$\begin{aligned} \mathcal{C} &= \text{monomorphisms,} \\ \mathcal{W} &= \{f \mid i_! f = p \circ i,\ p \in \mathcal{F}_{\mathbb{D}} \cap \mathcal{W}_{\mathbb{D}},\ i \in \mathcal{C}_{\mathbb{D}} \cap \mathcal{W}_{\mathbb{D}}\}\,, \\ \mathcal{F} &= (\mathcal{C} \cap \mathcal{W})^{\pitchfork}\,. \end{aligned}$$

The determination of $\mathcal{C}$ follows from the fact that $i_! : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{D}}$ is conservative.

To check the acyclicity condition,

$$(i_!^{-1}\mathcal{C}_{\mathbb{D}})^{\pitchfork} \;\subset\; i_!^{-1}\mathcal{W}_{\mathbb{D}}\,,$$

we know that $i_!^{-1}\mathcal{C}_{\mathbb{D}}$ consists of the monos in $\mathbb{C}$, so take $f : Y \longrightarrow X$ in $(i_!^{-1}\mathcal{C}_{\mathbb{D}})^{\pitchfork}$, apply $i_!$, and factor the result as $i_! f = p \circ m : i_! Y \longrightarrow Z \longrightarrow i_! X$ with $p \in \mathcal{F}_{\mathbb{D}} \cap \mathcal{W}_{\mathbb{D}}$ and $m : i_! Y \longrightarrow Z$ monic. We then need to show that m is in $\mathcal{W}_{\mathbb{D}}$.

We can apply Theorem 2.2.1 of [HKRS17], with $\mathsf{K} = \widehat{\mathbb{C}}$, $\mathsf{M} = \widehat{\mathbb{D}}$, $V = i_!$, $k = i^*$, and:

1. $QX = X$ and $\epsilon = 1_X : X \longrightarrow X$, so that $i_! 1_X = 1_{i_! X}$ and therefore in $\mathcal{W}_{\mathbb{D}}$, while all objects are cofibrant,

2. $Qf = f$ for any $f : X \longrightarrow Y$ in $\widehat{\mathbb{C}}$, so that the naturality condition is similarly trivial,

3. factor the codiagonal $X + X \longrightarrow X$ as $\pi_2 \circ j : X + X \longrightarrow \mathrm{I} \times X \longrightarrow X$ with $j = (\partial \mathrm{I} \times X) : X + X \longrightarrow \mathrm{I} \times X$.

It remains only to show that $i_! p : \mathrm{i}_!(\mathrm{I} \times X) \longrightarrow i_! X$ is in $\mathcal{W}_{\mathbb{D}}$ and $i_! j : i_!(X + X) \longrightarrow i_!(\mathrm{I} \times X)$ is in $\mathcal{C}_{\mathbb{D}}$. The latter is clear, since j is monic. To show the former, observe that for any $D \in \widehat{\mathbb{D}}$, the projection $\pi_2 : \mathrm{I}_{\mathbb{D}} \times D \longrightarrow D$ is in $\mathcal{W}_{\mathbb{D}}$ by 3-for-2, since the "cylinder end" inclusion $D \longrightarrow \mathrm{I}_{\mathbb{D}} \times D$, as a pullback of an endpoint inclusion, is a cofibration, and a strong deformation retract (using the connection on I), and hence is in $\mathcal{W}_{\mathbb{D}}$ by [GS17].

Thus we have shown:

Theorem 75. *There is a Quillen model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ on the category $\widehat{\mathbb{C}}$ of cartesian cubical sets, in which*

$$\begin{aligned} \mathcal{C} &= \textit{monomorphisms}, \\ \mathcal{W} &= \{ f \mid i_! f = p \circ i,\ p \in \mathcal{F}_{\mathbb{D}} \cap \mathcal{W}_{\mathbb{D}},\ i \in \mathcal{C}_{\mathbb{D}} \cap \mathcal{W}_{\mathbb{D}} \}, \\ \mathcal{F} &= (\mathcal{C} \cap \mathcal{W})^{\pitchfork}. \end{aligned}$$

where $i_! : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{D}}$ is the left adjoint of precomposition along the canonical map $i : \mathbb{C} \longrightarrow \mathbb{D}$ from Cartesian cubes to Dedekind cubes, and $(\mathcal{C}_{\mathbb{D}}, \mathcal{W}_{\mathbb{D}}, \mathcal{F}_{\mathbb{D}})$ is the Sattler model structure on $\widehat{\mathbb{D}}$.

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