

# A Quillen model structure on the category of cartesian cubical sets

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March 8, 2022

## Abstract

Add an abstract.

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## Introduction

*[need to fix long arrows and epi arrow commands]*  
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# 1 Cartesian cubical sets

In contrast to some other treatments of cubical sets [?, ?, ?, ?, ?, ?, ?, ?], we consider what may be termed the *cartesian* cube category  $\mathbb{C}$ , defined as the free finite product category on an interval  $\delta_0, \delta_1 : 1 \rightrightarrows I$ . As a classifying category for an algebraic theory with two constant symbols  $\mathbb{T} = \{0, 1\}$ , the category  $\mathbb{C}$  is dual to the full subcategory of finitely-generated, free  $\mathbb{T}$ -algebras  $\mathbf{Alg}(\mathbb{T})_{\text{fg}}$  (by Lawvere duality). In this case, the algebras are thus simply *bipointed sets*  $(A, a_0, a_1)$ , and the free ones are the *strictly* bipointed sets  $a_0 \neq a_1$ . Thus  $\mathbf{Alg}(\mathbb{T})_{\text{fg}}$  consists of the finite, strictly bipointed sets and all bipointed maps between them. We will use the following specific presentation.

**Definition 1.** The objects of the cartesian cube category  $\mathbb{C}$ , called  $n$ -cubes, will be written

$$[n] = \{0, x_1, \dots, x_n, 1\}.$$

The arrows,

$$f : [n] \rightarrow [m],$$

maybe taken to be  $m$ -tuples of elements drawn from the set  $\{0, x_1, \dots, x_n, 1\}$  regarded as formal terms representing composites of faces, degeneracies, permutations, and diagonals. Equivalently, the arrows  $[n] \rightarrow [m]$  are arbitrary bipointed maps  $[m] \rightarrow [n]$ .

See [?] for further details.

**Definition 2.** The category  $\mathbf{cSet}$  of *cubical sets* is the category of presheaves on the cartesian cube category  $\mathbb{C}$ ,

$$\mathbf{cSet} = \mathbf{Set}^{\mathbb{C}^{\text{op}}}.$$

It is generated by the representable presheaves  $y([n])$ , which will be written

$$I^n = y([n])$$

and called the *n-cubes*.

Note that the representables  $I^n$  are closed under finite products,  $I^n \times I^m = I^{n+m}$ . We of course write  $I$  for  $I^1$  and  $1$  for  $I^0$ . Later we will need to know the following basic fact about  $\mathbf{cSet}$ , which is easily proved.

**Proposition 3.** *The  $n$ -cubes  $I^n$  are tiny, in the sense that the functors  $(-)^{I^n} : \mathbf{cSet} \rightarrow \mathbf{cSet}$  have right adjoints.*  $\square$

## 2 The cofibration weak factorization system

**Cofibrations.** The *cofibrations* are a class  $\mathcal{C}$  of maps in  $\mathbf{cSet}$ , written

$$c : A \rightarrowtail B,$$

and are assumed to satisfy the following axioms:

- (C1) All isomorphisms are cofibrations.
- (C2) The composite of two cofibrations is a cofibration.
- (C3) Cofibrations are monomorphisms.
- (C4) Any pullback of a cofibration is a cofibration.

Moreover, we want the cofibrations to be classified by a subobject  $\Phi \hookrightarrow \Omega$  of the standard subobject classifier  $\top : 1 \rightarrow \Omega$  of  $\mathbf{cSet}$ . One way to ensure this is to further assume that they are closed under all colimits in the category of cartesian squares. An equivalent way is to just assume it from the start:

- (C0) There is a terminal object  $\Phi' \rightarrowtail \Phi$  in the category of cofibrations and cartesian squares.

It follows from (C1) that  $\Phi' = 1$  and the terminal cofibration  $1 \rightarrowtail \Phi$  is a factorization of  $\top : 1 \rightarrow \Omega$ . We call this map  $t : 1 \rightarrowtail \Phi$  the *cofibration classifier*. Note that we permit the case where  $\Phi = \Omega$ , i.e. all monos are cofibrations.

**Cofibrant partial map classifier.** The polynomial endofunctor  $[?]$  determined by the cofibration classifier  $t : 1 \rightarrowtail \Phi$  is defined on objects by

$$X \mapsto \Phi_! t_*(X) = \sum_{\varphi : \Phi} X^\varphi.$$

We shall write  $X^+ := \sum_{\varphi : \Phi} X^\varphi$ .

Observe that by the definition of  $X^+$  there is a pullback square,

$$\begin{array}{ccc} X & \xrightarrow{\quad} & X^+ \\ \downarrow \lrcorner & & \downarrow t_* X \\ 1 & \xrightarrow{t} & \Phi \end{array}$$

since  $t$  is monic. Let  $\eta : X \rightarrowtail X^+$  be the indicated top horizontal map; we call this map the *cofibrant partial map classifier* of  $X$ .

**Proposition 4.** *The map  $\eta : X \rightarrowtail X^+$  classifies partial maps into  $X$  with cofibrant domain, in the following sense.*

1. *The map  $\eta : X \rightarrowtail X^+$  is a cofibration.*
2. *For any object  $Z$  and any partial map  $(s, g) : Z \leftarrow S \rightarrow X$ , with  $s : S \rightarrowtail Z$  a cofibration, there is a unique  $f : Z \rightarrow X^+$  making a pullback square,*

$$\begin{array}{ccc} S & \xrightarrow{g} & X \\ \downarrow s & \lrcorner & \downarrow \eta \\ Z & \xrightarrow{f} & X^+ \end{array}.$$

*Proof.* The map  $\eta : X \rightarrowtail X^+$  is a cofibration since it is a pullback of  $t : 1 \rightarrow \Phi$ . Observe that  $(\eta, 1_X) : X^+ \leftarrow X \rightarrow X$  is therefore a partial map into  $X$  with cofibrant domain. The second statement follows directly from the definition of  $X^+$  as a polynomial (see [?], prop. 7).  $\square$

### The $+$ -Monad.

**Proposition 5.** *The pointed endofunctor determined by  $\eta_X : X \rightarrowtail X^+$  has a natural multiplication  $\mu_X : X^{++} \rightarrow X^+$  making it a monad.*

*Proof.* Since the cofibrations are closed under composition, the monad structure on  $X^+$  follows as in [?], proposition nm. Explicitly,  $\mu_X$  is determined as the unique map making the following a pullback diagram.

$$\begin{array}{ccc} X & \xrightarrow{=} & X \\ \eta_X \downarrow & & \downarrow \eta \\ X^+ & & \\ \eta_{X^+} \downarrow & & \\ X^{++} & \xrightarrow{\mu} & X^+ \end{array}$$

$\square$

**Relative partial map classifier.** For any object  $X \in \mathbf{cSet}$  the usual pullback functor

$$X^* : \mathbf{cSet} \rightarrow \mathbf{cSet}/X,$$

taking any  $A$  to the second projection  $A \times X \rightarrow X$ , not only preserves the subobject classifier  $\Omega$ , but also the cofibration classifier  $\Phi \hookrightarrow \Omega$ , where a map in  $\mathbf{cSet}/X$  is defined to be a cofibration if it is one in  $\mathbf{cSet}$ . Thus in  $\mathbf{cSet}/X$  the *(relative) cofibration classifier* is the map

$$t \times X : 1 \times X \rightarrow \Phi \times X \quad \text{over } X$$

which we may also write  $t_X : 1_X \rightarrow \Phi_X$ . Like  $t : 1 \rightarrow \Phi$ , this map determines a polynomial endofunctor

$$+_X : \mathbf{cSet}/X \rightarrow \mathbf{cSet}/X,$$

which commutes (up to natural isomorphism) with  $+$  :  $\mathbf{cSet} \rightarrow \mathbf{cSet}$  and  $X^* : \mathbf{cSet} \rightarrow \mathbf{cSet}/X$  in the evident way:

$$\begin{array}{ccc} \mathbf{cSet}/X & \xrightarrow{+_X} & \mathbf{cSet}/X \\ X^* \uparrow & & \uparrow X^* \\ \mathbf{cSet} & \xrightarrow{+} & \mathbf{cSet} \end{array} \quad (1)$$

The endofunctor  $+_X$  is also pointed  $\eta : Y \rightarrow Y^+$  and has a monad multiplication  $\mu_Y : Y^{++} \rightarrow Y^+$ , for any  $Y \rightarrow X$ , for the same reason that  $+$  has this structure. Summarizing, we may say that *the polynomial monad  $+$  :  $\mathbf{cSet} \rightarrow \mathbf{cSet}$  is fibered over  $\mathbf{cSet}$ .*

**Definition 6.** A  *$+$ -algebra* in  $\mathbf{cSet}$  is a cubical set  $A$  together with a retraction  $\alpha : A^+ \rightarrow A$  of  $\eta_A : A \rightarrow A^+$ , i.e. an algebra for the pointed endofunctor  $(+ : \mathbf{cSet} \rightarrow \mathbf{cSet}, \eta : 1 \rightarrow +)$ . Algebras for the monad  $(+, \eta, \mu)$  will be referred to specifically as  *$(+, \eta, \mu)$ -algebras*, or  *$+$ -monad algebras*.

A *relative  $+$ -algebra* in  $\mathbf{cSet}$  is a map  $A \rightarrow X$  together with an algebra structure for the pointed endofunctor  $+_X : \mathbf{cSet}/X \rightarrow \mathbf{cSet}/X$ .

**The factorization system.**

**Proposition 7.** *There is an (algebraic) weak factorization system on  $\mathbf{cSet}$  given by taking as the left class the cofibrations and as the right class the*

(maps underlying) the relative  $+$ -algebras. Thus a right map is a map  $f : A \rightarrow X$  for which there is a retract  $\alpha : A' \rightarrow A$  over  $X$  of the canonical map  $\eta_f : A \rightarrow A'$  over  $X$ ,

$$\begin{array}{ccccc} & & \overset{=}{\curvearrowright} & & \\ A & \xrightarrow{\eta_f} & A' & \xrightarrow{\alpha} & A \\ & \searrow f & \downarrow f^+ & \swarrow f & \\ & & X & & \end{array}$$

*Proof.* The factorization of any map  $f : Y \rightarrow X$  is given simply by applying the (relative)  $+$ -functor

$$\begin{array}{ccc} Y & \xrightarrow{\eta_f} & Y' \\ & \searrow f & \downarrow f^+ \\ & & X. \end{array}$$

We know that the unit  $\eta_f$  is always a cofibration, and since  $f^+$  is the free algebra for the  $+$ -monad, it is in particular a  $+$ -algebra.

For the lifting condition, consider a cofibration  $c : B \rightarrowtail C$ , a right map  $A \rightarrow X$ , with a  $+_X$ -algebra structure map  $\alpha : A^+ \rightarrow A$  over  $X$ , and a commutative square as indicated in the following.

$$\begin{array}{ccc} B & \xrightarrow{g} & A \\ \downarrow c & & \downarrow \eta \\ C & \xrightarrow{f} & X \end{array} \quad \begin{array}{c} \swarrow \alpha \\ A^+ \end{array}$$

Thus over  $X$ , we have the situation

$$\begin{array}{ccc} B & \xrightarrow{g} & A \\ \downarrow c & \nearrow d & \downarrow \eta \\ C & & A^+ \end{array} \quad \begin{array}{c} \swarrow \alpha \\ A^+ \end{array}$$

and we seek a diagonal filler as indicated. Since  $(c, g) : B \hookrightarrow C \rightarrow A$  is a cofibrant partial map into  $A$ , there is a map  $\varphi : C \rightarrow A^+$  (over  $X$ ) making a

(pullback) square,

$$\begin{array}{ccc} B & \xrightarrow{g} & A \\ \downarrow c & & \downarrow \eta \\ C & \xrightarrow[\varphi]{\dots\dots\dots} & A^+ \end{array} \quad \begin{array}{c} \curvearrowright \\ \alpha \end{array}$$

We thus have  $d := \alpha \circ \varphi : C \rightarrow A$  as the required diagonal filler.

The closure of the cofibrations under retracts follows from their classification by a universal object  $t : 1 \rightarrow \Phi$ , and the closure of the right maps under retracts follows from their being the algebras for a pointed endofunctor underlying a monad (cf. [?]). Algebraicity of this weak factorization system also follows directly, since  $+$  is a monad.  $\square$

Summarizing, we have a weak factorization system  $(\mathcal{L}, \mathcal{R})$  on the category  $\mathbf{cSet}$  of cubical sets, in which:

$$\begin{aligned} \mathcal{L} &= \mathcal{C} \quad (\text{the cofibrations}) \\ \mathcal{R} &= +\mathbf{Alg} \quad (\text{the relative } +\text{-algebras}) \end{aligned}$$

We shall call this the *cofibration weak factorization system*. As here, we will sometimes say that an object (or map) is a (relative)  $+$ -algebra when it can be equipped with a (relative)  $+$ -algebra structure; such maps will also be called *trivial fibrations* and the class of all such is denoted  $\mathbf{TFib}$ ,

$$\mathbf{TFib} = \mathcal{C}^\natural.$$

**Uniform filling structure.** It will be convenient to relate  $+$ -algebra structure with the more familiar diagonal filling condition of weak factorization systems, and specifically a special form of the latter that occurs in [CCHM16] under the name *uniform filling structure*.

Consider a generating subset of cofibrations, consisting of all those cofibrations  $c : C \rightarrowtail Z$  where  $Z$  is representable,  $Z = \mathbf{I}^n$ . Call these maps the *basic cofibrations*, and let

$$\mathbf{BCof} = \{c : C \rightarrowtail \mathbf{I}^n \mid c \in \mathcal{C}, n \geq 0\}. \quad (2)$$

**Proposition 8.** *For any object  $X$  in  $\mathbf{cSet}$  the following are equivalent:*

1.  $X$  is a  $+$ -algebra, i.e. there is a retraction  $\alpha : X^+ \rightarrow X$  of the unit  $\eta : X \rightarrow X^+$ .

2.  $X$  is  $\mathcal{C}$ -injective, in the sense that it has the right lifting property with respect to all cofibrations,

$$\mathcal{C} \pitchfork X.$$

3.  $X$  has a uniform filling structure: for each basic cofibration  $c : C \rightarrow I^n$  and map  $x : C \rightarrow X$  there is given an extension  $j(c, x)$ ,

$$\begin{array}{ccc} C & \xrightarrow{x} & X, \\ c \downarrow & \nearrow j(c, x) & \\ I^n & & \end{array} \quad (3)$$

and the choice is uniform in  $I^n$  in the following sense: given any cubical map  $u : I^m \rightarrow I^n$ , the pullback  $u^*c : u^*C \rightarrow I^m$  is again a basic cofibration and fits into a commutative diagram of the form

$$\begin{array}{ccccc} u^*C & \xrightarrow{c^*u} & C & \xrightarrow{x} & X. \\ u^*c \downarrow \lrcorner & & c \downarrow & \nearrow j(c, x) & \\ I^m & \xrightarrow{u} & I^n & & \end{array} \quad (4)$$

For the pair  $(u^*c, xc^*u)$  in (35) the chosen extension  $j(u^*c, xc^*u) : I^m \rightarrow X$ , is equal to  $j(c, x) \circ u$ ,

$$j(u^*c, xc^*u) = j(c, x)u. \quad (5)$$

*Proof.* Let  $(X, \alpha)$  be a  $+-$ -algebra and suppose given the span  $(c, x)$  as below, with  $c$  a cofibration.

$$\begin{array}{ccc} C & \xrightarrow{x} & X \\ c \downarrow & & \\ Z & & \end{array}$$

Let  $\chi(c, x) : Z \rightarrow X^+$  be the classifying map of the evident partial map  $(c, x) : Z \rightarrow X$ , so that we have a pullback square as follows.

$$\begin{array}{ccc} C & \xrightarrow{x} & X \\ c \downarrow \lrcorner & & \downarrow \eta \\ Z & \xrightarrow{\chi(c, x)} & X^+ \end{array} \quad (6)$$



Then set

$$j = \alpha \circ \chi(c, x) : Z \rightarrow X \quad (7)$$

to get a filler,

$$\begin{array}{ccc} C & \xrightarrow{x} & X \\ \downarrow c & \nearrow j & \downarrow \eta \\ Z & \xrightarrow{\chi(c, x)} & X^+ \end{array} \quad \alpha \quad (8)$$

since  $j \circ c = \alpha \circ \chi(c, x) \circ c = \alpha \circ \eta \circ x = x$ . Thus (1) implies (2). To see that it also implies (3), observe that in the case where  $Z = I^n$  and we specify, in (7), that

$$j(c, x) = \alpha \circ \chi(c, x) : I^n \rightarrow X, \quad (9)$$

then the assignment is natural in  $I^n$ . Indeed, given any  $u : I^m \rightarrow I^n$ , we have

$$j(c', xu') = \alpha \circ \chi(c', xu') = \alpha \circ \chi(c, x) \circ u = j(c, x)u, \quad (10)$$

by the uniqueness of classifying maps.

It is clear that (2) implies (1), since if  $\mathcal{C} \Vdash X$  then we can take as an algebra structure  $\alpha : X^+ \rightarrow X$  any filler for the span

$$\begin{array}{ccc} X & \xrightarrow{=} & X \\ \downarrow \eta & \nearrow \alpha & \\ X^+ & & \end{array}$$

To see that (3) implies (1), suppose that  $X$  has a uniform filling structure  $j$  and we want to define an algebra structure  $\alpha : X^+ \rightarrow X$ . By Yoneda, for every  $y : I^n \rightarrow X^+$  we need a map  $\alpha(y) : I^n \rightarrow X$ , naturally in  $I^n$ , in the sense that for any  $u : I^m \rightarrow I^n$ , we have

$$\alpha(yu) = \alpha(y)u. \quad (11)$$

Moreover, to ensure that  $\alpha\eta = 1_X$ , for any  $x : I^n \rightarrow X$  we must have  $\alpha(\eta \circ x) = x$ . So take  $y : I^n \rightarrow X^+$  and let

$$\alpha(y) = j(y^*\eta, y'),$$

as indicated on the right below.

$$\begin{array}{ccccc} u^*C & \xrightarrow{u'} & C & \xrightarrow{y'} & X \\ \downarrow u^*y^*\eta & \lrcorner & \downarrow y^*\eta & \lrcorner & \downarrow \eta \\ I^m & \xrightarrow{u} & I^n & \xrightarrow{y} & X^+ \end{array} \quad \begin{array}{c} \nearrow j(y^*\eta, y') \\ \nearrow \alpha \end{array} \quad (12)$$

Then for any  $u : I^m \rightarrow I^n$ , we indeed have

$$\alpha(yu) = j((yu)^*\eta, y'u') = j(y^*\eta, y') \circ u = \alpha(y)u,$$

by the uniformity of  $j$ . Finally, if  $y = \eta \circ x$  for some  $x : I^n \rightarrow X$  then

$$\alpha(\eta x) = j((\eta x)^*\eta, (\eta x)') = j(1_X, x) = x,$$

because the defining diagram for  $\alpha(\eta x)$ , i.e. the one on the right in (12), then factors as

$$\begin{array}{ccccc} I^n & \xrightarrow{x} & X & \xrightarrow{=} & X, \\ \downarrow \lrcorner & & \downarrow \lrcorner & & \downarrow \eta \\ I^n & \xrightarrow{x} & X & \xrightarrow{\eta} & X^+ \end{array} \quad (13)$$

and the only possible extension  $j(1_X, x)$  for the span  $(1_{I^n}, x)$  is  $x$  itself.  $\square$

The relative version of the foregoing is entirely analogous, since the  $+$ -functor is fibered over  $\mathbf{cSet}$  in the sense of diagram (1). We can therefore omit the entirely analogous proof. The statement is as follows.

**Proposition 9.** *For any map  $f : Y \rightarrow X$  in  $\mathbf{cSet}$  the following are equivalent:*

1.  $f : Y \rightarrow X$  is a (relative)  $+$ -algebra (over  $X$ ), i.e. there is a retraction  $\alpha : Y' \rightarrow Y$  over  $X$  of the unit  $\eta : Y \rightarrow Y'$  over  $X$ , where  $f^+ : Y' \rightarrow X$  is the result of the relative  $+_X$ -functor applied to  $f$ , as in definition 6.
2.  $f : Y \rightarrow X$  is trivial fibration in the sense that it has the right lifting property with respect to all cofibrations,

$$\mathcal{C} \pitchfork f.$$

3.  $f : Y \rightarrow X$  has a uniform filling structure: for each basic cofibration  $c : C \rightarrowtail I^n$  and maps  $x : C \rightarrow X$  and  $y : I^n \rightarrow Y$  making the square below commute, there is given a diagonal filler  $j(c, x, y)$ ,

$$\begin{array}{ccc} C & \xrightarrow{x} & X \\ c \downarrow & \nearrow j(c, x, y) & \downarrow f \\ I^n & \xrightarrow{y} & Y, \end{array} \quad (14)$$

and the choice is uniform in  $I^n$  in the following sense: given any cubical map  $u : I^m \rightarrow I^n$ , the pullback  $u^*c : u^*C \rightarrow I^m$  is again a basic cofibration and fits into a commutative diagram of the form

$$\begin{array}{ccccc}
 u^*C & \xrightarrow{c^*u} & C & \xrightarrow{x} & X \\
 \downarrow u^*c & \lrcorner & \downarrow c & \nearrow j(c,x,y) & \downarrow f \\
 I^m & \xrightarrow{u} & I^n & \xrightarrow{y} & Y.
 \end{array} \tag{15}$$

For the evident triple  $(u^*c, xc^*u, yu)$  in (15) the chosen diagonal filler

$$j(u^*c, xc^*u, yu) : I^m \rightarrow X$$

is equal to  $j(c, x, y) \circ u$ ,

$$j(u^*c, xc^*u, yu) = j(c, x, y)u. \tag{16}$$

We next collect some basic facts about  $+$ -algebras/trivial fibrations: they have sections, they are closed under composition and retracts, and they are closed under pullback and pushforward along all maps.

**Corollary 10.** *A  $+$ -algebra  $A \rightarrow X$  always has a section  $s : X \rightarrow A$ .*

*Proof.* It suffices to find a filler for the following diagram.

$$\begin{array}{ccc}
 0 & \longrightarrow & A \\
 \downarrow & \nearrow s & \downarrow \\
 X & \xrightarrow{=} & X
 \end{array} \tag{17}$$

But  $0 \rightarrow X$  is always a cofibration by assumption.  $\square$

**Corollary 11.** *If  $f : Y \rightarrow X$  is a  $+$ -algebra over  $X$  and  $g : Z \rightarrow Y$  is a  $+$ -algebra over  $Y$ , then there is a canonical way of making  $f \circ g : Z \rightarrow X$  into a  $+$ -algebra over  $X$ .*

*Proof.* One of several equivalent constructions is as follows. The square below has a canonical diagonal filler  $j$  as indicated, because  $\eta$  is a cofibration and  $f$  a  $+$ -algebra.

$$\begin{array}{ccc}
 Z & \xrightarrow{g} & Y \\
 \downarrow \eta & \nearrow j & \downarrow f \\
 (\Sigma_Y Z)^+ & \longrightarrow & X,
 \end{array} \tag{18}$$

Now use the fact that  $g$  is a  $+$ -algebra to get a canonical retraction of  $\eta$  over  $Y$ .  $\square$

**Corollary 12.** *If  $f : Y \rightarrow X$  is a  $+$ -algebra over  $X$  and  $f' : Y' \rightarrow X'$  is a retract of  $f$  in the arrow category, then  $f'$  is a  $+$ -algebra over  $X'$ .*

*Proof.* The right class of a weak factorization system is always closed under retracts.  $\square$

**Corollary 13.** *For any map  $f : Y \rightarrow X$  and any  $+$ -algebra  $A \rightarrow B$  over  $X$ , the pullback  $f^*A \rightarrow Y$  is a  $+$ -algebra over  $Y$ .*

$$\begin{array}{ccc} f^*A & \xrightarrow{\quad} & A \\ \downarrow \lrcorner & & \downarrow \\ Y & \xrightarrow{\quad f \quad} & X \end{array} \quad (19)$$

*Proof.* The right class of a weak factorization system is always stable under pullback, but one can also see directly that a pullback of a  $+$ -algebra is again a  $+$ -algebra, using the fact that the  $+$ -endofunctor commutes with pullbacks.  $\square$

The proof of the following is a straightforward application of the general fact that a left adjoint preserves the left class of a weak factorization system if and only if its right adjoint preserves the right class. We give a more detailed argument for this special case since we will need an analogous version later.

**Corollary 14.** *For any map  $f : Y \rightarrow X$  and any  $+$ -algebra  $A \rightarrow Y$  over  $Y$ , the pushforward  $f_*A \rightarrow X$  is a  $+$ -algebra over  $X$ .*

$$\begin{array}{ccc} A & & f_*A \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\quad f \quad} & X \end{array} \quad (20)$$

More generally, if  $B \rightarrow Y$  is any map and  $g : A \rightarrow B$  is a  $+$ -algebra over  $B$ , then the pushforward  $f_*g : f_*A \rightarrow f_*B$  is a  $+$ -algebra over  $f_*B$ .

$$\begin{array}{ccc} A & \xrightarrow{\quad g \quad} & B \\ \downarrow & \swarrow & \downarrow \\ Y & \xrightarrow{\quad f \quad} & X \end{array} \quad \begin{array}{ccc} f_*A & \xrightarrow{\quad f_*g \quad} & f_*B \\ \downarrow & \swarrow & \downarrow \\ X & \xrightarrow{\quad f \quad} & X \end{array} \quad (21)$$

*Proof.* Note that the first statement follows from the second one by taking  $B = Y$  and  $B \rightarrow Y$  the identity map.

To prove the second statement, by Proposition 9 it suffices to show that  $f_*g : f_*A \rightarrow f_*B$  has the right lifting property, in the slice category over  $X$ , against all cofibrations. So consider a lifting problem over  $X$ , with a cofibration on the left:

$$\begin{array}{ccc} C & \longrightarrow & f_*A \\ \downarrow c & & \downarrow f_*g \\ Z & \longrightarrow & f_*B \end{array} \quad (22)$$

Transposing across the adjunction  $f^* \dashv f_*$  results in a lifting problem over  $Y$  of the form

$$\begin{array}{ccc} f^*C & \longrightarrow & A \\ \downarrow f^*c & \nearrow j & \downarrow g \\ f^*Z & \longrightarrow & B \end{array}$$

in which  $f^*c$  is a cofibration, because these are preserved by pullbacks. Thus there is a filler  $j$  as indicated, since by assumption  $g : A \rightarrow B$  is a  $+$ -algebra over  $B$  and therefore a trivial fibration. Transposing the filler  $j$  then provides a filler for the original problem (22).  $\square$

### 3 The fibration weak factorization system

Our next goal is the specification of a second weak factorization system, the *fibration weak factorization system*, with a restricted class of “trivial” cofibrations on the left, and an expanded class of right maps, the fibrations.

As a warm-up, we first recall the specification of the trivial-cofibration/fibration WFS from [GS17], where the interval  $I$  is assumed to have connections. (In [?] it is shown that these fibrations agree with those specified in the “logical style” of [CCHM16, OP17]). We then modify the specifications in order to arrive at an “unbiased” version that is more appropriate to the cartesian setting, and results in more trivial cofibrations.

#### 3.1 Partial box filling (biased version)

A generating class of biased trivial cofibrations are all maps of the form

$$c \otimes \delta_\epsilon : D \rightarrow Z \times I, \quad (23)$$

where:

1.  $c : C \rightarrowtail Z$  is an arbitrary cofibration,
2.  $\delta_\epsilon : 1 \rightarrow I$  is one of the two “endpoint inclusions” where, recall,  $1 = y[0]$ , and  $I = y[1]$ , and for  $\epsilon = 0, 1$ , we have the maps  $\delta_\epsilon : 1 \rightarrow I$  corresponding to the two bipointed maps  $0, 1 : \{0, x, 1\} \rightarrow \{0, 1\}$ .
3.  $c \otimes \delta_\epsilon$  is the pushout-product (resp. “Leibniz tensor”) of the cofibration  $c : C \rightarrowtail Z$  and an endpoint  $\delta_\epsilon : 1 \rightarrow I$ , as indicated in the following diagram (in which the unlabelled maps are the expected ones).

$$\begin{array}{ccc}
 C \times 1 & \longrightarrow & C \times I \\
 \downarrow & & \downarrow \\
 Z \times 1 & \longrightarrow & Z +_C (C \times I) \\
 & \searrow & \swarrow \\
 & & Z \times I
 \end{array}
 \quad (24)$$

(Note: The diagram shows a pushout square with an additional map from  $C \times I$  to  $Z \times I$  labeled  $c \otimes \delta_\epsilon$  and a curved arrow from  $Z \times 1$  to  $Z \times I$ .)

4.  $D = Z +_C (C \times I)$  is the indicated pushout, the domain of  $c \otimes \delta_\epsilon$ .

In order to insure that such maps are indeed cofibrations, we assume two further axioms:

(C5) The endpoint inclusions  $\delta_\epsilon : 1 \rightarrow I$  are cofibrations.

(C6) The cofibrations are closed under pushout-products.

Note that since  $\delta_0$  and  $\delta_1$  are disjoint, by (C5) we have that  $0 \rightarrow 1$  is a cofibration, and hence that  $0 \rightarrow A$  is a cofibration, for all objects  $A$ .

In place of (C6), we could equivalently require the cofibrations to be closed under the join operation  $A \vee B$  in the lattice of subobjects of an object (as is done in [?, OP17]).

**Fibrations (biased version).** Let

$$\mathcal{C} \otimes \delta_\epsilon = \{c \otimes \delta_\epsilon : D \rightarrowtail Z \times I \mid c \in \mathcal{C}, \epsilon = 0, 1\}$$

be the class of all such pushout-products of arbitrary cofibrations  $c : C \rightarrowtail Z$  with endpoint inclusions  $\delta_\epsilon : 1 \rightarrowtail I$ . The *biased fibrations* are defined to be the right class of these generating trivial cofibrations,

$$(\mathcal{C} \otimes \delta_\epsilon)^\triangleright = \mathcal{F}.$$

Thus a map  $f : Y \rightarrow X$  is a biased fibration if for every commutative square of the form

$$\begin{array}{ccc} Z +_C (C \times I) & \longrightarrow & Y \\ c \otimes \delta_\epsilon \downarrow & \nearrow j & \downarrow f \\ Z \times I & \longrightarrow & X \end{array} \quad (25)$$

with a generating trivial cofibration on the left, there is a diagonal filler  $j$  as indicated. This condition can be seen as a generalized homotopy lifting property.

To relate this notion of fibration to the cofibration weak factorization system, fix any map  $u : A \rightarrow B$ , and recall (e.g. from [?]) that the pushout-product with  $u$  is a functor on the arrow category

$$(-) \otimes u : \mathbf{cSet}^2 \rightarrow \mathbf{cSet}^2.$$

This functor has a right adjoint, the *pullback-hom* (or “Leibniz exponential”), which for a map  $f : X \rightarrow Y$  we will write as

$$(u \Rightarrow f) : Y^B \rightarrow (X^B \times_{X^A} Y^A).$$

The pullback-hom is determined as indicated in the following diagram (in which the unlabelled maps are the expected ones).

$$\begin{array}{ccccc} Y^B & & & & \\ & \searrow^{u \Rightarrow f} & & \searrow & \\ & X^B \times_{X^A} Y^A & \longrightarrow & Y^A & \\ & \downarrow & & \downarrow & \\ & X^B & \longrightarrow & X^A & \end{array} \quad (26)$$

The  $\otimes \dashv \Rightarrow$  adjunction on the arrow category has the following useful relation to weak factorization systems (cf. [GS17, ?, ?]), where for any maps  $a : A \rightarrow B$  and  $f : X \rightarrow Y$  we write

$$a \dashv f$$

to mean that for every solid square of the form

$$\begin{array}{ccc} A & \longrightarrow & X \\ a \downarrow & \nearrow & \downarrow f \\ B & \longrightarrow & Y \end{array} \quad (27)$$

there exists a dotted diagonal filler as indicated.

Moreover, let us define the *set*

$$[a \pitchfork f] := \{j : B \longrightarrow X \mid f \circ j = j \circ a\}$$

Then there is a presheaf ...

*[fill this in and [?].]*

**Lemma 15.** *For any maps  $a : A_0 \longrightarrow A_1, b : B_0 \longrightarrow B_1, c : C_0 \longrightarrow C_1$  in  $\mathbf{cSet}$ , the following conditions are equivalent*

$$(a \otimes b) \pitchfork c \Leftrightarrow a \pitchfork (b \Rightarrow c).$$

Moreover, there is an isomorphism of the sets,

$$[(a \otimes b) \pitchfork c] \cong [a \pitchfork b \Rightarrow c],$$

which is natural in all three arguments, in the expected sense.

**Proposition 16.** *An object  $X$  is fibrant if and only if both of the endpoint projections  $X^1 \rightarrow X$  from the pathspace are (relative)  $+$ -algebras (over  $X$ ). More generally, a map  $f : Y \rightarrow X$  is a fibration iff both of the maps*

$$(\delta_\epsilon \Rightarrow f) : Y^I \rightarrow X^I \times_X Y$$

*are  $+$ -algebras (for  $\epsilon = 0, 1$ ).*

*Proof.* The first statement follows from the second, since the pathspace projections  $X^1 \rightarrow X$  are just the maps

$$(\delta_\epsilon \Rightarrow !_X) : X^I \rightarrow (1^I \times_1 X) \cong X,$$

for  $!_X : X \rightarrow 1$ .



By definition,  $f : X \rightarrow Y$  is a fibration iff every square of the form

$$\begin{array}{ccc} Z +_C (C \times I) & \longrightarrow & Y \\ c \otimes \delta_\epsilon \downarrow & \nearrow j & \downarrow f \\ Z \times I & \longrightarrow & X, \end{array} \quad (28)$$

with a generating trivial cofibration  $c \otimes \delta_\epsilon$  on the left, has a diagonal filler  $j$  as indicated. Briefly,

$$(c \otimes \delta_\epsilon) \pitchfork f \quad (\text{for } c \in \mathcal{C}, \epsilon = 0, 1).$$

By the  $\otimes \dashv \Rightarrow$  adjunction, this is equivalent to the condition

$$c \pitchfork (\delta_\epsilon \Rightarrow f) \quad (\text{for } c \in \mathcal{C}, \epsilon = 0, 1).$$

That is, for every square

$$\begin{array}{ccc} C & \longrightarrow & Y^I \\ c \downarrow & \nearrow k & \downarrow \delta_\epsilon \Rightarrow f \\ Z & \longrightarrow & X^I \times_X Y, \end{array}$$

with an arbitrary cofibration  $c : C \rightarrowtail Z$  on the left, there is a diagonal filler  $k$  as indicated, for  $\epsilon = 0, 1$ . But this is just to say that the maps  $\delta_\epsilon \Rightarrow f$  are in the right class of the cofibrations, which is equivalent to their being  $+$ -algebras, as claimed.  $\square$

**Fibration structure.** The  $\otimes \dashv \Rightarrow$  adjunction determines the fibrations in terms of the trivial fibrations, which in turn can be determined by *uniform* lifting against a *set* of basic cofibrations, by proposition 9. We can similarly determine the fibrations by uniform lifting against a *set* of trivial cofibrations, consisting of all those  $c \otimes \delta_\epsilon$  in  $\mathcal{C} \otimes \delta_\epsilon$  where  $c : C \rightarrowtail Z$  has a representable codomain  $Z = I^n$ . Call these maps the *basic (biased) trivial cofibrations*, and let

$$\mathcal{B} \otimes \delta_\epsilon = \{c \otimes \delta_\epsilon : C \rightarrowtail I^{n+1} \mid c : C \rightarrowtail I^n, \epsilon = 0, 1, n \geq 0\}, \quad (29)$$

where the pushout-product  $c \otimes \delta_\epsilon$  now takes the simpler form

$$\begin{array}{ccc}
 C & \longrightarrow & C \times I \\
 \downarrow & & \downarrow \\
 I^n & \longrightarrow & I^n +_C (C \times I) \\
 & \searrow & \downarrow c \otimes \delta_\epsilon \\
 & & I^n \times I
 \end{array}
 \quad (30)$$

for a cofibration  $c : C \hookrightarrow I^n$ , an endpoint  $\delta_\epsilon : 1 \rightarrow I$ , and with domain  $B = (I^n +_C (C \times I))$ . These subobjects  $B \hookrightarrow I^{n+1}$  can be seen geometrically as generalized open box inclusions.

For any map  $f : Y \rightarrow X$  a (uniform, biased) fibration structure on  $f$  is a choice of diagonal fillers  $j_\epsilon(c, x, y)$ ,

$$\begin{array}{ccc}
 I^n +_C (C \times I) & \xrightarrow{x} & X \\
 c \otimes \delta_\epsilon \downarrow & \nearrow j_\epsilon(c, x, y) & \downarrow f \\
 I^n \times I & \xrightarrow{y} & Y,
 \end{array}
 \quad (31)$$

for each basic trivial cofibration  $c \otimes \delta_\epsilon : B = (I^n +_C (C \times I)) \hookrightarrow I^{n+1}$  and maps  $x : B \rightarrow X$  and  $y : I^{n+1} \rightarrow Y$ , which is *uniform in  $I^n$*  in the following sense: given any cubical map  $u : I^m \rightarrow I^n$ , the pullback  $u^*c : u^*C \hookrightarrow I^m$  of  $c : C \hookrightarrow I^n$  along  $u$  determines another basic trivial cofibration

$$u^*c \otimes \delta_\epsilon : B' = (I^m +_{u^*C} (u^*C \times I)) \hookrightarrow I^{m+1},$$

which fits into a commutative diagram of the form

$$\begin{array}{ccccc}
 I^m +_{u^*C} (u^*C \times I) & \xrightarrow{(u \times I)'} & I^n +_C (C \times I) & \xrightarrow{x} & X \\
 u^*c \otimes \delta_\epsilon \downarrow & & c \otimes \delta_\epsilon \downarrow & \nearrow j_\epsilon(c, x, y) & \downarrow f \\
 I^m \times I & \xrightarrow{u \times I} & I^n \times I & \xrightarrow{y} & Y,
 \end{array}
 \quad (32)$$

by applying the functor  $(-) \otimes \delta_\epsilon$  to the pullback square relating  $u^*c$  to  $c$ . Now for the outer rectangle in (32) there is a chosen diagonal filler

$$j_\epsilon(u^*c, x(u \times I)', y(u \times I)) : I^m \times I \rightarrow X$$

and for this map we require that

$$j_\epsilon(u^*c, x(u \times \mathbf{I})', y(u \times \mathbf{I})) = j_\epsilon(c, x, y) \circ (u \times \mathbf{I}). \quad (33)$$

This is a reformulation of the logical specification given in [CCHM16] (see the appendix).

**Definition 17.** A *(uniform, biased) fibration structure* on a map  $f : Y \rightarrow X$  is a choice of fillers  $j_\epsilon(c, x, y)$  as in (34) satisfying (36) for all maps  $u : \mathbf{I}^m \rightarrow \mathbf{I}^n$ .

Essentially the same argument as that given for Proposition 16 also yields the following sharper formulation in terms of fibration structure.

**Corollary 18.** *Fibration structure on a map  $f : Y \rightarrow X$  is equivalent to a pair of  $+$ -algebra structures on the maps*

$$(\delta_\epsilon \Rightarrow f) : Y^{\mathbf{I}} \rightarrow X^{\mathbf{I}} \times_X Y$$

for  $\epsilon = 0, 1$ .

Finally, we have the analogue of proposition 8 for fibrant objects; we omit the analogous statement of proposition 9 for fibrations, as well as the entirely analogous proof.

**Corollary 19.** *For any object  $X$  in  $\mathbf{cSet}$  the following are equivalent:*

1.  $X$  is fibrant, i.e. every partial map to  $X$  with a generating trivial cofibration  $D \hookrightarrow Z \times \mathbf{I}$  as domain of definition extends to a total map  $Z \times \mathbf{I} \rightarrow X$ ,

$$\mathcal{C} \otimes \delta_\epsilon \dashv f$$

2. There are  $+$ -algebra structures on the canonical maps

$$(\delta_\epsilon \Rightarrow X) : X^{\mathbf{I}} \rightarrow X,$$

for  $\epsilon = 0, 1$ .

3.  $X \rightarrow 1$  has a (uniform, biased) fibration structure. Explicitly, for each basic trivial cofibration  $c \otimes \delta_\epsilon : B \hookrightarrow \mathbf{I}^{n+1}$  and map  $x : B \rightarrow X$ , there is given an extension  $j_\epsilon(c, x)$ ,

$$\begin{array}{ccc} B & \xrightarrow{x} & X, \\ c \otimes \delta_\epsilon \downarrow & \nearrow j_\epsilon(c, x) & \\ \mathbf{I}^{n+1} & & \end{array} \quad (34)$$

and the choice is uniform in  $I^n$  in the sense: given any cubical map  $u : I^m \rightarrow I^n$ , the pullback  $u^*c \otimes \delta_\epsilon : B' \rightarrow I^m \times I$  fits into a commutative diagram of the form

$$\begin{array}{ccccc} B' & \xrightarrow{(u \times I)'} & B & \xrightarrow{x} & X. \\ \downarrow u^*c \otimes \delta_\epsilon & \lrcorner & \downarrow c \otimes \delta_\epsilon & \nearrow j(c, x) & \\ I^m \times I & \xrightarrow{u \times I} & I^n \times I & & \end{array} \quad (35)$$

Then for the pair  $(u^*c \otimes \delta_\epsilon, x(u \times I)')$  in (35) the chosen extension

$$j(u^*c \otimes \delta_\epsilon, x(u \times I)') : I^m \times I \rightarrow X$$

is equal to  $j(c, x) \circ (u \times I)$ ,

$$j(u^*c \otimes \delta_\epsilon, x(u \times I)') = j(c, x)(u \times I). \quad (36)$$

### 3.2 Partial box filling (unbiased version)

Rather than building a weak factorization system based on the foregoing notion of biased fibration (as is done in [GS17]), we shall first eliminate the “bias” on a choice of endpoint  $\delta_\epsilon : 1 \rightarrow I$ , expressed by the indexing  $\epsilon = 0, 1$ . This will have the effect of adding more trivial cofibrations, and thus more weak equivalences, to our model structure. Consider first the simple path-lifting condition, which is a special case of (25) with  $c = ! : 0 \rightarrow 1$ , since  $! \otimes \delta_\epsilon = \delta_\epsilon$ :

$$\begin{array}{ccc} 1 & \longrightarrow & Y \\ \downarrow \delta_\epsilon & \nearrow j_\epsilon & \downarrow f \\ I & \longrightarrow & X. \end{array}$$

(Note that  $0 \rightarrow 1$  is a cofibration by axioms C4 and C5).

In topological spaces, rather than requiring lifts  $j_\epsilon$  for each of the endpoints  $\epsilon = 0, 1$ , we could instead require that there be a lift  $j_i$  for each point  $i : 1 \rightarrow I$  in the real interval  $I = [0, 1]$ . Such “unbiased path-lifting” can be formulated in  $\mathbf{cSet}$  by introducing a “generic point”  $\delta : 1 \rightarrow I$ , by passing to  $\mathbf{cSet}/I$ , and then requiring path-lifting with respect to  $\delta$ . The following specification implements that idea, while also adding partiality in the sense of the foregoing section. We need the following strengthening of axiom C5.

(C5') The diagonal map  $\delta : I \rightarrow I \times I$  is a cofibration.

**Definition 20** (Fibration). Let  $\delta : I \rightarrow I \times I$  be the diagonal map.

1. An object  $X$  is *(unbiased) fibrant* if the map

$$(\delta \Rightarrow X) = \langle \text{eval}, p_2 \rangle : X^I \times I \rightarrow X \times I$$

is a  $+$ -algebra.

2. A map  $f : Y \rightarrow X$  is an *(unbiased) fibration* if the map

$$(\delta \Rightarrow f) = \langle f^I \times I, \langle \text{eval}, p_2 \rangle \rangle : Y^I \times I \rightarrow (X^I \times I) \times_{(X \times I)} (Y \times I)$$

is a  $+$ -algebra.

Condition (1) above says that evaluation at the generic point  $\delta : 1 \rightarrow I$ , i.e. the map  $X^\delta : X^I \rightarrow X$  constructed in the slice category  $\mathbf{cSet}/I$ , is a  $+$ -algebra. Condition (2) says that the pullback-hom of the generic point  $\delta : 1 \rightarrow I$  with  $I^*f$ , constructed in the slice category  $\mathbf{cSet}/I$ , is a  $+$ -algebra. The latter can be reformulated as follows.

**Proposition 21.** *A map  $f : Y \rightarrow X$  is an (unbiased) fibration if and only if the canonical map  $c$  to the pullback, in the following diagram, is a  $+$ -algebra.*

$$\begin{array}{ccccc}
 Y^I \times I & & \xrightarrow{\text{eval}} & & Y \\
 & \searrow c & & \searrow & \\
 & & Y_{\text{eval}} & \xrightarrow{\quad} & Y \\
 & & \downarrow \lrcorner & & \downarrow f \\
 & & X^I \times I & \xrightarrow{\text{eval}} & X \\
 & \nearrow f^I \times I & & \nearrow & \\
 & & & & 
 \end{array} \tag{37}$$

*Proof.* We interpolate another pullback into the rectangle in (37) to obtain

$$\begin{array}{ccccc}
 Y_{\text{eval}} & \xrightarrow{\quad} & Y \times I & \xrightarrow{\quad} & Y \\
 \downarrow \lrcorner & & \downarrow \lrcorner & & \downarrow f \\
 X^I \times I & \xrightarrow{\quad} & X \times I & \xrightarrow{\quad} & X
 \end{array} \tag{38}$$

with the evident maps. The lefthand square is therefore a pullback, so we indeed have that

$$Y_{\text{eval}} = (X^I \times I) \times_{(X \times I)} (Y \times I)$$

and  $c = (\delta \Rightarrow f)$ . □

Now we can run the proof of Proposition 16 backwards in order to determine a class of generating trivial cofibrations for the unbiased case. We consider pairs of maps  $c : C \rightarrowtail Z$  and  $z : Z \rightarrow I$ , where the former is a cofibration and the latter is regarded as an “I-indexing”, so that

$$\begin{array}{ccc} C & \xrightarrow{c} & Z \\ & \searrow & \downarrow z \\ & & I \end{array}$$

can be regarded as an I-indexed family of cofibrations. Let

$$\mathbf{Gph}(z) : Z \rightarrow Z \times I,$$

be the graph of  $z : Z \rightarrow I$ , i.e.  $\mathbf{Gph}(z) = \langle 1_Z, z \rangle$ , and then let

$$c \otimes_z \delta := [\mathbf{Gph}(z), c \times I] : Z +_C (C \times I) \rightarrow Z \times I,$$

which is easily seen to be well-defined on the indicated pushout.

$$\begin{array}{ccc} C & \xrightarrow{\mathbf{Gph}(zc)} & C \times I \\ \downarrow c & & \downarrow \\ Z & \longrightarrow & Z +_C (C \times I) \\ & \searrow \mathbf{Gph}(z) & \downarrow c \otimes_z \delta \\ & & Z \times I \end{array} \quad \begin{array}{l} \text{curved arrow } c \times I \text{ from } C \times I \text{ to } Z \times I \\ \text{curved arrow } \mathbf{Gph}(z) \text{ from } Z \text{ to } Z \times I \end{array} \quad (39)$$

This specification differs from the similar (24) by using  $\mathbf{Gph}(z)$  for the inclusion  $Z \rightarrowtail Z \times I$ , rather than one of the “face maps” associated to the endpoint inclusions  $\delta_\epsilon : 1 \rightarrow I$ . (Note that a graph is always a cofibration by pulling back a diagonal.) The subobject  $c \otimes_z \delta \rightarrowtail Z \times I$  is the join of the subobjects  $\mathbf{Gph}(z) \rightarrowtail Z \times I$  and the cylinder  $C \times I \rightarrowtail Z \times I$ .

Observe that the endpoints  $\delta_\epsilon : 1 \rightarrow I$  are of the form  $c \otimes_z \delta$  by taking  $Z = 1$  and  $z = \delta_\epsilon$  and  $c = ! : 0 \rightarrow 1$ , so that biased filling is subsumed.

The maps of the form  $c \otimes_z \delta : Z +_C (C \times I) \rightarrowtail Z \times I$  now form a *class of generating trivial cofibrations* in the expected sense. Let

$$\mathcal{C} \otimes \delta = \{c \otimes_z \delta : D \rightarrowtail Z \times I \mid c : C \rightarrowtail Z, z : Z \rightarrow I\}, \quad (40)$$

then the fibrations are exactly the right class of these,

$$\mathcal{F} = (\mathcal{C} \otimes \delta)^\pitchfork.$$

**Proposition 22.** *A map  $f : Y \rightarrow X$  is an (unbiased) fibration iff for every pair of maps  $c : C \rightarrowtail Z$  and  $z : Z \rightarrow I$ , where the former is a cofibration, every commutative square of the following form has a diagonal filler, as indicated.*

$$\begin{array}{ccc} Z +_C (C \times I) & \longrightarrow & Y \\ c \otimes_z \delta \downarrow & \nearrow j & \downarrow f \\ Z \times I & \longrightarrow & X. \end{array} \quad (41)$$

*Proof.* Suppose that for all  $c : C \rightarrowtail Z$  and  $z : Z \rightarrow I$ , we have  $(c \otimes_z \delta) \pitchfork f$  in  $\mathbf{cSet}$ . Pulling  $f$  back over  $I$ , this is equivalent to the condition  $c \otimes \delta \pitchfork I^*f$  in  $\mathbf{cSet}/I$ , for all cofibrations  $c : C \rightarrowtail Z$  over  $I$ , which is equivalent to  $c \pitchfork (\delta \Rightarrow I^*f)$  in  $\mathbf{cSet}/I$  for all cofibrations  $c : C \rightarrowtail Z$ . But this in turn means that  $\delta \Rightarrow I^*f$  is a  $+$ -algebra, which by definition means that  $f$  is a fibration.  $\square$

**Unbiased fibration structure.** As in the biased case, the fibrations can also be determined by *uniform* right-lifting against a generating *set* of trivial cofibrations, now consisting of all those  $c \otimes_z \delta$  in  $\mathcal{C} \otimes \delta$  for which  $c : C \rightarrowtail Z$  has a representable codomain  $Z = I^n$ . Call these maps the *basic (unbiased) trivial cofibrations*, and let

$$\mathcal{B} \otimes \delta = \{c \otimes_z \delta : B \rightarrowtail I^{n+1} \mid c : C \rightarrowtail I^n, z : I^n \rightarrow I, n \geq 0\}, \quad (42)$$

where the pushout-product  $c \otimes_z \delta$  now has the form

$$\begin{array}{ccc} C & \xrightarrow{\text{Gph}(z)} & C \times I \\ c \downarrow & & \downarrow c \times I \\ I^n & \longrightarrow & I^n +_C (C \times I) \\ & \searrow c \otimes_z \delta & \downarrow \\ & & I^n \times I. \end{array} \quad (43)$$

$\text{Gph}(z)$

for a cofibration  $c : C \rightarrowtail I^n$ , an indexing map  $z : I^n \rightarrow I$ , and with domain  $B = (I^n +_C (C \times I))$ . These subobjects  $B \rightarrowtail I^{n+1}$  can again be seen geometrically as “generalized open box” inclusions, but now the floor or lid of the open box may be replaced by a “cross-section” given by the graph of a map  $z : I^n \rightarrow I$ .

For any map  $f : Y \rightarrow X$  a (uniform, unbiased) fibration structure on  $f$  is a choice of diagonal fillers  $j(c, z, x, y)$ ,

$$\begin{array}{ccc} B & \xrightarrow{x} & X \\ c \otimes_z \delta \downarrow & \nearrow j(c, z, x, y) & \downarrow f \\ \mathbb{I}^n \times \mathbb{I} & \xrightarrow{y} & Y, \end{array} \quad (44)$$

for each basic trivial cofibration  $c \otimes_z \delta : B \rightarrow \mathbb{I}^{n+1}$ , which is *uniform* in  $\mathbb{I}^n$  in the following sense: given any cubical map  $u : \mathbb{I}^m \rightarrow \mathbb{I}^n$ , the pullback  $u^*c : u^*C \rightarrow \mathbb{I}^m$  and the reindexing  $zu : \mathbb{I}^m \rightarrow \mathbb{I}^n \rightarrow \mathbb{I}$  determine another basic trivial cofibration  $u^*c \otimes_{zu} \delta : B' = (\mathbb{I}^m +_{u^*C} (u^*C \times \mathbb{I})) \rightarrow \mathbb{I}^{m+1}$  which fits into a commutative diagram of the form

$$\begin{array}{ccccc} B' & \xrightarrow{(u \times \mathbb{I})'} & B & \xrightarrow{x} & X \\ u^*c \otimes_{zu} \delta \downarrow \lrcorner & & c \otimes_z \delta \downarrow & \nearrow j(c, z, x, y) & \downarrow f \\ \mathbb{I}^m \times \mathbb{I} & \xrightarrow{u \times \mathbb{I}} & \mathbb{I}^n \times \mathbb{I} & \xrightarrow{y} & Y. \end{array} \quad (45)$$

For the outer rectangle in (45) there is a chosen diagonal filler

$$j(u^*c, zu, x(u \times \mathbb{I})', y(u \times \mathbb{I})) : \mathbb{I}^m \times \mathbb{I} \rightarrow X,$$

and for this map we require that

$$j(u^*c, zu, x(u \times \mathbb{I})', y(u \times \mathbb{I})) = j(c, z, x, y) \circ (u \times \mathbb{I}). \quad (46)$$

**Definition 23.** A (uniform, unbiased) fibration structure on a map

$$f : Y \rightarrow X$$

is a choice of fillers  $j(c, z, x, y)$  as in (44) satisfying (46) for all  $u : \mathbb{I}^m \rightarrow \mathbb{I}^n$ .

In these terms, we have following analogue of corollary 19.

**Proposition 24.** For any object  $X$  in  $\mathbf{cSet}$  the following are equivalent:

1. the canonical map  $X^{\mathbb{I}} \times \mathbb{I} \rightarrow X \times \mathbb{I}$  is a trivial fibration.
2.  $X$  has the right lifting property with respect to all generating trivial cofibrations,

$$(\mathcal{C} \otimes_z \delta) \pitchfork X.$$



3.  $X$  has a uniform fibration structure in the sense of Definition 23.

*Proof.* The equivalence between (1) and (2) is proposition 22. Suppose (1), i.e. that the map

$$(\delta \Rightarrow X) : X^I \times I \rightarrow X \times I$$

is a relative  $+$ -algebra over  $X \times I$ . By proposition 8, this means that  $(\delta \Rightarrow X)$ , as an object of  $\mathbf{cSet}/(X \times I)$ , has a uniform filling structure with respect to all cofibrations  $c : C \rightarrow I^n$  over  $(X \times I)$ . Transposing by the  $\otimes \dashv \Rightarrow$  adjunction and unwinding gives, equivalently, a uniform fibration structure on  $X$ .  $\square$

A statement analogous to the foregoing also holds for maps  $f : Y \rightarrow X$  in place of objects  $X$ . Indeed, as before, we have the following sharper formulation.

**Corollary 25.** *Fibration structures on a map  $f : Y \rightarrow X$  correspond uniquely to  $+$ -algebra structures on the map  $(\delta \Rightarrow f)$  (cf. definition 20),*

$$(\delta \Rightarrow f) : Y^I \times I \rightarrow (X^I \times I) \times_{(X \times I)} (Y \times I)$$

### 3.3 Factorization

**Definition 26.** Summarizing the foregoing definitions and results, we have the following classes of maps:

- The *generating trivial cofibrations* were determined in (40) to be

$$\mathcal{C} \otimes \delta = \{c \otimes_z \delta : D \rightarrow Z \times I \mid c : C \rightarrow Z, z : Z \rightarrow I\}, \quad (47)$$

where the pushout-product  $c \otimes_z \delta$  has the form

$$\begin{array}{ccc} C & \xrightarrow{\text{Gph}(zc)} & C \times I \\ \downarrow c & & \downarrow \\ Z & \longrightarrow & Z +_C (C \times I) \\ & \searrow \text{Gph}(z) & \downarrow c \times I \\ & & Z \times I \end{array} \quad (48)$$

$c \otimes_z \delta$  (dotted arrow from  $Z +_C (C \times I)$  to  $Z \times I$ )

for any cofibration  $c : C \rightarrow Z$  and indexing map  $z : Z \rightarrow I$ , with domain  $D = (Z +_C (C \times I))$ .

- The class  $\mathcal{F}$  of *fibrations*, written  $f : Y \twoheadrightarrow X$ , may be characterized as the right-lifting class of the generating trivial cofibrations,

$$(\mathcal{C} \otimes \delta)^\pitchfork = \mathcal{F}.$$

- The class of *trivial cofibrations* is defined to be left class of the fibrations,

$$\mathrm{TCof} = {}^\pitchfork \mathcal{F}.$$

It follows from the specification that the classes  $\mathrm{TCof}$  and  $\mathcal{F}$  are mutually weakly orthogonal,

$$\mathrm{TCof} \pitchfork \mathcal{F},$$

and are both closed under retracts, so in order to have a weak factorization system  $(\mathrm{TCof}, \mathcal{F})$  it just remains to show that every map  $f : X \rightarrow Y$  can be factored as  $f = g \circ h$  with  $g \in \mathcal{F}$  and  $h \in \mathrm{TCof}$ .

**Proposition 27.** *Every map  $f : X \rightarrow Y$  in  $\mathbf{cSet}$  can be factored as  $f = g \circ h$ ,*

$$\begin{array}{ccc} X & \xrightarrow{h} & X' \\ & \searrow f & \downarrow g \\ & & Y \end{array} \quad (49)$$

with  $h : X \rightarrowtail X'$  a trivial cofibration and  $g : X' \twoheadrightarrow Y$  a fibration.

*Proof.* This is a standard argument (cf. [?, GKR18]), which can be simplified a bit in this particular case. We sketch the proof for the case  $Y = 1$ ; the general case is not essentially different.

Thus let  $X$  be any object, and we wish to find a fibrant object  $X'$  and a trivial cofibration  $h : X \rightarrowtail X'$ . For each basic trivial cofibration  $\beta : B \rightarrowtail \mathbf{I}^k$ , we need to solve all extension problems of the form

$$\begin{array}{ccc} B & \xrightarrow{x} & X \\ \beta \downarrow & \nearrow \text{dotted} & \\ \mathbf{I}^k & & \end{array} \quad (50)$$

We first combine these into a single problem by taking a coproduct over all maps  $x : B \rightarrow X$ ,

$$\begin{array}{ccc} \coprod_x B & \xrightarrow{[x]} & X. \\ \downarrow \coprod_x \beta & \nearrow \text{dotted} & \\ \coprod_x I^k & & \end{array}$$

We then take the coproduct over all basic trivial cofibrations  $\beta : B \rightarrow I^k$ ,

$$\begin{array}{ccc} \coprod_\beta \coprod_x B & \xrightarrow{[[x]_\beta]} & X. \\ \downarrow \coprod_\beta \coprod_x \beta & \nearrow \text{dotted} & \\ \coprod_\beta \coprod_x I^k & & \end{array}$$

Note that a coproduct of trivial cofibrations is clearly a trivial cofibration.

Taking a pushout, the indicated map  $h_1$  is then also a trivial cofibration, because it is a pushout of one

$$\begin{array}{ccc} \coprod_\beta \coprod_x B & \xrightarrow{[[x]_\beta]} & X \\ \downarrow \coprod_\beta \coprod_x \beta & & \downarrow h_1 \\ \coprod_\beta \coprod_x I^k & \xrightarrow{\quad} & X_1 \end{array}$$

Now iterate the construction to get a sequence of trivial cofibrations, of which we take  $X'$  to be the colimit and  $h : X \rightarrow X'$  the canonical map,

$$h : X \xrightarrow{h_1} X_1 \xrightarrow{h_2} X_2 \xrightarrow{h_3} \dots \xrightarrow{\quad} \varinjlim X_n = X'. \quad (51)$$

To show that  $X'$  is fibrant, consider an extension problem of the form (50) with  $X'$  in place of  $X$ ,

$$\begin{array}{ccc} B & \xrightarrow{x} & \varinjlim X_n \\ \downarrow \beta & \nearrow \text{dotted} & \\ I^k & & \end{array}$$

The subobject  $B \twoheadrightarrow I^k$  has as domain an object  $B$  that is a *finite* colimit of maps  $I^m \rightarrow I^n$  of representables (as can be seen by considering sieves in the category of cubes), and is therefore finitely presented, in the sense that mapping out of it preserves filtered colimits. Thus the map  $x : B \rightarrow \varinjlim X_n$  must factor through some  $x_k : B \rightarrow X_k$ , giving rise to the problem

$$\begin{array}{ccc} B & \xrightarrow{x_k} & X_k \\ \beta \downarrow & & \downarrow \\ I^k & \dashrightarrow & \varinjlim X_n. \end{array}$$

But this has a solution in the next step, by the construction of  $X_{k+1}$ ,

$$\begin{array}{ccc} B & \xrightarrow{x_k} & X_k \\ \beta \downarrow & & \downarrow h_{k+1} \\ I^k & \longrightarrow & X_{k+1} \\ & \searrow j & \downarrow \\ & & \varinjlim X_n. \end{array}$$

Finally, we need to show the uniformity condition on the resulting fillers  $j = j(\beta, x)$ . For this to work, we must modify the colimit construction (51) by interleaving certain coequalizers, in order to identify fillers added at different stages. For details, see [GKR18, ?].  $\square$

**Proposition 28.** *There is a weak factorization system on the category  $\mathbf{cSet}$  in which the right maps are the fibrations and the left maps are the trivial cofibrations, both as specified in definition 26.*

This will be called the *fibration weak factorization system*. The following observation will be of use later on, the proof can be found in [GKR18, ?].

**Corollary 29.** *The construction given in (51) of the fibrant replacement,*

$$X' = \varinjlim_n X_n$$

*is functorial in  $X$ , and the canonical trivial cofibrations  $h : X \twoheadrightarrow X'$  are natural in  $X$ .*

## 4 Weak equivalences

**Definition 30** (Weak equivalence). A map  $f : X \rightarrow Y$  in  $\mathbf{cSet}$  will be called a *weak equivalence* if it can be factored as  $f = g \circ h$ ,

$$\begin{array}{ccc} X & \xrightarrow{h} & W \\ & \searrow f & \downarrow g \\ & & Y \end{array}$$

with  $h : X \rightarrow W$  a trivial cofibration and  $g : W \rightarrow Y$  a trivial fibration, i.e. a right map in the cofibration weak factorization system. Let

$$\mathcal{W} = \{f : X \rightarrow Y \mid f = g \circ h \text{ for } g \in \mathbf{TFib} \text{ and } h \in \mathbf{TCof}\}$$

be the class of weak equivalences.

Observe that every trivial fibration  $f \in \mathcal{C}^{\mathfrak{n}}$  is indeed a fibration, because the generating trivial cofibrations are indeed cofibrations; moreover, every trivial fibration is also a weak equivalence, since the identity maps are trivial cofibrations. Thus we have

$$\mathbf{TFib} \subseteq (\mathcal{F} \cap \mathcal{W}).$$

Thus, because the trivial fibrations are fibrations, every trivial cofibration  $g \in {}^{\mathfrak{n}}\mathcal{F}$  is a cofibration; moreover, every trivial cofibration is also a weak equivalence, since the identity maps are also trivial fibrations. Thus we also have

$$\mathbf{TCof} \subseteq (\mathcal{C} \cap \mathcal{W}).$$

**Lemma 31.**  $(\mathcal{C} \cap \mathcal{W}) \subseteq \mathbf{TCof}$ .

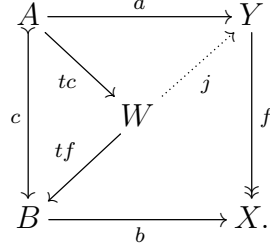
*Proof.* Let  $c : A \rightarrowtail B$  be a cofibration with a factorization

$$c = tf \circ tc : A \rightarrow W \rightarrow B$$

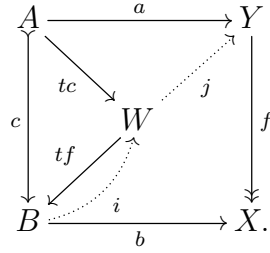
where  $tc \in \mathbf{TCof}$  and  $tf \in \mathbf{TFib}$ . Let  $f : Y \twoheadrightarrow X$  be a fibration and consider a commutative diagram,

$$\begin{array}{ccc} A & \xrightarrow{a} & Y \\ \downarrow c & & \downarrow f \\ B & \xrightarrow{b} & X. \end{array}$$

Inserting the factorization of  $c$ , we have  $j : W \rightarrow Y$  as indicated, with  $j \circ tc = a$  and  $f \circ j = b \circ tf$ , since  $tc \rhd f$ .



Moreover, since  $c \rhd tf$  there is an  $i : B \rightarrow W$  as indicated, with  $i \circ c = tc$  and  $tf \circ i = 1_B$ .



Let  $k = j \circ i$ . Then  $k \circ c = j \circ i \circ c = j \circ tc = a$ , and  $f \circ k = f \circ j \circ i = b \circ tf \circ i = b$ .  $\square$

The proof of the following is dual:

**Lemma 32.**  $(\mathcal{F} \cap \mathcal{W}) \subseteq \text{TFib}$ .

**Proposition 33.** For the three classes of maps  $\mathcal{C}, \mathcal{W}, \mathcal{F}$  in  $\text{cSet}$ , we have

$$\begin{aligned}\mathcal{F} \cap \mathcal{W} &= \text{TFib}, \\ \mathcal{C} \cap \mathcal{W} &= \text{TCof},\end{aligned}$$

and therefore two weak factorization systems:

$$(\mathcal{C}, \mathcal{W} \cap \mathcal{F}) \quad , \quad (\mathcal{C} \cap \mathcal{W}, \mathcal{F}).$$

It thus remains only to prove that the weak equivalences satisfy the 3-for-2 property.

## Weak homotopy equivalence

**Definition 34.** By a *homotopy* between parallel maps  $f, g : X \rightrightarrows Y$ , written  $\vartheta : f \sim g$ , we shall mean a map from the *cylinder of  $X$*  built using the (representable) interval  $I$ ,

$$\vartheta : I \times X \rightarrow Y,$$

and such that  $\vartheta \circ \iota_0 = f$  and  $\vartheta \circ \iota_1 = g$ ,

$$\begin{array}{ccccc} X & \xrightarrow{\iota_0} & I \times X & \xleftarrow{\iota_1} & X \\ & \searrow f & \downarrow \vartheta & \swarrow g & \\ & & Y & & \end{array}$$

where we write the canonical inclusions into the ends of the cylinder as

$$\iota_\epsilon = \mathbf{Gph}(\delta_\epsilon!) : X \rightarrow I \times X, \quad \epsilon = 0, 1.$$

**Proposition 35.** *If  $K$  is fibrant, then the relation of homotopy  $f \sim g$  between maps  $f, g : X \rightrightarrows K$  is an equivalence relation. Moreover, it is compatible with pre- and post-composition.*

*Proof.* For  $f, g : X \rightrightarrows Y$ , a homotopy  $f \overset{\vartheta}{\sim} g : X \times I \rightarrow Y$  is equivalent, under exponential transposition, to a path in the function space  $\vartheta : I \rightarrow Y^X$  with endpoints  $\vartheta_0 = \vartheta \circ \delta_0 = f : 1 \rightarrow Y^X$  and  $\vartheta_1 = g$ . Note that  $Y^X$  is fibrant if  $Y$  is fibrant, so we can use box-filling in  $Y^X$ .

The reflexivity of homotopy  $f \sim f$  is witnessed by  $\rho : I \rightarrow 1 \xrightarrow{f} Y^X$ .

For symmetry  $f \sim g \Rightarrow g \sim f$  take  $\vartheta : I \rightarrow Y^X$  with  $\vartheta_0 = f$  and  $\vartheta_1 = g$  and we want to build  $\vartheta' : I \rightarrow Y^X$  with  $\vartheta'_0 = g$  and  $\vartheta'_1 = f$ . Take an open 2-box in  $Y^X$  of the form

$$\begin{array}{ccc} & g & f \\ \vartheta \uparrow & & \uparrow \rho \\ f & \xrightarrow{\rho} & f \end{array}$$

This box is a map  $b : I +_1 I +_1 I \rightarrow Y^X$  with the indicated components, and it has a filler  $c : I \times I \rightarrow Y^X$ , i.e. an extension along the canonical map  $I +_1 I +_1 I \rightarrowtail I \times I$ , which is a trivial cofibration. Let  $t : I \rightarrow I \times I$  be the evident missing top face of the 2-cube. Then we can set  $\vartheta' = ct : I \rightarrow Y^X$  to get a homotopy  $\vartheta' : I \rightarrow Y^X$  with required endpoints.

For transitivity,  $f \stackrel{\vartheta}{\sim} g$  &  $g \stackrel{\varphi}{\sim} h \Rightarrow f \sim h$ , an analogous filling construction is used with the open box:

$$\begin{array}{ccc} & f & h \\ \rho \uparrow & & \uparrow \varphi \\ f & \xrightarrow{\vartheta} & g \end{array}$$

Compatibility under pre- and post-composition is shown by representing homotopy by mapping into the pathspace, for precomposition, and out of the cylinder, for post-composition.  $\square$

**Definition 36** (Connected components). The functor

$$\pi_0 : \mathbf{cSet} \rightarrow \mathbf{Set}$$

is defined, for any cubical set  $X$ , to be the coequalizer

$$X_1 \rightrightarrows X_0 \rightarrow \pi_0 X,$$

where the two parallel arrows are the maps  $X_{\delta_0}, X_{\delta_1} : X_1 \rightrightarrows X_0$  induced by the endpoints  $\delta_0, \delta_1 : 1 \rightrightarrows \mathbf{I}$ . For any Kan complex  $K$  we therefore have  $\pi_0 K = \text{Hom}(1, K)/\sim$ , that is,  $\pi_0 K$  is the set of points  $1 \rightarrow K$ , modulo the homotopy equivalence relation on them.

One can show that in fact  $\pi_0 X = \varinjlim_n X_n$ , the colimit being left adjoint to the constant presheaf functor  $\Delta : \mathbf{Set} \rightarrow \mathbf{cSet}$ . Since the category  $\mathbb{B}$  of finite strictly bipointed sets is sifted, we have:

**Corollary 37.** *The functor  $\pi_0 : \mathbf{cSet} \rightarrow \mathbf{Set}$  preserves finite products.*

As usual, a map  $f : X \rightarrow Y$  in  $\mathbf{cSet}$  will be called a *homotopy equivalence* if there is a *quasi-inverse*  $g : Y \rightarrow X$  and homotopies  $\vartheta : 1_X \sim g \circ f$  and  $\varphi : 1_Y \sim f \circ g$ .

**Definition 38** (Weak homotopy equivalence). A map  $f : X \rightarrow Y$  will be called a *weak homotopy equivalence* if for every fibrant object  $K$ , the “internal precomposition” map  $K^f : K^Y \rightarrow K^X$  is bijective on connected components, i.e.

$$\pi_0 K^f : \pi_0 K^Y \rightarrow \pi_0 K^X$$

is a bijection of sets.



**Lemma 39.** *A homotopy equivalence is weak homotopy equivalence.*

*Proof.* If  $f : X \rightarrow Y$  is a homotopy equivalence, then so is  $K^f : K^Y \rightarrow K^X$  for any  $K$ , since homotopy respects composition. Since  $K^X$  is always fibrant when  $K$  is,  $\pi_0$  is well defined, and it clearly takes homotopy equivalences to isomorphisms of sets.  $\square$

**Lemma 40.** *The weak homotopy equivalences  $f : X \rightarrow Y$  satisfy the 3-for-2 condition.*

*Proof.* Follows from the corresponding fact about bijections of sets.  $\square$

Our goal of showing that the weak equivalences satisfy 3-for-2 is now reduced to showing that a map is a weak equivalence (WE) if and only if it is a weak homotopy equivalence (WHE). This will be proved in four cases, showing that a (co)fibration is a WE if and only if it is a WHE.

**Lemma 41.** *A map  $f : X \rightarrow Y$  is a weak homotopy equivalence iff it satisfies the following two conditions.*

1. *For every fibrant object  $K$  and every map  $x : X \rightarrow K$  there is a map  $y : Y \rightarrow K$  such that  $y \circ f \sim x$ ,*

$$\begin{array}{ccc} X & \xrightarrow{x} & K \\ f \downarrow & \sim & \nearrow y \\ Y & & \end{array}$$

*We say that  $x$  “extends along  $f$  up to homotopy”.*

2. *For every fibrant object  $K$  and maps  $y, y' : Y \rightarrow K$  such that  $yf \sim y'f$ , there is a homotopy  $y \sim y'$ ,*

$$\begin{array}{ccc} X & \xrightarrow{\quad} & K^I \\ f \downarrow & \nearrow & \downarrow \\ Y & \xrightarrow{\langle y, y' \rangle} & K \times K \end{array}$$

*Proof.* Unwind the definition.  $\square$

**Lemma 42.** *A cofibration  $c : A \rightarrowtail B$  that is a WE is a WHE.*

*Proof.* A cofibration  $c : A \rightarrowtail B$  that is a WE is a trivial cofibration by proposition 33. So the result follows from Lemma 41, and the fact that  $K^1 \rightarrow K \times K$  is always a fibration when  $K$  is fibrant.  $\square$

**Lemma 43.** *A fibration  $p : Y \twoheadrightarrow X$  that is a WE is a WHE.*

*Proof.* A fibration weak equivalence  $f : Y \twoheadrightarrow X$  is a trivial fibration by proposition 33, and therefore has a section  $s : X \rightarrowtail Y$ , by the lifting problem

$$\begin{array}{ccc} 0 & \longrightarrow & Y \\ \downarrow & & \downarrow f \\ X & \xrightarrow{=} & X, \end{array}$$

since  $0 \rightarrow X$  is always a cofibration. Moreover, there is a homotopy  $\vartheta : sf \sim 1_Y$ , resulting from the lifting problem

$$\begin{array}{ccc} Y \amalg Y & \xrightarrow{[sf, 1]} & Y \\ \downarrow [\iota_0, \iota_1] & & \downarrow f \\ I \times Y & \xrightarrow{f\pi_2} & X. \end{array}$$

Thus  $f$  is a homotopy equivalence, and so a WHE by lemma 39.  $\square$

**Lemma 44.** *If  $K$  is fibrant, then any fibration  $f : Y \twoheadrightarrow K$  that is a HE is a WE.*

*Proof.* This is a standard argument, which we just sketch. It suffices to show that any diagram of the form

$$\begin{array}{ccc} C & \xrightarrow{y} & Y \\ \downarrow c & & \downarrow f \\ K & \xrightarrow{=} & K, \end{array} \tag{52}$$

with  $c : C \rightarrowtail X$  a cofibration, has a diagonal filler. Since  $f$  is a HE it has a quasi-inverse  $s : X \rightarrow Y$  with  $\vartheta : fs \sim 1_K$ , which we can correct to a section  $s' : K \rightarrow Y$ . Indeed, consider

$$\begin{array}{ccc} K & \xrightarrow{s} & Y \\ \downarrow \iota_0 & \nearrow \vartheta' & \downarrow f \\ K \times I & \xrightarrow{\vartheta} & K \\ \uparrow \iota_1 & \nearrow = & \\ K & & \end{array}$$

where  $\vartheta'$  results from  $\iota_0 \pitchfork f$ . Let  $s' = \vartheta' \iota_1$ , so that  $\vartheta' : s \sim s'$  and  $fs' = 1_K$ .

Thus we can assume that  $s = s' : K \rightarrow Y$  is a section, which fills the diagram (52) up to a homotopy in the upper triangle.

$$\begin{array}{ccc} C & \xrightarrow{y} & Y \\ \downarrow c & \sim \nearrow s & \downarrow f \\ K & \xrightarrow{=} & K, \end{array}$$

Now we can correct  $s : K \rightarrow Y$  to a homotopic  $t : K \rightarrow Y$  over  $f$  by using the homotopy  $\varphi : sc \sim y$  to get a map  $\varphi : C \rightarrow Y^I$  over  $f$ . Since  $f$  is a fibration, the projections  $p_0, p_1 : Y^I \rightarrow Y$  over  $f$  are trivial fibrations, and so there is a lift  $\varphi' : K \rightarrow Y^I$  for which  $t := p_1 \varphi'$  has  $tc = y$  and  $ft = 1_K$ , and so is a filler for (52).  $\square$

**Lemma 45.** *If  $K$  is fibrant, then any fibration  $f : Y \twoheadrightarrow K$  that is a WHE is a WE.*

*Proof.* Since  $K$  is fibrant, so is  $Y$ , and since  $f$  is a WHE, there is a map  $s : K \rightarrow Y$  and a homotopy  $\theta : sf \sim 1_Y$  by lemma 41(1). Thus, applying  $f$  again, we have a homotopy  $f\vartheta : fsf \sim f$ , forming the outer commutative square in

$$\begin{array}{ccc} Y & \xrightarrow{f\vartheta} & K^I \\ \downarrow f & \nearrow \varphi & \downarrow \\ K & \xrightarrow{\langle fs, 1_K \rangle} & K \times K. \end{array}$$

By lemma 41(2) there is a diagonal filler  $\varphi : fs \sim 1_K$ , and so  $f$  is a HE. Now apply lemma 44.  $\square$

**Lemma 46.** *If  $K$  is fibrant, then any cofibration  $c : A \rightarrowtail K$  that is a WHE is a WE.*

*Proof.* Let  $c : A \rightarrowtail K$  be a cofibration WHE and factor it into a trivial cofibration  $i : A \rightarrowtail Z$  followed by a fibration  $p : Z \twoheadrightarrow K$ . By lemma 41, it is clear that a trivial cofibration is a WHE. So both  $c$  and  $i$  are WHE, and therefore so is  $p$  by 3-for-2 for WHEs. Since  $K$  is fibrant,  $p$  is a trivial fibration by lemma 45, and thus  $c$  is a WE.  $\square$

**Lemma 47** ([?], x.n.m). *A cofibration  $c : A \rightarrowtail B$  WHE lifts against all fibrations  $f : Y \twoheadrightarrow K$  with fibrant codomain.*

*Proof.* Let  $c : A \rightarrowtail B$  be a cofibration WHE and  $f : Y \twoheadrightarrow K$  a fibration with fibrant codomain  $K$ , and consider a lifting problem

$$\begin{array}{ccc} A & \xrightarrow{a} & Y \\ c \downarrow & & \downarrow f \\ B & \xrightarrow{b} & K. \end{array}$$

Let  $\eta : B \rightarrowtail B'$  be a fibrant replacement of  $B$ , since  $K$  is fibrant,  $b$  extends along  $\eta$  to give  $b' : B' \rightarrow K$  as in:

$$\begin{array}{ccc} A & \xrightarrow{a} & Y \\ c \downarrow & & \downarrow f \\ B & \xrightarrow{b} & K \\ \eta \downarrow & \nearrow b' & \\ B' & & \end{array}$$

Since  $\eta$  is a trivial cofibration, it is a WHE. So the composite  $\eta c$  is also a WHE. But since  $B'$  is fibrant,  $\eta c$  is then a trivial cofibration by lemma 46. Thus there is a lift  $j : B' \rightarrow Y$ , and therefore also one  $k = j\eta : B \rightarrow Y$ .  $\square$

To complete the proof that a cofibration WHE is a WE we will use the fact that the fibration weak factorization system satisfies the fibration extension property, the proof of which is deferred to section 8.

**Definition 48** (Fibration extension). The fibration weak factorization system is said to satisfy the *fibration extension property* (FEP) if the following holds: Given a fibration  $f : Y \twoheadrightarrow X$  and a trivial cofibration  $\eta : X \rightarrowtail X'$ , there is a fibration  $f' : Y' \twoheadrightarrow X'$  such that  $f$  is a pullback of  $f'$  along  $\eta$ .

$$\begin{array}{ccc} Y & \xrightarrow{\quad} & Y' \\ f \downarrow & \lrcorner & \downarrow f' \\ X & \xrightarrow{\quad \eta \quad} & X'. \end{array} \tag{53}$$

**Lemma 49.** *Assuming the FEP, a cofibration that lifts against every fibration  $f : Y \twoheadrightarrow K$  with fibrant codomain is a WE.*

*Proof.* Let  $c : A \twoheadrightarrow B$  be a cofibration and consider a lifting problem against an arbitrary fibration  $f : Y \twoheadrightarrow X$ ,

$$\begin{array}{ccc} A & \xrightarrow{a} & Y \\ c \downarrow & & \downarrow f \\ B & \xrightarrow{b} & X. \end{array} \quad (54)$$

Let  $\eta : X \rightarrow X'$  be a fibrant replacement, so  $\eta$  is a trivial cofibration and  $X'$  is fibrant. By the fibration extension property of definition 48, there is a fibration  $f' : Y' \twoheadrightarrow X'$  such that  $f$  is a pullback of  $f'$  along  $\eta$ . So we can extend diagram (54) to obtain the following, in which the righthand square is a pullback.

$$\begin{array}{ccccc} A & \xrightarrow{a} & Y & \xrightarrow{y} & Y' \\ c \downarrow & & \downarrow f & \lrcorner & \downarrow f' \\ B & \xrightarrow{b} & X & \xrightarrow{\eta} & X'. \end{array} \quad (55)$$

By assumption, there is a lift  $j' : B \rightarrow Y'$  with  $f'j' = \eta b$  and  $j'c = yb$ . Therefore, since  $f$  is a pullback, there is a map  $j : B \rightarrow Y$  with  $fj = b$  and  $yj = j'$ .

$$\begin{array}{ccccc} A & \xrightarrow{a} & Y & \xrightarrow{y} & Y' \\ c \downarrow & j \nearrow & \downarrow f & \lrcorner & \downarrow f' \\ B & \xrightarrow{b} & X & \xrightarrow{\eta} & X'. \end{array} \quad (56)$$

Thus  $yjc = j'c = ya$ . But as a trivial cofibration,  $\eta$  is monic, and as a pullback of  $\eta$ ,  $y$  is also monic. So  $jc = a$ .  $\square$

Combining the previous two lemmas 47 and 49 we now have the following.

**Corollary 50.** *Assuming the FEP, a cofibration  $c : A \twoheadrightarrow B$  that is a WHE is a WE.*

The following is not required, but we state it anyway for the record:

**Lemma 51.** *Assuming the FEP, a fibration  $f : Y \twoheadrightarrow X$  that is a WHE is a WE.*

*Proof.* Factor  $f : Y \twoheadrightarrow X$  into a cofibration  $i : Y \twoheadrightarrow Z$  followed by a trivial fibration  $p : Z \twoheadrightarrow X$ . Then  $f$  is a trivial fibration if  $i \pitchfork f$ , for then  $f$  is a

retract of  $p$ . Since  $p$  is a trivial fibration, it is a WHE by lemma 43. Since  $f$  is also a WHE, so is  $i$  by 3-for-2. Thus  $i$  is a trivial cofibration by corollary 50. Since  $f$  is a fibration,  $i \pitchfork f$  as required.  $\square$

**Proposition 52.** *Assuming the FEP, a map  $f : X \rightarrowtail Y$  is a WHE if and only if it is a WE. Thus the weak equivalences  $\mathcal{W}$  satisfy the 3-for-2 condition.*

*Proof.* Let  $f : X \twoheadrightarrow Y$  be a WE and factor it into a trivial cofibration  $i : X \rightarrowtail Z$  followed by a trivial fibration  $p : Z \twoheadrightarrow Y$ . Then both  $i$  and  $p$  are WHE, whence so is  $f$ . Conversely, let  $f$  be a WHE and factor it into a cofibration  $i : X \rightarrowtail Z$  followed by a trivial fibration  $p : Z \twoheadrightarrow Y$ . Since  $p$  is then a WHE, as is  $f$ , it follows that  $i$  is as well. Thus  $i$  is also a WE, by lemma 50, hence a trivial fibration. So  $f$  is a WE.  $\square$

We summarize the results to this point in the following.

**Theorem 53.** *If the fibration weak factorization system of Definition 26 satisfies the fibration extension property of Definition 48, then the weak equivalences  $\mathcal{W}$  have the 3-for-2 property, and so by Proposition 33, the three classes  $(\mathcal{C}, \mathcal{W}, \mathcal{F})$  form a Quillen model structure.*

The proof of the fibration extension property will occupy the remainder of this paper. It requires several intermediate results: the equivalence extension property (Section 7), a universal fibration (section 6), and the Frobenius condition (section 5), to which we now turn.

## 5 The Frobenius condition

In this section, we show that the fibration WFS from section 3 has the *Frobenius property*: the left maps are stable under pullback along the right maps (see [?]). This will imply the *right properness* of our model structure: the weak equivalences are preserved by pullback along fibrations. The Frobenius property is also needed in the proof of the equivalence extension property in the next section. A proof of Frobenius in a related setting of cubical sets with connections can be found in [GS17]; however the type theoretic approach of [OP17, CCHM16] can be applied without connections and is also more direct. This approach proves the “dual” fact that the *pushforward* operation, which is right adjoint to pullback, and which always exists in a topos, when applied

along any *fibration*  $f : Y \twoheadrightarrow X$  preserves fibrations. This corresponds to the type-theoretic  $\Pi$ -formation rule.

Recall that a map  $f : A \rightarrow X$  is a fibration if (in the slice  $\mathbf{cSet}/I$ , where there is a generic point  $\delta : 1 \rightarrow I$ ) the map  $\delta \Rightarrow f$  admits a  $+$ -algebra structure (and so is a trivial fibration), where the definition of  $\delta \Rightarrow f$  is recalled below.

$$\begin{array}{ccc}
 A^I & \xrightarrow{A^\delta} & A \\
 \delta \Rightarrow f \searrow & & \downarrow f \\
 X^I \times_X A & \xrightarrow{\quad} & A \\
 \downarrow & & \downarrow \\
 X^I & \xrightarrow{X^\delta} & X.
 \end{array}
 \quad (57)$$

Let us write this condition schematically as follows:

$$\begin{array}{ccccc}
 A^I & \dashrightarrow & A_\epsilon & \longrightarrow & A \\
 & & \downarrow \lrcorner & & \downarrow f \\
 & & X^I & \xrightarrow{\epsilon} & X,
 \end{array}
 \quad (58)$$

where  $\epsilon = X^\delta$ ,  $A_\epsilon = X^I \times_X A$ , and the struck-through arrow indicates that it admits a  $+$ -algebra structure.

**Lemma 54.** *Let  $g : Y \rightarrow X$  be any map and  $f : A \rightarrow X$  a fibration, then the pullback  $g^*f : g^*A \rightarrow Y$  is also a fibration.*

*Proof.* This is clear, since fibrations are the right class of a weak factorization system, but let us see how the “algebraic” specification (61) is also stable under pullback ...  $\square$

**Lemma 55.** *Let  $\alpha : A \rightarrow X$  and  $\beta : B \rightarrow A$  be fibrations, then the composite  $\alpha \circ \beta : B \rightarrow X$  is also a fibration.*

*Proof.* We have the following for the fibration structures on  $B \rightarrow A$  and

$A \rightarrow X$  (with obvious notation).

$$\begin{array}{ccccc}
 B^I & \dashrightarrow & B_{\epsilon_A} & \longrightarrow & B \\
 & & \downarrow & & \downarrow \\
 & & A^I & \dashrightarrow & A_{\epsilon_X} \longrightarrow A \\
 & & & & \downarrow \\
 & & & & X^I \xrightarrow{\epsilon_X} X,
 \end{array} \tag{59}$$

Pulling back  $B \rightarrow A$  in two steps we therefore obtain,

$$\begin{array}{ccccccc}
 B^I & \dashrightarrow & B_{\epsilon_A} & \longrightarrow & B_{\epsilon_X} & \longrightarrow & B \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & A^I & \dashrightarrow & A_{\epsilon_X} & \longrightarrow & A \\
 & & & & \downarrow & & \downarrow \\
 & & & & X^I & \xrightarrow{\epsilon_X} & X,
 \end{array} \tag{60}$$

Now use the fact that trivial fibrations are closed under pullback along all maps, and under composition, to infer that the indicated composite map  $B^I \rightarrow B_{\epsilon_X}$  is also a trivial fibration, as required.  $\square$

**Proposition 56** (Frobenius). *Let  $\alpha : A \rightarrow X$  and  $\beta : B \rightarrow A$  be fibrations, then the pushforward  $\alpha_*\beta : \Pi_A B \rightarrow X$  is also a fibration.*

*Proof.* Given the fibrations  $\alpha : A \rightarrow X$  and  $\beta : B \rightarrow A$ , let  $p : A^I \rightarrow A_\epsilon$  and  $q : B^I \rightarrow p^*B_\epsilon$  be the associated  $+$ -algebras, so that we have the following situation, with all squares pullbacks.

$$\begin{array}{ccccccc}
 B^I & \xrightarrow{q} & p^*B_\epsilon & \longrightarrow & B_\epsilon & \longrightarrow & B \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & A^I & \xrightarrow{p} & A_\epsilon & \longrightarrow & A \\
 & & & & \downarrow & & \downarrow \\
 & & & & X^I & \xrightarrow{\epsilon} & X,
 \end{array} \tag{61}$$



Adding (some composites and) the relevant pushforward underneath, we have

$$\begin{array}{ccccccc}
 B^I & \xrightarrow{q} & p^*B_\epsilon & \longrightarrow & B_\epsilon & \longrightarrow & B \\
 & \searrow & \downarrow & & \downarrow & & \downarrow \\
 & & A^I & \xrightarrow{p} & A_\epsilon & \longrightarrow & A \\
 & & & \searrow & \downarrow & & \downarrow \\
 & & & & X^I & \xrightarrow{\epsilon} & X \\
 & & & \nearrow & \uparrow & & \uparrow \\
 (\Pi_A B)^I & \xrightarrow{r} & (\Pi_A B)_\epsilon & \longrightarrow & \Pi_A B & & 
 \end{array} \tag{62}$$

and we wish to show that the indicated map  $r : (\Pi_A B)^I \rightarrow (\Pi_A B)_\epsilon$  admits a  $+$ -algebra structure. We will do so by showing that it is a retract of a known  $+$ -algebra.

Indeed, let us apply the pushforward, along the indicated canonical map  $\alpha^I : A^I \rightarrow X^I$ , to the  $+$ -algebra  $q : B^I \rightarrow p^*B_\epsilon$ , regarded as an arrow over  $A^I$ . We obtain an arrow over  $X^I$  of the form

$$\Pi_{A^I} q : \Pi_{A^I} B^I \rightarrow \Pi_{A^I} p^*B_\epsilon$$

which is a  $+$ -algebra, because these are preserved by pushforward, according to Lemma 14.

Now observe that by the Beck-Chevalley condition, we have an isomorphism

$$(\Pi_A B)_\epsilon \cong \Pi_{A_\epsilon} B_\epsilon.$$

Moreover, since  $\Pi_{A^I} \cong \Pi_{A_\epsilon} \circ p_*$ , we have

$$\Pi_{A^I} p^*B_\epsilon \cong \Pi_{A_\epsilon} p_* p^*B_\epsilon.$$

Thus the image of the unit  $\eta : B_\epsilon \rightarrow p_* p^*B_\epsilon$  under  $\Pi_{A_\epsilon}$  is a map  $s = \Pi_{A_\epsilon} \eta$  over  $X^I$  of the form:

$$\begin{array}{ccc}
 & & X^I \\
 & \nearrow & \uparrow \\
 (\Pi_A B)^I & \xrightarrow{r} & \Pi_{A_\epsilon} B_\epsilon \\
 & & \downarrow s \\
 \Pi_{A^I} B^I & \xrightarrow{\Pi_{A^I} q} & \Pi_{A_\epsilon} p_* p^*B_\epsilon
 \end{array} \tag{63}$$

Our objective is now to fill in the further arrows  $\varphi, \psi, \vartheta$  indicated below in order to exhibit  $r$  as a retract of  $\Pi_{A^I} q$  in the arrow category over  $X^I$ .

$$\begin{array}{ccc}
 & & X^I \\
 & \nearrow & \uparrow \\
 (\Pi_A B)^I & \xrightarrow{r} & \Pi_{A_\epsilon} B_\epsilon \\
 \downarrow \varphi & & \downarrow s \\
 \Pi_{A^I} B^I & \xrightarrow{\Pi_{A^I} q} & \Pi_{A_\epsilon} p_* p^* B_\epsilon \\
 \downarrow \psi & & \downarrow \vartheta \\
 (\Pi_A B)^I & \xrightarrow{r} & \Pi_{A_\epsilon} B_\epsilon
 \end{array} \tag{64}$$

- For  $\varphi$ , we require a map

$$\varphi : (\Pi_A B)^I \rightarrow \Pi_{A^I} B^I \quad \text{over } X^I.$$

Consider the following diagram, which is based on (62).

$$\begin{array}{ccccccc}
 B^I & \xrightarrow{q} & p^* B_\epsilon & \longrightarrow & B_\epsilon & \longrightarrow & B \\
 & \searrow & \downarrow & & \downarrow & & \downarrow \\
 & & A^I & \xrightarrow{p} & A_\epsilon & \longrightarrow & A \\
 & \nearrow & \downarrow & & \downarrow & & \downarrow \\
 (\Pi_A B \times_X A)^I & & X^I & \xrightarrow{\epsilon} & X & \longleftarrow & \Pi_A B \\
 & \nearrow & \uparrow & & & & \uparrow \\
 & & (\Pi_A B)^I & \xrightarrow{\varphi} & \Pi_{A^I} B^I & & 
 \end{array} \tag{65}$$

The map  $c$  is the counit at  $B \rightarrow A$  of the pullback-pushforward adjunction along  $A \rightarrow X$ . The right-hand side of the diagram, including  $c$  and the associated pullback square, reappear on the left under the functor  $(-)^I$ , which preserves the pullback. Thus we can take  $\varphi$  to be the transpose of  $c^I$  under the pullback-pushforward adjunction along  $A^I \rightarrow X^I$ ,

$$\varphi = \tilde{c}^I.$$

A diagram chase involving the pullback-pushforward adjunction along  $A_\epsilon \rightarrow X^I$  shows that the upper square in (64) commutes.

- For  $\vartheta$ : referring to the diagram (62), since  $p : A^I \rightarrow A_\epsilon$  is a trivial fibration, it has a section  $o : A_\epsilon \rightarrow A^I$  by lemma 10. Pulling  $p^*B_\epsilon \rightarrow A^I$  back along  $o$  results in an iso over  $A_\epsilon$ ,

$$o^*p^*B_\epsilon \cong B_\epsilon,$$

and so by the adjunction  $o^* \dashv o_*$  there is a map over  $A^I$ ,

$$p^*B_\epsilon \rightarrow o_*B_\epsilon,$$

to which we can apply  $p_*$  to obtain a map,

$$\rho : p_*p^*B_\epsilon \rightarrow p_*o_*B_\epsilon \cong B_\epsilon \quad \text{over } A_\epsilon.$$

This is a retraction of the unit  $\eta : B_\epsilon \rightarrow p_*p^*B_\epsilon$  over  $A_\epsilon$ . Applying the functor  $\Pi_{A_\epsilon}$  therefore gives the desired retraction

$$\vartheta = \Pi_{A_\epsilon}\rho : \Pi_{A_\epsilon}p_*p^*B_\epsilon \rightarrow \Pi_{A_\epsilon}B_\epsilon$$

of  $s$ .

- For  $\psi$ , we require a map

$$\psi : \Pi_{A^I}B^I \rightarrow (\Pi_AB)^I \quad \text{over } X^I.$$

Consider the following diagram resulting from combining (62) and (64), and in which all solid arrows are those already introduced. The dotted arrow labelled  $\alpha$  is the evident composite.

$$\begin{array}{ccccc}
 & & X^I & \xrightarrow{\epsilon} & X \\
 & \nearrow & \uparrow & & \uparrow \\
 (\Pi_AB)^I & \longrightarrow & \Pi_{A_\epsilon}B_\epsilon & \longrightarrow & \Pi_AB \\
 \downarrow & & \downarrow & & \downarrow \\
 \Pi_{A^I}B^I & \longrightarrow & \Pi_{A_\epsilon}p_*p^*B_\epsilon & & = \\
 & \searrow \text{dotted } \alpha & \downarrow & & \downarrow \\
 (\Pi_AB)^I & \longrightarrow & \Pi_{A_\epsilon}B_\epsilon & \longrightarrow & \Pi_AB
 \end{array} \tag{66}$$

Now recall that we are working in the slice category over  $I$ , and the objects  $\Pi_{A^I}B^I$ ,  $\Pi_AB$ , and  $(\Pi_AB)^I$  are in the image of the base change  $I^*$ , and so are actually of the form  $I^*\Pi_{A^I}B^I$ ,  $I^*\Pi_AB$ , and  $I^*((\Pi_AB)^I)$ . Indeed, the latter is

$$I^*((\Pi_AB)^I) = I^*I_*I^*\Pi_AB.$$

Since the lower horizontal map is the counit  $\varepsilon$  of the base change  $I^* \dashv I_*$ , the map  $\alpha$  factors as  $\varepsilon \circ I^* \tilde{\alpha}$ , where  $\tilde{\alpha}$  is the adjoint transpose of  $\alpha$ , as shown in the following.

$$\begin{array}{ccc}
 I^* \Pi_{A^!} B^! & \longrightarrow & \Pi_{A_\epsilon} p_* p^* B_\epsilon \\
 I^* \tilde{\alpha} \downarrow & \searrow \alpha & \downarrow \\
 I^* I_* I^* \Pi_A B & \longrightarrow & \Pi_{A_\epsilon} B_\epsilon \longrightarrow I^* \Pi_A B \\
 & \searrow \varepsilon & \\
 & & 
 \end{array} \tag{67}$$

We set  $\psi = I^* \tilde{\alpha}$ , making the square commute.

We have now defined all the maps below, the squares involving  $\varphi$  and  $\psi$  commute, and the composite of  $\vartheta$  and  $s$  is the identity.

$$\begin{array}{ccccc}
 & & X^! & \longrightarrow & X \\
 & \nearrow & \uparrow & & \uparrow \\
 (\Pi_A B)^! & \longrightarrow & \Pi_{A_\epsilon} B_\epsilon & \longrightarrow & \Pi_A B \\
 \varphi \downarrow & & \downarrow s & & \downarrow \\
 \Pi_{A^!} B^! & \longrightarrow & \Pi_{A_\epsilon} p_* p^* B_\epsilon & & = \\
 \psi \downarrow & & \downarrow \vartheta & & \downarrow \\
 (\Pi_A B)^! & \longrightarrow & \Pi_{A_\epsilon} B_\epsilon & \longrightarrow & \Pi_A B \\
 & \searrow \varepsilon & & & 
 \end{array} \tag{68}$$

To see that  $\psi \circ \varphi = 1$ , observe that each map is in the image of  $I^*$ , say:

$$\begin{aligned}
 \varphi &= I^* f \\
 \psi &= I^* g,
 \end{aligned}$$

where  $g = \tilde{\alpha}$ . Recall that in general the unit  $\varepsilon$  satisfies,

$$\varepsilon \circ I^*(h) = \tilde{h}$$

for any map  $h : X \rightarrow I_* Y$ . Thus

$$\begin{aligned}
 \varepsilon \circ \psi \circ \varphi &= \varepsilon \circ I^* g \circ I^* f \\
 &= \varepsilon \circ I^*(g \circ f) \\
 &= \widetilde{(g \circ f)}.
 \end{aligned}$$

On the other hand, a diagram chase on (68) shows that

$$\varepsilon \circ \psi \circ \varphi = \varepsilon.$$

Therefore  $g \circ f = \tilde{\varepsilon} = 1$ , so  $\psi \circ \varphi = I^*g \circ I^*f = I^*(g \circ f) = I^*1 = 1$ . □

## 6 The universe

In this section, we define a universal small fibration  $\dot{\mathcal{U}} \rightarrow \mathcal{U}$ . In section 8 we will show that the base  $\mathcal{U}$  is a fibrant object, using the equivalence extension property of section 7.

### 6.1 Classifying families

Let  $\kappa$  be an inaccessible cardinal number, and call the sets of size strictly less than  $\kappa$  *small*. Write  $\mathbf{Set}_\kappa$  for the category of small sets and  $\mathbf{cSet}_\kappa = \mathbf{Set}_\kappa^{\mathbf{C}^{\text{op}}}$  for the category of small set valued presheaves on the cube category  $\mathbf{C}$ . By a *small fibration* we mean a fibration in the category of small cubical sets, which we identify with the evident subcategory  $\mathbf{cSet}_\kappa \subseteq \mathbf{cSet}$ . Finally, let  $\dot{\mathbf{Set}}_\kappa$  be the category of small pointed sets, i.e. the coslice category  $1/\mathbf{Set}_\kappa$ . There is an evident forgetful functor  $U : \dot{\mathbf{Set}}_\kappa \rightarrow \mathbf{Set}_\kappa$ .

**Definition 57.** The  $(\kappa\text{-})\text{universe } p : \dot{\mathcal{V}} \rightarrow \mathcal{V}$  in  $\mathbf{cSet}$  is defined:

1.  $\mathcal{V}_n = \{A : \mathbf{C}/[n] \rightarrow \mathbf{Set}_\kappa^{\text{op}}\}$ , the *set* of small presheaves on  $\mathbf{C}/[n]$ .

The action of a map  $h : [m] \rightarrow [n]$  in  $\mathbf{C}$  is given by precomposition with postcomposition: from  $h : [m] \rightarrow [n]$  we have  $\mathbf{C}/h : \mathbf{C}/[m] \rightarrow \mathbf{C}/[n]$ , which we precompose with any  $A : \mathbf{C}/[n] \rightarrow \mathbf{Set}_\kappa^{\text{op}}$  to get  $A.h = A \circ \mathbf{C}/h$ ,

$$\begin{array}{ccc} [n] & \mathbf{C}/[n] & \xrightarrow{A} \mathbf{Set}_\kappa^{\text{op}} \\ h \uparrow & \mathbf{C}/h \uparrow & \nearrow A.h \\ [m] & \mathbf{C}/[m] & \end{array} \quad (69)$$

2.  $\dot{\mathcal{V}}_n = \{a : \mathbf{C}/[n] \rightarrow \dot{\mathbf{Set}}_\kappa^{\text{op}}\}$ , the *set* of small pointed presheaves on  $\mathbf{C}/[n]$ , with the corresponding action.

3. For  $a \in \dot{\mathcal{V}}_n$ , let  $p_n(a) = U(a) \in \mathcal{V}_n$ , where  $U : \dot{\mathbf{Set}}_\kappa \rightarrow \mathbf{Set}_\kappa$ .

Functoriality of  $\mathcal{V}$  and  $\dot{\mathcal{V}}$  and naturality of  $p : \dot{\mathcal{V}} \rightarrow \mathcal{V}$  are immediate.

**Lemma 58.** *For each  $A : \mathbf{I}^n \rightarrow \mathcal{V}$  there is a canonical choice of a small family  $p_A : E_A \rightarrow \mathbf{I}^n$  and a map  $q_A : E_A \rightarrow \dot{\mathcal{V}}$  making a pullback square as follows.*

$$\begin{array}{ccc} E_A & \xrightarrow{q_A} & \dot{\mathcal{V}} \\ p_A \downarrow \lrcorner & & \downarrow p \\ \mathbf{I}^n & \xrightarrow[A]{} & \mathcal{V} \end{array} \quad (70)$$

*Proof.* Since  $\mathbf{I}^n \cong y[n]$  is representable, there is a distinguished associated presheaf  $A : (\mathbb{C}/[n])^{op} \rightarrow \mathbf{Set}_\kappa$ . Define  $p_A : E_A \rightarrow \mathbf{I}^n$  by

$$(E_A)_k = \coprod_{h \in \mathbb{C}(k,n)} A(h) \quad \ni (h, a)$$

with first projection  $(p_A)_k(h, a) = h$ . Note that  $(E_A)_k$  is small. Then let  $q_A : E_A \rightarrow \dot{\mathcal{V}}$  be defined on  $(h, a) : \mathbf{I}^k \rightarrow E_A$  by

$$(q_A) \circ (h, a) = a \in Ah$$

as illustrated below.

$$\begin{array}{ccccc} & & E_A & \xrightarrow{\quad q_A \quad} & \dot{\mathcal{V}} \\ & \nearrow (h,a) & \downarrow p_A & \nearrow a & \downarrow p \\ \mathbf{I}^k & \xrightarrow[h]{} & \mathbf{I}^n & \xrightarrow[A]{} & \mathcal{V} \end{array} \quad (71)$$

The proof that the square is a pullback is left to the reader.  $\square$

**Lemma 59.** *For each small family  $p_E : E \rightarrow \mathbf{I}^n$  there is a canonical map  $\chi_E : \mathbf{I}^n \rightarrow \mathcal{V}$  and a map  $q_E : E \rightarrow \dot{\mathcal{V}}$  making a pullback square as follows.*

$$\begin{array}{ccc} E & \xrightarrow{q_E} & \dot{\mathcal{V}} \\ p_E \downarrow \lrcorner & & \downarrow p \\ \mathbf{I}^n & \xrightarrow[\chi_E]{} & \mathcal{V} \end{array} \quad (72)$$

*Proof.* It suffices to give a small set  $(\chi_E)_k(h)$  for each  $h : [k] \rightarrow [n]$  in a way that is functorial in  $h \in \mathbb{C}/[n]$  and natural in  $[k]$ . Thus let

$$(\chi_E)_k(h) := \Gamma(h, E) = \{e : \mathbf{I}^k \rightarrow E \mid p_E \circ e = h\}.$$

$$\begin{array}{ccc} & E & \\ & \downarrow p_E & \\ \mathbf{I}^k & \xrightarrow{h} & \mathbf{I}^n \end{array}$$

*(Note: A dotted arrow labeled  $e$  points from  $\mathbf{I}^k$  to  $E$  in the original diagram.)*

which is small if each  $E_k$  is.

To define  $q_E : E \rightarrow \dot{\mathcal{V}}$ , take any  $e : \mathbf{I}^k \rightarrow E$  and first compose with  $p_E$  and observe that  $e \in \Gamma(p_E e, E)$ . Thus the assignment gives a map  $\dot{e} : \mathbf{I}^k \rightarrow \dot{\mathcal{V}}$  making the solid arrows in the following commute.

$$\begin{array}{ccccc} & & E & \xrightarrow{q_E} & \dot{\mathcal{V}} \\ & \nearrow e & \downarrow p_E & \nearrow \dot{e} & \downarrow p \\ \mathbf{I}^k & \xrightarrow{p_E e} & \mathbf{I}^n & \xrightarrow{E} & \dot{\mathcal{V}} \end{array} \quad (73)$$

Since the assignment of  $\dot{e}$  to  $e$  is natural in  $[k]$ , we get the required map  $q_E : E \rightarrow \dot{\mathcal{V}}$ . The proof that the square is a pullback is again left to the reader.  $\square$

**Corollary 60.** *Given a small family  $p_E : E \rightarrow \mathbf{I}^n$  there is a unique isomorphism  $E \cong E_{\chi_E}$  over  $\mathbf{I}^n$  making a commutative diagram as follows.*

$$\begin{array}{ccccc} & & E & \xrightarrow{q_E} & \dot{\mathcal{V}} \\ & \nearrow \cong & E_{\chi_E} & \xrightarrow{q_{\chi_E}} & \downarrow p \\ & \searrow p_E & \downarrow p_{\chi_E} & \nearrow \chi_E & \dot{\mathcal{V}} \\ & & \mathbf{I}^n & \xrightarrow{\chi_E} & \end{array} \quad (74)$$

**Proposition 61.** *For any cubical set  $X$  and any small family  $p_E : E \rightarrow X$  there are canonical maps  $\chi_E : X \rightarrow \mathcal{V}$  and  $q_E : E \rightarrow \dot{\mathcal{V}}$  making a pullback square as follows.*

$$\begin{array}{ccc} E & \xrightarrow{q_E} & \dot{\mathcal{V}} \\ p_E \downarrow \lrcorner & & \downarrow p \\ X & \xrightarrow{\chi_E} & \mathcal{V} \end{array} \quad (75)$$

Moreover,  $\chi_E$  and  $q_E$  are uniquely determined by the equations (77) below.

*Proof.* Write  $X = \varinjlim_x I^n$  as a colimit of a cocone of maps  $x : I^n \rightarrow X$  from representables, over the canonical index category  $([n], x) \in \int_{\mathbb{C}} X$ . Form the family of pullback squares below, where the arrows with a dot represent cocones, and the cocone consisting of the  $q_x : E_x \rightarrow E$  is determined by taking pullbacks along  $p_E$ , and is therefore also a colimit.

$$\begin{array}{ccccc} & & q_{E_x} & & \\ & \curvearrowright & & \curvearrowright & \\ E_x & \xrightarrow{q_x} & E & \xrightarrow{q_E} & \dot{\mathcal{V}} \\ p_{E_x} \downarrow \lrcorner & & \downarrow p_E & & \downarrow p \\ I^n & \xrightarrow{x} & X & \xrightarrow{\chi_E} & \mathcal{V} \\ & \curvearrowleft & & \curvearrowleft & \\ & & \chi_{E_x} & & \end{array} \quad (76)$$

The maps  $\chi_{E_x}$  and  $q_{E_x}$  are determined by lemma 59, since the families  $p_{E_x}$  are small if  $p_E : E \rightarrow X$  is. Thus we can define the indicated maps  $\chi_E$  and  $q_E$  from the colimits as those uniquely determined by the equations:

$$\chi_E \circ x = \chi_{E_x} \quad (77)$$

$$q_E \circ q_x = q_{E_x} \quad (78)$$

The square on the right is a pullback because the outer squares are all pullbacks, the family of left-hand squares are pullbacks, and the family of maps  $x : I^n \rightarrow X$  covers  $X$ .  $\square$

*Remark 62.* Note that the classification operation

$$\chi : \mathbf{cSet}_{\kappa}/X \rightarrow \mathbf{cSet}(X, \mathcal{V})$$



again has the evident “pullback of  $p : \dot{\mathcal{V}} \rightarrow \mathcal{V}$ ” operation

$$E : \mathbf{cSet}(X, \mathcal{V}) \rightarrow \mathbf{cSet}_\kappa/X$$

as a left (quasi-)inverse  $E \cong E_{\chi_E}$ , which is (pseudo-)natural in  $X$ . But there is no corresponding uniqueness of classifying maps, relating  $A : X \rightarrow \mathcal{V}$  and  $\chi_{E_A} : X \rightarrow \mathcal{V}$ . (This is what is provided, in a suitable sense, by the *univalence* of the universe  $\dot{\mathcal{U}} \rightarrow \mathcal{U}$  of fibrations, to be established in section 7.)

## 6.2 Classifying trivial fibrations

Recall from section 2 that (uniform) trivial fibration structures on a map  $A \rightarrow X$  correspond bijectively to relative  $+$ -algebra structures over  $X$  (definition 6). A relative  $+$ -algebra structure on  $A \rightarrow X$  is an algebra structure for the pointed endofunctor  $+_X : \mathbf{cSet}/X \rightarrow \mathbf{cSet}/X$ , where

$$A^+ = \sum_{\varphi : \Phi} A^\varphi \quad \text{over } X.$$

A  $+$ -algebra structure is then a retract  $\alpha : A^+ \rightarrow A$  over  $X$  of the canonical map  $\eta_A : A \rightarrow A^+$ ,

$$\begin{array}{ccccc} & & \overset{=}{\curvearrowright} & & \\ A & \xrightarrow{\eta_A} & A^+ & \xrightarrow{\alpha} & A \\ & \searrow & \downarrow & \swarrow & \\ & & X & & \end{array}$$

In more detail, let us write  $A \rightarrow X$  as a family  $\sum_{x:X} A_x \rightarrow X$  over  $X$ . Since the  $+$ -functor acts fiberwise, the  $A^+$  in the diagram above is then the indexing projection

$$\sum_{x:X} A_x^+ \rightarrow X.$$

Working in the slice  $\mathbf{cSet}/X$ , we make the (relative) exponentials (internal Hom’s)  $[A^+, A]$  and  $[A, A]$  with the “precomposition by  $\eta_A$ ” map  $[\eta_A, A]$ , which fit into the following pullback diagram

$$\begin{array}{ccc} +\mathbf{Alg}(A) & \longrightarrow & [A^+, A] \\ \downarrow \lrcorner & & \downarrow [\eta_A, A] \\ 1 & \xrightarrow{\quad 'id_A'\quad} & [A, A]. \end{array}$$

The constructed object  $+Alg(A) \rightarrow X$  over  $X$  is the *object of +-algebra structures on  $A \rightarrow X$* , in the sense that sections  $X \rightarrow +Alg(A)$  correspond isomorphically to +-algebra structures on  $A \rightarrow X$ . Moreover,  $+Alg(A) \rightarrow X$  is stable under pullback in the sense that for any  $f : Y \rightarrow X$ , we have two pullback squares,

$$\begin{array}{ccc} f^*A & \longrightarrow & A \\ \downarrow & & \downarrow \\ Y & \xrightarrow{f} & X \\ \uparrow & & \uparrow \\ +Alg(f^*A) & \longrightarrow & +Alg(A) \end{array} \quad (79)$$

because the +-functor, exponentials and pullbacks occurring in the construction of  $+Alg(A) \rightarrow X$  are themselves all stable.

It follows from proposition 61 that if  $A \rightarrow X$  is small, then  $+Alg(A) \rightarrow X$  is itself a pullback of the analogous object  $+Alg(\dot{\mathcal{V}}) \rightarrow \mathcal{V}$  constructed from the universal small family  $\dot{\mathcal{V}} \rightarrow \mathcal{V}$ ,

$$\begin{array}{ccc} A & \longrightarrow & \dot{\mathcal{V}} \\ \downarrow & & \downarrow \\ X & \xrightarrow{\chi_A} & \mathcal{V} \\ \uparrow & & \uparrow \\ +Alg(A) & \longrightarrow & +Alg(\dot{\mathcal{V}}) \end{array} \quad (80)$$

**Proposition 63.** *There is a universal small trivial fibration*

$$TFib \rightarrow TFib.$$

*Every small trivial fibration  $A \rightarrow X$  is a pullback of  $TFib \rightarrow TFib$  along a canonically determined classifying map  $X \rightarrow TFib$ .*

$$\begin{array}{ccc} A & \longrightarrow & TFib \\ \downarrow \lrcorner & & \downarrow \\ X & \longrightarrow & TFib \end{array} \quad (81)$$

*Proof.* We can take  $TFib = +Alg(\dot{\mathcal{V}})$ , which comes with its projection  $+Alg(\dot{\mathcal{V}}) \rightarrow$

$\mathcal{V}$  as in diagram (80). Now define  $p_t : \mathbf{TFib} \rightarrow \mathbf{TFib}$  by pulling back the universal small family,

$$\begin{array}{ccc} \mathbf{TFib} & \longrightarrow & \dot{\mathcal{V}} \\ p_t \downarrow & \lrcorner & \downarrow p \\ \mathbf{TFib} & \longrightarrow & \mathcal{V}. \end{array}$$

Consider the following diagram, in which all the squares (including the distorted ones) are pullbacks, with the outer one coming from proposition 61 and the lower one from (80).

$$\begin{array}{ccccc} A & \xrightarrow{q_A} & \dot{\mathcal{V}} & & \\ & \searrow & \downarrow p & & \\ & & \mathbf{TFib} & \xrightarrow{p_t} & \mathbf{TFib} \\ & \swarrow & \downarrow p & & \\ \mathbf{TFib}(A) & \xrightarrow{\quad} & \mathbf{TFib} & \xrightarrow{\quad} & \mathcal{V} \\ & \searrow \alpha & \swarrow \alpha' & & \\ & & X & \xrightarrow{\chi_A} & \mathcal{V} \end{array} \quad (82)$$

A trivial fibration structure  $\alpha$  on  $A \rightarrow X$  is a section the object of  $+$ -algebra structures on  $A$ , occurring in the diagram as  $\mathbf{TFib}(A)$ , the pullback of  $\mathbf{TFib}$ . Such sections correspond uniquely to factorizations  $\alpha'$  of  $\chi_A$  as indicated, which in turn induce pullback squares of the required kind (81).

Note that the map  $p_t : \mathbf{TFib} \rightarrow \mathbf{TFib}$  has a canonical trivial fibration structure. Indeed, consider the following diagram, in which both squares are pullbacks.

$$\begin{array}{ccc} \mathbf{TFib} & \longrightarrow & \dot{\mathcal{V}} \\ p_t \downarrow & & \downarrow \\ \mathbf{TFib} & \longrightarrow & \mathcal{V} \\ \uparrow & & \uparrow \\ \mathbf{TFib}(\mathbf{TFib}) & \longrightarrow & \mathbf{TFib}(\dot{\mathcal{V}}) \end{array} \quad (83)$$

$\mathbf{TFib}(\dot{\mathcal{V}})$  is the object of trivial fibration structures on  $\dot{\mathcal{V}} \rightarrow \mathcal{V}$ , and its pullback  $\mathbf{TFib}(\mathbf{TFib})$  is therefore the object of trivial fibration structures on  $p_t : \mathbf{TFib} \rightarrow \mathbf{TFib}$ . Thus we seek a section of  $\mathbf{TFib}(\mathbf{TFib}) \rightarrow \mathbf{TFib}$ . But recall

that  $\mathbf{TFib} = \mathbf{TFib}(\dot{\mathcal{V}})$  by definition, so the lower pullback square is the pullback of  $\mathbf{TFib}(\dot{\mathcal{V}}) \rightarrow \mathcal{V}$  against itself, which does indeed have a distinguished section, namely the diagonal

$$\Delta : \mathbf{TFib}(\dot{\mathcal{V}}) \rightarrow \mathbf{TFib}(\dot{\mathcal{V}}) \times_{\mathcal{V}} \mathbf{TFib}(\dot{\mathcal{V}}).$$

□

### 6.3 Classifying fibrations

In order to classify fibrations  $A \rightarrow X$ , we shall proceed as for trivial fibrations by constructing an object  $\mathbf{Fib}(A) \rightarrow X$  of fibration structures on  $A \rightarrow X$  which, moreover, is stable under pullback. We then apply the construction to the universal small family  $\dot{\mathcal{V}} \rightarrow \mathcal{V}$  to get a universal small fibration.

The construction of  $\mathbf{Fib}(A) \rightarrow X$  is a bit more involved than that of  $\mathbf{TFib}(A) \rightarrow X$ . Recall from section ?? the characterization of (uniform, unbiased) fibration structures on a map  $p_A : A \rightarrow X$  in terms of  $+$ -algebra structures:

1. First, pass to  $\mathbf{cSet}/\mathbf{I}$  where there is a generic point  $\delta : 1 \rightarrow \mathbf{I}$ ,
2. Form the pullback-hom  $\delta \Rightarrow p_A : A^{\mathbf{I}} \rightarrow X^{\mathbf{I}} \times_X A$  as indicated in the following diagram.

$$\begin{array}{ccc}
 A^{\mathbf{I}} & \xrightarrow{A^\delta} & A \\
 \delta \Rightarrow p_A \searrow & & \downarrow p_A \\
 X^{\mathbf{I}} \times_X A & \longrightarrow & A \\
 \downarrow & & \downarrow p_A \\
 X^{\mathbf{I}} & \xrightarrow{X^\delta} & X
 \end{array}
 \quad (84)$$

$(p_A)^{\mathbf{I}}$  (curved arrow from  $A^{\mathbf{I}}$  to  $X^{\mathbf{I}}$ )

3. A fibration structure on  $p_A : A \rightarrow X$  is then a relative  $+$ -algebra structure on  $\delta \Rightarrow p_A$  in the slice category over the codomain.

We next construct an object  $\mathbf{Fib}(A) \rightarrow X$  classifying such structures. For convenience, let us relabel the objects and arrows in the previous

diagram as follows:

$$\begin{aligned}
p &:= p_A \\
\epsilon &:= X^\delta : X^I \rightarrow X \\
A_\epsilon &:= X^I \times_X A \\
\epsilon^A &:= \delta \Rightarrow p_A
\end{aligned}$$

so that (the working part of) our diagram becomes:

$$\begin{array}{c}
A^I \\
\searrow \epsilon^A \\
\begin{array}{ccc}
A_\epsilon & \longrightarrow & A \\
p_\epsilon \downarrow \lrcorner & & \downarrow p \\
X^I & \xrightarrow{\epsilon} & X
\end{array}
\end{array} \tag{85}$$

4. A  $+$ -algebra structure on  $\epsilon^A$  is a retract  $\alpha$  over  $A_\epsilon$  of the unit  $\eta$  as indicated below, where  $D$  is the domain of the map  $(\epsilon^A)^+$ :

$$\begin{array}{c}
\begin{array}{ccc}
A^I & \xrightarrow[\eta]{\alpha} & D \\
\searrow \epsilon^A & & \downarrow (\epsilon^A)^+ \\
A_\epsilon & \longrightarrow & A \\
p_\epsilon \downarrow \lrcorner & & \downarrow p \\
X^I & \xrightarrow{\epsilon} & X
\end{array}
\end{array} \tag{86}$$

5. Thus, as in the previous section, there is an object  $+\mathbf{Alg}(\epsilon^A)$  over  $A_\epsilon$  of  $+$ -algebra structures on  $\epsilon^A$ , the sections of which correspond uniquely to  $+$ -algebra structures on  $\epsilon^A$  (and thus fibration structures on  $A$ ).

$$\begin{array}{c}
\begin{array}{ccc}
A^I & \xrightarrow[\eta]{\alpha} & D \\
\epsilon^A \downarrow & \nearrow (\epsilon^A)^+ & \\
+\mathbf{Alg}(\epsilon^A) \longrightarrow A_\epsilon & \longrightarrow & A \\
p_\epsilon \downarrow \lrcorner & & \downarrow p \\
X^I & \xrightarrow{\epsilon} & X
\end{array}
\end{array} \tag{87}$$

6. Sections of  $+Alg(\epsilon^A) \rightarrow A_\epsilon$  correspond to sections of its push-forward along  $p_\epsilon$ , which we shall call  $F_A$ :

$$F_A := (p_\epsilon)_*(+Alg(\epsilon^A)).$$

$$\begin{array}{ccccc}
 & & A^I & \xrightarrow[\eta]{\alpha} & D \\
 & & \downarrow \epsilon^A & \swarrow (\epsilon^A)^+ & \\
 +Alg(\epsilon^A) & \longrightarrow & A_\epsilon & \longrightarrow & A \\
 & & \downarrow p_\epsilon & \lrcorner & \downarrow p \\
 F_A & \longrightarrow & X^I & \xrightarrow{\epsilon} & X
 \end{array} \tag{88}$$

7. We might now think of taking another pushforward of  $F_A \rightarrow X^I$  along  $\epsilon : X^I \rightarrow X$  to get the object  $Fib(A) \rightarrow X$  that we seek, but unfortunately, this would not be stable under pullback along arbitrary maps  $Y \rightarrow X$ , because  $\epsilon : X^I \rightarrow X$  is not stable in this way. Instead we will use the *root* functor, i.e. the “amazing right adjoint” to the pathspace (see [?]).

$$(-)^I \dashv (-)_I$$

Let  $f : F_A \rightarrow X^I$  be the map indicated in (88), and let  $\eta_X : X \rightarrow (X^I)_I$  be the unit of the root adjunction. Then define  $Fib(A) \rightarrow X$  as the following pullback.

$$\begin{array}{ccc}
 Fib(A) & \longrightarrow & (F_A)_I \\
 \downarrow \lrcorner & & \downarrow f_I \\
 X & \xrightarrow[\eta]{} & (X^I)_I
 \end{array} \tag{89}$$

By adjointness, sections of  $Fib(A) \rightarrow X$  correspond uniquely to sections of  $f : F_A \rightarrow X^I$ .

8. Finally, we are still working in the slice  $\mathbf{cSet}/I$  and need to get back to  $\mathbf{cSet}$  by applying the functor  $I_* : \mathbf{cSet}/I \rightarrow \mathbf{cSet}$ . Call the map  $Fib(A) \rightarrow X$  constructed over  $I$  in the last step  $Fib(A)_i \rightarrow I^*X$  and apply  $I_*$  to get,

$$I_*(Fib(A)_i) = \Pi_{i:I} Fib(A)_i \rightarrow X^I$$

in  $\mathbf{cSet}$ . We then define the desired map  $\mathbf{Fib}(A) \rightarrow X$  as the pullback along the unit  $\rho : X \rightarrow X^I$  of  $I^* \dashv I_*$ , as indicated below.

$$\begin{array}{ccc} \mathbf{Fib}(A) & \longrightarrow & \prod_{i:I} \mathbf{Fib}(A)_i \\ \downarrow \lrcorner & & \downarrow \\ X & \xrightarrow{\rho} & X^I \end{array} \quad (90)$$

It follows directly from the adjunction  $I^* \dashv I_*$  that sections of  $\mathbf{Fib}(A) \rightarrow X$  correspond bijectively to sections of  $\mathbf{Fib}(A)_i \rightarrow I^*X$  over  $I$ .

**Proposition 64.** *The map*

$$\mathbf{Fib} \rightarrow \mathbf{Fib}$$

*just constructed is a universal small fibration. Every small fibration  $A \rightarrow X$  is a pullback of  $\mathbf{Fib} \rightarrow \mathbf{Fib}$  along a canonically determined classifying map  $X \rightarrow \mathbf{Fib}$ .*

$$\begin{array}{ccc} A & \longrightarrow & \mathbf{Fib} \\ \downarrow \lrcorner & & \downarrow \\ X & \longrightarrow & \mathbf{Fib} \end{array} \quad (91)$$

*Proof.* First, we need to show that the construction of  $\mathbf{Fib}(A) \rightarrow X$  as the object of fibration structures on a map  $A \rightarrow X$  is stable under pullback along all maps  $f : Y \rightarrow X$ . The relevant parts of the construction diagram (92) are repeated below,

$$\begin{array}{ccccc} & & A^I & & \\ & & \downarrow \epsilon^A & & \\ +\mathbf{Alg}(\epsilon^A) & \longrightarrow & A_\epsilon & \longrightarrow & A \\ & & \downarrow p_\epsilon \lrcorner & & \downarrow p \\ F_A & \longrightarrow & X^I & \xrightarrow{\epsilon} & X \end{array} \quad (92)$$

Now consider the following in which the front face of the central cube is a

pullback.

$$\begin{array}{ccccc}
 B^I & \xrightarrow{\quad} & A^I & & \\
 \downarrow \epsilon^B & & \downarrow \epsilon^A & & \\
 +\mathbf{Alg}(\epsilon^B) \longrightarrow B_\epsilon & \xrightarrow{\quad \cdots \quad} & A_\epsilon \longleftarrow +\mathbf{Alg}(\epsilon^A) & & \\
 \downarrow & \searrow & \swarrow & \downarrow & \\
 F_B \longrightarrow Y^I & & B \longrightarrow A & & X^I \longleftarrow F_A \\
 & \searrow & \downarrow & \swarrow & \\
 \mathbf{Fib}(B) \longrightarrow Y & \xrightarrow{f} & X & \longleftarrow & \mathbf{Fib}(A)
 \end{array} \tag{93}$$

On the left side we will repeat the construction with  $B \rightarrow Y$  in place of  $A \rightarrow X$ . The left face is thus a pullback, whence the back (dotted) face is a pullback. The two-story square in back is the image of the front square under the right adjoint  $(-)^I$  and is therefore a pullback, therefore the top rectangle in the back is a pullback. It follows that  $+\mathbf{Alg}(\epsilon^B)$  is a pullback of  $+\mathbf{Alg}(\epsilon^A)$  along the upper dotted arrow, as in diagram (79), and so the pushforward  $F_B$  is a pullback of the corresponding  $F_A$ , along the lower dotted arrow (which is  $f^I$ ), by the Beck-Chevalley condition. Thus we have shown

$$F_B \cong (f^I)^* F_A. \tag{94}$$

It remains to show that  $\mathbf{Fib}(B)$  is a pullback of  $\mathbf{Fib}(A)$  along  $f : Y \rightarrow X$ , and now it is good that we did not take these to be pushforwards of  $F_B$  and  $F_A$ , because the floor of the cube is not a pullback, and so the Beck-Chavalley condition would not apply. Instead, consider the following diagram.

$$\begin{array}{ccccc}
 & \mathbf{Fib}(B) & \longrightarrow & \mathbf{Fib}(A) & \\
 & \swarrow & & \searrow & \\
 (F_B)_I & \xrightarrow{\quad} & (F_A)_I & & \\
 \downarrow & & \downarrow & & \downarrow \\
 & Y & \xrightarrow{f} & X & \\
 \swarrow \eta & & \searrow \eta & & \\
 (Y^I)_I & \xrightarrow{(f^I)_I} & (X^I)_I & & 
 \end{array} \tag{95}$$



The sides of the cube are pullbacks by the construction of  $\mathbf{Fib}(A)$  and  $\mathbf{Fib}(B)$ . The front face is the root of the pullback (94) and is thus also a pullback, since the root is a right adjoint. The base commutes by naturality of the unit, and so the back face is also a pullback as required. Finally, the base change along  $I_* : \mathbf{cSet}/I \rightarrow \mathbf{cSet}$  in step 8 above clearly also preserves the pullback.

Now we can take  $\mathbf{Fib} = \mathbf{Fib}(\dot{\mathcal{V}})$ , which comes with its projection  $\mathbf{Fib}(\dot{\mathcal{V}}) \rightarrow \mathcal{V}$ , and define the universal small fibration  $\dot{\mathbf{Fib}} \rightarrow \mathbf{Fib}$  by pulling back the universal small family,

$$\begin{array}{ccc} \mathbf{Fib} & \longrightarrow & \dot{\mathcal{V}} \\ \downarrow \lrcorner & & \downarrow p \\ \mathbf{Fib} & \longrightarrow & \mathcal{V}. \end{array}$$

The remainder of the proof is just as for proposition 63.  $\square$

**Definition 65.** Write  $\dot{\mathcal{U}} \rightarrow \mathcal{U}$  for the universal small fibration  $\dot{\mathbf{Fib}} \rightarrow \mathbf{Fib}$  constructed in proposition 64.

## 7 The equivalence extension property

The equivalence extension property (EEP) is closely related to the *univalence* of the universal fibration  $\dot{\mathcal{U}} \rightarrow \mathcal{U}$  constructed in section 6 (see [?]). We shall use it in section 8 to show that the base object  $\mathcal{U}$  is fibrant, which implies the fibration extension property. Our proof of the EEP is a reformulation of a type-theoretic argument due to Coquand [CCHM16], which in turn is a modification of the original argument of Voevodsky [?]. See [Sat17] for another reformulation.

We first recall some basic facts and make some simple observations. For any map  $f : X \rightarrow Y$ , recall the *pathspace factorization*  $f = t \circ s$  indicated

below.

$$\begin{array}{ccccc}
 X & \xrightarrow{f} & Y & & \\
 \downarrow s & \lrcorner & \downarrow r & \searrow = & \\
 P_f & \xrightarrow{p_0^* f} & Y^I & \xrightarrow{p_1} & Y \\
 \downarrow p_f & \lrcorner & \downarrow p_0 & \nearrow t & \\
 X & \xrightarrow{f} & Y & & 
 \end{array}
 \quad (96)$$

Here  $p_0, p_1$  are the evaluations  $Y^{\delta_0}, Y^{\delta_1}$  at the endpoints  $\delta_0, \delta_1 : 1 \rightarrow I$ , and let  $r := Y^!$  for  $! : I \rightarrow 1$ . Note that  $p_0 r = p_1 r = 1_Y$ . Then let  $p_f := f^* p_0 : P_f \rightarrow Y$ , the pullback of  $p_0$  along  $f$ , and  $s := f^* r : X \rightarrow P_f$  (over  $X$ ). Finally, let  $t := p_1 \circ p_0^* f : P_f \rightarrow Y$  be the indicated horizontal composite.

We make the following well-known observations.

1. If  $f : X \rightarrow Y$  is over a base  $Z$ , then the factorization  $t \circ s : X \rightarrow P_f \rightarrow Y$  is stable under pullback along any map  $g : Z' \rightarrow Z$ , in the sense that  $g^* P_f = P_{g^* f} : g^* X \rightarrow g^* Y$ , and similarly for  $g^* s$  and  $g^* t$ . Note that in this case we form the pathspace  $Y^I$  as an exponential in the slice category over  $Z$ .
2. The retraction  $p_0 \circ r = 1_Y$  pulls back along  $f$  to a retraction  $p_f \circ s = 1_X$ .
3. If  $Y$  is fibrant (either as an object, or over a base  $Y \rightarrow Z$ ), then  $p_0 : Y^I \rightarrow Y$  is a trivial fibration (as is  $p_1$ ). In that case, its pullback  $p_f : P_f \rightarrow X$  is also a trivial fibration.
4. If  $X$  and  $Y$  are both fibrant, then  $t = p_1 \circ p_0^* f : P_f \rightarrow Y$  is a fibration. This can be seen by factoring the maps  $p_0, p_1 : Y^I \rightrightarrows Y$  through the product projections as  $\pi_0 \circ p, \pi_1 \circ p : Y^I \rightarrow Y \times Y \rightrightarrows Y$ , with  $p = (p_0, p_1)$ , and then interpolating the pullback along the map  $(f, 1_X) : X \times Y \rightarrow$

$Y \times Y$  into (96) as indicated below.

$$\begin{array}{ccccc}
 X & \xrightarrow{f} & Y & & \\
 \downarrow s & \lrcorner & \downarrow r & & \\
 P_f & \xrightarrow{p_0^* f} & Y^I & & \\
 \swarrow t & \downarrow f^* p & \downarrow p & \searrow p_1 & \\
 Y & \xleftarrow{\pi'_1} X \times Y & \xrightarrow{(f, 1_Y)} Y \times Y & \xrightarrow{\pi_1} Y & \\
 & \downarrow \pi'_0 & \downarrow \pi_0 & & \\
 & X & \xrightarrow{f} & Y & 
 \end{array} \tag{97}$$

The second factor  $t = p_1 \circ p_0^* f : P_f \rightarrow Y$  now appears also as  $\pi_1 \circ (f, 1_Y) \circ f^* p$ , which is the pullback  $f^* p : P_f \rightarrow X \times Y$  followed by the second projection  $\pi'_1 : X \times Y \rightarrow Y$  (which is not a pullback). But if  $Y$  is fibrant, then  $p : Y^I \rightarrow Y \times Y$  is a fibration, and then so is  $f^* p$ . And if  $X$  is fibrant, then the projection  $\pi'_1 : X \times Y \rightarrow Y$  is a fibration. Thus in this case,  $t = \pi'_1 \circ f^* p : P_f \rightarrow Y$  is a fibration, as claimed.

5. Summarizing (1)-(4), for any map  $f : X \rightarrow Y$ , we have a stable factorization  $f = t \circ s : X \rightarrow P_f \rightarrow Y$ , in which  $s$  has a retraction  $p_f$ , which is a trivial fibration when  $Y$  is fibrant, and  $t$  is a fibration when both  $X$  and  $Y$  are fibrant.

$$\begin{array}{ccc}
 & p_f & \\
 & \curvearrowright & \\
 X & \xrightarrow{s} & P_f \\
 \searrow f & & \downarrow t \\
 & & Y
 \end{array} \tag{98}$$

Note that the retraction  $p_f : P_f \rightarrow X$  is not over  $Y$ .

The following simple fact concerning just the cofibration weak factorization system will also be needed.

**Lemma 66.** *Let  $p : E \twoheadrightarrow B$  be a trivial fibration and  $c : C \rightarrowtail B$  a cofibration. Then the unit  $\eta : E \rightarrow c_* c^* E$  of the base change  $c^* \dashv c_*$  along  $c$  is a trivial fibration.*

*Proof.* The unit map  $\eta : E \rightarrow c_*c^*E$  is the pullback-hom  $c \Rightarrow p$ , as is easily checked. By lemma 15, for any map  $a : A \rightarrow Z$  we have the equivalence of diagonal filling conditions,

$$a \dashv c \Rightarrow p \quad \text{iff} \quad a \otimes c \dashv p.$$

But since  $c : C \rightarrow B$  is a cofibration,  $a \otimes c$  is also a cofibration if  $a : A \rightarrow Z$  is one by axiom (C6), which says that cofibrations are closed under pushout-products. So  $a \otimes c \dashv p$  indeed holds, since  $p$  is a trivial fibration.  $\square$

**Proposition 67** (EEP). *Weak equivalences extended along cofibrations in the following sense: given a cofibration  $c : C' \rightarrow C$  and fibrations  $A' \rightarrow C'$  and  $B \rightarrow C$ , and a weak equivalence  $w' : A' \simeq c^*B$  over  $C'$ ,*

$$\begin{array}{ccccc} A' & \xrightarrow{\quad} & A & & \\ \downarrow & \searrow w' & \downarrow & \searrow w & \\ & \simeq & c^*B & \xrightarrow{\quad} & B \\ & \swarrow & \downarrow & \swarrow & \\ C' & \xrightarrow{\quad c \quad} & C & & \end{array} \quad (99)$$

*there is a fibration  $A \rightarrow C$  and a weak equivalence  $w : A \simeq B$  over  $C$  that pulls back along  $c : C' \rightarrow C$  to  $w'$ , so that  $c^*w = w'$ .*

*Proof.* Call the given fibration  $q : B \rightarrow C$  and let  $b := q^*c : c^*B \rightarrow B$  be the indicated pullback, which is thus also a cofibration. Let  $w := b_*w' : A \rightarrow B$  be the pushforward of  $w'$  along  $b$ . Composing with  $q$  gives the map  $p := q \circ w : A \rightarrow C$ . Since  $b$  is monic, we indeed have  $b^*w = w'$ , thus filling in all the dotted arrows in (99). Note moreover that  $c^*w = b^*w = w'$ , as required. It remains to show that  $p : A \rightarrow C$  is a fibration and  $w : A \rightarrow B$  is a weak equivalence.

$$\begin{array}{ccccccc} A' & \xrightarrow{\quad} & A & \xrightarrow{s} & P_w & & \\ \downarrow & \searrow w' & \downarrow & \searrow w & \downarrow t & & \\ & \simeq & c^*B & \xrightarrow{b} & B & & \\ & \swarrow & \downarrow & \swarrow & \downarrow q & & \\ C' & \xrightarrow{\quad c \quad} & C & & & & \end{array} \quad (100)$$

Let us write  $p' := c^*p : A' \rightarrow C'$  and  $B' := c^*B$  and  $q' := c^*q$ . Now let  $w = t \circ s$  be the pathspace factorization (96) of  $w$ , as a map over  $C$ . Since  $q : B \rightarrow C$  is a fibration, by the foregoing remarks on pathspace factorizations, we know that  $s : A \rightarrow P_w$  has a retraction  $p_w : P_w \rightarrow A$  which is a trivial fibration. The retraction  $p_w$  is a map over  $C$ .

The pathspace factorization  $w = t \circ s : A \rightarrow P_w \rightarrow B$  is stable under pullback along  $c$ , providing a pathspace factorization  $w' = t' \circ s' : A' \rightarrow P_{w'} \rightarrow B'$  over  $C'$ . Since both  $p'$  and  $q'$  are fibrations, the retract  $p_{w'} : P_{w'} \rightarrow A'$  is a trivial fibration, and now  $t' : P_{w'} \rightarrow B'$  is a fibration.

$$\begin{array}{ccc}
 A' & \xrightarrow{s'} & P_{w'} \\
 \downarrow p' & \searrow w' & \downarrow t' \\
 & \sim & B' \\
 & \swarrow q' & \\
 C' & & 
 \end{array}
 \quad (101)$$

Thus the composite  $q' \circ t' : P_{w'} \rightarrow B' \rightarrow C'$  is a fibration and therefore, by the retraction over  $C'$  with the trivial fibration  $p_{w'}$ , we have that  $s' : A' \rightarrow P_{w'}$  is a weak equivalence, by 3-for-2 for weak equivalences between fibrations. For the same reason,  $t'$  is then a weak equivalence, and therefore a trivial fibration.

Since  $t' = c^*t = b^*t$  is a trivial fibration, its pushforward  $b_*b^*t$  along  $b$  is also one. Moreover,  $b_*b^*t : b_*b^*P_w \rightarrow B$  admits a unit  $\eta : P_w \rightarrow b_*b^*P_w$

(over  $B$ ).

$$\begin{array}{ccccc}
 A' & \xrightarrow{\quad} & A & \xrightarrow{s} & P_w & \xrightarrow{\eta} & b_*b^*P_w \\
 \searrow w' & & \downarrow & \searrow w & \downarrow t & \nearrow b_*b^*t & \\
 & & B' & \xrightarrow{b} & B & & \\
 \downarrow & \nearrow & \downarrow p & & \downarrow q & & \\
 C' & \xrightarrow{c} & C & & & & 
 \end{array}
 \quad (102)$$

We now *claim* that  $\eta : P_w \rightarrow b_*b^*P_w$  is a trivial fibration. Given that, the composite  $t = b_*b^*t \circ \eta$  is also a trivial fibration, whence  $q \circ t : P_w \rightarrow C$  is a fibration, and so its retract  $p : A \rightarrow C$  is a fibration. Moreover, since  $s$  is a section of the trivial fibration  $p_w : P_w \rightarrow A$  between fibrations, as before it is also a weak equivalence. Thus  $w = t \circ s$  is a weak equivalence, and we are finished.

To prove the remaining claim that  $\eta : P_w \rightarrow b_*b^*P_w$  is a trivial fibration, we shall use lemma 66. But it does not apply directly since  $t : P_w \rightarrow B$  is not yet known to be a trivial fibration. Instead, we show that  $\eta$  is a pullback of the corresponding unit at the trivial fibration  $p_1 : B^I \rightarrow B$ .

Consider the following cube (viewed with  $b : B' \rightarrow B$  at the front).

$$\begin{array}{ccccc}
 P_{w'} & \xrightarrow{\bar{a}} & P_w & & \\
 \searrow (p'_0)^*w' & & \searrow p_0^*w & & \\
 & & B^I & \xrightarrow{\bar{b}} & B^I \\
 \downarrow p_{w'} & & \downarrow p_w & & \downarrow p_0 \\
 A' & \xrightarrow{a} & A & & \\
 \searrow w' & & \searrow w & & \\
 & & B' & \xrightarrow{b} & B
 \end{array}
 \quad (103)$$

The right hand face is a pullback by definition, and the remainder results from pulling the right face back along  $b$ . Thus all faces are pullbacks. The base is also a pushforward,  $b_*w' = w$ , again by definition. Thus the top is also a

pushforward,  $\bar{b}_*((p'_0)^*w') = p_0^*w$ . Indeed, since the front face is a pullback, the Beck-Chevalley condition applies and we have  $\bar{b}_*(p'_0)^*(w') = p_0^*b_*(w') = p_0^*w$ .

Now consider the following, in which the top square remains the same as in (103), but  $p_0$  has been relaxed by  $p_1 : B^I \rightarrow B$ , so the composite at right is by definition  $t = p_1 \circ p_0^*w$ .

$$\begin{array}{ccccc}
 P_{w'} & \xrightarrow{\bar{a}} & P_w & & \\
 & \searrow (p'_0)^*w' & \searrow p_0^*w & & \\
 & & B^I & \xrightarrow{\bar{b}} & B^I \\
 & \searrow t' & \downarrow p'_1 & \searrow t & \downarrow p_1 \\
 & & B' & \xrightarrow{b} & B
 \end{array} \tag{104}$$

The horizontal direction is still pullback along  $b$ ; let us rename  $p_0^*w =: u$  so that  $(p'_0)^*w' = b^*u$  and  $t' = b^*t$  and  $p'_1 = b^*p_1$  to make this clear. We then add the pushforward along  $b$  on the right, in order to obtain the two units  $\eta$ .

$$\begin{array}{ccccccc}
 b^*P_w & \xrightarrow{\bar{a}} & P_w & \xrightarrow{\eta_t} & b_*b^*P_w & & \\
 & \searrow b^*u & & \searrow u & \searrow b_*b^*u & & \\
 & & b^*B^I & \xrightarrow{\bar{b}} & B^I & \xrightarrow{\eta_{p_1}} & b_*b^*B^I \\
 & \searrow b^*t & \downarrow b^*p_1 & \searrow t & \downarrow p_1 & \searrow b_*b^*t & \downarrow b_*b^*p_1 \\
 & & B' & \xrightarrow{b} & B & \xrightarrow{=} & B
 \end{array} \tag{105}$$

By the usual calculation of pushforwards in slice categories,  $\bar{b}_* \cong \eta_{p_1}^* \circ b_*$ , and so for  $b^*u$  we have  $\bar{b}_*b^*u = \eta_{p_1}^*b_*b^*u$ . But as we just determined in (103) the top left square is already a pushforward, and therefore  $u = \eta_{p_1}^*b_*b^*u$ , so the top right naturality square is a pullback.

To finish the proof as planned,  $p_1 : B^I \rightarrow B$  is a trivial fibration because  $q : B \rightarrow C$  is a fibration, and  $b : B' \rightarrow B$  is a cofibration because it is a pullback of  $c : C' \rightarrow C$ . Thus by lemma 66, we have that  $\eta_{p_1} : B^I \rightarrow b_*b^*B^I$  is a trivial fibration, and so its pullback  $\eta_t : P_w \rightarrow b_*b^*P_w$  is a trivial fibration, as claimed.  $\square$

*Remark 68.* Note that  $p : A \rightarrow C$  is small if  $q : B \rightarrow C$  is small.

[ *Expected to use alignment here. Also expected to need closure of cofibrations under  $\Pi_1$ .* ]

## 8 The fibration extension property

In the presence of a universal fibration  $\dot{\mathcal{U}} \rightarrow \mathcal{U}$ , as was constructed in section 6, the fibration extension property (Definition 48) is closely related to the statement that the base object  $\mathcal{U}$  is fibrant. For Kan simplicial sets, Voevodsky proved the latter directly using minimal fibrations [?]. Shulman [?] gives a proof from univalence (in the form of the equivalence extension property as stated in section 7) in a more general setting, but it uses the 3-for-2 property for weak equivalences, which is what we are trying to prove. In [CCHM16], Coquand uses the equivalence extension property to prove that  $\mathcal{U}$  is fibrant, without assuming 3-for-2 for weak equivalences, by a neat argument using a reduction of general box-filling to a condition called “Kan-composition”. We shall prove that  $\mathcal{U}$  is fibrant using the equivalence extension property via a different argument than that in [CCHM16], avoiding the reduction of filling to composition, which we therefore do not require.

Returning to the relation between the fibration extension property and the condition that the base object  $\mathcal{U}$  is fibrant, it is easily seen that the latter implies the former. Indeed, let  $t : X \rightarrowtail X'$  be a trivial cofibration and  $Y \twoheadrightarrow X$  a fibration. To extend  $Y$  along  $t$ , take a classifying map  $y : X \rightarrow \mathcal{U}$ , so that  $Y \cong y^* \dot{\mathcal{U}}$  over  $X$ . If  $\mathcal{U}$  is fibrant then we can extend  $y$  along  $t : X \rightarrowtail X'$  to get  $y' : X' \rightarrow \mathcal{U}$  with  $y = y' \circ t$ . The pullback  $Y' = (y')^* \dot{\mathcal{U}} \twoheadrightarrow X'$  is then a (small) fibration such that  $t^* Y' \cong t^* (y')^* \dot{\mathcal{U}} \cong y^* \dot{\mathcal{U}} \cong Y$  over  $X$ . Thus for the record.

**Proposition 69.** *If the base object  $\mathcal{U}$  of the universal fibration  $\dot{\mathcal{U}} \rightarrow \mathcal{U}$  is fibrant, then the fibration weak factorization system has the fibration extension property.*

Conversely, we merely note that the FEP clearly implies the fibrancy of  $\mathcal{U}$ , given the following “alignment” lemma, which will also be required below (cf.(2') of [?]).

**Lemma 70.** *Given a fibration  $Y \twoheadrightarrow X$  with classifying map  $y : X \rightarrow \mathcal{U}$ , a cofibration  $t : X \rightarrowtail X'$ , and a (small) fibration  $Y' \twoheadrightarrow X'$  with  $Y \cong t^* Y'$  over*



$X$ , there is a classifying map  $y' : X' \rightarrow \mathcal{U}$  for  $Y'$  with  $y' \circ t = y$ .

$$\begin{array}{ccc}
 Y & \xrightarrow{\quad} & \dot{\mathcal{U}} \\
 \downarrow & \searrow & \uparrow \text{dotted} \\
 & Y' & \\
 \downarrow & \downarrow & \downarrow \\
 X & \xrightarrow{y} & \mathcal{U} \\
 \searrow t & & \nearrow \text{dotted } y' \\
 & X' & 
 \end{array}$$

*Proof.* This is routine using Yoneda and assuming that cofibrations  $A \rightarrowtail B$  have *complemented* monos as components  $A[n] \rightarrowtail B[n]$ .<sup>1</sup> In more detail, since  $Y' \twoheadrightarrow X'$  is small, there is a classifying map  $z : X' \rightarrow \mathcal{U}$ , perhaps not commuting with  $t$ . Nonetheless, we can use  $z$  to define the desired  $y' : X' \rightarrow \mathcal{U}$  objectwise as follows: Take any map from a representable  $x' : I^n \rightarrow X'$  and consider whether it factors through  $t$ , say as  $x' = t \circ x$  for some (necessarily unique)  $x : I^n \rightarrow X$ . If  $x'$  does factor, set  $y' \circ x' = y \circ x$ ; if not, set  $y' \circ x' = z \circ x'$ . This specification is clearly natural in  $I^n$ , so it defines  $y' : X' \rightarrow \mathcal{U}$ , and the specification ensures that  $y' \circ t = y$ , and that the pullback of  $\dot{\mathcal{U}}$  along  $y'$  is the same as that along  $z$ , namely  $Y' \twoheadrightarrow X'$ .  $\square$

**Corollary 71.** *The fibration extension property implies that the base  $\mathcal{U}$  of the universal fibration is a fibrant object.*

Now we use the equivalence extension property to show that  $\mathcal{U}$  is fibrant.

**Proposition 72.** *The base  $\mathcal{U}$  of the universal fibration  $\dot{\mathcal{U}} \rightarrow \mathcal{U}$  constructed in section 6 is a fibrant object.*

*Proof.* We need to solve the following filling problem for an arbitrary cofibration  $c : C \rightarrowtail Z$ , thus showing that  $\mathcal{U}^{\delta_0} : \mathcal{U}^I \rightarrow \mathcal{U}$  is a trivial fibration (for, say, the point  $\delta_0 : 1 \rightarrow I$ ). [*say something about using  $\delta_0$* ]

$$\begin{array}{ccc}
 C & \xrightarrow{\tilde{a}} & \mathcal{U}^I \\
 \downarrow c & \nearrow \text{dotted} & \downarrow \mathcal{U}^{\delta_0} \\
 Z & \xrightarrow{b} & \mathcal{U}
 \end{array} \tag{106}$$

---

<sup>1</sup>This of course holds if the base category  $\mathbf{Set}$  is classical; otherwise, one needs to take this as a further axiom on  $\mathcal{C}$ .

Transposing  $\tilde{a}$  to  $a : C \times I \rightarrow \mathcal{U}$  and taking pullbacks of  $\dot{\mathcal{U}} \twoheadrightarrow \mathcal{U}$  along  $a$  and  $b$  to get corresponding fibrations  $A \twoheadrightarrow C \times I$  and  $B \twoheadrightarrow Z$ , we have the following equivalent condition. Letting

$$C_0 : C \cong C \times 1 \rightarrow C \times I$$

be the evident inclusion of the 0-end of the cylinder, let  $A_0 = (C_0)^* A \twoheadrightarrow C$  be the “slice of  $A$  over  $C_0$ ”. We then have  $c^* B \cong A_0$  over  $C$  by the outer square of (106).

$$\begin{array}{ccccc}
 & A_0 & \xrightarrow{\quad} & A & \\
 & \downarrow & & \downarrow & \\
 B & \swarrow & & \searrow & \\
 & C & \xrightarrow{c_0} & C \times I & \\
 & \downarrow c & & \downarrow & \\
 Z & \xrightarrow{z_0} & Z \times I & & 
 \end{array}$$

The diagonal filler in (106) corresponds, again by transposition and pullback of  $\dot{\mathcal{U}} \twoheadrightarrow \mathcal{U}$ , to a fibration  $D \twoheadrightarrow Z \times I$  with  $(c \times I)^* D \cong A$  over  $C \times I$  and  $(Z_0)^* D \cong B$  over  $Z$ , as indicated below.

$$\begin{array}{ccccc}
 & A_0 & \xrightarrow{\quad} & A & \\
 & \downarrow & & \downarrow & \\
 B & \swarrow & & \searrow & \\
 & C & \xrightarrow{c_0} & C \times I & \\
 & \downarrow c & & \downarrow c \times I & \\
 Z & \xrightarrow{z_0} & Z \times I & & 
 \end{array}$$

We construct  $D \twoheadrightarrow Z \times I$  by the equivalence extension property as follows. Apply the functor  $(-) \times I$  to the left (pullback) face of the above cube to get the following with a new pullback square on the right, with the indicated

fibrations.

$$\begin{array}{ccccc}
 & A_0 & \xrightarrow{\quad} & A & \\
 & \downarrow & & \downarrow & \\
 & C & \xrightarrow{c_0} & C \times I & \\
 & \downarrow & & \downarrow & \\
 B & \xrightarrow{c} & D & \xrightarrow{c \times I} & B \times I \\
 \downarrow & & \downarrow & & \downarrow \\
 Z & \xrightarrow{Z_0} & Z \times I & & \\
 & & & & A_0 \times I
 \end{array}
 \tag{107}$$

We claim there is a weak equivalence  $e : A \simeq A_0 \times I$  over  $C \times I$ , from which follow by the EEP:

- (i) a fibration  $D \twoheadrightarrow Z \times I$  with  $(c \times I)^* D \cong A$  over  $C \times I$ , and
- (ii) a weak equivalence  $f : D \simeq B \times I$  over  $Z \times I$  with  $(c \times I)^* f \cong e$  over  $C \times I$ .

It then remains only to show that  $B \cong (Z_0)^* D$  over  $Z$ .

To get  $e$ , consider the following square, in which the top map is  $A_0 \times \delta_0$  (after  $A_0 \cong A_0 \times 1$ ) and the others are those from the previous diagram.

$$\begin{array}{ccc}
 A_0 & \xrightarrow{\quad} & A_0 \times I \\
 \downarrow & & \downarrow \\
 A & \twoheadrightarrow & C \times I
 \end{array}
 \tag{108}$$

The square is easily seen to commute, and the maps with  $A_0$  as domain are trivial cofibrations by Frobenius (proposition 56), because each is the pullback of a trivial cofibration along a fibration. Applying a simple lemma (given below as 73) gives the required weak equivalence  $e : A \simeq A_0 \times I$  over  $C \times I$ .

To see that  $B \cong (Z_0)^* D$  over  $Z$ , recall from the proof of the EEP that the map  $f : B \cong (Z_0)^* D$  is the pushforward of  $e : A \simeq A_0 \times I$  along the cofibration  $d_0 \times I : A_0 \times I \rightarrow B \times I$ , calling the evident map  $d_0 : A_0 \rightarrow B$  in (107). That is, by construction  $f = (d_0 \times I)_* e$ . We can apply the Beck-Chevalley condition for the pushforward using the pullback square on the left

below.

$$\begin{array}{ccccc}
 A_0 & \xrightarrow{\quad} & A_0 \times I & \xleftarrow{e} & A \\
 \downarrow \lrcorner & & \downarrow & & \\
 B & \xrightarrow{\quad} & B \times I & \xleftarrow{f} & D
 \end{array} \tag{109}$$

The pullback of  $e$  along the top of the square is the identity on  $A_0$ , as can be seen by pulling back  $e$  as a map over  $C \times I$  along  $C_0 : C \rightarrow C \times I$ . Thus the same is true (up to isomorphism) for the pullback of  $f$  along the bottom.

An application of the alignment lemma 70 along the trivial cofibration  $c \otimes \delta_0$  completes the proof.  $\square$

**Lemma 73.** *Suppose the following square commutes and the indicated cofibrations are trivial.*

$$\begin{array}{ccc}
 A & \xrightarrow{\quad} & C \\
 \downarrow & & \downarrow \\
 B & \twoheadrightarrow & D
 \end{array} \tag{110}$$

*Then there is a weak equivalence  $e : B \simeq C$  over  $D$  (and under  $A$ ).*

*Proof.* Use the fact that any two diagonal fillers are homotopic to get a homotopy equivalence  $e : B \simeq C$  filling the square.  $\square$

Applying proposition 69 therefore yields the following.

**Corollary 74.** *The fibration weak factorization system has the fibration extension property (definition 48).*

By Theorem 53, finally, we have the following.

**Theorem 75.** *There is a Quillen model structure  $(\mathcal{C}, \mathcal{W}, \mathcal{F})$  on the category of cubical sets, where:*

1. *the cofibrations  $\mathcal{C}$  are any class of monomorphisms satisfying axioms (C0)-(C6),*
2. *the fibrations  $\mathcal{F}$  are the maps  $f : Y \rightarrow X$  for which the canonical map*

$$(f^I \times I, \text{eval}_Y) : Y^I \times I \rightarrow (X^I \times I) \times_X Y$$

*lifts on the right against  $\mathcal{C}$ .*

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