Characterization of bicategories of stacks

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Introduction

Although the paper [11] was written in the setting of 2-categories, it was pointed out in the introduction of that paper how to modify the work in order to make it bicategorical. The purpose of the present paper is to make these modifications precise and to give an application.

The main theorem is a characterization of bicategories of stacks (= *champs* in French) in terms of limit, colimit, exactness, and size conditions on the bicategories: a bicategorical version of Giraud's characterization of categories of sheaves [1].

On the way to this result a formula is given for the associated stack. The existence of the associated stack on a categorical site was proved by Giraud [5] using the associated category-valued sheaf construction and a strictification construction on fibrations. Exactness properties of the associated stack construction were not obvious from Giraud's formula. The formula we give uses the obvious generalization to bicategories of the functor L used in [1] for the associated sheaf. Exactness of L is immediate: it preserves all finite bicategorical limits (biterminal objects, bipullbacks, and bicotensoring with finite categories in the sense of [9]). The associated stack is given by <u>three</u> applications of L (recall that two are needed in the sheaf case).

The application we wish to present is really an application of the formula for the associated stack. We give an easy proof of the relationship between *torsors* and *Čech cocyles*. Combining this with a very general theorem giving the classifying property of torsors (they classify objects locally structure isomophic to some member of a given family of mathematical structures), we are able to deduce information about local structures in mathematics; for example, about vector bundles, locally finite objects in a topos, Azumaya algebras, and so on.

§1. Regular and exact bicategories.

The notion of bicategory, homomorphism of bicategories, strong transformation and modification are those of Bênabou [3]. We write $\mathrm{Hom}(A,\mathcal{B})$ for the bicategory of homomorphisms, strong transformations and modifications from the bicategory A to the bicategory $\mathcal{B}$.

The notion of limit for bicategories is taken from Street [9]. For homomorphisms $F : \mathcal{A} \to \mathrm{Cat}$, $S : \mathcal{A} \to \mathcal{K}$, the *F-indexed bilimit of* S is an object $\{F,S\}$ of $\mathcal{K}$ satisfying an equivalence of homomorphisms:

$$\mathcal{K}(-,\{F,S\}) \simeq \mathrm{Hom}(\mathcal{A},\mathrm{Cat})(F,\mathcal{K}(-,S)).$$

As special cases we have biterminal objects, bipullback, biproduct, bicotensor product.

Suppose $\mathcal{K}$ is a bicategory with finite bilimits. An arrow $m : X \to Y$ in $\mathcal{K}$ is called *f.f.* when the functor

$$\mathcal{K}(K,m) : \mathcal{K}(K,X) \to \mathcal{K}(K,Y)$$

is fully faithful for all K.

An arrow $e : A \to B$ is called *e.s.o.* (short for "essentially surjective on objects") when the following diagram is a bipullback for all f.f. arrows $m : X \to Y$.

$$\begin{tikzcd}
\mathcal{K}(B,X) \arrow[r, "\mathcal{K}(B,m)"] \arrow[d, "\mathcal{K}(e,X)"'] \arrow[dr, phantom, "\simeq"] & \mathcal{K}(B,Y) \arrow[d, "\mathcal{K}(e,Y)"] \\
\mathcal{K}(A,X) \arrow[r, "\mathcal{K}(A,m)"'] & \mathcal{K}(A,Y)
\end{tikzcd}$$

More generally, one can define what it means for a family of arrows into B to be e.s.o. using a many-legged bipullback.

A *weak category* T in $\mathcal{K}$ is a homomorphism of bicategories from the sketch (= Gabriel theory) for the theory of categories into $\mathcal{K}$ which takes the distinguished cones to bilimits. There is an obvious notion of *weak functor* between weak categories.

Given an arrow $f : A \to B$ in $\mathcal{K}$ we can form the following diagrams in which the squares containing 2-cells are bicomma object diagrams and the squares containing isomorphisms are bipullbacks.

$$\begin{tikzcd}
E_2^2 \arrow[r, "d_2"] \arrow[d, "d_o"'] \arrow[dr, phantom, "\cong"] & E_1^2 \arrow[r, "d_1"] \arrow[d, "d_o"] \arrow[dr, phantom, "\Rightarrow"] & A \arrow[d, "f"] \\
E_1^2 \arrow[r, "d_1"] \arrow[d, "d_o"'] \arrow[dr, phantom, "\Rightarrow"] & A \arrow[r, "f"] \arrow[d, "f"] & B \\
A \arrow[r, "f"'] & B &
\end{tikzcd}
\qquad
\begin{tikzcd}
E_1^1 \arrow[r, "d_1"] \arrow[d, "d_o"'] \arrow[dr, phantom, "\cong"] & A \arrow[d, "f"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

We obtain two weak categories $E^2 : E_2^2 \rightrightarrows E_1^2 \rightleftarrows E_0^2$ and $E^1 : E_1^1 \rightleftarrows E_0^1$, and a weak functor $j : E^1 \to E^2$ with the following properties:

(a) $E_o^1 = E_o^2 = A$, j_o is an identity, and $j_1 : E_1^1 \to E_1^2$ is f.f.;

(b) the span (d_o, E_1^2, d_1) from A to A is a bidiscrete fibration in the sense of Street [9];

(c) E^1 is a weak equivalence relation on A.

This leads us to define a *congruence on* A to be a weak functor $j : E^1 \to E^2$ satisfying (a), (b), (c). So each arrow $f : A \to B$ has a congruence associated with it.

A quotient for a congruence $j : E^1 \to E^2$ consists of an arrow $g : A \to X$ and a 2-cell $\gamma : gd_o \Rightarrow gd_1$ such that:

$$\begin{tikzcd} E_1^1 \arrow[r, "j_1"] & E_1^2 \arrow[r, "d_1"] \arrow[d, "d_o"'] \arrow[dr, phantom, "\overset{\gamma}{\Rightarrow}"] & A \arrow[d, "g"] \\ & A \arrow[r, "g"'] & X \end{tikzcd}$$

is invertible, and

$$\begin{tikzcd} E_2^2 \arrow[r, "d_2"] \arrow[d, "d_o"'] \arrow[dr, phantom, "\cong"] & E_1^2 \arrow[r, "d_1"] \arrow[d] \arrow[dr, phantom, "\Rightarrow"] & A \arrow[d, "g"] \\ E_1^2 \arrow[r, "d_1"'] \arrow[d] \arrow[dr, phantom, "\overset{\gamma}{\Rightarrow}"] & A \arrow[r, "g"'] \arrow[d, "g"] \arrow[dr, phantom, "\cong"] & X \arrow[d, "1"] \\ A \arrow[r, "g"'] & X \arrow[r, "1"'] & X \end{tikzcd} = \begin{tikzcd} E_2^2 \arrow[r, "d_1"] & E_1^2 \arrow[r, "d_1"] \arrow[d, "d_o"'] \arrow[dr, phantom, "\overset{\gamma}{\Rightarrow}"] & A \arrow[d, "g"] \\ & A \arrow[r, "g"'] & X \end{tikzcd} ;$$

and which is biuniversal with these properties.

If K has finite colimits then every congruence has a quotient.

An arrow $q : A \to Q$ is called a quotient map when there exists a congruence E on A, and a 2-cell $\tau : qd_o \Rightarrow qd_1$ such that Q, q, τ form a quotient for E.

Proposition. *Every quotient map is e.s.o.* (Compare [11; (1.17)].) □

Call K *regular* when the following properties hold:

— all finite bilimits exist;

— each arrow f is isomorphic to a composite me where m is f.f. and e is e.s.o.;

— each bipullback of an e.s.o. is e.s.o.

Theorem. *In a regular bicategory, every e.s.o. is a quotient map.* (Compare [11; (1.22)].) □

Call K *exact* when it is regular and each congruence is the congruence associated with some arrow. It follows that every congruence has a quotient in an exact bicategory. For all bicategories C, the bicategory $\mathrm{Hom}(C^{op}, \mathrm{Cat})$ is exact.

§2. Bitoposes.

A topology on a bicategory C assigns to each object U of C, a set $\mathrm{Cov}\,U$ of f.f. arrows $R \to C(-,U)$ in $\mathrm{Hom}(C^{op},\mathrm{Cat})$ satisfying the following conditions:

T 0. the identity of $C(-,U)$ is in $\mathrm{Cov}\,U$;

T 1. for all $R \to C(-,U)$ in $\mathrm{Cov}\,U$ and all arrows $u : V \to U$ in C, there exists a bipullback

$$\begin{tikzcd} S \arrow[r] \arrow[d] \arrow[dr, phantom, "\cong"] & C(-,V) \arrow[d, "C(-,u)"] \\ R \arrow[r] & C(-,U) \end{tikzcd}$$

in which the top arrow is in $\mathrm{Cov}\;V$;

T 2. if $R' \to C(-,U)$ is in $\mathrm{Cov}\,U$ and $R \to C(-,U)$ is f.f. with the property that for each $u : V \to U$ in the image of $R'V \to C(V,U)$ there exists a bipullback as in T1 with the top arrow in $\mathrm{Cov}\,V$, then $R \to C(-,U)$ is equivalent to an arrow in $\mathrm{Cov}\,U$.

A *bisite* is a bicategory together with a topology. A *stack* for such a bisite is a homomorphism of bicategories $F : C^{op} \to \mathrm{Cat}$ such that, for each $R \to C(-,U)$ in $\mathrm{Cov}\,U$, an equivalence of categories is induced as follows:

$$\mathrm{Hom}(C^{op},\mathrm{Cat})(C(-,U),F) \simeq \mathrm{Hom}(C^{op},\mathrm{Cat})(R,F).$$

The canonical topology on a bicategory is the largest topology for which the representable homomorphisms are all stacks.

Write $\mathrm{Stack}\;C$ for the full sub-bicategory of $\mathrm{Hom}(C^{op},\mathrm{Cat})$ consisting of the stacks for the bisite C.

A bicategory K is called a *bitopos* when there exists a bisite C with small underlying bicategory such that there is a biequivalence: $K \sim \mathrm{Stack}\;C$.

For a bisite C, regard $\mathrm{Cov}\,U$ as an ordered set by taking $R \leq S$ when there exists a diagram:

$$\begin{tikzcd} R \arrow[rr] \arrow[dr] & \cong & S \arrow[dl] \\ & C(-,U) & \end{tikzcd}.$$

If C is small then, for each homomorphism $P : C^{op} \to \mathrm{Cat}$, we can define a homomorphism $LP : C^{op} \to \mathrm{Cat}$ by:

$$(LP)U = \operatorname*{colim}_{R} \mathrm{Hom}(C^{op},\mathrm{Cat})(R,P)$$

where R runs over the directed set $(\mathrm{Cov}\,U)^{op}$.

A homomorphism of bicategories which preserves finitary indexed bilimits will be called *left exact*.

Since filtered colimits in Cat commute with finitary indexed bilimits, L is a left exact homomorphism from $\mathrm{Hom}(C^{op},\mathrm{Cat})$ to itself.

Theorem. *For any small bisite* C*, the left biadjoint of the inclusion*

$$\mathrm{Stack}\, C \to \mathrm{Hom}(C^{op},\mathrm{Cat})$$

is obtained by applying L *three times and is hence left exact. If* $P \to F$ *is faithful and* F *is a stack then* L^2P *is the associated stack of* P. *If* $P \to F$ *is fully faithful and* F *is a stack then* LP *is the associated stack of* P. (Compare [11;(3.8)].) □

§3. Characterization theorem.

A set of objects of a bicategory C is called *e.s.o. generating* when, for each object U of C, the set of arrows into U with sources in the set, is e.s.o.

A bicategory K is called *lex-total* when it has small homcategories and the Yoneda embedding

$$Y : K \to \mathrm{Hom}(K^{op},\mathrm{Cat})$$

has a left-exact left biadjoint.

Bicoproducts in a bicategory are *universal* when they are preserved by bipullbacks. When any two distinct coprojections into a bicoproduct have a bi-initial bicomma object then the bicoproduct is *disjoint*.

A set whose cardinality is no greater than the cardinality of the set of small sets is called *moderate*.

Theorem. *The following conditions on a bicategory* K *with small homcategories are equivalent*:

(i) K *is a bitopos;*

(ii) K *is lex-total and there exists a moderate set* M *of objects of* K *such that, for all* X *in* K*, there exists an e.s.o.* $M \to X$ *with* M *in* M*;*

(iii) *every canonical stack on* K *is representable and* K *has an e.s.o. generating small set of objects;*

(iv) K *is an exact bicategory which has disjoint universal small bicoproducts and has an e.s.o. generating small set of objects;*

(v) *there exists a small canonical bisite* C *with finitary indexed bilimits such that* $K \sim \mathrm{Stack}\, C$. (Compare [11;(4.11)].) □

§4. Application to torsors.

Let E denote a finitely complete category with coequalizers and such that each of the categories E/U is cartesian closed. Let K denote the 2-category of categories in E. Let $F = \mathrm{Hom}(E^{op}, \mathrm{Cat})$.

Regard E as a site by taking single regular epimorphisms into U as covers of U and generating the usual *regular epimorphism topology* on E. Regarding E as a bicategory with only identity 2-cells, we obtain a bisite. The objects of F which are stacks for this bisite will simply be called *stacks* in this section.

Regard E as contained in K by taking objects of E as discrete categories. Regard K as contained in F by taking each category A in E to the representable $E(-,A)$.

An object X of F is called *admissible* when, for all $x : U \to X$, $y : V \to X$ with U,V in E, there is a bicomma object x/y in E.

Define $S \in F$ by $SU = E/U$ and S on arrows is given by pulling back along them.

For each X in F there exists PX in F satisfying:

$$F(Y,PX) \simeq F(X^{op} \times Y, S) \ .$$

For A in K, we can identify $(PA)U$ with the full subcategory of the spans $A \xleftarrow{p} E \xrightarrow{q} U$ in K from U to A consisting of those spans for which the following is a pullback.

$$\begin{tikzcd} E_1 \arrow[r, "d_1"] \arrow[d, "p_1"'] & E_o \arrow[d, "p_o"] \\ A_1 \arrow[r, "d_1"'] & A_o \end{tikzcd}$$

In standard topos terminology, $(PA)U \simeq E^{A^{op} \times U}$.

For any admissible X in F, there is a *yoneda arrow* $y_X : X \to PX$. For A in K, the yoneda arrow y_A has component $y_A U : E(U,A) \to (PA)U$ that functor which takes $a : U \to A$ to the span $A \leftarrow A/a \to U$.

Suppose A is admissible and $E \in (PA)U$. The E-*indexed colimit* $\mathrm{colim}(E,f)$ of $f : A \to X$ is the pointwise left extension of fp along q as below:

$$\begin{tikzcd} E \arrow[r, "p"] \arrow[d, "p"'] & U \arrow[d, "\mathrm{colim}(E,f)"] \\ A \arrow[r, "f"'] & X \ . \end{tikzcd}$$

Here pointwiseness means that the left (Kan) extension property is stable under pullback along an arrow into U.

Call X *cocomplete* when it admits $\mathrm{colim}(E,f)$ for all E and $f : A \to X$ with A in K. In particular, PB is cocomplete for all B in K (see [8], [10]).

An object $z \in XU$ is *locally isomorphic to a value of* $f : A \to X$ when there exists a regular epimorphism $e : V \to U$, an object a of AV, and an isomorphism $(Xe)z \cong f_V a$.

$$\begin{tikzcd} V \arrow[r, "e"] \arrow[d, "a"'] \arrow[dr, phantom, "\cong"] & U \arrow[d, "z"] \\ A \arrow[r, "f"'] & X \end{tikzcd}$$

Let $\mathrm{Loc}_X(f)U$ be the full subcategory of XU consisting of such z. Since the pullback of a regular epic is a regular epic, this defines an object $\mathrm{Loc}_X(f)$ of F which is a subhomomorphism of X.

For $A \in K$, an object $E \in (PA)U$ which is locally isomorphic to a value of $y_A : A \to PA$ is called an A-<u>torsor</u>.

$$\begin{tikzcd} & A/a \arrow[r, "d_1"] \arrow[dl, "d_o"'] \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & V \arrow[d] \\ A \arrow[r, "p"'] & E \arrow[r, "q"'] & U \end{tikzcd}$$

Put $\mathrm{Tor}\, A = \mathrm{Loc}_{PA}(y_A)$.

<u>Proposition.</u> *An object* X *of* F *is a stack if and only if it admits all colimits indexed by torsors. In particular,* PB *is a stack for all* B *in* K. □

<u>Theorem on classification by torsors.</u> *Suppose* $X \in F$ *is an admissible stack. Each* $x : W \to X$ *in* F *with* W *in* E *factors up to isomorphism as a composite of an arrow* $W \to X[x]$ *in* K *which is the identity on objects and an arrow* $i : X[x] \to X$ *in* F *whose components are fully faithful. The functor* $E \to \mathrm{colim}(E,i)$ *provides an equivalence:*

$$\mathrm{Tor}\, X[x] \simeq \mathrm{Loc}_X(x) \ . \quad \square$$

<u>Theorem relating torsors and Čech cocycles.</u> *For each object* A *of* K *and* U *of* E *there is an equivalence*

$$(\mathrm{Tor}\, A)\, U \simeq \operatorname*{colim}_{V \xrightarrow{e} U} K(er_U(e), A)$$

where $e : V \to U$ *runs over the regular epics into* U *and* $er_U(e)$

denotes the category in E *determined by the kernel pair of* e.

Proof. Bunge [4] has shown that $\mathrm{Tor}\,A$ is the associated stack of A. The colimit of the Theorem is precisely the formula for LA as given in §2. There is a fully faithful arrow $y_A : A \to PA$ with PA a stack. So only one application of L is needed to obtain the associated stack. So $\mathrm{Tor}\,A \simeq LA$ as required. □

§5. Finiteness in a topos.

Take E to be a topos with natural numbers object N. In the terminology of the last section, let $\mathrm{Fin} \in SN = E/N$ denote the object $N \times N \xrightarrow[+]{} N \xrightarrow[suc]{} N$ of E/N.

The objects Z of S1 which are locally isomorphic to a value of $\mathrm{Fin} : N \to S$ are the locally finite (= Kuratowski-finite decidable) objects of E.

The category S[Fin] in E is the usual category E_{fin} of cardinal-finite objects. The last two theorems give:

$$\mathrm{Tor}(E_{fin}) \simeq \mathrm{Loc}(\mathrm{Fin})$$

$$\mathrm{Tor}(E_{fin})1 \simeq \operatorname*{colim}_{R \twoheadrightarrow 1} K(R_c, E_{fin})$$

where R_c denotes the chaotic category on the object R of E.

Since E_{fin} is a topos in E [6] and topos is an essentially algebraic notion, the filtered colimit on the right above is a topos. This gives another proof that the locally finite objects in E form a topos showing that the ideas involved are basically cohomological (provided we allow cohomology with category-valued coefficients and not merely abelian-group-valued coefficients).

§6. Vector bundles.

In the situation of §4, take E to be a nice category of topological spaces. Restrict the regular epics to local homeomorphisms.

Take $X \in F$ to be the internalization of the theory of vector spaces over $\mathbb{R}$; that is, XU is the category of modules in E/U over the ring $\mathbb{R} \times U \to U$.

Take $\mathrm{Euc} : \mathbb{N} \to X$ to be the family $\mathbb{R}^n$, $n \in \mathbb{N}$, of finite dimensional vector spaces.

Objects Z of XU locally isomorphic to a value of Euc are vector bundles over U.

X[Euc] is the category $\mathrm{Mat}(\mathbb{R})$ of matrices over $\mathbb{R}$ as a category in E.

The two theorems of §4 give equivalences:

$$\mathrm{Tor}\ (\mathrm{Mat}(\mathbb{R})) \simeq \mathrm{Loc}(\mathrm{Euc}) = (\text{vector bundles})$$

$$\mathrm{Tor}\ (\mathrm{Mat}(\mathbb{R}))U \simeq \operatorname*{colim}_{V \xrightarrow{e} U} K(er_U(e), \mathrm{Mat}(\mathbb{R}))$$

where e runs over surjective local homeomorphisms into U.

Thus we obtain an equivalence between the category of vector bundles over U and the colimit of $K(er_U(e), \mathrm{Mat}(\mathbb{R}))$ as e runs over surjective local homeomorphisms into U. Now $\mathrm{Mat}(\mathbb{R})$ is a compact symmetric closed monoidal additive category with finite products and splitting idempotents. These properties are therefore inherited by the category of vector bundles . The result is a precise formulation of the *clutching construction* for vector bundles from which we can immediately deduce the property:

$$S \oplus T \cong S' \oplus T \ \text{ implies } \ S \cong S'$$

necessary for the construction of K-theory. The usual colimit involving the general linear group $GL(n,\mathbb{R})$ is also a consequence. The equivalence therefore brings together much of the introductory K-theory appearing in books such as [2],[7] as an aspect of category-valued cohomology.

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