A note on Hofmann-Streicher universes

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Let $\mathbb{C}$ be a small category and $\widehat{\mathbb{C}} = \mathsf{Set}^{\mathbb{C}^{\mathrm{op}}}$ the category of presheaves on $\mathbb{C}$.

1. The Hofmann-Streicher universe

In [HS97] the authors define a (type-theoretic) *universe*

$$\mathsf{E}l \to U \tag{1}$$

in $\widehat{\mathbb{C}}$ as follows. For $I \in \mathbb{C}$, set

$$\begin{aligned} U(I) &= \mathsf{ob}(\widehat{\mathbb{C}/_I})\,, & (2)\\ \mathsf{E}l(\langle I, A\rangle) &= A(id_I)\,, & (3)\end{aligned}$$

with an evident associated action on morphisms, which need not concern us for the moment. A few comments are required:

1. Since $U : \mathbb{C}^{\mathrm{op}} \longrightarrow \mathsf{Set}$, we have taken the underlying *set* $\mathsf{ob}(\widehat{\mathbb{C}/_I})$ of objects of the category $\widehat{\mathbb{C}/_I}$ in (2).

2. In (3), and throughout, the authors steadfastly adopt the "categories with families" point of view in describing the morphism $\mathsf{E}l \to U$ in $\widehat{\mathbb{C}}$ as an object in
$$\widehat{\textstyle\int_{\mathbb{C}} U} \;\simeq\; \widehat{\mathbb{C}}/_U\,, \tag{4}$$
and thus as a presheaf on the *category of elements* $\int_{\mathbb{C}} U$ (rather than specifying the object $\mathsf{E}l$ in $\widehat{\mathbb{C}}$). Thus the argument $\langle I, A\rangle \in \int_{\mathbb{C}} U$ in (3) consists of an object $I \in \mathbb{C}$ and an element $A \in U(I)$.

3. In order to account for size issues, the authors assume a Grothendieck universe $\mathcal{U}$ in Set, the elements of which are called *small*. The category $\mathbb{C}$ is then assumed to be small, as are the values of the presheaves (unless otherwise stated).

The presheaf U, which is not small, is regarded as the Grothendieck universe $\mathcal{U}$ "lifted" from Set to $\mathsf{Set}^{\mathbb{C}^{\mathrm{op}}}$. We will analyse the construction of (1) from a slightly different perspective in order to arrive at its basic property as a classifier for small families in $\widehat{\mathbb{C}}$.

2. An unused adjunction

For a presheaf X on $\mathbb{C}$, recall that the category of elements is the comma category,

$$\textstyle\int_{\mathbb{C}} X \;=\; \mathsf{y}_{\mathbb{C}}/_X\,,$$

where $\mathsf{y}_{\mathbb{C}} : \mathbb{C} \to \mathsf{Set}^{\mathbb{C}^{\mathrm{op}}}$ is the Yoneda embedding, which we may supress and write simply $\mathbb{C}/_X$. While the category of elements $\int_{\mathbb{C}} X$ is used in the specification of the Hofmann-Streicher universe $\mathsf{E}l \to U$ at the point (4), the authors seem to have missed a trick, which can be used to simplify things:

Proposition 1. *The category of elements functor* $\int_{\mathbb{C}} : \widehat{\mathbb{C}} \longrightarrow \mathsf{Cat}$ *has a right adjoint, which we denote*

$$\nu_{\mathbb{C}} : \mathsf{Cat} \longrightarrow \widehat{\mathbb{C}}\,.$$

For a small category $\mathbb{A}$*, we call the presheaf* $\nu_{\mathbb{C}}(\mathbb{A})$ *the* $\mathbb{C}$-nerve *of* $\mathbb{A}$*.*

Proof. As suggested by the name, the adjunction $\int_{\mathbb{C}} \dashv \nu_{\mathbb{C}}$ can be seen as the familiar "realization $\dashv$ nerve" construction with respect to the covariant functor $\mathbb{C}/- : \mathbb{C} \to \mathsf{Cat}$. In detail, for $\mathbb{A} \in \mathsf{Cat}$ and $c \in \mathbb{C}$ define $\nu_{\mathbb{C}}(\mathbb{A})(c)$ to be the Hom-set,

$$\nu_{\mathbb{C}}(\mathbb{A})(c) = \mathsf{Cat}\big(\mathbb{C}/_c\,,\, \mathbb{A}\big)\,,$$

with contravariant action on $h : d \to c$ given by precomposing a functor $P : \mathbb{C}/_c \to \mathbb{A}$ with the postcomposition functor

$$\mathbb{C}/_h : \mathbb{C}/_d \longrightarrow \mathbb{C}/_c\,.$$

For the adjunction, observe that the slice category $\mathbb{C}/_c$ is the category of elements of the representable functor $\mathsf{y}c$,

$$\textstyle\int_{\mathbb{C}} \mathsf{y}c \;\cong\; \mathbb{C}/_c\,.$$

Thus for all representables $\mathsf{y}c$, we have the required natural isomorphism

$$\widehat{\mathbb{C}}\big(\mathsf{y}c\,,\, \nu_{\mathbb{C}}(\mathbb{A})\big) \;\cong\; \nu_{\mathbb{C}}(\mathbb{A})(c) \;=\; \mathsf{Cat}\big(\mathbb{C}/_c\,,\, \mathbb{A}\big) \;\cong\; \mathsf{Cat}\big(\textstyle\int_{\mathbb{C}} \mathsf{y}c\,,\, \mathbb{A}\big)\,.$$

For arbitrary presheaves X, one uses the presentation of X as a colimit of representables over the index category $\int_{\mathbb{C}} X$, and the easy to prove fact that $\int_{\mathbb{C}}$ itself preserves colimits. Indeed, for any category $\mathbb{D}$, we have an isomorphism in Cat,

$$\varinjlim_{d \in \mathbb{D}} \mathbb{D}/_d \;\cong\; \mathbb{D}\,.$$

□

When $\mathbb{C}$ is fixed, as here, we may omit the subscript from the notation $\int_{\mathbb{C}}$ and $\mathsf{y}_{\mathbb{C}}$ and $\nu_{\mathbb{C}}$. The unit and counit maps of the adjunction $\int \dashv \nu$, vis.

$$\begin{aligned} \eta : X &\longrightarrow \nu \textstyle\int X\,, \\ \epsilon : \textstyle\int \nu\mathbb{A} &\longrightarrow \mathbb{A}\,, \end{aligned}$$

are as follows. At $c \in \mathbb{C}$, for $x : \mathsf{y}c \to X$, the functor $(\eta_X)_c(x) : \mathbb{C}/_c \to \mathbb{C}/_X$ is just composition with x,

$$(\eta_X)_c(x) = \mathbb{C}/_x : \mathbb{C}/_c \longrightarrow \mathbb{C}/_X\,. \tag{5}$$

For $\mathbb{A} \in \mathsf{Cat}$, the functor $\epsilon : \int \nu\mathbb{A} \to \mathbb{A}$ takes a pair $(c \in \mathbb{C}, f : \mathbb{C}/_c \to \mathbb{A})$ to the object $f(1_c) \in \mathbb{A}$,

$$\epsilon(c, f) = f(1_c).$$

Lemma 2. *For any $f : Y \to X$, the naturality square below is a pullback.*

$$\begin{tikzcd} Y \arrow[r, "\eta_Y"] \arrow[d, "f"'] & \nu\int Y \arrow[d, "\nu\int f"] \\ X \arrow[r, "\eta_X"'] & \nu\int X. \end{tikzcd} \tag{6}$$

Proof. It suffices to prove it for the case $f : X \to 1$. Thus consider the square

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d] & \nu\int X \arrow[d] \\ 1 \arrow[r, "\eta_1"'] & \nu\int 1. \end{tikzcd} \tag{7}$$

Evaluating at $c \in \mathbb{C}$ and applying (5) then gives the following square in Set.

$$\begin{tikzcd} Xc \arrow[r, "\mathbb{C}/_{-}"] \arrow[d] & \mathsf{Cat}\big(\mathbb{C}/_c\,, \mathbb{C}/_X\big) \arrow[d] \\ 1c \arrow[r, "\mathbb{C}/_{-}"'] & \mathsf{Cat}\big(\mathbb{C}/_c\,, \mathbb{C}/_1\big) \end{tikzcd} \tag{8}$$

The image of $* \in 1c$ along the bottom is the forgetful functor $U_c : \mathbb{C}/_c \to \mathbb{C}$, and its fiber under the map on the right is therefore the set of functors $F : \mathbb{C}/_c \to \mathbb{C}/_X$ such that $U_X \circ F = U_c$, where $U_X : \mathbb{C}/_X \to \mathbb{C}$ is also a forgetful functor. But any such F is easily seen to be uniquely of the form $\mathbb{C}/_x$ for $x = F(1_c) : \mathsf{y}c \to X$. $\square$

3. Classifying families

For the terminal presheaf $1 \in \widehat{\mathbb{C}}$, we have $\int 1 \cong \mathbb{C}$, so for every $X \in \widehat{\mathbb{C}}$ there is a canonical projection $\int X \to \mathbb{C}$, which is easily seen to be a discrete fibration. It follows that for any map $Y \to X$ of presheaves, the associated map $\int Y \to \int X$ is also a discrete fibration. Ignoring size issues for the moment, recall that discrete fibrations in Cat are classified by the forgetful functor $\dot{\mathsf{Set}}^{\mathrm{op}} \to \mathsf{Set}^{\mathrm{op}}$ from (the opposites of) the category of pointed sets to that of sets. For every presheaf $X \in \widehat{\mathbb{C}}$, we therefore have a pullback diagram in Cat,

$$\begin{tikzcd} \int X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathsf{Set}}^{\mathrm{op}} \arrow[d] \\ \mathbb{C} \arrow[r, "X"'] & \mathsf{Set}^{\mathrm{op}}. \end{tikzcd} \tag{9}$$

Transposing by the adjunction $\int \dashv \nu$ then gives a commutative square in $\widehat{\mathbb{C}}$,

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \nu\dot{\mathsf{Set}}^{\mathrm{op}} \arrow[d] \\ 1 \arrow[r, "\tilde{X}"'] & \nu\mathsf{Set}^{\mathrm{op}}. \end{tikzcd} \tag{10}$$

Lemma 3. *The square (10) is a pullback in $\widehat{\mathbb{C}}$. More generally, for any map $Y \to X$ in $\widehat{\mathbb{C}}$, there is a pullback square*

$$\begin{tikzcd} Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \nu\dot{\mathsf{Set}}^{\mathrm{op}} \arrow[d] \\ X \arrow[r] & \nu\mathsf{Set}^{\mathrm{op}}. \end{tikzcd} \tag{11}$$

Proof. Apply the right adjoint ν to the pullback square (9) and paste the naturality square (6) from Lemma 2 on the left, to obtain the transposed square (11) as a pasting of two pullbacks. $\square$

Let us write $\dot{\mathcal{V}} \to \mathcal{V}$ for the vertical map on the right in (11), so that

$$\begin{aligned} \dot{\mathcal{V}} &= \nu\dot{\mathsf{Set}}^{\mathrm{op}} \\ \mathcal{V} &= \nu\mathsf{Set}^{\mathrm{op}}. \end{aligned} \tag{12}$$

We can summarize our results so far as follows.

Proposition 4. *The nerve $\dot{\mathcal{V}} \to \mathcal{V}$ of the classifier for discrete fibrations $\dot{\mathsf{Set}}^{\mathrm{op}} \to \mathsf{Set}^{\mathrm{op}}$, as defined in (12), classifies natural transformations $Y \to X$ in $\widehat{\mathbb{C}}$, in the sense that there is always a pullback square,*

$$\begin{tikzcd} Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathcal{V}} \arrow[d] \\ X \arrow[r, "\tilde{Y}"'] & \mathcal{V}. \end{tikzcd} \tag{13}$$

The classifying map $\tilde{Y} : X \to \mathcal{V}$ is determined by the adjunction $\int \dashv \nu$ as the transpose of the classifiyng map of the discrete fibration $\int X \to \int Y$.

Of course, $\dot{\mathcal{V}} \to \mathcal{V}$ itself cannot be a map in $\widehat{\mathbb{C}}$, for reasons of size.

4. Small maps

Let α be a cardinal number and call the sets that are strictly smaller *α-small*. Let $\mathsf{Set}_\alpha \hookrightarrow \mathsf{Set}$ be the full subcategory of α-small sets. Call a presheaf $X : \mathbb{C}^{\mathrm{op}} \to \mathsf{Set}$ α-small if all of its values are α-small sets, and thus if, and only if, it factors through $\mathsf{Set}_\alpha \hookrightarrow \mathsf{Set}$. Call a map $f : Y \to X$ of presheaves α-small if all of the fibers $f_c^{-1}\{x\} \subseteq Yc$ are α-small sets (for all $c \in \mathbb{C}$ and $x \in Xc$). The latter condition is of course equivalent to saying that, in the pullback square over the element $x : \mathsf{y}c \to X$,

$$\begin{tikzcd} Y_x \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "f"] \\ \mathsf{y}c \arrow[r, "x"'] & X, \end{tikzcd} \tag{14}$$

the presheaf Y_x is α-small.

Now let us restrict the specification (12) of $\dot{\mathcal{V}} \to \mathcal{V}$ to the α-small sets:

$$\begin{aligned} \dot{\mathcal{V}}_\alpha &= \nu\dot{\mathsf{Set}}_\alpha^{\mathsf{op}} \\ \mathcal{V}_\alpha &= \nu\mathsf{Set}_\alpha^{\mathsf{op}}. \end{aligned} \tag{15}$$

Then the evident forgetful map $\dot{\mathcal{V}}_\alpha \to \mathcal{V}_\alpha$ *is* a map in the category $\widehat{\mathbb{C}}$ of presheaves, and it is in fact α-small. Moreover, it has the following basic property, which is just a restriction of the basic property of $\dot{\mathcal{V}} \to \mathcal{V}$ stated in Proposition 4.

Proposition 5. *The map $\dot{\mathcal{V}}_\alpha \to \mathcal{V}_\alpha$ classifies α-small maps $f : Y \to X$ in $\widehat{\mathbb{C}}$, in the sense that there is always a pullback square,*

$$\begin{tikzcd}
Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathcal{V}}_\alpha \arrow[d] \\
X \arrow[r, "\tilde{Y}"'] & \mathcal{V}_\alpha .
\end{tikzcd} \tag{16}$$

The classifying map $\tilde{Y} : X \to \mathcal{V}_\alpha$ is determined by the adjunction $\int \dashv \nu$ as (the factorization of) the transpose of the classifiyng map of the discrete fibration $\int X \to \int Y$.

Proof. If $Y \to X$ is small, its classifying map $\tilde{Y} : X \to \mathcal{V}$ factors through $\mathcal{V}_\alpha \hookrightarrow \mathcal{V}$, as indicated below,

$$\begin{tikzcd}
Y \arrow[r] \arrow[d] \arrow[rr, bend left] & \nu\dot{\mathsf{Set}}_\alpha{}^{\mathrm{op}} \arrow[r, hook] \arrow[d] & \nu\dot{\mathsf{Set}}^{\mathrm{op}} \arrow[d] \\
X \arrow[r] \arrow[rr, bend right, "\tilde{Y}"'] & \nu\mathsf{Set}_\alpha{}^{\mathrm{op}} \arrow[r, hook] & \nu\mathsf{Set}^{\mathrm{op}},
\end{tikzcd} \tag{17}$$

in virtue of the following adjoint transposition,

$$\begin{tikzcd}
\int Y \arrow[r] \arrow[d] \arrow[rr, bend left] & \dot{\mathsf{Set}}_\alpha{}^{\mathrm{op}} \arrow[r, hook] \arrow[d] & \dot{\mathsf{Set}}^{\mathrm{op}} \arrow[d] \\
\int X \arrow[r] \arrow[rr, bend right] & \mathsf{Set}_\alpha{}^{\mathrm{op}} \arrow[r, hook] & \mathsf{Set}^{\mathrm{op}}.
\end{tikzcd} \tag{18}$$

Note that the square on the right is evidently a pullback, and the one on the left therefore is, too, because the outer rectangle is the classifying pulback of the discrete fibration $\int Y \to \int X$, as stated. Thus the left square in (17) is a pullback. □

5. Examples

1. Let $\alpha = \kappa$ a strongly inaccessible cadinal, so that $\mathsf{ob}(\mathsf{Set}_\kappa)$ is a Grothendieck universe. Then the Hofmann-Streicher universe of (1) is recovered in the present setting as the κ-small map classifier

$$\mathsf{E}l \;\cong\; \dot{\mathcal{V}}_\kappa \longrightarrow \mathcal{V}_\kappa \;\cong\; U$$

in the sense of Proposition 5. Indeed, for $c \in \mathbb{C}$, we have

$$\begin{aligned}
\mathcal{V}_\kappa c \;&=\; \nu(\mathsf{Set}_\kappa^{\mathsf{op}})(c) = \mathsf{Cat}\big(\mathbb{C}/_c\,,\, \mathsf{Set}_\kappa^{\mathsf{op}}\big) = \mathsf{ob}(\widehat{\mathbb{C}/_c}) \;=\; Uc \\
\dot{\mathcal{V}}_\kappa c \;&=\; \nu(\,\dot{\mathsf{Set}}_\kappa^{\mathsf{op}})(c) = \mathsf{Cat}\big(\mathbb{C}/_c\,,\, \dot{\mathsf{Set}}_\kappa^{\mathsf{op}}\big) = \textstyle\int_{\mathbb{C}/_c} \mathsf{E}l(\langle c, A\rangle)\,.
\end{aligned} \tag{19}$$

2. By functoriality of $\nu : \mathsf{Cat} \to \widehat{\mathbb{C}}$, a sequence of Grothendieck universes

$$\mathcal{U} \subseteq \mathcal{U}' \subseteq ...$$

in Set gives rise to a (cumulative) sequence of type-theoretic universes

$$\mathcal{V} \rightarrowtail \mathcal{V}' \rightarrowtail ...$$

in $\widehat{\mathbb{C}}$. More precisely, there is a sequence of cartesian squares,

$$\begin{tikzcd}
\dot{\mathcal{V}} \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathcal{V}}' \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \dots \\
\mathcal{V} \arrow[r, tail] & \mathcal{V}' \arrow[r, tail] & \dots\,,
\end{tikzcd} \tag{20}$$

in the image of $\nu : \mathsf{Cat} \longrightarrow \widehat{\mathbb{C}}$, classifying small maps in $\widehat{\mathbb{C}}$ of increasing size, in the sense of Proposition 5.

3. Let $\alpha = 2$ so that $1 \to 2$ is the subobject classifier of Set, and

$$\mathbb{1} = \dot{\mathsf{Set}}_2^{\mathsf{op}} \longrightarrow \mathsf{Set}_2^{\mathsf{op}} = \mathbb{2}$$

is then a classifier in Cat for full subcategories $\mathbb{S} \hookrightarrow \mathbb{A}$ that are closed under the domains of arrows $a \to s$ for $s \in \mathbb{S}$ ("total sieves"). The "lifted universe" $\dot{\mathcal{V}}_2 \to \mathcal{V}_2$ is then the subobject classifier $1 \to \Omega$ of $\widehat{\mathbb{C}}$,

$$1 = \nu\mathbb{1} = \dot{\mathcal{V}}_2 \longrightarrow \mathcal{V}_2 = \nu\mathbb{2} = \Omega\,.$$

4. Let $i : \mathbb{2} \hookrightarrow \mathsf{Set}_\kappa$ and $p : \mathsf{Set}_\kappa \to \mathbb{2}$ be the embedding-retraction pair with $i : \mathbb{2} \hookrightarrow \mathsf{Set}_\kappa$ the inclusion of the full subcategory on the sets $\{0,1\}$ and $p : \mathsf{Set}_\kappa \to \mathbb{2}$ the retraction that takes $0 = \emptyset$ to itself, and everything else (i.e. the non-empty sets) to $1 = \{\emptyset\}$. There is a retraction (of arrows) in Cat,

$$\begin{tikzcd} \mathbb{1} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathsf{Set}}_\kappa \arrow[r] \arrow[d] & \mathbb{1} \arrow[d] \\ \mathbb{2} \arrow[r, hook, "i"'] & \mathsf{Set}_\kappa \arrow[r, two heads, "p"'] & \mathbb{2} \end{tikzcd} \tag{21}$$

where the left square is a pullback.

By the functoriality of $\nu : \mathsf{Cat} \to \widehat{\mathbb{C}}$ we then have a retract diagram in $\widehat{\mathbb{C}}$, again with a pullback on the left,

$$\begin{tikzcd} 1 \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathcal{V}}_\kappa \arrow[r] \arrow[d] & 1 \arrow[d] \\ \Omega \arrow[r, hook, "\{-\}"'] & \mathcal{V}_\kappa \arrow[r, two heads, "{[-]}"'] & \Omega \end{tikzcd} \tag{22}$$

where for any $\phi : X \to \Omega$ the subobject $\{\phi\} \rightarrowtail X$ is classified as a small map by the composite $\{\phi\} : X \to \mathcal{V}_\kappa$, and for any small map $A \to X$, the image $[A] \rightarrowtail X$ is classified as a subobject by the composite $[\alpha] : X \to \mathcal{V}_\kappa \to \Omega$, where $\alpha : X \to \mathcal{V}_\kappa$ classifies $A \to X$. The idempotent composite

$$\|-\| = [\{-\}] : \mathcal{V}_\kappa \longrightarrow \mathcal{V}_\kappa$$

is the *propositional truncation modality* in the natural model of type theory given by $\dot{\mathcal{V}}_\kappa \to \mathcal{V}_\kappa$ (see [AGH21]).

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References

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