COHERENT GROUPOIDS (MODIFIED VERSION)

STEVE AWODEY, JACOPO EMMENEGGER, AND GIUSEPPE ROSOLINI

1. Compact and coherent objects

Let $\mathbb{C}$ be a category with finite limits and $\mathcal{E} = [\mathbb{C}^{\mathsf{op}}, \mathsf{Set}]$ the topos of presheaves on $\mathbb{C}$. Call an object K of $\mathcal{E}$ **compact** (Johnstone: *supercompact*) if it is covered by a representable, $\mathsf{y}c \twoheadrightarrow K$. Call an arrow $f\colon A \to B$ **compact** if, for every compact K and arrow $K \to B$, the pullback object $K \times_B A$ is compact. Note that an object is compact if and only if its terminal projection is compact, and compact objects are closed under finite products. The following is also easily shown.

Lemma 1.1. *The compact arrows include all isomorphisms and are closed under composition and pullback along arbitrary maps.*

Let K be compact, with cover $\mathsf{y}c \twoheadrightarrow K$, and suppose that the diagonal map $\Delta_K : K \to K \times K$ is also compact. Then the evident pullback $K' \rightarrowtail \mathsf{y}c \times \mathsf{y}c$ of Δ_K to $\mathsf{y}c \times \mathsf{y}c$ is also compact, and K' is therefore a compact object, with cover $\mathsf{y}c' \twoheadrightarrow K'$.

$$\begin{tikzcd}
\mathsf{y}c' \arrow[r, two heads] \arrow[dr] & K' \arrow[r, two heads] \arrow[d] & K \arrow[d, "\Delta_K"] \\
& \mathsf{y}c \times \mathsf{y}c \arrow[r, two heads] & K \times K
\end{tikzcd}$$

It follows that K is a coequalizer of representables, $\mathsf{y}c' \rightrightarrows \mathsf{y}c \twoheadrightarrow K$, and that $c' \rightrightarrows c$ is, moreover, a pseudo-equivalence relation (in the sense of [?]) in $\mathbb{C}$. This motivates the following.

Definition 1.2. An object C in $\mathcal{E}$ is called **coherent** if it is compact and the diagonal $\Delta_C\colon C \to C \times C$ is a compact map.

Lemma 1.3.
1. *A coherent object C is the coequalizer of a pseudo-equivalence relation of representables $\mathsf{y}c' \rightrightarrows \mathsf{y}c \twoheadrightarrow C$.*
2. *Moreover, the coherent objects are closed under finite limits.*

Proof. We just showed the first statement, except for the pseudo-equivalence relation part. For that, observe that $K' \rightarrowtail \mathsf{y}c \times \mathsf{y}c$ in the previous displayed diagram is an actual equivalence relation with a projective cover $\mathsf{y}c' \twoheadrightarrow K'$, so there are maps $\rho : \mathsf{y}c \to \mathsf{y}c'$ and $\sigma : \mathsf{y}c' \to \mathsf{y}c'$ witnessing reflexivity and symmetry, as well as $\tau : \mathsf{y}c' \times_{\mathsf{y}c} \mathsf{y}c' \to \mathsf{y}c'$ for transitivity. It follows that $c' \rightrightarrows c$ is also a pseudo-equivalence relation in $\mathbb{C}$, as stated above.

The terminal object $1 = \mathsf{y}1$ is clearly coherent. For closure under products, let A, B be coherent, therefore compact, so $A \times B$ is compact. The diagonal $\Delta_{A\times B} : A \times B \to (A \times B) \times (A \times B) \cong (A \times A) \times (B \times B)$ is isomorphic to $\Delta_A \times \Delta_B$, and therefore also compact, since the product of compact maps is easily seen to be compact, using Lemma 1.1. Let $f, g : C \rightrightarrows C'$ with C, C' coherent. The equalizer $E \rightarrowtail C$ is the pullback of the compact map $\Delta_{C'}$ along $\langle f, g\rangle : C \to C' \times C'$, and so E is compact, since C is. Finally, Δ_E is the pullback of the compact map Δ_C along the mono $E \times E \rightarrowtail C \times C$, and is therefore also compact. □

2. Coherent groupoids

Everything we say in this section holds with groupoids replaced by categories.

A groupoid $\mathbb{A} = (A_0, A_1, \ldots)$ internal to $\mathcal{E}$ is **coherent** if $\langle \mathsf{d}_{\mathbb{A}}, \mathsf{c}_{\mathbb{A}}\rangle\colon A_1 \to A_0 \times A_0$ is coherent. This means that:

(1) A_0 is compact,

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(2) $\langle \mathsf{d}_{\mathbb{A}}, \mathsf{c}_{\mathbb{A}} \rangle \colon A_1 \to A_0 \times A_0$ is compact, and
(3) $\mathrm{pr}_{1,1} \colon A_1 \hookrightarrow A_1 \underset{A_0 \times A_0}{\times} A_1$ is compact.

The two legs $\mathsf{d}_{\mathbb{A}}$ and $\mathsf{c}_{\mathbb{A}}$ will not be coherent in general. However, they are always compact since the two product projections from $A_0 \times A_0$ are compact by **??**.

We shall be interested in the full sub-2-category $\mathbf{CohGpd}(\mathcal{E})$ of $\mathbf{Gpd}(\mathcal{E})$ on the coherent groupoids. **??** entails that it will not be possible to show in general that $\mathbf{CohGpd}(\mathcal{E})$ has finite limits: since the object of objects of a coherent groupoid is not a coherent object, we should not expect to be able to cover every (finite) limit of coherent groupoids.

On the other hand, $\mathbf{CohGpd}(\mathcal{E})$ does have certain (2-)limits.

Let us begin considering the object of (not necessarily) commutative squares in an internal groupoid $\mathbb{A}$. It can be computed as the pullback on the left below,

$$\begin{tikzcd} \mathrm{Sq}_{\mathbb{A}} \arrow[r, "\langle \mathsf{l},\mathsf{r} \rangle"] \arrow[d, "\langle \mathsf{t},\mathsf{b} \rangle"'] & A_1 \times A_1 \arrow[d, "\langle \mathsf{d},\mathsf{c} \rangle \times \langle \mathsf{d},\mathsf{c} \rangle"] \\ A_1 \times A_1 \arrow[r, "\langle \mathsf{d} \times \mathsf{d}, \mathsf{c} \times \mathsf{c} \rangle"'] & A_0 \times A_0 \times A_0 \times A_0 \end{tikzcd} \qquad \begin{tikzcd} A_1 \times_{A_0} A_1 \arrow[r, "p_2"] \arrow[d, "p_1"'] & A_1 \arrow[d, "\mathsf{d}"] \\ A_1 \arrow[r, "\mathsf{c}"'] & A_0 \end{tikzcd} \tag{1}$$

where think of the first leg as providing the top and bottom arrows in the square, and of the second leg as providing the left and right arrows. The two (non trivial) composable pairs in the square can be obtained as the two arrows into the pullback on the right induced by the pairs $\langle \mathsf{l}, \mathsf{b} \rangle \colon \mathrm{Sq}_{\mathbb{A}} \to A_1 \times A_1$ and $\langle \mathsf{t}, \mathsf{r} \rangle \colon \mathrm{Sq}_{\mathbb{A}} \to A_1 \times A_1$, respectively. We shall write $\langle \mathsf{l}, \mathsf{b} \rangle$ and $\langle \mathsf{t}, \mathsf{r} \rangle$ for the induced arrows into $A_1 \times_{A_0} A_1$ as well.

Lemma 2.1. *The two legs $\langle \mathsf{t}, \mathsf{b} \rangle$ and $\langle \mathsf{l}, \mathsf{r} \rangle$ of the pullback (1) that defines $\mathrm{Sq}_{\mathbb{A}}$ are coherent.*

Proof. This follows from **??** once we observe that $\langle \mathsf{d}, \mathsf{c} \rangle \times \langle \mathsf{d}, \mathsf{c} \rangle$ is a product of coherent arrows, and $\langle \mathsf{d} \times \mathsf{d}, \mathsf{c} \times \mathsf{c} \rangle$ factors through it via the automorphism $\mathrm{pr}_{1,3,2,4}$ of $A_0 \times A_0 \times A_0 \times A_0$ (which is a compact object). □

Given an internal groupoid $\mathbb{A}$, the object of internal commutative squares in $\mathrm{CSq}_{\mathbb{A}}$ is the subobject of $\mathrm{Sq}_{\mathbb{A}}$ that fits into the pullback below,

$$\begin{tikzcd} \mathrm{CSq}_{\mathbb{A}} \arrow[rr] \arrow[d, hook, "m"'] & & A_1 \arrow[d, hook, "\mathrm{pr}_{1,1}"] \\ \mathrm{Sq}_{\mathbb{A}} \arrow[rr, "\langle \mathsf{cmp}_{\mathbb{A}} \circ \langle \mathsf{l},\mathsf{b} \rangle, \mathsf{cmp}_{\mathbb{A}} \circ \langle \mathsf{t},\mathsf{r} \rangle \rangle"'] & & A_1 \underset{A_0 \times A_0}{\times} A_1 \end{tikzcd} \tag{2}$$

where the object in the bottom right is the kernel of $\langle \mathsf{d}_{\mathbb{A}}, \mathsf{c}_{\mathbb{A}} \rangle \colon A_1 \to A_0 \times A_0$. The bottom arrow clearly exists.

The object $\mathrm{CSq}_{\mathbb{A}}$ is the object of arrows of the internal groupoid $\mathbb{A}^{\to}$ of arrows of $\mathbb{A}$, and the object of objects is A_1. For the domain and codomain structure map of $\mathbb{A}^{\to}$ we could pick either of the two arrows below.

$$\begin{tikzcd} \mathrm{CSq}_{\mathbb{A}} \arrow[rr, "\langle \mathsf{t},\mathsf{b} \rangle"] \arrow[dr, hook, "m"'] & & A_1 \times A_1 \\ & \mathrm{Sq}_{\mathbb{A}} \arrow[ur, "\langle \mathsf{t},\mathsf{b} \rangle"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathrm{CSq}_{\mathbb{A}} \arrow[rr, "\langle \mathsf{l},\mathsf{r} \rangle"] \arrow[dr, hook, "m"'] & & A_1 \times A_1 \\ & \mathrm{Sq}_{\mathbb{A}} \arrow[ur, "\langle \mathsf{l},\mathsf{r} \rangle"'] & \end{tikzcd} \tag{3}$$

The two choices give rise of course to isomorphic groupoids. We take the leg $\langle \mathsf{l}, \mathsf{r} \rangle$ to be the structure map. The rest of the structure is induced in the obvious way from that of $\mathbb{A}$ via the pullbacks in (1) and (2), or more directly using the fact that the pair of arrows $\langle \mathsf{l}, \mathsf{b} \rangle \circ m, \langle \mathsf{t}, \mathsf{r} \rangle \circ m \colon \mathrm{CSq}_{\mathbb{A}} \to A_1 \times_{A_0} A_1$ is the kernel pair of the internal composition $\mathsf{cmp}_{\mathbb{A}} \colon A_1 \times_{A_0} A_1 \to A_1$.

Corollary 2.2. *If $\mathbb{A}$ is a coherent groupoid, then its groupoid of arrows $\mathbb{A}^{\to}$ is coherent.*

Proof. The diagonal in (2) is compact, and thus coherent by **??**(**??**). Therefore, the arrow m is coherent as well by **??**. It follows from Lemma 2.1 and **??**(**??**) that the two composites in (3) are both coherent. In particular, the groupoid of arrows $\mathbb{A}^{\to}$ is coherent. □

Proposition 2.3. *Let $f\colon \mathbb{A} \to \mathbb{C}$ and $g\colon \mathbb{B} \to \mathbb{C}$ be functors between coherent groupoids. Then the comma groupoid $f{\downarrow}g$ is coherent.*

Proof. The objects of objects and of arrows of the comma groupoid $f{\downarrow}g$ can be constructed taking the pullback squares below.

$$\begin{tikzcd} (f{\downarrow}g)_0 \arrow[r, "\mathsf{c}_0"] \arrow[d, "\langle \mathsf{l}_0,\mathsf{r}_0\rangle"'] & C_1 \arrow[d, "\langle \mathsf{d}_{\mathbb{C}},\mathsf{c}_{\mathbb{C}}\rangle"] \\ A_0 \times B_0 \arrow[r, "f_0\times g_0"] & C_0 \times C_0 \end{tikzcd} \qquad \begin{tikzcd} S \arrow[r, "\langle l,r\rangle"] \arrow[d, "\langle t,b\rangle"'] & A_1 \times B_1 \arrow[d, "\langle \mathsf{d}_{\mathbb{A}},\mathsf{c}_{\mathbb{A}}\rangle\times\langle \mathsf{d}_{\mathbb{B}},\mathsf{c}_{\mathbb{B}}\rangle"] \\ (f{\downarrow}g)_0 \times (f{\downarrow}g)_0 \arrow[r, "\langle \mathsf{l}_0\times\mathsf{l}_0,\, \mathsf{r}_0\times\mathsf{r}_0\rangle"] & A_0\times A_0\times B_0\times B_0 \end{tikzcd}$$

$$\begin{tikzcd} (f{\downarrow}g)_1 \arrow[r] \arrow[d, hook, "m"'] & C_1 \arrow[d, hook, "\mathrm{pr}_{1,1}"] \\ S \arrow[r, "\langle \mathsf{cmp}_{\mathbb{C}}\circ\langle l,b\rangle,\, \mathsf{cmp}_{\mathbb{C}}\circ\langle t,r\rangle\rangle"] & C_1 \underset{C_0\times C_0}{\times} C_1 \end{tikzcd}$$

We need to show that the arrow $\langle t,b\rangle \circ m\colon (f{\downarrow}g)_1 \to (f{\downarrow}g)_0 \times (f{\downarrow}g)_0$ is coherent. This follows from the following applications of **??**: the arrow $\langle \mathsf{l}_0, \mathsf{r}_0\rangle$ is coherent, in particular, $(f{\downarrow}g)_0$ is compact; the arrow $\langle t,b\rangle$ is coherent, in particular S is compact; and, finally, the arrow m is coherent. ☐

Remark 2.4. The arrow $\langle \mathsf{l}_0 \times \mathsf{l}_0, \mathsf{r}_0 \times \mathsf{r}_0\rangle$ is also coherent, since it factors through $\langle \mathsf{l}_0, \mathsf{r}_0\rangle \times \langle \mathsf{l}_0, \mathsf{r}_0\rangle$ via an automorphism of their codomain. Therefore, both legs from S are coherent. we shall write

$$\langle \mathsf{l}_1, \mathsf{r}_1\rangle \coloneqq \langle l, r\rangle \circ m \qquad \text{and} \qquad \langle \mathsf{d}_{f\downarrow g}, \mathsf{c}_{f\downarrow g}\rangle \coloneqq \langle t, b\rangle \circ m$$

for the corresponding coherent legs from $(f{\downarrow}g)_1$.

The pairs $(\mathsf{l}_0, \mathsf{l}_1)$ and $(\mathsf{r}_0, \mathsf{r}_1)$ are internal functors from $f{\downarrow}g$ to $\mathbb{A}$ and $\mathbb{B}$, respectively.

There is an algebraic weak factorisation system (L, R) on $\mathbf{Gpd}(\mathcal{E})$ whose algebras are split isofibrations, and whose 2-category of pseudo-algebras and pseudo-morphisms is equivalent to the category of cloven isofibrations (where morphisms simply preserve cartesian arrows).

Given an internal functor $f\colon \mathbb{A} \to \mathbb{B}$, the factorisation of f is taken through an object which can be computed as the comma groupoid $f{\downarrow}\mathrm{id}_{\mathbb{B}}$. If $\mathbb{A}$ and $\mathbb{B}$ are coherent groupoids, then so is $f{\downarrow}\mathrm{id}_{\mathbb{B}}$ by Proposition 2.3. We thus have the following.

Proposition 2.5. *The awfs (L, R) on $\mathbf{Gpd}(\mathcal{E})$ restricts to an awfs $(\mathsf{L}', \mathsf{R}')$ on $\mathbf{CohGpd}(\mathcal{E})$ such that*

(1) the 2-category of algebras of R' and their pseudo-morphisms is equivalent to the 2-category of split isofibrations between coherent groupoids, and
(2) the 2-category of pseudo-algebras of R' and their pseudo-morphisms is equivalent to the 2-category of cloven isofibrations between coherent groupoids.

Or rather, internally it makes sense to define the 2-category of internal (split) isofibrations between coherent groupoids as the 2-category of (strict) algebras and pseudo morphisms $\mathbf{Alg}(\mathsf{R}')$. This is fibred over $\mathbf{CohGpd}(\mathcal{E})$, and the fibration is the one that supports the Hofmann–Streicher groupoid model.