A model structure on the cartesian cubical sets

May 25, 2018

1 The cartesian cube category

We consider the cartesian cube category $\mathbb{C}$, defined as the free finite product category on an interval $\delta_0, \delta_1 : 1 \rightrightarrows \mathrm{I}$. As a classifying category for an algebraic theory $\mathbb{T} = \{0, 1\}$, $\mathbb{C}$ has a covariant presentation by Lawvere duality, namely as the dual of the full subcategory of finitely-generated, free $\mathbb{T}$-algebras $\mathsf{Alg}(\mathbb{T})_{\mathrm{fg}}$. In this case, the algebras are simply *bipointed sets* (A, a_0, a_1), and the free ones are the *strictly* bipointed sets $a_0 \neq a_1$. Thus $\mathsf{Alg}(\mathbb{T})_{\mathrm{fg}}$ consists of the finite, strictly bipointed sets and all bipointed maps between them.

Definition 1. The objects of the cartesian cube category $\mathbb{C}$ are themselves called cubes, and will be written

$$[n] = \{x_1, ..., x_n\},$$

where the x_i may be regarded as coordinate axes. The arrows,

$$f : [n] \longrightarrow [m]\,,$$

are then taken to be m-tuples of elements drawn from the set

$$[n]^+ = \{0, x_1, ..., x_n, 1\},$$

regarded as formal terms representing composites of faces, degeneracies, permutations, and diagonals (alternately, the arrows are arbitrary bipointed maps $[m]^+ \longrightarrow [n]^+$).

2 Cubical sets

The category cSet of *cubical sets* is the category of presheaves on the cartesian cube category $\mathbb{C}$. It is generated by the representable presheaves $y([n])$, which will be written $\mathrm{I}^n = y([n])$ and called the *standard n-cubes*.

3 Partial map classification and the +-algebra weak factorization system

Cofibrations, partial map classification, the functor X^+, the awfs of +-algebras.

4 Partial path lifting (biased version)

Given a type $P : \mathrm{I} \longrightarrow \mathsf{Set}$, the type of (0-biased) partial path-lifting structures $L^0(P)$ may be defined in the "logical style" of [**?**] as:

$$L^0(P) \;=\; \prod_{\phi:\Phi}\; \prod_{s:\prod_{i:I}(Pi)^\phi}\; \prod_{a_0:P0} a_0|\phi = s0 \;\longrightarrow \sum_{a:\prod_{i:I}Pi} (a0 = a_0) \times (a|\phi = s)\,. \tag{1}$$

The data involved in this type can be represented as follows:

$$\begin{tikzcd}
& P0 \arrow[rr] \arrow[dd] && P \arrow[dd] \\
{[\phi]} \arrow[ur, "s0"] \arrow[rr] \arrow[dd, tail] && {[\phi]\times\mathrm{I}} \arrow[ur, "s"'] \arrow[dd, tail] & \\
& 1 \arrow[rr, "\delta_0"'] && \mathrm{I} \\
\mathrm{I}^n \arrow[ur] \arrow[rr] \arrow[uuur, "a_0"] && \mathrm{I}^n\times\mathrm{I} \arrow[ur, "\pi_2"'] \arrow[uuur, dotted, "a"] &
\end{tikzcd} \tag{2}$$

Here the left-hand vertical square is understood to be a pullback of the right-hand one along the chosen endpoint $\delta_0 : 1 \longrightarrow \mathrm{I}$ (the "bias").

Now write

$$\widetilde{P} \;=\; \prod_{i:I} Pi$$

for the type of sections of the projection $P = \sum_{i:I} Pi \longrightarrow \mathrm{I}$, and write

$$\pi_0 : \widetilde{P} \longrightarrow P0$$

for the 0^{th}-projection (i.e. the evaluation of $P : \mathrm{I} \longrightarrow \mathsf{Set}$ at $0 : \mathrm{I}$).

Then the (0-biased) partial path-lifting structures on P correspond to $+$-algebra structures on the projection $\pi_0 : \widetilde{P} \longrightarrow P0$, as follows.

Proposition 2. *For any $P : \mathsf{Set}^{\mathrm{I}}$, there is an isomorphism*

$$L^0(P) \;\cong\; {}^{+}\mathsf{Alg}(\pi_0 : \widetilde{P} \longrightarrow P0)\,.$$

Proof. Consider the following diagram,

$$\begin{tikzcd}
 & \widetilde{P} \arrow[rr] \arrow[dd, "\pi_0"] & & \widetilde{P} \times \mathrm{I} \arrow[dd, "\varepsilon"] \\
 & & & \\
 & P0 \arrow[rr] \arrow[dd] & & P \arrow[dd] \\
{[\phi]} \arrow[uuur, "\overline{s}_0"] \arrow[ur, "s0"] \arrow[rr] \arrow[dd, tail] & & {[\phi]} \times \mathrm{I} \arrow[uuur, "\overline{s}"] \arrow[ur, "s"] \arrow[dd, tail] & \\
 & 1 \arrow[rr, "\delta_0"'] & & \mathrm{I} \\
\mathrm{I}^n \arrow[ur] \arrow[uuur, "a_0"] \arrow[rr] & & \mathrm{I}^n \times \mathrm{I} \arrow[ur, "\pi_2"'] \arrow[uuur, dotted, "a"'] &
\end{tikzcd} \tag{3}$$

which is (2), extended by the counit (evaluation) $\varepsilon : \widetilde{P} \times \mathrm{I} \longrightarrow P$ over I on the right. The pullback of ε over I along δ_0 is just $\pi_0 : \widetilde{P} \longrightarrow P0$.

Given an $L^0(P)$-structure we construct a ${}^{+}\mathsf{Alg}$-structure on $\pi_0 : \widetilde{P} \longrightarrow P0$ as follows: for any I^n and cofibration $i_\phi : [\phi] \rightarrowtail \mathrm{I}^n$ and any commutative square,

$$\begin{tikzcd}
{[\phi]} \arrow[r, "s"] \arrow[d, tail, "i_\phi"'] & \widetilde{P} \arrow[d, "\pi_0"] \\
\mathrm{I}^n \arrow[r, "a_0"'] & P0\,,
\end{tikzcd}$$

we require a diagonal filler,

$$\begin{tikzcd} {[\phi]} \arrow[r, "s"] \arrow[d, tail, "i_\phi"'] & \widetilde{P} \arrow[d, "\pi_0"] \\ \mathrm{I}^n \arrow[r, "a_0"'] \arrow[ur, dotted, "j"] & P0 \,, \end{tikzcd}$$

uniformly in I^n and ϕ. Transposing the span formed by i_ϕ and s along the adjunction $\mathrm{I}^* \dashv \prod_{\mathrm{I}}$ gives the right-hand square in (**??**), and the commutative square formed by a_0 and π_0 gives the rest of the data in that diagram. Thus the $L^0(P)$-structure gives an $a : \mathrm{I}^n \times \mathrm{I} \longrightarrow P$ as indicated. Looking at (3), we see that a lifts across ε to a unique map $\overline{a} : \mathrm{I}^n \times \mathrm{I} \longrightarrow \widetilde{P} \times \mathrm{I}$ over I, by the universal property of $\varepsilon : \widetilde{P} \times \mathrm{I} \longrightarrow P$. We can therefore set

$$j = \delta_0^*(\overline{a}) : \mathrm{I}^n \longrightarrow \widetilde{P}\,.$$

Suppose conversely that we have a $^+\mathsf{Alg}$-structure on $\pi_0 : \widetilde{P} \longrightarrow P0$, and we want to build a (0-biased) partial path-lifting structure on P. Take any $\mathrm{I}^n, \phi, s, a_0$ as indicated and we require an $a : \mathrm{I}^n \times \mathrm{I} \longrightarrow P$ over I. From s we get $\overline{s}$ by the universal property of ε, and therefore we get $\overline{s}_0$ by pullback. From $\overline{s}_0$ and a_0 and the $^+\mathsf{Alg}$ structure on π_0 we get a map $j : \mathrm{I}^n \longrightarrow \widetilde{P}$ over $P0$ which is a diagonal filler of the indicated square formed by $i_\phi, \overline{s}_0, a_0$ and π_0. We then get the required map $a : \mathrm{I}^n \times \mathrm{I} \longrightarrow P$ over I as the $(\mathrm{I}^* \dashv \prod_{\mathrm{I}})$-transpose of j,

$$a = \varepsilon \circ (j \times \mathrm{I})\,.$$

We leave to the reader the verification that these assignments are mutually inverse. $\square$

$$
\begin{tikzcd}
 & & B_1 \arrow[dd, "q"] & & \\
 \cdot \arrow[urr] \arrow[rr] \arrow[dd, "q_X"'] & & & G \arrow[ul] \arrow[dd, "q'"] & \\
 & & B_0 & & \\
 X \times_{A_0} A_1 \arrow[urr, "b"] \arrow[rrr, "\bar{b} \times_{A_0} A_1"'] \arrow[d, "p_X"'] & & & B_0^{A_1} \times_{A_0} A_1 \arrow[ul, "\mathrm{ev}"'] \arrow[r] \arrow[d, "p'"] & A_1 \arrow[d, "p"] \\
 X \arrow[rrr, "\bar{b}"] \arrow[rrrr, bend right, "a"'] \arrow[uuu, bend left, "{(a',b')}"] & & & B_0^{A_1} \arrow[r] & A_0
\end{tikzcd}
$$

$$
\begin{tikzcd}
 B_1 \arrow[d, "q"] & & \\
 B_0 & G \arrow[ul] \arrow[d, "q'"] \arrow[dd, bend right] & \\
 & B_0^{A_1} \times_{A_0} A_1 \arrow[ul] \arrow[r] \arrow[d, "p'"] & A_1 \arrow[d, "p"] \\
 & B_0^{A_1} \arrow[r] & A_0.
\end{tikzcd}
$$

5 Unbiased partial path lifting

6 A left-induced model structure on the Cartesian cubical sets

We make use of the Sattler model structure [**?**] on the *Dedekind cubical sets* $\widehat{\mathbb{D}} = \mathsf{Set}^{\mathbb{D}^{\mathrm{op}}}$, where $\mathbb{D}$ is the category of *Dedekind cubes*, defined as the Lawvere theory of distributive lattices. The unique product-preserving functor

$$i : \mathbb{C} \longrightarrow \mathbb{D}$$

classifying the Dedekind interval $\mathrm{I}_{\mathbb{D}} \in \mathbb{D}$ induces an adjunction,

$$i_! \dashv i^* \dashv i_* \ : \ \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{D}} \,,$$

where $i^*(Q) = Q \circ i$, for $Q \in \mathbb{D}$.

Lemma 3. *Observe that $i_!$ is left exact since the Dedekind interval $\mathrm{I}_{\mathbb{D}}$ is* strict*, $0 \neq 1 : 1 \rightrightarrows \mathrm{I}_{\mathbb{D}}$. Thus we have geometric morphisms:*

$$(i_! \dashv i^*) : \ \widehat{\mathbb{D}} \longrightarrow \widehat{\mathbb{C}} \,,$$

classifying the bipointed object $i_!(\mathrm{I}_{\mathbb{C}}) = \mathrm{I}_{\mathbb{D}}$,

$$(i^* \dashv i_*) : \ \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{D}} \,,$$

classifying the dLat $i^*(\mathrm{I}_{\mathbb{D}}) := \mathbb{I}$, where $\eta : \mathrm{I}_{\mathbb{C}} \longrightarrow \mathbb{I}$ can be described pointwise as the distributive lattice completion of the corresponding bipointed set.

Also, since i is faithful so is $i_!$, and since i is surjective on objects i^* is also faithful.

It follows that:

- *$\widehat{\mathbb{C}}$ is $(i_! \circ i^*)$-coalgebras on $\widehat{\mathbb{D}}$,*
- *$\widehat{\mathbb{D}}$ is $(i^* \circ i_*)$-coalgebras on $\widehat{\mathbb{C}}$,*
- *$\widehat{\mathbb{D}}$ is $(i^* \circ i_!)$-algebras on $\widehat{\mathbb{C}}$.*

We will use the following transfer theorem for QMSs from [**?**, **?**]:

Theorem ([**?**, **?**]). *Suppose $\widehat{\mathbb{D}}$ has a (cofibrantly generated) model structure $(\mathcal{C}_{\mathbb{D}}, \mathcal{W}_{\mathbb{D}}, \mathcal{F}_{\mathbb{D}})$. Given an adjunction*

$$i_! \dashv i^* \ : \ \widehat{\mathbb{D}} \longrightarrow \widehat{\mathbb{C}} \,,$$

there is a left-induced model structure *on $\widehat{\mathbb{C}}$ if the following* acyclicity condition *holds:*

$$(i_!^{-1}\mathcal{C}_{\mathbb{D}})^{\pitchfork} \ \subset \ i_!^{-1}\mathcal{W}_{\mathbb{D}} \,.$$

For the left-induced model structure $(\mathcal{C}_{\mathbb{C}}, \mathcal{W}_{\mathbb{C}}, \mathcal{F}_{\mathbb{C}})$ on $\widehat{\mathbb{C}}$ we then have:

$$\begin{aligned} \mathcal{C}_{\mathbb{C}} &= i_!^{-1}\mathcal{C}_{\mathbb{D}} \,, \\ \mathcal{W}_{\mathbb{C}} &= i_!^{-1}\mathcal{W}_{\mathbb{D}} \,. \end{aligned}$$

The Sattler model structure on $\widehat{\mathbb{D}}$ is given as follows (for a constructive treatment a smaller class of "pointwise decidable cofibrations" is used, but we consider the classical case first):

$$\begin{aligned}\mathcal{C} &= \text{monomorphisms}\,,\\ \mathcal{W} &= \{f \mid f = p \circ i,\ p \in \mathcal{F} \cap \mathcal{W},\ i \in \mathcal{C} \cap \mathcal{W}\}\,,\\ \mathcal{F} &= (\mathcal{C} \otimes \delta)^{\pitchfork}\,.\end{aligned}$$

where $\delta : 1 \longrightarrow \mathrm{I}$ is either endpoint inclusion.

For the left-induced model structure on $\widehat{\mathbb{C}}$ we therefore have the following specification:

$$\begin{aligned}\mathcal{C} &= \text{monomorphisms},\\ \mathcal{W} &= \{f \mid i_!f = p \circ i,\ p \in \mathcal{F}_{\mathbb{D}} \cap \mathcal{W}_{\mathbb{D}},\ i \in \mathcal{C}_{\mathbb{D}} \cap \mathcal{W}_{\mathbb{D}}\}\,,\\ \mathcal{F} &= (\mathcal{C} \cap \mathcal{W})^{\pitchfork}\,.\end{aligned}$$

The determination of $\mathcal{C}$ follows from the fact that $i_! : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{D}}$ is conservative.

To check the acyclicity condition,

$$(i_!^{-1}\mathcal{C}_{\mathbb{D}})^{\pitchfork} \ \subset\ i_!^{-1}\mathcal{W}_{\mathbb{D}}\,,$$

we know that $i_!^{-1}\mathcal{C}_{\mathbb{D}}$ consists of the monos in $\mathbb{C}$, so take $f : Y \longrightarrow X$ in $(i_!^{-1}\mathcal{C}_{\mathbb{D}})^{\pitchfork}$, apply $i_!$, and factor the result as $i_!f = p \circ m : i_!Y \longrightarrow Z \longrightarrow i_!X$ with $p \in \mathcal{F}_{\mathbb{D}} \cap \mathcal{W}_{\mathbb{D}}$ and $m : i_!Y \longrightarrow Z$ monic. We then need to show that m is in $\mathcal{W}_{\mathbb{D}}$.

We can apply Theorem 2.2.1 of [**?**], with $\mathsf{K} = \widehat{\mathbb{C}}$, $\mathsf{M} = \widehat{\mathbb{D}}$, $V = i_!$, $k = i^*$, and:

1. $QX = X$ and $\epsilon = 1_X : X \longrightarrow X$, so that $i_!1_X = 1_{i_!X}$ and therefore in $\mathcal{W}_{\mathbb{D}}$, while all objects are cofibrant,
2. $Qf = f$ for any $f : X \longrightarrow Y$ in $\widehat{\mathbb{C}}$, so that the naturality condition is similarly trivial,
3. factor the codiagonal $X + X \longrightarrow X$ as $\pi_2 \circ j : X + X \longrightarrow \mathrm{I} \times X \longrightarrow X$ with $j = (\partial\mathrm{I} \times X) : X + X \longrightarrow \mathrm{I} \times X$.

It remains only to show that $i_!p : \mathrm{i}_!(\mathrm{I} \times X) \longrightarrow i_!X$ is in $\mathcal{W}_{\mathbb{D}}$ and $i_!j : i_!(X + X) \longrightarrow i_!(\mathrm{I} \times X)$ is in $\mathcal{C}_{\mathbb{D}}$. The latter is clear, since j is monic. To show the former, observe that for any $D \in \widehat{\mathbb{D}}$, the projection $\pi_2 : \mathrm{I}_{\mathbb{D}} \times D \longrightarrow D$ is in $\mathcal{W}_{\mathbb{D}}$ by 3-for-2, since an endpoint inclusion $D \longrightarrow \mathrm{I}_{\mathbb{D}} \times D$ is a cofibration and a strong deformation retract, hence in $\mathcal{W}_{\mathbb{D}}$.

Thus we have shown:

Theorem 4. *There is a Quillen model structure* $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ *on the category* $\widehat{\mathbb{C}}$ *of cartesian cubical sets, in which*

$$\begin{aligned} \mathcal{C} &= \textit{monomorphisms}, \\ \mathcal{W} &= \{ f \mid i_! f = p \circ i,\ p \in \mathcal{F}_{\mathbb{D}} \cap \mathcal{W}_{\mathbb{D}}\,,\ i \in \mathcal{C}_{\mathbb{D}} \cap \mathcal{W}_{\mathbb{D}} \}\,, \\ \mathcal{F} &= (\mathcal{C} \cap \mathcal{W})^{\pitchfork}\,. \end{aligned}$$

where $i_! : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{D}}$ *is the left adjoint of precomposition along the canonical map* $i : \mathbb{C} \longrightarrow \mathbb{D}$ *from Cartesian cubes to Dedekind cubes, and* $(\mathcal{C}_{\mathbb{D}}, \mathcal{W}_{\mathbb{D}}, \mathcal{F}_{\mathbb{D}})$ *is the Sattler model structure on* $\widehat{\mathbb{D}}$*.*

References:
- Gambino-Sattler
- Sattler
- Hess, Kedziorek, Riehl, Shipley
- Garner, Kedziorek, Riehl