

A model structure on the cartesian cubical sets

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1 The cartesian cube category

We consider the cartesian cube category \mathbb{C} , defined as the free finite product category on an interval $\delta_0, \delta_1 : 1 \rightrightarrows I$. As a classifying category for an algebraic theory $\mathbb{T} = \{0, 1\}$, \mathbb{C} has a covariant presentation by Lawvere duality, namely as the dual of the full subcategory of finitely-generated, free \mathbb{T} -algebras $\text{Alg}(\mathbb{T})_{\text{fg}}$. In this case, the algebras are simply *bipointed sets* (A, a_0, a_1) , and the free ones are the *strictly* bipointed sets $a_0 \neq a_1$. Thus $\text{Alg}(\mathbb{T})_{\text{fg}}$ consists of the finite, strictly bipointed sets and all bipointed maps between them.

Definition 1. The objects of the cartesian cube category \mathbb{C} are themselves called cubes, and will be written

$$[n] = \{x_1, \dots, x_n\},$$

where the x_i may be regarded as coordinate axes. The arrows,

$$f : [n] \longrightarrow [m],$$

are then taken to be m -tuples of elements drawn from the set

$$[n]^+ = \{0, x_1, \dots, x_n, 1\},$$

regarded as formal terms representing composites of faces, degeneracies, permutations, and diagonals (alternately, the arrows are arbitrary bipointed maps $[m]^+ \longrightarrow [n]^+$).

2 Cubical sets

The category \mathbf{cSet} of *cubical sets* is the category of presheaves on the cartesian cube category \mathbb{C} . It is generated by the representable presheaves $y([n])$, which will be written $I^n = y([n])$ and called the *standard n -cubes*.

3 Partial map classification and the $+-$ -algebra weak factorization system

Cofibrations, partial map classification, the functor X^+ , the awfs of $+-$ -algebras.

4 Partial path lifting (directed version)

We first recall the specification of the trivial-cofibration/fibration WFS from [?], and show that the resulting fibrations are equivalent to those specified in the “logical style” given in [?, ?].

The generating trivial cofibrations are all maps of the form

$$m \otimes \delta_\epsilon : U \longrightarrow I^{n+1}, \quad (1)$$

where:

1. $n \geq 0$,
2. $\delta_\epsilon : 1 \longrightarrow I$ is one of the two endpoint inclusions, where $\epsilon = 0, 1$,
3. $m \otimes \delta_\epsilon$ is the push-out product, resp. “Leibniz tensor”, of any cofibration $m : M \longrightarrow I^n$ and a $\delta_\epsilon : 1 \longrightarrow I$,
4. U is $I^n +_M (M \times I)$, the domain of $m \otimes \delta_\epsilon$.

Let $\mathcal{C} \otimes \delta_\epsilon$ be the set of all such maps; the *fibrations* are defined to be the elements of the right class of these,

$$\mathcal{F} = (\mathcal{C} \otimes \delta_\epsilon)^\pitchfork.$$

A *fibration structure* on a map $f : Y \longrightarrow X$ is a choice of diagonal fillers,

$$\begin{array}{ccc} I^n +_M (M \times I) & \longrightarrow & X \\ m \otimes \delta_\epsilon \downarrow & \nearrow & \downarrow f \\ I^n \times I & \longrightarrow & Y. \end{array} \quad (2)$$

that is uniform with respect to arbitrary pullbacks of the cofibration m , as in the case of the $+$ algebra factorization system.

Fixing the argument δ_ϵ , the Leibniz tensor functor

$$(-) \otimes \delta_\epsilon : \widehat{\mathbb{C}}^2 \longrightarrow \widehat{\mathbb{C}}^2$$

has a right adjoint, the “Leibniz exponential”, which for a map $f : X \longrightarrow Y$ we will write as,

$$(\delta_\epsilon \Rightarrow f) : X^{\mathbf{I}} \longrightarrow (Y^{\mathbf{I}} \times_Y X).$$

Using this adjunction on arrow categories, one can easily show the following:

Proposition 2. *An object X is fibrant if and only if both of the pathspace projections $X^{\delta_\epsilon} : X^{\mathbf{I}} \longrightarrow X$ are $+$ algebras.*

An analogous statement also holds for maps $f : X \longrightarrow Y$ in place of objects X .

4.1 Logical specification of partial path lifting

To make the connection to the logical style of presentation used in [?, ?], suppose we want to describe a (uniform) filling structure on an arbitrary $f : X \longrightarrow Y$ with respect to all generating trivial cofibrations $m \otimes \delta_\epsilon : \mathbf{I}^n +_M (M \times \mathbf{I}) \longrightarrow \mathbf{I}^{n+1}$,

$$\begin{array}{ccc} \mathbf{I}^n +_M (M \times \mathbf{I}) & \xrightarrow{\quad} & X \\ m \otimes \delta_\epsilon \downarrow & \nearrow & \downarrow f \\ \mathbf{I}^n \times \mathbf{I} & \xrightarrow{\quad c \quad} & Y. \end{array} \quad (3)$$

By pulling back along c , it suffices to consider just the case $Y = \mathbf{I}^n \times \mathbf{I}$ and c the identity map. Moreover, since we shall internalize the quantification over all cofibrations $m : M \hookrightarrow \mathbf{I}^n$ using the classifier Φ , it suffices to consider the following case internally,

$$\begin{array}{ccc} 1 + [\phi] ([\phi] \times \mathbf{I}) & \xrightarrow{[a_0, s]} & X \\ \phi \otimes \delta_\epsilon \downarrow & \nearrow a & \downarrow \\ 1 \times \mathbf{I} & \xrightarrow{\cong} & \mathbf{I} \end{array} \quad (4)$$

where $[\phi] \hookrightarrow 1$ is classified by $\phi : 1 \rightarrow \Phi$.

Using a universe \mathbf{Set} in the internal language of $\widehat{\mathbb{C}}$, we can regard the family $X \rightarrow \mathbf{I}$ internally as a map $P : \mathbf{I} \rightarrow \mathbf{Set}$ (switching notation from X to P to agree with [?]). Thus we arrive at the following “logical” specification, in the internal language of $\widehat{\mathbb{C}}$, of the object of “(0-directed) lifting structures” $L^0(P)$ on a family $P : \mathbf{I} \rightarrow \mathbf{Set}$:

$$L^0(P) = \prod_{\phi:\Phi} \prod_{s:\prod_{i:I}(Pi)^\phi} \prod_{a_0:P0} a_0|_\phi = s0 \longrightarrow \sum_{a:\prod_{i:I} Pi} (a0 = a_0) \times (a|_\phi = s). \quad (5)$$

Here the variables $s : \prod_{i:I}(Pi)^\phi$ and $a_0 : P0$, and the condition $a_0|_\phi = s0$, give the domain $1 +_{[\phi]} ([\phi] \times \mathbf{I})$ of the arrow $[a_0, s]$ in (4), and $a : \prod_{i:I} Pi$ is the diagonal filler, with $(a0 = a_0) \times (a|_\phi = s)$ expressing the commutativity of the top triangle.

There is an analogous condition $L^1(P)$ in which 1 replaces 0 everywhere, describing (“directed”) filling from the other end of the interval. Note that [?, ?] derive the “filling” conclusion

$$\sum_{a:\prod_{i:I} Pi} (a0 = a_0) \times (a|_\phi = s)$$

from (connections on \mathbf{I} and) a weaker “composition operation”

$$\sum_{a_1:P1} a_1|_\phi = s_1,$$

but we will not take this approach.

The specification of the type $L^0(P)$ of (5) can also be represented diagrammatically as follows:

$$\begin{array}{ccc} P0 & \xrightarrow{\quad} & P \\ \uparrow s0 & & \uparrow s \\ [\phi] & \xrightarrow{\quad} & [\phi] \times \mathbf{I} \\ \downarrow a_0 & & \downarrow a \\ 1 & \xrightarrow{\delta_0} & \mathbf{I} \\ \downarrow & & \downarrow \pi_2 \\ 1 & \xrightarrow{\quad} & 1 \times \mathbf{I} \end{array} \quad (6)$$

Here the left-hand vertical square is determined as a pullback of the right-hand one along the endpoint $\delta_0 : 1 \rightarrow \mathbf{I}$.

Now write

$$\tilde{P} = \prod_{i:I} P_i$$

for the type of sections of the projection $P = \sum_{i:I} P_i \longrightarrow I$, and write

$$\pi_0 : \tilde{P} \longrightarrow P0$$

for the 0^{th} -projection (i.e. the evaluation of $P : I \longrightarrow \mathbf{Set}$ at $0 : I$).

Then the (0-directed) partial path-lifting structures on P correspond to $+$ -algebra structures on the projection $\pi_0 : \tilde{P} \longrightarrow P0$, as follows.

Proposition 3. *For any $P : \mathbf{Set}^I$, there is an isomorphism*

$$L^0(P) \cong {}^+\mathbf{Alg}(\pi_0 : \tilde{P} \longrightarrow P0).$$

Proof. Consider the following diagram,

$$\begin{array}{ccccc}
 & \tilde{P} & \xrightarrow{\quad} & \tilde{P} \times I & \\
 & \downarrow \pi_0 & & \downarrow \varepsilon & \\
 & P0 & \xrightarrow{\quad} & P & \\
 \uparrow s_0 & & & & \uparrow s \\
 [\phi] & \xrightarrow{\quad} & [\phi] \times I & & \\
 \downarrow a_0 & & \downarrow a & & \\
 I^n & \xrightarrow{\quad} & I^n \times I & \xrightarrow{\quad} & I \\
 & \downarrow \delta_0 & & \downarrow \pi_2 & \\
 & 1 & \xrightarrow{\quad} & I &
 \end{array} \tag{7}$$

which is (6), extended by the counit (evaluation) $\varepsilon : \tilde{P} \times I \longrightarrow P$ over I on the right, and with I^n replacing 1 in order to reason externally. The pullback of ε over I along δ_0 is just $\pi_0 : \tilde{P} \longrightarrow P0$.

Given an $L^0(P)$ -structure we construct a ${}^+\mathbf{Alg}$ -structure on $\pi_0 : \tilde{P} \longrightarrow P0$ as follows: for any I^n and cofibration $i_\phi : [\phi] \hookrightarrow I^n$ and any commutative square,

$$\begin{array}{ccc}
 [\phi] & \xrightarrow{s} & \tilde{P} \\
 i_\phi \downarrow & & \downarrow \pi_0 \\
 I^n & \xrightarrow{a_0} & P0,
 \end{array}$$

we require a diagonal filler,

$$\begin{array}{ccc}
 [\phi] & \xrightarrow{s} & \tilde{P} \\
 i_\phi \downarrow & \nearrow j & \downarrow \pi_0 \\
 I^n & \xrightarrow{a_0} & P0,
 \end{array}$$

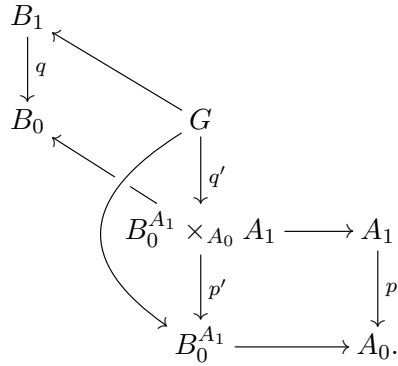
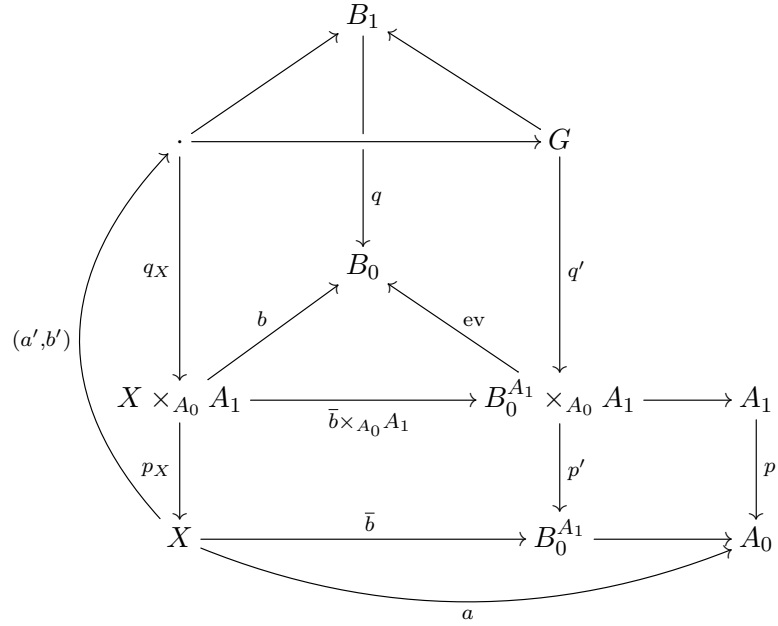
uniformly in I^n and ϕ . Transposing the span formed by i_ϕ and s along the adjunction $I^* \dashv \prod_I$ gives the right-hand square in (??), and the commutative square formed by a_0 and π_0 gives the rest of the data in that diagram. Thus the $L^0(P)$ -structure gives an $a : I^n \times I \longrightarrow P$ as indicated. Looking at (7), we see that a lifts across ε to a unique map $\bar{a} : I^n \times I \longrightarrow \tilde{P} \times I$ over I , by the universal property of $\varepsilon : \tilde{P} \times I \longrightarrow P$. We can therefore set

$$j = \delta_0^*(\bar{a}) : I^n \longrightarrow \tilde{P}.$$

Suppose conversely that we have a ${}^+\mathbf{Alg}$ -structure on $\pi_0 : \tilde{P} \longrightarrow P0$, and we want to build a (0-directed) partial path-lifting structure on P . Take any I^n, ϕ, s, a_0 as indicated and we require an $a : I^n \times I \longrightarrow P$ over I . From s we get \bar{s} by the universal property of ε , and therefore we get \bar{s}_0 by pullback. From \bar{s}_0 and a_0 and the ${}^+\mathbf{Alg}$ structure on π_0 we get a map $j : I^n \longrightarrow \tilde{P}$ over $P0$ which is a diagonal filler of the indicated square formed by i_ϕ, \bar{s}_0, a_0 and π_0 . We then get the required map $a : I^n \times I \longrightarrow P$ over I as the $(I^* \dashv \prod_I)$ -transpose of j ,

$$a = \varepsilon \circ (j \times I).$$

We leave to the reader the verification that these assignments are mutually inverse. \square



5 Undirected partial path lifting

6 A left-induced model structure on the Cartesian cubical sets

We make use of the Sattler model structure [?] on the *Dedekind cubical sets* $\widehat{\mathbb{D}} = \mathbf{Set}^{\mathbb{D}^{\text{op}}}$, where \mathbb{D} is the category of *Dedekind cubes*, defined as

the Lawvere theory of distributive lattices. The unique product-preserving functor

$$i : \mathbb{C} \longrightarrow \mathbb{D}$$

classifying the Dedekind interval $I_{\mathbb{D}} \in \mathbb{D}$ induces an adjunction,

$$i_! \dashv i^* \dashv i_* : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{D}},$$

where $i^*(Q) = Q \circ i$, for $Q \in \mathbb{D}$.

Lemma 4. *Observe that $i_!$ is left exact since the Dedekind interval $I_{\mathbb{D}}$ is strict, $0 \neq 1 : 1 \Rightarrow I_{\mathbb{D}}$. Thus we have geometric morphisms:*

$$(i_! \dashv i^*) : \widehat{\mathbb{D}} \longrightarrow \widehat{\mathbb{C}},$$

classifying the bipointed object $i_!(I_{\mathbb{C}}) = I_{\mathbb{D}}$,

$$(i^* \dashv i_*) : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{D}},$$

classifying the dLat $i^(I_{\mathbb{D}}) := \mathbb{I}$, where $\eta : I_{\mathbb{C}} \longrightarrow \mathbb{I}$ can be described pointwise as the distributive lattice completion of the corresponding bipointed set.*

Also, since i is faithful so is $i_!$, and since i is surjective on objects i^ is also faithful.*

It follows that:

- $\widehat{\mathbb{C}}$ is $(i_! \circ i^*)$ -coalgebras on $\widehat{\mathbb{D}}$,
- $\widehat{\mathbb{D}}$ is $(i^* \circ i_*)$ -coalgebras on $\widehat{\mathbb{C}}$,
- $\widehat{\mathbb{D}}$ is $(i^* \circ i_!)$ -algebras on $\widehat{\mathbb{C}}$.

We will use the following transfer theorem for QMSs from [?, ?]:

Theorem ([?, ?]). *Suppose $\widehat{\mathbb{D}}$ has a (cofibrantly generated) model structure $(\mathcal{C}_{\mathbb{D}}, \mathcal{W}_{\mathbb{D}}, \mathcal{F}_{\mathbb{D}})$. Given an adjunction*

$$i_! \dashv i^* : \widehat{\mathbb{D}} \longrightarrow \widehat{\mathbb{C}},$$

there is a left-induced model structure on $\widehat{\mathbb{C}}$ if the following acyclicity condition holds:

$$(i_!^{-1} \mathcal{C}_{\mathbb{D}})^{\heartsuit} \subset i_!^{-1} \mathcal{W}_{\mathbb{D}}.$$

For the left-induced model structure $(\mathcal{C}_{\mathbb{C}}, \mathcal{W}_{\mathbb{C}}, \mathcal{F}_{\mathbb{C}})$ on $\widehat{\mathbb{C}}$ we then have:

$$\begin{aligned} \mathcal{C}_{\mathbb{C}} &= i_!^{-1} \mathcal{C}_{\mathbb{D}}, \\ \mathcal{W}_{\mathbb{C}} &= i_!^{-1} \mathcal{W}_{\mathbb{D}}. \end{aligned}$$

The Sattler model structure on $\widehat{\mathbb{D}}$ is given as follows (for a constructive treatment a smaller class of “pointwise decidable cofibrations” is used, but we consider the classical case first):

$$\begin{aligned}\mathcal{C} &= \text{monomorphisms}, \\ \mathcal{W} &= \{f \mid f = p \circ i, p \in \mathcal{F} \cap \mathcal{W}, i \in \mathcal{C} \cap \mathcal{W}\}, \\ \mathcal{F} &= (\mathcal{C} \otimes \delta)^\#.\end{aligned}$$

where $\delta : 1 \longrightarrow \mathbf{I}$ is either endpoint inclusion.

For the left-induced model structure on $\widehat{\mathbb{C}}$ we therefore have the following specification:

$$\begin{aligned}\mathcal{C} &= \text{monomorphisms}, \\ \mathcal{W} &= \{f \mid i_! f = p \circ i, p \in \mathcal{F}_{\mathbb{D}} \cap \mathcal{W}_{\mathbb{D}}, i \in \mathcal{C}_{\mathbb{D}} \cap \mathcal{W}_{\mathbb{D}}\}, \\ \mathcal{F} &= (\mathcal{C} \cap \mathcal{W})^\#.\end{aligned}$$

The determination of \mathcal{C} follows from the fact that $i_! : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{D}}$ is conservative.

To check the acyclicity condition,

$$(i_!^{-1} \mathcal{C}_{\mathbb{D}})^\# \subset i_!^{-1} \mathcal{W}_{\mathbb{D}},$$

we know that $i_!^{-1} \mathcal{C}_{\mathbb{D}}$ consists of the monos in \mathbb{C} , so take $f : Y \longrightarrow X$ in $(i_!^{-1} \mathcal{C}_{\mathbb{D}})^\#$, apply $i_!$, and factor the result as $i_! f = p \circ m : i_! Y \longrightarrow Z \longrightarrow i_! X$ with $p \in \mathcal{F}_{\mathbb{D}} \cap \mathcal{W}_{\mathbb{D}}$ and $m : i_! Y \longrightarrow Z$ monic. We then need to show that m is in $\mathcal{W}_{\mathbb{D}}$.

We can apply Theorem 2.2.1 of [?], with $\mathbf{K} = \widehat{\mathbb{C}}$, $\mathbf{M} = \widehat{\mathbb{D}}$, $V = i_!$, $k = i^*$, and:

1. $QX = X$ and $\epsilon = 1_X : X \longrightarrow X$, so that $i_! 1_X = 1_{i_! X}$ and therefore in $\mathcal{W}_{\mathbb{D}}$, while all objects are cofibrant,
2. $Qf = f$ for any $f : X \longrightarrow Y$ in $\widehat{\mathbb{C}}$, so that the naturality condition is similarly trivial,
3. factor the codiagonal $X + X \longrightarrow X$ as $\pi_2 \circ j : X + X \longrightarrow \mathbf{I} \times X \longrightarrow X$ with $j = (\partial \mathbf{I} \times X) : X + X \longrightarrow \mathbf{I} \times X$.

It remains only to show that $i_! p : i_!(\mathbf{I} \times X) \longrightarrow i_! X$ is in $\mathcal{W}_{\mathbb{D}}$ and $i_! j : i_!(X + X) \longrightarrow i_!(\mathbf{I} \times X)$ is in $\mathcal{C}_{\mathbb{D}}$. The latter is clear, since j is monic. To show the former, observe that for any $D \in \widehat{\mathbb{D}}$, the projection $\pi_2 : \mathbf{I}_{\mathbb{D}} \times D \longrightarrow D$ is in $\mathcal{W}_{\mathbb{D}}$ by 3-for-2, since the “cylinder end” inclusion $D \longrightarrow \mathbf{I}_{\mathbb{D}} \times D$, as a pullback of an endpoint inclusion, is a cofibration, and a strong deformation retract (using the connection on \mathbf{I}), and hence is in $\mathcal{W}_{\mathbb{D}}$ by [?].

Thus we have shown:

Theorem 5. *There is a Quillen model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ on the category $\widehat{\mathbb{C}}$ of cartesian cubical sets, in which*

$$\begin{aligned}\mathcal{C} &= \text{monomorphisms,} \\ \mathcal{W} &= \{f \mid i_! f = p \circ i, \ p \in \mathcal{F}_{\mathbb{D}} \cap \mathcal{W}_{\mathbb{D}}, \ i \in \mathcal{C}_{\mathbb{D}} \cap \mathcal{W}_{\mathbb{D}}\}, \\ \mathcal{F} &= (\mathcal{C} \cap \mathcal{W})^{\text{th}}.\end{aligned}$$

where $i_! : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{D}}$ is the left adjoint of precomposition along the canonical map $i : \mathbb{C} \longrightarrow \mathbb{D}$ from Cartesian cubes to Dedekind cubes, and $(\mathcal{C}_{\mathbb{D}}, \mathcal{W}_{\mathbb{D}}, \mathcal{F}_{\mathbb{D}})$ is the Sattler model structure on $\widehat{\mathbb{D}}$.

References:

- Gambino-Sattler
- Sattler
- Hess, Kedziorek, Riehl, Shipley
- Garner, Kedziorek, Riehl