On the cubical model of homotopy type theory*

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The main goal of these notes is to establish the following:

Theorem. *There is an algebraic weak factorization system* (L, R) *on the category of cartesian cubical sets such that for any* R*-object* A*, the factorization of the diagonal map,*

$$A \longrightarrow A^{\mathrm{I}} \longrightarrow A \times A\,,$$

determined by the 1-cube I*, is an* (L, R)*-factorization.*

It follows that there is a cubical model of homotopy type theory in which the identity type of a type A is the path-object A^{I}, a choice with some advantages.

We begin by reviewing the basic idea of homotopical semantics of type theory in weak factorization systems, including the somewhat technical issue of coherence that motivates the use of algebraic weak factorization systems.

1 The basic homotopical interpretation

Definition 1. A *weak factorization system* on a category $\mathbb{C}$ consists of two classes of arrows,

$$\mathcal{L} \hookrightarrow \mathbb{C}_1 \hookleftarrow \mathcal{R}$$

satisfying the following conditions:

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1. Every arrow $f : X \longrightarrow Y$ in $\mathbb{C}$ factors as a left map followed by a right map,

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\mathcal{L}\ni"'] & & Y \\ & \cdot \arrow[ur, "\in\mathcal{R}"'] & \end{tikzcd}$$

2. Given any commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] \arrow[ur, dotted] & D \end{tikzcd}$$

with an $\mathcal{L}$-map on the left and an $\mathcal{R}$-map on the right, there is a "diagonal filler" making both triangles commute, as indicated.

3. Each of the classes $\mathcal{L}$ and $\mathcal{R}$ is closed under retracts in the arrow category $\mathbb{C}^{\rightarrow}$.

Examples include (i) Groupoids (or categories), with $\mathcal{R}$ = isofibrations and $\mathcal{L}$ = injective equivalences; (ii) Simplicial sets, with $\mathcal{R}$ = Kan fibrations and $\mathcal{L}$ = acyclic cofibrations. A Quillen model structure on a category involves two such interrelated weak factorization systems, so this provides the basic examples and the homotopical intuition. In a WFS, we think of the $\mathcal{R}$-maps as "fibrations" – that is, good families of objects indexed by the codomain. The basic idea of the homotopy interpretation is to use these as the dependent types.

1.1 Interpreting Id-types

Let $\mathbb{C}$ be a category with finite limits and a WFS. Closed types are interpreted as $\mathcal{R}$-objects A, i.e. those for which $A \to 1$ is in $\mathcal{R}$. Dependent types $x : A \vdash B : \mathsf{type}$ are interpreted as $\mathcal{R}$-maps $B \to A$. Terms $x : A \vdash b : B$ are sections $b : A \to B$ of the chosen $\mathcal{R}$-map $B \to A$.

The formation rule for Id-types says that each type has an identity type:

$$\frac{A\ \mathsf{type}}{x, y : A\ \vdash\ \mathsf{Id}_A(x, y)\ \mathsf{type}} \qquad (\mathsf{Id}\text{-Formation})$$

We model this by factoring the diagonal map of (the object interpreting) A

as an $\mathcal{L}$-map followed by an $\mathcal{R}$-map, using axiom 1 for the WFS:

$$\begin{tikzcd} & \mathsf{Id}_A \arrow[dr] & \\ A \arrow[ur] \arrow[rr] & & A \times A \end{tikzcd}$$

The $\mathcal{R}$-map $\mathsf{Id}_A \to A \times A$ interprets the dependent type $x, y : A \vdash \mathsf{Id}_A(x, y)$ type. Th $\mathcal{L}$-factor $A \longrightarrow \mathsf{Id}_A$ interprets the reflexivity term $\mathsf{refl}(x)$ in the Id-introduction rule:

$$x : A \ \vdash \ \mathsf{refl}(x) : \mathsf{Id}_A(x, x) \qquad (\mathsf{Id}\text{-Introduction})$$

The Id-elimination rule has the form:

$$\frac{x, y : A, z : \mathsf{Id}_A(x, y) \vdash B(x, y, z) \ \mathsf{type}, \quad x : A \vdash b(x) : B(x, x, \mathsf{refl}(x))}{\mathsf{J}(x, y, z, b) : B(x, y, z)} \qquad (\mathsf{Id}\text{-Elimination})$$

with associated computation rule:

$$\mathsf{J}(x, x, \mathsf{refl}(x), b) = b(x) : B(x).$$

The data above the line in Id-elimination are interpreted as a commutative square as on the outside of the diagram:

$$\begin{tikzcd} A \arrow[r, "b"] \arrow[d, "\mathsf{refl}"'] & B \arrow[d] \\ \mathsf{Id}_A \arrow[r] \arrow[ur, dotted, "\mathsf{J}"'] & \mathsf{Id}_A \end{tikzcd}$$

Since refl is an $\mathcal{L}$-map, and $B \to \mathsf{Id}_A$ is an $\mathcal{R}$-map (as the interpretation of a dependent type), there is a diagonal filler J as indicated. Commuting of the lower triangle is just the conclusion of the Id-elimination rule. The commuting of the upper triangle is exactly the J-computation rule.

1.2 Coherence

The interpretation just sketched is required to respect the result of substituting a term into a context, since the rules have this property. Substitution into dependent types is interpreted as pullback, and substitution into terms as (roughly) composition. There are then three separate issues involved in giving a strict interpretation of type theory with Id-types, and all three are called "coherence":

1. Using the fact that $\mathcal{R}$-maps are closed under retracts, one can show that they are also stable under pullback along any map, so the interpretation of dependent types as $\mathcal{R}$-maps works right with the interpretation of substitution as pullback. However, the fact that the pullback operation is only defined up to isomorphism means that the interpretation must be "strictified" in order to model substitution stritly. This is a known iddue in the semantics of dependent type theory, with known solutions (including a recent one by Lumsdaine-Warren), and will not concern us further here.

2. The choice of factorization of the diagonal,

$$\begin{tikzcd} & \mathsf{Id}_A \arrow[dr] & \\ A \arrow[ur] \arrow[rr] & & A \times A\,, \end{tikzcd}$$

must be stable under pulback. Specifically, if A is a type in context $\Gamma \vdash A : \mathsf{type}$ interpreted as an $\mathcal{R}$-map $A \to \Gamma$, then there is a factorization of the diagonal over Γ of the form

$$\begin{tikzcd} & \mathsf{Id}_A \arrow[dr] & \\ A \arrow[ur] \arrow[rr] \arrow[dr] & & A \times_\Gamma A\,, \arrow[dl] \\ & \Gamma & \end{tikzcd}$$

Pulling back along any $f : \Delta \longrightarrow \Gamma$ preserves the diagonal, but not necessarily the $\mathcal{L}$-$\mathcal{R}$ factorization,

$$\begin{tikzcd} & f^*\mathsf{Id}_A \arrow[dr] & \\ f^*A \arrow[ur] \arrow[rr] \arrow[dr] & & f^*A \times_\Delta f^*A\,. \arrow[dl] \\ & \Delta & \end{tikzcd}$$

Choosing an $\mathcal{L}$-$\mathcal{R}$ factorization of the pulled-back diagonal gives an interpretation of Id_{f^*A} that need not agree (even up to isomorphism) with $f^*\mathsf{Id}_A$. A choice of factorizations, for all diagonals, that respects pullback in this sense is said to be *stable*. One way such a stable choice of factorizations can arise is when it is determined by exponentiating by a fixed “interval” object I, so that $\mathsf{Id}_A = A^{\mathrm{I}}$. This is what happens, for example, in the groupoid model, where as an interval one can take the groupoid with two objects and two, mutually inverse, non-identity arrows.

3. Assuming a stable choice of factorizations of the diagonal, we have made a choice of diagonal fillers J in order to interpret the Id-elimination rule. Again, there is no reason why these choices of diagonal fillers should “respect substitution” in the way required for the interpretation of type theory. More specifically, given a diagonal filling problem as on the right below, and a square on the left with $g \in \mathcal{L}$,

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "g"'] & A \arrow[r] \arrow[d] & C \arrow[d] \\ B' \arrow[r, "f"'] \arrow[urr, dotted, "\psi"] & B \arrow[r] \arrow[ur, dotted, "\phi"'] & D \end{tikzcd} \tag{1}$$

we may have the two different diagonal fillers for the outer filling problem, namely ψ and $\phi \circ f$. Under certain conditions, we want to have these two solutions be equal. This leads to a strengthening of the notion of weak factorization system which implies the existence of natural choices of diagonal fillers.

Definition 2. A *functorial factorization* on a category $\mathbb{C}$ is a functor

$$(L, E, R) : \mathbb{C}^{\rightarrow} \longrightarrow \mathbb{C}^{\rightarrow\cdot\rightarrow}$$

taking each arrow $f : X \longrightarrow Y$ to a factorization $f = R(f) \circ L(f)$,

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "L(f)"'] & & Y \\ & E(f) \arrow[ur, "R(f)"'] & \end{tikzcd}$$

in a functorial way. Specifically, given any $h : f \longrightarrow f'$ in $\mathbb{C}^{\rightarrow}$ we have a commutative diagram:

$$\begin{tikzcd}
X \arrow[rrr, "h_0"] \arrow[ddd, "f"'] \arrow[dr, "L(f)"] & & & X' \arrow[ddd, "f'"] \arrow[dl, "L(f')"'] \\
& E(f) \arrow[r] \arrow[dl, "R(f)"] & E(f') \arrow[dr, "R(f')"'] & \\
& & & \\
Y \arrow[rrr, "h_1"'] & & & Y'
\end{tikzcd}$$

and we write $E(h) : E(f) \longrightarrow E(f')$ for the evident map.

Observe that, for each fixed Y, the functorial factorization determines an endofunctor,

$$R : \mathbb{C}/Y \longrightarrow \mathbb{C}/Y$$

taking $f : X \longrightarrow Y$ to $R(f) : E(f) \longrightarrow Y$, and that this endofunctor is pointed by $L : 1 \longrightarrow R$ (abusing notation slightly).

Dually, for each fixed X, the functorial factorization determines an endofunctor,

$$L : X/\mathbb{C} \longrightarrow X/\mathbb{C}$$

taking $f : X \longrightarrow Y$ to $L(f) : X \longrightarrow E(f)$, and that this endofunctor is copointed by $R : L \longrightarrow 1$.

Definition 3. An *algebraic weak factorization system* on $\mathbb{C}$ consists of a functorial factorization (L, E, R) together with:

1. a multiplication $\mu : R^2 \longrightarrow R$ making (R, μ, L) a monad,
2. a comultiplication $\nu : L \longrightarrow L^2$ making (L, ν, R) a comonad.

Some authors also a distributive law for the monad over the comonad, however we shall not need this.

Remark 4. Let us show that an AWFS determines a WFS. The factorization axiom is is satisfied by the functorial factorization $f = R(f) \circ L(f)$. We then know that $R(f)$ is an R-algebra and $L(f)$ is an L-coalgebra by the laws of monads. Suppose given a diagonal filling problem such as the outer square below, in which f is an L-coalgebra and g is an R-algebra:

$$\begin{tikzcd}
X \arrow[rrr, "h_0"] \arrow[ddd, "f"'] \arrow[dr, "L(f)"] & & & Z \arrow[ddd, "g"] \arrow[dl, "L(g)"'] \\
& E(f) \arrow[r, "E(h)"] \arrow[dl, "R(f)"] & E(g) \arrow[dr, "R(g)"'] & \\
& & & \\
Y \arrow[rrr, "h_1"'] & & & W
\end{tikzcd}$$

Applying the factorizations of f and g, we obtain an L-coalgebra structure map $\phi : Y \longrightarrow E(f)$ and an R-algebra structure map $\psi : E(g) \longrightarrow Z$. We can then set $j = \psi \circ E(h) \circ \phi$ to obtain the required diagonal filler $j : Y \longrightarrow Z$. Finally, to ensure closure under retracts we let $\mathcal{R}$ be the retract closure of the R-algebras and $\mathcal{L}$ the retract closure of the L-coalgebras. The factorization axiom still holds trivially, and the filling axiom is also easily seen to still hold. Thus every AWFS determines a WFS with the left and right classes being the retracts of the L- and R- (co)algebras respectively.

Remark 5. A morphism of L-coalgebras $h : (f', \phi') \longrightarrow (f, \phi)$ is a commutative square $fh_0 = h_1 f'$ such that $E(h) \circ \phi' = \phi \circ h_1$,

$$\begin{tikzcd}
X' \arrow[rrr, "h_0"] \arrow[ddd, "f'"'] \arrow[dr, "L(f')"] & & & X \arrow[ddd, "f"] \arrow[dl, "L(f)"'] \\
& E(f') \arrow[r, "E(h)"] \arrow[dl, shift left, "R(f')"] & E(f) \arrow[dr, shift right, "R(f)"'] & \\
& & & \\
Y' \arrow[rrr, "h_1"'] \arrow[ur, shift left, "\phi'"] & & & Y . \arrow[ul, shift right, "\phi"']
\end{tikzcd}$$

It is easy to see that the naturality condition for diagonal fillers mentioned in (1) is satisfied for the fillers constructed algebraically as in Remark 4, when the left-hand square is a morphism of L-algebras in this sense.

We summarize the result of this section with the following.

Proposition 6. *Let $\mathbb{C}$ be a category with finite limits, an algebraic weak factorization system, and a stable choice of factorizations for all diagonal maps. Then $\mathbb{C}$ admits a model of type theory with Id-types.*

2 Cubical sets

Out goal is to make an algebraic weak factorization system on the cubical sets, but let us first recall *why* the cubical sets is a good setting for a model of Id-types.