Cartesian cubical model categories

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Abstract

Add an abstract.

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Introduction

1 Cartesian cubical sets

There are now many treatments of cubical sets in the literature, including [?, ?, ?, ?, ?, ?, ?, ?]. Our construction is intended to work in *all* of these, insofar as the axioms in Definition **??** below are satisfied. For the sake of concreteness, however, we shall consider what may be called the *cartesian* cube category $\square$, defined as the free finite product category on an interval $\delta_0, \delta_1 : 1 \rightrightarrows \mathrm{I}$.

Definition 1. The objects of the cartesian cube category $\square$, called n-cubes, are finite sets of the form

$$[n] = \{0, x_1, ..., x_n, 1\},$$

where $x_1, ..., x_n$, are formal *generators*. The arrows,

$$f : [n] \to [m],$$

may be taken to be m-tuples of elements drawn from the set $\{0, x_1, ..., x_n, 1\}$, regarded as formal terms representing composites of faces, degeneracies, permutations, and diagonals. Equivalently, the arrows $[n] \to [m]$ are arbitrary bipointed maps $[m] \to [n]$.

See [**?**] for further details.

Definition 2. The category cSet of *cubical sets* is the category of presheaves on the cartesian cube category $\square$,

$$\mathsf{cSet} \;=\; \mathsf{Set}^{\square^{\mathrm{op}}}.$$

It is of course generated by the representable presheaves $\mathsf{y}[n]$, to be written

$$\mathrm{I}^n = \mathsf{y}[n]$$

and called the *geometric n-cubes.*

Note that the representables I^n are closed under finite products, $\mathrm{I}^n \times \mathrm{I}^m = \mathrm{I}^{n+m}$. We of course write I for I^1 and 1 for I^0. We will need the following basic fact about the cubes I^n in cSet.

Proposition 3. *For each n, the n-cube I^n is* tiny*, in the sense that the exponential (or "internal* Hom*") functor $(-)^{\mathrm{I}^n} : \mathsf{cSet} \longrightarrow \mathsf{cSet}$ has a right adjoint.*

(See [**?**] for more on such "amazing right adjoints".)

Proof. It clearly suffices to prove the claim for $n = 1$. For any cubical set X, the exponential X^{I} is a "shift by one dimension",

$$X^{\mathrm{I}}(n) \cong \mathrm{Hom}(\mathrm{I}^n, X^{\mathrm{I}}) \;\cong\; \mathrm{Hom}(\mathrm{I}^{n+1}, X) \cong\; X(n+1).$$

Thus X^{I} is given by precomposition with the "successor" functor $\square \to \square$ with $[n] \mapsto [n+1]$. Precomposition always has a right adjoint, which in this case we write as

$$X^{\mathrm{I}} \dashv X_{\mathrm{I}}$$

and call the I-*root*. We can calculate the values of the I-root to be,

$$\begin{aligned} X_{\mathrm{I}}(n) &\cong \mathrm{Hom}(\mathrm{I}^n, X_{\mathrm{I}}) \\ &\cong \mathrm{Hom}((\mathrm{I}^n)^{\mathrm{I}}, X) \\ &\cong \mathrm{Hom}((\mathrm{I}^{\mathrm{I}})^n, X) \\ &\cong \mathrm{Hom}((\mathrm{I}+1)^n, X) \\ &\cong \mathrm{Hom}\big(\textstyle\sum_{k=0}^{n} \binom{n}{k} \mathrm{I}^k, X\big) \\ &\cong \textstyle\prod_{k=0}^{n} X(k)^{\binom{n}{k}}, \end{aligned}$$

using the fact that $\mathrm{I}^{\mathrm{I}} \cong (\mathrm{I}+1)$, much as in [**?**]. □

The exponential object X^{I} will be called the *pathobject* of X, and plays a special role. As we have just seen, it classifies "paths" in X, i.e. maps $p : \mathrm{I} \to X$, corresponding to 1-cubes $p \in X_1$ between the endpoints (0-cubes) $p_0, p_1 \in X_0$. More generally, higher cubes $c \in X_{n+1}$ correspond to maps $c : \mathrm{I}^{n+1} \cong \mathrm{I}^n \times \mathrm{I} \to X$, which can be regarded as paths between the n-cubes $c_0, c_1 : \mathrm{I}^n \to X$, respectively $c_0, c_1 \in X_n$.

We conclude this section with two facts that will be useful later on concerning the base change functors,

$$f_! \dashv f^* \dashv f_* : \mathsf{cSet}/_X \longrightarrow \mathsf{cSet}/_Y,$$

associated to a map $f : X \to Y$ in cSet.

Lemma 4. *The pulled-back interval* $\mathrm{I}^*\mathrm{I} = \mathrm{I} \times \mathrm{I} \to \mathrm{I}$ *in* $\mathsf{cSet}/_{\mathrm{I}}$ *is also tiny.*

Proof. Since the interval $\mathrm{I} = \mathsf{y}[1]$ is representable, the slice category $\mathsf{cSet}/_{\mathrm{I}}$ is also a category of presheaves, namely over the sliced cube category $\square/_{[1]}$,

$$\mathsf{cSet}/_{\mathrm{I}} \;=\; \mathsf{Set}^{\square^{\mathrm{op}}}/_{\mathsf{y}[1]} \;\cong\; \mathsf{Set}^{(\square/_{[1]})^{\mathrm{op}}} \;=\; \widehat{\square/_{[1]}}\,.$$

However, since $\square$ does not have all finite limits, the sliced index category does not have all finite products, and so we cannot simply repeat the proof from Proposition 3. But as in that proof, we do have a "successor" functor

$$s_{[1]} : \square/_{[1]} \to \square/_{[1]} \,,$$

resulting from the "predecessor" natural transformation $s \Rightarrow 1_{\square}$ given by the projection $\mathrm{I} \times X \to X$. Evaluating s at each object $f : [n] \to [1]$ in $\square/_{[1]}$, we obtain a commutative diagram:

$$\begin{tikzcd} s[n] \arrow[r, "\cong"] \arrow[d, "sf"'] & {[1]\times[n]} \arrow[r, "p_n"] & {[n]} \arrow[d, "f"] \\ s[1] \arrow[r, "\cong"'] & {[1]\times[1]} \arrow[r, "p_1"'] & {[1]} \end{tikzcd} \tag{1}$$

We can then set $s_{[1]}(f) = p_1 \circ sf = f \circ p_n$. As in the foregoing proof, we can then calculate the values of the adjoints on presheaves, associated to $s_{[1]}$,

$$s_{[1]_!} \dashv s_{[1]}{}^* : \widehat{\square/_{[1]}} \longrightarrow \widehat{\square/_{[1]}}$$

to be, successively,

$$\begin{aligned} s_{[1]_!}(X) &= \mathrm{I}^*\mathrm{I} \times X \,, \\ s_{[1]}{}^*(X) &= X^{\mathrm{I}^*\mathrm{I}} \,. \end{aligned}$$

The first equation follows from the observation that the diagram (1) is a pullback, and so the object $s_{[1]}(f) : s[n] \to [1]$ of $\widehat{\square/_{[1]}}$ given by the evident composite is just $\mathrm{I}^*\mathrm{I} \times f$, and the diagram itself represents the counit map $(\mathrm{I}^*\mathrm{I} \times f) \to f$ over I. The second line then follows by adjointness, as does the fact that we have a further right adjoint, namely, the $\mathrm{I}^*\mathrm{I}$-*root*:

$$s_{[1]_*}(X) =: X_{\mathrm{I}^*\mathrm{I}} \,.$$

$\square$

Lemma 5. *The pushforward functor along any map $f : X \to Y$ preserves pathspaces; for any object $A \to X$ over X, the pathobject of the pushforward f_*A is (canonically isomorphic to) the pushforward of the pathobject,*

$$(f_*A)^{\mathrm{I}} \cong f_*(A^{\mathrm{I}})$$

over Y.

Proof. This true for any constant family $X^*C = X \times C \to X$ with C in place of I, as the reader can easily verify, using the Beck-Chevalley condition. $\square$

Cubical sets as a classifying topos

(This subsection collects some basic facts about cubical sets that are not needed in the remainder of the lectures.)

The objects of the Cartesian cube category $\square$ may be taken concretely to be finite sets of the form

$$[n] = \{\bot, x_1, ..., x_n, \top\},$$

and the arrows $f : [n] \to [m]$ to be all bipointed maps $[m] \to [n]$ (note the direction). These maps are evidently just m-tuples of elements from the set $[n]$, which are easily shown to be composites of faces, degeneracies, permutations, and diagonals.

The category of *(Cartesian) cubical sets* is the presheaf topos

$$\mathsf{cSet} = \mathsf{Set}^{\square^{\mathrm{op}}}.$$

It is of course generated by the representable presheaves $\mathrm{I}^n := \mathsf{y}[n]$, called the *n-cubes*. The 0-cube is $\mathrm{I}^0 = \mathsf{y}[0] = 1$; the 1-cube is $\mathrm{I} = \mathsf{y}[1]$; and $\mathrm{I}^n \times \mathrm{I}^m \cong \mathrm{I}^{n+m}$ in virtue of preservation of products by the Yoneda embedding $\mathsf{y} : \square^{\mathrm{op}} \hookrightarrow \mathsf{cSet}$. For a cubical set $X : \square^{\mathrm{op}} \to \mathsf{Set}$ we write $X_n = X[n]$ and call this the *set of n-cubes in X*, for which we have the usual Yoneda correspondence,

$$\{c \in X_n\} \;\cong\; \{c : \mathrm{I}^n \to X\}.$$

In particular, $\mathrm{I}^n_m = \mathsf{cSet}([m], [n])$ is the set of m-cubes in the n-cube.[1]

Proposition 6. *The category* cSet *of Cartesian cubical sets is the classifying topos for* strictly bipointed objects*: objects A with points $a_0, a_1 : 1 \to A$ the equalizer of which is* $0 \to 1$.

Proof. Consider the covariant presentation $\mathsf{cSet} = \mathsf{Set}^{\mathbb{B}}$ where $\mathbb{B}$ is the category of finite, strictly bipointed sets and bipointed maps. We can extend $\mathbb{B} \hookrightarrow \mathbb{B}_=$ by freely adjoining coequalizers, making $\mathbb{B}_=$ the free finite *colimit* category on a co-bipointed object. An concrete presentation of $\mathbb{B}_=$ is the finite bipointed sets, including those with $\bot = \top$. Let us write (n) for the

[1] Note that the cardinality of I^n_m is therefore just $(m+2)^n$, in comparison to the *Dedekind* cubes, for which *e.g.* $\mathsf{cSet}([1], [n])$ the n^{th} *Dedekind number*, the number of elements in the free distributive lattice on n generators, which in general is a number so large that it is unknown for values of $n > 7$.

bipointed set $\{x_1, ..., x_n, \bot = \top\}$, with n (non-constant) elements and a further element $\bot = \top$. There is an evident coequalizer $[1] \rightrightarrows [n] \to (n)$, which just identifies the distinguished points, and every coqualizer has either the form $[m] \rightrightarrows [n] \to [k]$ or $[m] \rightrightarrows [n] \to (k)$, for a suitable choice of k. Note that there are no maps of the form $(m) \to [n]$, and that every map $[m] \to (n)$ factors uniquely as $[m] \to (m) \to (n)$ with $[m] \to (m)$ the canonical coequalizer of $\bot$ and $\top$. The category $\mathbb{B}_=$ can therefore be decomposed into two "levels", the upper one of which is essentially $\mathbb{B}$, the lower one consisting of just the objects (n) and thus essentially the finite *pointed* sets, and for each n, there is the canonical coequalizer $[n] \to (n)$ going from the upper level to the lower one.

$$\begin{tikzcd}
\dots \arrow[r] & {[m]} \arrow[r] \arrow[d, two heads] & {[n]} \arrow[r] \arrow[d, two heads] & \dots \\
\dots \arrow[r] & (m) \arrow[r] & (n) \arrow[r] & \dots
\end{tikzcd}$$

Write $u : \mathbb{B} \to \mathbb{B}_=$ for the upper inclusion, which is the classifying functor of generic co-bipointed object in $\mathbb{B}_=$ (which is strict).

Now consider the induced geometric morphism:

$$\begin{tikzcd}
\mathsf{Set}^{\mathbb{B}} \arrow[r, shift left=2, "u_*"] \arrow[r, shift right=2, "u_!"'] & \mathsf{Set}^{\mathbb{B}_=} \arrow[l, "u^*" description]
\end{tikzcd} \qquad u_! \dashv u^* \dashv u_*$$

Since u^* is the restriction along u, the right adjoint u_* must be "prolongation by 1",

$$u_*(P)[n] = P[n],$$
$$u_*(P)(n) = \{*\},$$

with the obvious maps,

$$\begin{tikzcd}
\dots \arrow[r] & P[m] \arrow[r] \arrow[d, two heads] & P[n] \arrow[r] \arrow[d, two heads] & \dots \\
\dots \arrow[r] & \{*\} \arrow[r] & \{*\} \arrow[r] & \dots
\end{tikzcd}$$

as is easily seen by considering maps in $\mathsf{Set}^{\mathbb{B}_=}$ of the form

$$\begin{tikzcd}
Q[n] \arrow[r] \arrow[d, "\cdot"'] & P[n] \arrow[d, two heads, "\cdot"] \\
Q(n) \arrow[r] & \{*\}.
\end{tikzcd}$$

Since $u_* : \mathsf{Set}^{\mathbb{B}} \to \mathsf{Set}^{\mathbb{B}_=}$ is evidently full and faithful, it is the inclusion part of a sheaf subtopos $\mathsf{sh}(\mathbb{B}_=^{\mathsf{op}}, j) \hookrightarrow \mathsf{Set}^{\mathbb{B}_=}$ for a suitable Grothendieck topology j on $\mathbb{B}_=^{\mathsf{op}}$. We claim that j is the closed complement topology of the subobject $[\bot = \top] \rightarrowtail 1$ represented by the coequalizer $[0] \to (0)$. Indeed, in $\mathsf{Set}^{\mathbb{B}_=}$ we have the representable functors:

$$
\begin{aligned}
\mathrm{I} &= y[1], \\
1 &= y[0], \\
[\bot = \top] &= y(0)
\end{aligned}
$$

fitting into an equalizer $[\bot = \top] \to 1 \rightrightarrows \mathrm{I}$, which is the image under Yoneda of the canonical coequalizer $[1] \rightrightarrows [0] \to (0)$ in $\mathbb{B}_=$. The closed complement topology for $[\bot = \top] \rightarrowtail 1$ is generated by the single cover $0 \to [\bot = \top]$, which can be described logically as forcing the sequent $(\bot = \top \vdash \bot)$ to hold. Recall from [**?**], Proposition 3.53, the following simple characterization of the sheaves for a closed topology generated by an object $U \rightarrowtail 1$: an object X is a sheaf iff $X \times U \cong U$. In the present case, it therefore suffices to show that for any $P : \mathbb{B}_= \to \mathsf{Set}$ we have:

$$
P \times [\bot = \top] \cong [\bot = \top] \quad \text{iff} \quad P(n) = 1 \text{ for all } n.
$$

For any object $b \in \mathbb{B}_=$, consider the map

$$
\mathrm{Hom}(yb, P \times [0 = 1]) \cong \mathrm{Hom}(yb, P) \times \mathrm{Hom}(yb, [\bot = \top]) \to \mathrm{Hom}(yb, [\bot = \top]).
$$

If $b = [k]$, then $\mathrm{Hom}(yb, [\bot = \top]) \cong \mathrm{Hom}_{\mathbb{B}_=}((0), [k]) \cong 0$, and so we always have an iso

$$
\begin{aligned}
\mathrm{Hom}(yb, P \times [\bot = \top]) &\cong \mathrm{Hom}(yb, P) \times \mathrm{Hom}(yb, [\bot = \top]) \\
&\cong \mathrm{Hom}(yb, P) \times 0 \cong 0.
\end{aligned}
$$

If $b = (k)$, then $\mathrm{Hom}(y(k), [\bot = \top]) \cong \mathrm{Hom}_{\mathbb{B}_=}((0), (k)) \cong 1$, and we have an iso

$$
\begin{aligned}
\mathrm{Hom}(y(k), P \times [\bot = \top]) &\cong \mathrm{Hom}(y(k), P) \times \mathrm{Hom}(y(k), [\bot = \top]) \\
&\cong \mathrm{Hom}(y(k), P) \times 1 \cong \mathrm{Hom}(y(k), P) \cong P(k).
\end{aligned}
$$

Thus we will have an iso $P \times [\bot = \top] \cong [\bot = \top]$ iff $P(k) \cong 1$.

Thus the presheaf topos $\mathsf{Set}^{\mathbb{B}}$ is the closed complement of the open subtopos

$$
\mathsf{Set}^{\mathbb{B}_=}/_{[\bot=\top]} \hookrightarrow \mathsf{Set}^{\mathbb{B}_=}
$$

given by forcing the proposition $\bot \neq \top$. Since $\mathsf{Set}^{\mathbb{B}_=}$ is clearly the classifying topos for *arbitrary* bipointed objects $\bot, \top : 1 \to B$, the subtopos $\mathsf{Set}^{\mathbb{B}}$ indeed classifies *strictly* bipointed objects, as claimed. $\square$

Corollary 7. *The geometric realization functor to topological spaces*

$$R : \mathsf{cSet} \to \mathsf{Top}$$

preserves cartesian products, $R(X \times Y) \cong R(X) \times R(Y)$*.*[2]

Proof. This can of course be shown directly, but it follows immediately by composing the inverse image of the classifying geometric morphism $\mathsf{sSets} \to \mathsf{cSet}$ of the 1-simplex Δ^1 with the standard geometric realization $\mathsf{sSets} \to \mathsf{Top}$, each of which preserves finite products. $\square$

Definition 8. Let $\square \to \mathsf{Cat}$ be the unique product-preserving functor taking the interval $[1]$ to the one arrow category $\mathbb{2} = (0 \leq 1)$. This functor then takes $[n]$ to $\mathbb{2}^n$, the n-fold product in Cat, and maps $[m] \to [n]$ to the corresponding monotone functions of the posets $\mathbb{2}^n$. The *cubical nerve* functor

$$N : \mathsf{Cat} \to \mathsf{cSet}$$

is then defined by:

$$N(\mathbb{C})_n = \mathsf{Cat}(\mathbb{2}^n, \mathbb{C}).$$

Thus $N(\mathbb{C})_0$ is the set of objects of $\mathbb{C}$; $N(\mathbb{C})_1$ is the set of arrows; $N(\mathbb{C})_2$ consists of all commutative squares; $N(\mathbb{C})_3$ all commutative cubes, etc.

Proposition 9. *The nerve functor* $N : \mathsf{Cat} \to \mathsf{cSet}$ *is full and faithful.*

Proof. Given categories $\mathbb{C}$ and $\mathbb{D}$ and functors $F, G : \mathbb{C} \to \mathbb{D}$, suppose $F(f) \neq G(f)$ for some $f : A \to B$ in $\mathbb{C}$. Take $f^\sharp : \mathbb{2} \to \mathbb{C}$ with image f. Then $N(F)_1(f^\sharp) = F(f) \neq G(f) = N(G)_1(f^\sharp)$, and so $N(F) \neq N(G) : N(\mathbb{C}) \to N(\mathbb{D})$. So N is faithful.

For fullness, let $\varphi : N(\mathbb{C}) \to N(\mathbb{D})$ be a natural transformation, and define a proposed functor $F : \mathbb{C} \to \mathbb{D}$ by

$$F_0 = \varphi_0 : \mathbb{C}_0 = N(\mathbb{C})_0 \to N(\mathbb{D})_0 = \mathbb{D}_0$$
$$F_1 = \varphi_1 : \mathbb{C}_1 = N(\mathbb{C})_1 \to N(\mathbb{D})_1 = \mathbb{D}_1.$$

[2] This fact and the next one are to be contrasted with the case of monoidal cubical sets, *e.g.* as studied by [**?**, **?**]

We just need to show that F preserves identity arrows and composition. Consider the following diagram.

$$\begin{tikzcd} \mathsf{Cat}(\mathbb{2}^1, \mathbb{C}) = N(\mathbb{C})_1 \arrow[r, "F_1"] & N(\mathbb{D})_1 = \mathsf{Cat}(\mathbb{2}^1, \mathbb{D}) \\ \mathsf{Cat}(\mathbb{2}^0, \mathbb{C}) = N(\mathbb{C})_0 \arrow[r, "F_0"'] \arrow[u, "!^*"] & N(\mathbb{D})_0 = \mathsf{Cat}(\mathbb{2}^0, \mathbb{D}). \arrow[u, "!^*"'] \end{tikzcd}$$

Here $!^* : \mathsf{Cat}(\mathbb{2}^0, \mathbb{C}) \to \mathsf{Cat}(\mathbb{2}, \mathbb{C})$ is precomposition with $! : \mathbb{2} = \mathbb{2}^1 \to \mathbb{2}^0 = \mathbb{1}$, so the diagram commutes. But since $! : \mathbb{2} \to \mathbb{1}$ is a functor,

$$\mathbb{C}_0 = \mathsf{Cat}(\mathbb{1}, \mathbb{C}) \xrightarrow{!^*} \mathsf{Cat}(\mathbb{2}, \mathbb{C}) = \mathbb{C}_1$$

takes objects in $\mathbb{C}$ to their identity arrows. Thus F preserves identity arrows. Similarly, for composition, consider

$$\begin{tikzcd} \mathsf{Cat}(\mathbb{2}^2, \mathbb{C}) = N(\mathbb{C})_2 \arrow[r, "\varphi_2"] \arrow[d, "d^*"'] & N(\mathbb{D})_2 = \mathsf{Cat}(\mathbb{2}^2, \mathbb{D}) \arrow[d, "d^*"] \\ \mathsf{Cat}(\mathbb{2}, \mathbb{C}) = N(\mathbb{C})_1 \arrow[r, "F_1"'] & N(\mathbb{D})_1 = \mathsf{Cat}(\mathbb{2}, \mathbb{D}). \end{tikzcd}$$

where $\varphi_2 : N(\mathbb{C})_2 \to N(\mathbb{D})_2$ is the action of φ on commutative squares of arrows, and $d^* : \mathsf{Cat}(\mathbb{2}^2, \mathbb{C}) \to \mathsf{Cat}(\mathbb{2}, \mathbb{C})$ is precomposition with the diagonal map $d : \mathbb{2} \to \mathbb{2}^2 = \mathbb{2} \times \mathbb{2}$, so the diagram commutes. For any composable pair of arrows $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathbb{C}$ there is a commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ B \arrow[r, "g"'] & C, \end{tikzcd}$$

and the effect of $d^* : \mathsf{Cat}(\mathbb{2}^2, \mathbb{C}) \to \mathsf{Cat}(\mathbb{2}, \mathbb{C})$ on this square is exactly $g \circ f : A \to C$, and similarly for $d^* : \mathsf{Cat}(\mathbb{2}^2, \mathbb{D}) \to \mathsf{Cat}(\mathbb{2}, \mathbb{D})$. Thus the commutativity of the above diagram implies that F preserves composition. Since clearly $N(F) = \varphi$, we indeed have that N is also full. $\square$

Proposition 10. *For any cubical set X, the exponential X^{I} is calculated as the "shift by one dimension",*

$$X^{\mathrm{I}}(n) \cong X(n+1)\,.$$

Proof.

$$\begin{aligned} X^{\mathrm{I}}(n) \cong \mathrm{Hom}(y[n], X^{\mathrm{I}}) \cong \mathrm{Hom}(\mathrm{I}^n, X^{\mathrm{I}}) \ &\cong\ \mathrm{Hom}(\mathrm{I}^n \times \mathrm{I}, X) \\ \cong\ \mathrm{Hom}(\mathrm{I}^{n+1}, X) \cong\ \mathrm{Hom}(y[n+1], X) \ &\cong\ X(n+1). \end{aligned}$$

□

Corollary 11. *The functor $X \mapsto X^{\mathrm{I}}$ has a* right *adjoint.*

Proof. The functor $X \mapsto X^{\mathrm{I}}$ is given by precomposition with the "successor" functor $S : \Box \to \Box$ with $S[n] = [n+1]$. Thus $X^{\mathrm{I}}([n]) = X(S[n]) = (S^*(X))([n])$. Precomposition always has a right adjoint $S^* \dashv S_*$, which can be calculated as:

$$S_*(X)(n) \cong \mathrm{Hom}(y[n], S_*X) \cong \mathrm{Hom}(S^*(y[n]), X) \ \cong \mathrm{Hom}(\Box(S(-), [n]), X).$$

□

We need the following fact in order to calculate the right adjoint further.

Lemma 12. *In* cSet*, we have* $\mathrm{I}^{\mathrm{I}} \ \cong\ \mathrm{I} + 1$*.*

Proof. For any $[n] \in \Box$ we have:

$$(\mathrm{I}^{\mathrm{I}})(n) \cong \mathrm{I}(n+1) \cong \mathrm{Hom}(\mathrm{I}^{(n+1)}, \mathrm{I}) \cong \Box([n+1], [1]) \cong \mathbb{B}([1], [n+1]) \cong n+3.$$

On the other hand,

$$(\mathrm{I}+1)(n) \cong \mathrm{I}(n) + 1(n) \cong \mathrm{Hom}(\mathrm{I}^n, \mathrm{I}) + 1 \cong \mathbb{B}([1], [n]) + 1 \cong (n+2) + 1.$$

The isomorphism is natural in n. □

We mention that a similar fact holds for the generic object in the object classifier topos, and in the Schanuel topos, and is used in the theory of "abstract higher-order syntax" [**?**, **?**].

Definition 13. Let us write

$$X_{\mathrm{I}} = S_*(X)$$

for the right adjoint of the path object functor $X^{\mathrm{I}} = S^*X$.

Corollary 14. *We have the following calculation for the right adjoint X_{I}:*

$$\begin{aligned} X_{\mathrm{I}}(n) &\cong \mathrm{Hom}(\mathrm{I}^n, X_{\mathrm{I}}) \\ &\cong \mathrm{Hom}((\mathrm{I}^n)^{\mathrm{I}}, X) \\ &\cong \mathrm{Hom}((\mathrm{I}^{\mathrm{I}})^n, X) \\ &\cong \mathrm{Hom}((\mathrm{I}+1)^n, X) \\ &\cong \mathrm{Hom}(\mathrm{I}^n + C^n_{n-1}\mathrm{I}^{n-1} + \cdots + C^n_1 \mathrm{I} + 1, X) \\ &\cong X(n) \times X(n-1)^{C^n_{n-1}} \times \cdots \times X(1)^{C^n_1} \times X(0), \end{aligned}$$

where $C^n_k = \binom{n}{k}$ is the usual binomial coefficient.

Corollary 15. *There is a natural transformation $X_{\mathrm{I}} \to X$, given by the first projection from $X_{\mathrm{I}}(n) \cong X(n) \times X(n-1)^{C^n_{n-1}} \times \cdots \times X(1)^{C^n_1} \times X(0)$.*

Finally, we observe that that the path object functor X^{I} itself, as a left adjoint, preserves all *co*limits. This does not hold in general in type theory, but will be a special property of the cubical model. (Cf. Lawvere [**?**] on the notion of "tiny" objects and the "amazing right adjoint".)

Example. (P. Aczel) The cubical set P of polynomials (over the integers, say), is defined by:

$$P_n = \{p(x_1, ..., x_n) \mid \text{polynomials in at most } x_1, ..., x_n\}$$

with the evident maps $P_m \to P_n$ for each function $[m] \to [n]$.

This is a ring object in the category of cubical sets, and the interval $\mathrm{I} = y[1]$ embeds into P. The same is true for any algebraic theory $\mathbb{T}$ with two constants, such as boolean algebras: there is a cubical $\mathbb{T}$-algebra A and a monic $\mathrm{I} \rightarrowtail A$.

Let $\Box[\mathrm{I}] = \Box$ be the cube category, classifying intervals, and $\Box[\mathbb{T}]$ the classifying category for $\mathbb{T}$-algebras. There is an interval J in $\Box_{\mathbb{T}}$ consisting of the generic $\mathbb{T}$-algebra and its two constants. This J has a classifying functor $J : \Box_{\mathrm{I}} \to \Box_{\mathbb{T}}$, inducing functors on presheaves

$$J_! \dashv J^* \dashv J_* : \mathsf{Set}^{\Box_{\mathrm{I}}^{\mathrm{op}}} \to \mathsf{Set}^{\Box_{\mathbb{T}}^{\mathrm{op}}}$$

as usual, where $J_! \circ \mathsf{y}_{\Box_{\mathrm{I}}} = \mathsf{y}_{\Box_{\mathbb{T}}} \circ J$, with y the respective Yoneda embeddings.

We can calculate:

$$\begin{aligned} J^*J_!(\mathrm{I})([n]) &= J^*J_!(Y[1])([n]) \\ &= J^*Y(J[1])([n]) = Y(J[1])(J[n]) \\ &= \Box_{\mathbb{T}}(J[n], J[1]) = \mathbb{T}\mathrm{-Alg}(J[1], J[n]) \\ &= \mathbb{T}\mathrm{-Alg}(F(1), F(n)) = |F(n)|, \end{aligned} \tag{2}$$

where $F(n)$ is the free $\mathbb{T}$-algebra on n generators. So in the case of polynomials we indeed have

$$P = J^*J_!(\mathrm{I}).$$

The unit of the adjunction $\mathrm{I} \to J^*J_!(\mathrm{I})$ is faithful, since J itself is faithful and therefore the left adjoint $J_!$ is faithful. P is a ring in $\mathsf{Set}^{\Box_{\mathrm{I}}^{\mathrm{op}}}$ since $J_!(\mathrm{I})$ is a ring in $\mathsf{Set}^{\Box_{\mathbb{T}}^{\mathrm{op}}}$ and J^* is left exact.

A closely related example is the cubical set of "boolean polynomials",

$$B_n = \{\varphi(p_1, ..., p_n) \mid \text{propositional formulas in at most } p_1, ..., p_n\}$$

which is the free boolean algeba 2^{2^n}.

2 The cofibration weak factorization system

Definition 16 (Cofibration)**.** The *cofibrations*, written

$$c : A \rightarrowtail B\,,$$

are any class $\mathcal{C}$ of monomorphisms in cSet satisfying the following axioms:

(C0) The map $0 \to C$ is always a cofibration.

(C1) All isomorphisms are cofibrations.

(C2) The composite of two cofibrations is a cofibration.

(C3) Any pullback of a cofibration is a cofibration.

We also require the cofibrations to be classified by a subobject $\Phi \hookrightarrow \Omega$ of the standard subobject classifier $\top : 1 \to \Omega$ of cSet:

(C4) There is a terminal object $t : 1 \rightarrowtail \Phi$ in the category of cofibrations and cartesian squares.

Two further axioms for cofibrations will be added in Section 3.1, one in Section 3.2, and a final one in Section 6.4 (see Appendix 8 for a summary). Note that we also permit the case $\Phi = \Omega$, so that all monos are cofibrations, in which case no axioms are required.

The cofibrant partial map classifier. Write

$$X^+ := \sum_{\varphi:\Phi} X^{[\varphi]} \;=\; \Phi_!\, t_*(X)\,, \tag{3}$$

for the polynomial endofunctor $\mathsf{cSet} \longrightarrow \mathsf{cSet}$ determined by the cofibration classifier $t : 1 \rightarrowtail \Phi$ (see [?]). The reader familiar with type theory will recognize the similarity to the "partiality" or "lifting" monad.

Observe that since t is monic there is a pullback square,

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X^+ \arrow[d, "t_* X"] \\ 1 \arrow[r, "t"'] & \Phi. \end{tikzcd}$$

Let $\eta : X \rightarrowtail X^+$ be the indicated top horizontal map; we call this map the *cofibrant partial map classifier* of X.

Proposition 17. *The map $\eta : X \rightarrowtail X^+$ classifies partial maps into X with cofibrant domain, in the following sense.*

1. *The map $\eta : X \rightarrowtail X^+$ is a cofibration.*

2. *For any object Z and any partial map $(s, g) : Z \leftarrowtail S \to X$, with $s : S \rightarrowtail Z$ a cofibration, there is a unique $f : Z \to X^+$ making a pullback square as follows.*

$$\begin{tikzcd} S \arrow[r, "g"] \arrow[d, tail, "s"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, tail, "\eta"] \\ Z \arrow[r, "f"'] & X^+ \end{tikzcd}$$

Proof. The map $\eta : X \rightarrowtail X^+$ is a cofibration since it is a pullback of $t : 1 \to \Phi$. Observe that $(\eta, 1_X) : X^+ \leftarrowtail X \to X$ is therefore a partial map into X with cofibrant domain. The second statement is the universal property of X^+ as a polynomial (see [?], prop. 7). $\square$

Proposition 18. *The pointed endofunctor determined by $\eta_X : X \rightarrowtail X^+$ has a natural multiplication $\mu_X : X^{++} \to X^+$ making it a monad.*

Proof. Since the cofibrations are closed under composition, the monad structure on X^+ follows as in [?], proposition XY. Explicitly, μ_X is determined by proposition 17 as the unique map making the following a pullback diagram.

$$
\begin{tikzcd}
X \arrow[r, "="] \arrow[d, tail, "\eta_X"'] & X \arrow[dd, tail, "\eta"] \\
X^+ \arrow[d, tail, "\eta_{X^+}"'] & \\
X^{++} \arrow[r, dotted, "\mu"'] & X^+
\end{tikzcd}
$$

□

Relative partial map classifier. For any object $X \in \mathsf{cSet}$ the usual pullback functor

$$X^* : \mathsf{cSet} \to \mathsf{cSet}/_X \,,$$

taking any A to the second projection $A \times X \to X$, not only preserves the subobject classifier Ω, but also the cofibration classifier $\Phi \hookrightarrow \Omega$, where a map in $\mathsf{cSet}/_X$ is defined to be a cofibration if it is one in cSet. Thus in $\mathsf{cSet}/_X$ the *(relative) cofibration classifier* is the map

$$t \times X : 1 \times X \to \Phi \times X \quad \text{over } X$$

which we may also write $t_X : 1_X \to \Phi_X$. Like $t : 1 \to \Phi$, this map determines a polynomial endofunctor

$$+_X : \mathsf{cSet}/_X \to \mathsf{cSet}/_X \,,$$

which commutes (up to natural isomorphism) with $+ : \mathsf{cSet} \to \mathsf{cSet}$ and $X^* : \mathsf{cSet} \to \mathsf{cSet}/_X$ in the evident way:

$$
\begin{tikzcd}
\mathsf{cSet}/_X \arrow[r, "+_X"] & \mathsf{cSet}/_X \\
\mathsf{cSet} \arrow[u, "X^*"] \arrow[r, "+"'] & \mathsf{cSet} \arrow[u, "X^*"']
\end{tikzcd}
\tag{4}
$$

The endofunctor $+_X$ is also pointed $\eta_Y : Y \to Y^+$ and has a natural monad multiplication $\mu_Y : Y^{++} \to Y^+$, for any $Y \to X$, for the same reason that $+$ has this structure. Summarizing, we may say that *the polynomial monad* $+ : \mathsf{cSet} \to \mathsf{cSet}$ *is indexed (or fibered) over* cSet.

Definition 19. A $+$-*algebra* in cSet is a cubical set A together with a retraction $\alpha : A^+ \to A$ of $\eta_A : A \to A^+$, i.e. an algebra for the pointed endofunctor $(+ : \mathsf{cSet} \to \mathsf{cSet},\ \eta : 1 \to +)$. Algebras for the monad $(+, \eta, \mu)$ will be referred to specifically as $(+, \eta, \mu)$-*algebras*, or $+$-*monad algebras*.

A *relative* $+$-*algebra* in cSet is a map $A \to X$ together with an algebra structure over the codomain X for the pointed endofunctor $+_X : \mathsf{cSet}/_X \to \mathsf{cSet}/_X$.

The cofibration weak factorization system.

Proposition 20. *There is an (algebraic) weak factoriation system on* cSet *with the cofibrations as the left class and as the right class, the maps underlying the relative* $+$*-algebras. Thus a right map is one* $f : A \to X$ *for which there is a retract* $\alpha : A' \to A$ *over* X *of the canonical map* $\eta_f : A \to A'$*,*

$$\begin{tikzcd}
A \arrow[r, "\eta_f"] \arrow[rr, bend left, "="] \arrow[dr, "f"'] & A' \arrow[r, "\alpha"] \arrow[d, "f^+"] & A \arrow[dl, "f"] \\
& X. &
\end{tikzcd}$$

Proof. The factorization of a map $f : Y \to X$ is given by applying the relative $+$-functor over the codomain,

$$\begin{tikzcd}
Y \arrow[r, tail, "\eta_f"] \arrow[dr, "f"'] & Y' \arrow[d, "f^{+_X}"] \\
& X.
\end{tikzcd}$$

We know by proposition 17 that the unit η_f is always a cofibration, and since f^{+_X} is the free algebra for the $+_X$-monad, it is in particular a $+_X$-algebra.

For the lifting condition, consider a cofibration $c : B \rightarrowtail C$, a right map $A \to X$, with $+_X$-algebra structure map $\alpha : A^{+_X} \to A$ over X, and a commutative square as indicated in the following.

$$\begin{tikzcd}
B \arrow[r, "g"] \arrow[dd, tail, "c"'] & A \arrow[dd] \arrow[dr, "\eta"] & \\
& & A^{+X} \arrow[ul, bend right, "\alpha"'] \arrow[dl] \\
C \arrow[r, "f"'] & X &
\end{tikzcd}$$

Thus over X, we have the situation

$$\begin{tikzcd}
B \arrow[r, "g"] \arrow[d, tail, "c"'] & A \arrow[d, "\eta"] \\
C \arrow[ur, dotted, "d"'] & A^+ \arrow[u, bend right=60, "\alpha"']
\end{tikzcd}$$

and we seek a diagonal filler d as indicated. Since $(c, g) : B \leftarrowtail C \to A$ is a cofibrant partial map into A, there is a map $\varphi : C \to A^+$ (over X) making a (pullback) square,

$$\begin{tikzcd}
B \arrow[r, "g"] \arrow[d, tail, "c"'] & A \arrow[d, "\eta"] \\
C \arrow[r, dotted, "\varphi"'] & A^+ \arrow[u, bend right=60, "\alpha"']
\end{tikzcd}$$

We thus have $d := \alpha \circ \varphi : C \to A$ as the required diagonal filler.

The closure of the cofibrations under retracts follows from their classification by a universal object $t : 1 \to \Phi$, and the closure of the right maps under retracts follows from their being the algebras for a pointed endofunctor underlying a monad (cf. [?]). Algebraicity of this weak factorization system is immediate, since $+$ is a monad. $\square$

Summarizing, we have an algebraic weak factorization system $(\mathcal{C}, \mathcal{C}^{\pitchfork})$ on the category cSet of cubical sets, where:

$$\begin{aligned}
\mathcal{C} &= \text{ the cofibrations} \\
\mathcal{C}^{\pitchfork} &= \text{ the maps underlying relative } +\text{-algebras}
\end{aligned}$$

We shall call this the *cofibration weak factorization system.* The right maps will be denoted

$$\mathsf{TFib} = \mathcal{C}^{\pitchfork}$$

and called *trivial fibrations*.

The cofibration algebraic weak factorization system is a refinement of the one defined in [**?**] and mentioned in [**?**].

Uniform filling structure. It is convenient to relate relative +-algebra structure with the more familiar diagonal filling condition of cofibrantly generated weak factorization systems, and specifically the special form occuring in [CCHM16] under the name *uniform filling structure*.

Consider a generating sub*set* of cofibrations consisting of those with representable codomain $c : C \rightarrowtail \mathrm{I}^n$, and call these the *basic cofibrations*.

$$\mathsf{BCof} = \{c : C \rightarrowtail \mathrm{I}^n \mid c \in \mathcal{C}, n \geq 0\}. \tag{5}$$

Proposition 21. *For any object X in* cSet *the following are equivalent:*

1. *X admits a* +-algebra structure: *a retraction $\alpha : X^+ \to X$ of the unit $\eta : X \to X^+$.*
2. *$X \to 1$ is a* trivial fibration: *it has the right lifting property with respect to all cofibrations,*
$$\mathcal{C} \pitchfork X.$$
3. *X admits a* uniform filling structure: *for each basic cofibration $c : C \rightarrowtail \mathrm{I}^n$ and map $x : C \to X$ there is given an extension $j(c, x)$,*
$$\begin{tikzcd} C \arrow[r, "x"] \arrow[d, tail, "c"'] & X, \\ \mathrm{I}^n \arrow[ur, dotted, "j(c,x)"'] & \end{tikzcd} \tag{6}$$
and the choice is uniform in I^n *in the following sense.*

   Given any cubical map $u : \mathrm{I}^m \to \mathrm{I}^n$, the pullback $u^*c : u^*C \rightarrowtail \mathrm{I}^m$, which is again a basic cofibration, fits into a commutative diagram of the form
$$\begin{tikzcd} u^*C \arrow[r, "c^*u"] \arrow[d, tail, "u^*c"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[r, "x"] \arrow[d, tail, "c"'] & X. \\ \mathrm{I}^m \arrow[r, "u"'] & \mathrm{I}^n \arrow[ur, dotted, "j(c,x)"'] & \end{tikzcd} \tag{7}$$
   For the pair $(u^*c,\ x \circ c^*u)$ in (7), the chosen extension $j(u^*c, x \circ c^*u) : \mathrm{I}^m \to X$, is required to be equal to $j(c, x) \circ u$,
$$j(u^*c, x \circ c^*u) = j(c, x) \circ u. \tag{8}$$

Proof. Let (X, α) be a $+$-*algebra* and suppose given the span (c, x) as below, with c a cofibration.

$$\begin{tikzcd} C \arrow[r, "x"] \arrow[d, tail, "c"'] & X \\ Z \end{tikzcd}$$

Let $\chi(c, x) : Z \to X^+$ be the classifying map of the evident partial map $(c, x) : Z \rightharpoonup X$, so that we have a pullback square as follows.

$$\begin{tikzcd} C \arrow[rr, "x"] \arrow[d, tail, "c"'] \arrow[drr, phantom, "\lrcorner", very near start] & & X \arrow[d, "\eta"] \\ Z \arrow[rr, "\chi(c,x)"'] & & X^+ \end{tikzcd} \tag{9}$$

Then set

$$j = \alpha \circ \chi(c, x) : Z \to X \tag{10}$$

to get a filler,

$$\begin{tikzcd} C \arrow[rr, "x"] \arrow[d, tail, "c"'] & & X \arrow[d, "\eta"] \\ Z \arrow[rr, "\chi(c,x)"'] \arrow[urr, dotted, "j"] & & X^+ \arrow[u, bend right, "\alpha"'] \end{tikzcd} \tag{11}$$

since

$$j \circ c = \alpha \circ \chi(c, x) \circ c = \alpha \circ \eta \circ x = x.$$

Thus (1) implies (2). To see that it also implies (3), observe that in the case where $Z = I^n$ and we specify, in (10), that

$$j(c, x) = \alpha \circ \chi(c, x) : \mathrm{I}^n \to X, \tag{12}$$

then the assignment is natural in I^n. Indeed, given any $u : \mathrm{I}^m \to \mathrm{I}^n$, we have

$$j(c', xu') = \alpha \circ \chi(c', xu') = \alpha \circ \chi(c, x) \circ u = j(c, x)u, \tag{13}$$

by the uniqueness of the classifying maps.

It is clear that (2) implies (1), since if $\mathcal{C} \pitchfork X$ then we can take as an algebra structure $\alpha : X^+ \to X$ any filler for the span

$$\begin{tikzcd} X \arrow[r, "="] \arrow[d, tail, "\eta"'] & X. \\ X^+ \arrow[ur, dotted, "\alpha"'] \end{tikzcd}$$

To see that (3) implies (1), suppose that X has a uniform filling structure j and we want to define an algebra structure $\alpha : X^+ \to X$. By Yoneda, for every $y : \mathrm{I}^n \to X^+$ we need a map $\alpha(y) : \mathrm{I}^n \to X$, naturally in I^n, in the sense that for any $u : \mathrm{I}^m \to \mathrm{I}^n$, we have

$$\alpha(yu) = \alpha(y)u. \tag{14}$$

Moreover, to ensure that $\alpha\eta = 1_X$, for any $x : \mathrm{I}^n \to X$ we must have $\alpha(\eta \circ x) = x$. So take $y : \mathrm{I}^n \to X^+$ and let

$$\alpha(y) = j(y^*\eta, y'),$$

as indicated on the right below.

$$\begin{tikzcd} u^*C \arrow[r, "u'"] \arrow[d, tail, "u^*y^*\eta"'] \arrow[dr, phantom, very near start, "\lrcorner"] & C \arrow[rr, "y'"] \arrow[d, tail, "y^*\eta"'] \arrow[dr, phantom, very near start, "\lrcorner"] & & X. \arrow[d, "\eta"] \\ \mathrm{I}^m \arrow[r, "u"'] & \mathrm{I}^n \arrow[rr, "y"'] \arrow[urr, dotted, "j(y^*\eta, y')"'] & & X^+ \end{tikzcd} \tag{15}$$

Then for any $u : \mathrm{I}^m \to \mathrm{I}^n$, we indeed have

$$\alpha(yu) = j\big((yu)^*\eta, y'u'\big) = j(y^*\eta, y') \circ u = \alpha(y)u,$$

by the uniformity of j. Finally, if $y = \eta \circ x$ for some $x : \mathrm{I}^n \to X$ then

$$\alpha(\eta x) = j\big((\eta x)^*\eta, (\eta x)'\big) = j(1_X, x) = x,$$

because the defining diagram for $\alpha(\eta x)$, i.e. the one on the right in (15), then factors as

$$\begin{tikzcd} \mathrm{I}^n \arrow[r, "x"] \arrow[d, tail, "="'] \arrow[dr, phantom, very near start, "\lrcorner"] & X \arrow[r, "="] \arrow[d, tail, "="'] \arrow[dr, phantom, very near start, "\lrcorner"] & X, \arrow[d, "\eta"] \\ \mathrm{I}^n \arrow[r, "x"'] & X \arrow[r, "\eta"'] & X^+ \end{tikzcd} \tag{16}$$

and the only possible extension $j(1_X, x)$ for the span $(1_{\mathrm{I}^n}, x)$ is x itself. $\square$

Remark 22. Observe that the uniformilty condition (3) can be extended to the *class of all* cofibrations, in the form:

4. X admits a *(large) uniform filling structure:* for each cofibration $c : C \rightarrowtail Z$ and map $x : C \to X$ there is given an extension $j(c, x)$,

$$\begin{tikzcd} C \arrow[r, "x"] \arrow[d, tail, "c"'] & X, \\ Z \arrow[ur, dotted, "j(c,x)"'] & \end{tikzcd} \tag{17}$$

   and the choice is *uniform in Z* in the following sense: Given any map $u : Y \to Z$, the pullback $u^*c : u^*C \rightarrowtail Y$, which is again a cofibration, fits into a commutative diagram of the form

$$\begin{tikzcd} u^*C \arrow[r, "c^*u"] \arrow[d, tail, "u^*c"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[r, "x"] \arrow[d, tail, "c"'] & X. \\ Y \arrow[r, "u"'] & z \arrow[ur, dotted, "j(c,x)"'] & \end{tikzcd} \tag{18}$$

   For the pair $(u^*c,\ x \circ c^*u)$ in (18), the chosen extension $j(u^*c, x \circ c^*u) : \mathrm{I}^m \to X$, is required to be equal to $j(c, x) \circ u$,

$$j(u^*c, x \circ c^*u) = j(c, x) \circ u. \tag{19}$$

Indeed, the proof that (1) implies (2) and (3) works just as well to infer (4), which in turn implies (2) and (3) as special cases.

The relative version of the foregoing is entirely analogous, since the +-functor is fibered over cSet in the sense of diagram (4). We can therefore omit the entirely analogous proof of the following.

Proposition 23. *For any map $f : Y \to X$ in* cSet *the following are equivalent:*

1. *$f : Y \to X$ admits a* relative +-algebra structure over X, *i.e. there is a retraction $\alpha : Y' \to Y$ over X of the unit $\eta : Y \to Y'$, where $f^+ : Y' \to X$ is the result of the relative +-functor applied to f, as in definition 19.*

2. *$f : Y \to X$ is a* trivial fibration,

$$\mathcal{C} \pitchfork f.$$

3. $f : Y \to X$ *admits a (small)* uniform filling structure*: for each basic cofibration* $c : C \rightarrowtail \mathrm{I}^n$ *and maps* $x : C \to X$ *and* $y : \mathrm{I}^n \to Y$ *making the square below commute, there is given a diagonal filler* $j(c, x, y)$*,*

$$\begin{tikzcd} C \arrow[r, "x"] \arrow[d, tail, "c"'] & X \arrow[d, "f"] \\ \mathrm{I}^n \arrow[r, "y"'] \arrow[ur, dotted, "j(c,x,y)"'] & Y, \end{tikzcd} \tag{20}$$

and the choice is uniform in I^n *in the following sense: given any cubical map* $u : \mathrm{I}^m \to \mathrm{I}^n$*, the pullback* $u^*c : u^*C \rightarrowtail \mathrm{I}^m$ *is again a basic cofibration and fits into a commutative diagram of the form*

$$\begin{tikzcd} u^*C \arrow[r, "c^*u"] \arrow[d, tail, "u^*c"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[r, "x"] \arrow[d, tail, "c"'] & X \arrow[d, "f"] \\ \mathrm{I}^m \arrow[r, "u"'] & \mathrm{I}^n \arrow[r, "y"'] \arrow[ur, dotted, "j(c,x,y)"'] & Y. \end{tikzcd} \tag{21}$$

For the evident triple $(u^*c,\ x \circ c^*u, y \circ u)$ *in* (21) *the chosen diagonal filler*

$$j(u^*c, x \circ c^*u, y \circ u) : \mathrm{I}^m \to X$$

is equal to $j(c, x, y) \circ u$*,*

$$j(u^*c, x \circ c^*u, y \circ u) = j(c, x, y) \circ u. \tag{22}$$

We next collect some basic facts about trivial fibrations: they have sections, they are closed under composition and retracts, and they are closed under pullback and pushforward along all maps.

Corollary 24. 1. *Every trivial fibration* $A \to X$ *has a section* $s : X \to A$*.*

2. *If* $a : A \to X$ *is a trivial fibration and* $b : B \to A$ *is a trivial fibration, then* $a \circ b : B \to X$ *is a trivial fibration.*

3. *If* $a : A \to X$ *is a trivial fibration and* $a' : A' \to X'$ *is a retract of* a *in the arrow category, then* a' *is a trivial fibration.*

4. *For any map* $f : X \to Y$ *and any trivial fibration* $B \to Y$*, the pullback* $f^*B \to X$ *is a trivial fibration.*

5. *For any map $f : X \to Y$ and any trivial fibration $A \to X$, the push-forward $f_*A \to Y$ is a trivial fibration.*

Proof. (1) holds because all objects are cofibrant by (C0). (5) is a consequence of (C3), stability of cofibrations under pullback, by a standard argument using the adjunction $f^* \dashv f_*$. The rest hold for the right maps in any weak factorization system. □

Remark 25. The structured notion of trivial fibration, *vis.* relative +-algebra, can also be shown algebraically (i.e. not using Proposition 23) to be preserved by pullback, pushforward, composition, and to be closed under retracts. We do just the case of pushforward as an example. Thus consider the following situation with $A \to X$ a +-algebra with structure α, as indicated.

$$\begin{tikzcd}
A \arrow[dd] \arrow[dr, "\eta_A"'] & & & f_*A \arrow[dd] \arrow[dr, "\eta_{f_*A}"'] & \\
& A^+ \arrow[ul, bend right, "\alpha"'] \arrow[dl] & & & (f_*A)^+ \arrow[ul, dotted, bend right, "\beta"'] \arrow[dl] \\
X \arrow[rrr, "f"'] & & & Y &
\end{tikzcd} , \tag{23}$$

A +-algebra structure for $f_*A \to Y$ would be a retract $\beta : (f_*A)^+ \to f_*A$ of $\eta_{f_*A} : f_*A \to (f_*A)^+$ over Y, which corresponds under $f^* \dashv f_*$ to a map $\tilde{\beta} : f^*((f_*A)^+) \to A$ over X with

$$\tilde{\beta} \circ f^*\eta_{f_*A} = \epsilon_A$$

as indicated below.

$$\begin{tikzcd}
f^*f_*A \arrow[r, "f^*\eta_{f_*A}"'] \arrow[dr, "\epsilon_A"'] & f^*((f_*A)^+) \arrow[d, dotted, "\tilde{\beta}"'] & \\
& A \arrow[r, "\eta_A"'] & A^+ . \arrow[l, bend right, "\alpha"']
\end{tikzcd} \tag{24}$$

But since pullback f^* commutes with $+$, there is a canonical iso $c : f^*((f_*A)^+) \cong$

$(f^*f_*A)^+$ with $c \circ f^*\eta_{f_*A} = \eta_{f^*f_*A}$. So we can set $\tilde{\beta} := \alpha \circ (\epsilon_A)^+ \circ c$.

$$\begin{tikzcd}
f^*f_*A \arrow[r, "f^*\eta_{f_*A}"'] \arrow[rr, bend left, "\eta_{f^*f_*A}"] \arrow[dr, "\epsilon_A"'] & f^*((f_*A)^+) \arrow[r, "\sim", "c"'] \arrow[d, dotted, "\tilde{\beta}"'] & (f^*f_*A)^+ \arrow[d, "(\epsilon_A)^+"] \\
& A \arrow[r, "\eta_A"'] & A^+ \arrow[l, bend right, "\alpha"']
\end{tikzcd} \tag{25}$$

3 The fibration weak factorization system

We now specify a second weak factorization system, with a restricted class of "trivial" cofibrations on the left, and an expanded class of right maps, the *fibrations*. As explained in the introduciton, we first recall what we call a "biased" version of the trivial-cofibration/fibration weak factorization system from [GS17], before giving the unbiased one to be used in our model structure. A proof of a model structure based on the biased version is given in [?], and makes use of *connections*,

$$\vee, \wedge : \mathrm{I} \times \mathrm{I} \longrightarrow \mathrm{I},$$

on the cubes, which we do not assume (in [AGH21] it is shown that the fibrations of [GS17] agree with those specified in the "logical style" of [CCHM16, OP17]).

3.1 Partial box filling (biased version)

A *generating class of biased trivial cofibrations* are all maps of the form

$$c \otimes \delta_\epsilon : D \rightarrowtail Z \times \mathrm{I}\,, \tag{26}$$

where:

1. $c : C \rightarrowtail Z$ is an arbitrary cofibration,
2. $\delta_\epsilon : 1 \to \mathrm{I}$ is one of the two *endpoint inclusions*, for $\epsilon = 0, 1$.

3. $c \otimes \delta_\epsilon$ is the *pushout-product* indicated in the following diagram.

$$\begin{tikzcd}
C \times 1 \arrow[r, "C\times\delta_\epsilon"] \arrow[d, "C\times 1"'] & C \times \mathrm{I} \arrow[d] \arrow[ddr, bend left, "c\times\mathrm{I}"] & \\
Z \times 1 \arrow[r] \arrow[drr, bend right, "Z\times\delta_\epsilon"'] & Z +_C (C \times \mathrm{I}) \arrow[dr, dotted, "c\otimes\delta_\epsilon"'] & \\
 & & Z \times \mathrm{I}
\end{tikzcd} \tag{27}$$

4. $D = Z +_C (C \times \mathrm{I})$ is the indicated domain of $c \otimes \delta_\epsilon$.

In order to ensure that such maps are indeed cofibrations, we assume two further axioms:

(C5) The endpoint inclusions $\delta_\epsilon : 1 \to \mathrm{I}$ are cofibrations.

(C6) The cofibrations are closed under pushout-products.

Note that if we assume δ_0 and δ_1 are disjoint (as they are in most categories of cubical sets), then by (C5) we have that $0 \to 1$ is a cofibration, and hence that $0 \to A$ is a cofibration, for all objects A, so that (C0) is no longer required. In place of (C6), we could require the cofibrations to be closed under the join operation $A \vee B$ in the lattice of subobjects of an object (as is done in [CCHM16, OP17]).

Fibrations (biased version). Let

$$\mathcal{C} \otimes \delta_\epsilon \;=\; \{c \otimes \delta_\epsilon : D \rightarrowtail Z \times \mathrm{I} \mid c \in \mathcal{C},\ \epsilon = 0, 1\}$$

be the class of all such generating biased trivial cofibrations. The *biased fibrations* are defined to be the right class of these maps,

$$(\mathcal{C} \otimes \delta_\epsilon)^{\pitchfork} \;=\; \mathcal{F}\,.$$

Thus a map $f : Y \to X$ is a biased fibration if for every commutative square of the form

$$\begin{tikzcd}
Z +_C (C \times \mathrm{I}) \arrow[r] \arrow[d, tail, "c\otimes\delta_\epsilon"'] & Y \arrow[d, "f"] \\
Z \times \mathrm{I} \arrow[r] \arrow[ur, dotted, "j"] & X
\end{tikzcd} \tag{28}$$

with a generating biased trivial cofibration on the left, there is a diagonal filler j as indicated.

To relate this notion of fibration to the cofibration weak factorization system, fix any map $u : A \to B$, and recall (e.g. from [?]) that the pushout-product with u is a functor on the arrow category

$$(-)\otimes u : \mathsf{cSet}^{\mathbb{2}} \to \mathsf{cSet}^{\mathbb{2}}\,.$$

This functor has a right adjoint, the *pullback-hom*, which for a map $f : X \to Y$ we shall write as

$$(u \Rightarrow f) : Y^B \longrightarrow (X^B \times_{X^A} Y^A)\,.$$

The pullback-hom is determined as indicated in the following diagram.

$$\begin{tikzcd} Y^B \arrow[dr, dotted, "u\Rightarrow f"] \arrow[drr, bend left, "Y^u"] \arrow[ddr, bend right, "f^B"'] & & \\ & X^B \times_{X^A} Y^A \arrow[r] \arrow[d] & Y^A \arrow[d, "f^A"] \\ & X^B \arrow[r, "X^u"'] & X^A \end{tikzcd} \tag{29}$$

The $\otimes \dashv \Rightarrow$ adjunction on the arrow category has the following useful relation to weak factorization systems (cf. [GS17, Rie14, ?]), where, as usual, for any maps $a : A \to B$ and $f : X \to Y$ we write

$$a \pitchfork f$$

to mean that for every solid square of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, "a"'] & X \arrow[d, "f"] \\ B \arrow[r] \arrow[ur, dotted, "j"] & Y \end{tikzcd} \tag{30}$$

there exists a diagonal filler j as indicated.

Lemma 26. *For any maps $a : A_0 \to A_1, b : B_0 \to B_1, c : C_0 \to C_1$ in* cSet*,*

$$(a \otimes b) \pitchfork c \quad \textit{iff} \quad a \pitchfork (b \Rightarrow c)\,.$$

The following is now a direct corollary.

Proposition 27. *An object X is fibrant if and only if both of the endpoint projections $X^{\mathrm{I}} \to X$ from the pathspace are trivial fibrations. More generally, a map $f : Y \to X$ is a fibration iff both of the maps*

$$(\delta_\epsilon \Rightarrow f) : Y^I \to X^I \times_X Y$$

are trivial fibrations (for $\epsilon = 0, 1$).

Fibration structure (biased version). The $\otimes \dashv \Rightarrow$ adjunction determines the fibrations in terms of the trivial fibrations, which in turn can be determined by *uniform* lifting against a *set* of basic cofibrations, by proposition 23. The fibrations are similarly determined by *uniform* lifting against the *set* of biased trivial cofibrations consisting of all those $c \otimes \delta_\epsilon$ in $\mathcal{C} \otimes \delta_\epsilon$ where $c : C \rightarrowtail \mathrm{I}^n$ is a basic cofibration. Call these maps the *basic biased trivial cofibrations*, and let

$$\mathsf{BCof} \otimes \delta_\epsilon = \{c \otimes \delta_\epsilon : B \rightarrowtail \mathrm{I}^{n+1} \mid c : C \rightarrowtail \mathrm{I}^n,\ \epsilon = 0, 1,\ n \geq 0\}, \tag{31}$$

where the pushout-product $c \otimes \delta_\epsilon$ now takes the simpler form

$$\begin{tikzcd}
C \arrow[r] \arrow[d, tail] & C \times \mathrm{I} \arrow[d] \arrow[ddr, bend left] & \\
\mathrm{I}^n \arrow[r] \arrow[drr, bend right] & \mathrm{I}^n +_C (C \times \mathrm{I}) \arrow[dr, tail, "c \otimes \delta_\epsilon"'] & \\
& & \mathrm{I}^n \times \mathrm{I}
\end{tikzcd} \tag{32}$$

for a basic cofibration $c : C \rightarrowtail \mathrm{I}^n$, an endpoint $\delta_\epsilon : 1 \to \mathrm{I}$, and with domain $B = \big(\mathrm{I}^n +_C (C \times \mathrm{I})\big)$. These subobjects $B \rightarrowtail \mathrm{I}^{n+1}$ can be seen geometrically as generalized open box inclusions.

For any map $f : Y \to X$ a *uniform, biased fibration structure* on f is a choice of diagonal fillers $j_\epsilon(c, x, y)$,

$$\begin{tikzcd}
\mathrm{I}^n +_C (C \times \mathrm{I}) \arrow[r, "x"] \arrow[d, "c \otimes \delta_\epsilon"'] & X \arrow[d, "f"] \\
\mathrm{I}^n \times \mathrm{I} \arrow[r, "y"'] \arrow[ur, dotted, "j_\epsilon(c,x,y)"'] & Y,
\end{tikzcd} \tag{33}$$

for each basic biased trivial cofibration $c \otimes \delta_\epsilon : B = (\mathrm{I}^n +_C (C \times \mathrm{I})) \rightarrowtail \mathrm{I}^{n+1}$ and maps $x : B \to X$ and $y : \mathrm{I}^{n+1} \to Y$, which is *uniform in* I^n in the following sense: Given any cubical map $u : \mathrm{I}^m \to \mathrm{I}^n$, the pullback $u^*c : u^*C \rightarrowtail \mathrm{I}^m$ of $c : C \rightarrowtail \mathrm{I}^n$ along u determines another basic biased trivial cofibration

$$u^*c \otimes \delta_\epsilon : B' = (\mathrm{I}^m +_{u^*C} (u^*C \times \mathrm{I})) \rightarrowtail \mathrm{I}^{m+1},$$

which fits into a commutative diagram of the form

$$\begin{tikzcd}
\mathrm{I}^m +_{u^*C} (u^*C \times \mathrm{I}) \arrow[r, "(u\times \mathrm{I})'"] \arrow[d, "u^*c \otimes \delta_\epsilon"'] & \mathrm{I}^n +_C (C \times \mathrm{I}) \arrow[r, "x"] \arrow[d, "c \otimes \delta_\epsilon"'] & X \arrow[d, "f"] \\
\mathrm{I}^m \times \mathrm{I} \arrow[r, "u\times \mathrm{I}"'] & \mathrm{I}^n \times \mathrm{I} \arrow[r, "y"'] \arrow[ur, dotted, "j_\epsilon(c,x,y)"'] & Y,
\end{tikzcd} \tag{34}$$

by applying the functor $(-) \otimes \delta_\epsilon$ to the pullback square relating u^*c to c. For the outer rectangle in (36) there is then a chosen diagonal filler

$$j_\epsilon(u^*c, x \circ (u \times \mathrm{I})', y \circ (u \times \mathrm{I})) : \mathrm{I}^m \times \mathrm{I} \to X$$

and for this map we require that

$$j_\epsilon(u^*c, x \circ (u \times \mathrm{I})', y \circ (u \times \mathrm{I})) = j_\epsilon(c, x, y) \circ (u \times \mathrm{I}). \tag{35}$$

This can be shown to be a reformulation of the logical specification given in [CCHM16] (see [AGH21]).

Definition 28. A *uniform, biased fibration structure* on a map $f : Y \to X$ is a choice of fillers $j_\epsilon(c, x, y)$ as in (33) satisfying (35) for all maps $u : \mathrm{I}^m \to \mathrm{I}^n$.

Finally, we have the analogue of proposition 21 for fibrant objects; we omit the analogous statement of proposition 23 for fibrations, as well as the entirely analogous proof.

Corollary 29. *For any object X in* cSet *the following are equivalent:*

1. *X is* biased fibrant, *i.e. every partial map to X with a generating biased trivial cofibration $D \rightarrowtail Z \times \mathrm{I}$ as domain of definition extends to a total map $Z \times \mathrm{I} \to X$,*
$$\mathcal{C} \otimes \delta_\epsilon \pitchfork X\,.$$
2. *The canonical maps $(\delta_\epsilon \Rightarrow X) : X^I \to X$ are trivial fibrations.*

3. $X \to 1$ *admits a* uniform biased fibration structure*. Explicitly, for each basic biased trivial cofibration* $c \otimes \delta_\epsilon : B \rightarrowtail \mathrm{I}^{n+1}$ *and map* $x : B \to X$*, there is given an extension* $j_\epsilon(c, x)$*,*

$$\begin{tikzcd} B \arrow[r, "x"] \arrow[d, tail, "c\otimes\delta_\epsilon"'] & X, \\ \mathrm{I}^{n+1} \arrow[ur, dotted, "j_\epsilon(c,x)"'] & \end{tikzcd} \tag{36}$$

and the choice is uniform in I^n *in the following sense: Given any cubical map* $u : \mathrm{I}^m \to \mathrm{I}^n$*, the pullback* $u^*c \otimes \delta_\epsilon : B' \rightarrowtail \mathrm{I}^m \times \mathrm{I}$ *fits into a commutative diagram of the form*

$$\begin{tikzcd} B' \arrow[r, "(u\times \mathrm{I})'"] \arrow[d, tail, "u^*c\otimes\delta_\epsilon"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[r, "x"] \arrow[d, tail, "c\otimes\delta_\epsilon"'] & X. \\ \mathrm{I}^m \times \mathrm{I} \arrow[r, "u\times \mathrm{I}"'] & \mathrm{I}^n \times \mathrm{I} \arrow[ur, dotted, "j(c,x)"'] & \end{tikzcd} \tag{37}$$

For the pair $(u^*c \otimes \delta_\epsilon,\ x \circ (u \times \mathrm{I})')$ *in* (37) *the chosen extension*

$$j(u^*c \otimes \delta_\epsilon, x \circ (u \times \mathrm{I})') : \mathrm{I}^m \times \mathrm{I} \to X$$

is equal to $j(c, x) \circ (u \times \mathrm{I})$*,*

$$j(u^*c \otimes \delta_\epsilon, x \circ (u \times \mathrm{I})') = j(c, x)(u \times \mathrm{I}). \tag{38}$$

3.2 Partial box filling (unbiased version)

Rather than building a weak factorization system based on the foregoing notion of biased fibration (as is done in [GS17]), we shall first eliminate the "bias" on a choice of endpoint $\delta_\epsilon : 1 \to \mathrm{I}$, expressed by the indexing $\epsilon = 0, 1$. This will have the effect of adding more trivial cofibrations, and thus more weak equivalences, to our model structure. Consider first the simple path-lifting condition for a map $f : Y \to X$, which is a special case of (28) with $c = ! : 0 \rightarrowtail 1$, since $! \otimes \delta_\epsilon = \delta_\epsilon$:

$$\begin{tikzcd} 1 \arrow[r] \arrow[d, tail, "\delta_\epsilon"'] & Y \arrow[d, "f"] \\ \mathrm{I} \arrow[r] \arrow[ur, dotted, "j_\epsilon"] & X \end{tikzcd}$$

In topological spaces, for instance, rather than requiring lifts j_ϵ for each of the endpoints $\epsilon = 0, 1$ of the real interval $\mathrm{I} = [0, 1]$, one could instead require there to be a lift j_i *for each point* $i : 1 \to \mathrm{I}$. Such "unbiased path-lifting" can be formulated in cSet by introducing a "generic point" $\delta : 1 \to \mathrm{I}$ by passing to cSet/I via the pullback functor $\mathrm{I}^* : \mathsf{cSet} \to \mathsf{cSet}/\mathrm{I}$, and then requiring path-lifting for $\mathrm{I}^* f$ with respect to δ. The following specification implements that idea, while also adding cofibrant partiality, as in the biased case. We first replace axiom (C5) with the following stronger assumption.

(C7) The diagonal map $\delta : \mathrm{I} \to \mathrm{I} \times \mathrm{I}$ is a cofibration.

The unbiased notion of a fibration is now as follows.

Condition (**??**) above, which is of course a special case of (**??**), says that evaluation at the generic point $\delta : 1 \to \mathrm{I}$, i.e. the map $X^\delta : X^{\mathrm{I}} \to X$ constructed in the slice category $\mathsf{cSet}/_{\mathrm{I}}$, is a trivial fibration. Condition (**??**) says that the pullback-hom of the generic point $\delta : 1 \to \mathrm{I}$ with $\mathrm{I}^* f$, constructed in the slice category $\mathsf{cSet}/_{\mathrm{I}}$, is a trivial fibration. Thus a map $f : Y \to X$ is an *unbiased* fibration just if its base change $I^* f$ is a δ-*biased* fibration in the slice category $\mathsf{cSet}/_{\mathrm{I}}$. The latter condition can also be reformulated as follows.

Proposition 30. *A map $f : Y \to X$ is a fibration if and only if the canonical map c to the pullback, in the following diagram in* cSet*, is a trivial fibration.*

$$\begin{tikzcd}
Y^{\mathrm{I}} \times \mathrm{I} \arrow[dr, dotted, "c"'] \arrow[drr, bend left, "\mathsf{eval}"] \arrow[ddr, bend right, "f^{\mathrm{I}} \times \mathrm{I}"'] & & \\
& Y_{\mathsf{eval}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "f"] \\
& X^{\mathrm{I}} \times \mathrm{I} \arrow[r, "\mathsf{eval}"'] & X.
\end{tikzcd} \tag{39}$$

Proof. We interpolate another pullback into the rectangle in (39) to obtain

$$\begin{tikzcd}
Y_{\mathsf{eval}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \times \mathrm{I} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "f"] \\
X^{\mathrm{I}} \times \mathrm{I} \arrow[r] & X \times \mathrm{I} \arrow[r] & X
\end{tikzcd} \tag{40}$$

with the evident maps. The left hand square is therefore a pullback, so we indeed have that

$$Y_{\mathsf{eval}} \;=\; (X^{\mathrm{I}} \times \mathrm{I}) \times_{(X\times\mathrm{I})} (Y \times \mathrm{I})$$

and $c = (\delta \Rightarrow f)$. $\square$

Now we can run the proof of Proposition 27 backwards in order to determine a class of generating trivial cofibrations for the unbiased case. Consider pairs of maps $c : C \rightarrowtail Z$ and $z : Z \to \mathrm{I}$, where the former is a cofibration and the latter is regarded as an "I-indexing", so that

$$\begin{tikzcd} C \arrow[r, tail, "c"] \arrow[dr] & Z \arrow[d, "z"] \\ & \mathrm{I} \end{tikzcd}$$

is regarded as an I-indexed family of cofibrations. Let

$$\mathsf{Gph}(z) = \langle 1_Z, z\rangle : Z \longrightarrow Z \times \mathrm{I}\,,$$

be the graph of $z : Z \to \mathrm{I}$, and define

$$c \otimes_z \delta := [\mathsf{Gph}(z), c \times \mathrm{I}] : Z +_C (C \times \mathrm{I}) \to Z \times \mathrm{I}\,,$$

which is easily seen to be well-defined on the indicated pushout.

$$\begin{tikzcd} C \arrow[r, "\mathsf{Gph}(zc)"] \arrow[d, tail, "c"'] & C \times \mathrm{I} \arrow[d] \arrow[ddr, bend left, "c\times\mathrm{I}"] & \\ Z \arrow[r] \arrow[drr, bend right, "\mathsf{Gph}(z)"'] & Z +_C (C \times \mathrm{I}) \arrow[dr, dotted, "c\otimes_z\delta"] & \\ & & Z \times \mathrm{I}\,. \end{tikzcd} \tag{41}$$

This specification differs from the similar (27) by using $\mathsf{Gph}(z)$ for the inclusion $Z \rightarrowtail Z \times \mathrm{I}$, rather than one of the "face maps" associated to the endpoint inclusions $\delta_\epsilon : 1 \to \mathrm{I}$. (Note that a graph is always a cofibration by pulling back a diagonal.) The subobject $c \otimes_z \delta \rightarrowtail Z \times \mathrm{I}$ is the join of the subobjects $\mathsf{Gph}(z) \rightarrowtail Z \times \mathrm{I}$ and the cylinder $C \times \mathrm{I} \rightarrowtail Z \times \mathrm{I}$.

Note that the endpoints $\delta_\epsilon : 1 \to \mathrm{I}$ are of the form $c \otimes_z \delta$ by taking $Z = 1$ and $z = \delta_\epsilon$ and $c = \,! : 0 \to 1$, so that biased filling is subsumed.

The maps of the form $c \otimes_z \delta : Z +_C (C \times \mathrm{I}) \rightarrowtail Z$ now form a *class of generating trivial cofibrations* in the expected sense. Let

$$\mathcal{C} \otimes \delta \;=\; \{c \otimes_z \delta : D \rightarrowtail Z \times \mathrm{I} \mid c : C \rightarrowtail Z, z : Z \to \mathrm{I}\}\,. \tag{42}$$

The fibrations are exactly the right class of these,

$$(\mathcal{C} \otimes \delta)^{\pitchfork} = \mathcal{F}.$$

Proposition 31. *A map $f : Y \to X$ is a fibration iff for every pair of maps $c : C \rightarrowtail Z$ and $z : Z \to \mathrm{I}$, where the former is a cofibration, every commutative square of the following form has a diagonal filler, as indicated.*

$$\begin{tikzcd}
Z +_C (C \times \mathrm{I}) \arrow[r] \arrow[d, tail, "c \otimes_z \delta"'] & Y \arrow[d, "f"] \\
Z \times \mathrm{I} \arrow[r] \arrow[ur, dotted, "j"] & X.
\end{tikzcd} \tag{43}$$

Proof. Suppose that for all $c : C \rightarrowtail Z$ and $z : Z \to \mathrm{I}$, we have $(c \otimes_z \delta) \pitchfork f$ in cSet. Pulling f back over I, this is equivalent to the condition $c \otimes \delta \pitchfork \mathrm{I}^* f$ in cSet/I, for all cofibrations $c : C \rightarrowtail Z$ over I, which is equivalent to $c \pitchfork (\delta \Rightarrow \mathrm{I}^* f)$ in cSet/I for all cofibrations $c : C \rightarrowtail Z$. But this in turn means that $\delta \Rightarrow \mathrm{I}^* f$ is a trivial fibration, which by definition means that f is a fibration. $\square$

Unbiased fibration structure. As in the biased case, the fibrations can be determined by *uniform* right-lifting against a generating *set* of trivial cofibrations, now consisting of all those $c \otimes_z \delta$ in $\mathcal{C} \otimes \delta$ for which $c : C \rightarrowtail \mathrm{I}^n$ is basic. Call these maps the *basic (unbiased) trivial cofibrations*, and let

$$\mathsf{BCof} \otimes \delta = \{c \otimes_z \delta : B \rightarrowtail \mathrm{I}^{n+1} \mid c : C \rightarrowtail \mathrm{I}^n,\, z : \mathrm{I}^n \to \mathrm{I},\, n \geq 0\}\,, \tag{44}$$

where the pushout-product $c \otimes_z \delta$ now has the form

$$\begin{tikzcd}
C \arrow[r, "\mathsf{Gph}(zc)"] \arrow[d, tail, "c"'] & C \times \mathrm{I} \arrow[d] \arrow[ddr, bend left, "c \times \mathrm{I}"] & \\
\mathrm{I}^n \arrow[r] \arrow[drr, bend right, "\mathsf{Gph}(z)"'] & \mathrm{I}^n +_C (C \times \mathrm{I}) \arrow[dr, dotted, "c \otimes_z \delta"] & \\
 & & \mathrm{I}^n \times \mathrm{I}
\end{tikzcd} \tag{45}$$

for a basic cofibration $c : C \rightarrowtail \mathrm{I}^n$, an indexing map $z : \mathrm{I}^n \to \mathrm{I}$, and with domain $B = \big(\mathrm{I}^n +_C (C \times \mathrm{I})\big)$. These subobjects $B \rightarrowtail \mathrm{I}^{n+1}$ can again be seen geometrically as "generalized open box inclusions", but now the floor or lid of the open box may be replaced by a "cross-section" given by the graph of a map $z : \mathrm{I}^n \to \mathrm{I}$.

For any map $f : Y \to X$ a (uniform, unbiased) fibration structure on f is a choice of diagonal fillers $j(c, z, x, y)$,

$$\begin{tikzcd} B \arrow[r, "x"] \arrow[d, tail, "c\otimes_z\delta"'] & X \arrow[d, "f"] \\ \mathrm{I}^n \times \mathrm{I} \arrow[r, "y"'] \arrow[ur, dotted, "j(c,z,x,y)"'] & Y, \end{tikzcd} \tag{46}$$

for each basic trivial cofibration $c \otimes_z \delta : B \rightarrowtail \mathrm{I}^{n+1}$, which is *uniform* in I^n in the following sense: Given any cubical map $u : \mathrm{I}^m \to \mathrm{I}^n$, the pullback $u^*c : u^*C \rightarrowtail \mathrm{I}^m$ and the reindexing $zu : \mathrm{I}^m \to \mathrm{I}^n \to \mathrm{I}$ determine another basic trivial cofibration $u^*c \otimes_{zu} \delta : B' = (\mathrm{I}^m +_{u^*C} (u^*C \times \mathrm{I})) \rightarrowtail \mathrm{I}^{m+1}$ which fits into a commutative diagram of the form

$$\begin{tikzcd} B' \arrow[r, "(u\times\mathrm{I})'"] \arrow[d, tail, "u^*c\otimes_{zu}\delta"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[r, "x"] \arrow[d, tail, "c\otimes_z\delta"'] & X \arrow[d, "f"] \\ \mathrm{I}^m \times \mathrm{I} \arrow[r, "u\times\mathrm{I}"'] & \mathrm{I}^n \times \mathrm{I} \arrow[r, "y"'] \arrow[ur, dotted, "j(c,z,x,y)"'] & Y. \end{tikzcd} \tag{47}$$

For the outer rectangle in (47) there is a chosen diagonal filler

$$j(u^*c, zu, x(u \times \mathrm{I})', y(u \times \mathrm{I})) : \mathrm{I}^m \times \mathrm{I} \to X,$$

and for this map we require that

$$j(u^*c, zu, x(u \times \mathrm{I})', y(u \times \mathrm{I})) = j(c, z, x, y) \circ (u \times \mathrm{I}). \tag{48}$$

Definition 32. A *(uniform, unbiased) fibration structure* on a map

$$f : Y \to X$$

is a choice of fillers $j(c, z, x, y)$ as in (46) satisfying (48) for all $u : \mathrm{I}^m \to \mathrm{I}^n$.

In these terms, we have the following analogue of corollary 29.

Proposition 33. *For any object X in cSet the following are equivalent:*

1. *the canonical map $X^{\mathrm{I}} \times \mathrm{I} \to X \times \mathrm{I}$ is a trivial fibration.*

2. *X has the right lifting property with respect to all generating trivial cofibrations,*
$$(\mathcal{C} \otimes_z \delta) \pitchfork X.$$

3. *X has a uniform fibration structure in the sense of Definition 32.*

Proof. The equivalence between (1) and (2) is proposition 31. Suppose (1), i.e. that the map
$$(\delta \Rightarrow X) : X^{\mathrm{I}} \times \mathrm{I} \to X \times \mathrm{I}$$
is a relative +-algebra over $X \times \mathrm{I}$. By proposition 21, this means that $(\delta \Rightarrow X)$, as an object of $\mathsf{cSet}/(X \times \mathrm{I})$, has a uniform filling structure with respect to all cofibrations $c : C \rightarrowtail \mathrm{I}^n$ over $(X \times \mathrm{I})$. Transposing by the $\otimes \dashv \Rightarrow$ adjunction and unwinding gives, equivalently, a uniform fibration structure on X. $\square$

A statement analogous to the foregoing also holds for maps $f : Y \to X$ in place of objects X. Indeed, as before, we have the following sharper formulation.

Corollary 34. *Fibration structures on a map $f : Y \to X$ correspond uniquely to relative +-algebra structures on the map $(\delta \Rightarrow f)$ (cf. definition **??**),*
$$(\delta \Rightarrow f) : Y^I \times \mathrm{I} \longrightarrow (X^I \times \mathrm{I}) \times_{(X \times \mathrm{I})} (Y \times \mathrm{I})$$

3.3 Factorization

Definition 35. Summarizing the foregoing definitions and results, we have the following classes of maps:

- The *generating trivial cofibrations* were determined in (42) to be
$$\mathcal{C} \otimes \delta = \{c \otimes_z \delta : D \rightarrowtail Z \times \mathrm{I} \mid c : C \rightarrowtail Z,\ z : Z \to \mathrm{I}\}, \tag{49}$$
where the pushout-product $c \otimes_z \delta$ has the form
$$\begin{tikzcd}
C \arrow[r, "\mathsf{Gph}(zc)"] \arrow[d, tail, "c"'] & C \times \mathrm{I} \arrow[d] \arrow[ddr, bend left, "c \times \mathrm{I}"] & \\
Z \arrow[r] \arrow[drr, bend right, "\mathsf{Gph}(z)"'] & Z +_C (C \times \mathrm{I}) \arrow[dr, dotted, tail, "c \otimes_z \delta"] & \\
& & Z \times \mathrm{I}\,.
\end{tikzcd} \tag{50}$$

for any cofibration $c : C \rightarrowtail Z$ and indexing map $z : Z \to \mathrm{I}$, with domain $D = \big(Z +_C (C \times \mathrm{I})\big)$.

- The class $\mathcal{F}$ of *fibrations*, written $f : Y \twoheadrightarrow X$, may be characterized as the right-lifting class of the generating trivial cofibrations,

$$(\mathcal{C} \otimes \delta)^{\pitchfork} \;=\; \mathcal{F}.$$

- The class of *trivial cofibrations* is defined to be left class of the fibrations,

$$\mathsf{TCof} \;=\; {}^{\pitchfork}\mathcal{F}.$$

It follows that the classes TCof and $\mathcal{F}$ are mutually weakly orthogonal,

$$\mathsf{TCof} \pitchfork \mathcal{F},$$

and are closed under retracts. Thus to have a weak factorization system $(\mathsf{TCof}, \mathcal{F})$ it just remains to show that every map $f : X \to Y$ can be factored as $f = g \circ h$ with $g \in \mathcal{F}$ and $h \in \mathsf{TCof}$.

Proposition 36. *Every map $f : X \to Y$ in cSet can be factored as $f = p \circ i$,*

$$\begin{tikzcd} X \arrow[r, tail, "i"] \arrow[dr, "f"'] & X' \arrow[d, two heads, "p"] \\ & Y \end{tikzcd} \tag{51}$$

with $i : X \rightarrowtail X'$ a trivial cofibration and $p : X' \twoheadrightarrow Y$ a fibration.

Proof. We can use a standard argument (the “algebraic small objects argument”, cf. [GKR18]), further simpified by the fact that the codomains of the basic trivial cofibrations $c \otimes_z \delta : B \rightarrowtail \mathrm{I}^{n+1}$ are not just representable, but *tiny* in the sense of Proposition 3, while the domains are not merely “small”, but finitely presented. The reader is referred to [**?**] for details (in a similar case). □

Proposition 37. *There is a weak factorization system on the category* cSet *in which the right maps are the fibrations and the left maps are the trivial cofibrations, both as specified in definition 35.*

This will be called the *fibration weak factorization system*. The following observation will be of use later on; a proof can be found in [GKR18, **?**].

Corollary 38. *The* fibrant replacement *of a map* $f : X \to Y$

$$\begin{tikzcd} X \arrow[r, tail, "i_f"] \arrow[dr, "f"'] & X' \arrow[d, two heads, "f'"] \\ & Y , \end{tikzcd} \tag{52}$$

can be given as an ω*-colimit in the slice category over* Y*,*

$$f' = \varinjlim_n f_n$$

so that it is functorial, and the canonical trivial cofibrations $i_f : X \rightarrowtail X'$ *over* Y *are natural, in* $f : X \to Y$*.*

4 The weak equivalences

Definition 39 (Weak equivalence)**.** A map $f : X \to Y$ in cSet is a *weak equivalence* if it can be factored as $f = g \circ h$,

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[dr, "f"'] & W \arrow[d, "g"] \\ & Y \end{tikzcd}$$

with $h : X \to W$ a trivial cofibration and $g : W \to Y$ a trivial fibration. Let

$$\mathcal{W} = \{f : X \to Y \mid f = g \circ h \text{ for } g \in \mathsf{TFib} \text{ and } h \in \mathsf{TCof}\}$$

be the class of weak equivalences.

Observe that every trivial fibration $f \in \mathcal{C}^{\pitchfork}$ is indeed a fibration, because the generating trivial cofibrations are cofibrations; moreover, every trivial fibration is also a weak equivalence, since the identity maps are trivial cofibrations. Thus we have

$$\mathsf{TFib} \subseteq (\mathcal{F} \cap \mathcal{W}).$$

Thus, because the trivial fibrations are fibrations, every trivial cofibration $g \in {}^{\pitchfork}\mathcal{F}$ is a cofibration; moreover, every trivial cofibration is also a weak equivalence, since the identity maps are also trivial fibrations. Thus we also have

$$\mathsf{TCof} \subseteq (\mathcal{C} \cap \mathcal{W}).$$

Lemma 40. $(\mathcal{C} \cap \mathcal{W}) \subseteq \mathsf{TCof}$.

Proof. Let $c : A \rightarrowtail B$ be a cofibration with a factorization

$$c = tf \circ tc : A \to W \to B$$

where $tc \in \mathsf{TCof}$ and $tf \in \mathsf{TFib}$. Let $f : Y \twoheadrightarrow X$ be a fibration and consider a commutative diagram,

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, tail, "c"'] & Y \arrow[d, two heads, "f"] \\ B \arrow[r, "b"'] & X. \end{tikzcd}$$

Inserting the factorization of c, we have $j : W \to Y$ as indicated, with $j \circ tc = a$ and $f \circ j = b \circ tf$, since $tc \pitchfork f$.

$$\begin{tikzcd} A \arrow[rr, "a"] \arrow[dd, tail, "c"'] \arrow[dr, "tc"'] & & Y \arrow[dd, two heads, "f"] \\ & W \arrow[ur, dotted, "j"'] \arrow[dl, "tf"] & \\ B \arrow[rr, "b"'] & & X. \end{tikzcd}$$

Moreover, since $c \pitchfork tf$ there is an $i : B \to W$ as indicated, with $i \circ c = tc$ and $tf \circ i = 1_B$.

$$\begin{tikzcd} A \arrow[rr, "a"] \arrow[dd, tail, "c"'] \arrow[dr, "tc"'] & & Y \arrow[dd, two heads, "f"] \\ & W \arrow[ur, dotted, "j"'] \arrow[dl, "tf"] & \\ B \arrow[rr, "b"'] \arrow[ur, dotted, bend right, "i"'] & & X. \end{tikzcd}$$

Let $k = j \circ i$. Then $k \circ c = j \circ i \circ c = j \circ tc = a$, and $f \circ k = f \circ j \circ i = b \circ tf \circ i = b$. $\square$

The proof of the following is dual:

Lemma 41. $(\mathcal{F} \cap \mathcal{W}) \subseteq \mathsf{TFib}$.

Proposition 42. *For the three classes of maps $\mathcal{C}, \mathcal{W}, \mathcal{F}$ in* cSet*, we have*

$$\mathcal{F} \cap \mathcal{W} = \mathsf{TFib},$$
$$\mathcal{C} \cap \mathcal{W} = \mathsf{TCof},$$

and therefore two weak factorization systems:

$$(\mathcal{C},\, \mathcal{W} \cap \mathcal{F}) \quad , \quad (\mathcal{C} \cap \mathcal{W},\, \mathcal{F}).$$

Corollary 43. *The following are equivalent for a map $f : X \to Y$.*

1. *$f : X \to Y$ is a weak equivalence*
2. *the first factor $\eta : X \to X'$ of the cofibration factorization of f is a trivial cofibration.*
3. *the second factor $p : Y' \to Y$ of the fibration factorization of f is a trivial fibration.*

Weak homotopy equivalence. To show that the weak equivalences satisfy the 3-for-2 condition, we shall follow the approach of [**?**], verifying that many of the same arguments go through in the current setting – up to a certain point.

Definition 44. By a *homotopy* between parallel maps $f, g : X \rightrightarrows Y$, written $\vartheta : f \sim g$, we mean a map from the *cylinder of* X built using the (representable) interval I,

$$\vartheta : \mathrm{I} \times X \to Y,$$

and such that $\vartheta \circ \iota_0 = f$ and $\vartheta \circ \iota_1 = g$,

$$\begin{tikzcd} X \arrow[r, "\iota_0"] \arrow[dr, "f"'] & \mathrm{I} \times X \arrow[d, "\vartheta"] & X, \arrow[l, "\iota_1"'] \arrow[dl, "g"] \\ & Y & \end{tikzcd}$$

where we write the canonical inclusions into the ends of the cylinder as

$$\iota_\epsilon = \mathsf{Gph}(\delta_\epsilon !) : X \to \mathrm{I} \times X\,, \qquad \epsilon = 0, 1.$$

Proposition 45. *If K is fibrant, then the relation of homotopy $f \sim g$ between maps $f, g : X \rightrightarrows K$ is an equivalence relation. Moreover, it is compatible with pre- and post-composition.*

Proof. For $f, g : X \rightrightarrows Y$, a homotopy $f \overset{\vartheta}{\sim} g : X \times \mathrm{I} \to Y$ is equivalent, under exponential transposition, to a path in the function space $\vartheta : \mathrm{I} \to Y^X$ with endpoints $\vartheta_0 = \vartheta \circ \delta_0 = f : 1 \to Y^X$ and $\vartheta_1 = g$. Note that Y^X is fibrant if Y is fibrant, since the generating trivial cofibrations are closed under taking the product with a fixed object. So we can use box-filling in Y^X.

The reflexivity of homotopy $f \sim f$ is witnessed by $\rho : \mathrm{I} \to 1 \xrightarrow{f} Y^X$.

For symmetry $f \sim g \Rightarrow g \sim f$ take $\vartheta : \mathrm{I} \to Y^X$ with $\vartheta_0 = f$ and $\vartheta_1 = g$ and we want to build $\vartheta' : \mathrm{I} \to Y^X$ with $\vartheta'_0 = g$ and $\vartheta'_1 = f$. Take an open 2-box in Y^X of the form

$$\begin{tikzcd} g & f \\ f \arrow[u, "\vartheta"] \arrow[r, "\rho"'] & f \arrow[u, "\rho"'] \end{tikzcd}$$

This box is a map $b : \mathrm{I} +_1 \mathrm{I} +_1 \mathrm{I} \to Y^X$ with the indicated components, and it has a filler $c : \mathrm{I} \times \mathrm{I} \to Y^X$, i.e. an extension along the canonical map $\mathrm{I} +_1 \mathrm{I} +_1 \mathrm{I} \rightarrowtail \mathrm{I} \times \mathrm{I}$, which is a trivial cofibration. Let $t : \mathrm{I} \to \mathrm{I} \times \mathrm{I}$ be the evident missing top face of the 2-cube. We can set $\vartheta' = c \circ t : \mathrm{I} \to Y^X$ to get a homotopy $\vartheta' : \mathrm{I} \to Y^X$ with required endpoints.

For transitivity, $f \overset{\vartheta}{\sim} g \;\&\; g \overset{\varphi}{\sim} h \Rightarrow f \sim h$, an analogous filling construction can be used with the open box:

$$\begin{tikzcd} f & h \\ f \arrow[u, "\rho"] \arrow[r, "\vartheta"'] & g \arrow[u, "\varphi"'] \end{tikzcd}$$

Compatibility under pre- and post-composition is shown by representing homotopies by mapping into the pathspace, for precomposition, and out of the cylinder, for post-composition. $\square$

Definition 46 (Connected components)**.** The functor

$$\pi_0 : \mathsf{cSet} \to \mathsf{Set}$$

is defined on a cubical set X as the coequalizer

$$X_1 \rightrightarrows X_0 \to \pi_0 X,$$

where the two parallel arrows are the maps $X_{\delta_0}, X_{\delta_1} : X_1 \rightrightarrows X_0$ induced by the endpoints $\delta_0, \delta_1 : 1 \rightrightarrows \mathrm{I}$. For any fibrant object K we therefore have

$\pi_0 K = \mathrm{Hom}(1, K)/\!\sim$, that is, $\pi_0 K$ is the set of points $1 \to K$, modulo the homotopy equivalence relation on them.

One can show that in fact $\pi_0 X = \varinjlim_n X_n$, the colimit being left adjoint to the constant presheaf functor $\Delta : \mathsf{Set} \to \mathsf{cSet}$. Since the category $\mathbb{B}$ of finite strictly bipointed sets is sifted, we have:

Corollary 47. *The functor* $\pi_0 : \mathsf{cSet} \to \mathsf{Set}$ *preserves finite products.*

As usual, a map $f : X \to Y$ in cSet will be called a *homotopy equivalence* if there is a *quasi-inverse* $g : Y \to X$ and homotopies $\vartheta : 1_X \sim g \circ f$ and $\varphi : 1_Y \sim f \circ g$.

Definition 48 (Weak homotopy equivalence)**.** A map $f : X \to Y$ is called a *weak homotopy equivalence* if for every fibrant object K, the "internal precomposition" map $K^f : K^Y \to K^X$ is bijective on connected components,

$$\pi_0 K^f : \pi_0 K^Y \cong \pi_0 K^X .$$

Lemma 49. *A homotopy equivalence is a weak homotopy equivalence.*

Proof. If $f : X \to Y$ is a homotopy equivalence, then so is $K^f : K^Y \to K^X$ for any K, since homotopy respects composition. Since K^X is always fibrant when K is, π_0 is well defined, and it clearly takes homotopy equivalences to isomorphisms of sets. □

Lemma 50. *The weak homotopy equivalences* $f : X \to Y$ *satisfy the 3-for-2 condition.*

Proof. This follows from the corresponding fact about bijections of sets. □

Our goal of showing that the weak equivalences satisfy 3-for-2 is now reduced to showing that a map is a weak equivalence (WE) if and only if it is a weak homotopy equivalence (WHE). This will be proved in four cases, showing that a (co)fibration is a WE if and only if it is a WHE.

Lemma 51. *A map* $f : X \to Y$ *is a* weak homotopy equivalence *iff it satisfies the following two conditions.*

1. *For every fibrant object K and every map $x : X \to K$ there is a map $y : Y \to K$ such that $y \circ f \sim x$,*

$$\begin{tikzcd} X \arrow[r, "x"] \arrow[d, "f"'] & K. \\ Y \arrow[ur, dotted, "y"'] \arrow[u, phantom, shift left=20pt, "\sim"] & \end{tikzcd}$$

   We say that x "extends along f up to homotopy".

2. *For every fibrant object K and maps $y, y' : Y \to K$ such that $yf \sim y'f$, there is a homotopy $y \sim y'$,*

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & K^{\mathrm{I}} \arrow[d] \\ Y \arrow[r, "\langle y, y' \rangle"'] \arrow[ur, dotted] & K \times K. \end{tikzcd}$$

Proof. Condition (1) says exactly that the internal precomposition map $K^f : K^Y \to K^X$ is surjective on connected components, while (2) says just that it is injective. $\square$

Lemma 52. *A cofibration $c : A \rightarrowtail B$ that is a WE is a WHE.*

Proof. A cofibration $c : A \rightarrowtail B$ that is a WE is a trivial cofibration by proposition 42. So the result follows from Lemma 51, together with the fact that $K^\partial : K^{\mathrm{I}} \to K^{1+1} \cong K \times K$ is a fibration whenever K is fibrant, since $\partial : 1 + 1 \rightarrowtail \mathrm{I}$ is a cofibration, $\square$

Lemma 53. *A fibration $p : Y \twoheadrightarrow X$ that is a WE is a WHE.*

Proof. A fibration weak equivalence $f : Y \twoheadrightarrow X$ is a trivial fibration by proposition 42, and therefore has a section $s : X \rightarrowtail Y$, by the lifting problem

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, tail] & Y \arrow[d, two heads, "f"] \\ X \arrow[r, "="'] & X, \end{tikzcd}$$

since $0 \to X$ is a cofibration. Moreover, there is a homotopy $\vartheta : sf \sim 1_Y$, resulting from the lifting problem

$$\begin{tikzcd} Y+Y \arrow[r, "{[sf,1]}"] \arrow[d, tail, "{[\iota_0,\iota_1]}"'] & Y \arrow[d, two heads, "f"] \\ \mathrm{I} \times Y \arrow[r, "f\pi_2"'] & X. \end{tikzcd}$$

Thus f is a homotopy equivalence, and so a WHE by lemma 49. ☐

Corollary 54. *A WE $e : X \simeq Y$ is a WHE, since e can be factored into a trivial cofibration followed by a trivial fibration, each of which is a WHE, and these are closed under composition.*

Lemma 55. *If K is fibrant, then any fibration $f : Y \twoheadrightarrow K$ that is a HE is a WE.*

Proof. This is a standard argument, which we just sketch. It suffices to show that any diagram of the form

$$\begin{tikzcd} C \arrow[r, "y"] \arrow[d, tail, "c"'] & Y \arrow[d, two heads, "f"] \\ K \arrow[r, "="'] & K, \end{tikzcd} \tag{53}$$

with $c : C \rightarrowtail X$ a cofibration, has a diagonal filler. Since f is a HE it has a quasi-inverse $s : X \to Y$ with $\vartheta : fs \sim 1_K$, which we can correct to a section $s' : K \to Y$. Indeed, consider

$$\begin{tikzcd} K \arrow[r, "s"] \arrow[d, tail, "\iota_0"'] & Y \arrow[d, two heads, "f"] \\ K \times \mathrm{I} \arrow[r, "\vartheta"'] \arrow[ur, dotted, "\vartheta'"] & K \\ K \arrow[u, tail, "\iota_1"] \arrow[ur, "="'] & \end{tikzcd}$$

where ϑ' results from $\iota_0 \pitchfork f$. Let $s' = \vartheta' \iota_1$, so that $\vartheta' : s \sim s'$ and $fs' = 1_K$.

Thus we can assume that $s = s' : K \to Y$ is a section, which fills the diagram (53) up to a homotopy in the upper triangle.

$$\begin{tikzcd} C \arrow[r, "y"] \arrow[d, tail, "c"'] & Y \arrow[d, two heads, "f"] \\ K \arrow[r, "="'] \arrow[ur, "s"'] \arrow[ur, phantom, shift left=3, "\sim"] & K, \end{tikzcd}$$

Now we can correct $s : K \to Y$ to a homotopic $t : K \to Y$ over f by using the homotopy $\varphi : sc \sim y$ to get a map $\varphi : C \to Y^{\mathrm{I}}$ over f. Since f is a fibration, the projections $p_0, p_1 : Y^{\mathrm{I}} \to Y$ over f are trivial fibrations, and so there is a lift $\varphi' : K \to Y^{\mathrm{I}}$ for which $t := p_1\varphi'$ has $tc = y$ and $ft = 1_K$, and so is a filler for (53). □

Lemma 56. *If K is fibrant, then any fibration $f : Y \twoheadrightarrow K$ that is a WHE is a WE.*

Proof. Since K is fibrant, so is Y, and since f is a WHE, there is a map $s : K \to Y$ and a homotopy $\theta : sf \sim 1_Y$ by lemma 51(1). Thus, applying f again, we have a homotopy $f\vartheta : fsf \sim f$, forming the outer commutative square in

$$\begin{tikzcd}
Y \arrow[r, "f\vartheta"] \arrow[d, "f"'] & K^{\mathrm{I}} \arrow[d] \\
K \arrow[r, "\langle fs, 1_K \rangle"'] \arrow[ur, dotted, "\varphi"'] & K \times K.
\end{tikzcd}$$

By lemma 51(2) there is a diagonal filler $\varphi : fs \sim 1_K$, and so f is a HE. Now apply lemma 55. □

Lemma 57. *If K is fibrant, then any cofibration $c : A \rightarrowtail K$ that is a WHE is a WE.*

Proof. Let $c : A \rightarrowtail K$ be a cofibration WHE and factor it into a trivial cofibration $i : A \rightarrowtail Z$ followed by a fibration $p : Z \twoheadrightarrow K$. By lemma 51, it is clear that a trivial cofibration is a WHE. So both c and i are WHE, and therefore so is p by 3-for-2 for WHEs. Since K is fibrant, p is a trivial fibration by lemma 56, and thus c is a WE. □

Lemma 58 ([?], x.n.m)**.** *A cofibration $c : A \rightarrowtail B$ WHE lifts against all fibrations $f : Y \twoheadrightarrow K$ with fibrant codomain.*

Proof. Let $c : A \rightarrowtail B$ be a cofibration WHE and $f : Y \twoheadrightarrow K$ a fibration with fibrant codomain K, and consider a lifting problem

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, tail, "c"'] & Y \arrow[d, two heads, "f"] \\
B \arrow[r, "b"'] & K.
\end{tikzcd}$$

Let $\eta : B \rightarrowtail B'$ be a fibrant replacement of B, since K is fibrant, b extends along η to give $b' : B' \to K$ as in:

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, tail, "c"'] & Y \arrow[d, two heads, "f"] \\
B \arrow[r, "b"'] \arrow[d, "\eta"'] & K \\
B' \arrow[ur, "b'"'] &
\end{tikzcd}$$

Since η is a trivial cofibration, it is a WHE. So the composite ηc is also a WHE. But since B' is fibrant, ηc is then a trivial cofibration by lemma 57. Thus there is a lift $j : B' \to Y$, and therefore also one $k = j\eta : B \to Y$. $\square$

To complete the proof that a cofibration WHE is a WE, we use the following *fibration extension property* (FEP), the proof of which is deferred to section 8.

Definition 59 (Fibration extension property)**.** For any fibration $f : Y \twoheadrightarrow X$ and trivial cofibration $\eta : X \to X'$, there is a fibration $f' : Y' \twoheadrightarrow X'$ of which f is a pullback along η,

$$\begin{tikzcd}
Y \arrow[r] \arrow[d, two heads, "f"'] \arrow[dr, phantom, very near start, "\lrcorner"] & Y' \arrow[d, two heads, "f'"] \\
X \arrow[r, tail, "\eta"'] & X'.
\end{tikzcd} \tag{54}$$

Lemma 60. *Assuming the FEP, a cofibration that lifts against every fibration $f : Y \twoheadrightarrow K$ with fibrant codomain is a WE.*

Proof. Let $c : A \rightarrowtail B$ be a cofibration and consider a lifting problem against an arbitrary fibration $f : Y \twoheadrightarrow X$,

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, tail, "c"'] & Y \arrow[d, two heads, "f"] \\
B \arrow[r, "b"'] & X.
\end{tikzcd} \tag{55}$$

Let $\eta : X \to X'$ be a fibrant replacement, so η is a trivial cofibration and X' is fibrant. By the fibration extension property of definition 59, there is a fibration $f' : Y' \twoheadrightarrow X'$ such that f is a pullback of f' along η. So we can extend diagram (55) to obtain the following, in which the righthand square is a pullback.

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, tail, "c"'] & Y \arrow[r, "y"] \arrow[d, two heads, "f"] \arrow[dr, phantom, "\lrcorner", very near start] & Y' \arrow[d, two heads, "f'"] \\ B \arrow[r, "b"'] & X \arrow[r, "\eta"'] & X'. \end{tikzcd} \tag{56}$$

By assumption, there is a lift $j' : B \to Y'$ with $f'j' = \eta b$ and $j'c = yb$. Therefore, since f is a pullback, there is a map $j : B \to Y$ with $fj = b$ and $yj = j'$.

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, tail, "c"'] & Y \arrow[r, "y"] \arrow[d, two heads, "f"] \arrow[dr, phantom, "\lrcorner", very near start] & Y' \arrow[d, two heads, "f'"] \\ B \arrow[r, "b"'] \arrow[ur, dotted, "j"] \arrow[urr, dotted, "j'"' near end] & X \arrow[r, "\eta"'] & X'. \end{tikzcd} \tag{57}$$

Thus $yjc = j'c = ya$. But as a trivial cofibration, η is monic, and as a pullback of η, y is also monic. So $jc = a$. $\square$

Combining the previous two lemmas 58 and 60 we now have the following.

Corollary 61. *Assuming the FEP, a cofibration $c : A \rightarrowtail B$ that is a WHE is a WE.*

The following is not required, but we state it anyway for the record:

Lemma 62. *Assuming the FEP, a fibration $f : Y \twoheadrightarrow X$ that is a WHE is a WE.*

Proof. Factor $f : Y \twoheadrightarrow X$ into a cofibration $i : Y \rightarrowtail Z$ followed by a trivial fibration $p : Z \twoheadrightarrow X$. Then f is a trivial fibration if $i \pitchfork f$, for then f is a retract of p. Since p is a trivial fibration, it is a WHE by lemma 53. Since f is also a WHE, so is i by 3-for-2. Thus i is a trivial cofibration by corollary 61. Since f is a fibration, $i \pitchfork f$ as required. $\square$

Proposition 63. *Assuming the FEP, a map $f : X \rightarrowtail Y$ is a WHE if and only if it is a WE. Thus the weak equivalences $\mathcal{W}$ satisfy the 3-for-2 condition.*

Proof. Let $f : X \twoheadrightarrow Y$ be a WE and factor it into a trivial cofibration $i : X \rightarrowtail Z$ followed by a trivial fibration $p : Z \twoheadrightarrow Y$. Then both i and p are WHE, whence so is f. Conversely, let f be a WHE and factor it into a cofibration $i : X \rightarrowtail Z$ followed by a trivial fibration $p : Z \twoheadrightarrow Y$. Since p is then a WHE, as is f, it follows that i is as well. Thus i is also a WE, by lemma 61, hence a trivial cofibration. So f is a WE. $\square$

Our results thus far can now be summarized as follows.

Theorem 64. *Assume the fibration weak factorization system of Definition 35 satisfies the fibration extension property of Definition 59 (as will be shown in Corollary 93). Then the weak equivalences $\mathcal{W}$ have the 3-for-2 property, and so by Proposition 42, the three classes $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ determined by Definition 35 form a Quillen model structure on the category* cSet *of cubical sets.*

The weak equivalences are those maps $f : X \to Y$ for which $K^f : K^Y \to K^X$ is bijective on connected components whenever K is fibrant.

The proof of the fibration extension property will occupy the second half of these lectures, concluding in Section 8. It requires several intermediate results, namely the equivalence extension property (Section 7), a universal fibration (Section 6.1), and the Frobenius condition (Section 5), to which we now turn.

5 The Frobenius condition

In this section, we show that the (unbiased) fibration weak factorization system from section 3 satisfies what has been called the *Frobenius condition*: the left maps are stable under pullback along the right maps (see [**?**]). This will imply the *right properness* of our model structure: the weak equivalences are preserved by pullback along fibrations. In our setting, it then follows that the entire model structure is stable under such a base change. The Frobenius condition will be used in the proof of the equivalence extension property in Section 7.

A proof of Frobenius in the related setting of cubical sets *with connections* was given in [GS17] using conventional, functorial methods (which we shall call *algebraic*). By contrast, the type theoretic approach of [CCHM16] provides a proof that is much more direct, and can also be modified to work without connections (as in [**?**]). That approach proves the dual fact that the *pushforward* operation, which is right adjoint to pullback and always exists in a topos, preserves fibrations when applied along a fibration. This corresponds to the type-theoretic Π-formation rule, and the proof given in *op.cit.* is entirely in type theory. It employs a reduction of box filling (in all dimensions) to an apparently weaker condition of *Kan composition* (in all dimensions), which merely “puts a lid on” the open box, rather than filling it. This aspect of the type theoretic proof can also be described algebraically, but is not used in the algebraic proof given here, and so it will not be discussed further (see [?] for an algebraic description of Kan composition with connections, and [?, ?, ?, ?] for the same without connections).

This section applies the method of *generalization* explained in the introduction, which procedes by first establishing a structure or proposition in the generic biased setting, and then transferring it to the unbiased setting by pullback along the base change $\mathsf{cSet} \to \mathsf{cSet}/_{\mathrm{I}}$. In order to emphasize this method, we give the second step first, in the form of a conditional statement.

Proposition 65. *Suppose the δ-biased fibration weak factorization system on $\mathsf{cSet}/_{\mathrm{I}}$ satisfies the Frobenius condition. Then the unbiased fibration weak factorization system on cSet also satisfies the Frobenius condition.*

Proof. This follows directly from the fact that the pullback functor $\mathrm{I}^* : \mathsf{cSet} \to \mathsf{cSet}/_{\mathrm{I}}$ preserves the locally cartesian closed structure, and also creates fibrations. In detail, let unbiased fibrations $B \twoheadrightarrow A$ and $A \twoheadrightarrow X$ in cSet be given, and we wish to find $C \twoheadrightarrow X$ and $e : A \times_X C \to B$ over A, universal in the way recalled in the diagram below.

$$\begin{tikzcd}
A \times_X C \arrow[r, dotted] \arrow[d, dotted, "e"] & C \arrow[dd, dotted, two heads] \\
B \arrow[d, two heads] & \\
A \arrow[r, two heads] & X
\end{tikzcd} \tag{58}$$

Take the pushforward $C := A_*B \to X$, and its associated map $e : A \times_X C \to B$, in the locally cartesian closed category cSet. Since fibrations are stable under (all) pullbacks, it then suffices to show that $C \to X$ is a fibration.

By definition, $C \to X$ is an unbiased fibration in cSet just in case the base change $\mathrm{I}^*C \to \mathrm{I}^*X$ is a δ-biased fibration in the slice category $\mathsf{cSet}/_{\mathrm{I}}$. Since the pullback functor $\mathrm{I}^* : \mathsf{cSet} \to \mathsf{cSet}/_{\mathrm{I}}$ preserves all lcc structure, over I^*X we have an iso,

$$\mathrm{I}^*C = \mathrm{I}^*(A_*B) \cong (\mathrm{I}^*A)_*\mathrm{I}^*B\,,$$

where the pushforward $(\mathrm{I}^*A)_*\mathrm{I}^*B$ is taken in the topos $\mathsf{cSet}/_{\mathrm{I}}$. But $\mathrm{I}^*B \to \mathrm{I}^*A$ and $\mathrm{I}^*A \to \mathrm{I}^*X$ are δ-biased fibrations in $\mathsf{cSet}/_{\mathrm{I}}$ because $B \to A$ and $A \to X$ were assumed to be unbiased fibrations in cSet. Since we are assuming the Frobenius condition for δ-biased fibrations in $\mathsf{cSet}/_{\mathrm{I}}$, the pushforward $\mathrm{I}^*C \cong (\mathrm{I}^*A)_*\mathrm{I}^*B \to \mathrm{I}^*X$ is also a δ-biased fibration, as required. □

Frobenius for biased fibrations. The results proved in this section will be applied to the slice category $\mathsf{cSet}/_{\mathrm{I}}$ and the generic point $\delta : 1 \to \mathbb{I} = \mathrm{I}^*\mathrm{I}$, but nothing in their proofs depends on this particular case, and so we shall write simply $\delta : 1 \to \mathrm{I}$ for a chosen pointed object in an arbitrary topos $\mathcal{E}$. (Indeed, in this section $\mathcal{E}$ may even be taken to be just a locally cartesian closed category with a representable class of cofibrations.)

Recall from Definition **??** that a map $f : A \to X$ is a δ-biased fibration just if the map $\delta \Rightarrow f$ admits a relative +-algebra structure, and is therefore a trivial fibration. The definition of the pullback-hom $\delta \Rightarrow f$ is recalled below.

$$\begin{tikzcd}
A^{\mathrm{I}} \arrow[dr, "\delta\Rightarrow f"] \arrow[drr, bend left, "A^{\delta}"] \arrow[ddr, bend right, "f^{\mathrm{I}}"'] & & \\
& X^{\mathrm{I}} \times_X A \arrow[r] \arrow[d] & A \arrow[d, "f"] \\
& X^{\mathrm{I}} \arrow[r, "X^{\delta}"'] & X
\end{tikzcd} \tag{59}$$

Let us write this condition schematically as follows:

$$\begin{tikzcd}
A^{\mathrm{I}} \arrow[r, "+" marking] & A_{\epsilon} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
& X^{\mathrm{I}} \arrow[r, "\epsilon"'] & X
\end{tikzcd} \tag{60}$$

where $\epsilon = X^{\delta}$, $A_{\epsilon} = X^{\mathrm{I}} \times_X A$, and the struck-through arrow indicates that it admits a +-algebra structure.

Lemma 66. *Let $A \to X$ be a δ-biased fibration and $t : Y \to X$ any map, then the pullback $t^*A \to Y$ is also a δ-biased fibration.*

Proof. This is of course clear, since δ-biased fibrations are the right class of a weak factorization system by Proposition **??**, but it is still instructive to see how the structure indicated in (60) is itself stable under pullback. Indeed, consider the following commutative diagram, in which the front face of the cube is the pullback in question, and the right and left sides are the respective versions of the construction in (60).

$$
\begin{tikzcd}
(t^*A)^{\mathrm{I}} \arrow[rr] \arrow[d] & & A^{\mathrm{I}} \arrow[d] & \\
(t^*A)_\epsilon \arrow[rr, dotted] \arrow[dr] \arrow[dd] & & A_\epsilon \arrow[dr] \arrow[dd] & \\
& t^*A \arrow[rr] \arrow[dd] & & A \arrow[dd] \\
Y^{\mathrm{I}} \arrow[rr] \arrow[dr, "\epsilon"'] & & X^{\mathrm{I}} \arrow[dr, "\epsilon"'] & \\
& Y \arrow[rr, "t"'] & & X
\end{tikzcd}
\tag{61}
$$

The rear square of solid arrows is the image of the front face under the pathobject functor and is therefore also a pullback. The base commutes by the naturality of the maps ϵ, as does a corresponding top square involving further such ϵ's not shown. Note that these naturality squares need not be pullbacks, but the vertical squares on the sides are, by construction. It follows that there is a dotted arrow as shown, making the resulting lower rear square commute. That lower square is then also a pullback, since the other vertical faces of the resulting cube are pullbacks, and thus finally, the upper rear square is also a pullback.

Now if $A \to X$ is a δ-biased fibration, then $A^{\mathrm{I}} \to A_\epsilon$ is a trivial fibration, and then so is its pullback $(t^*A)^{\mathrm{I}} \to (t^*A)_\epsilon$ since relative $+$-algebras are stable under pullback. Therefore the pullback $t^*A \to Y$ is also a δ-biased fibration. □

Remark 67. In this way we can show algebraically that the pullback of a δ-biased fibration is again one by pulling back the structure that makes it so. In Section 6.3, the pullback stability of the fibration structure will be used in the construction of a universal fibration via a closely related argument.

Lemma 68. *Let $\alpha : A \to X$ and $\beta : B \to A$ be δ-biased fibrations, then the composite $\alpha \circ \beta : B \to X$ is also a δ-biased fibration.*

Proof. Again for maps in the right class of a weak factorization system this is immediate. But let us see how the fibration structures also compose. We have the following diagram for the fibration structures on $B \to A$ and $A \to X$ (with obvious notation).

$$\begin{tikzcd}
B^{\mathrm{I}} \arrow[r, "+" description] & B_{\epsilon_A} \arrow[rr] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & & B \arrow[d] \\
& A^{\mathrm{I}} \arrow[r, "+" description] & A_{\epsilon_X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d] \\
& & X^{\mathrm{I}} \arrow[r, "\epsilon_X"'] & X,
\end{tikzcd} \tag{62}$$

Pulling back $B \to A$ in two steps we therefore obtain the intermediate map $B_{\epsilon_X} \to A_{\epsilon_X}$ indicated in the following diagram.

$$\begin{tikzcd}
B^{\mathrm{I}} \arrow[r, "+" description] & B_{\epsilon_A} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B_{\epsilon_X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d] \\
& A^{\mathrm{I}} \arrow[r, "+" description] & A_{\epsilon_X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d] \\
& & X^{\mathrm{I}} \arrow[r, "\epsilon_X"'] & X
\end{tikzcd} \tag{63}$$

Now use the fact that a trivial fibration structure (*i.e.* a +-algebra structure) has a canonical pull-back along any map, and that two such structures have a canonical composition (cf. Remark **??**), to obtain a trivial fibration structure for the indicated composite map $B^{\mathrm{I}} \to B_{\epsilon_X}$, which is then a fibration structure for the composite $B \to A \to X$. □

Proposition 69 (Biased Frobenius)**.** *If $\alpha : A \to X$ and $\beta : B \to A$ are δ-biased fibrations, then the pushforward $\alpha_*\beta : \Pi_A B \to X$ is also a δ-biased fibration.*

Proof. Given the (δ-biased) fibrations $\alpha : A \to X$ and $\beta : B \to A$, let $a : A^{\mathrm{I}} \to A_\epsilon$ and $b : B^{\mathrm{I}} \to a^* B_\epsilon$ be the associated trivial fibrations, so that we have the situation of diagram (63), with all three squares pullbacks.

$$\begin{tikzcd}
B^{\mathrm{I}} \arrow[r, "b", "|" marking] \arrow[dr, "\beta^{\mathrm{I}}"'] & a^*B_\epsilon \arrow[r] \arrow[d] & B_\epsilon \arrow[r] \arrow[d] & B \arrow[d, "\beta"] \\
& A^{\mathrm{I}} \arrow[r, "a", "|" marking] \arrow[dr, "\alpha^{\mathrm{I}}"'] & A_\epsilon \arrow[r] \arrow[d] & A \arrow[d, "\alpha"] \\
& & X^{\mathrm{I}} \arrow[r, "\epsilon"'] & X.
\end{tikzcd} \tag{64}$$

Taking the pushforward of the righthand vertical column,

$$\gamma := \alpha_*\beta : \Pi_A B \to X \,,$$

and placing it underneath, along with the corresponding construction from (60), we then have the following commutative diagram.

$$\begin{tikzcd}
B^{\mathrm{I}} \arrow[r, "b", "|" marking] \arrow[dr, "\beta^{\mathrm{I}}"'] & a^*B_\epsilon \arrow[r] \arrow[d] & B_\epsilon \arrow[r] \arrow[d] & B \arrow[d, "\beta"] \\
& A^{\mathrm{I}} \arrow[r, "a", "|" marking] \arrow[dr, "\alpha^{\mathrm{I}}"'] & A_\epsilon \arrow[r] \arrow[d] & A \arrow[d, "\alpha"] \\
& & X^{\mathrm{I}} \arrow[r, "\epsilon"'] & X \\
& (\Pi_A B)^{\mathrm{I}} \arrow[r, "c"'] \arrow[ur, "\gamma^{\mathrm{I}}"] & (\Pi_A B)_\epsilon \arrow[r] \arrow[u] & \Pi_A B \arrow[u, "\gamma"']
\end{tikzcd} \tag{65}$$

We wish to show that the indicated map $c : (\Pi_A B)^{\mathrm{I}} \to (\Pi_A B)_\epsilon$ admits a +-algebra structure. This we will do by showing that it is a retract of a known +-algebra. Namely, we can apply the pushforward along the map $\alpha^{\mathrm{I}} : A^{\mathrm{I}} \to X^{\mathrm{I}}$ to the +-algebra $b : B^{\mathrm{I}} \to a^*B_\epsilon$ regarded as an arrow over A^{I}. We obtain an arrow over X^{I} of the form

$$\Pi_{A^{\mathrm{I}}}\, b : \Pi_{A^{\mathrm{I}}}\, B^{\mathrm{I}} \longrightarrow \Pi_{A^{\mathrm{I}}}\, a^*B_\epsilon \tag{66}$$

which is indeed a +-algebra, since these are preserved under pushing forward, by Remark 25.

Next, observe that by the Beck-Chevalley condition for the central pullback, for the codomain of c we have an isomorphism

$$(\Pi_A B)_\epsilon \;\cong\; \Pi_{A_\epsilon} B_\epsilon \qquad \text{over } X^{\mathrm{I}}.$$

And since $\Pi_{A^{\mathrm{I}}} \cong \Pi_{A_\epsilon} \circ a_*$, for the codomain of our $+$-algebra $\Pi_{A^{\mathrm{I}}}\, b$ from (66) we also have

$$\Pi_{A^{\mathrm{I}}}\, a^* B_\epsilon \ \cong\ \Pi_{A_\epsilon} a_* a^* B_\epsilon\,.$$

Thus the image of the unit $\eta : B_\epsilon \to a_* a^* B_\epsilon$ under Π_{A_ϵ} provides a map $\sigma := \Pi_{A_\epsilon}\eta$ over X^{I} of the form:

$$\begin{tikzcd}
 & X^{\mathrm{I}} \\
(\Pi_A B)^{\mathrm{I}} \arrow[ur] \arrow[r, "c"'] & \Pi_{A_\epsilon} B_\epsilon \arrow[u] \arrow[d, "\sigma"] \\
\Pi_{A^{\mathrm{I}}} B^{\mathrm{I}} \arrow[r, "\Pi_{A^{\mathrm{I}}}\, b"'] & \Pi_{A_\epsilon} a_* a^* B_\epsilon
\end{tikzcd} \tag{67}$$

Our goal is now to determine further arrows φ, ψ, τ as indicated below, exhibiting c as a retract of $\Pi_{A^{\mathrm{I}}}\, b$ in the arrow category over X^{I}.

$$\begin{tikzcd}
 & X^{\mathrm{I}} \\
(\Pi_A B)^{\mathrm{I}} \arrow[ur] \arrow[r, "c"'] \arrow[d, dotted, "\varphi"'] & \Pi_{A_\epsilon} B_\epsilon \arrow[u] \arrow[d, "\sigma"] \\
\Pi_{A^{\mathrm{I}}} B^{\mathrm{I}} \arrow[r, "\Pi_{A^{\mathrm{I}}}\, b"'] \arrow[d, dotted, "\psi"'] & \Pi_{A_\epsilon} a_* a^* B_\epsilon \arrow[d, dotted, "\tau"] \\
(\Pi_A B)^{\mathrm{I}} \arrow[r, "c"'] & \Pi_{A_\epsilon} B_\epsilon
\end{tikzcd} \tag{68}$$

- For φ, we require a map

$$\varphi : (\Pi_A B)^{\mathrm{I}} \to \Pi_{A^{\mathrm{I}}} B^{\mathrm{I}} \qquad \text{over } X^{\mathrm{I}}.$$

Consider the following diagram, which is based on (65).

$$
\begin{tikzcd}
B^{\mathrm{I}} \arrow[r, "\shortmid" marking, "b"'] \arrow[dr, "\beta^{\mathrm{I}}"'] & a^*B_\epsilon \arrow[r] \arrow[d] & B_\epsilon \arrow[r] \arrow[d] & B \arrow[d, "\beta"] & \\
& A^{\mathrm{I}} \arrow[r, "\shortmid" marking, "a"'] \arrow[dr, "\alpha^{\mathrm{I}}"'] & A_\epsilon \arrow[r] \arrow[d] & A \arrow[d, "\alpha"] & \Pi_A B \times_X A \arrow[l] \arrow[d] \arrow[ul, dotted, "e"'] \\
(\Pi_A B \times_X A)^{\mathrm{I}} \arrow[uu, dotted, "e^{\mathrm{I}}"] \arrow[ur] \arrow[dr] & & X^{\mathrm{I}} \arrow[r, "\epsilon"'] & X & \Pi_A B \arrow[l] \\
& (\Pi_A B)^{\mathrm{I}} \arrow[ur] \arrow[r, dotted, "\varphi"'] & \Pi_{A^{\mathrm{I}}} B^{\mathrm{I}} \arrow[u] & &
\end{tikzcd}
\tag{69}
$$

The map e is the counit at $\beta : B \to A$ of the pullback-pushforward adjunction along $\alpha : A \to X$. The right-hand side of the diagram, including e and the associated pullback square, reappears (mirrored) on the left under the functor $(-)^{\mathrm{I}}$, which preserves the pullback. Thus we can take φ to be the transpose of e^{I} under the pullback-pushforward adjunction along $\alpha^{\mathrm{I}} : A^I \to X^{\mathrm{I}}$,

$$\varphi := \widetilde{e^{\mathrm{I}}}\,.$$

An easy diagram chase involving the pullback-pushforward adjunction along $A_\epsilon \to X^{\mathrm{I}}$ shows that the upper square in (68) then commutes.

• For τ: referring to the diagram (65), since $a : A^{\mathrm{I}} \to A_\epsilon$ is a trivial fibration, it has a section $o : A_\epsilon \to A^{\mathrm{I}}$ by lemma 24. Pulling $a^*B_\epsilon \to A^{\mathrm{I}}$ back along o results in an iso,

$$o^*a^*B_\epsilon \cong B_\epsilon \quad \text{over } A_\epsilon$$

and so by the adjunction $o^* \dashv o_*$ there is an associated map,

$$a^*B_\epsilon \to o_*B_\epsilon \quad \text{over } A^{\mathrm{I}}$$

to which we can apply a_* to obtain a map,

$$t : a_*a^*B_\epsilon \to a_*o_*B_\epsilon \cong B_\epsilon \quad \text{over } A_\epsilon\,.$$

This map t is evidently a retraction of the unit $\eta : B_\epsilon \to a_*a^*B_\epsilon$ over A_ϵ. Applying the functor Π_{A_ϵ} therefore gives the desired retraction of σ,

$$\tau := \Pi_{A_\epsilon} t : \Pi_{A_\epsilon} a_*a^*B_\epsilon \to \Pi_{A_\epsilon} B_\epsilon\,.$$

- For ψ, we require a map

$$\psi : \Pi_{A^{\mathrm{I}}} B^{\mathrm{I}} \to (\Pi_A B)^{\mathrm{I}} \qquad \text{over } X^{\mathrm{I}}.$$

Consider the following diagram resulting from combining (65) and (68), in which all solid arrows are those already introduced. The dotted arrow labelled p is the evident composite.

$$\begin{tikzcd}
 & X^{\mathrm{I}} \arrow[r, "\epsilon"] & X \\
(\Pi_A B)^{\mathrm{I}} \arrow[ur] \arrow[r] \arrow[d] & \Pi_{A_\epsilon} B_\epsilon \arrow[r] \arrow[u] \arrow[d] & \Pi_A B \arrow[u] \arrow[dd, "="] \\
\Pi_{A^{\mathrm{I}}} B^{\mathrm{I}} \arrow[r] \arrow[drr, dotted, "p"] & \Pi_{A_\epsilon} a_* a^* B_\epsilon \arrow[d] & \\
(\Pi_A B)^{\mathrm{I}} \arrow[r] & \Pi_{A_\epsilon} B_\epsilon \arrow[r] & \Pi_A B
\end{tikzcd} \tag{70}$$

The lower horizontal composite is the evaluation of the pathobject $(\Pi_A B)^{\mathrm{I}}$ at the point $\delta : 1 \to \mathrm{I}$,

$$\epsilon_{\Pi_A B} = (\Pi_A B)^{\delta} : (\Pi_A B)^{\mathrm{I}} \longrightarrow (\Pi_A B)^{1} \cong \Pi_A B\,.$$

This is constructed from the (cartesian closed) evaluation,

$$\mathsf{eval} : \mathrm{I} \times (\Pi_A B)^{\mathrm{I}} \longrightarrow \Pi_A B$$

which is the counit of $\mathrm{I} \times (-) \dashv (-)^{\mathrm{I}}$, as the composite shown below.

$$\begin{tikzcd}
(\Pi_A B)^{\mathrm{I}} \arrow[r, "\epsilon_{\Pi_A B}"] \arrow[d, "\cong"'] & \Pi_A B \\
1 \times (\Pi_A B)^{\mathrm{I}} \arrow[r, "\delta \times (\Pi_A B)^{\mathrm{I}}"'] & \mathrm{I} \times (\Pi_A B)^{\mathrm{I}} \arrow[u, "\mathsf{eval}"']
\end{tikzcd} \tag{71}$$

Let us analyse this evaluation at δ further, in terms of the *locally* cartesian closed structure associated to the base changes along the section $\delta : 1 \to \mathrm{I}$ and retraction $\mathrm{I} \to 1$ in $\mathcal{E}$. Since $\mathsf{id} \cong \delta^* \mathrm{I}^* : \mathcal{E} \to \mathcal{E}/_{\mathrm{I}} \to \mathcal{E}$, the map $\epsilon_{\Pi_A B}$ can be rewritten as follows.

$$
\begin{tikzcd}
(\Pi_A B)^{\mathrm{I}} \arrow[r, "\epsilon_{\Pi_A B}"] \arrow[d, "\cong"'] & \Pi_A B \arrow[d, "\cong"] \\
\delta^* \mathrm{I}^*((\Pi_A B)^{\mathrm{I}}) \arrow[r, "\delta^* \mathrm{I}^* \epsilon_{\Pi_A B}"] \arrow[d, "\cong"'] & \delta^* \mathrm{I}^* \Pi_A B \arrow[d, "="] \\
\delta^* \mathrm{I}^* \mathrm{I}_* \mathrm{I}^* \Pi_A B \arrow[r, "\delta^* \varepsilon"'] & \delta^* \mathrm{I}^* \Pi_A B
\end{tikzcd}
\tag{72}
$$

where the map $\delta^*\varepsilon$ across the bottom is the counit of the adjunction $\mathrm{I}^* \dashv \mathrm{I}_*$, taken at $\mathrm{I}^*\Pi_A B$, and then pulled back along $\delta : 1 \to \mathrm{I}$. Before taking the pullback, we therefore have the following iso over I between that counit $\varepsilon_{\mathrm{I}^*}$ and the image under I^* of the previously considered evaluation $\epsilon : (\Pi_A B)^{\mathrm{I}} \to \Pi_A B$ from (71).

$$
\begin{tikzcd}
\mathrm{I}^*((\Pi_A B)^{\mathrm{I}}) \arrow[r, "I^* \epsilon"] \arrow[d, "\cong"'] & \mathrm{I}^* \Pi_A B \arrow[d, "="] \\
\mathrm{I}^* \mathrm{I}_* \mathrm{I}^* \Pi_A B \arrow[r, "\varepsilon_{\mathrm{I}^*}"'] & \mathrm{I}^* \Pi_A B \,.
\end{tikzcd}
\tag{73}
$$

Now let us apply I^* to (70) to get the map I^*p in the diagram below, which therefore factors (up to (73)) through the counit $\varepsilon_{\mathrm{I}^*}$ as $\varepsilon_{\mathrm{I}^*} \circ \mathrm{I}^*(\widetilde{\mathrm{I}^*p})$, where $\widetilde{\mathrm{I}^*p}$ is the adjoint transpose of I^*p, as shown.

$$
\begin{tikzcd}
\mathrm{I}^* \Pi_{A^{\mathrm{I}}} B^{\mathrm{I}} \arrow[r] \arrow[d, "\mathrm{I}^*(\widetilde{\mathrm{I}^*p})"'] \arrow[drr, dotted, "\mathrm{I}^*p"] & \mathrm{I}^* \Pi_{A_\epsilon} a_* a^* B_\epsilon \arrow[d] & \\
\mathrm{I}^* \mathrm{I}_* \mathrm{I}^* \Pi_A B \arrow[r] \arrow[rr, bend right, "\varepsilon_{\mathrm{I}^*}"'] & \mathrm{I}^* \Pi_{A_\epsilon} B_\epsilon \arrow[r] & \mathrm{I}^* \Pi_A B
\end{tikzcd}
\tag{74}
$$

We can therefore set

$$
\psi := \widetilde{\mathrm{I}^*p}\,,
$$

and we obtain $\epsilon \circ \psi = p$, from which it follows that the square in (74) commutes by the definition of $\Pi_{A_\epsilon} B_\epsilon$ as a pullback. The same square without I^* then also commutes by applying the retraction δ^*.

We have now defined all the maps indicated below, the squares involving φ and ψ commute, and the composite of σ and τ is the identity.

$$
\begin{tikzcd}
 & X^{\mathrm{I}} \arrow[r] & X \\
(\Pi_A B)^{\mathrm{I}} \arrow[ur] \arrow[r] \arrow[d, "\varphi"'] & \Pi_{A_\epsilon} B_\epsilon \arrow[u] \arrow[r] \arrow[d, "\sigma"] & \Pi_A B \arrow[u] \arrow[dd, "="] \\
\Pi_{A^{\mathrm{I}}} B^{\mathrm{I}} \arrow[r] \arrow[d, "\psi"'] \arrow[drr, dotted, "p"] & \Pi_{A_\epsilon} p_* p^* B_\epsilon \arrow[d, "\tau"] & \\
(\Pi_A B)^{\mathrm{I}} \arrow[r] \arrow[rr, bend right, "\epsilon"'] & \Pi_{A_\epsilon} B_\epsilon \arrow[r] & \Pi_A B
\end{tikzcd}
\tag{75}
$$

To see that $\psi \circ \varphi = 1$, an easy chase through the diagram (75) shows that

$$\epsilon \circ \psi \circ \varphi = p \circ \varphi = \epsilon \,.$$

Thus by applying I^* and using (73) we have $\varepsilon_{\mathrm{I}^*} \circ \mathrm{I}^*(\psi \circ \varphi) = \varepsilon_{\mathrm{I}^*}$, and so $\psi \circ \varphi = \widetilde{\varepsilon_{\mathrm{I}^*}} = 1$.

$\square$

From Proposition 65 we therefore have:

Corollary 70 (Unbiased Frobenius)**.** *The unbiased fibration weak factorization system on* cSet *satisfies the Frobenius condition.*

Remark 71. We note in passing that the proof just given for the δ-biased case of Frobenius, Proposition 69, made no use of the fact that $\delta : 1 \to \mathrm{I}$ is generic, nor even that we were working in the slice category over I. Indeed the same algebraic argument works for p-biased fibrations for any point $p : 1 \to \mathrm{I}$ of any object I in any topos $\mathcal{E}$.

6 A universal fibration

In this section we construct a *universal small fibration* $\dot{\mathcal{U}} \to \mathcal{U}$. It will then be shown in Section 8 that the base object $\mathcal{U}$ is fibrant, using the equivalence extension property to be proved in Section 7. Our construction of $\dot{\mathcal{U}} \to \mathcal{U}$

makes use, first of all, of a new description of the well-known Hofmann-Streicher universe in a category $\widehat{\mathbb{C}} = [\mathbb{C}^{\mathrm{op}}, \mathsf{Set}]$ of presheaves on a small category $\mathbb{C}$, which was used in [HS97] to interpret dependent type theory. See [?] for further details.

6.1 Classifying families

Definition 72 ([HS97])**.** Let $\mathbb{C}$ be a small category. A (type-theoretic) *universe* $(U, \mathsf{E}l)$ consists of $U \in \widehat{\mathbb{C}}$ and $\mathsf{E}l \in \widehat{\int_{\mathbb{C}} U}$ with:

$$U(c) \;=\; \mathsf{Cat}\big(\mathbb{C}/_c{}^{\mathrm{op}}, \mathsf{Set}\big) \tag{76}$$

$$\mathsf{E}l(c, A) \;=\; A(id_c) \tag{77}$$

with the evident associated action on morphisms.

A few comments are required:

- In contrast to [HS97], in (76) we take the underlying set of objects of the functor category $\widehat{\mathbb{C}/_c} = [\mathbb{C}/_c{}^{\mathrm{op}}, \mathsf{Set}]$.

- As in [HS97], (77) adopts the "categories with families" point of view in describing an arrow $E \to U$ in $\widehat{\mathbb{C}}$ equivalently as a presheaf on the category of elements $\int_{\mathbb{C}} U$, using

$$\widehat{\mathbb{C}}/_U \;\simeq\; \widehat{\int_{\mathbb{C}} U} \tag{78}$$

  where

$$E(c) \;=\; \textstyle\coprod_{A \in U(c)} \mathsf{E}l(c, A).$$

  The argument $(c, A) \in \int_{\mathbb{C}} U$ in (77) thus consists of an object $c \in \mathbb{C}$ and an element $A \in U(c)$.

- To account for size issues, the authors of [HS97] assume a Grothendieck universe u in Set, the elements of which are called *small*. The category $\mathbb{C}$ is assumed to be small, as are the values of the presheaves, unless otherwise stated.

The presheaf U, which is not small, is then regarded as the Grothendieck universe u "lifted" from Set to $[\mathbb{C}^{\mathrm{op}}, \mathsf{Set}]$. We first analyse this specification of $(U, \mathsf{E}l)$ from a different perspective, in order to establish its basic property as a classifier for small families in $\widehat{\mathbb{C}}$.

A realization-nerve adjunction. For a presheaf X on $\mathbb{C}$, recall that the category of elements is the comma category,

$$\textstyle\int_{\mathbb{C}} X \;=\; \mathsf{y}_{\mathbb{C}}/_{X}\,,$$

where $\mathsf{y}_{\mathbb{C}} : \mathbb{C} \to \mathsf{Set}^{\mathbb{C}^{\mathrm{op}}}$ is the Yoneda embedding, which we sometimes supress and write simply $\mathbb{C}/_{X}$ for $\mathsf{y}_{\mathbb{C}}/_{X}$.

Proposition 73 ([Gro83],§28)**.** *The category of elements functor*

$$\textstyle\int_{\mathbb{C}} : \widehat{\mathbb{C}} \longrightarrow \mathsf{Cat}$$

has a right adjoint,

$$\nu_{\mathbb{C}} : \mathsf{Cat} \longrightarrow \widehat{\mathbb{C}}\,.$$

For a small category $\mathbb{A}$, we shall call the presheaf $\nu_{\mathbb{C}}(\mathbb{A})$ the ($\mathbb{C}$-)nerve *of $\mathbb{A}$.*

Proof. The adjunction $\int_{\mathbb{C}} \dashv \nu_{\mathbb{C}}$ is an instance of the usual "realization/nerve" adjunction, here with respect to the covariant slice category functor $\mathbb{C}/- : \mathbb{C} \to \mathsf{Cat}$, as indicated below.

$$\begin{tikzcd}
\widehat{\mathbb{C}} \arrow[r, "\int_{\mathbb{C}}"'] & \mathsf{Cat} \arrow[l, bend right, "\nu_{\mathbb{C}}"'] \\
\mathbb{C} \arrow[u, hook, "\mathsf{y}"] \arrow[ur, "\mathbb{C}/_"']
\end{tikzcd} \tag{79}$$

In detail, for $\mathbb{A} \in \mathsf{Cat}$ and $c \in \mathbb{C}$, let $\nu_{\mathbb{C}}(\mathbb{A})(c)$ be the Hom-set of functors,

$$\nu_{\mathbb{C}}(\mathbb{A})(c) = \mathsf{Cat}\big(\mathbb{C}/_{c}\,,\, \mathbb{A}\big)\,,$$

with contravariant action on $h : d \to c$ given by pre-composing a functor $P : \mathbb{C}/_{c} \to \mathbb{A}$ with the post-composition functor

$$\mathbb{C}/_{h} : \mathbb{C}/_{d} \longrightarrow \mathbb{C}/_{c}\,.$$

For the adjunction, observe that the slice category $\mathbb{C}/_{c}$ is the category of elements of the representable functor $\mathsf{y}c$,

$$\textstyle\int_{\mathbb{C}} \mathsf{y}c \;\cong\; \mathbb{C}/_{c}\,.$$

Thus for representables $\mathsf{y}c$, we have the required natural isomorphism

$$\widehat{\mathbb{C}}\big(\mathsf{y}c\,,\,\nu_{\mathbb{C}}(\mathbb{A})\big) \;\cong\; \nu_{\mathbb{C}}(\mathbb{A})(c) \;=\; \mathsf{Cat}\big(\mathbb{C}/_c\,,\,\mathbb{A}\big) \;\cong\; \mathsf{Cat}\big(\textstyle\int_{\mathbb{C}}\mathsf{y}c\,,\,\mathbb{A}\big)\,.$$

For arbitrary presheaves X, one uses the presentation of X as a colimit of representables over the index category $\int_{\mathbb{C}} X$, and the easy to prove fact that $\int_{\mathbb{C}}$ itself preserves colimits. Indeed, for any category $\mathbb{D}$, we have an isomorphism in Cat,

$$\varinjlim_{d\in\mathbb{D}} \mathbb{D}/_d \;\cong\; \mathbb{D}\,.$$

□

When $\mathbb{C}$ is fixed, we may omit the subscript in the notation $\mathsf{y}_{\mathbb{C}}$ and $\int_{\mathbb{C}}$ and $\nu_{\mathbb{C}}$. The unit and counit maps of the adjunction $\int \dashv \nu$,

$$\eta : X \longrightarrow \nu\textstyle\int X\,,$$
$$\epsilon :\; \textstyle\int\nu\mathbb{A} \longrightarrow \mathbb{A}\,,$$

are then as follows. At $c \in \mathbb{C}$, for $x : \mathsf{y}c \to X$, the functor $(\eta_X)_c(x) : \mathbb{C}/_c \to \mathbb{C}/_X$ is just composition with x,

$$(\eta_X)_c(x) = \mathbb{C}/_x : \mathbb{C}/_c \longrightarrow \mathbb{C}/_X\,. \tag{80}$$

For $\mathbb{A} \in \mathsf{Cat}$, the functor $\epsilon : \int \nu\mathbb{A} \to \mathbb{A}$ takes a pair $(c \in \mathbb{C}, f : \mathbb{C}/_c \to \mathbb{A})$ to the object $f(1_c) \in \mathbb{A}$,

$$\epsilon(c, f) = f(1_c).$$

Lemma 74. *For any $f : Y \to X$, the naturality square below is a pullback.*

$$\begin{tikzcd} Y \arrow[r, "\eta_Y"] \arrow[d, "f"'] & \nu\textstyle\int Y \arrow[d, "\nu\int f"] \\ X \arrow[r, "\eta_X"'] & \nu\textstyle\int X. \end{tikzcd} \tag{81}$$

Proof. It suffices to prove this for the case $f : X \to 1$. Thus consider the square

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d] & \nu\textstyle\int X \arrow[d] \\ 1 \arrow[r, "\eta_1"'] & \nu\textstyle\int 1. \end{tikzcd} \tag{82}$$

Evaluating at $c \in \mathbb{C}$ and applying (80) gives the following square in Set.

$$\begin{tikzcd} Xc \arrow[r, "\mathbb{C}/_{-}"] \arrow[d] & \mathsf{Cat}\big(\mathbb{C}/_c\,,\,\mathbb{C}/_X\big) \arrow[d] \\ 1c \arrow[r, "\mathbb{C}/_{-}"'] & \mathsf{Cat}\big(\mathbb{C}/_c\,,\,\mathbb{C}/_1\big) \end{tikzcd} \tag{83}$$

The image of $* \in 1c$ along the bottom is the forgetful functor $U_c : \mathbb{C}/_c \to \mathbb{C}$, and its fiber under the map on the right is the set of functors $F : \mathbb{C}/_c \to \mathbb{C}/_X$ such that $U_X \circ F = U_c$, where $U_X : \mathbb{C}/_X \to \mathbb{C}$ is also a forgetful functor. But any such F is uniquely of the form $\mathbb{C}/_x$ for $x = F(1_c) : \mathsf{y}c \to X$. $\square$

A universal family. For the terminal presheaf $1 \in \widehat{\mathbb{C}}$ we have an iso $\int 1 \cong \mathbb{C}$, so for every $X \in \widehat{\mathbb{C}}$ there is a canonical projection $\int X \to \mathbb{C}$, which is a discrete fibration. It follows that for any map $Y \to X$ of presheaves, the associated map $\int Y \to \int X$ is also a discrete fibration. Ignoring size issues temporarily, recall that discrete fibrations in Cat are classified by the forgetful functor $\dot{\mathsf{Set}}^{\mathrm{op}} \to \mathsf{Set}^{\mathrm{op}}$ from (the opposites of) the category of pointed sets to that of sets (cf. [Web07]). For every presheaf $X \in \widehat{\mathbb{C}}$, we therefore have a pullback diagram in Cat,

$$\begin{tikzcd} \int X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathsf{Set}}^{\mathrm{op}} \arrow[d] \\ \mathbb{C} \arrow[r, "X"'] & \mathsf{Set}^{\mathrm{op}}. \end{tikzcd} \tag{84}$$

Using $\mathbb{C} \cong \int 1$ and transposing by the adjunction $\int \dashv \nu$ then gives a commutative square in $\widehat{\mathbb{C}}$ of the form:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \nu\dot{\mathsf{Set}}^{\mathrm{op}} \arrow[d] \\ 1 \arrow[r, "\tilde{X}"'] & \nu\mathsf{Set}^{\mathrm{op}}. \end{tikzcd} \tag{85}$$

Lemma 75. *The square* (85) *is a pullback in* $\widehat{\mathbb{C}}$. *More generally, for any* *map $Y \to X$ in $\widehat{\mathbb{C}}$, there is a canonical pullback square*

$$\begin{tikzcd} Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \nu\dot{\mathsf{Set}}^{\mathrm{op}} \arrow[d] \\ X \arrow[r] & \nu\mathsf{Set}^{\mathrm{op}}. \end{tikzcd} \tag{86}$$

Proof. Apply the right adjoint ν to the pullback square (84) and paste the naturality square (81) from Lemma 74 on the left, to obtain the transposed square (86) as a pasting of two pullbacks. □

Let us write $\dot{\mathcal{V}} \to \mathcal{V}$ for the vertical map on the right in (86), setting

$$\begin{aligned} \dot{\mathcal{V}} &:= \nu\dot{\mathsf{Set}}^{\mathrm{op}} \\ \mathcal{V} &:= \nu\mathsf{Set}^{\mathrm{op}}. \end{aligned} \tag{87}$$

We summarize our results so far as follows.

Proposition 76. *The nerve $\dot{\mathcal{V}} \to \mathcal{V}$ of the classifier for discrete fibrations $\dot{\mathsf{Set}}^{\mathrm{op}} \to \mathsf{Set}^{\mathrm{op}}$, as defined in (87), classifies natural transformations $Y \to X$ in $\widehat{\mathbb{C}}$, in the sense that there is always a pullback square,*

$$\begin{tikzcd} Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathcal{V}} \arrow[d] \\ X \arrow[r, "\tilde{Y}"'] & \mathcal{V}. \end{tikzcd} \tag{88}$$

The classifying map $\tilde{Y} : X \to \mathcal{V}$ is determined by the adjunction $\int \dashv \nu$ as the transpose of the classifying map of the discrete fibration $\int Y \to \int X$.

Given a natural transformation $Y \to X$, the classifying map $\tilde{Y} : X \to \mathcal{V}$ is of course not in general unique. Nonetheless, we can use the construction of $\dot{\mathcal{V}} \to \mathcal{V}$ as the nerve of the discrete fibration classifier $\dot{\mathsf{Set}}^{\mathrm{op}} \to \mathsf{Set}^{\mathrm{op}}$, for which classifying functors $\mathbb{C} \to \mathsf{Set}^{\mathrm{op}}$ are unique up to natural isomorphism, to infer the following proposition, which will be required below (cf. [Shu15, GSS22]).

Proposition 77 (Realignment for families)**.** *Given a monomorphism $c : C \rightarrowtail X$ and a family $Y \to X$, let $y_c : C \to \mathcal{V}$ classify the pullback $c^*Y \to C$.*

Then there is a classifying map $y : X \to \mathcal{V}$ for $Y \to X$ with $y \circ c = y_c$.

$$
\begin{tikzcd}
c^*Y \arrow[rr] \arrow[dr] \arrow[dd] & & \dot{\mathcal{V}} \arrow[dd] \\
& Y \arrow[ur, dotted] \arrow[dd] & \\
C \arrow[rr, "y_c"] \arrow[dr, tail, "c"'] & & \mathcal{V} \\
& X \arrow[ur, dotted, "y"'] &
\end{tikzcd}
\tag{89}
$$

Proof. Transposing the realignment problem (89) for presheaves across the adjunction $\int \dashv \nu$ results in the following realignment problem for discrete fibrations.

$$
\begin{tikzcd}
\int c^*Y \arrow[rr] \arrow[dr] \arrow[dd] & & \dot{\mathsf{Set}}^{\mathrm{op}} \arrow[dd] \\
& \int Y \arrow[ur, dotted] \arrow[dd] & \\
\int C \arrow[rr, "\widetilde{y_c}"] \arrow[dr, tail, "\int c"'] & & \mathsf{Set}^{\mathrm{op}} \\
& \int X \arrow[ur, dotted, "\tilde{y}"'] &
\end{tikzcd}
\tag{90}
$$

The category of elements functor $\int$ is easily seen to preserve pullbacks, hence monos; thus let us consider the general case of a functor $C : \mathbb{C} \rightarrowtail \mathbb{D}$ which is monic in Cat, a pullback of discrete fibrations as on the left below, and a presheaf $E : \mathbb{C} \to \mathsf{Set}^{\mathrm{op}}$ with $\int E \cong \mathbb{E}$ over $\mathbb{C}$.

$$
\begin{tikzcd}
\mathbb{E} \arrow[rr] \arrow[dr] \arrow[dd] & & \dot{\mathsf{Set}}^{\mathrm{op}} \arrow[dd] \\
& \mathbb{F} \arrow[ur, dotted] \arrow[dd] & \\
\mathbb{C} \arrow[rr, "E"] \arrow[dr, tail, "C"'] & & \mathsf{Set}^{\mathrm{op}} \\
& \mathbb{D} \arrow[ur, dotted, "F"'] &
\end{tikzcd}
\tag{91}
$$

We seek $F : \mathbb{D} \to \mathsf{Set}^{\mathrm{op}}$ with $\int F \cong \mathbb{F}$ over $\mathbb{D}$ and $F \circ C = E$. Let $F_0 : \mathbb{D} \to \mathsf{Set}^{\mathrm{op}}$ with $\int F_0 \cong \mathbb{F}$ over $\mathbb{D}$, which exists since $\mathbb{F} \to \mathbb{D}$ is a discrete fibration. Since $F_0 \circ C$ and E both classify $\mathbb{E}$, there is a natural iso $e : F_0 \circ C \cong E$. Consider the following diagram

$$
\begin{tikzcd}
\mathbb{C} \arrow[r, "e"] \arrow[d, tail, "C"'] & (\mathsf{Set}^{\cong})^{\mathrm{op}} \arrow[r, "p_2"'] \arrow[d, "p_1"] & \mathsf{Set}^{\mathrm{op}} \\
\mathbb{D} \arrow[r, "F_0"'] \arrow[ur, dotted, "f"] & \mathsf{Set}^{\mathrm{op}} &
\end{tikzcd}
\tag{92}
$$

where $\mathsf{Set}^{\cong}$ is the category of isos in Set, with p_1, p_2 the (opposites of the) domain and codomain projections. There is a well-known weak factorization system on Cat (part of the "canonical model structure") with injective-on-objects functors on the left and isofibrations on the right. Thus there is a diagonal filler f as indicated. The functor $F := p_2 \circ f : \mathbb{D} \to \mathsf{Set}^{\mathrm{op}}$ is then the one we seek. $\square$

Small maps. Of course, as defined in (87), the classifier $\dot{\mathcal{V}} \to \mathcal{V}$ cannot be a map in $\widehat{\mathbb{C}}$, for reasons of size; we now address this. Let α be a cardinal number, and call the sets strictly smaller than it *α-small*. Let $\mathsf{Set}_\alpha \hookrightarrow \mathsf{Set}$ be the full subcategory of α-small sets. Call a presheaf $X : \mathbb{C}^{\mathrm{op}} \to \mathsf{Set}$ α-small if all of its values are α-small sets, and thus if, and only if, it factors through $\mathsf{Set}_\alpha \hookrightarrow \mathsf{Set}$. Call a map $f : Y \to X$ of presheaves α-small if all of the fibers $f_c^{-1}\{x\} \subseteq Yc$ are α-small sets (for all $c \in \mathbb{C}$ and $x \in Xc$). The latter condition is of course equivalent to saying that, in the pullback square over the element $x : \mathsf{y}c \to X$,

$$\begin{tikzcd} Y_x \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "f"] \\ \mathsf{y}c \arrow[r, "x"'] & X, \end{tikzcd} \tag{93}$$

the presheaf Y_x is α-small.

Now let us restrict the specification (87) of $\dot{\mathcal{V}} \to \mathcal{V}$ to the α-small sets:

$$\begin{aligned} \dot{\mathcal{V}}_\alpha &:= \nu \dot{\mathsf{Set}}_\alpha^{\mathsf{op}} \\ \mathcal{V}_\alpha &:= \nu \mathsf{Set}_\alpha^{\mathsf{op}}. \end{aligned} \tag{94}$$

Then the evident forgetful map $\dot{\mathcal{V}}_\alpha \to \mathcal{V}_\alpha$ *is* a map in the category $\widehat{\mathbb{C}}$ of presheaves, and it is in fact α-small. Moreover, it has the following basic property, which is just a restriction of the basic property of $\dot{\mathcal{V}} \to \mathcal{V}$ stated in Proposition 76.

Proposition 78. *The map $\dot{\mathcal{V}}_\alpha \to \mathcal{V}_\alpha$ classifies α-small maps $f : Y \to X$ in $\widehat{\mathbb{C}}$, in the sense that there is always a pullback square,*

$$\begin{tikzcd} Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathcal{V}}_\alpha \arrow[d] \\ X \arrow[r, "\tilde{Y}"'] & \mathcal{V}_\alpha. \end{tikzcd} \tag{95}$$

The classifying map $\tilde{Y} : X \to \mathcal{V}_\alpha$ is determined by the adjunction $\int \dashv \nu$ as (the factorization of) the transpose of the classifiyng map of the discrete fibration $\int X \to \int Y$.

Proof. If $Y \to X$ is α-small, its classifying map $\tilde{Y} : X \to \mathcal{V}$ factors through $\mathcal{V}_\alpha \hookrightarrow \mathcal{V}$, as indicated below,

$$\begin{tikzcd} Y \arrow[r] \arrow[d] \arrow[rr, bend left] & \nu \dot{\mathsf{Set}_\alpha}^{\mathrm{op}} \arrow[r, hook] \arrow[d] & \nu \dot{\mathsf{Set}}^{\mathrm{op}} \arrow[d] \\ X \arrow[r] \arrow[rr, bend right, "\tilde{Y}"'] & \nu \mathsf{Set}_\alpha{}^{\mathrm{op}} \arrow[r, hook] & \nu \mathsf{Set}^{\mathrm{op}}, \end{tikzcd} \tag{96}$$

in virtue of the following adjoint transposition,

$$
\begin{tikzcd}
\int Y \arrow[r] \arrow[d] \arrow[rr, bend left] & \dot{\mathsf{Set}}_\alpha{}^{\mathrm{op}} \arrow[r, hook] \arrow[d] & \dot{\mathsf{Set}}{}^{\mathrm{op}} \arrow[d] \\
\int X \arrow[r] \arrow[rr, bend right] & \mathsf{Set}_\alpha{}^{\mathrm{op}} \arrow[r, hook] & \mathsf{Set}^{\mathrm{op}}.
\end{tikzcd}
\tag{97}
$$

Note that the square on the right is evidently a pullback, and so the one on the left is, too, because the outer rectangle is the classifying pulback of the discrete fibration $\int Y \to \int X$, as stated. Thus the left square in (96) is also a pullback. $\square$

Examples of universal families $\dot{\mathcal{V}}_\alpha \longrightarrow \mathcal{V}_\alpha$.

1. Let $\alpha = \kappa$ a strongly inaccessible cadinal, so that $\mathsf{ob}(\mathsf{Set}_\kappa)$ is a Grothendieck universe. Then the Hofmann-Streicher universe of Definition 72 is recovered as the κ-small map classifier

   $$E \cong \dot{\mathcal{V}}_\kappa \longrightarrow \mathcal{V}_\kappa \cong U$$

   in the sense of Proposition 78. Indeed, for $c \in \mathbb{C}$, we have

   $$\mathcal{V}_\kappa c \;=\; \nu(\mathsf{Set}_\kappa^{\mathsf{op}})(c) = \mathsf{Cat}\big(\mathbb{C}/_c\,,\, \mathsf{Set}_\kappa^{\mathsf{op}}\big) \;=\; \mathsf{ob}(\widehat{\mathbb{C}/_c}) \;=\; Uc\,. \tag{98}$$

   For $\dot{\mathcal{V}}_\kappa$ we then have,

   $$\begin{aligned} \dot{\mathcal{V}}_\kappa c \;=\; \nu(\,\dot{\mathsf{Set}}_\kappa^{\mathsf{op}})(c) \;&=\; \mathsf{Cat}\big(\mathbb{C}/_c\,,\; \dot{\mathsf{Set}}_\kappa^{\mathsf{op}}\big) \\ &\cong\; \textstyle\coprod_{A \in \mathcal{V}_\kappa c} \mathsf{Cat}_{\mathbb{C}/_c}\big(\mathbb{C}/_c\,,\, A^*\mathsf{Set}_\kappa^{\mathsf{op}}\big) \end{aligned} \tag{99}$$

   where the A-summand in (99) is defined by taking sections of the pullback indicated below.

   $$
   \begin{tikzcd}
   A^*\mathsf{Set}_\kappa^{\mathsf{op}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathsf{Set}}_\kappa^{\mathsf{op}} \arrow[d] \\
   \mathbb{C}/_c \arrow[r, "A"'] \arrow[u, dotted, bend left] \arrow[ur, dotted] & \mathsf{Set}_\kappa^{\mathsf{op}}
   \end{tikzcd}
   \tag{100}
   $$

But $A^*\mathsf{Set}_\kappa^{\mathsf{op}} \cong \int_{\mathbb{C}/_c} A$ over $\mathbb{C}/_c$, and sections of this discrete fibration in Cat correspond uniquely to natural maps $1 \to A$ in $\widehat{\mathbb{C}/_c}$. Since 1 is representable in $\widehat{\mathbb{C}/_c}$ we can continue (99) by

$$\begin{aligned}
\dot{\mathcal{V}}_\kappa c &\cong \textstyle\coprod_{A \in \mathcal{V}_\kappa c} \mathsf{Cat}_{\mathbb{C}/_c}\big(\mathbb{C}/_c\,,\, A^*\mathsf{Set}_\kappa^{\mathsf{op}}\big) \\
&\cong \textstyle\coprod_{A \in \mathcal{V}_\kappa c} \widehat{\mathbb{C}/_c}(1, A) \\
&\cong \textstyle\coprod_{A \in \mathcal{V}_\kappa c} A(1_c) \\
&= \textstyle\coprod_{A \in \mathcal{V}_\kappa c} \mathsf{E}l(\langle c, A\rangle) \\
&= Ec\,.
\end{aligned}$$

2. By functoriality of the nerve $\nu : \mathsf{Cat} \to \widehat{\mathbb{C}}$, a sequence of Grothendieck universes

$$\mathsf{Set}_\alpha \subseteq \mathsf{Set}_\beta \subseteq \ldots$$

in Set gives rise to a (cumulative) sequence of type-theoretic universes

$$\mathcal{V}_\alpha \rightarrowtail \mathcal{V}_\beta \rightarrowtail \ldots$$

in $\widehat{\mathbb{C}}$. More precisely, there is a sequence of cartesian squares,

$$\begin{tikzcd}
\dot{\mathcal{V}}_\alpha \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathcal{V}}_\beta \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \ldots \\
\mathcal{V}_\alpha \arrow[r, tail] & \mathcal{V}_\beta \arrow[r, tail] & \ldots\,,
\end{tikzcd} \tag{101}$$

in the image of $\nu : \mathsf{Cat} \longrightarrow \widehat{\mathbb{C}}$, classifying small maps in $\widehat{\mathbb{C}}$ of increasing size, in the sense of Proposition 78.

3. Let $\alpha = 2$ so that $1 \to 2$ is the subobject classifier of Set, and

$$\mathbb{1} = \dot{\mathsf{Set}}_2^{\mathsf{op}} \longrightarrow \mathsf{Set}_2^{\mathsf{op}} = \mathbb{2}$$

is then a classifier in Cat for *sieves*, i.e. full subcategories $\mathbb{S} \hookrightarrow \mathbb{A}$ closed under the domains of arrows $a \to s$ for $s \in \mathbb{S}$. The nerve $\dot{\mathcal{V}}_2 \to \mathcal{V}_2$ is then the usual subobject classifier $1 \to \Omega$ of $\widehat{\mathbb{C}}$,

$$1 = \nu\mathbb{1} = \dot{\mathcal{V}}_2 \longrightarrow \mathcal{V}_2 = \nu\mathbb{2} = \Omega\,.$$

4. For any $X \in \widehat{\mathbb{C}}$, we have an equivalence

$$\widehat{\mathbb{C}}/_X \;\simeq\; \widehat{\textstyle\int_{\mathbb{C}} X} \;\simeq\; \mathsf{dFib}/_{\int_{\mathbb{C}} X}$$

where, generally, $\mathsf{dFib}/_{\mathbb{D}}$ is the category of discrete fibrations over a category $\mathbb{D}$. This equivalence commutes with composition along discrete fibrations, in the sense that the forgetful functor

$$X_! : \widehat{\mathbb{C}}/_X \to \widehat{\mathbb{C}}$$

given by composition along $X \to 1$ agrees (up to canonical isomorphism) with the base change $(p_X)_! \dashv (p_X)^*$ of presheaves along the projection $p_X : \int_{\mathbb{C}} X \to \mathbb{C}$, and with composition along the discrete fibration p_X, as indicated in:

$$\begin{tikzcd}
\widehat{\mathbb{C}}/_X \arrow[r, "\sim"] \arrow[d, "X_!"'] & \widehat{\int_{\mathbb{C}} X} \arrow[r, "\sim"] \arrow[d, "(p_X)_!"'] & \mathsf{dFib}/_{\int_{\mathbb{C}} X} \arrow[d, "p_X \circ (-)"] \\
\widehat{\mathbb{C}} \arrow[r, "\sim"'] & \widehat{\mathbb{C}} \arrow[r, "\sim"'] & \mathsf{dFib}/_{\mathbb{C}}.
\end{tikzcd} \tag{102}$$

It follows that the pullback functor $X^* : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}/_X$ commutes with the corresponding right adjoints (one of which is the nerve), and therefore preserves the respective universes,

$$X^*\mathcal{V}_{\mathbb{C}} \;\cong\; (p_X)^*\nu_{\mathbb{C}}(\mathsf{Set}^{\mathsf{op}}) \;\cong\; \nu_{\int_{\mathbb{C}} X}(\mathsf{Set}^{\mathsf{op}}) \;\cong\; \mathcal{V}_{\int_{\mathbb{C}} X}\,.$$

6.2 Classifying trivial fibrations

Returning now to the presheaf category $\mathsf{cSet} = \mathsf{Set}^{\square^{\mathrm{op}}}$ of cubical sets, recall from section 2 that (uniform) trivial fibration structures on a map $A \to X$ correspond bijectively to relative $+$-algebra structures over X (definition 19). A relative $+$-algebra structure on $A \to X$ is an algebra structure for the pointed endofunctor $+_X : \mathsf{cSet}/X \to \mathsf{cSet}/X$, where recall from (3),

$$A^+ = \sum_{\varphi:\Phi} A^{[\varphi]} \quad \text{over } X.$$

A $+$-algebra structure is then a retract $\alpha : A^+ \to A$ over X of the canonical map $\eta_A : A \to A^+$,

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[rr, bend left, "="] \arrow[dr] & A^+ \arrow[r, "\alpha"] \arrow[d] & A \arrow[dl] \\ & X. & \end{tikzcd} \tag{103}$$

In more detail, let us write $A \to X$ as a family $(A_x)_{x \in X}$, so that $A = \sum_{x:X} A_x \to X$. Since the $+$-functor acts fiberwise, the object A^+ in (103) is then the indexing projection

$$\sum_{x:X} A_x^+ \to X.$$

Working in the slice cSet/X, the (relative) exponentials (internal Hom's) $[A^+, A]$ and $[A, A]$ and the "precomposition by η_A" map $[\eta_A, A]$, fit into the following pullback diagram

$$\begin{tikzcd} +\mathsf{Alg}(A) \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & {[A^+, A]} \arrow[d, "{[\eta_A, A]}"] \\ 1 \arrow[r, "{'id_A'}"'] & {[A, A]}. \end{tikzcd} \tag{104}$$

The constructed object $+\mathsf{Alg}(A) \to X$ over X is then the *object of $+$-algebra structures on* $A \to X$, in the sense that sections $X \to +\mathsf{Alg}(A)$ correspond uniquely to $+$-algebra structures on $A \to X$. Moreover, $+\mathsf{Alg}(A) \to X$ is stable under pullback, in the sense that for any $f : Y \to X$, we have two pullback squares,

$$\begin{tikzcd} f^*A \arrow[r] \arrow[d] & A \arrow[d] \\ Y \arrow[r, "f"'] & X \\ +\mathsf{Alg}(f^*A) \arrow[r] \arrow[u] & +\mathsf{Alg}(A) \arrow[u] \end{tikzcd} \tag{105}$$

because the $+$-functor, exponentials and pullbacks occurring in the construction of $+\mathsf{Alg}(A) \to X$ are themselves all stable.

It then follows from Proposition 78 that, if $A \to X$ is small, then $+\mathsf{Alg}(A) \to X$ is itself a pullback of the analogous object $+\mathsf{Alg}(\dot{\mathcal{V}}) \to \mathcal{V}$ constructed from the universal small family $\dot{\mathcal{V}} \to \mathcal{V}$ of Proposition 78, so there are two pullback squares:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \dot{\mathcal{V}} \arrow[d] \\ X \arrow[r, "\chi_A"'] & \mathcal{V} \\ +\mathsf{Alg}(A) \arrow[r] \arrow[u] & +\mathsf{Alg}(\dot{\mathcal{V}}) \arrow[u] \end{tikzcd} \tag{106}$$

Proposition 79. *There is a* universal small trivial fibration

$$\dot{\mathsf{TFib}} \to \mathsf{TFib}.$$

Every small trivial fibration $A \to X$ is a pullback of $\dot{\mathsf{TFib}} \to \mathsf{TFib}$ along a canonically determined classifying map $X \to \mathsf{TFib}$.

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathsf{TFib}} \arrow[d] \\ X \arrow[r] & \mathsf{TFib} \end{tikzcd} \tag{107}$$

Proof. We can take

$$\mathsf{TFib} := +\mathsf{Alg}(\dot{\mathcal{V}}),$$

which comes with its projection $+\mathsf{Alg}(\dot{\mathcal{V}}) \to \mathcal{V}$ as in diagram (106). Now define $p_t : \dot{\mathsf{TFib}} \to \mathsf{TFib}$ by pulling back the universal small family,

$$\begin{tikzcd} \dot{\mathsf{TFib}} \arrow[r] \arrow[d, "p_t"'] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathcal{V}} \arrow[d, "p"] \\ \mathsf{TFib} \arrow[r] & \mathcal{V}. \end{tikzcd}$$

Consider the following diagram, in which all the squares (including the distorted ones) are pullbacks, with the outer one coming from proposition 78 and the lower one from (106).

$$
\begin{tikzcd}
 & A \arrow[rr, "q_A"] \arrow[dr, dotted] \arrow[ddd] & & \dot{\mathcal{V}} \arrow[ddd, "p"] \\
 & & \mathsf{T\dot{F}ib} \arrow[ur] \arrow[d, "p_t"] & \\
\mathsf{TFib}(A) \arrow[rr] \arrow[dr] & & \mathsf{TFib} \arrow[dr] & \\
 & X \arrow[rr, "\chi_A"'] \arrow[ur, dotted, "\alpha'"'] \arrow[ul, dotted, bend left, "\alpha"] & & \mathcal{V}.
\end{tikzcd}
\tag{108}
$$

A trivial fibration structure α on $A \to X$ is a section the object of $+$-algebra structures on A, occurring in the diagram as

$$
\mathsf{TFib}(A) := +\mathsf{Alg}(A),
$$

the pullback of $\mathsf{TFib} = +\mathsf{Alg}(\dot{\mathcal{V}})$ along the classifying map $\chi_A : X \to \mathcal{V}$ for the small family $A \to X$. Such sections correspond uniquely to factorizations α' of χ_A as indicated, which in turn induce pullback squares of the required kind (107).

Note that the map $p_t : \mathsf{T\dot{F}ib} \to \mathsf{TFib}$ has a canonical trivial fibration structure. Indeed, consider the following diagram, in which both squares are pullbacks.

$$
\begin{tikzcd}
\mathsf{T\dot{F}ib} \arrow[r] \arrow[d, "p_t"'] & \dot{\mathcal{V}} \arrow[d] \\
\mathsf{TFib} \arrow[r] & \mathcal{V} \\
\mathsf{TFib}(\mathsf{T\dot{F}ib}) \arrow[r] \arrow[u] & \mathsf{TFib}(\dot{\mathcal{V}}) \arrow[u]
\end{tikzcd}
\tag{109}
$$

$\mathsf{TFib}(\dot{\mathcal{V}})$ is the object of trivial fibration structures on $\dot{\mathcal{V}} \to \mathcal{V}$, and its pullback $\mathsf{TFib}(\mathsf{T\dot{F}ib})$ is therefore the object of trivial fibration structures on $p_t : \mathsf{T\dot{F}ib} \to \mathsf{TFib}$. Thus we seek a section of $\mathsf{TFib}(\mathsf{T\dot{F}ib}) \to \mathsf{TFib}$. But recall that $\mathsf{TFib} = \mathsf{TFib}(\dot{\mathcal{V}})$ by definition, so the lower pullback square is the pullback of $\mathsf{TFib}(\dot{\mathcal{V}}) \to \mathcal{V}$ against itself, which does indeed have a distinguished section, namely the diagonal

$$
\Delta : \mathsf{TFib}(\dot{\mathcal{V}}) \to \mathsf{TFib}(\dot{\mathcal{V}}) \times_{\mathcal{V}} \mathsf{TFib}(\dot{\mathcal{V}}).
$$

□

We record the following notation and corresponding fact from the foregoing proof for future reference:

Lemma 80. *The classifying type* $\mathsf{TFib}(A) := {+}\mathsf{Alg}(A) \to X$ *for trivial fibration structures on a map* $A \to X$ *is stable under pullback, in the sense that for any* $f : Y \to X$*, we have two pullback squares,*

$$\begin{tikzcd} f^*A \arrow[r] \arrow[d] & A \arrow[d] \\ Y \arrow[r, "f"'] & X \\ \mathsf{TFib}(f^*A) \arrow[r] \arrow[u] & \mathsf{TFib}(A) \arrow[u] \end{tikzcd} \tag{110}$$

6.3 Classifying fibrations

In order to classify fibrations $A \twoheadrightarrow X$, we shall proceed as for trivial fibrations by constructing an object $\mathsf{Fib}(A) \to X$ of fibration structures on a map $A \to X$ which, moreover, is stable under pullback. We then apply the construction to the universal small family $\dot{\mathcal{V}} \to \mathcal{V}$ of Proposition 78 to obtain a universal small fibration.

The construction of $\mathsf{Fib}(A) \to X$ is a bit more involved than that of $\mathsf{TFib}(A) \to X$. Recall from section 3.2 the characterization of (uniform, unbiased) fibration structures on a map $p_A : A \to X$ in terms of +-algebra structures on $\delta \Rightarrow p_A$:

1. First, pull the map $p_A : A \to X$ back to $\mathsf{cSet}/_{\mathrm{I}}$ by applying the functor $\mathrm{I} \times (-) : \mathsf{cSet} \to \mathsf{cSet}/_{\mathrm{I}}$. We may continue to write $p_A : A \to X$ for the resulting map over I.

2. Form the pullback-hom $\delta \Rightarrow p_A : A^{\mathrm{I}} \to X^{\mathrm{I}} \times_X A$ of p_A with the generic

point $\delta : \mathrm{I} \to \mathrm{I} \times \mathrm{I}$ over I, as indicated in the following diagram.

$$\begin{tikzcd}
A^{\mathrm{I}} \arrow[dr, "\delta \Rightarrow p_A"] \arrow[drr, bend left, "A^{\delta}"] \arrow[ddr, bend right, "(p_A)^{\mathrm{I}}"'] & & \\
& X^{\mathrm{I}} \times_X A \arrow[r] \arrow[d] & A \arrow[d, "p_A"] \\
& X^{\mathrm{I}} \arrow[r, "X^{\delta}"'] & X
\end{tikzcd} \tag{111}$$

3. A fibration structure on $p_A : A \to X$ is then a relative $+$-algebra structure on the map $\delta \Rightarrow p_A$ in the slice category over the codomain $X^{\mathrm{I}} \times_X A$ (formed in the slice over I).

   In order to construct the object $\mathsf{Fib}(A) \to X$ classifying such structures, let us first relabel the objects and arrows in diagram (111) as follows:

$$\begin{aligned}
p &:= p_A \\
\epsilon &:= X^{\delta} : X^{\mathrm{I}} \to X \\
A_\epsilon &:= X^{\mathrm{I}} \times_X A \\
\epsilon_A &:= \delta \Rightarrow p_A
\end{aligned}$$

   so that (the working part of) our diagram becomes:

$$\begin{tikzcd}
A^{\mathrm{I}} \arrow[dr, "\epsilon_A"'] & & \\
& A_\epsilon \arrow[r] \arrow[d, "p_\epsilon"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "p"] \\
& X^{\mathrm{I}} \arrow[r, "\epsilon"'] & X
\end{tikzcd} \tag{112}$$

4. A $+$-algebra structure on ϵ_A is a retract α over A_ϵ of the unit η as

indicated below, where D is the domain of the map $(\epsilon_A)^+$,

$$
\begin{tikzcd}
A^{\mathrm{I}} \arrow[r, "\eta"'] \arrow[dr, "\epsilon_A"'] & D \arrow[l, dotted, bend right, "\alpha"'] \arrow[d, "(\epsilon_A)^+"] & \\
& A_\epsilon \arrow[r] \arrow[d, "p_\epsilon"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "p"] \\
& X^{\mathrm{I}} \arrow[r, "\epsilon"'] & X
\end{tikzcd}
\tag{113}
$$

5. As in the construction (104), there is an object $+\mathsf{Alg}(\epsilon_A)$ over A_ϵ of $+$-algebra structures on ϵ_A, the sections of which correspond uniquely to $+$-algebra structures on ϵ_A (and thus fibration structures on A).

$$
\begin{tikzcd}
& A^{\mathrm{I}} \arrow[r, "\eta"'] \arrow[d, "\epsilon_A"'] & D \arrow[l, dotted, bend right, "\alpha"'] \arrow[dl, "(\epsilon_A)^+"] \\
+\mathsf{Alg}(\epsilon_A) \arrow[r] & A_\epsilon \arrow[r] \arrow[d, "p_\epsilon"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "p"] \\
& X^{\mathrm{I}} \arrow[r, "\epsilon"'] & X
\end{tikzcd}
\tag{114}
$$

6. Sections of $+\mathsf{Alg}(\epsilon_A) \to A_\epsilon$ then correspond to sections of its pushforward along p_ϵ, which we shall call F_A:

$$
F_A := (p_\epsilon)_*(+\mathsf{Alg}(\epsilon_A)) .
$$

$$
\begin{tikzcd}
& A^{\mathrm{I}} \arrow[r, "\eta"'] \arrow[d, "\epsilon_A"'] & D \arrow[l, dotted, bend right, "\alpha"'] \arrow[dl, "(\epsilon_A)^+"] \\
+\mathsf{Alg}(\epsilon_A) \arrow[r] & A_\epsilon \arrow[r] \arrow[d, "p_\epsilon"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "p"] \\
F_A \arrow[r] & X^{\mathrm{I}} \arrow[r, "\epsilon"'] & X
\end{tikzcd}
\tag{115}
$$

7. One might now think of taking another pushforward of $F_A \to X^{\mathrm{I}}$ along $\epsilon : X^{\mathrm{I}} \to X$ to get the object $\mathsf{Fib}(A) \to X$ that we seek, but unfortunately, this would not be stable under pullback along arbitrary maps $Y \to X$, because $\epsilon : X^{\mathrm{I}} \to X$ is not stable in that way. Instead we use the *root* functor of Proposition (3), i.e. the right adjoint of the pathspace, $(-)^{\mathrm{I}} \dashv (-)_{\mathrm{I}}$. (Recall that the interval I in the slice $\mathsf{cSet}/_{\mathrm{I}}$, namely $\mathrm{I}^*\mathrm{I} = \mathrm{I} \times \mathrm{I} \to \mathrm{I}$, is still tiny, by Lemma 4.)

   Let $f : F_A \to X^{\mathrm{I}}$ be the map indicated in (115), and $\eta_X : X \to (X^{\mathrm{I}})_{\mathrm{I}}$ the unit of the root adjunction. Then define $\mathsf{Fib}(A) \to X$ by

$$\mathsf{Fib}(A) := \eta^* f_{\mathrm{I}}$$

   as indicated in the following pullback diagram.

$$\begin{tikzcd} \mathsf{Fib}(A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (F_A)_{\mathrm{I}} \arrow[d, "f_{\mathrm{I}}"] \\ X \arrow[r, "\eta"'] & (X^{\mathrm{I}})_{\mathrm{I}} \end{tikzcd} \tag{116}$$

   By adjointness, sections of $\mathsf{Fib}(A) \to X$ then correspond bijectively to sections of $f : F_A \to X^{\mathrm{I}}$.

8. Finally, recall that we are still working in the slice $\mathsf{cSet}/_{\mathrm{I}}$ and need to get back to cSet, which we will do by applying the pushforward $\mathrm{I}_* : \mathsf{cSet}/_{\mathrm{I}} \to \mathsf{cSet}$. Let us rename the map $\mathsf{Fib}(A) \to X$ constructed over I in the last step to $\mathsf{Fib}_i(A) \to \mathrm{I}^*X$, and then apply I_* to get the map,

$$\mathrm{I}_*(\mathsf{Fib}_i(A)) = \Pi_{i:\mathrm{I}}\mathsf{Fib}_i(A) \to X^{\mathrm{I}}$$

   in cSet. Finally, we define the desired map $\mathsf{Fib}(A) \to X$ as the pullback along the unit $\rho : X \to X^{\mathrm{I}}$ of $\mathrm{I}^* \dashv \mathrm{I}_*$, as indicated below.

$$\begin{tikzcd} \mathsf{Fib}(A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Pi_{i:\mathrm{I}}\mathsf{Fib}_i(A) \arrow[d] \\ X \arrow[r, "\rho"'] & X^{\mathrm{I}} \end{tikzcd} \tag{117}$$

   It then follows directly from the adjunction $\mathrm{I}^* \dashv \mathrm{I}_*$ that sections of $\mathsf{Fib}(A) \to X$ correspond bijectively to sections of $\mathsf{Fib}_i(A) \to \mathrm{I}^*X$ over I.

Now apply the foregoing construction to the universal family $\dot{\mathcal{V}} \to \mathcal{V}$ to get $\mathsf{Fib}(\dot{\mathcal{V}}) \to \mathcal{V}$, and define the universal small fibration by setting $\mathsf{Fib} := \mathsf{Fib}(\dot{\mathcal{V}})$ and $\dot{\mathsf{Fib}} \twoheadrightarrow \mathsf{Fib}$ by pulling back the universal family,

$$\begin{tikzcd} \dot{\mathsf{Fib}} \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \dot{\mathcal{V}} \arrow[d, "p"] \\ \mathsf{Fib} \arrow[r] & \mathcal{V}. \end{tikzcd} \tag{118}$$

Proposition 81. *The map*

$$\dot{\mathsf{Fib}} \to \mathsf{Fib}$$

just constructed is a universal small fibration*: every small fibration $A \twoheadrightarrow X$ is a pullback of $\dot{\mathsf{Fib}} \twoheadrightarrow \mathsf{Fib}$ along a canonically determined classifying map $X \to \mathsf{Fib}$.*

$$\begin{tikzcd} A \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, very near start, "\lrcorner"] & \dot{\mathsf{Fib}} \arrow[d, two heads] \\ X \arrow[r] & \mathsf{Fib} \end{tikzcd} \tag{119}$$

Proof. First, we show that the construction of $\mathsf{Fib}(A) \to X$ as the classifying type of fibration structures on a map $A \to X$ is stable under pullback along all maps $f : Y \to X$. To that end, the relevant parts of the construction given in steps (1)-(7) are recalled schematically below.

$$\begin{tikzcd} & A^{\mathrm{I}} \arrow[d, "\epsilon_A"'] & & \\ +\mathsf{Alg}(\epsilon_A) \arrow[r] & A_\epsilon \arrow[r] \arrow[d, "p_\epsilon"'] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, "p"] & \\ F_A \arrow[r] & X^{\mathrm{I}} \arrow[r, "\epsilon"'] & X & \mathsf{Fib}_i(A) \arrow[l] \end{tikzcd} \tag{120}$$

Now consider the following diagram, in which the right hand side consists of the data from (120), and the front, central square is a pullback.

$$\begin{tikzcd}
 & B^{\mathrm{I}} \arrow[rrr] \arrow[d, "\epsilon_B"'] & & & A^{\mathrm{I}} \arrow[d, "\epsilon_A"] & \\
+\mathsf{Alg}(\epsilon_B) \arrow[r] & B_\epsilon \arrow[rrr, dotted] \arrow[d] \arrow[dr] & & & A_\epsilon \arrow[d] \arrow[dl] & +\mathsf{Alg}(\epsilon_A) \arrow[l] \\
F_B \arrow[r] & Y^{\mathrm{I}} \arrow[rrr, dotted, bend right] \arrow[dr] & B \arrow[r] \arrow[d] & A \arrow[d] & X^{\mathrm{I}} \arrow[dl] & F_A \arrow[l] \\
 & \mathsf{Fib}_i(B) \arrow[r] & Y \arrow[r, "f"'] & X & \mathsf{Fib}_i(A) \arrow[l] &
\end{tikzcd} \tag{121}$$

As in the proof of Lemma 66, on the left side we repeat the construction with $B \to Y$ in place of $A \to X$. The left face of the indicated cube is then also a pullback, whence the back (dotted) face is a pullback, since the two-story square in back is the image of the front square under the right adjoint $(-)^{\mathrm{I}}$. Finally, the top rectangle in the back is therefore a pullback.

It follows that $+\mathsf{Alg}(\epsilon_B)$ is a pullback of $+\mathsf{Alg}(\epsilon_A)$ along the upper dotted arrow, as in diagram (105), and so the pushforward F_B is a pullback of the corresponding F_A, along the lower dotted arrow (which is f^{I}), by the Beck-Chevalley condition for the dotted pullback square. Let us record this for later reference:

$$F_B \cong (f^{\mathrm{I}})^* F_A. \tag{122}$$

It remains to show that $\mathsf{Fib}_i(B)$ is a pullback of $\mathsf{Fib}_i(A)$ along $f : Y \to X$, and now it is good that we did not take these to be pushforwards of F_B and F_A, because the floor of the cube need not be a pullback, and so the Beck-Chavalley condition would not apply. Instead, consider the following diagram.

$$\begin{tikzcd}
 & \mathsf{Fib}_i(B) \arrow[r] \arrow[dl] \arrow[dd] & \mathsf{Fib}_i(A) \arrow[dr] \arrow[dd] & \\
(F_B)_{\mathrm{I}} \arrow[rrr] \arrow[dd] & & & (F_A)_{\mathrm{I}} \arrow[dd] \\
 & Y \arrow[r, "f"'] \arrow[dl, "\eta"'] & X \arrow[dr, "\eta"] & \\
(Y^{\mathrm{I}})_{\mathrm{I}} \arrow[rrr, "(f^{\mathrm{I}})_{\mathrm{I}}"'] & & & (X^{\mathrm{I}})_{\mathrm{I}}
\end{tikzcd} \tag{123}$$

The sides of the cube are pullbacks by the construction of $\mathsf{Fib}_i(A)$ and $\mathsf{Fib}_i(B)$. The front face is the root of the pullback (122) and is thus also a pullback, since the root is a right adjoint. The base commutes by naturality of the unit of the adjunction, and so the back face is also a pullback, as required.

Finally, the base change along $\mathrm{I}_* : \mathsf{cSet}/_{\mathrm{I}} \to \mathsf{cSet}$ in step 8 above clearly also preserves the pullback.

Thus we can indeed use $\mathsf{Fib} := \mathsf{Fib}(\dot{\mathcal{V}})$ to define the universal small fibration $\dot{\mathsf{Fib}} \to \mathsf{Fib}$ as in (118), and the proof can conclude just as in that for proposition 79. $\square$

Remark 82. Recall from Example 3 of Section 6.1 that the universe in the slice category $\mathsf{cSet}/_{\mathrm{I}}$ is the pullback of the universe $\mathcal{V}$ from cSet along the base change $\mathrm{I}^* : \mathsf{cSet} \to \mathsf{cSet}/_{\mathrm{I}}$. Thus in Step 8 above we are first building the classifying type

$$\mathsf{Fib}_i(\mathrm{I}^*\dot{\mathcal{V}}) \to \mathrm{I}^*\mathcal{V}$$

for $\mathbb{I}$-fibration structures on the universal family in $\mathsf{cSet}/_{\mathrm{I}}$, and then taking a pushforward $\mathrm{I}_* : \mathsf{cSet}/_{\mathrm{I}} \to \mathsf{cSet}$ to obtain the classifier for fibrations as the pullback along the unit:

$$\begin{tikzcd}
\mathsf{Fib}(\dot{\mathcal{V}}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Pi_{i:\mathrm{I}}\mathsf{Fib}_i(\mathrm{I}^*\dot{\mathcal{V}}) \arrow[d] \\
\mathcal{V} \arrow[r, "\rho"'] & \mathcal{V}^{\mathrm{I}}
\end{tikzcd} \tag{124}$$

We remark for later reference that this classifying type $\mathsf{Fib} = \mathsf{Fib}(\dot{\mathcal{V}}) \to \mathcal{V}$ for fibration structures can also be constructed as the pushforward of the classifier $\mathsf{Fib}_i(\mathrm{I}^*\dot{\mathcal{V}}) \to \mathrm{I}^*\mathcal{V}$ for $\mathbb{I}$-fibration structures, along the projection $q : \mathrm{I}^*\mathcal{V} = \mathrm{I} \times \mathcal{V} \to \mathcal{V}$ indicated below.

$$\begin{tikzcd}
\mathsf{Fib}_i(\mathrm{I}^*\dot{\mathcal{V}}) \arrow[d] & \mathsf{Fib}(\dot{\mathcal{V}}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Pi_{i:\mathrm{I}}\mathsf{Fib}_i(\dot{\mathcal{V}}) \arrow[d] \\
\mathrm{I}^*\mathcal{V} \arrow[r, "q"'] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{V} \arrow[r, "\rho"'] \arrow[d] & \mathcal{V}^{\mathrm{I}} \\
\mathrm{I} \arrow[r] & 1 &
\end{tikzcd} \tag{125}$$

6.4 Realignment for fibration structure

The realignment for families of Proposition 77 will need to be extended to (structured) fibrations. Our approach makes use of the notion of a *weak proposition*. Informally, a map $P \to X$ may be said to be a weak proposition if it is "conditionally contractible", in the sense that it is contractible if it has a section (recall that a proposition may be defined as a fibration that is "contractible if inhabited"). More formally, we have the following.

Definition 83. A map $P \to X$ is said to be a *weak proposition* if the projection $P \times_X P \to P$ is a trivial fibration.

$$\begin{tikzcd} P^2 \arrow[r] \arrow[d, two heads, "\sim"'] \arrow[dr, phantom, "\lrcorner", very near start] & P \arrow[d] \\ P \arrow[r] & X. \end{tikzcd} \tag{126}$$

Note that if either projection is a trivial fibration, then both are.

As an object over the base, a weak proposition is thus one that "thinks it is contractible". The key fact needed for realignment is the following.

Lemma 84. *For any $Y \to X$, the classifying type $\mathsf{TFib}(Y) \to X$ is a weak proposition. Moreover, the same is true for $\mathsf{Fib}(Y) \to X$ if the cofibrations are closed under exponentiation by the interval* I.

Proof. Let $Y \to X$ and consider the following diagram, in which we have written $Y' = \mathsf{TFib}(Y) \times_X Y$ and $\mathsf{TFib}(Y)^2 = \mathsf{TFib}(Y) \times_X \mathsf{TFib}(Y)$.

$$\begin{tikzcd} Y' \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & Y \arrow[dd] & \\ & \mathsf{TFib}(Y)^2 \arrow[rr] \arrow[dl] \arrow[dr, phantom, "\lrcorner", very near start] & & \mathsf{TFib}(Y) \arrow[dl] \\ \mathsf{TFib}(Y) \arrow[rr] & & X & \end{tikzcd} \tag{127}$$

Since TFib is stable under pullback (by Lemma 80), we have $\mathsf{TFib}(Y)^2 \cong \mathsf{TFib}(Y')$, and since $\mathsf{TFib}(Y)^2$ has a canonical section, $Y' \to \mathsf{TFib}(Y)$ is therefore a trivial fibration. Inspecting the definition of $\mathsf{TFib}(A) = {+}\mathsf{Alg}(A)$ in (104), we see that if a map $A \to X$ is a trivial fibration, then so is

$\mathsf{TFib}(A) \to X$ (since $A \to A^+$ is always a cofibration). Thus $\mathsf{TFib}(Y)^2 \cong \mathsf{TFib}(Y') \to \mathsf{TFib}(Y)$ is also a trivial fibration.

For $\mathsf{Fib}(Y) \to X$, we use the foregoing to infer that $\mathsf{TFib}(Y^I) \to X^I \times_X Y$ is a weak proposition, and so therefore is its pushforward $p_*\mathsf{TFib}(Y^{\mathrm{I}}) \to X^{\mathrm{I}}$ along the first projection $p : X^I \times_X Y \to X^{\mathrm{I}}$, since pushforward clearly preserve weak propositions. Applying the root $(-)_{\mathrm{I}}$ preserves trivial fibrations, by the assumption that its left adjoint $(-)^{\mathrm{I}}$ preserves cofibrations, so as a right adjoint, it also preserves weak propositions. Therefore $(p_*\mathsf{TFib}(Y^{\mathrm{I}}))_{\mathrm{I}} \to (X^{\mathrm{I}})_{\mathrm{I}}$ is a weak proposition, but then so is its pullback along the unit $X \to (X^{\mathrm{I}})_{\mathrm{I}}$, which is $\mathsf{Fib}(Y) \to X$. □

In light of Lemma 84 we assume as a final axiom on cofibrations:

(C8) The map $c^{\mathrm{I}} : A^{\mathrm{I}} \to B^{\mathrm{I}}$ is a cofibration if $c : A \to B$ is a cofibration.

Now by Proposition 81 we have a universal (small) fibration $\dot{\mathcal{U}} \twoheadrightarrow \mathcal{U}$, with $\mathcal{U} = \mathsf{Fib}(\dot{\mathcal{V}})$, where $\dot{\mathcal{V}} \to \mathcal{V}$ is the universal (small) family. Type theoretically, we therefore have

$$\mathcal{U} = \Sigma_{E:\mathcal{V}}\mathsf{Fib}(E)\,,$$

with canonical projection

$$\mathcal{U} = \Sigma_{E:\mathcal{V}}\mathsf{Fib}(E) \to \mathcal{V}\,,$$

so that a fibration $E \twoheadrightarrow X$ is a pair $\langle E, e\rangle$, consisting of the underlying family $E \to X$, equipped with a fibration structure $e : \mathsf{Fib}(E)$. Lemma 84 then allows us to establish the following, which was first isolated in [**?**] (as condition (2'), also see [Shu15]). It will be needed in the sequel to correct the fibration structure on certain maps.

Lemma 85 (Realignment for fibrations)**.** *Given a fibration $F \twoheadrightarrow X$ and a cofibration $c : C \rightarrowtail X$, let $f_c : C \to \mathcal{U}$ classify the pullback $c^*F \twoheadrightarrow C$. Then there is a classifying map $f : X \to \mathcal{U}$ for F with $f \circ c = f_c$.*

$$\begin{tikzcd}
c^*F \arrow[rr] \arrow[dr] \arrow[dd, two heads] & & \dot{\mathcal{U}} \arrow[dd, two heads] \\
& F \arrow[ur, dotted] \arrow[dd, two heads] & \\
C \arrow[rr, "f_c"] \arrow[dr, tail, "c"'] & & \mathcal{U} \\
& X \arrow[ur, dotted, "f"'] &
\end{tikzcd} \tag{128}$$

Proof. First, let $|f_c| : C \to \mathcal{V}$ be the composite of $f_c : C \to \mathcal{U}$ with the canonical projection $\mathcal{U} \to \mathcal{V}$, thus classifying the underlying family $c^*F \to C$. Next, let $f_0 : X \to \mathcal{V}$ classify the underlying family $F \to X$. We may assume that $f_0 \circ c = |f_c|$ by realignment for families, Proposition 77.

$$
\begin{tikzcd}
c^*F \arrow[rr] \arrow[dr] \arrow[dd, two heads] & & \dot{\mathcal{U}} \arrow[r] \arrow[dd, two heads] & \dot{\mathcal{V}} \arrow[dd] \\
& F \arrow[urr, dotted] \arrow[dd, two heads] & & \\
C \arrow[rr, "f_c"] \arrow[dr, tail, "c"'] & & \mathcal{U} \arrow[r] & \mathcal{V} \\
& X \arrow[urr, dotted, "f_0"'] & &
\end{tikzcd}
\tag{129}
$$

Since $F \twoheadrightarrow X$ is a fibration, there is a lift $f_1 : X \to \mathcal{U}$ of f_0 classifying the fibration structure. We thus have the following commutative diagram in the base of (129).

$$
\begin{tikzcd}
C \arrow[r, "f_c"'] \arrow[rr, bend left, "|f_c|"] \arrow[d, tail, "c"'] & \mathcal{U} \arrow[r] & \mathcal{V} \arrow[d, equal] \\
X \arrow[r, "f_1"] \arrow[rr, bend right, "f_0"'] & \mathcal{U} \arrow[r] & \mathcal{V}
\end{tikzcd}
\tag{130}
$$

Now pull $\mathcal{U} \to \mathcal{V}$ back against itself and rearrange the previous data to give (the solid part of) the following, which also commutes.

$$
\begin{tikzcd}
C \arrow[r, "\langle f_1 c, f_c \rangle"'] \arrow[rr, bend left, "f_c"] \arrow[d, tail, "c"'] & \mathcal{U} \times_{\mathcal{V}} \mathcal{U} \arrow[r, "\pi_2"'] \arrow[d, two heads, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{U} \arrow[d] \\
X \arrow[r, "f_1"'] \arrow[ur, dotted, "f_2"] \arrow[rr, bend right, "f_0"'] & \mathcal{U} \arrow[r] & \mathcal{V}
\end{tikzcd}
\tag{131}
$$

Since $\mathcal{U} = \mathsf{Fib}(\dot{\mathcal{V}}) \to \mathcal{V}$ is a weak proposition by Lemma 84 and (C8), the projection $\pi_1 : \mathcal{U} \times_{\mathcal{V}} \mathcal{U} \twoheadrightarrow \mathcal{U}$ is a trivial fibration, so there is a diagonal filler

$f_2 : X \to \mathcal{U} \times_{\mathcal{V}} \mathcal{U}$ as indicated. Taking $f := \pi_2 \circ f_2 : X \to \mathcal{U} \times_{\mathcal{V}} \mathcal{U} \to \mathcal{U}$ gives another classifying map for the fibration structure on $F \to X$, for which $f \circ c = f_c$ as required. $\square$

7 The equivalence extension property

The equivalence extension property (EEP) is closely related to the *univalence* of the universal fibration $\dot{\mathcal{U}} \to \mathcal{U}$ constructed in section 6.3 (see [?]). We shall use it in section 8 to show that the base object $\mathcal{U}$ is fibrant, which implies the fibration extension property. Our proof of the EEP is a reformulation of a type-theoretic argument due to Coquand [CCHM16], which in turn is a modification of the original argument of Voevodsky [?]. See [Sat17] for another reformulation.

We first recall some basic facts and make some simple observations. For any map $f : X \to Y$, recall the *pathspace factorization* $f = t \circ s$ indicated below.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "s"] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dd, bend right=60, "="'] & Y \arrow[d, "r"] \arrow[dr, "="] & \\
P_f \arrow[r, "p_0^* f"] \arrow[d, "p_f"] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[rr, dotted, bend right, "t"'] & Y^{\mathrm{I}} \arrow[r, "p_1"] \arrow[d, "p_0"] & Y \\
X \arrow[r, "f"'] & Y &
\end{tikzcd} \tag{132}$$

Here p_0, p_1 are the evaluations $Y^{\delta_0}, Y^{\delta_1}$ at the endpoints $\delta_0, \delta_1 : 1 \to \mathrm{I}$, and let $r := Y^!$ for $! : \mathrm{I} \to 1$, so that $p_0 r = p_1 r = 1_Y$. Then let $p_f := f^* p_0 : P_f \to Y$, the pullback of p_0 along f, and $s := f^* r : X \to P_f$ (over X). Finally, let $t := p_1 \circ p_0^* f : P_f \to Y$ be the indicated horizontal composite.

We make the following well-known observations.

1. If $f : X \to Y$ is over a base Z, then the factorization $t \circ s : X \to P_f \to Y$ is stable under pullback along any map $g : Z' \to Z$, in the sense that $g^* P_f = P_{g^* f} : g^* X \to g^* Y$, and similarly for $g^* s$ and $g^* t$. Note that in this case we form the pathspace Y^{I} as an exponential in the slice category over Z.

2. The retraction $p_0 \circ r = 1_Y$ pulls back along f to a retraction $p_f \circ s = 1_X$.

3. If Y is fibrant (either as an object, or over a base $Y \to Z$), then $p_0 : Y^{\mathrm{I}} \to Y$ is a trivial fibration (as is p_1). In that case, its pullback $p_f : P_f \to X$ is also a trivial fibration.

4. If X and Y are both fibrant, then $t = p_1 \circ p_0^* f : P_f \to Y$ is a fibration. This can be seen by factoring the maps $p_0, p_1 : Y^{\mathrm{I}} \rightrightarrows Y$ through the product projections as $\pi_0 \circ p, \pi_1 \circ p : Y^{\mathrm{I}} \to Y \times Y \rightrightarrows Y$, with $p = (p_0, p_1)$, and then interpolating the pullback along the map $(f, 1_X) : X \times Y \to Y \times Y$ into (132) as indicated below.

$$\begin{tikzcd}
& X \arrow[r, "f"] \arrow[d, "s"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "r"] & \\
& P_f \arrow[r, "p_0^* f"] \arrow[d, "f^* p"] \arrow[dl, "t"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y^{\mathrm{I}} \arrow[d, "p"'] \arrow[dr, "p_1"] & \\
Y & X \times Y \arrow[l, "\pi_1'"] \arrow[r, "(f, 1_Y)"'] \arrow[d, "\pi_0'"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \times Y \arrow[d, "\pi_0"] \arrow[r, "\pi_1"'] & Y \\
& X \arrow[r, "f"'] & Y &
\end{tikzcd} \tag{133}$$

The second factor $t = p_1 \circ p_0^* f : P_f \to Y$ now appears also as $\pi_1 \circ (f, 1_Y) \circ f^* p$, which is the pullback $f^* p : P_f \to X \times Y$ followed by the second projection $\pi_1' : X \times Y \to Y$ (which is not a pullback). But if Y is fibrant, then $p : Y^{\mathrm{I}} \to Y \times Y$ is a fibration, and then so is $f^* p$. And if X is fibrant, then the projection $\pi_1' : X \times Y \to Y$ is a fibration. Thus in this case, $t = \pi_1' \circ f^* p : P_f \to Y$ is a fibration, as claimed.

5. Summarizing (1)-(4), for any map $f : X \to Y$, we have a stable factorization $f = t \circ s : X \to P_f \to Y$, in which s has a retraction p_f, which is a trivial fibration when Y is fibrant, and t is a fibration when both X and Y are fibrant.

$$\begin{tikzcd}
X \arrow[r, "s"] \arrow[dr, "f"'] & P_f \arrow[l, bend right, dotted, "p_f"'] \arrow[d, "t"] \\
& Y
\end{tikzcd} \tag{134}$$

Note that the retraction $p_f : P_f \to X$ is not over Y.

The following simple fact concerning just the cofibration weak factorization system will also be needed.

Lemma 86. *Let $p : E \twoheadrightarrow B$ be a trivial fibration and $c : C \rightarrowtail B$ a cofibration. Then the unit $\eta : E \to c_* c^* E$ of the base change $c^* \dashv c_*$ along c is a trivial fibration.*

Proof. The unit map $\eta : E \to c_* c^* E$ is the pullback-hom $c \Rightarrow p$, as is easily checked. By lemma 26, for any map $a : A \to Z$ we have the equivalence of diagonal filling conditions,

$$a \pitchfork c \Rightarrow p \quad \text{iff} \quad a \otimes c \pitchfork p.$$

But since $c : C \rightarrowtail B$ is a cofibration, $a \otimes c$ is also a cofibration if $a : A \rightarrowtail Z$ is one, by axiom (C6), which says that cofibrations are closed under pushout-products. So $a \otimes c \pitchfork p$ indeed holds, since p is a trivial fibration. $\square$

Proposition 87 (EEP)**.** *Weak equivalences extended along cofibrations in the following sense: given a cofibration $c : C' \rightarrowtail C$ and fibrations $A' \twoheadrightarrow C'$ and $B \twoheadrightarrow C$, and a weak equivalence $w' : A' \simeq c^* B$ over C',*

$$\begin{tikzcd}
A' \arrow[rr, dotted] \arrow[dr, "w'", "\sim"'] \arrow[dd, two heads] & & A \arrow[dr, dotted, "w", "\sim"'] \arrow[dd, dotted, two heads] & \\
& c^* B \arrow[rr] \arrow[dl] & & B \arrow[dl] \\
C' \arrow[rr, tail, "c"'] & & C &
\end{tikzcd} \tag{135}$$

there is a fibration $A \twoheadrightarrow C$ and a weak equivalence $w : A \simeq B$ over C that pulls back along $c : C' \rightarrowtail C$ to w', so $c^* w = w'$.

Proof. Call the given fibration $q : B \to C$ and let $b := q^* c : c^* B \to B$ be the indicated pullback, which is thus also a cofibration. Let $w := b_* w' : A \to B$ be the pushforward of w' along b. Composing with q gives the map $p := q \circ w : A \to C$. Since b is monic, we indeed have $b^* w = w'$, thus filling in all the dotted arrows in (135). Note moreover that $c^* w = b^* w = w'$, as required. It remains to show that $p : A \to C$ is a fibration and $w : A \to B$ is a weak equivalence.

$$
\begin{tikzcd}
A' \arrow[rr, tail] \arrow[dr, "w'", "\sim"'] \arrow[dd, two heads] & & A \arrow[r, "s"] \arrow[dr, "w"] \arrow[dd, "p"' near end] & P_w \arrow[l, dotted, bend right, two heads, "p_w"'] \arrow[d, "t"] \\
& c^*B \arrow[rr, tail, "b"' near start] \arrow[dl, two heads] & & B \arrow[dl, two heads, "q"] \\
C' \arrow[rr, tail, "c"'] & & C &
\end{tikzcd}
\tag{136}
$$

Let us rename $p' := c^*p : A' \to C'$ and $B' := c^*B$ and $q' := c^*q$. Now let $w = t \circ s$ be the pathspace factorization (132) of w, as a map over C. Since $q : B \to C$ is a fibration, by the foregoing remarks on pathspace factorizations, we know that $s : A \to P_w$ has a retraction $p_w : P_w \to A$ which is a trivial fibration. The retraction p_w is a map over C.

The pathspace factorization $w = t \circ s : A \to P_w \to B$ is stable under pullback along c, providing a pathspace factorization $w' = t' \circ s' : A' \to P_{w'} \to B'$ over C'. Since both p' and q' are fibrations, the retraction $p_{w'} : P_{w'} \to A'$ is a trivial fibration, and now $t' : P_{w'} \to B'$ is a fibration.

$$
\begin{tikzcd}
A' \arrow[r, "s'"] \arrow[dr, "w'", "\sim"'] \arrow[dd, two heads, "p'"'] & P_{w'} \arrow[l, dotted, bend right, two heads, "p_{w'}"'] \arrow[d, two heads, "t'"] \\
& B' \arrow[dl, two heads, "q'"] \\
C' &
\end{tikzcd}
\tag{137}
$$

Thus the composite $q' \circ t' : P_{w'} \to B' \to C'$ is a fibration and therefore, by the retraction over C' with the trivial fibration $p_{w'}$, we have that $s' : A' \to P_{w'}$ is a weak equivalence, by 3-for-2 for weak equivalences between fibrations. For the same reason, t' is then a weak equivalence, and therefore a trivial fibration.

Since $t' = c^*t = b^*t$ is a trivial fibration, its pushforward b_*b^*t along b is also one by by Corollary 24. Moreover, $b_*b^*t : b_*b^*P_w \to B$ admits a unit $\eta : P_w \to b_*b^*P_w$ (over B).

$$\begin{tikzcd}
A' \arrow[rr, tail] \arrow[dr, "w'", "\sim"'] \arrow[dd, two heads] & & A \arrow[r, "s"] \arrow[dr, "w"] \arrow[dd, "p"' near end] & P_w \arrow[r, "\eta"] \arrow[d, "t"] \arrow[l, dotted, bend right, "p_w"'] & b_*b^*P_w \arrow[dl, "b_*b^*t"] \\
& B' \arrow[rr, tail, "b"' near start] \arrow[dl] & & B \arrow[dl, two heads, "q"] & \\
C' \arrow[rr, tail, "c"'] & & C & &
\end{tikzcd} \tag{138}$$

We now *claim* that $\eta : P_w \to b_*b^*P_w$ is a trivial fibration. Given that, the composite $t = b_*b^*t \circ \eta$ is also a trivial fibration, whence $q \circ t : P_w \to C$ is a fibration, and so its retract $p : A \to C$ is a fibration. Moreover, since s is a section of the trivial fibration $p_w : P_w \to A$ between fibrations, as before it is also a weak equivalence. Thus $w = t \circ s$ is a weak equivalence, and we are finished.

To prove the remaining claim that $\eta : P_w \to b_*b^*P_w$ is a trivial fibration, we shall use lemma 86. But it does not apply directly since $t : P_w \to B$ is not yet known to be a trivial fibration. Instead, we show that η is a pullback of the corresponding unit at the trivial fibration $p_1 : B^{\mathrm{I}} \to B$.

Consider the following cube (viewed with $b : B' \to B$ at the front).

$$\begin{tikzcd}
P_{w'} \arrow[rr, tail, "\bar{a}"] \arrow[dr, "(p_0')^*w'"] \arrow[dd, two heads, "p_{w'}"'] & & P_w \arrow[dr, "p_0^*w"] \arrow[dd, "p_w" near start] & \\
& B'^{\mathrm{I}} \arrow[rr, tail, "\bar{b}" near end] \arrow[dd, two heads, "p_0'" near end] & & B^{\mathrm{I}} \arrow[dd, two heads, "p_0"] \\
A' \arrow[rr, tail, "a" near end] \arrow[dr, "w'"'] & & A \arrow[dr, "w"] & \\
& B' \arrow[rr, tail, "b"'] & & B
\end{tikzcd} \tag{139}$$

The right hand face is a pullback by definition, and the remainder results from pulling the right face back along b, by the stability of the pathspace factorization (134). Thus all faces are pullbacks. The base is also a pushforward, $b_* w' = w$, again by definition. Thus the top face is also a pushforward, $\overline{b}_*((p_0')^*w') = p_0^*w$. Indeed, since the front face is a pullback, the Beck-Chevalley condition applies, and so we have $\overline{b}_*(p_0')^*(w') = p_0^*\, b_*(w') = p_0^*w$.

Now consider the following, in which the top square remains the same as in (139), but p_0 has been relaced by $p_1 : B^{\mathrm{I}} \to B$, so the composite at right is by definition $t = p_1 \circ p_0^*w$.

$$
\begin{tikzcd}
P_{w'} \arrow[rr, tail, "\overline{a}"] \arrow[dr, "(p_0')^*w'"'] \arrow[ddr, two heads, "t'"'] & & P_w \arrow[dr, "p_0^*w"] \arrow[ddr, two heads, "t"'] & \\
& B'^{\mathrm{I}} \arrow[rr, tail, "\overline{b}"] \arrow[d, two heads, "p_1'"] & & B^{\mathrm{I}} \arrow[d, two heads, "p_1"] \\
& B' \arrow[rr, tail, "b"'] & & B
\end{tikzcd}
\tag{140}
$$

The horizontal direction is still pullback along b; let us rename $p_0^*w =: u$ so that $(p_0')^*w' = b^*u$ and $t' = b^*t$ and $p_1' = b^*p_1$ to make this clear. We then add the pushforward along b on the right, in order to obtain the two units η.

$$
\begin{tikzcd}
b^*P_w \arrow[rr, tail, "\overline{a}"] \arrow[dr, "b^*u"] \arrow[ddr, two heads, "b^*t"'] & & P_w \arrow[rr, "\eta_t"] \arrow[dr, "u"] \arrow[ddr, two heads, "t"'] & & b_*b^*P_w \arrow[dr, "b_*b^*u"] \arrow[ddr, "b_*b^*t"'] & \\
& b^*B^{\mathrm{I}} \arrow[rr, tail, "\overline{b}"] \arrow[d, two heads, "b^*p_1"] & & B^{\mathrm{I}} \arrow[rr, "\eta_{p_1}"] \arrow[d, two heads, "p_1"] & & b_*b^*B^{\mathrm{I}} \arrow[d, "b_*b^*p_1"] \\
& B' \arrow[rr, tail, "b"'] & & B \arrow[rr, "="'] & & B
\end{tikzcd}
\tag{141}
$$

By the usual calculation of pushforwards in slice categories, $\overline{b}_* \cong \eta_{p_1}^* \circ b_*$, and so for b^*u we have $\overline{b}_* b^*u = \eta_{p_1}^* b_* b^*u$. But as we just determined in (139) the top left square is already a pushforward, and therefore $u = \eta_{p_1}^* b_* b^*u$, so the top right naturality square is a pullback.

To finish the proof as planned, $p_1 : B^{\mathrm{I}} \to B$ is a trivial fibration because $q : B \to C$ is a fibration, and $b : B' \rightarrowtail B$ is a cofibration because it is a pullback of $c : C' \rightarrowtail C$. Thus by lemma 86, we have that $\eta_{p_1} : B^{\mathrm{I}} \to b_* b^* B^{\mathrm{I}}$ is a trivial fibration, and so its pullback $\eta_t : P_w \to b_* b^* P_w$ is a trivial fibration, as claimed. $\square$

Remark 88. Note that $p : A \to C$ is small if $q : B \to C$ is small.

8 The fibration extension property

In the presence of a universal fibration $\dot{\mathcal{U}} \twoheadrightarrow \mathcal{U}$, as given by Proposition 81, the fibration extension property (Definition 59) is closely related to the statement that the base object $\mathcal{U}$ is fibrant. For Kan simplicial sets, Voevodsky proved the latter directly using minimal fibrations [?]. Shulman [?] gives a proof from univalence (in the form of the equivalence extension property as stated in section 7) in a more general setting, but it uses the 3-for-2 property for weak equivalences, which is what we are trying to prove. In [CCHM16], Coquand uses the equivalence extension property to prove that $\mathcal{U}$ is fibrant, without assuming 3-for-2 for weak equivalences, by a neat argument using a reduction of general box-filling to a condition called "Kan-composition". We shall prove that $\mathcal{U}$ is fibrant using the equivalence extension property via a different argument than that in [CCHM16], avoiding the reduction of filling to composition, which we therefore do not require.

Returning to the relation between the fibration extension property and the condition that the base object $\mathcal{U}$ is fibrant, it is easily seen that the latter implies the former. Indeed, let $t : X \rightarrowtail X'$ be a trivial cofibration and $Y \twoheadrightarrow X$ a fibration. To extend Y along t, take a classifying map $y : X \to \mathcal{U}$, so that $Y \cong y^* \dot{\mathcal{U}}$ over X. If $\mathcal{U}$ is fibrant then we can extend y along $t : X \rightarrowtail X'$ to get $y' : X' \to \mathcal{U}$ with $y = y' \circ t$. The pullback $Y' = (y')^* \dot{\mathcal{U}} \twoheadrightarrow X'$ is then a (small) fibration such that $t^* Y' \cong t^* (y')^* \dot{\mathcal{U}} \cong y^* \dot{\mathcal{U}} \cong Y$ over X.

$$\begin{tikzcd}
Y \arrow[rr] \arrow[dr] \arrow[dd, two heads] & & \dot{\mathcal{U}} \arrow[dd, two heads] \\
& Y' \arrow[ur, dotted] \arrow[dd, two heads] & \\
X \arrow[rr, "y"'] \arrow[dr, tail, "t"'] & & \mathcal{U} \\
& X' \arrow[ur, dotted, "y'"'] &
\end{tikzcd}$$

Thus for the record, we have:

Proposition 89. *If the base object $\mathcal{U}$ of the universal fibration $\dot{\mathcal{U}} \twoheadrightarrow \mathcal{U}$ is fibrant, then the fibration weak factorization system has the fibration extension property.*

Conversely, given the Realignment Lemma 85, the FEP also implies the fibrancy of $\mathcal{U}$:

Corollary 90. *The fibration extension property implies that the base $\mathcal{U}$ of the universal fibration $\dot{\mathcal{U}} \twoheadrightarrow \mathcal{U}$ is fibrant: given any $y : X \to \mathcal{U}$ and trivial cofibration $t : X \rightarrowtail X'$, there is a map $y' : X' \to \mathcal{U}$ with $y' \circ t = y$.*

Proof. Take the pullback of $\dot{\mathcal{U}} \twoheadrightarrow \mathcal{U}$ along $y : X \to \mathcal{U}$ to get a (small) a fibration $Y \twoheadrightarrow X$, which extends along the (trivial) cofibration $t : X \rightarrowtail X'$ by the FEP, to a (small) fibration $Y' \twoheadrightarrow X'$ with $Y \cong t^*Y'$ over X. By realignment there is a classifying map $y' : X' \to \mathcal{U}$ for Y' with $y' \circ t = y$. $\square$

We shall now show that $\mathcal{U}$ is indeed fibrant in the following two steps:

1. We will show that, in the slice category $\mathsf{cSet}/_{\mathrm{I}}$, the total space of the classifying type
$$\mathsf{Fib}_i(\mathrm{I}^*\dot{\mathcal{V}}) \to \mathrm{I}^*\mathcal{V}$$
for $\mathbb{I}$-fibration structures on the universal family $\mathrm{I}^*\dot{\mathcal{V}} \to \mathrm{I}^*\mathcal{V}$ from Remark 82 is a fibrant object. This uses the equivalence extension property, Proposition 87, applied with respect to $\mathbb{I}$-fibrations in the slice category $\mathsf{cSet}/_{\mathrm{I}}$, [*as justified by ...*]

2. We will show that the total space of the pushforward of $\mathsf{Fib}_i(\mathrm{I}^*\dot{\mathcal{V}}) \to \mathrm{I}^*\mathcal{V}$ along the projection $q : \mathrm{I} \times \mathcal{V} \to \mathcal{V}$ in (125) is a fibrant object in cSet. This is a general argument not depending on the specific objects involved.

First, the reader may find it illuminating to reconsider the construction of the universal small fibration $\dot{\mathcal{U}} \twoheadrightarrow \mathcal{U}$ in more type theoretic terms. It was determined in Remark 82 to be $\dot{\mathsf{Fib}} \twoheadrightarrow \mathsf{Fib} = \mathsf{Fib}(\dot{\mathcal{V}})$, for the universal family $\dot{\mathcal{V}} \to \mathcal{V}$, with $\dot{\mathsf{Fib}}$ the pullback of $\dot{\mathcal{V}} \to \mathcal{V}$ along the canonical projection $\mathsf{Fib}(\dot{\mathcal{V}}) \to \mathcal{V}$. Type theoretically, we have $\dot{\mathcal{V}} = \Sigma_{A:\mathcal{V}} A$, and then:

$$\dot{\mathcal{U}} \;=\; \Sigma_{A:\mathcal{V}}\mathsf{Fib}(A) \times A \;\longrightarrow\; \Sigma_{A:\mathcal{V}}\mathsf{Fib}(A) \;=\; \mathcal{U}\,.$$

We show that $\mathrm{I}^*\dot{\mathcal{U}} \twoheadrightarrow \mathrm{I}^*\mathcal{U}$ is a universal fibration in $\mathsf{cSet}/_{\mathrm{I}}$ in two steps:

1. $\mathrm{I}^*\dot{\mathcal{V}} \to \mathrm{I}^*\mathcal{V}$ is a universal small family in $\mathsf{cSet}/_\mathrm{I}$. This follows immediately from the fact that both of the adjoint functors

$$\Sigma_\mathrm{I} \dashv \mathrm{I}^* : \mathsf{cSet} \to \mathsf{cSet}/_\mathrm{I}$$

preserve pullbacks. Indeed, a small family $A \to X \xrightarrow{\xi} \mathrm{I}$ indexed over I, is classified by $\langle \xi, a\rangle : X \to \mathrm{I}^*\mathcal{V} = \mathrm{I} \times \mathcal{V}$ over I, where $a : X \to \mathcal{V}$ classifies $A \to X$ in cSet.

$$\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{I} \times \dot{\mathcal{V}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathcal{V}} \arrow[d] \\
X \arrow[r, "\langle \xi, a\rangle"] \arrow[dr, "\xi"'] & \mathrm{I} \times \mathcal{V} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{V} \arrow[d] \\
 & \mathrm{I} \arrow[r] & 1
\end{tikzcd}$$

2. By definition a map $A \to X$ in cSet is a fibration just if its base change $\mathrm{I}^*A \to \mathrm{I}^*X$ is one in $\mathsf{cSet}/_\mathrm{I}$ (with respect to $\delta : 1 \to \mathrm{I}^*\mathrm{I}$). Moreover, the classifying type for fibration structures $\mathsf{Fib}(A) \to X$ is then constructed by pushing the classifying type $\mathsf{Fib}_\mathrm{I}(\mathrm{I}^*A) \to \mathrm{I}^*X$ forward along the projection $\mathrm{I}^*X = \mathrm{I} \times X \to X$,

$$\mathsf{Fib}(A) = \Pi_\mathrm{I}\mathsf{Fib}_\mathrm{I}(\mathrm{I}^*A) \to X\,.$$

We therefore have

$$\mathcal{U} = \mathsf{Fib}(\dot{\mathcal{V}}) = \Pi_\mathrm{I}\mathsf{Fib}_\mathrm{I}(\mathrm{I}^*\dot{\mathcal{V}}).$$

Proposition 91. *The base object* $\mathcal{U}$ *of the universal fibration* $\dot{\mathcal{U}} \twoheadrightarrow \mathcal{U}$ *in* cSet *constructed in section 6.1 is fibrant.*

Proof. Moving $\mathcal{U}$ to the slice category cSet/I by (silently) applying the base change $\mathrm{I}^* : \mathsf{cSet} \to \mathsf{cSet}/\mathrm{I}$, we need to solve the following filling problem there, for $\delta : 1 \to \mathrm{I}$ the generic point and $c : C \rightarrowtail Z$ an arbitrary cofibration (over I),

$$\begin{tikzcd}
C \arrow[r, "\tilde{a}"] \arrow[d, tail, "c"'] & \mathcal{U}^\mathrm{I} \arrow[d, "\mathcal{U}^\delta"] \\
Z \arrow[r, "b"'] \arrow[ur, dotted] & \mathcal{U}
\end{tikzcd} \tag{142}$$

This shows that the map $\delta \Rightarrow \mathcal{U} = \mathcal{U}^\delta : \mathcal{U}^{\mathrm{I}} \longrightarrow \mathcal{U}$ over I is a trivial fibration in cSet/I, and so $\mathcal{U}$ is fibrant in cSet, by Definition **??**. The remainder of the proof occurs in the Cartesian cubical presheaf category cSet/I, for which we have all the same results of sections 1 through 7 as for cSet. In particular, by Lemma **??** there is a universal fibration $\dot{\mathcal{U}} \to \mathcal{U}$ resulting from the base change.

Transposing $\tilde{a}$ to $a : C \times \mathrm{I} \to \mathcal{U}$ and taking pullbacks of $\dot{\mathcal{U}} \twoheadrightarrow \mathcal{U}$ along a and b to get corresponding fibrations $A \twoheadrightarrow C \times \mathrm{I}$ and $B \twoheadrightarrow Z$, we have the following equivalent condition. Letting

$$C_\delta : C \cong C \times 1 \to C \times \mathrm{I}$$

be the evident inclusion of the "δ-section" of the cylinder, let $A_\delta = (C_\delta)^* A \twoheadrightarrow C$ be the "section of A over C_δ". We then have $c^* B \cong A_\delta$ over C by the outer square of (142).

$$\begin{tikzcd}
 & A_\delta \arrow[rr] \arrow[d, two heads] \arrow[ddl] & & A \arrow[d, two heads] \\
 & C \arrow[rr, "C_\delta"'] \arrow[ddl, "c"] & & C \times \mathrm{I} \arrow[ddl] \\
B \arrow[d, two heads] & & & \\
Z \arrow[rr, "Z_\delta"'] & & Z \times \mathrm{I} &
\end{tikzcd}$$

The diagonal filler sought in (142) corresponds, again by transposition and pullback of $\dot{\mathcal{U}} \twoheadrightarrow \mathcal{U}$, to a fibration $D \twoheadrightarrow Z \times \mathrm{I}$ with $(c \times \mathrm{I})^* D \cong A$ over $C \times \mathrm{I}$ and $(Z_\delta)^* D \cong B$ over Z, as indicated below.

$$\begin{tikzcd}
 & A_\delta \arrow[rr] \arrow[d, two heads] \arrow[ddl] & & A \arrow[d, two heads] \arrow[ddl, dotted] \\
 & C \arrow[rr, "C_\delta"'] \arrow[ddl, "c"] & & C \times \mathrm{I} \arrow[ddl, "c \times \mathrm{I}"] \\
B \arrow[d, two heads] \arrow[rr, dotted] & & D \arrow[d, dotted, two heads] & \\
Z \arrow[rr, "Z_\delta"'] & & Z \times \mathrm{I} &
\end{tikzcd} \tag{143}$$

We can construct such a $D \twoheadrightarrow Z \times \mathrm{I}$ using the equivalence extension property, as follows. First apply the functor $(-) \times \mathrm{I}$ to the left (pullback) face of the cube in (143) to get the following, with a new pullback square on the right, involving the indicated fibrations.

$$
\begin{tikzcd}
 & A_\delta \arrow[rr] \arrow[d, two heads] \arrow[ddl] & & A \arrow[d, two heads] \arrow[ddl, dotted] & & A_\delta \times \mathrm{I} \arrow[dll, two heads] \arrow[ddl] \\
 & C \arrow[rr, "C_\delta"'] \arrow[ddl, "c"'] & & C \times \mathrm{I} \arrow[ddl, "c\times \mathrm{I}"] & & \\
B \arrow[rr, dotted] \arrow[d, two heads] & & D \arrow[d, dotted, two heads] & & B \times \mathrm{I} \arrow[dll, two heads] & \\
Z \arrow[rr, "Z_\delta"'] & & Z \times \mathrm{I} & & &
\end{tikzcd}
\tag{144}
$$

We next *claim* that there is a weak equivalence $e : A \simeq A_\delta \times \mathrm{I}$ over $C \times \mathrm{I}$, from which follow by the EEP:

(i) a fibration $D \twoheadrightarrow Z \times \mathrm{I}$ with $(c \times \mathrm{I})^* D \cong A$ over $C \times \mathrm{I}$, and

(ii) a weak equivalence $f : D \simeq B \times \mathrm{I}$ over $Z \times \mathrm{I}$ with $(c \times \mathrm{I})^* f \cong e$ over $C \times \mathrm{I}$.

It then remains only to show that $B \cong (Z_\delta)^* D$ over Z to complete the proof.

To obtain e, consider the following square, in which the top map is $A_\delta \times \delta$ (after $A_\delta \cong A_\delta \times 1$) and the others are those from diagram (144).

$$
\begin{tikzcd}
A_\delta \arrow[r, tail] \arrow[d, tail] & A_\delta \times \mathrm{I} \arrow[d, two heads] \\
A \arrow[r, two heads] & C \times \mathrm{I}
\end{tikzcd}
\tag{145}
$$

The square is easily seen to commute, and the maps with A_δ as domain are trivial cofibrations by Frobenius (Proposition 69), because each is the pullback of a trivial cofibration along a fibration. A simple lemma (given below as Lemma 92) provides the claimed weak equivalence $e : A \simeq A_\delta \times \mathrm{I}$ over $C \times \mathrm{I}$.

To see that $B \cong (Z_\delta)^* D$ over Z, recall from the proof of the EEP that the map $f : B \cong (Z_\delta)^* D$ is the pushforward of $e : A \simeq A_\delta \times \mathrm{I}$ along the cofibration $b_\delta \times \mathrm{I} : A_\delta \times \mathrm{I} \rightarrowtail B \times \mathrm{I}$, where we have named the evident map in (144) $b_\delta : A_\delta \rightarrowtail B$. That is, by construction $f = (b_\delta \times \mathrm{I})_* e$. We can then apply the Beck-Chevalley condition for the pushforward using the pullback square on the left below.

$$\begin{tikzcd} A_\delta \arrow[r, rightarrowtail] \arrow[d, rightarrowtail] \arrow[dr, phantom, "\lrcorner", very near start] & A_\delta \times \mathrm{I} \arrow[d, twoheadrightarrow] & A \arrow[l, "e"'] \\ B \arrow[r, rightarrowtail] & B \times \mathrm{I} & D \arrow[l, "f"] \end{tikzcd} \tag{146}$$

The pullback of e along the top of the square is the identity on A_δ, as can be seen by pulling back e as a map over $C \times \mathrm{I}$ along $C_\delta : C \to C \times \mathrm{I}$. Thus the same is true (up to isomorophism) for the pullback of f along the bottom.

An application of the Realignment Lemma 85 along the trivial cofibration $c \otimes \delta$ completes the proof. □

Lemma 92. *Suppose the following square commutes and the indicated cofibrations are trivial.*

$$\begin{tikzcd} A \arrow[r, rightarrowtail] \arrow[d, rightarrowtail] & C \arrow[d, twoheadrightarrow] \\ B \arrow[r, twoheadrightarrow] & D \end{tikzcd} \tag{147}$$

Then there is a weak equivalence $e : B \simeq C$ over D (and under A).

Proof. Use the fact that any two diagonal fillers are homotopic to get a homotopy equivalence $e : B \simeq C$ filling the square. □

Applying proposition 89 now yields the following.

Corollary 93. *The fibration weak factorization system has the fibration extension property (definition 59).*

By Theorem 64, finally, we have the following.

Theorem 94. *There is a Quillen model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ on the category of cubical sets* cSet*, where:*

1. *the cofibrations $\mathcal{C}$ are any class of maps satisfying (C0)-(C8) (equivalently, the simplified axioms in the Appendix),*

2. *the fibrations $\mathcal{F}$ are the maps $f : Y \to X$ for which the canonical map*
$$(f^{\mathrm{I}} \times \mathrm{I}, \mathrm{eval}_Y) : Y^{\mathrm{I}} \times \mathrm{I} \to (X^{\mathrm{I}} \times \mathrm{I}) \times_X Y$$
lifts on the right against $\mathcal{C}$.

3. *the weak equivalences $\mathcal{W}$ are the maps $w : X \to Y$ for which the internal precomposition $K^w : K^Y \to K^X$ is bijective on connected components for every fibrant object K.*

Appendix: Axioms for Cartesian cofibrations

A system of maps satisfying the axioms (C0)-(C8) above for the cofibrations in a cartesian cubical model category will be called *cartesian cofibrations.* The axioms for cartesian cofibrations can be reformulated equivalently as follows.

(C0) All cofibrations are monomorphisms.

(C1) All isomorphisms are cofibrations.

(C2) The composite of two cofibrations is a cofibration.

(C3) Any pullback of a cofibration is a cofibration.

(C4) The category of cofibrations and cartesian squares has a terminal object.

(C5) The join of two cofibrant subobjects is a cofibration.

(C6) The diagonal map $\delta : \mathrm{I} \to \mathrm{I} \times \mathrm{I}$ is a cofibration.

(C7) If $c : A \to B$ is a cofibration, then so is $c^{\mathrm{I}} : A^{\mathrm{I}} \to B^{\mathrm{I}}$.

We have the following non-trivial example.

Proposition 95. *The locally decidable subobjects in any topos satisfy the axioms for cartesian cofibrations.*

Proof. [*fill in ...*] □

Appendix B: Semantics of HoTT

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