Pseudomonadicity and 2-stack completions

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PSEUDOMONADICITY AND 2-STACK COMPLETIONS

Dedicated to Michael Makkai on his 70th birthday

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ABSTRACT. The notion of a 1-stack (or simply, a stack), originally due to Grothendieck and Giraud [15], is taken here in the (intrinsic) sense of Lawvere [25, 26], that is, relative to the class of all epimorphisms in a topos $\mathbf{S}$. We extend to dimension 2 the result of [7] on the fibrational 1-stack completion of (the externalization of) a category object $\mathbb{C}$ in a Grothendieck topos $\mathbf{S}$. In dimension 1, the monadicity and descent theorems of [2] and [4] are employed in [11] to show that $\mathbf{S}$ (or 0-**Stack**) is a 1-stack over itself. As one of the several applications, Diaconescu's theorem[22] on the classification of G-torsors for a groupoid G is derived. Likewise, in dimension 2, we resort to the pseudomonadicity and 2-descent theorems of [19] in order to prove that **Stack** (or 1-**Stack**) is a 2-stack over $\mathbf{S}$. Applied to a 2-gerbe $\mathbb{G}$, a classification theorem for $\mathbb{G}$-2-torsors is derived. An axiom of stack completions (ASC) stating, for a topos $\mathbf{S}$, that the fibrational stack completion of any category object in $\mathbf{S}$ is representable, holds for any Grothendieck topos $\mathbf{S}$, as a general argument [22] shows, and similarly for 2-stack completions. In the first case, the Quillen model structure on $\mathbf{Cat}(\mathbf{S})$ given in [20] for $\mathbf{S}$ a Grothendieck topos gives an alternative proof of this result. The question of giving a similar construction in the 2-dimensional case is left open. The passages from dimension 1 to dimension 2 pave the way for similar results in higher dimensions.

Introduction

Section 1 discusses the beautiful theory of stacks and non-abelian topos cohomology. This material is considered by many to be one of the pinnacles of 20th century mathematics. The theory of stacks (or *champs*) was developed first by Grothendieck [16] and Giraud [15] in terms of sites, and then recast in an intrinsic fashion and clarified considerably in a second wave by the topos theory community based mostly in North America.

Section 1, which is largely expository, is based on [11] and [7], though formulated in terms of fibrations over a given topos $\mathbf{S}$, as opposed to $\mathbf{S}$-indexed categories as in those sources. We use the formal definition-theorem-proof environment just in order to pave the wave for an extension to the 2-dimensional case, which is the purpose of this paper. However, an informal description of the contents of these two sections that should be sufficient as a motivation for our program, is given next. The notion of an intrinsic intrinsic stack was proposed by Lawvere [25, 26]. The intrinsic (or canonical) topology on $\mathbf{S}$ is the large site consisting of the class of all epimorphisms in $\mathbf{S}$ [1]. Denote by Fib($\mathbf{S}$)

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the 2-category of fibrations over $\mathbf{S}$, which is cartesian closed as a bicategory.

Fundamental to the theory of intrinsic (1-)stacks are two key results, both from [11]. These are recalled in Section 1. The first of these, Theorem 1.9, is the characterization of stacks in terms of the weak equivalences recalled in Definition 1.3. This result says that $\mathscr{A}$ in $\mathrm{Fib}(\mathbf{S})$ is a stack iff $\mathscr{A}^{(-)}$ sends weak equivalences to equivalences. Whether one takes this characterization as the definition of a stack (a suggestion of Joyal) or not, is irrelevant, as the explicit definition of a stack (Definition 1.1) cast in terms of (1-) descent, is also needed for the deep applications of stacks theory.

From this characterization of stacks, it is immediate that any weak equivalence from a fibration to a stack enjoys the universal property of a stack completion. The facts that stacks are closed in $\mathrm{Fib}(\mathbf{S})$ under exponentiation with arbitrary objects, that weakly equivalent fibrations have equivalent stack completions (when such exist), and other results mentioned, all follow pretty immediately from the universal properties.

It is common to regard $\mathbf{S}$ as a fibration over itself via the codomain fibration

$$\mathrm{cod} : \mathbf{S}^{\rightarrow} \longrightarrow \mathbf{S},$$

and to denote this fibration also as $\mathbf{S}$. The second key result, Corollary 1.12, is that $\mathbf{S}$ is a stack. This result is easily obtained from the monadicity and descent theorems of [2] and [4].

This stack $\mathbf{S}$ plays the role of the 'object of sets' giving rise to a Yoneda structure on $\mathrm{Fib}(\mathbf{S})$ (see [35], [36] and [37] for a discussion of this Yoneda structure), and so all the presheaf objects $\mathbf{S}^{\mathscr{A}^{\mathrm{op}}}$ are stacks too. Using the above results it is established in [7] that, for any locally internal (= admissible for the Yoneda structure) fibration $\mathscr{A}$, one has a canonical embedding of it into a stack

$$y_{\mathscr{A}} : \mathscr{A} \longrightarrow \mathbf{S}^{\mathscr{A}^{\mathrm{op}}}$$

namely the Yoneda embedding. Factoring this through its 'weakly-essential image' gives the stack completion of $\mathscr{A}$ of Theorem 1.13.

As non-trivial consequences of the results just mentioned, one obtains, for a non-empty and connected groupoid G in $\mathbf{S}$, both the fact that the fibration $\mathrm{Tors}^1(G)$ over $\mathbf{S}$ is the stack completion of G, and that the latter is classified by the topos $\mathscr{B}(G) = \mathbf{S}^{G^{\mathrm{op}}}$ (Diaconescu's theorem [22]). This application of stacks [7] shows how non-abelian cohomology includes Galois theory, thus making stacks such a terrific unifying perspective. By restricting to the case of discrete (localic) groupoids, we also recover a classification and Morita theorems given in [8], with applications in [9].

In Section 2 we present, in a unified manner, examples of stack completions in Commutative Algebra, all related to the Zariski topos and motivated by an example of [25]. In particular we obtain, as applications of stack completions, conceptual proofs of Kaplansky's theorem and Swan's theorem in addition to those given by Mulvey [28]. Other examples had already been given at the end of Section 2 of [7].

It is well known [13, 22] that, if $\mathbf{S}$ is a Grothendieck topos, then the fibrational stack completion of a category object $\mathbb{C}$ in $\mathbf{S}$ is representable on account of the existence of a generating set for $\mathbf{S}$. A direct proof of this result is given in [20] by means of a Quillen model structure on $\mathbf{Cat}(\mathbf{S})$ whose fibrant objects are the (so called) strong stacks. In particular, this gives an internal (strong) stack completion of any category object $\mathbb{C}$ in $\mathbf{S}$. In Section 3 we interpret this to say that any Grothendieck topos $\mathbf{S}$ satisfies (ASC), an 'axiom of stack completions'. Also in Section 3, we discuss another Quillen model structure on $\mathbf{Cat}(\mathbf{S})$ given by Lack in [23, 24]. In $\mathbf{Cat}(Set^{\mathbf{2}})$, and as shown independently in [11] and [23, 24], the iso-fibrations, that is, those internal functors with the isomorphism lifting property, are precisely the stacks. We argue then that, for a general Grothendieck topos $\mathbf{S}$, the data for the constructions of [20] and [23, 24] need not agree. We expand further on this issue.

Sections 4 and 5 then develop the 2-dimensional analogue of the theory of (1-)stacks. 2-stacks are in particular 2-fibrations, and defined in terms of 2-descent – involving the three-(rather than two)-truncated simplicial nerve of the kernel groupoid of an epi, discussed in [19]. The two main results of the paper are the 2-dimensional analogues of the two fundamental results on stacks isolated above. The analogue of Theorem 1.9 is Theorem 4.9, and that of Corollary 1.12 is Theorem 5.7.

With these results established, the theory of 2-stack completions then unfolds as in dimension 1. The notion of an intrinsic 2-stack is introduced here via 2-descent and shown to be equivalent to a condition of inverting weak 2-equivalence functors. The characterization of intrinsic 2-stacks is given directly in Section 4 after stating the notion of a weak 2-equivalence. In Section 5 we adapt the pseudomonadicity and 2-descent theorems of [19] in order to prove that the 2-fibration cod : $\mathbf{Stack}^{\to} \longrightarrow \mathbf{S}$ is a 2-stack. The representability of stack completions in $\mathbf{S}$ is a crucial ingredient in the proof.

A notable application is then given in Section 6, in which we construct the 2-stack completion of any 'generalized 2-gerbe', a name we give to any 2-category object $\mathbf{C}$ in $\mathbf{S}$ which is furthermore 'hom-by-hom' a stack. Restricting to 2-groupoids, we obtain, for a 2-gerbe $\mathbf{G}$, the 'classification' of $\mathbf{G}$-torsors by the '2-topos' [32, 34] $\mathscr{B}^2(\mathbf{G}) = \mathbf{Stack}^{\mathbf{G}^{\mathrm{op}}}$. To make this precise we need to specify a notion of morphism of 2-toposes, this task being left at the moment as an open question.

In Section 7, on equiv-fibrations, we exhibit an example of a 2-stack completion in connection with the Quillen model structure on 2-$\mathbf{Cat}(\mathbf{S})$ for $\mathbf{S}$ a Grothendieck topos as given by Lack in [23, 24], applied to the Grothendieck topos $\mathbf{S} = Set^{\mathbf{2}}$. This Quillen model structure is then not what we are looking for in the general case. Explicitly, by an argument involving the existence of a generating set, it can be shown in a manner analogous to that of the 1-dimensional case that every 2-gerbe in a Grothendieck topos S admits a representable 2-stack completion when regarded as a 2-fibration. The question of giving a suitable Quillen model structure on $2-\mathbf{Cat}(S)$ whose fibrant objects are the '(strong) 2-stacks' is still open.

Finally, Section 8 gives an outline of how one would, in the same spirit as this paper, proceed to higher dimensions for a theory of n-stacks and their completions. We argue that, in principle, the results obtained in dimensions 1 and 2 pave the way for similar theorems about intrinsic n-stacks in higher dimensions $n > 2$, to be shown by induction by analogy with the passage from dimension 1 to dimension 2. Alternative outlines have been given in [14] and [33].

1. Intrinsic stacks

In this largely expository section we review the definition of a stack, state the two fundamental results of the theory of stacks (described informally in the Introduction), and recall an application of stack completions to non-abelian cohomology [11] and [7]. Throughout $\mathbf{S}$ is an elementary topos in the sense of Lawvere and Tierney [22], We assume familiarity with fibrations over a topos [26] and [30].

1.1. Definition. *[11]* A fibration $\mathscr{A} \longrightarrow \mathbf{S}$ is a *stack* (for the intrinsic topology on $\mathbf{S}$) if

1. given any I-indexed family $\{J_i \mid i \in I\}$ of objects of $\mathbf{S}$, the canonical functor

$$\mathscr{A}^{\sqcup_{i\in I} J_i} \longrightarrow \textstyle\prod_{i\in I} \mathscr{A}^{J_i}$$

is an equivalence of categories, and

2. for every epimorphism $e : J \longrightarrow I$ in $\mathbf{S}$, the functor $e^* : \mathscr{A}^I \longrightarrow \mathscr{A}^J$ is of effective descent. This means that the canonical functor Φ_e in the diagram below, is an equivalence.

$$\begin{tikzcd} \mathscr{A}^I \arrow[rr, "\Phi_e"] \arrow[dr, "e^*"'] & & \mathrm{Des}_{\mathscr{A}}(e) \arrow[dl, "U"] \\ & \mathscr{A}^J & \end{tikzcd}$$

1.2. Remark. A category object $\mathbb{A}$ in $\mathbf{S}$ is a stack iff the second condition in Definition 1.1 holds for the fibration $[\mathbb{A}] \longrightarrow \mathbf{S}$ which is the *externalization* of $\mathbb{A}$.

1.3. Definition. Let $F : \mathscr{C} \longrightarrow \mathscr{D}$ be a functor between fibrations over $\mathbf{S}$. It is said to be a *weak equivalence* if the following conditions hold.

1. (*essentially surjective*) For each $I \in \mathbf{S}$, $F^I : |\mathscr{D}^I| \longrightarrow |\mathscr{C}^I|$, and $c \in |\mathscr{C}^I|$, there exists an epimorphism $e : J \longrightarrow I$ in $\mathbf{S}$, $b \in |\mathscr{D}^I|$, and an isomorphism $\theta : F^J(b) \longrightarrow e^*(c)$.

2. (*fully faithful*) $\forall I \in \mathbf{S}\ \forall x, x' \in |\mathscr{D}^I|$, the function

$$\mathrm{Hom}_{\mathscr{D}^I}(x, x') \xrightarrow{F_{x,x'}} \mathrm{Hom}_{\mathscr{C}^I}(Fx, Fx')$$

is an isomorphism.

An explicit construction in $\mathbf{Cat}(\mathbf{S})$ of the *2-pullback (or iso-comma-object)* of two functors $F : \mathbb{A} \longrightarrow \mathbb{B}$ and $G : \mathbb{B}' \longrightarrow \mathbb{B}$ is given [11] by the data

$$\begin{tikzcd} F||G \arrow[r, "F'"] \arrow[d, "G'"'] \arrow[dr, phantom, "\Rightarrow t"] & \mathbb{B}' \arrow[d, "G"] \\ \mathbb{A} \arrow[r, "F"'] & \mathbb{B} \end{tikzcd}$$

where $F||G$ is explicitly described as the category whose objects are triples (a, b', θ), $\theta : Fa \longrightarrow Gb'$ an isomorphism, and whose morphisms are pairs of morphisms making the obvious diagrams commute. The functors F' and G' are given, on objects by $F'(a, b', \theta) = b'$ and $G'(a, b', \theta) = a$, and similarly for morphisms. The natural isomorphism $t : GF' \longrightarrow GF'$ is defined by $t(a, b', \theta) = \theta$.

1.4. Remark. This construction, which is easily extended to fibrations over $\mathbf{S}$, satisfies the required universal property. If the codomain $\mathbb{B}$ is discrete, the square is simply a pullback.

The following results are taken from [11].

1.5. Proposition. *Weak equivalences between fibrations over* $\mathbf{S}$ *are*

1. *stable under 2-pullbacks, and*
2. *if, in a commutative triangle*

$$\begin{tikzcd} \mathscr{A} \arrow[rr, "F"] \arrow[dr, "H"'] & & \mathscr{B} \arrow[dl, "G"] \\ & \mathscr{C} & \end{tikzcd}$$

any two of F, G, H are wef, then so is the third.

Given a morphism $e : J \longrightarrow I$ in $\mathbf{S}$, its 1-kernel pair is the 2-truncated simplicial complex

$$\begin{tikzcd} J \times_I J \times_I J \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & J \times_I J \arrow[r, shift left=2] \arrow[r, shift right=2] & J \arrow[l] \end{tikzcd}$$

where the arrows are labelled by π for the projections and δ for the diagonal.

1.6. Definition.

For a morphism $e : J \longrightarrow I$ in $\mathbf{S}$, and any fibration $\mathscr{A}$ over $\mathbf{S}$, *the category* $\mathrm{Des}_{\mathscr{A}}(e)$ *of descent objects and morphisms in* $\mathscr{A}$ *with respect to* e, has, as *objects*, pairs (a, θ) with $a \in \mathscr{A}^J$, and $\theta : {\pi_0}^*(a) \cong {\pi_1}^*(a)$, satisfying the

1. (normalization condition)
$$\delta^*(\theta) = 1_a$$

.

2. (cocycle condition)

$$\pi_{12}{}^*(\theta).\pi_{01}{}^*(\theta) :=: \pi_{02}{}^*(\theta),$$

and whose *morphisms* $f : (a, \theta) \longrightarrow (b, \sigma)$ are morphisms $f : a \longrightarrow b$ in $\mathbf{S}$ such that

$$\pi_1{}^*(f).\theta = \sigma.\pi_1{}^*(f).$$

It is understood that canonical isomorphisms are to be inserted wherever necessary.

Any epimorphism $e : J \twoheadrightarrow I$ in $\mathbf{S}$ gives rise to an internal functor $F_e : \mathbb{J}_e \longrightarrow \mathbb{I}_{\text{dis}}$ where $\mathbb{J}_e$ is the 1-kernel of e and $\mathbb{I}_{\text{dis}}$ is the discrete category on I.

1.7. Proposition. *Let $\mathscr{A}$ be a fibration over $\mathbf{S}$. Then, $\mathscr{A}$ is a stack if and only if the internal functor $F_e : \mathbb{J}_e \longrightarrow \mathbb{I}_{\text{dis}}$ induces an equivalence of categories*

$$\mathscr{A}^{F_e} : \mathscr{A}^{\mathbb{I}_{\text{dis}}} \longrightarrow \mathscr{A}^{\mathbb{J}_e}$$

Proof. The diagram

$$\begin{tikzcd}
\mathscr{A}^I \arrow[r, "\Phi_e"] \arrow[d, "\kappa_I"'] & \mathrm{Des}_{\mathscr{A}}(e) \arrow[d, "\Psi_e"] \\
\mathscr{A}^{\mathbb{I}_{\text{dis}}} \arrow[r, "\mathscr{A}^{F_e}"'] & \mathscr{A}^{\mathbb{J}_e}
\end{tikzcd}$$

is commutative, with κ_I an equivalence [30]. That Ψ_e is an equivalence is shown in [11]. Therefore, for any epimorphism $e : J \longrightarrow I$ in $\mathbf{S}$, Φ_e is an equivalence (hence $\mathscr{A}$ is a stack), if and only of $\mathscr{A}^{F_e}$ is an equivalence, as claimed. □

1.8. Lemma. *For any epimorphism e in $\mathbf{S}$, the internal functor $F_e : \mathbb{J}_e \longrightarrow \mathbb{I}_{\text{dis}}$ is a weak equivalence functor.*

1.9. Theorem. *A fibration $p : \mathscr{A} \longrightarrow \mathbf{S}$ is a stack if and only if for every weak equivalence functor $F : \mathscr{B} \longrightarrow \mathscr{C}$, the functor*

$$\mathscr{A}^F : \mathscr{A}^{\mathscr{C}} \longrightarrow \mathscr{A}^{\mathscr{B}}$$

is an equivalence of fibrations over $\mathbf{S}$.

Proof. It follows from Proposition 1.7 and Lemma 1.8 that the condition of the theorem is sufficient. The converse is shown by localization from the fact that, for $\mathscr{A}$ a stack and $F : \mathbb{B} \longrightarrow \mathbb{C}$ a weak equivalence functor, the S-indexed functor $\mathscr{A}^I$ is an equivalence 'at 1'. This is done by a standard descent argument. □

1.10. Corollary. *Let $\mathscr{A}$ be a fibration over $\mathbf{S}$, and let $F : \mathscr{A} \longrightarrow \mathscr{B}$ be a weak equivalence functor, with $\mathscr{B}$ a stack over $\mathbf{S}$. Then, the pair $(\mathscr{B}, F)$ is the* stack completion *of $\mathscr{A}$ in the sense of satisfying the obvious universal property. Stack completions of a given $\mathscr{A}$ are unique up to equivalence.*

We next recall the construction [7] of the fibrational stack completion of any category object $\mathbb{C}$ in an elementary topos $\mathbf{S}$. The crucial result is Theorem 1.12 below, proved in [11].

Recall the following characterization of stacks, itself a corollary of a joint application of the following two basic theorems.

1. (Beck Monadicity Criterion) [2, 13]. If a fibration $\mathscr{A} \longrightarrow \mathbf{S}$ has Σ subject to the Beck-Chevalley condition (BCC), then for each epimorphism $e : J \longrightarrow I$ in $\mathbf{S}$, the canonical functor

$$\begin{tikzcd} \mathscr{A}^I \arrow[rr, "\Phi"] \arrow[dr, "e^*"'] & & (\mathscr{A}^J)^{\mathbb{T}_e} \arrow[dl, "U_{\mathbb{T}_e}"] \\ & \mathscr{A}^J & \end{tikzcd}$$

   where $\mathbb{T}_e$ is the monad on $\mathscr{A}^J$ induced by the adjoint pair $\Sigma_e \dashv e^*$, is an equivalence of categories.

2. (Monadicity and Descent Theorem) [4] For any fibration $\mathscr{A} \longrightarrow \mathbf{S}$ with Σ subject to the Beck-Chevalley condition (BCC), then for each morphism $e : J \longrightarrow I$ in $\mathbf{S}$, the canonical functor

$$\begin{tikzcd} \mathrm{Des}_{\mathscr{A}}(e) \arrow[rr, "\Psi"] \arrow[dr, "U"'] & & (\mathscr{A}^J)^{\mathbb{T}_e} \arrow[dl, "U_{\mathbb{T}_e}"] \\ & \mathscr{A}^J & \end{tikzcd}$$

   is an equivalence of categories.

1.11. Remark. The theorem of Bénabou and Roubaud can be stated directly as follows. Let $\mathscr{A}$ be a fibration over $\mathbf{S}$, with Σ subject to the Beck-Chevalley condition (BCC). Then $\mathscr{A}$ is a stack (over $\mathbf{S}$) if and only if for each epimorphism $e : J \twoheadrightarrow I$ in $\mathbf{S}$, the adjoint pair

$$\Sigma_e \dashv e^* : \mathscr{A}^I \longrightarrow \mathscr{A}^J$$

is monadic.

1.12. Corollary. *The fibration*

$$\mathrm{cod} : \mathbf{S}^{\rightarrow} \longrightarrow \mathbf{S}$$

is a stack.

Proof. An exact category is a regular category in which equivalence relations are kernel pairs. Any topos $\mathbf{S}$ is exact and so are all of its slices. For $e : J \longrightarrow I$ a morphism in $\mathbf{S}$, $e^* : \mathbf{S}/I \longrightarrow \mathbf{S}/J$ is exact [1]. Since there is a left adjoint $\Sigma_e \dashv e^*$ that satisfies the (BCC), we need only verify monadicity as follows from Remark 1.11. For e an epimorphism, e^* is faithful so it reflects isomorphisms. A refinement of Beck's Monadicity Criterion given in [12] applies and so the adjoint pair in question is indeed monadic. $\square$

1.13. THEOREM. *The (fibrational) stack completion of a category $\mathbb{C}$ in $\mathbf{S}$ is identified with the first factor in the factorization of* $\mathrm{yon} : \mathbb{C} \longrightarrow \mathbf{S}^{\mathbb{C}^{\mathrm{op}}}$ *given by*

$$[\mathbb{C}] \xrightarrow{\mathrm{yon}} \mathbf{LR}(\mathbf{S}^{\mathbb{C}^{\mathrm{op}}}) \hookrightarrow \mathbf{S}^{\mathbb{C}^{\mathrm{op}}}$$

where $[\mathbb{C}]$ *denotes the externalization of* $\mathbb{C}$*, and* $\mathbf{LR}(\mathbf{S}^{\mathbb{C}^{\mathrm{op}}})$ *denotes the full subcategory of* $\mathbf{S}^{\mathbb{C}^{\mathrm{op}}}$ *whose objects are the locally representable presheaves.*

Proof. The argument given in [7] uses Theorem 1.12 and the characterization of stacks in Proposition 1.9. It follows from these two that the fibration $\mathbf{S}^{\mathbb{C}^{\mathrm{op}}}$ is a stack for any category object $\mathbb{C}$ in $\mathbf{S}$. Further, the Yoneda embedding

$$\mathrm{yon} : \mathbb{C} \hookrightarrow \mathbf{S}^{\mathbb{C}^{\mathrm{op}}}$$

is already fully faithful, but it need not be essentially surjective. By cutting down to the full subcategory of locally representable objects, the first factor in the factorization

$$[\mathbb{C}] \xrightarrow{\mathrm{yon}} \mathbf{LR}(\mathbf{S}^{\mathbb{C}^{\mathrm{op}}}) \hookrightarrow \mathbf{S}^{\mathbb{C}^{\mathrm{op}}}$$

is not only fully faithful but also essentially surjective, hence a weak equivalence. It is easy to see that $\mathbf{LR}(\mathbf{S}^{\mathbb{C}^{\mathrm{op}}})$ is a stack, hence the stack completion of the category $\mathbb{C}$ in $\mathbf{S}$ (equivalently, of the representable fibration $[\mathbb{C}] \longrightarrow \mathbf{S}$). □

Denote by $\mathbf{Top}_{\mathbf{S}}$ the 2-category whose objects are $\mathbf{S}$-bounded toposes, its 1-cells are geometric morphisms over $\mathbf{S}$, and where the 2-cells are natural isomorphisms between (inverse image parts) of geometric morphisms. For any pair of $\mathbf{S}$-bounded toposes $\mathscr{E}$ and $\mathscr{F}$ there is then a category denoted $\mathbf{Top}_{\mathbf{S}}[\mathscr{E}, \mathscr{F}]$, whose 1-cells and 2-cells are as described above.

This structure is to be distinguished from that of the bicategory of $\mathbf{S}$-bounded toposes, whose 2-cells are arbitrary natural transformations subject to coherence conditions. We work with the former and not the latter since it is what is suitable in dealing with non-abelian topos cohomology, in particular, taking into account the meaning of a (locally trivial) covering [9, 10].

We end this review section with some important connections between stacks theory and Galois theory. We recall some applications of stack completions to first degree non-abelian cohomology of a topos from [8] and [9].

In [16], Grothendieck defines the *fundamental groupoid* $\Pi_1(\mathscr{G})$ of a Galois topos $\mathscr{E}$ to be the fibration $\mathbf{Points}(\mathscr{E})$ over Set. We have argued coincidentally elsewhere [9] that this is the correct choice (also) over an arbitrary base topos $\mathbf{S}$, whereas a natural candidate for the *Galois groupoid* of $\mathscr{E}$ is the groupoid $G = \mathrm{Aut}(A)$ where A is a universal cover in $\mathscr{E}$. The Galois topos $\mathscr{E}$ is the classifying topos $\mathscr{B}(G)$ of G, and the fundamental groupoid of $\mathscr{E}$ is then identified with the stack completion of the Galois groupoid G. We refer to [9] for a proof of the following result.

1.14. Theorem. *Let $\mathscr{E}$, $\mathscr{F}$ be Galois toposes bounded over* $\mathbf{S}$*, considered as classifying toposes of the (etale complete) discrete groupoids G, K. Then we have the following facts.*

1. There is an equivalence of categories

$$\mathrm{Hom}(G, K) \simeq \mathbf{Top}_{\mathbf{S}}[\mathscr{E}, \mathscr{F}]_{+}$$

where the symbol $+$ indicates commutation with the canonical (bags of) points.

2. There is an equivalence of categories

$$\mathrm{Hom}(G^{*}, K^{*}) \simeq \mathbf{Top}_{\mathbf{S}}[\mathscr{E}, \mathscr{F}]$$

where G^{*} and K^{*} are the (representable) stack completions of G and K, respectively.

By adding multiplicity (in the stack completions) one eliminates the dependence on the chosen points.

1.15. Proposition. *Let $\mathscr{E}$ be an* $\mathbf{S}$*-bounded topos. There is an* $\mathbf{S}$*-fibration* $\mathbf{Points}(\mathscr{E})$ *of points of $\mathscr{E}$. For G an* etale complete *groupoid in* $\mathbf{S}$*, the stack completion of G may be identified with*

$$[G] \xrightarrow{\Phi} \mathbf{Points}(\mathbf{S}^{G^{\mathrm{op}}})$$

Proof. This follows by restricting Proposition 3.1 and Theorem 3.2 of [8] to the discrete case. □

Recall that *1-dimensional cohomology* of $\mathbf{S}$ with coefficients in an etale complete groupoid $\mathbb{G}$ is given by the formula

$$\mathrm{H}^{1}(\mathbf{S}; G) = \Pi_{0}(\mathrm{Tors}^{1}(G))$$

where Π_0 denotes 'isomorphism classes'.

1.16. Proposition. *Let G be any non-empty and connected groupoid in* $\mathbf{S}$*. Then there is an equivalence*

$$\mathbf{LR}(\mathbf{S}^{G^{\mathrm{op}}}) \cong \mathrm{Tors}^{1}(G)$$

as $\mathbf{S}$*-fibrations.*

Proof. The proof can be easily adapted from that of [8] Proposition 4.9 in the discrete case. □

1.17. Theorem. *Let G be any etale complete groupoid in* $\mathbf{S}$*.*

1. *Then the canonical morphism*

$$G \xrightarrow{\text{triv}} \mathrm{Tors}^1(G)$$

exhibits $\mathrm{Tors}^1(G)$ *as the stack completion of* G*.*

2. *(Diaconescu's theorem [22]) Furthermore, the topos* $\mathbf{S}^{G^{\mathrm{op}}}$ *classifies G-torsors in the sense that there is an equivalence*

$$\mathrm{Tors}^1(G) \cong \mathbf{Points}(\mathbf{S}^{G^{\mathrm{op}}})$$

Proof. Theorem 1.13 applies to G. That is, we have that

$$[G] \xrightarrow{\text{yon}} \mathbf{LR}(\mathbf{S}^{G^{\mathrm{op}}})$$

is 'the' stack completion of G. We now apply Proposition 1.16 to prove the first assertion. The second assertion follows from the first assertion and Proposition 1.15. $\square$

2. Stack completions in Algebra and Analysis

In this section we apply the criterion from Corollary 1.10 directly in order to find examples of stack completions in mathematics in addition to those given at the end of the second section of [7], as well as the main examples in non-abelian cohomology.

We illustrate this first by deriving a result of [26] involving the Zariski topos **Zar** from a more basic theorem in Commutative Algebra, and we then use the fact that **Zar** classifies local rings in order to derive further examples, such as Kaplansky's theorem and Swan's theorem. Different conceptual proofs of these theorems were given by Mulvey [28].

2.1. Theorem. *[5] For a ring A and a finitely generated A-module M, M is projective if and only if there exists a finite family $(a_i \mid i \in I)$, with $a_i \in A$, $1 = \sum c_i a_1$ for some $c_i \in A$, such that for each $i \in I$, the A_{a_i}-module M_{a_i} is free of finite rank. (By A_a and M_a it is meant localization at the multiplicative subset $\{1, a, a^2, ...\} \subseteq A$.)*

2.2. Corollary. *[26] In the* Zariski topos **Zar***, for U the generic local ring, the canonical functor*

$$\alpha_U : \mathbb{F}_U \longrightarrow \mathbb{P}_U,$$

where $\mathbb{F}_U$ is the internal category of free U-modules of finite rank, and $\mathbb{P}_U$ is the internal category of finitely generated projective U-modules, is a weak equivalence.

Proof. It is a simple exercise to show that Theorem 2.1 in Commutative Algebra can be interpreted as claimed in the Zariski topos in terms of the given site for it. $\square$

2.3. Corollary. *(Kaplansky's theorem.) For a local ring L in Set, the canonical functor*

$$\alpha_L : \mathbb{F}_L \longrightarrow \mathbb{P}_L$$

is an equivalence.

Proof. This follows from the fact that $L = \varphi^*(U)$ for a (unique) geometric morphism $\varphi : \mathrm{Set} \longrightarrow \mathrm{Zar}$, that any inverse image part of a geometric morphism preserves weak equivalence functors, and that in any topos satisfying the axiom of choice, every weak equivalence is an equivalence. □

2.4. Corollary. *(Swan's theorem.) Let X be a paracompact topological space. Then, there is an equivalence between the category of* real vector bundles over X *and that of* finitely generated projective $\mathrm{Cont}(X, \mathbb{R})$-modules.

Proof. In $\mathrm{Sh}(X)$, with X paracompact, and $\mathscr{C}_\mathbb{R}$ the sheaf of germs of $\mathbb{R}$-valued continuous functions, we have the following commutative diagram:

$$\begin{tikzcd} \mathbb{F}_{\mathscr{C}_\mathbb{R}} \arrow[r, "\alpha_{\mathscr{C}_\mathbb{R}}"] \arrow[d, "F"'] & \mathbb{P}_{\mathscr{C}_\mathbb{R}} \arrow[d, "P"] \\ \widetilde{\mathbb{F}_{\mathscr{C}_\mathbb{R}}} \arrow[r, "\widetilde{\alpha_{\mathscr{C}_\mathbb{R}}}"'] & \widetilde{\mathbb{P}_{\mathscr{C}_\mathbb{R}}} \end{tikzcd}$$

where $\alpha_{\mathscr{C}_\mathbb{R}}$ is a wef (same argument as for Kaplansky's theorem), and where P and F are weak equivalence functors into the stack completions, so that the induced $\widetilde{\alpha_{\mathscr{C}_\mathbb{R}}}$ is also one by properties of weak equivalence functors but, between stacks, any weak equivalence is an equivalence. In view of classical theorems from Analysis, this equivalence translates in turn into the statement that there is an equivalence between the categories of *real vector bundles over* X and that of *finitely generated projective* $\mathrm{Cont}(X, \mathbb{R})$*-modules.* □

2.5. Remark. A different proof of Swan's theorem is given in [28]. The strategy is to obtain first a constructive proof of Kaplansky's theorem and then interpreting it in the topos $\mathrm{Sh}(X)$, with X paracompact, and with $L = \mathscr{C}_\mathbb{R}$ the sheaf of germs of $\mathbb{R}$-valued continuous functions.

3. The Isomorphism Lifting Property

Joyal and Tierney [20] have shown that, for $\mathbf{S}$ a Grothendieck topos, there is a Quillen Model structure [31, 21] on $\mathbf{Cat}(\mathbf{S})$ whose weak equivalences are the weak equivalence functors, whose cofibrations are the internal functors injective on objects, and whose fibrations are those functors with the right lifting property with respect to all trivial cofibrations. The fibrant objects are called 'strong stacks' and are shown to be stacks via a cone construction applied to a weak equivalence functor. In particular, this development provides a second justification for the assertion in Proposition 3.2.

It follows that any category object $\mathbb{C}$ in a Grothendieck topos $\mathbf{S}$ has an internal stack completion. Factor the unique functor $\mathbb{C} \longrightarrow \mathbf{1}$ into

$$\mathbb{C} \longrightarrow \mathbb{C}^* \longrightarrow \mathbf{1}$$

where $\mathbb{C} \longrightarrow \mathbb{C}^*$ is a trivial cofibration and $\mathbb{C}^* \longrightarrow \mathbf{1}$ is a fibration. Then,

$$\mathbb{C} \longrightarrow \mathbb{C}^*$$

is the strong stack completion of $\mathbb{C}$, in particular, the stack completion of $\mathbb{C}$ in $\mathbf{Cat}(\mathbf{S})$.

The stack completion of a category object in an elementary topos $\mathbf{S}$ always exists as a fibration, but it need not be representable if $\mathbf{S}$ is not a Grothendieck topos. Notice also that the Grothendieck topos assumption is crucial for the Quillen model structure on $\mathbf{Cat}(\mathbf{S})$ to exist, more precisely, because of the 'small object argument'.

3.1. Definition. Let $\mathbf{S}$ be an elementary topos. We say that an elementary topos $\mathbf{S}$ satisfies the *axiom of stack completions* (or *(ASC)* for short) if, for any a category object $\mathbb{C}$ in $\mathbf{S}$, there there is a representable stack completion of the fibration $[\mathbb{C}] \longrightarrow \mathbf{S}$ over $\mathbf{S}$.

3.2. Proposition. *Any Grothendieck topos* $\mathbf{S}$ *satisfies the axiom of stack completions.*

Proof. That the fibrational stack completion of the externalization of any category object $\mathbb{C}$ in $\mathbf{S}$ is representable can be shown in the style of [22] (Lemma 8.35) using the existence of a generating set (see also [13]). □

Another Quillen Model structure on $\mathbf{Cat}(\mathbf{S})$ is that of lack [23] whose fibrations are those morphisms with the isomorphism lifting propety. It has nothing to do with stacks in $\mathbf{Cat}(\mathbf{S})$, but it provides examples of stacks in $\mathbf{Cat}(\mathbf{S})^2$, as constructed already in [11].

3.3. Definition. A functor $F : \mathbb{A} \longrightarrow \mathbb{B}$ in $\mathbf{Cat}(\mathbf{S})$ is said to satisfy the *isomorphism lifting property* (*(ILP)* for short), if the following holds.

- For every object A in $\mathbb{A}$, and every isomorphism $B \cong FA$ in $\mathbb{B}$ consisting of $b : B \longrightarrow FA$ and $b' : FA \longrightarrow B$ such that $bb' = 1$ and $b'b = 1$, there is given an isomorphism $A' \cong A$ in $\mathbb{A}$ consisting of $a : A' \longrightarrow A$ and $a' : A \longrightarrow A'$ such that $aa' = 1$ and $a'a = 1$, with $F(A') = B$, $Fa = b$, and $F(a') = b'$. (The indicated liftings are unique up to iso, hence cartesian.)

3.4. Remark. The stability of wef in $\mathbf{Cat}(\mathbf{S})$ under 2-pullbacks (or iso-comma-objects), shown directly in [11], can alternatively be seen as a consequence a more basic fact, to wit, stability under pullback along any functor with the (ILP). In the diagram below, the projection $\tilde{G} : \mathbb{B}||G \longrightarrow \mathbb{B}$ is the *free (ILP) functor* on G. The left square is the usual pullback of $\tilde{G}$ along F.

$$\begin{tikzcd}
\mathbb{A}' \arrow[r, "\tilde{G}^* F"] \arrow[d, "G'"'] & \mathbb{B}||G \arrow[r, "H"] \arrow[d, "\tilde{G}"] \arrow[dr, phantom, "\Rightarrow"] & \mathbb{B}' \arrow[d, "G"] \\
\mathbb{A} \arrow[r, "F"'] & \mathbb{B} \arrow[r, "\mathrm{id}"'] & \mathbb{B}
\end{tikzcd}$$

We know from [34] that the other projection $H : \mathbb{B}||G \longrightarrow \mathbb{B}'$ has a left adjoint left inverse, in fact, a pseudoinverse K, so that H is an adjoint equivalence. If F is a wef, then, since $\tilde{G}$ has the (ILP), $\tilde{G}^* F$ is a wef, and therefore the composite $F' = G^*(F) = H \cdot \tilde{G}^* F$ is a wef.

3.5. Lemma. *[23] A functor $F : \mathbb{A} \longrightarrow \mathbb{B}$ in $\mathbf{Cat}(\mathbf{S})$ satisfies the (ILP) iff it has the right lifting property with respect to the trivial cofibration $j : \mathbf{1} \longrightarrow \mathbf{I}$, with $\mathbf{I}$ the 'free-living isomorphism' .*

3.6. Remark. As shown in [11] as a consequence of Corollary 1.10, a functor $F : \mathbb{A} \longrightarrow \mathbb{B}$ in *Set* satisfying the (ILP) is a stack as a category object in the topos $\mathit{Set}^{\mathbf{2}}$. Furthermore, for an arbitrary category object $F : \mathbb{A} \longrightarrow \mathbb{B}$ in $\mathit{Set}^{\mathbf{2}}$, its stack completion exists internally and is given by the *free* (ILP) functor on F. Indeed, in the commutative diagram

$$\begin{tikzcd}
\mathbb{A} \arrow[r, "K"] \arrow[d, "F"'] & \mathbb{B}||F \arrow[d, "\tilde{F}"] \\
\mathbb{B} \arrow[r, "\mathrm{id}"'] & \mathbb{B}
\end{tikzcd}$$

we have $\tilde{F} : \mathbb{B}||F \longrightarrow \mathbb{B}$ an isofibration, and both K and id are equivalence functors. The pair $(K, \mathrm{id}) : F \longrightarrow \tilde{F}$ is a wef in $\mathbf{Cat}^{\mathbf{2}}$ and so, by Corollary 1.10, it exhibits $\tilde{F}$ as the stack completion of F.

We now give an example of a stack completion that shows in particular that for an arbitrary topos $\mathbf{S}$ not every internal category in it is a stack.

3.7. Example. This example is an instance of Remark 3.6. Let $\mathbf{S} = \mathit{Set}^{\mathbf{2}}$. Let G be a group (in *Set*). Consider the category object in $\mathbf{S}$ given by the functor

$$H : 1 \longrightarrow G$$

which picks up the unit element e of G, that is, $H(0) = e$.

We claim that $\mathbb{C}$ is not a stack in $\mathbf{S}$. The best way to illustrate this is to exhibit its stack completion $\tilde{\mathbb{C}}$ given by

$$\tilde{H} : \mathrm{Ind}(G) \longrightarrow G$$

where $\mathrm{Ind}(G)$ has as objects all elements $g \in G$, identified with triplets $(0,0,g)$, and as morphisms $k : (0,0,g) \longrightarrow (0,0,h)$ any element $k \in G$ such that $g = kh$ in G, that is $k = g.h^{-1}$. The functor $\tilde{H}$ is defined by $\tilde{H}(0,0,g) = 0$ amd $\tilde{H}(k) = k$. We have a commutative square

$$\begin{tikzcd} 1 \arrow[r, "\Phi_0"] \arrow[d, "H"'] & \mathrm{Ind}(G) \arrow[d, "\tilde{H}"] \\ G \arrow[r, "\Phi_1"'] & G \end{tikzcd}$$

where Φ_1 is the identity functor, and $\Phi_1(0) = (0,0,e)$. Notice that both Φ_0 and Φ_1 are equivalences in *Set*, yet the pair (Φ_0, Φ_1) is not an equivalence.

3.8. Remarks. The fibrant objects of the Quillen Model structure of [20] are those functors of the form $\mathbb{C} \longrightarrow \mathbf{1}$ with the right lifting property with respect to all trivial cofibrations. These are in turn identified with the category objects $\mathbb{C}$ which are stacks.

Contrast this with the notion of a category object $\mathbb{C}$ such that $\mathbb{C} \longrightarrow \mathbf{1}$ has the right lifting property with respect to the trivial cofibration $j : \mathbf{1} \longrightarrow \mathbb{I}$, where $\mathbb{I}$ is the free-living isomorphism, that is, such that $\mathbb{C} \longrightarrow \mathbf{1}$ satisfies (ILP).

For $\mathbf{S}$ an arbitrary Grothendieck topos $\mathbf{S}$, although every internal functor $\mathbb{C} \longrightarrow \mathbf{1}$ satisfies the (ILP), not every such functor is a fibration for the Quillen model structure on $\mathbf{Cat}(\mathbf{S})$ of [20] – equivalently, not every category object $\mathbb{C}$ in $\mathbf{S}$ is a stack. As argued in [11], a topos $\mathbf{S}$ satisfies the axiom of choice if and only if every internal category in it is a stack.

One may ask then what is the precise connection between the Quillen model structures of [20] and of [23]. It is clear from the above discussion that every fibration of [20] is also one for [23], that is, it satisfies the (ILP). This can also be inferred from the fact that orthogonality with respect to every trivial cofibration of [20] is in particular orthogonal to the trivial cofibration $j : \mathbf{1} \longrightarrow \mathbb{I}$ and from the possibility in principle to find trivial cofibrations other than $j : \mathbf{1} \longrightarrow \mathbb{I}$ that would force an additional injectivity requirement that is non-trivial without Choice. For instance, for any monomorphism $m : X \rightarrowtail Y$ with $X \twoheadrightarrow 1$, the codiscrete groupoid functor $G(m) : G(X) \longrightarrow G(Y)$ is one such trivial cofibration.

It is claimed in [20] that the given Quillen model structure on $\mathbf{Cat}(\mathbf{S})$, for $\mathbf{S}$ a Grothendieck topos, is cofibrantly generated. The above shows that it is not the case that in general $j : \mathbf{1} \longrightarrow \mathbb{I}$, where $\mathbb{I}$ is the free-living isomorphism, can be the single generating trivial cofibration. An explicit description of a generating set of trivial cofibrations is not, however, given therein. A general argument seems to be the only way to assert this fact.

4. Intrinsic 2-stacks

Let $\mathbf{S}$ be an elementary topos. When passing from the 0-level to level 1, we should note that the trivial identifications $\mathbf{Cat}^0(\mathbf{S}) = \mathbf{S} = \mathbf{Stack}^0(\mathbf{S})$ get differentiated. We shall denote by $\mathbf{Stack}(\mathbf{S})$ the full sub 2-category

$$\mathbf{Stack} \hookrightarrow \mathbf{Cat}$$

whose objects are category objects in $\mathbf{S}$ which are (intrinsic) 1-stacks. We omit the superscript 1 in this case. More generally we want an inclusion of fibrations.

4.1. Definition. A 2-functor $P : F \longrightarrow E$ is a 2-*fibration* if:

- for any object X in F and any 1-cell $u : I \longrightarrow PX$ in E, there is a 2-cartesian 1-cell $\bar{u} : u^*(X) \longrightarrow X$ with $P\bar{u} = u$.
- For any pair of objects X, Y in F, the induced functor $P_{X,Y} : F(X,Y) \longrightarrow E(PX, PY)$ is a fibration, stable under precomposition: for every 1-cell $h : Z \longrightarrow X$ in F, the functor $F(h, Y) : F(X,Y) \longrightarrow F(Z,Y)$ preserves 2-cartesian morphisms (from $P_{X,Y}$ to $P_{Z,Y}$).

4.2. Definition. Let $\mathrm{cod} : \mathbf{Stack}^{\rightarrow} \longrightarrow \mathbf{S}$ be given by the following data.

1. The fiber $\mathbf{Stack}^I$ above an object I of $\mathbf{S}$ is given by the 2-category $\mathbf{Stack}/\mathbb{I}_{dis}$,

   - whose *objects* are pairs $\langle \mathbb{C}, P\rangle$, where $\mathbb{C}$ is a category object that is a stack in $\mathbf{S}$ and $P : \mathbb{C} \longrightarrow \mathbb{I}_{dis}$ a functor (necessarily both a fibration and a cofibration since $\mathbb{I}_{dis}$ is discrete),
   - whose *morphisms* are functors F which fit into a commutative diagram

$$\begin{tikzcd} \mathbb{C} \arrow[rr, "F"] \arrow[dr, "P"'] & & \mathbb{D} \arrow[dl, "Q"] \\ & \mathbb{I}_{\mathrm{dis}} & \end{tikzcd}$$

   - and whose *2-cells* are natural transformations $\alpha : F \Rightarrow F' : \mathbb{C} \longrightarrow \mathbb{D}$ over $\mathbb{I}_{\mathrm{dis}}$.

2. Change of base along a morphism $u : J \longrightarrow I$ in $\mathbf{S}$ is given by the 2-functor

$$u^* : \mathbf{Stack}/\mathbb{I}_{dis} \longrightarrow \mathbf{Stack}/\mathbb{J}_{dis},$$

   pullback in $\mathbf{Cat}$ along the functor $u_{dis} : \mathbb{J}_{dis} \longrightarrow \mathbb{I}_{dis}$.

4.3. Proposition. *With the data as in Definition 4.2,* cod : $\mathbf{Stack}^{\rightarrow} \longrightarrow \mathbf{S}$ *is a 2-fibration.*

Proof. The proof is straightforward. □

Given a morphism $e : J \longrightarrow I$ in $\mathbf{S}$, its 2-kernel is the 2-groupoid given by the 3-truncated simplicial complex partially depicted by the following diagram whose horizontal right arrows are labelled d with subscripts, and whose horizontal left arrows are labelled s with subscripts.

$$\begin{tikzcd} J \times_I J \times_I J \times_I J \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & J \times_I J \times_I J \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & J \times_I J \arrow[r, shift left=2] \arrow[r, shift right=2] & J \arrow[l] \end{tikzcd} .$$

4.4. Definition. Let $\mathscr{A}$ be a 2-fibration. We denote by 2-Des$_{\mathscr{A}}(e)$ the 2-fibration of *2-descent data* on $\mathscr{A}$ with respect to $e : J \longrightarrow I$, a morphism in $\mathbf{S}$.

- An *object* of 2-Des$_{\mathscr{A}}(e)$ is a 4-tuple

$$(a, f, \gamma, \alpha)$$

with a an object of $\mathscr{A}^J$, together with an equivalence $f : {d_0}^*(a) \simeq {d_1}^*(a)$, and isomorphisms $\gamma : \mathrm{id}_a \cong {s_0}^*(f)$ and $\alpha : {d_1}^*(f) \cong {d_0}^*(f){d_2}^*(f)$ satisfying the coherence conditions ${d_0}^*(\gamma).f = {s_1}^*(\alpha)$, $f.{d_1}^*(\gamma) = {s_0}^*(\alpha)$.

- A *1-cell*

$$(a, f, \gamma, \alpha) \longrightarrow (b, g, \eta, \beta)$$

consists of a 1-cell $h : a \longrightarrow b$, and a 2-cell $\theta : {d_0}^*(a).f \Rightarrow g.{d_1}^*(h)$, satisfying ${s_0}^*(\theta).\gamma = \eta.h$ and the commutative tetrahedron condition (see [19]).

- A *2-cell*

$$(h, \theta) \Rightarrow (k, \kappa)$$

is given by a 2-cell $\varphi : h \Rightarrow k$ such that ${d_0}^*(\varphi).\kappa = \theta.{d_1}^*(\varphi)$.

4.5. Definition.

- Let $\mathbf{S}$ be an elementary topos. A 2-fibration $\mathscr{A} \longrightarrow \mathbf{S}$ is said to be an intrinsic *2-stack* if

  1. given any I-indexed family $\{J_i \mid i \in I\}$ of objects of $\mathbf{S}$, the canonical 2-functor

  $$\mathscr{A}^{\sqcup_{i\in I} J_i} \longrightarrow \textstyle\prod_{i\in I} \mathscr{A}^{J_i}$$

  is an equivalence of 2-categories, and

  2. for every epimorphism $e : J \longrightarrow I$ in $\mathbf{S}$, the 2-functor $e^* : \mathscr{A}^I \longrightarrow \mathscr{A}^J$ is of effective 2-descent.

- Let $\mathscr{A} \longrightarrow \mathbf{S}$ be a 2-fibration. Then, the 2-*stack completion* of $\mathscr{A}$ (if it exists) is given a pair $(\tilde{\mathscr{A}}, F)$, where $\tilde{\mathscr{A}} \longrightarrow \mathbf{S}$ is a 2-stack, and $F : \mathscr{A} \longrightarrow \tilde{\mathscr{A}}$ is a morphism of 2-fibrations, satisfying the obvious universal property among such pairs.

4.6. Definition. Let $F : \mathscr{B} \longrightarrow \mathscr{C}$ be a 2-functor between 2-fibrations over $\mathbf{S}$. It is said to be a *weak 2-equivalence* if the following conditions hold.

1. For each $I \in \mathbf{S}$, $F^I : \mathscr{B}^I \longrightarrow \mathscr{C}^I$, and $c \in \mathscr{C}^I$, there exists an epimorphism $e : J \longrightarrow I$ in $\mathbf{S}$, $b \in \mathscr{B}^J$, and an equivalence $\theta : F^J(b) \longrightarrow e^*(c)$.

2. $\forall I \in \mathbf{S}\ \forall b, b' \in \mathscr{B}^I$, the morphism

$$\mathrm{Hom}_{\mathscr{B}^I}(b, b') \xrightarrow{{F^I}_{b,b'}} \mathrm{Hom}_{\mathscr{C}^I}(F^I b, F^I b')$$

   of 1-fibrations is a weak 1-equivalence functor.

4.7. Proposition. *Let $\mathscr{A}$ be a 2-fibration over $\mathbf{S}$. Then, $\mathscr{A}$ is a 2-stack if and only if the internal 2-functor $F_e : {\mathbb{J}^{(2)}}_e \longrightarrow {\mathbb{I}^{(2)}}_{\mathrm{dis}}$, from the 2-kernel groupoid of e to the discrete 2-category on I, induces an equivalence of 2-categories*

$$\mathscr{A}^{F_e} : \mathscr{A}^{{\mathbb{I}^{(2)}}_{\mathrm{dis}}} \longrightarrow \mathscr{A}^{{\mathbb{J}^{(2)}}_e}$$

Proof. The diagram

$$\begin{tikzcd}
\mathscr{A}^I \arrow[r, "\Phi_e"] \arrow[d, "\kappa_I"'] & \text{2-Des}_{\mathscr{A}}(e) \arrow[d, "\Psi_e"] \\
\mathscr{A}^{{\mathbb{I}^{(2)}}_{\mathrm{dis}}} \arrow[r, "\mathscr{A}^{F_e}"'] & \mathscr{A}^{{\mathbb{J}^{(2)}}_e}
\end{tikzcd}$$

is commutative, with κ_I a 2-equivalence. That Ψ_e is a 2-equivalence is shown by an argument analogous to that employed in the 1-dimensional case (exercise). Therefore, for any epimorphism $e : J \longrightarrow I$ in $\mathbf{S}$, Φ_e is a 2-equivalence (hence $\mathscr{A}$ is a 2-stack), if and only of $\mathscr{A}^{F_e}$ is a 2-equivalence, as claimed. ▫

4.8. Lemma. *For any epimorphism e in $\mathbf{S}$, the internal 2-functor $F_e : {\mathbb{J}^{(2)}}_e \longrightarrow {\mathbb{I}^{(2)}}_{\mathrm{dis}}$, from the 2-kernel groupoid of e to the discrete 2-category on I, is a weak 2-equivalence functor.*

Proof. The first condition in Definition 4.6 is obviously satisfied since $e : J \longrightarrow I$ is an epimorphism. The second condition requires that

$$\mathrm{HOM}_{{\mathbb{J}^{(2)}}_e}(x, x') \xrightarrow{(F_e)x,x'} \mathrm{HOM}_{{\mathbb{I}^{(2)}}_{\mathrm{dis}}}(Fx, Fx')$$

be a weak 1-equivalence of categories. This is trivial since ${\mathbb{I}^{(2)}}_{\mathrm{dis}}$ is a discrete 2-category. ▫

4.9. THEOREM. *A 2-fibration $\mathscr{A}$ over $\mathbf{S}$ is a 2-stack iff for every weak 2-equivalence $\Gamma : \mathscr{B} \longrightarrow \mathscr{C}$, the induced $\mathscr{A}^\Gamma : \mathscr{A}^\mathscr{C} \longrightarrow \mathscr{A}^\mathscr{B}$ is an equivalence of 2-fibrations.*

Proof. It follows from Proposition 4.7 and Lemma 4.8 that the condition of the theorem is sufficient. We now prove necessity. Let $\mathscr{A}$ be an intrinsic 2-stack (Definition 4.5), and $\Gamma : \mathscr{B} \longrightarrow \mathscr{C}$ a weak 2-equivalence (Definition 4.6).

Our goal is to define a 2-functor

$$H : \mathscr{A}^\mathscr{B} \longrightarrow \mathscr{A}^\mathscr{C}$$

that is 'inverse' to $G = \mathscr{A}^\Gamma : \mathscr{A}^\mathscr{C} \longrightarrow \mathscr{A}^\mathscr{B}$. Since all concepts involved are stable under localization, it will be enough to argue with the fibers over $1 \in \mathbf{S}$.

1. Let $\Phi : \mathscr{B} \longrightarrow \mathscr{A}$ be a 2-functor. In what follows we define the data for a 2-functor $\Psi : \mathscr{C} \longrightarrow \mathscr{A}$, and let $H(\Phi)) = \Psi$. We then extend the definition to 1-cells and 2-cells.

   - Let $I \in \mathbf{S}$ and $c \in \mathscr{C}^I$. By the first condition on the weak 2-equivalence $\Gamma : \mathscr{B} \longrightarrow \mathscr{C}$, there is an epimorphism $e : J \longrightarrow I$ in $\mathbf{S}$, $b \in \mathscr{B}^J$, and an equivalence $\theta : \Gamma^J(b) \cong e^\star(c)$ in $\mathscr{C}^J$. Since $e^\star(c)$ has 2-descent data for e (see Definition 4.4), then so does $\Gamma^J(b)$.

     Simplify the notation for the 2-kernel of e as the following 3-truncated simplicial complex:

     $$J'''' \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}\; J'' \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; J' \;\substack{\longrightarrow\\\longleftarrow\\\longrightarrow}\; J\,.$$

     Since $\Gamma : \mathscr{B} \longrightarrow \mathscr{C}$ is a weak 2-equivalence, all three of the $\Gamma^{J'}$, $\Gamma^{J''}$ and $\Gamma^{J''''}$ are fully faithful by condition 2 in Definition 4.6. Since $\Gamma^J(b)$ has 2-descent data for e, so does b by the above observation. Therefore, also $\Phi^J(b)$ has 2-descent data for e. Since $\mathscr{A}$ is a 2-stack, there exists $a \in \mathscr{A}^I$ such that $e^\star(a) \cong \Phi^J(b)$. The choice of a is unique up to equivalence compatible with 2-descent data. If we choose some other e and b, by taking a common refinement, we see that there will be a unique equivalence between the new a and the old one, compatible with the 2-descent data. So, choose one such a and let $\Psi^I(c) = a$.

   - If $h : c \longrightarrow c'$ is a 1-cell in $\mathscr{C}^I$ then, by choosing a common refinement, we may assume that the same epimorphism $e : J \longrightarrow I$ works for both c and c'. Then there exist $b, b' \in \mathscr{B}^J$ and equivalences $\theta : \Gamma^J(b) \cong e^\star(c)$ and $\theta' : \Gamma^J(b') \cong e^\star(c')$. Using that

     $$\mathrm{Hom}_{\mathscr{B}^J}(b, b') \xrightarrow{\Gamma^J{}_{b,b'}} \mathrm{Hom}_{\mathscr{C}^J}(\Gamma^J(b), \Gamma^J(b'))$$

is a (1-)weak equivalence functor, and in particular full and faithful, there exists a unique $g : b \longrightarrow b'$ such that the diagram

$$\begin{tikzcd} \Gamma^J(b) \arrow[r, "\theta"] \arrow[d, "\Gamma^J(g)"'] & e^\star(c) \arrow[d, "e^\star(h)"] \\ \Gamma^J(b') \arrow[r, "\theta'"'] & e^*(c') \end{tikzcd}$$

commutes (up to equivalence) with the horizontal arrows the given equivalences in the 2-category $\mathscr{C}^J$. Further, g is compatible with the 2-descent data on b and b', so that

$$\Phi^J(g) : \Phi^J(b) \longrightarrow \Phi^J(b')$$

is also compatible with the corresponding 2-descent data. Therefore there exists a unique $f : a \longrightarrow a'$ such that

$$\begin{tikzcd} \Phi^J(b) \arrow[r] \arrow[d, "\Phi^J(g)"'] & e^\star(a) \arrow[d, "e^\star(f)"] \\ \Phi^J(b') \arrow[r] & e^*(a') \end{tikzcd}$$

commutes (up to equivalence). Define $\Psi^I(h) = f$.

- For a 2-cell $\gamma : h \Rightarrow h'$, get $\beta : g \Rightarrow g'$ and then $\alpha : f \Rightarrow f'$, by arguments of the sort employed above. Define $\Psi^I(\gamma) = \alpha$. It is a routine verification to show that Ψ is a 2-functor. We let $H(\Phi) = \Psi$.
- In a similar manner we define $H(t)$ for a 1-cell $t : \Phi \longrightarrow \Phi'$, and $H(\delta)$ for a 2-cell $\delta : t \Rightarrow t'$. Verify that with these definitions H is a morphism of 2-fibrations.

2. Claim. $(G \cdot H) \cong 1_{\mathscr{A}\mathscr{B}}$. By construction,

$$(G \cdot H)(\Phi) = \mathscr{A}^\Gamma(H(\Phi)) = \mathscr{A}^\Gamma(\Psi) = \Psi\Gamma.$$

Let $I \in \mathbf{S}$ and $b \in \mathscr{B}^I$. To calculate $\Psi^I\Gamma^I(b)$ as above, we may chose $e = 1_I : I \longrightarrow I$ since $\Gamma^I(b) = (1_I)^\star\Gamma^I(b)$. Then, $\Psi^I\gamma^I(b) = a$ for some $a \in \mathscr{A}^I$ such that ${1_I}^\star(a) \cong \Phi^I(b)$, thus $\Psi^I\Gamma^I(b) \cong \Phi^I(b)$. We argue in a similar manner for 1-cells $g : b \longrightarrow b'$ and for 2-cells $t : g \Rightarrow g'$. The conclusion reached says that

$$\Psi\Gamma \cong \Phi$$

hence (part of) the claim. One has to complete this with arguments involving 1-cells $\alpha : \Phi \longrightarrow \Phi'$ and 2-cells $s : \alpha \Rightarrow \alpha' : \Phi \longrightarrow \Phi'$. This is lenghtly but entirely routine.

3. Claim. $(H \cdot G) \cong 1_{\mathscr{A}^{\mathscr{C}}}$. Consider any 2-functor $\Psi' : \mathscr{C} \longrightarrow \mathscr{A}$ such that $H(\mathscr{A}^\Gamma(\Psi')) = \Psi$. This time, to calculate $\Psi^I(c)$ we choose, as above, an epimorphism $e : J \longrightarrow I$ and $b \in \mathscr{B}^I$ such that $\Gamma^J(b) \cong e^\star(c)$, and then find the 'unique' $a \in \mathscr{A}^J$ such that $\Phi'^J\Gamma^J(b) \cong e^\star(a)$. But
$$\Psi'^J\Gamma^J(b) \cong \Psi'^J(e^\star(c)) \cong e^\star\Psi'^I(c)$$
so $\Psi^I(c) \cong \Psi'^I(c)$. This equivalence is natural, so $\Psi \cong \Psi'$. This proves the claim.

Hence, $\mathscr{A}^\Gamma$ is an equivalence. Modulo the many routine details left to the patient reader, this finishes the proof.

□

4.10. Corollary. *If $F : \mathscr{A} \longrightarrow \mathscr{B}$ is a morphism of 2-fibrations that is a weak 2-equivalence, and $\mathscr{B}$ is a 2-stack, then the pair $(\mathscr{B}, F)$ is the 2-stack completion of $\mathscr{A}$.*

5. **Stack** is a 2-stack

In the construction [7] of the fibrational 1-stack completion of a category object in an elementary topos $\mathbf{S}$, a crucial step was to prove that the fibration cod : $\mathbf{S}^{\rightarrow} \longrightarrow \mathbf{S}$ is a 1-stack [11]. Recall that this was shown using the monadicity and descent theorems of [2] and [4].

It is our aim to construct the 2-stack completion of a 2-category object $\mathbf{C}$ in $\mathbf{S}$, possibly with certain assumptions on $\mathbf{C}$. By analogy with the case of dimension 1, we first show that cod : $\mathbf{Stack}^{\rightarrow} \longrightarrow \mathbf{S}$ is a 2-stack.

In the case of *groupoid stacks*, a direct proof of this fact is given in [29], but only for the fibers over 1. We shall, instead, resort to 2-dimensional analogues of the monadicity and descent theorems of [19] and prove the full 2-fibrational result. We assume, just as in [29], that $\mathbf{S}$ is a Grothendieck topos or, more generally, that $\mathbf{S}$ is an elementary topos satisfying the axiom of 1-stack completions.

We begin by recalling some definitions and theorems from [19].

5.1. Definition. *[19]* A 2-fibration $P : \mathscr{A} \longrightarrow \mathbf{Cat}$ is said *to have* Σ *with the BCC for comma objects* if for every $u : \mathbb{B} \longrightarrow \mathbb{A}$ in $\mathbf{Cat}$, the change of base $u^* : \mathscr{A}^{\mathbb{A}} \longrightarrow \mathscr{A}^{\mathbb{B}}$ admits a left 2-adjoint $\Sigma_u \dashv u^*$ such that, for every comma square

$$\begin{tikzcd}
(v \downarrow u) \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\Rightarrow \lambda"] & \mathbb{B} \arrow[d, "u"] \\
\mathbb{K} \arrow[r, "v"'] & \mathbb{A}
\end{tikzcd}$$

in $\mathbf{Cat}$, the induced
$$\widetilde{\lambda} : \Sigma_p \cdot q^* \Rightarrow v^* \cdot \Sigma_u$$
is an equivalence.

5.2. Theorem. *[19] If a 2-fibration $P : \mathscr{A} \longrightarrow$ **Cat** has Σ subject to the BCC for comma objects in* **Cat** *along a cofibration, then, given $q : \mathbb{T} \longrightarrow \mathbb{Q}$ in* **Cat**, *there is a canonical biequivalence*

$$\text{2-Des}_{\mathscr{A}}(e)_q(\mathscr{A}) \longrightarrow \mathrm{Ps}(q^*\Sigma_q)\mathrm{Alg}$$

where $\mathrm{Ps}(q^*\Sigma_q)\mathrm{Alg}$ *is the 2-category of pseudo-algebras of the pseudo-monad $q^*\Sigma$, induced by q.*

5.3. Definition. *[19]* A functor $q : \mathbb{O} \longrightarrow \mathbb{Q}$ in Cat is said to be a *regular 2-epi* if the comma object

$$\begin{tikzcd}
(q \downarrow q) \arrow[r, "d"] \arrow[d, "c"'] \arrow[dr, phantom, "\Rightarrow \lambda"] & \mathbb{O} \arrow[d, "q"] \\
\mathbb{O} \arrow[r, "q"'] & \mathbb{Q}
\end{tikzcd}$$

exhibits (q, λ) as the representable Kleisi object of the bimodule monad $(d, c) : \mathbb{O} \rightharpoonup \mathbb{Q}$.

5.4. Remark. *The standard factorization of a functor in* **Cat** *is the one into a regular 2-epi followed by a fully faithful functor. Using an explicit construction of the representable Kleisli object of a monad in bimodules in* $\mathbf{Cat}(\mathbf{S})$ *[18], one can identify the ingredients of this factorization for any (regular) epi $e : J \longrightarrow I$ in* $\mathbf{S}$, *regarded as $e_{\text{dis}} : \mathbb{J}_{\text{dis}} \longrightarrow \mathbb{I}_{\text{dis}}$ in* **Cat**, *into a regular 2-epi $\bar{e}$ followed by a fully faithful functor E*

$$\begin{tikzcd}
\mathbb{J}_{\text{dis}} \arrow[rr, "\bar{e}"] \arrow[dr, "e_{dis}"'] & & \mathbb{J}_e \arrow[dl, "E"] \\
& \mathbb{I}_{\text{dis}} &
\end{tikzcd}$$

where $E : \mathbb{J}_e \longrightarrow \mathbb{I}_{\text{dis}}$ is the 1-kernel of e. The regular 2-epi $\bar{e} : \mathbb{J}_{\text{dis}} \longrightarrow \mathbb{J}_e$ is the left square in the diagram

$$\begin{tikzcd}
J \arrow[r, equal] \arrow[d, "\text{diag}"'] & J \arrow[r, equal] \arrow[d, shift left] \arrow[d, shift right] & J \arrow[d, "e"] \\
J \times_I J \arrow[r, "\pi"] & J \arrow[r, "e"] & I
\end{tikzcd}$$

so that it is the identity at the level of objects and the diagonal at the level of morphisms.

Just as a regular epi (i.e., the coequalizer if its own kernel pair) in any regular category $\mathbf{R}$ is of effective 1-descent, a regular 2-epi in any strongly 2-regular 2-category $\mathscr{R}$ (in the sense of [19]) is of effective 2-descent. We state a particular case of this result, using that **Cat** is a strongly regular 2-category.

5.5. Theorem. *[19] Any regular 2-epi in* **Cat** *is of effective 2-descent for the basic 2-fibration*

$$cod : \mathbf{Fib} \longrightarrow \mathbf{Cat}.$$

5.6. Proposition. *Let* **S** *be a Grothendieck topos. Then* **Stack** *is a reflective subcategory of* **Cat**. *In particular, it has all colimits that exist in* **Cat**.

Proof. This is a consequence of the existence of representable stack completions of category objects in any Grothendieck topos (Proposition 3.2). The colimit of a diagram in **Stack** is the stack completion of the colimit of the diagram regarded in **Cat**. ▫

5.7. Theorem. *Let* **S** *be a Grothendieck topos. Then, the 2-fibration* cod : $\mathbf{Stack}^{\rightarrow} \longrightarrow \mathbf{S}$ *is an intrinsic 2-stack. Equivalently, any regular epi in* **S** *is of effective 2-descent for* cod : $\mathbf{Stack}^{\rightarrow} \longrightarrow \mathbf{S}$.

Proof. Since **S** is assumed to be a Grothendieck topos, the 2-category **Stack** has all bicolimits that exist in $\mathbf{Cat}(\mathbf{S})$, by Proposition 5.6. The same is true of the fiber of **Stack** at any I (not just at 1) and, for any $u : J \longrightarrow I$, $u^* : \mathbf{Stack}/\mathbb{I}_{dis} \longrightarrow \mathbf{Stack}/\mathbb{J}_{dis}$ preserves such bicolimits (in particular, pseudo-coequalizers). Moreover, since the 2-fibration cod : $\mathbf{Fib} \longrightarrow \mathbf{Cat}$ has Σ with the BCC for comma objects, the same is true for cod : $(\mathbf{Stack}^{\rightarrow}) \longrightarrow \mathbf{S}$.

By Theorem 5.2, in order to prove that **Stack** is a 2-stack, it is enough (by the pseudomonadicity theorem of [19]) to prove that, for any epimorphism $e : J \longrightarrow I$ in **S**, $e^* : \mathbf{Stack}/\mathbb{I}_{dis} \longrightarrow \mathbf{Stack}/\mathbb{J}_{dis}$ reflects equivalences.

Recall from Remark 5.4 that the regular 2-epi /fully faithful factorization of $e_{\mathrm{dis}} : \mathbb{J}_{\mathrm{dis}} \longrightarrow \mathbb{I}_{\mathrm{dis}}$ is given by

$$\begin{tikzcd} \mathbb{J}_{\mathrm{dis}} \arrow[rr, "\bar{e}"] \arrow[dr, "e_{dis}"'] & & \mathbb{J}_e \arrow[dl, "E"] \\ & \mathbb{I}_{\mathrm{dis}} & \end{tikzcd}$$

where $E : \mathbb{J}_e \longrightarrow \mathbb{I}_{dis}$ is the 1-kernel of e. Let x be such that $e^*x = (\bar{e}^*E^*)x$ is an equivalence. Since $\bar{e}$ is a regular 2-epi, it is of effective 2-descent for *cod* : $\mathbf{Fib} \longrightarrow \mathbf{Cat}$ by Theorem 5.5, hence, it reflects equivalences. It follows then that E^*x is an equivalence.

It remains to show that x is an equivalence. For this, we use the fact that $E : \mathbb{J}_e \longrightarrow \mathbb{I}_{dis}$ is a wef as follows. Consider the prism diagram

$$\begin{tikzcd} & E^*\mathbb{C} \arrow[dl] \arrow[dr] \arrow[dd] & \\ E^*\mathbb{D} \arrow[rr] \arrow[dd] & & \mathbb{J}_e \arrow[dd, "E"] \\ & \mathbb{C} \arrow[dl, "x"] \arrow[dr, "c"] & \\ \mathbb{D} \arrow[rr, "d"'] & & \mathbb{I} \end{tikzcd}$$

where the front and back square faces are pullbacks, and where $E^*x : E^*\mathbb{C} \longrightarrow E^*\mathbb{D}$ is an equivalence. The morphisms $c^*E : E^*\mathbb{C} \longrightarrow \mathbb{C}$ and $d^*E : E^*\mathbb{D} \longrightarrow \mathbb{D}$ are weak equivalence functors, since E is one and the codomains of the pullbacks are discrete, so that they are 2-pullbacks in the sense of [11]. (Recall that weak equivalence functors are stable under 2-pullbacks, by Proposition 1.5.) Therefore, $d^*E \cdot E^*x = x \cdot c^*E$ (the lhs commuting square) is a weak equivalence functor since wef are composable and any equivalence is a wef. Now, since both the composite $x \cdot c^*E$ and c^*E are wef, so is x. But $\mathbb{C}$ is a stack, hence the wef x is an equivalence. This concludes the proof.

□

5.8. Remark. The argument that, since E is a wef so are c^*E and d^*E, may be given two alternative justifications using that c and d are cofibrations (as their codomain is discrete) and wef are pullback stable along cofibrations, or else that E is a swef (surjective weak equivalence, i.e., with the object part an epi), and those are are pullback stable, so that c^*E and d^*E are in fact swef.

6. Applications to second degree non-abelian cohomology of a topos

The notion of a 2-gerbe [13, 6] is normally reserved for 2-groupoids. We are interested in a generalization of this notion which includes, as instances, both the 2-gerbes and the 2-stacks. In particular, the 2-stack completion of a generalized 2-gerbe, which we will show exists, is a generalized 2-gerbe.

6.1. Definition. A 2-category $\mathbf{C}$ in $\mathbf{S}$ is said to be a *generalized 2-gerbe* if it is locally a 1-stack, in the sense that all the Hom-categories $\mathbf{C}(x, y)$ are 1-stacks in $\mathbf{Cat}(\mathbf{S})$.

Denote by 2-**Gerb** the full subcategory of $2-\mathbf{Cat}(\mathbf{S})$ whose objects are the generalized 2-gerbes. For any generalized 2-gerbe $\mathbf{C}$, we have a Yoneda 2-functor

$$\mathbf{C} \xrightarrow{Y} \mathbf{Stack}^{\mathbf{C}^{\mathrm{op}}}.$$

6.2. Remark.

1. *Any 2-gerbe is a generalized 2-gerbe.*
2. *Any 2-stack in* $\mathbf{S}$ *is a generalized 2-gerbe.*

6.3. Definition. Let $F : A \longrightarrow B$ be a 2-functor between 2-fibrations over $\mathbf{S}$. It is said to be a *special weak 2-equivalence* if the following conditions hold.

1. For each $I \in \mathbf{S}$, $F^I : |A^I| \longrightarrow |B^I|$, and $b \in |B^I|$, there exists an epimorphism $e : J \longrightarrow I$ in $\mathbf{S}$, $a \in |A^I|$, and a (strong) equivalence $\theta : F^J(e) \longrightarrow e^*(b)$.
2. $\forall I \in \mathbf{S}\ \forall x, x' \in |A^I|$, the morphism

$$\mathrm{Hom}_{AA^I}(x, x') \xrightarrow{F_{x,x'}} \mathrm{Hom}_{B^I}(Fx, Fx')$$

of fibrations is a (strong) equivalence functor.

6.4. Remark. Any weak 2-equivalence in $2-\mathbf{Gerb}$ is special since, at the level of the homs, it is a strong (not just weak) equivalence. In fact, since the fiber functors (at $I \in \mathbf{S}$)

$$\mathrm{Hom}(x,x') \xrightarrow{\mathrm{Y}_{x,x'}} \mathrm{Hom}(Y(x),Y(x'))$$

of fibrations is a weak equivalence functor with domain a stack, it is necessarily a strong equivalence.

6.5. Definition. Let $\mathbb{C}$ be a generalized 2-gerbe. An I-indexed family X of the 2-category $(\mathbf{Stack}^{\mathbf{C}^{\mathrm{op}}})^I$ is said to be *locally representable* if here exists an epimorphism $k : K \twoheadrightarrow I$ in $\mathbf{S}$, as well as an object $a \in \mathscr{A}^K$ and an equivalence $\mathrm{Y}(a) \cong k^*(X)$.

6.6. Theorem. *Let* $\mathbf{S}$ *be a Grothendieck topos. Let* $\mathbf{C}$ *be a generalized 2-gerbe in* $\mathbf{S}$*. Then the 2-stack completion of* $\mathbf{C}$ *can be identified with the weak equivalence 2-functor*

$$\mathbf{C} \xrightarrow{\mathrm{Y}} \mathbf{LR}(\mathbf{Stack}^{\mathbf{C}^{\mathrm{op}}})$$

where $\mathbf{LR}(\mathbf{Stack}^{\mathbf{C}^{\mathrm{op}}})$ *is the full sub 2-fibration of* $\mathbf{Stack}^{\mathbf{C}^{\mathrm{op}}}$ *determined by the locally representable objects in each fiber.*

Proof. Since $\mathbf{Stack}$ is a 2-stack (over $\mathbf{S}$), so is $\mathbf{Stack}^{\mathbf{C}^{\mathrm{op}}}$ by Theorem 4.9. Factor the yoneda embedding as

$$\mathbf{C} \xrightarrow{Y} \mathbf{LR}(\mathbf{Stack}^{\mathbf{C}^{\mathrm{op}}}) \hookrightarrow \mathbf{Stack}^{\mathbf{C}^{\mathrm{op}}}$$

and prove (just as in the case $n = 1$ from [11] that, since $\mathbf{Stack}^{\mathbf{C}^{\mathrm{op}}}$ is a 2-stack, so is $\mathbf{LR}(\mathbf{Stack}^{\mathbf{C}^{\mathrm{op}}})$. Furthermore, the first factor is a (special) weak 2-equivalence. □

We shall now restrict our attention from categories to groupoids and from 2-categories to 2-groupoids. By a *2-groupoid* $\mathbf{G}$ we understand here a 2-category whose 1-cells are *equivalences* and whose 2-cells are *isomorphisms*. There is a different notion of 2-groupoid with isos at both levels, that we shall not consider here. What we call a 2-groupoid has been called a pseudogroupoid in [23].

6.7. Definition. Denote by $\mathrm{cod} : stack^{\rightarrow} \longrightarrow \mathbf{S}$ the corresponding data of Definition 4.2 restricted to *groupoids* that are stacks.

6.8. Corollary. *[29] Let* $\mathbf{S}$ *be a Grothendieck topos. Then, the 2-fibration* $\mathrm{cod} : stack^{\rightarrow} \longrightarrow \mathbf{S}$ *is a 2-stack. Equivalently, any regular epi in* $\mathbf{S}$ *is of effective 2-descent for* $\mathrm{cod} : \mathbf{Stack} \longrightarrow \mathbf{S}$.

Proof. The proof of Theorem 5.7 is valid also if restricted to groupoids, in fact, with some simplifications in the construction of Σ from [19], as there is then no need to invert certain morphisms. □

6.9. Remark. A particular case of the notion of a generalized 2-gerbe in **Gpd** is that of a 2-gerbe [12, 6]. A 2-groupoid $\mathbf{G}$ in $\mathbf{S}$ is said to be a *2-gerbe* if for some ('non-empty' and 'connected') 1-groupoid 1-stack $\mathbb{A}$ (a *bouquet*), there is a 2-equivalence

$$\mathbf{G} \simeq \mathrm{Equ}(\mathbb{A}).$$

We shall define a 2-fibration $\mathrm{Tors}^2(\mathbf{G}) \longrightarrow \mathbf{S}$ for any 2-gerbe $\mathbf{G}$ in $2-\mathbf{Gpd}$. For each $I \in \mathbf{S}$ the 2-category $(\mathrm{Tors}^2(\mathbf{G}))^{\mathbf{I}}$ has:

1. as *objects* the $\mathbf{G}$-2-torsors over I, where a 2-torsor over I is a groupoid stack $\mathbf{T}$ equipped with a structure $\mathbf{T} \times \mathbf{G} \xrightarrow{a} \mathbf{T}$ of a right $\mathbf{G}$-object, and with an epi $p : T_0 \longrightarrow I$, such that
$$\mathbf{T} \times \mathbf{G} \xrightarrow{<\pi_1, a>} \mathbf{T} \times_I \mathbf{T}$$
is an equivalence.
2. as *1-cells* the $\mathbf{G}$-equivariant 2-functors $h : \mathbf{T} \longrightarrow \mathbf{R}$ (over I), in the sense that the diagram
$$\begin{tikzcd} \mathbf{T} \times \mathbf{G} \arrow[r, "h\times \mathrm{id}"] \arrow[d, "<\pi_1,a>"'] & \mathbf{R} \times \mathbf{G} \arrow[d, "<\pi_1,b>"] \\ \mathbf{T} \times_I \mathbf{T} \arrow[r, "h\times h"'] & \mathbf{R} \times_I \mathbf{R} \end{tikzcd}$$
is commutative,
3. and as *2-cells*, natural isomorphisms $h \Rightarrow k : \mathbf{T} \longrightarrow \mathbf{R}$ over I.

6.10. Proposition. $\mathrm{Tors}^2(\mathbf{G})$ *is a 2-groupoid.*

Proof. This is shown in basically the same way as in [1] (or [8] in the discrete case), using the properties of $\mathbf{S}$ as a regular (exact) category. □

6.11. Theorem. *Let* $\mathbf{S}$ *be a Grothendieck topos, and* $\mathbf{G}$ *a 2-gerbe in* $2-\mathbf{Gpd}$. *Then there is a biequivalence of 2-fibrations over* $\mathbf{S}$

$$\mathbf{LR}(stack^{\mathbf{G}^{\mathrm{op}}}) \cong \mathrm{Tors}^2(\mathbf{G}).$$

Proof.

The proof is entirely analogous to the same statement in [8] but for 2-groupoids and discrete spaces. That any 2-torsor is locally representable is inmediate. The converse uses properties of regular epis, and that pulling back along an epi reflects equivalences.

□

6.12. Corollary. *Let* $\mathbf{G}$ *be a 2-gerbe in* $\mathbf{S}$. *The canonical morphism*

$$[\mathbf{G}] \xrightarrow{\text{triv}} \mathrm{Tors}^{\mathbf{2}}(\mathbf{G})$$

exhibits $\mathrm{Tors}^2(\mathbf{G})$ *as the 2-stack completion of* $[\mathbf{G}]$.

Proof. Apply Theorem 6.6 together with Theorem 6.11 in the case of a 2-gerbe $\mathbf{G}$, which is a particular case of a generalized 2-gerbe. □

Recall that 2-dimensional cohomology of $\mathbf{S}$ with coefficients in a 2-gerbe $\mathbf{G}$ is given by the formula

$$\mathrm{H}^2(\mathbf{S}; \mathbf{G}) = \mathbf{\Pi_0}(\mathrm{Tors}^2(\mathbf{G}))$$

where $\mathbf{\Pi_0}$ in this case denotes 'equivalence classes'.

6.13. Remark. By analogy with the case of dimension 1, we interpret Theorem 6.11 to say that, for any 2-gerbe $\mathbf{G}$, the 2-topos $\mathit{stack}^{\mathbf{G}^{\mathrm{op}}}$ *classifies* $\mathbf{G}$-2-torsors in $\mathbf{S}$. This remark can, in principle, be made precise using the notion of a 2-topos (or a bitiopos) from [32]. To this end we need to isolate a notion of morphism of 2-toposes and the analogue of the notion of a point of a topos. We leave this matter open for the time being.

7. The Equivalences Lifting Property

We begin this section by briefly recalling a definition and a result from [23, 24] about the existence of a Quillen model structure on 2-$\mathbf{Cat}$(Set). The definition (Definition 7.1) makes sense for any elementary topos $\mathbf{S}$.

7.1. Definition. A 2-functor $F : \mathbf{A} \longrightarrow \mathbf{B}$ in 2-$\mathbf{Cat}(\mathbf{S})$ is said to have the *equivalence lifting property* (*(ELP)* for short) if

- Given $A \in A_0$, and an *adjoint equivalence* in $\mathbf{B}$, consisting of $b : B \longrightarrow FA$, $b' : FA \longrightarrow B$, $\beta_1 : bb' \cong 1$, and $\beta_2 : b'b \cong 1$, there is an adjoint equivalence in $\mathbf{A}$, consisting of $a : A' \longrightarrow A$, $a' : A \longrightarrow A'$, $\alpha_1 : aa' \cong 1$ and $\alpha_2 : a'a \cong 1$, with $FA' = B$, $Fa = b$, $Fa' = b'$, $F\alpha_1 = \beta_1$, and $F\alpha_2 = \beta_2$.

- For any 1-cell $a : A \longrightarrow A'$ in $\mathbf{A}$ and every invertible 2-cell $\beta : b \longrightarrow Fa$ in $\mathbf{B}$, there is a 1-cell $a' : A \longrightarrow A'$ with $F(a') = b$ and an invertible 2-cell $\alpha : a' \longrightarrow a$ with $F(\alpha) = \beta$.

In fact, the result (Theorem 7.2) shown in [23, 24] only for $\mathbf{S} = \mathit{Set}$, is valid for any Grothendieck topos $\mathbf{S}$, provided the notions involved are interpreted internally.

7.2. Theorem. *(Lack)[24] Let* $\mathbf{S}$ *be a Grothendieck topos. There is a cofibrantly generated Quillen model structure on the category 2-*$\mathbf{Cat}(\mathbf{S})$ *for which the weak equivalences in the model structure are the weak 2-equivalences, and the fibrations (in the sense of model structures [31]) are the 2-functors with the (ELP).*

We can now use this result in order to exhibit internal 2-stack completions in the topos $\mathbf{S} = \mathit{Set}^{\mathbf{2}}$.

7.3. PROPOSITION. *A 2-functor $F : \mathbf{A} \longrightarrow \mathbf{B}$ in* **Cat** *satisfying the (ELP) is a 2-stack regarded as a 2-category object in the topos* $\mathbf{S} = Set^{\mathbf{2}}$.

Proof. By Proposition 3.8 we can already deduce, from condition (2) in Definition 7.1, that the Homs are 1-stacks. The proof of the fact that condition (1) in this definition implies the full 2-stack property is entirely analogous to that of the same theorem given in [11]. The basic fact is that item (1) in Definition 4.6 of a weak 2-equivalence is a geometric notion, hence true for any 2-functor $F : \mathbf{A} \longrightarrow \mathbf{B}$ iff it is 'pointwise true', that is, if for every object c in $\mathbf{C}$, the functor $F(c) : \mathbf{A(c)} \longrightarrow \mathbf{B(c)}$ is a weak equivalence functor in *Set*, that is, a strong equivalence functor since, as an elementary topos, *Set* satisfies the Axiom of Choice. □

7.4. COROLLARY. *Let $F : \mathbf{A} \longrightarrow \mathbf{B}$ be any 2-category object in $Set^{\mathbf{2}}$. Factor the unique morphism $F \longrightarrow \mathbf{1}$ into*

$$F \longrightarrow F^* \longrightarrow \mathbf{1}$$

where $F \longrightarrow \mathbf{F}^*$ is a trivial cofibration and $F^* \longrightarrow \mathbf{1}$ is a fibration for the Quillen model structure on $2 - \mathbf{Cat}(\mathbf{S})$, where $\mathbf{S} = Set^{\mathbf{2}}$. Then,

$$F \longrightarrow F^*$$

is the 2-stack completion of F in $\mathbf{Cat}(\mathbf{S})$.

Proof. The unique $F^* \longrightarrow \mathbf{1}$ has the (ELP) and so, by Proposition 7.3, F^* is a 2-stack. Since $F \longrightarrow F^*$ is a trivial cofibration, it is, in particular, a weak 2-equivalence, hence by Corollary 4.10, it is the 2-stack completion of F. □

An explicit description of the stack completion of a 2-functor $F : \mathbf{A} \longrightarrow \mathbf{B}$ is possible in the manner of the analogous 1-dimensional case [11]. We leave this as an interesting exercise.

8. An outline of higher stack completions

In order to deal with higher dimensions we would first need to choose a suitable notion of n-category [27] and, relative to it, a notion of weak n-equivalence n-functor. The latter is done as follows, by induction.

8.1. DEFINITION. Let $\mathbf{S}$ be a topos. We define the notion of a *weak n-equivalence n-functor* in $\mathbf{S}$ by induction on n.

- (n = 0) A 0-functor $F : A \longrightarrow B$ in $\mathbf{S}$ is a weak 0-equivalence if it is an isomorphism.
- $(n \geq 1)$ An n-functor $F : A \longrightarrow B$ between n-categories in $\mathbf{S}$ is a weak n-equivalence if

1. $F_0 : A_0 \longrightarrow B_0$ is essentially (n-1)-surjective, and
2. For all $x, x' \in A_0$, the (n-1)-functor

$$F_{x,x'} : Hom_A(x, x') \longrightarrow Hom_B(Fx, Fx')$$

is a weak (n-1)-equivalence of (n-1)-categories.

We state the following result without proof.

8.2. Lemma. *Let $e : J \longrightarrow I$ be an epimorphism in $\mathbf{S}$. Then the the n-functor*

$$F^{(n)}{}_e : J_e^{(n)} \longrightarrow I_{\text{dis}}^{(n)}$$

from the n-kernel of e to the disccrete n-category on I is a weak n-equivalence.

The enterprise of dealing with higher dimensional stack completions is well beyond our possibilities here. Nevertheless, we wish to indicate an outline of how to proceed once these first steps are done.

8.3. Remark.

- The construction of the (n+1)-stack completion of a generalized (n+1)-gerbe $\mathbf{C}$ as given by the yoneda embedding

$$\mathbf{C} \xrightarrow{Y} \mathbf{LR}(\mathbf{n} - \mathbf{Stack}^{\mathbf{C}^{\text{op}}})$$

  requires proving first that the (n + 1)-fibration cod : $n - \mathbf{Stack} \longrightarrow \mathbf{S}$ be an (n + 1)-stack. The complicated coherence conditions involved constitute a technical obstruction to such a program. However, assuming this to be the case, it would follow from it that the above is the (n+1)-stack completion of $\mathbf{C}$.

- By contrast, we conjecture that, in higher dimensions, a construction of the n-stack completion of a generalized n-gerbe by means of a Quillen model structure on $n - \mathbf{Gerb}$ exists, whose weak equivalences are the (special) weak n-equivalence functors, and whose fibrant objects are the 'strong' n-stacks.

- We have seen that, in the cases $n = 1$ and $n = 2$, there are, in principle, two different constructions of the n-stack completion of an n-gerbe. The first, for $\mathbf{S}$ a Grothendieck topos, is given by a suitable Quillen model structure on $n - \mathbf{Cat}(\mathbf{S})$. This is the one that relates to *homotopy* and gives a precise meaning to the assertion that $\mathbf{S}$ satisfies an axiom of n-stack completions. The second one carves out of a larger n-stack by cutting down to the locally representables. This is the one that exhibits an 'n-topos' as classifier of the n-torsors, and that relates to *non-abelian cohomology.* A comparison between the two would give rise to an appropriate classification and Morita equivalence theorems.

- Alternative outlines of non-abelian higher cohomology have been given in [14, 33] with similar conclusions as ours in Remark 8.3 for higher dimensions.

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