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DIRECTED COLIMITS OF FREE COMMUTATIVE SEMIGROUPS

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Our main result is an elementary characterization of directed colimits of free commutative semigroups (with or without identity elements).

Specifically, we show the equivalence of three conditions on a commutative monoid S. The first condition is that S be a directed colimit of free commutative monoids. The second condition is that every inclusion map $T \to S$, where T is finitely generated, can be factored through some free commutative monoid. The equivalence of these two conditions is proved by an argument due to Lazard [7], which is not included in this paper since Shannon [9] has extended the proof to all varieties. The second condition is named the killing interpolation property since in the terminology of [5] it means that S kills all its finitely generated subsemigroups. It can be restated as follows: given any finitely many elements of S, all relations satisfied in S by these elements are trivial consequences of (are "killed" by) some suitable presentation of these elements within the monoid S. Our main result is the equivalence of this and our third condition: S is cancellative, has no units, and, whenever $a, b, c, d \in S$ satisfy $na + b = nc + d$ with $n > 0$, then $a = u + v$, $b = nw + z$, $c = u + w$, $d = nv + z$ for some $u, v, w, z \in S$. This last condition kills the very simple relation $na + b = nc + d$ (i.e. makes it a trivial consequence of the presentation $a = u + v$, etc.); proving the main result consists in using this simple condition to kill increasingly complex relations. This is done in Section 2. The first section gives various recalls and simpler properties of our conditions.

In Section 3, we extend the main result to commutative semigroups without identity element, by showing that S is a directed colimit of free commutative semigroups if and only if adjunction of an identity element to S yields a directed colimit of free commutative monoids. Finally, we show in Section 4 that a commutative semigroup can be embedded into a directed colimit of free commutative monoids if and only if it satisfies the obvious necessary conditions, i.e. is cancellative, power-cancellative and has no units (other than an identity element). An embedding can then be produced which has a weak universal property. Two open questions complete the paper.

All these results were announced in [4] and [6], but the present version has greatly benefited from a number of remarks and suggestions by the referee, not all of which are mentioned in the text, and by the developments in Shannon's work [9].

Our notation generally follows Clifford and Preston [1], with the following exceptions. All semigroups under consideration are commutative (a fact we recall only occasionally); and we use the additive notation throughout. In particular, identity elements are denoted by 0; adjoining an identity element to S (if S has none) yields S^0; when $n>0$, $na = a + \dots + a$. No exponents will be used; all superscripts in the paper are indices.

1. Interpolation properties

1.1. Let $x_1, \dots, x_n$ be elements of a [commutative] monoid $S = S^0$; assume that there is a presentation $x_i = \Sigma_j m_i^j y_j$ of each x_i in terms of elements $y_1, \dots, y_m$ of S (with $m_i^j \geq 0$). The existence of such a presentation usually implies certain relations between the elements x_i: if for example the elements a, b, c, d can be presented in terms of u, v, w, z by: $a = u + v$, $b = nw + z$, $c = u + w$, $d = nv + z$ (where $n>0$), then the relation $na + b = nc + d$ must hold; in the general case, whenever the nonnegative integers r_i, s_i satisfy $\Sigma_i r_i m_i^j = \Sigma_i s_i m_i^j$ for every j, then the relation $\Sigma_i r_i x_i = \Sigma_i s_i x_i$ must hold in S. There may be relations between the elements x_i which hold in S but cannot be obtained in that fashion; if there are not, then the given presentations of $x_1, \dots, x_n$ are said to constitute a *killing presentation* of these elements (and also of the subsemigroup they generate). Then every relation $\Sigma_i r_i x_i = \Sigma_i s_i x_i$ which holds in S yields a relation between the elements y_j, namely $\Sigma_j(\Sigma_i r_i m_i^j) y_j = \Sigma_j(\Sigma_i s_i m_i^j) y_j$, which is trivial, so that the given relation is "killed" by the presentation. For instance this is always the case if S is freely generated by the elements y_j.

We need to recall, from [5], a few results about this mode of murder. We say that S *kills* its subsemigroup T in case T has a finite generating subset for which there exists a killing presentation in S. In this case, every finite generating subset of T has a killing presentation. Furthermore:

Fact 1.1. *When S kills T, every semigroup containing S also kills T; S kills T if and only if S kills T^0 [$\subseteq S$].*

Fact 1.2. *When S kills T and φ is a homomorphism of S which is injective on T, then $\varphi(S)$ kills $\varphi(T)$.*

Fact 1.3. *A free semigroup with identity kills all its finitely generated subsemigroups.*

Killing can also be expressed as follows. Let T be the subsemigroup of S generated by $x_1, \dots, x_n$ and (P) $x_i = \Sigma_j m_i^j y_j$ be a presentation of the elements x_i in terms of $y_1, \dots, y_m \in S$. Let F be the free [commutative] monoid on m generators

$a_1, \ldots, a_m$ and $\varphi: F \to S$ be the homomorphism such that $\varphi(a_j) = y_j$ for all j. We see that (P) is a killing presentation if and only if the mapping $x_i \mapsto \Sigma_j m_i^j a_j$ extends to a homomorphism $\iota: T \to F$; then the inclusion map $T \to S$ factors (as $\varphi \circ \iota$) through F. Then it follows from 1.3, 1.2, 1.1 that S kills T if and only if the inclusion map $T \to S$ factors through some free monoid.

This implies that not every finitely generated semigroup can be killed. More precisely, if T can be killed, then the map ι above is injective and hence T inherits certain properties from F: T is cancellative, *power-cancellative* (i.e. $a, b \in T$, $na = nb$, $n > 0$ implies $a = b$) and *reduced* (= without identity element or with trivial group of units).

If conversely T is finitely generated with these properties, then [5] provides a semigroup which kills T and a way to produce the free monoid F above in case S kills T. Namely, the free envelope of T constructed in [5] is a free monoid $F(T)$ together with a homomorphism $\alpha: T \to F(T)$ with the weak universal property that every homomorphism of T into a free monoid factors through α (though perhaps not uniquely); $F(T)$ also has minimality properties given in [5]. The map α is injective if and only if T is cancellative, power-cancellative and reduced. The weak universal property of $F(T)$ yields:

Fact 1.4. *For a finitely generated subsemigroup T of a monoid S the following are equivalent*: 1) *S kills T*; 2) *the inclusion map $T \to S$ factors through some free monoid*; 3) *the inclusion map $T \to S$ factors through the free envelope of T.*

1.2. A [commutative] monoid S has the *killing interpolation property* (hereafter abbreviated as KIP) in case it kills all its finitely generated subsemigroups. For instance, 1.3 states that all free monoids have the KIP. If conversely S has the KIP, then all its finitely generated subsemigroups are cancellative, power-cancellative and reduced; since these are local properties, S is also cancellative, power-cancellative and reduced.

The KIP can be defined in any variety $\mathcal{V}$ of universal algebras, as follows: an algebra $A \in \mathcal{V}$ has the KIP if and only if, for any finitely many elements $x_1, \ldots, x_n$ of A and any finitely many relations which hold in A between these elements, there exists a presentation of $x_1, \ldots, x_n$ in A which kills all the given relations. This definition is due to Shannon [9]. In the variety of commutative monoids, it follows from Rédei's theorem [1] that all the relations between $x_1, \ldots, x_n$ which hold in A are trivial consequences of finitely many such relations; therefore the general definition of the KIP agrees with ours. In general, it is equivalent to define the KIP by the killing of just one (arbitrary) relation between the elements x_i; that any finitely many relations can then be killed is shown by an easy induction on the number of relations [9].

The main result about the KIP is:

Theorem 1.5 (Shannon's Theorem). *In any variety, an algebra is a directed colimit of free algebras if and only if it has the killing interpolation property.* □

This was first established by Lazard [7] for the variety of left R-mocules (where the KIP is a classical characterization of flatness); then by the author [6] for the varieties of commutative semigroups and commutative monoids (using, essentially, Lazard's argument); and, finally, by Shannon [9], in full generality (by a similar but much clearer proof).

1.3. The rest of this section deals with the simpler interpolation property

$$R(n)\colon\ na+b=nc+d \text{ implies } a=u+v,\ b=nw+z,$$
$$c=u+w,\ d=nv+z \text{ for some } u,v,w,z\in S,$$

where $n>0$ and S is a [commutative] monoid. The interpolation property $R(1)$ is the Riesz interpolation property (in its semigroup form, as in [3]), so called because of 1.9 below. We say that S has the *strong Riesz interpolation property* (hereafter abbreviated as: strong RIP) if $R(n)$ holds in S for all $n>0$.

Proposition 1.6. *The positive cone of a lattice-ordered abelian group A always has the strong* RIP.

Proof. Take $a,b,c,d\in A$, $a,b,c,d\geqslant 0$, and $n>0$, such that $na+b=nc+d$. Let $u=a\wedge c$, $v=a-u$, $w=c-u$, so that $u,v,w\geqslant 0$. Since $b,d\geqslant 0$, we also have $na-d=nc-b\leqslant na,nc$; hence $na-d=nc-b\leqslant na\wedge nc=n(a\wedge c)=nu$ and $na-d=nc-b=nu-z$ with $z\geqslant 0$. Thus $a=u+v$, $b=nc-(nu-z)=nw+z$, $c=u+w$ and $d=na-(nu-z)=nv+z$ with $u,v,w,z\geqslant 0$. □ (For a more detailed result, see 2.7 below.)

Corollary 1.7. *Free monoids have the strong* RIP. □

Corollary 1.8. *The* KIP *implies the strong* RIP.

Proof. This follows from 1.5, 1.7, as the strong RIP clearly is inherited by directed colimits. □

1.4. Our main result is the converse of 1.8, for cancellative reduced monoids. This would be a nicer result if the arbitrary integer n in the strong RIP could be dispensed with. For this and other reasons we now compare RIP and strong RIP.

The analogue of 1.6 for the RIP is:

Proposition 1.9. *Let A be a partially ordered abelian group. Then A is Riesz ordered if and only if its positive cone has the* RIP.

Proof. Assume that A is Riesz ordered, i.e. whenever $p,q\leqslant r,s$ in A then $p,q\leqslant u\leqslant r,s$ for some $u\in A$ (cf. [2]). Let $a,b,c,d\geqslant 0$ satisfy $a+b=c+d$. Then

$a - d = c - b$; since $a - d = c - b \leqslant a, c$, $0 \leqslant a, c$ there exists $u \in A$ with $0 \leqslant u \leqslant a, c$, $a - d = c - b \leqslant u$. Put $a = u + v$, $c = u + w$, $a - d = c - b = u - z$; then $u, v, w, z \geqslant 0$, $b = c - (u - z) = w + z$, $d = a - (u - z) = v + z$ and thus the positive cone of A has the RIP. Conversely, assume that the positive cone of A has the RIP and that $p, q, r, s \in A$ satisfy $p, q \leqslant r, s$. Then $a = r - p$, $b = s - q$, $c = r - q$, $d = s - p$ satisfy $a, b, c, d \geqslant 0$, $a + b = c + d$; therefore $a = u + v$, $b = w + z$, $c = u + w$, $d = v + z$ for some $u, v, w, z \geqslant 0$; and we see that $x = r - u = s - z = p + v = q + w \in A$ satisfies $p, q \leqslant x \leqslant r, s$, so that A is Riesz ordered. □

This result was mentioned in [3] and justifies the names given our interpolation properties. The proof is given here purely for the sake of self-completeness. With reference to 1.6, 1.9 it should be recalled that any cancellative reduced monoid S can be described as the positive cone of a partially ordered, directed abelian group A: if A is the universal group of S (or group of quotients, here written as a group of differences), then $S \subseteq A$ satisfies $S \cap -S = \{0\}$, since S is reduced, so that the binary relation $x \geqslant y \Leftrightarrow y - x \in S$ on A makes A a partially ordered abelian group; A is directed since S generates A (cf. [2]).

We continue our comparison with:

Proposition 1.10. *The strong* RIP *is equivalent to the* RIP *together with all conditions $R(p)$ with p prime.*

Proof. It suffices to show that $R(m)$ and $R(n)$ imply $R(mn)$. Thus, assume $R(m)$, $R(n)$ hold and $mna + b = mnc + d$. By $R(m)$, $na = u + v$, $b = mw + z$, $nc = u + w$, $d = mv + z$ for some u, v, w, z. We see that also $na + w = nc + v$. By $R(n)$, $a = u' + v'$, $w = nw' + z'$, $c = u' + w'$, $v = nv' + z'$ for some u', v', w', z'. Hence $a = u' + v'$, $b = mnw' + (mz' + z)$, $c = u' + w'$ and $d = mnv' + (mz' + z)$, which shows that $R(mn)$ holds. □

Next we note that a reduced monoid S with the strong RIP is, necessarily, power-cancellative: if $na + 0 = nc + 0$ (with $n > 0$), then $a = u + v$, $c = u + w$, $0 = nw + z = nv + z$ for some $u, v, w, z \in S$; since S is reduced, $v = w = z = 0$ and hence $a = u = c$.

Proposition 1.11. *In a divisible power-cancellative monoid, the* RIP *and the strong* RIP *are equivalent.*

Proof. Let S be divisible (i.e., for every $a \in S$, $n > 0$, the equation $nx = a$ has a solution in S) and power-cancellative (so the solution of $nx = a$ is unique). Assume $na + b = nc + d$, $n > 0$. By the RIP, $na = u + v$, $b = w + z$, $nc = u + w$, $d = v + z$ for some $u, v, w, z \in S$. Put $u = nu'$, $v = nv'$, $w = nw'$; then $na = nu' + nv'$ implies $a = u' + v'$, similarly $c = u' + w'$, whereas $b = nw' + z$, $d = nv' + z$. □

The next result is similar to the main result of [3] and requires the following lemma.

Lemma 1.12. *In a monoid S with the* RIP, $a_1 + \ldots + a_n = b_1 + \ldots + b_m$ *implies* $a_i = \sum_{j=1}^{j=m} u_{i,j}$, $b_j = \sum_{i=1}^{i=n} u_{i,j}$ *for all i, j, for some elements $u_{i,j} \in S$.*

Proof [3]. There is nothing to show if $n = 1$, if $m = 1$, or if $n = m = 2$. We proceed by induction on $n + m$ and may therefore start with $n + m \geqslant 5$, $n, m \geqslant 2$, so that, say, $n \geqslant 3$. Then $2 + m < n + m$: by the induction hypothesis, $(a_1 + \ldots + a_{n-1}) + a_n = b_1 + \ldots + b_m$ implies $a_1 + \ldots + a_{n-1} = \sum_{j=1}^{j=m} v_{1,j}$, $a_n = \sum_{j=1}^{j=m} v_{2,j}$, $b_j = v_{1,j} + v_{2,j}$ for all j, with $v_{1,j}, v_{2,j} \in S$. Similarly, $(n-1) + m < n + m$, so that $a_1 + \ldots + a_{n-1} = v_{1,1} + \ldots + v_{1,m}$ implies $a_i = \sum_{j=1}^{j=m} w_{i,j}$ for all $i < n$, $v_{1,j} = \sum_{i=1}^{i=n-1} w_{i,j}$ for all j, with $w_{i,j} \in S$. Let $u_{i,j} = w_{i,j}$ if $i < n$, $u_{i,j} = v_{2,j}$ if $i = n$; then $a_i = \sum_{j=1}^{j=m} u_{i,j}$, $b_j = \sum_{i=1}^{i=n} u_{i,j}$ for all i, j, which completes the induction. $\square$

Proposition 1.13. *Let S be a cancellative reduced monoid with a minimal generating subset X. If S has the* RIP *then S is free (and hence has the strong* RIP).

Proof. First we show that every $x \in X$ is irreducible, i.e. $x = a + b$ in S implies $a = 0$ or $b = 0$. Put $a = \sum_{y \in X} r_y y$, $b = \sum_{y \in X} s_y y$, so that $x = a + b$ reads: $x = \sum_{y \in X} (r_y + s_y) y$, with r_y, s_y nonnegative integers. If $r_x + s_x = 0$, this last relation presents x in terms of the other elements of X, contradicting the minimality of X; therefore $r_x + s_x \geqslant 1$. Cancelling x from both sides then yields $0 = (r_x + s_x - 1)x + \sum_{y \neq x} (r_y + s_y) y$. Since S is reduced, this implies $r_x + s_x = 1$, $r_y + s_y = 0$ for all $y \neq x$, so that either r_x or s_x equals 1 and all other r_y, s_y are 0. Thus $a = 0$ or $b = 0$.

We now show, as in [3], that S is free on X. Assume that there is a non-trivial relation in S between the elements of X; we can write this relation $x_1 + \ldots + x_n = y_1 + \ldots + y_m$, where the elements x_i, y_j are in X and not necessarily distinct. We may also assume that $x_i \neq y_j$ for all i, j (since S is cancellative) and that $n, m > 0$ (since S is reduced). The lemma then yields $u_{i,j} \in S$ with $x_i = \sum_j u_{i,j}$, $y_j = \sum_i u_{i,j}$ for all i, j. Since x_1 is irreducible, there exists $k \leqslant m$ such that $u_{1,j} = 0$ for all $j \neq k$, $u_{1,k} = x_1$. Since y_k is also irreducible, only one of the $u_{i,k}$ is nonzero; this must be $u_{1,k}$, so that $y_k = u_{1,k} = x_1$, a contradiction. $\square$

It follows from 1.11, 1.13 that, in a cancellative, power-cancellative reduced monoid which is either divisible or with minimal generating subset (e.g. finitely generated), the RIP and strong RIP are equivalent. To show that they are no longer equivalent without divisibility or a minimal generating subset makes counterexamples none too easy to come by. The following example was suggested by the referee.

Example 1.14. *A cancellative, power-cancellative reduced monoid with the* RIP *which does not have the strong* RIP.

Let S be the additive submonoid of $\mathbf{Q}\times\mathbf{Z}$ consisting of all pairs (p,q) such that either $p>0$, or $p=0$, $q\geqslant 0$, q even. On the additive group $\mathbf{Q}\times\mathbf{Z}$ we see that S is cancellative, power-cancellative and reduced. Also, $2(1,1)+0=2(1,0)+(0,2)$ holds in S, whereas $(1,1)=u+v$, $0=2w+z$, $(1,0)=u+w$, $(0,2)=2v+z$ is impossible with $u,v,w,z\in S$: it would imply $w=z=0$ and $(0,2)=2v$, but S contains no such element v. Thus S does not have the strong RIP.

On the other hand, assume $a+b=c+d$ holds in S. To produce $u,v,w,z\in S$ with $a=u+v$, $b=w+z$, $c=u+w$, $d=v+z$, we consider several cases. Write $a=(p_a,q_a)$, etc.

Case I: $p_a,p_b,p_c,p_d>0$. Then $p_a-p_d=p_c-p_b<p_a,p_c$; let r be a rational number such that $r>0$, $r>p_a-p_d=p_c-p_b$, $r<p_a$, $r<p_c$. Let $u=(r,0)$, $v=(p_a-r,q_a)$, $w=(p_c-r,q_c)$, $z=(r-p_c+p_b,q_b-q_c)=(r-p_a+p_d,q_d-q_a)$. Then $u,v,w,z\in S$ (with $p_u,p_v,p_w,p_z>0$) and $a=u+v$, $b=w+z$, $c=u+w$, $d=v+z$.

Case II: some, but not all, of p_a,p_b,p_c,p_d are 0. Then $p_a+p_b=p_c+p_d\neq 0$, and we may assume $p_a=0$, $p_c>0$. Then $u=a$, $v=0$, $z=d$ and $w=b-d=c-a$ are in S (as $p_w=p_c-p_a>0$) and serve.

Case III: $p_a=p_b=p_c=p_d=0$. Then a,b,c,d lie, up to isomorphism, in the additive semigroup of even nonnegative integers, so the existence of u,v,w,z follows from 1.7. □

2. The main theorem

2.1. In this section we prove our main result, namely:

Theorem 2.1. *A commutative monoid is a directed colimit of free commutative monoids if and only if it is cancellative, reduced, and has the strong Riesz interpolation property.*

In view of 1.5, 1.8, it suffices to prove the converse of 1.8, more precisely, that in a cancellative reduced [commutative] monoid S the strong RIP implies the KIP. This amounts to using the strong RIP sufficiently many times to produce a presentation of any finitely many elements of S which kills all the relations between these elements. Most of the work is actually spent on killing just one relation. We note that the strong RIP can be used (e.g. through 1.12) to kill simple types of relations, and we proceed through a sequence of lemmas which kill increasingly complicated relations. One may think of 1.12 as the first such lemma. It is assumed throughout that S is a cancellative reduced monoid.

We begin with a lemma which does not really belong in that sequence. It aims at avoiding repeated presentations of the same element of S, at least in one case (repeated presentations are not allowed in the KIP).

Lemma 2.2. *Let S have the* RIP; *assume that $a=\Sigma_{j=1}^{j=q_i}u_{i,j}$ holds in S for $i=$*

$1, 2, \ldots, n$. *Then there exist* $v_{k_1 \ldots k_n} \in S$, $1 \leq k_i \leq q_i$, *such that* $a = \sum v_{k_1 \ldots k_n}$ *and* $u_{i,j} = \sum_{k_i = j} v_{k_1 \ldots k_n}$ *for all* i, j.

Proof. We proceed by induction on n. There is nothing to show if $n = 1$. Now assume $n \geq 2$, the property holds for $n - 1$ and $a, u_{i,j}$ are as in the statement. Then there exist $w_{k_1 \ldots k_{n-1}} \in S$ $(1 \leq k_i \leq q_i)$ such that $u_{i,j} = \sum_{k_i = j} w_{k_1 \ldots k_{n-1}}$ for all i, j $(i \neq n)$, and $\sum w_{k_1 \ldots k_{n-1}} = a = \sum_j u_{n,j}$. Applying 1.12 to this last relation yields elements $v_{k_1 \ldots k_{n-1} k_n} \in S$ $(1 \leq k_n \leq q_n)$ such that $w_{k_1 \ldots k_{n-1}} = \sum_{j=1}^{j=q_n} v_{k_1 \ldots k_{n-1} j}$ for all $k_1, \ldots, k_{n-1}$ and $u_{n,j} = \sum_{k_n = j} v_{k_1 \ldots k_n}$ for all j. Then also $u_{i,j} = \sum_{k_i = j} v_{k_1 \ldots k_n}$ for all $i < n$, and $a = \sum v_{k_1, \ldots, k_n}$, i.e. the property holds for n. □

2.2. We now begin killing relations.

Lemma 2.3. *Let S have the* RIP; *assume that* $na + a_2 + \ldots + a_q = b_1 + \ldots + b_r$ *holds in* S, *with* $n > 0$. *Then there exist* $u_{i,j} \in S$ $(2 \leq i \leq q, 1 \leq j \leq r)$ *and* $w_{k_1 \ldots k_n} \in S$ $(1 \leq k_i \leq r)$ *such that* $a = \sum w_{k_1 \ldots k_n}$, $a_i = \sum_{j=1}^{j=r} u_{i,j}$ *and* $b_j = \sum_{i=2}^{i=q} u_{i,j} + \sum_{s=1}^{s=n} \sum_{k_s = j} w_{k_1 \ldots k_n}$ *for all* $i \geq 2$ *and all* j.

Proof. By 1.12, there are elements $u_{i,j} \in S$ $(1 \leq i \leq q, 1 \leq j \leq r)$ such that $na = \sum_{j=1}^{j=r} u_{1,j}$, $a_i = \sum_{j=1}^{j=r} u_{i,j}$ when $i \geq 2$ and $b_j = u_{1,j} + \sum_{i=2}^{i=q} u_{i,j}$ for all j. Similarly, $a + a + \ldots + a = u_{1,1} + \ldots + u_{1,r}$ yields $v_{s,j} \in S$ $(1 \leq s \leq n, 1 \leq j \leq r)$ such that $a = \sum_{j=1}^{j=r} v_{s,j}$ for all s and $u_{1,j} = \sum_{s=1}^{s=n} v_{s,j}$ for all j. The first s relations and 2.2 then yields $w_{k_1 \ldots k_n} \in S$ $(1 \leq k_i \leq r)$ such that $a = \sum w_{k_1 \ldots k_n}$ and $v_{s,j} = \sum_{k_s = j} w_{k_1 \ldots k_n}$ for all s, j. The result follows. □

Lemma 2.4. *Let S have the* RIP; *assume that* $p_1 a_1 + \ldots + p_q a_q = b_1 + \ldots + b_r$ *holds in* S, *where all* $p_i > 0$. *Then there exist* $z_k \in S$ *and integers* $n_i^k, m_j^k \geq 0$ $(1 \leq i \leq q, 1 \leq j \leq r)$ *such that* $a_i = \sum_k n_i^k z_k$, $b_j = \sum_k m_j^k z_k$ *and* $\sum_{i=1}^{i=q} p_i n_i^k = \sum_{j=1}^{j=r} m_j^k$ *for all* i, j, k.

Proof. We use induction on q: if $q = 1$, the result follows from 2.3. Assume $q \geq 2$ and the property holds for $q - 1$. If $p_1 a_1 + \ldots + p_q a_q = b_1 + \ldots + b_r$, then by 2.3 there exist $w_\alpha, u_j \in S$ such that $a_1 = \sum_\alpha w_\alpha$, $p_2 a_2 + \ldots + p_q a_q = \sum_{j=1}^{j=r} u_j$ and $b_j = u_j + \sum_\alpha m_j^\alpha w_\alpha$, where $m_j^\alpha \geq 0$ is the number of times that j appears as component of $\alpha = (k_1, \ldots, k_{p_1})$ (i.e. the number of times that w_α appears in the summation $\sum_{s=1}^{s=n} \sum_{k_s = j} w_{k_1 \ldots k_{p_1}}$). We see that $\sum_{j=1}^{j=r} m_j^\alpha = p_1$, the total number of components of α. Next, the induction hypothesis, applied to $p_2 a_2 + \ldots + p_q a_q = u_1 + \ldots + u_r$ yields $x_\beta \in S$ and integers $n_i^\beta, m_j^\beta \geq 0$ $(2 \leq i \leq q, 1 \leq j \leq r)$ such that $a_i = \sum_\beta n_i^\beta x_\beta$, $u_j = \sum_\beta m_j^\beta x_\beta$ and $\sum_{i=2}^{i=q} p_i n_i^\beta = \sum_{j=1}^{j=r} m_j^\beta$ for all $i \geq 2$, j and β. Then $a_1 = \sum_\alpha w_\alpha$, $a_i = \sum_\beta n_i^\beta x_\beta$ $(i \geq 2)$, $b_j = \sum_\beta m_j^\beta x_\beta + \sum_\alpha m_j^\alpha w_\alpha$ is a presentation of the a's and b's in terms of $\{z_k\} = \{w_\alpha, x_\beta\}$ with the desired properties. □

Lemma 2.5. *Let S have the strong* RIP; *assume that* $pa = q_1 b_1 + \ldots + q_r b_r$ *holds in*

S. Then there exist $z_k \in S$ *and integers* $n^k, m_j^k \geq 0$ $(1 \leq j \leq r)$ *such that* $a = \Sigma_k n^k z_k$, $b_j = \Sigma_k m_j^k z_k$ *and* $pn^k = \Sigma_j q_j m_j^k$ *for all* j, k.

Proof. Since S is reduced, the given relation, and the result, are trivial if $p = 0$; if $p = 1$, the result follows from 2.4. We assume $p \geq 2$ and proceed by induction on p, i.e. also assume that $nx = \Sigma_j m_j y_j$ can be killed (as in the statement) whenever $n < p$. We now kill all (R): $pa = q_1 b_1 + \ldots + q_r b_r$. Note that it suffices to do so when all $q_j > 0$: if, say, $q_r = 0$, then we kill $pa = q_1 b_1 + \ldots + q_{r-1} b_{r-1}$, present b_r with $m_r^k = 0$ for all k, and so obtain a presentation which evidently kills (R).

First consider the case when $q_j < p$ for all j. We can kill (R), by 2.4, if $q_j = 1$ for all j. We now show by induction on s that (R) can be killed if $q_j = 1$ for all $j \geq s$: this is true if $s = 1$, and letting $s = r + 1$ kills (R). Assume $pa = q_1 b_1 + \ldots + q_s b_s + b_{s+1} + \ldots + b_r$ (where $1 \leq s \leq r$, so that $q_j = 1$ for all $j \geq s + 1$). By the induction hypothesis on s, there exist $z_k \in S$ and $n^k, m_j^k \geq 0 (1 \leq j \leq r)$ such that $a = \Sigma_k n^k z_k$, $b_j = \Sigma_k m_j^k z_k$ for all $j \neq s$, $q_s b_s = \Sigma_k m_s^k z_k$ and for all k, $pn^k = q_1 m_1^k + \ldots + q_{s-1} m_{s-1}^k + m_s^k + \ldots + m_r^k$. Since $q_s < p$, the induction hypothesis on p yields $x_i \in S$, $c^i, d_k^i \geq 0$ such that $b_s = \Sigma_i c^i x_i$, $z_k = \Sigma_i d_k^i x_i$ and $q_s c^i = \Sigma_k m_s^k d_k^i$, for all k, i. This yields $a = \Sigma_i (\Sigma_k n^k d_k^i) x_i$, $b_j = \Sigma_i (\Sigma_k m_j^k d_k^i) x_i$ if $j \neq s$, $b_s = \Sigma_i c^i x_i$, which kills $pa = q_1 b_1 + \ldots + q_s b_s + b_{s+1} + \ldots + b_r$ since $p(\Sigma_k n^k d_k^i) = q_1(\Sigma_k m_1^k d_k^i) + \ldots + q_{s-1}(\Sigma_k m_{s-1}^k d_k^i) + q_s c^i + \Sigma_k m_{s+1}^k d_k^i + \ldots + \Sigma_k m_r^k d_k^i [= q_1(\Sigma_k m_1^k d_k^i) + \ldots + q_{s-1}(\Sigma_k m_{s-1}^k d_k^i) + \Sigma_k m_s^k d_k^i + \ldots + \Sigma_k m_r^k d_k^i]$, for all i. This completes the induction on s and thus (R) can be killed when $q_j < p$ for all j.

Now consider the general case (only assume $q_j > 0$). Put $q_j = pm_j + s_j$, where $0 \leq s_j < p$. Then $pa = p(\Sigma_j m_j b_j) + pa = p(\Sigma_j m_j b_j) + (\Sigma_j s_j b_j)$. The strong RIP yields $u, v, w, z \in S$ such that $a = u + v$, $0 = pw + z$, $\Sigma_j m_j b_j = u + w$, $\Sigma_j s_j b_j = pv + z$; since S is reduced, we have $w = z = 0$ and hence $a = u + v$, $\Sigma_j m_j b_j = u$, $\Sigma_j s_j b_j = pv$. Since $s_j < p$ for all j, the first part of the proof yields $z_k \in S$, n^k, $m_j^k \geq 0$ such that $v = \Sigma_k n^k z_k$, $b_j = \Sigma_k m_j^k z_k$ and $pn^k = \Sigma_j s_j m_j^k$, for all i, j. Then $a = \Sigma_j m_j b_j + v = \Sigma_k (n^k + \Sigma_j m_j m_j^k) z_k$, $b_j = \Sigma_k m_j^k z_k$ is a presentation which kills (R), since $p(n^k + \Sigma_j m_j m_j^k) = \Sigma_j s_j m_j^k + \Sigma_j p m_j m_j^k = \Sigma_j q_j m_j^k$, for all k $\square$

Lemma 2.6. *Let* S *have the strong* RIP; *assume that* $p_1 a_1 + \ldots + p_n a_n = q_1 b_1 + \ldots + q_m b_m$ *holds in* S. *Then there exist* $z_k \in S$ *and integers* $n_i^k, m_j^k \geq 0$ $(1 \leq i \leq n,\ 1 \leq j \leq m)$ *such that* $a_i = \Sigma_k n_i^k z_k$, $b_j = \Sigma_k m_j^k z_k$ *and* $\Sigma_i p_i n_i^k = \Sigma_j q_j m_j^k$, *for all* i, j, k.

Proof. Again we may always assume $q_j > 0$ for all j. If $q_j = 1$ for all j, the result follows from 2.4; we shall prove by induction on s that the given relation can be killed if $q_j = 1$ for all $j > s$. Assume that this holds for s and that $\Sigma_i p_i a_i = q_1 b_1 + \ldots + q_s b_s + b_{s+1} + \ldots + b_m$, where $1 \leq s \leq r$ (so that $q_j = 1$ when $j \geq s + 1$). The induction hypothesis yields $x_\alpha \in S$ and $n_i^\alpha, m_j^\alpha \geq 0$ such that $a_i = \Sigma_\alpha n_i^\alpha x^\alpha$ for all i, $b_j = \Sigma_\alpha m_j^\alpha x_\alpha$ for all $j \neq s$, $q_s b_s = \Sigma_\alpha m_s^\alpha x^\alpha$ and $\Sigma_i p_i n_i^\alpha = q_1 m_1^\alpha + \ldots + a_{s-1} m_{s-1}^\alpha + m_s^\alpha + \ldots + m_m^\alpha$ for all α. Next, 2.5 and $q_s b_s = \Sigma_\alpha m_s^\alpha x_\alpha$ yield

$z_k \in S$ and n^k, $m^k_\alpha \geqslant 0$ such that $b_s = \Sigma_k n^k z_k$, $x_\alpha = \Sigma_k m^k_\alpha z_k$ and $q_s n^k = \Sigma_\alpha m^\alpha_s m^k_\alpha$ for all k, α. This yields a presentation $a_i = \Sigma_k(\Sigma_\alpha n^\alpha_i m^k_\alpha)z_k$, $b_j = \Sigma_k(\Sigma_\alpha m^\alpha_j m^k_\alpha)z_k$ if $j \neq s$, $b_s = \Sigma_k n^k z_k$ which kills $\Sigma_i p_i a_i = q_1 b_1 + \ldots + q_s b_s + b_{s+1} + \ldots + b_m$ since, for all k, $\Sigma_i(\Sigma_\alpha p_i n^\alpha_i m^k_\alpha) = \Sigma_\alpha q_1 m^\alpha_1 m^k_\alpha + \ldots + \Sigma_\alpha s_{s-1} m^\alpha_{s-1} m^k_\alpha + \Sigma_\alpha m^\alpha_s m^k_\alpha + \ldots + \Sigma_\alpha m^\alpha_m m^k_\alpha = q_1(\Sigma_\alpha m^\alpha_1 m^k_\alpha) + \ldots + q_{s-1}(\Sigma_\alpha m^\alpha_{s-1} m^k_\alpha) + q_s n^k + \Sigma_\alpha m^k_{s+1} m^k_\alpha + \ldots + \Sigma_\alpha m^\alpha_m m^k_\alpha$. □ Note that the strong RIP was used in this proof only through 2.5.)

2.3. With Lemma 2.6 the proof of the theorem is essentially complete. Indeed 2.6 tells us that any one relation between finitely many elements of S can always be killed: since S is cancellative, a non-trivial relation between elements of a finite set $X \subseteq S$ can always be written as $\Sigma_i p_i a_i = \Sigma_j q_j b_j$, where $a_i, b_j \in X$ and the a_i, b_j are all distinct; we can arrange that in fact $X = \{a_i, b_j\}$, by putting all the elements of X which are not of this form in one side of the relation, with zero coefficients. Then 2.6 yields a presentation of each element of X which kills the given relation.

The fact that we can kill any one relation between the elements of X implies (as shown by Shannon [9]) that we can kill any finitely many relations between the elements of X (and hence, by Rédei's theorem, all relations); and thus S has the KIP and the theorem is proved. [We outline the proof of Shannon's result in case the reader would prefer a more direct argument. The killing of any n relations between the elements of any finite subset $X \subseteq S$ is done by induction on n, as follows. We already know the case $n = 1$. If $n > 1$, the induction hypothesis yields a presentation of X in terms of the elements of a finite subset $Y \subseteq S$, which kills the first $n - 1$ relations. The last relation then yields a relation between the elements of Y, which can in turn be killed by presenting Y in terms of the elements of some $Z \subseteq S$. Combining the two yields a presentation of X in terms of Z, which is readily seen to kill all n relations.]

2.4. A slight improvement of our main theorem can be readily obtained by a careful look at its proof. The strong RIP (as against the RIP) is quoted for use in the proof only in the proof of Lemma 2.5; and there the full strength of the implication $na + b = nc + d \implies a = u + v$ etc., is not used, since we only use this implication in a case where $b = 0$, i.e. (as S is reduced) we only use the implication (C): $na = nc + d \implies a = c + v$, $na = nc + d \implies a = c + v, d = nv$ for some $u, v \in S$ (where $n > 0$). It follows that, in the cancellative reduced monoid S, this implication, together with the RIP, implies the KIP and hence is equivalent to the strong RIP.

A reduced monoid which satisfies (C) must be power-cancellative (since $d = 0$ in (C) then forces $v = 0$) and satisfy (D): $na = nc + d \implies d$ is divisible by n. Conversely, a power-cancellative monoid which satisfies (D) is readily seen to satisfy (C). Therefore, in a cancellative reduced monoid, the strong RIP is equivalent to the conjunction of the RIP, power-cancellativity and condition (D). (The main theorem could have been stated in this fashion, but we preferred to state it with the single condition $R(n)$.)

We state this improvement to the main result as part of the following proposition (which, except for the implication b) $\Longrightarrow$ a) which we have just shown, and was only conjectured, is due entirely to our referee):

Proposition 2.7. *Let S be a cancellative reduced monoid. The following conditions on S are equivalent:*

a) *S has the strong* RIP;

b) *S has the* RIP, *is power-cancellative, and $na = nc + d$ implies $d \in nS$;*

c) *S is the positive cone of a Riesz ordered abelian group A in which $nx \geq 0$ implies $x \geq 0$ (when $n > 0$).*

Proof. The equivalence of a) and b) was shown above. Now assume b) and let A be the universal group of S (or group of fractions of S, now written as a group of differences). We can order A with S as positive cone, and then A is a Riesz group (by 1.9 and the remark following its proof). Also, A is torsion-free, since S is power-cancellative. Now assume $x \in A$, $nx \geq 0$ (where $n > 0$), i.e. $nx \in S$. Put $x = a - b$, where $a, b \in S$. Then $na = nb + (nx)$, with $a, b, nx \in S$; by b), $nx = nc$ for some $c \in S$; hence $x = c \in S$, i.e. $x \geq 0$. Thus b) implies c).

Conversely, assume that c) holds. Then S has the RIP, by 1.9. When $n > 0$, $nx \geq 0$ implies $x \geq 0$, so $nx = 0$ implies $x \geq 0$, $-x \geq 0$ and $x = 0$; thus A is torsion-free, and S is power-cancellative. Finally, assume $na = nc + d$, with $a, c, d \in S$, $n > 0$. Then $n(a - c) = d \geq 0$, so that $a - c \geq 0$, i.e. $a - c \in S$, and $d = n(a - c) \in nS$. □

Note that 1.6, 1.11 follows immediately from ?.7.

3. From monoids to semigroups

3.1. In this section we complete the main theorem by:

Proposition 3.1. *Let S be a [commutative] semigroup without identity element. Then S is a directed colimit of free semigroups if and only if S^0 is a directed colimit of free monoids.*

Proof. First it is clear that, if S is a directed colimit of free semigroups F_i, then S^0 is directed colimit of the free monoids $(F_i)^0$. Now assume that S^0 is the colimit of the free monoids F_i $(i \in I)$, where I is a directed preordered set, and homomorphisms $f_i^j: F_i \to F_j$ $(i \leq j)$ (where f_i^i is the identity on F_i and $f_k^j \circ f_j^i = f_k^i$ whenever $i \leq j \leq k$); let $f^i: F_i \to S^0$ $(i \in I)$ be the colimiting cone.

For each $i \in I$, let $K_i = (f_i)^{-1}(\{0\})$. We see that K_i is a subsemigroup of F_i. Since S^0 is reduced, $u + v \in K_i$ implies $u, v \in K_i$. Therefore K_i is free; more precisely, if B_i is the basis of F_i, then K_i is the submonoid of F_i generated by $B_i \cap K_i$. Let $\mathscr{C}_i$ be the least cancellative congruence on F_i which identifies K_i to 0; i.e. $u\mathscr{C}_i v$ if and only if $u+p=v+q$ for some $p,q\in K_i$. The quotient $G_i=F_i/\mathscr{C}_i$ is again a free monoid, whose basis may be identified with $B_i\backslash K_i$. Let $p_i:F_i\to G_i$ be the projection.

On the description of $\mathscr{C}_i$ we see that $u\mathscr{C}_i v$ implies $f^i u=f^i v$. Therefore there is a homomorphism $g^i:G_i\to S^0$ unique such that $f^i=g^i\circ p_i$. Note that $(p_i)^{-1}(\{0\})=K_i$; hence $(g^i)^{-1}(\{0\})=\{0\}$; we say that g^i is *pure*. Now assume $i\leqslant j$. We see that $f^i_j(K_i)\subseteq K_j$; hence $u\mathscr{C}_i v$ implies $f^i_j(u)\mathscr{C}_j f^i_j(v)$; therefore there exists a homomorphism g^i_j: $G_i\to G_j$ unique such that $g^i_j\circ p_i=p_j\circ f^i_j$. From the uniqueness we see that g^i_i is the identity on G_i and $g^j_k\circ g^i_j=g^i_k$ whenever $i\leqslant j\leqslant k$; also, $g^i=g^j\circ g^i_j$ whenever $i\leqslant j$. In fact, $(g^i)_{i\in I}$ is a colimiting cone. To see this, first note that $S^0=\bigcup f^i(F_i)=\bigcup g^i(G_i)$ (since each p_i is surjective). Further assume that $g^i(p_i(u))=g^i(p_i(v))$ for some $p_i(u),p_i(v)\in G_i$. Then $f^i(u)=f^i(v)$, whence $f^i_j(u)=f^i_j(v)$ for some $j\geqslant i$. Applying p_j yields $g^i_j(p_i(u))=g^i_j(p_i(v))$, for some $j\geqslant i$. Thus S^0 is the directed colimit of the free monoids G_i and homomorphisms g^i_j.

What has been gained in this new description of S^0 is that all the colimiting maps g^i are pure. Since $g^i_j(u)=0$ implies $g^i(u)=0$, all maps g^i_j are pure too. Therefore they induce homomorphisms h^i: $G_i\backslash\{0\}\to S$, h^i_j: $G_i\backslash\{0\}\to G_j\backslash\{0\}$, and we see that S is the directed colimit of the free semigroups $G_i\backslash\{0\}$ (and homomorphisms h^i_j). □

3.2. This suggests that we extend the definition of the strong RIP as follows: if S does not have an identity element, then S has the strong RIP if and only if S^0 does. Then we can state:

Corollary 3.2. *A [commutative] semigroup is a directed colimit of free semigroups if and only if it is cancellative without identity element and has the strong* RIP. □

4. The embedding theorem

4.1. Let S be a directed colimit of free monoids. Then we saw that S is cancellative, power-cancellative and reduced, every submonoid of S has these properties. Conversely, we shall show that a cancellative, power-cancellative reduced monoid S can always be embedded into a directed colimit of free monoids. (By 3.1, the same result is then true for semigroups without identity element.) Since the main theorem describes directed colimits of free monoids as an implicative class, as defined by McAlister [8], the result can be established by McAlister's technique: the embedding is then constructed recursively, each step consisting in the adjunction of elements u, v, w, z for each a, b, c, d, n satisfying $na+b=nc+d$, such that $u+v=a$, etc. We give a different proof, whose principle is not too different, but which is based on a pushout lemma for free envelopes that may be of interest in itself.

4.2. Lemma 4.1. *Let S be a cancellative, power-cancellative, reduced monoid, A be* *a finitely generated submonoid of S and $\alpha : A \to F(A)$ be the free envelope of A. In the category of cancellative and power-cancellative [commutative] monoids, the pushout:*

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & F(A) \\ \downarrow & & \downarrow \\ S & \longrightarrow & S * A \end{array}$$

has the following properties: $S \to S * A$ is injective, and $S * A$ is reduced.

Proof. Recall that the pushout may be constructed as follows. Let $T = S \times F(A)$ $[= S \sqcup F(A)]$. Let $\mathscr{C}$ be the binary relation on T defined by: $(s, u)\mathscr{C}(t, v)$ if and only if $s + b = t + a$, $u + \alpha(a) = v + \alpha(b)$ for some $a, b \in A$. It is easy to verify that $\mathscr{C}$ is the least cancellative congruence on T such that $(a, 0)\mathscr{C}(0, \alpha(a))$ for all $a \in A$. However, $\mathscr{C}$ might not be power-cancellative as well. Let $P = T/\mathscr{C}$ and $\mathscr{P}$ be the least power-cancellative congruence on P; i.e., $x\mathscr{P}y$ if and only if $nx = ny$ for some $n > 0$. It is then readily seen that $S * A = P/\mathscr{P}$ is the quotient of T by the least cancellative and power-cancellative congruence which identifies $(a, 0)$ and $(0, \alpha(a))$ for each $a \in A$. The map $S \to S * A$ is obtained by composing the injection $s \mapsto (s, 0)$ and the projections $T \to P \to P/\mathscr{C}$. The map $F(A) \to S * A$ is defined similarly.

To prove the lemma, we first show that $S \to P$ is injective and P is reduced (this amounts to proving the similar lemma for the cancellative pushout). Since A is also cancellative, power-cancellative and reduced, α is injective. Hence $(s, 0)\mathscr{C}(t, 0)$ implies, for some $a, b \in A$, $s + b = t + a$ and $0 + \alpha(a) = 0 + \alpha(b)$, whence $\alpha(a) = \alpha(b)$, $a = b$ and $s = t$. Thus $S \to P$ is injective. Next, assume $(s, u)\mathscr{C}(0, 0)$, so that $s + b = a$, $u + \alpha(a) = \alpha(b)$ for some $a, b \in A$. Let B be the submonoid of S generated by A and s, and β: $B \to F(B)$ be its free envelope. Since B is also cancellative, power-cancellative and reduced, β is injective; furthermore, by the weak universal property of free envelopes, the restriction $A \to F(B)$ of β yields a homomorphism γ: $F(A) \to F(B)$ such that $\gamma(\alpha(a)) = \beta(a)$ for all $a \in A$. Then $\beta(s) + \beta(b) = \beta(a)$, $\gamma(u) + \beta(a) = \beta(b)$ hold in $F(B)$; since $F(B)$ is free, this implies $\beta(a) = \beta(b)$, whence $a = b$, and $s = 0$, $u = 0$. Thus, $(s, u)\mathscr{C}(0, 0)$ implies $s = 0$, $u = 0$. It follows that P is reduced: if (s, u) yields a unit in P, then $(s, u) + (r, v)\mathscr{C}(0, 0)$ for some $r \in S$, $v \in F(A)$, whence $s + r = 0$, $u + v = 0$ and $s = 0$, $u = 0$ since S and $F(A)$ are reduced.

Recall that $S \to S * A$ is the composition $S \to P \to P/\mathscr{P} = S * A$, where $x\mathscr{P}y \Leftrightarrow nx = ny$ for some $n > 0$. Since S is power-cancellative, we see that $\mathscr{P}$ is the equality on the range of $S \to P$ (which is isomorphic to S); thus $S \to S * A$ is injective. If furthermore $p \in P$ yields a unit of $S * A$, then $p + q\mathscr{P}0$ for some $q \in P$, whence $np + nq = 0$ for some $n > 0$, and $p = q = 0$ since P is reduced; therefore $S * A$ is reduced. □

4.3. Now let S be a cancellative, power-cancellative reduced monoid. We build a semigroup S_1 as follows. First, we well-order the set of all finitely generated submonoids of S, which can then be written as a family (A_σ) indexed by all ordinal numbers less than some ordinal υ. An ascending tower $(T_\tau)_{\tau<\upsilon}$ is then built as follows: $T_1 = S * A_1$; if $\tau = \sigma + 1$, then $T_\tau = T_\sigma * A_\tau$; if τ is a limit ordinal, then $T_\tau = (\bigcup_{\sigma<\tau} T_\sigma) * A_\tau$ (in this construction we identify each $S \to S * A$ to an inclusion map). Then $S_1 = \bigcup_{\tau<\upsilon} T_\tau$. By induction, S_1 is cancellative, power-cancellative and reduced; furthermore, $S \subseteq S_1$ and, by 1.4, 1.1, S_1 kills every finitely generated submonoid of S.

From this we build another ascending tower $S \subseteq S_1 \subseteq S_2 \subseteq \ldots$, with $S_{n+1} = (S_n)_1$ for all $n \geqslant 1$. Let $\hat{S} = \bigcup S_n$. Each finitely generated submonoid of $\hat{S}$ is contained in some S_n and hence is killed by $S_{n+1} \subseteq \hat{S}$. In other words, $\hat{S}$ has the KIP.

In addition, $\hat{S}$ has a weak universal property. Let φ be any homomorphism of S into a monoid K with the KIP. Let A be a finitely generated submonoid of S. Then K kills $\varphi(A)$, so that the inclusion map $\varphi(A) \to K$ factors through the free envelope $\varphi(A) \to F(\varphi(A))$. But the composite $A \to \varphi(A) \to F(\varphi(A))$ in turn factors through the free envelope $A \to F(A)$, since $F(\varphi(A))$ is a free monoid; hence φ can be extended from A to $F(A)$. The universal property of the pushout then shows that φ can be extended to $S * A$. It follows, by induction, that φ can be extended to every T_τ and hence to S_1; and then φ can be extended to $S_2, S_3, \ldots$ and eventually to $\hat{S}$. We have proved:

Theorem 4.2. *Let S be a cancellative, power-cancellative reduced commutative monoid. Then S can be embedded into a directed colimit $\hat{S}$ of free commutative monoids, so that every homomorphism of S into a directed colimit of free commutative monoids extends to $\hat{S}$.* □

The result evidently extends to semigroups without identity element.

Theorem 4.2 is much easier to prove if one only wants an embedding, without the weak universal property. One can let T be a maximal reduced submonoid, containing S, of the universal group A of S. If $x \notin T$, then by maximality the submonoid of A generated by T and x is not reduced, hence must contain a unit $t + nx \neq 0$, with $n \geqslant 0$; since T is reduced, $n > 0$. Since $t + nx$ is a unit of T, we have $t + nx + u + mx = 0$ for some $u \in T$, $m \geqslant 0$, and thus $kx \in (-T)$ for some $k > 0$. This is impossible if $px \in T$ for some $p > 0$, lest $0 \neq kpx \in T \cap -T$; hence $py \in T$, $p > 0$ implies $y \in T$. By symmetry, $-T$ also has this property, and it follows that $x \in (-T)$. Thus $A = T \cup -T$. Since T is reduced, we can order A with T as positive cone, and then $A = T \cup -T$ shows that A is totally ordered. By 1.6, T has the strong RIP, and by our main result S is now embedded into a monoid T with the KIP. (We are endebted to the referee for this observation.)

4.4. If S is finitely generated, the free envelope of S can serve as $\hat{S}$ in Theorem 4.2.

However, it is not known if in general an embedding of S can be found as in Theorem 4.2, which also has any of the minimality properties of free envelopes.

Another open question is whether a directed colimit S of free monoids must actually be locally free: i.e. must any finitely many elements of S lie in a free submonoid of S? The author could only show that the answer is yes under the additional hypothesis that S is divisible and finite-dimensional [6], a severely particular case. No answer is known outside of that case.

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