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Finite-Product-Preserving Functors, Kan Extensions, and Strongly-Finitary 2-Monads

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Abstract. We study those 2-monads on the 2-category **Cat** of categories which, as endofunctors, are the left Kan extensions of their restrictions to the sub-2-category of finite discrete categories, describing their algebras syntactically. Showing that endofunctors of this kind are closed under composition involves a lemma on left Kan extensions along a coproduct-preserving functor in the context of cartesian closed categories, which is closely related to an earlier result of Borceux and Day.

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1. Introduction

Kelly and Power [6] studied finitary $\mathcal{V}$-monads on a locally-finitely-presentable $\mathcal{V}$-category $\mathcal{A}$, and presentations of these; here $\mathcal{V}$ was to be locally finitely presentable as a closed category, in the sense of Kelly [5]. The monad (T, η, μ) is *finitary* if its endofunctor-part $T : \mathcal{A} \to \mathcal{A}$ is so; recall that a $\mathcal{V}$-functor $T : \mathcal{A} \to \mathcal{B}$ is said to be finitary if it preserves filtered colimits. When $\mathcal{A}$ is locally finitely presentable. T is finitary precisely when it is the left Kan extension of its restriction to $\mathcal{A}_f$, the full subcategory of $\mathcal{A}$ determined by the finitely-presentable objects: see [5, Proposition 7.6].

The important special case of a finitary 2-monad T on the 2-category **Cat** of (small) categories is that where $\mathcal{V}$ is the cartesian-closed category **Cat** and $\mathcal{A}$ is the 2-category $\mathcal{V}$. The results of [6] show that in this case a T-algebra for some such T is precisely a category A endowed with a structure which admits a description of the following sort. There are *basic operations* of two kinds: first, functors $A^c \to A$, where c is a finitely-presentable category, and secondly, natural transformations between such functors; then there are *equations*, in general at the levels both of objects and of morphisms, between *derived operations*. Categories with (chosen) limits and colimits of various kinds are examples of such algebras; to discuss equalizers, for instance, we need an operation $A^c \to A$ where c is $\bullet \rightrightarrows \bullet$.

Such structures as monoidal closed or cartesian closed categories escape this description; they are instead algebras for monads on the groupoid-enriched category

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$\mathbf{Cat}_g$ of categories, functors, and natural *isomorphisms* – a concept of wider scope (in that every 2-monad on **Cat** gives such a monad on $\mathbf{Cat}_g$), but still covered by the results of [6]. The present article is concerned with a concept of *narrower* scope, not so covered: namely the subclass of the finitary 2-monads on **Cat** consisting of those whose algebras may be described using only functors $A^n \to A$ where n is a natural number (as well as natural transformations between these and equations between derived operations). The reason for our interest in this subclass is that some important properties of T-algebras, such as those studied in the following article on categories of fractions, hold only for such T. Examples of structures corresponding to monads in this class are monoidal or symmetric monoidal categories, categories with finite products or finite coproducts or both, distributive categories, and so on.

A rational study of such 2-monads needs a better description of them than that above, which in effect defines them by the existence of a presentation of a certain kind. We shall show that they are precisely those 2-monads whose endo-2-functor part $T : \mathbf{Cat} \to \mathbf{Cat}$ is *strongly finitary*. By this we mean that T is the left Kan extension (in the **Cat**-enriched sense, of course) of its restriction to the full sub-2-category $\mathbf{S}$ of **Cat** determined by the finite sets $n \in \mathbf{N}$ seen as discrete categories; or equally that T is in the replete image of the full embedding $L : [\mathbf{S}, \mathbf{Cat}] \to [\mathbf{Cat}, \mathbf{Cat}]$ (of 2-categories) given by left Kan extension along the full inclusion $J : \mathbf{S} \to \mathbf{Cat}$. Since $\mathbf{S} \subset \mathbf{Cat}_f$, the strongly-finitary endo-2-functors are *a fortiori* finitary. (We shall in future abbreviate "2-functor" or "$\mathcal{V}$-functor" to "functor", where the context makes the meaning clear; and sometimes even "2-category" or "$\mathcal{V}$-category" to "category".) By [4, (4.18)] we have for L the explicit formula

$$L(R)A = \int^{n\in\mathbf{S}} A^n \times Rn \,, \tag{1.1}$$

which can be seen indifferently as a 2-coend or as a mere unenriched coend, since $\mathbf{S}$ has only trivial 2-cells; here of course $L(R)A$ stands for $(L(R))\,(A)$.

As the left adjoint of the restriction, L of course preserves colimits. A central observation below is the fact that it preserves finite products. (Equivalently, the strongly-finitary endofunctors are closed under finite products; contrast this with the fact that, since finite (weighted) limits commute with filtered colimits in **Cat**, the finitary endofunctors are closed under finite limits.) The consequence of this, that L preserves finite powers, gives a simple proof, as we shall see below, that strongly-finitary endofunctors (like finitary ones) are closed under composition; we find that

$$L(Q)\,L(R) \cong L(Q \circ R)\,, \tag{1.2}$$

where

$$(Q \circ R)\,n = \int^{m\in\mathbf{S}} (Rn)^m \times Qm\,. \tag{1.3}$$

Moreover, the identity endofunctor $1 : \mathbf{Cat} \to \mathbf{Cat}$ is strongly finitary, since

$$L(J) \cong 1 \tag{1.4}$$

by an easy application of the (enriched) Yoneda isomorphism; which is just to say that the 2-functor $J : \mathbf{S} \to \mathbf{Cat}$ is dense. So $([\mathbf{S}, \mathbf{Cat}], \circ, J)$ is a monoidal 2-category, monoidally equivalent to the strict monoidal 2-category $SFin[\mathrm{Cat}, \mathrm{Cat}]$ of strongly-finitary 2-functors with composition as the tensor product. In fact $[\mathbf{S}, \mathbf{Cat}]$ is right-closed, since we easily verify that

$$[\mathbf{S}, \mathbf{Cat}]\,(Q \circ R, P) \cong [\mathbf{S}, \mathbf{Cat}]\,(Q, < R, P >) \tag{1.5}$$

where

$$< R, P > \; n = \int_{m \in \mathbf{S}} [(Rm)^n, Pm] \tag{1.6}$$

and where $[X, Y]$ like Y^X denotes the internal hom in **Cat**.

Thus we may identify the strongly-finitary 2-monads with the monoids in $[\mathbf{S}, \mathbf{Cat}]$; in fact the monoids in a monoidal category form only an ordinary category, so that we are dealing with the category $Mon(\mathbf{S}, \mathbf{Cat})$ of monoids in the ordinary category $(\mathbf{S}, \mathbf{Cat})$ underlying the 2-category $[\mathbf{S}, \mathbf{Cat}]$. We have the forgetful functors

$$Mon(\mathbf{S}, \mathbf{Cat}) \to (\mathbf{S}, \mathbf{Cat}) \to (\mathbf{N}, \mathbf{Cat})\,, \tag{1.7}$$

the second induced by the inclusion $\mathbf{N} \to \mathbf{S}$ of the discrete category $\mathbf{N}$; and we can imitate here the arguments of [6, Section 5]. Each of the arrows in (1.7) is finitary and monadic, while their composite is of descent type; it follows as in [6] that the strongly-finitary monads are precisely those whose algebras are presentable as in the third paragraph above. We give more details below.

In fact an analysis of this kind is available with any complete and cocomplete cartesian-closed category $\mathcal{V}$ in place of **Cat**; it is just a little more complicated in that $\mathbf{S}$ no longer sits directly inside $\mathcal{V}$ as a full sub-$\mathcal{V}$-category. Moreover, to deal with many-sorted theories, one wants to consider $\mathcal{V}$-monads not only on $\mathcal{V}$ itself, but at least on a power such as $\mathcal{V}^X$; for instance, the strongly-finitary 2-monads on $\mathbf{Cat} \times \mathbf{Cat}$ would be those which are the left Kan extensions of their restrictions to $\mathbf{S} \times \mathbf{S}$, their algebras having operations such as $A^n \times B^m \to A$ where $n, m \in \mathbf{N}$. A suitable setting might be that where $\mathcal{V}$ is cartesian closed and we consider $\mathcal{V}$-monads on a cartesian-closed $\mathcal{V}$-category $\mathcal{A}$; except that unfortunately we do not know what subcategory of a general such $\mathcal{A}$ should replace $\mathbf{S} \subset \mathbf{Cat}$ or $\mathbf{S} \times \mathbf{S} \subset \mathbf{Cat} \times \mathbf{Cat}$. In Section 2 below, on product-preserving Kan extensions, we do work in this generality, so that we have what is needed for many-sorted theories; but in Section 3, where we give the details of the results sketched above on the nature of strongly-finitary monads, we restrict ourselves for brevity to the one-sorted case $\mathcal{A} = \mathcal{V}$, leaving the many-sorted case to the reader.

The central result we need, that L above preserves finite products, in an easy consequence of a result on weighted colimits. Recall from [4, Section 3.1] the notation $H * R$ for the colimit of the $\mathcal{V}$-functor $R : \mathcal{C} \to \mathcal{A}$ weighted by $H : \mathcal{C}^{\mathrm{op}} \to \mathcal{V}$, where $\mathcal{C}$ is a small $\mathcal{V}$-category. Suppose now that $\mathcal{V}$ is cartesian closed and $\mathcal{A}$ is a cartesian-closed $\mathcal{V}$-category. Our main lemma in Section 2 below asserts that $H * -$ preserves finite products when H does so, and $- * R$ preserves finite products when R does so; these are of course the *same* assertion when $\mathcal{A} = \mathcal{V}$. Note the analogy with *flatness*: H is said to be flat when $H * -$ is left exact, and it is shown in [5, Theorem 6.11] that (for $\mathcal{V}$ and $\mathcal{A}$ locally finitely presentable and $\mathcal{C}^{\mathrm{op}}$ finitely complete) H is flat precisely when it is left exact.

We may now appeal to the formula

$$Lan_J R = \tilde{J} - *R \tag{1.8}$$

of [4, (4.17)] for the left Kan extension of $R : \mathcal{C} \to \mathcal{A}$ along $J : \mathcal{C} \to \mathcal{B}$, where $\tilde{J} : \mathcal{B} \to [\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ is given by $\tilde{J}B = \mathcal{B}(J-, B)$. When J preserves finite coproducts (as does our $J : \mathbf{S} \to \mathbf{Cat}$), $\tilde{J}B$ preserves finite products, and hence (for $\mathcal{V}$ and $\mathcal{A}$ cartesian closed) Lan_J does so too, as required. While this does not seem to have been noticed previously as such, it is closely related to a result given in Appendix 2 of Day's thesis [2] and also contained in Borceux–Day [1]; taking $\mathcal{V}$ cartesian closed and $\mathcal{A} = \mathcal{V}$, they observe that, for any J, if R preserves finite products then $Lan_J R$ does so. This not only follows from the second assertion of the last paragraph, but is in fact equivalent to it, as we see on taking for J the Yoneda embedding $\mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$. So our main lemma is less original than it might be, except for our allowing $\mathcal{A}$ to be different from $\mathcal{V}$ (which gives the useful possibility of taking $\mathcal{V}$ to be **Set** and $\mathcal{A}$ to be any cartesian-closed category), and except for our observing that Lan_J preserves finite products when J preserves finite coproducts. We give the lemma nevertheless, since our proof is very simple and direct.

2. Finite-Product-Preserving Functors and Kan Extensions

For the first observation, which seems to be new, let $\mathcal{V}$ be any symmetric monoidal closed category (which as usual we always suppose to be complete and cocomplete).

LEMMA 2.1. *Consider a $\mathcal{V}$-functor $T : \mathcal{C}^{\mathrm{op}} \otimes \mathcal{B} \to \mathcal{A}$ and a $\mathcal{V}$-adjunction η, ϵ: $Q \dashv P : \mathcal{C} \to \mathcal{B}$. Then we have*

$$\int^{C \in \mathcal{C}} T(C, PC) \;\cong\; \int^{B \in \mathcal{B}} T(QB, B)\,, \tag{2.1}$$

either side existing if the other does.

Proof. By the definition [4, (3.67)] of the $\mathcal{V}$-coend, we are to show that

$$\int_{C \in \mathcal{C}} \mathcal{A}(T(C, PC), A) \;\cong\; \int_{B \in \mathcal{B}} \mathcal{A}(T(QB, B), A)$$

for all A. These ends in $\mathcal{V}$, however, are defined (see [4, Section 2.1]) purely in terms of universal $\mathcal{V}$-natural transformations. Given a $\mathcal{V}$-natural family $\alpha_C : X \to \mathcal{A}(T(C, PC), A)$ we get a $\mathcal{V}$-natural family $\beta_B : X \to \mathcal{A}(T(QB, B), A)$ by defining β_B as the composite

$$X \xrightarrow[\alpha_{QB}]{} \mathcal{A}(T(QB, PQB), A) \xrightarrow[\mathcal{A}(T(QB,\eta_B),A)]{} \mathcal{A}(T(QB, B), A)\,;$$

similarly we get a family (α_C) from a family (β_B) by using the counit $\epsilon_C : QPC \to C$; and that these two processes are inverse follows easily from the naturality and the triangular equations for the adjunction. □

REMARK 2.2. Some may prefer the following argument, which requires $\mathcal{A}$ to be tensored. Since $\mathcal{B}(B, PC) \cong \mathcal{C}(QB, C)$ we have

$$\int^{B,C} \mathcal{B}(B, PC) \otimes T(B, C) \;\cong\; \int^{B,C} \mathcal{C}(QB, C) \otimes T(B, C)\,;$$

two applications of the Yoneda isomorphism [4, (3.71)] now give (2.1).

For the rest of this section *we suppose that $\mathcal{V}$ is cartesian closed and that $\mathcal{A}$ is a cocomplete cartesian-closed $\mathcal{V}$-category.* So for $X \in \mathcal{V}$ and $A \in \mathcal{A}$, both $X \times - : \mathcal{V} \to \mathcal{V}$ and $A \times - : \mathcal{A} \to \mathcal{A}$ preserve all colimits. We write $X \otimes A$ for the tensor product of $X \in \mathcal{V}$ and $A \in \mathcal{A}$; by its definition, this is cocontinuous in each of its variables. Note that $(X \times Y) \otimes A \cong X \otimes (Y \otimes A)$.

For a small $\mathcal{V}$-category $\mathcal{C}$ and functors $H : \mathcal{C}^{\mathrm{op}} \to \mathcal{V}$ and $R : \mathcal{C} \to \mathcal{A}$ we have the weighted colimit

$$H * R = \int^{C \in \mathcal{C}} HC \otimes RC\,. \tag{2.2}$$

When we consider $H * (R \times Q)$, what we mean by $R \times Q$ is of course the product in $[\mathcal{C}, \mathcal{A}]$, sending C to $RC \times QC$. Let $(R, Q) : \mathcal{C} \times \mathcal{C} \to \mathcal{A}$ denote the $\mathcal{V}$-functor sending $(B, D) \in \mathcal{C} \times \mathcal{C}$ to $RB \times QD$; it is the composite of the *external* product "$R \times Q$" : $\mathcal{C} \times \mathcal{C} \to \mathcal{A} \times \mathcal{A}$ with the product-functor $\times : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$. So $R \times Q : \mathcal{C} \to \mathcal{A}$ is $(R, Q)\,\Delta$, where $\Delta : \mathcal{C} \to \mathcal{C}$ is the diagonal.

For the first result we suppose that $\mathcal{C}$ has finite coproducts $B + D$, with initial object 0. So $\Delta : \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ has $+ : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ as a left $\mathcal{V}$-adjoint.

LEMMA 2.3. *If $H : \mathcal{C}^{\mathrm{op}} \to \mathcal{V}$ preserves finite products, so does $H * - : [\mathcal{C}, \mathcal{A}] \to \mathcal{A}$.*

Proof. Consider binary products first. we have

$$H * (R \times Q) = H * ((R, Q)\Delta)$$

$$= \int^{C} HC \otimes (R,Q)\Delta C$$

$$\cong \int^{B,D} H(B+D) \otimes (R,Q)\,(B,D) \qquad \text{by Lemma 2.1}$$

$$\cong \int^{B,D} (HB \times HD) \otimes (RB \times QD) \qquad \text{by the hypothesis on } H$$

$$\cong \int^{B,D} HB \otimes (HD \otimes (RB \times QD))$$

$$\cong \int^{B} HB \otimes \int^{D} HD \otimes (RB \times QD) \qquad \text{since } HB \otimes - \text{ is cocontinuous}$$

$$\cong \int^{B} HB \otimes (RB \times \int^{D} HD \otimes QD) \qquad \text{since } RB \times - \text{ is cocontinuous}$$

$$\cong (\int^{B} HB \otimes RB) \times (\int^{D} HD \otimes QD)$$

$$\text{since } - \times \int^{D} HD \otimes QD \text{ is cocontinuous}$$

$$\cong (H * R) \times (H * Q)\,.$$

A similar but easier proof applies to nullary products; we could appeal to Lemma 2.1 again; or just observe that the terminal object 1 of $[\mathcal{C}, \mathcal{A}]$ sends C to $\mathcal{C}(0, C) \otimes 1$, so that

$$H * 1 = \int^{C} HC \otimes (\mathcal{C}(0,C) \otimes 1)$$

$$\cong \int^{C} ((HC \times \mathcal{C}(0,C)) \otimes 1)$$

$$\cong (\int^{C} (HC \times \mathcal{C}(0,C))) \otimes 1$$

$$\cong H0 \otimes 1 \qquad \text{by the Yoneda isomorphism}$$

$\cong 1 \otimes 1$ by the hypothesis on H

$\cong 1\,.$ □

The proof of the second result (due to Borceux and Day [1], and not needed for our purposes below) is so similar that we leave it to the reader. Here $\mathcal{C}$ is to have finite products; and the result is:

LEMMA 2.4. *If $R : \mathcal{C} \to \mathcal{A}$ preserves finite products, so does $-*R : [\mathcal{C}^{\mathrm{op}}, \mathcal{V}] \to \mathcal{A}$.* □

The original form of the Borceux–Day result was in fact:

PROPOSITION 2.5. *If $R : \mathcal{C} \to \mathcal{A}$ preserves finite products, so does its left Kan extension $Lan_J R : \mathcal{B} \to \mathcal{A}$ along any $J : \mathcal{C} \to \mathcal{B}$.*

Proof. This is immediate from (1.8), since $\tilde{J} : \mathcal{B} \to [\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ given by $\tilde{J}B = \mathcal{B}(J-, B)$ clearly preserves all limits. □

REMARK 2.6. As we said in the Introduction, Lemma 2.4 and Proposition 2.5 are really the same assertion; for $-*R = Lan_Y R$ where Y is the Yoneda embedding $\mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$.

The result we need, however, is the following:

PROPOSITION 2.7. *Let $\mathcal{C}$ have finite coproducts and let $J : \mathcal{C} \to \mathcal{B}$ preserve them. Then $Lan_J : [\mathcal{C}, \mathcal{A}] \to [\mathcal{B}, \mathcal{A}]$ preserves finite products.*

Proof. Products in $[\mathcal{B}, \mathcal{A}]$ being formed pointwise, we want $R \mapsto (Lan_J R)B = \tilde{J}B * R$ to preserve finite products. However $\tilde{J}B = \mathcal{B}(J-, B)$ sends finite products in $\mathcal{C}^{\mathrm{op}}$ (or coproducts in $\mathcal{C}$) to finite products in $\mathcal{V}$, by the hypothesis on J; so that the result follows by Lemma 2.3. □

3. Strongly Finitary Endofunctors

Once again we suppose $\mathcal{V}$ to be a complete and cocomplete cartesian-closed category. As we said in the Introduction, we restrict our considerations here to endo-$\mathcal{V}$-functors of $\mathcal{V}$, leaving to the reader the extension to endo-$\mathcal{V}$-functors of $\mathcal{V}^X$, needed for many-sorted theories. A discussion in this generality of strongly-finitary endofunctors of $\mathcal{V}$ is of necessity a little more complicated than in the case $\mathcal{V} = \mathbf{Cat}$ of the Introduction, essentially because $\mathbf{S}$ no longer sits directly inside $\mathcal{V}$ as a full sub-$\mathcal{V}$-category. Clarity in this context demands a careful distinction between a $\mathcal{V}$-category $\mathcal{A}$ and its underlying ordinary category $\mathcal{A}_0$.

Write $V : \mathcal{V}_0 \to \mathbf{Set}$ for the "underlying-set" functor $\mathcal{V}_0(1, -)$, recalling that $\mathcal{A}_0(A, B) \cong V\mathcal{A}(A, B)$; this V has the left adjoint $F : \mathbf{Set} \to \mathcal{V}_0$ sending the set X to the coproduct $X \bullet 1$ of X copies of 1. Since $\mathbf{Set}(X, Y) = Y^X$ while

$\mathcal{V}_0(X \bullet 1, Y \bullet 1) = (\mathcal{V}_0(1, Y \bullet 1))^X$, we see that F is fully faithful precisely when $Y \mapsto \mathcal{V}_0(1, Y \bullet 1) = V(Y \bullet 1)$ is a bijection; which, since $V1 = \mathcal{V}_0(1,1) = 1$, it certainly is if V also has a *right* adjoint, as it does when $\mathcal{V}$ is **Cat** or, say, compactly-generated (not necessarily Hausdorff) topological spaces. In general, however, it is not so; for a typical topos $\mathcal{V}$, the map $Y \mapsto \mathcal{V}_0(1, Y \bullet 1)$ is not surjective, even for finite Y; it is more often injective, but not when $\mathcal{V}$ is a Heyting algebra, such as $\mathbf{2} = (0 \to 1)$.

However (see [4, Section 2.5]) the 2-functor $(\ )_0 : \mathcal{V}\text{-}\mathbf{CAT} \to \mathbf{CAT}$ has a left adjoint F_* ; for an ordinary category $\mathcal{C}$, the $\mathcal{V}$-category $F_*\mathcal{C}$ has the same objects, and its hom-object $(F_*\mathcal{C})(C, D)$ is $F\mathcal{C}(C, D) = \mathcal{C}(C, D) \bullet 1$. To give an ordinary functor $T : \mathcal{C} \to \mathcal{B}_0$ for a $\mathcal{V}$-category $\mathcal{B}$ is to give a $\mathcal{V}$-functor $\tilde{T} : F_*\mathcal{C} \to \mathcal{B}$, and so on.

Write $\mathbf{S}$ again for the full subcategory of **Set** given by the finite sets $n \in \mathbf{N}$. We have the ordinary functor $K : \mathbf{S} \to \mathcal{V}_0$ given by $Kn = n \bullet 1$; as we have seen, this is in general neither full nor faithful. However the corresponding $\mathcal{V}$-functor $J : F_*\mathbf{S} \to \mathcal{V}$ *is* fully faithful; for $[m \bullet 1, n \bullet 1] \cong (n \bullet 1)^m \cong n^m \bullet 1$, since $\mathcal{V}$ is cartesian closed and m is finite. Note that J preserves finite coproducts, its image in $\mathcal{V}$ being closed under these.

We call an endo-$\mathcal{V}$-functor $T : \mathcal{V} \to \mathcal{V}$ *strongly finitary* if it is the left Kan extension of its restriction to $F_*\mathbf{S}$; which is to say that T is in the replete image of the full embedding $\bar{L} : (F_*\mathbf{S}, \mathcal{V}) \to (\mathcal{V}, \mathcal{V})$ given by left Kan extension along J. Here $(F_*\mathbf{S}, \mathcal{V})$ denotes the ordinary category underlying the $\mathcal{V}$-category $[F_*\mathbf{S}, \mathcal{V}]$; since $F_* \dashv (\ )_0$, it is isomorphic to $(\mathbf{S}, \mathcal{V}_0)$; let us write $L : (\mathbf{S}, \mathcal{V}_0) \to (\mathcal{V}, \mathcal{V})$ for the composite of $\bar{L}$ with this isomorphism. By [4, (4.18)] we have for $\bar{L}$ the explicit formula

$$\bar{L}(\bar{R})A = \int^{n \in F_*\mathbf{S}} A^n \times \bar{R}n$$

where $\bar{R} : F_*\mathbf{S} \to \mathcal{V}$ is the $\mathcal{V}$-functor corresponding to an ordinary functor $R : \mathbf{S} \to \mathcal{V}_0$. It is easy, however, to see that a family $(\alpha_n : A^n \times \bar{R}n \to B)$ is $\mathcal{V}$-natural in $n \in F_*\mathbf{S}$ if and only if $(\alpha_n : A^n \times Rn \to B)$ is (merely) natural in n; so that we have

$$L(R)A = \int^{n \in \mathbf{S}} A^n \times Rn\,, \tag{3.1}$$

where this is an ordinary coend.

By Proposition 2.7, the functor $\bar{L}$ and hence L preserves finite products; recall that these are formed pointwise in $(\mathcal{V}, \mathcal{V})$. So

$$L(Q)\,L(R)A = \int^{m \in \mathbf{S}} (L(R)A)^m \times Qm$$

$$= \int^{m} (L(R))^m A \times Qm$$

$$\cong \int^{m} L(R^m)A \times Qm$$

$$= \int^{m} (\int^{n} A^n \times (Rn)^m) \times Qm$$

$$\cong \int^{n} A^n \times (\int^{m} (Rn)^m \times Qm) ,$$

the last transformation using the fact that $B \times -$ and $- \times C$ preserve colimits. So

$$L(Q)\, L(R) = L(Q \circ R) \tag{3.2}$$

where

$$(Q \circ R)n = \int^{m} (Rn)^m \times Qm = L(Q)\,(Rn)\,. \tag{3.3}$$

Thus the strongly-finitary endofunctors of $\mathcal{V}$ are closed under composition. They also contain the identity, $L(K)$ being 1 because, $1 \in \mathcal{V}$ being dense, J is *a fortiori* so.

So $((\mathbf{S}, \mathcal{V}_0), \circ, K)$ is a monoidal category, monoidally equivalent to the strict monoidal category of strongly-finitary endofunctors of $\mathcal{V}$ with composition as the tensor product. In fact $(\mathbf{S}, \mathcal{V}_0)$ is right closed, since we easily verify (1.5) with $\mathcal{V}$ in place of **Cat**. *A fortiori*, $- \circ R$ preserves colimits. On the other hand each $Q \circ -$ preserves filtered colimits (that is, is finitary) by (3.3) and (3.1), since, $\mathcal{V}$ being cartesian closed, each n-th power functor $(\ )^n : \mathcal{V}_0 \to \mathcal{V}_0$ preserves these; see [5, (3.8)].

It is convenient to *identify* the category of strongly-finitary endofunctors of $\mathcal{V}$ with $(\mathbf{S}, \mathcal{V}_0)$; and thus the category of strongly-finitary monads on $\mathcal{V}$ with the category $Mon(\mathbf{S}, \mathcal{V}_0)$ of monoids in $(\mathbf{S}, \mathcal{V}_0)$. It now follows from [3, Theorem 23.3] that the forgetful functor $W : Mon(\mathbf{S}, \mathcal{V}_0) \to (\mathbf{S}, \mathcal{V}_0)$ has a left adjoint H, which by [3, (23.2)] sends R to $HR = S$ given inductively by

$$S_0 = K\ , \quad S_{n+1} = K + R \circ S_n\ , \quad S = \mathrm{colim}_{n<\omega} S_n\ , \tag{3.4}$$

with an evident monoid-structure. Moreover, W is monadic and finitary, by arguments identical to those in [6, Section 4]. Accordingly $Mon(\mathbf{S}, \mathcal{V}_0)$ is cocomplete by [3, Theorem 25.4].

We further have the forgetful $Z : (\mathbf{S}, \mathcal{V}_0) \to (\mathbf{N}, \mathcal{V}_0)$ induced by the inclusion into $\mathbf{S}$ of the discrete category $\mathbf{N}$; this has a left adjoint G given by

$$(GB)_n = \sum_{m \in \mathbf{N}} n^m \bullet Bm \,, \tag{3.5}$$

and a right adjoint as well. So Z too is monadic and finitary.

The forgetful functor of central interest here is the composite $U = ZW : Mon(\mathbf{S}, \mathcal{V}_0) \to (\mathbf{N}, \mathcal{V}_0)$, with its left adjoint, say, $F = HG$; note that U sends a strongly-finitary monad T to the mere sequence (Tn) of objects of $\mathcal{V}$; we may call Tn the *$\mathcal{V}$-object of n-ary operations.* Its left adjoint F is easily seen to send $B \in (\mathbf{N}, \mathcal{V}_0)$ to S where, combining (3.4) and (3.5), we find that

$$S_{k+1}n = n + \sum_{m \in \mathbf{N}} (S_k n)^m \times Bm \,. \tag{3.6}$$

Here we regard Bn as the $\mathcal{V}$-object of *basic n-ary operations*; and now $S = FB$ is the *free monad* on B, with Sn being called the object of *derived n-ary operations.*

The argument of [6, Section 5] goes over unchanged to show that U is of descent type; so every strongly-finitary monad T admits a presentation as a coequalizer

$$FE \underset{\tau}{\overset{\sigma}{\rightrightarrows}} FB \xrightarrow[\rho]{} T \,. \tag{3.7}$$

Now an FB-algebra is of course just an object A of $\mathcal{V}$ together with maps $A^n \times Bn \to A$ for each n; so a T-algebra is such an algebra for which these "operations" satisfy the equations given by σ and τ above, or equally by the corresponding $\bar{\sigma}, \bar{\tau} \; : \; E \to UFB$; these are the equations between derived operations.

In the case $\mathcal{V} = \mathbf{Cat}$ with which we began, Bn is the *category* of basic n-ary operations; an object of it gives, for each T-algebra A, a functor $A^n \to A$, while a morphism in it gives a natural transformation between such functors. This completes our promised syntactic description of the algebras for strongly-finitary 2-monads.

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