LAZARD'S THEOREM IN ALGEBRAIC CATEGORIES

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It is well known [1], [3] that flatness of an R-module A is equivalent to each of the following conditions: (a) Each relation in A is a consequence of relations in R. (b) Each finite set of relations in A is a consequence of relations in R. (c) A is a directed colimit of finitely generated free modules. It is shown that the generalizations of these conditions are equivalent in any algebraic category.

For any small category C let $\hat{C}$ denote the category of functors $C^{\mathrm{op}} \to Ens$ and natural transformations. Let $h: C \to \hat{C}$ be the Yoneda embedding

$$(f: x \to y) \to (h_f: h_x \to h_y).$$

For $A \in \hat{C}$ and $f: x \to y$ in C denote $A(f)$ by f^*. If $A \in \hat{C}$, the comma category (h, A) has as objects (h_x, h_a) with $x \in C$ and $a \in A(x)$ and its morphisms are $h_f: (h_x, h_a) \to (h_y, h_b)$ with $f: x \to y$ and $f^*(b) = a$.

LEMMA 1. *For $A \in \hat{C}$, A is a filtered colimit of representable functors if and only if (h, A) is filtered.*

Proof. Suppose (h, A) is filtered. Then A is the colimit of the forgetful functor $(h, A) \to \hat{C}$ [2, p. 105].

Suppose $A = \varinjlim h_u$ is a filtered colimit of representable functors. Let (h_x, h_a) and (h_y, h_b) be objects of (h, A). Since the colimit is constructed agumentwise $a \in A(x)$ is the image of some $f \in h_u(x)$. If the canonical morphism $h_u \to A$ is h_d, then $f^*(d) = a$. Similarly there is $g \in h_v(y)$ with $h_e: h_v \to A$ and $g^*(e) = b$. Since the system is filtered there are $h_k: h_u \to h_w$ and $h_{\bar{k}}: h_v \to h_w$. If $h_c: h_w \to A$ is the canonical morphism, $h_d = h_c h_k$ and $h_e = h_c h_{\bar{k}}$. Then $k^*(c) = d$ and $\bar{k}^*(c) = e$, so $a = f^*(d) = f^* k^*(c) = (kf)^*(c)$ and $b = (\bar{k}g)^*(c)$. Hence $h_{kf}: (h_x, h_a) \to (h_w, h_c)$ and $h_{\bar{k}g}: (h_y, h_b) \to (h_w, h_c)$ in (h, A).

Let $h_k: (h_y, h_b) \to (h_x, h_a)$ and $h_g: (h_y, h_b) \to (h_x, h_a)$ be in (h, A). As above, there is $f \in h_u(x)$ with $h_d(x)\,(f) = a$. Since h_d is a natural transformation, $h_d(y)\,(fk) = h_d(y)\,(h_u(k)\,(f)) = k^*\,(h_d(x)\,(f)) = k^*(a) = \mathrm{b} = g^*(a) = h_d(y)\,(fg)$. Hence there is $h_\theta: (h_u, h_d) \to (h_v, h_c)$ with $h_\theta(y)\,(fk) = h_\theta(y)\,(fg)$. Then $a = h_d(x)\,(f) = h_c(x)\ h_\theta(x)\,(f) = h_c(x)\ (\theta f) = (\theta f)^*\ (c)$, so $h_{\theta f}: (h_x, h_a) \to (h_v, h_c)$ in (h, A). Furthermore, $h_{\theta f} h_k = h_{\theta f} h_g$ since $\theta f k = \theta f g$.

LEMMA 2. *If J is a small filtered category there is a cofinal functor $I \to J$ with I a directed set.*

Proof. Let J be a small filtered category. Let H be the set whose objects are pairs

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(j, θ) with $j \in J$ and θ a finite set of morphisms of J having distinct domains and j as codomain. Define $(j, \theta) \leq (k, \lambda)$ if $(j, \theta) = (k, \lambda)$ or $\theta = \{\theta_i : d_i \to j \mid i = 1, \ldots, n\}$, λ contains a morphism $\bar{\lambda}$ with domain j and for each $i = 1, \ldots, n$ a morphism λ_i with domain d_i, and $\bar{\lambda}\theta_i = \lambda_i$, $i = 1, \ldots, n$. This relation is reflexive and transitive, making H a category. Let I be a skeletal subcategory of H. I is a poset.

Let (j, θ) and (k, λ) be in I. Let D be the set of objects $d \in J$ such that d is the domain of a morphism θ_d in θ and a morphism λ_d in λ. Since this set is finite and J is filtered there are $f : j \to \bar{j}$ and $g : k \to \bar{j}$ with $f\theta_d = g\lambda_d$ for $d \in D$, $f\alpha = f$ for $\alpha : j \to j \in \theta$ and $g\beta = g$ for $\beta : k \to k \in \lambda$. Then $(j, \theta) \leq (\bar{j}, \gamma)$ and $(k, \lambda) \leq (\bar{j}, \gamma)$ with

$$\gamma = \{f, g, f\theta_1, \ldots, f\theta_n, g\lambda_1, \ldots, g\lambda_m\}$$

where $\theta = \{\theta_1, \ldots, \theta_n\}$ and $\lambda = \{\lambda_1, \ldots, \lambda_m\}$, so I is a directed set. The functor $I \to J$ which sends $(j, \theta) < (k, \lambda)$ to the unique morphism in λ with domain j is cofinal.

DEFINITION. A theory is a category T with coproducts such that every object is a coproduct of a finite number of copies of a fundamental object $[1]$. The coproduct of n copies of $[1]$ is denoted by $[n]$. A T-model is a product preserving functor from T^{op} into sets. The category of T-models is denoted by T^b.

If T is a theory and $A \in \hat{T}$ is a T-model, let A also denote $A([1])$. Then $A([n]) = A^n$. A representable functor in $\hat{T}$ is just a finitely generated free T-model. Since T^b has colimits and the construction of colimits in both T^b and $\hat{T}$ is augumentwise, the following corollary is true.

COROLLARY. *If T is a theory, a T-model A is a directed colimit, in T^b, of finitely generated free T-models if and only if (h, A) is filtered.*

The following generalizes condition (a).

DEFINITION. A T-algebra A has the Killing Interpolation Property (KIP) if whenever $\theta^* a = \mu^* a$ with $a \in A^n$, and $\theta, \mu \in T([1], [n])$ there are $c \in A^k$ and $\lambda \in T([n], [k])$ with $a = \lambda^* c$ and $\lambda\theta = \lambda\mu$.

THEOREM 1. *A T-model A has the* KIP *if and only if whenever $\theta^* a = \lambda^* a$ with $a \in A^n$ and $\theta, \mu \in T([m], [n])$ there are $c \in A^k$ and $\lambda \in T([n], [k])$ with $a = \lambda^* c$ and $\lambda\theta = \lambda\mu$.*

Proof. Let $\theta, \mu \in T([m], [n])$. Recalling that $[m]$ is a coproduct, let $\theta_1, \ldots, \theta_m$ and $\mu_1, \ldots, \mu_m$ be the components of θ and μ, respectively. If $a \in A^n$ and $\theta^* a = \mu^* a$, then $\theta_m^* a = \mu_m^* a$ so there are $d \in A^p$ and $\tau \in T([n], [p])$ with $\tau^* d = a$ and $\tau\theta_m = \tau\mu_m$. Then $(\tau\theta_i)^* d = (\tau\mu_i)^* d$, $i = 1, \ldots, m-1$, so, by induction, there are $c \in A^k$ and $\lambda \in T([p], [k])$ with $\lambda^* c = d$ and $\lambda\tau\theta_i = \lambda\tau\mu_i$, $i = 1, \ldots, m-1$. Then $(\lambda\tau)^* c = a$ and $(\lambda\tau)\theta = (\lambda\tau)\mu$.

THEOREM 2. *A T-model A is a directed colimit of finitely generated free T-models if and only if A has the* KIP.

Proof. We show (h, A) is filtered if and only if A has the KIP. For any T-model A, if $(h_{[n]}, h_a)$ and $(h_{[m]}, h_b)$ are objects in (h, A) then

$$\sigma_n^*: (h_{[n]}, h_a) \to (h_{[n+m]}, h_c)$$

and

$$\sigma_m^*: (h_{[m]}, h_b) \to (h_{[n+m]}, h_c)$$

where

$$c = (a, b) \in A^n \times A^m = A^{n+m},\ \sigma_n: [n] \to [n] \coprod [m]$$

and

$$\sigma_m: [m] \to [n] \coprod [m].$$

Since

$$h_\theta: (h_{[m]}, h_b) \to (h_{[n]}, h_a)$$

and

$$h_\mu: (h_{[m]}, h_b) \to (h_{[n]}, h_a)$$

if and only if $\theta^* a = \mu^* a$, there is, by Theorem 1,

$$h_\lambda: (h_{[n]}), h_a) \to (h_{[k]}, h_c)$$

with $h_\lambda h_\theta = h_\lambda h_\mu$ if and only if A has the KIP.

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