Report on *Algebraic Type Theory Part 1: Martin-Löf Algebras*

for: Mathematical Structures in Computer Science

I apologise for my tardiness in returning this report.

1 Overview

This manuscript concerns an *algebraic* viewpoint on Martin-Löf type theory that the author has been developing over the past decade or so. Here I mean "algebraic" not just in the sense that Martin-Löf type theory can be described as a (generalised) algebraic theory, but also in the more interesting sense that the structure of terms over types closed under internal sums and internal products can be described in terms of polynomials in a category and various algebraic operations that can be expressed in polynomial terms, such as the substitution tensor product, *etc.* As suggested by the Awodey, this perspective on Martin-Löf type theory is akin to the viewpoint of *algebraic set theory*.

Section 1 concerns Awodey's natural models (a categorical reformulation of Dybjer's *categories with families*), describing how to express the closure of a model of the judgemental structure of Martin-Löf type theory under internal sums and internal products in terms of cartesian morphisms of polynomials in presheaves on the category of contexts. Most of this material is also exposed in Awodey's prior paper, *Natural models of homotopy type theory*.

In Sections 2–3, the *elementary* aspect of the structure above is abstracted away from categories of presheaves to an arbitrary locally cartesian closed category. The resulting notion is called a *Martin-Löf algebra*. The representability criterion for a natural model is replaced here (optionally) by *tininess*. This is elegant indeed, and restricts to the representability notion as soon as the category of contexts is closed under splitting of idempotents. (It seems like this fact could have been used to give a considerably shorter proof of Proposition 9, since any category with finite limits is Cauchy complete.)

Perhaps it should be noted that elementary descriptions of models of Martin-Löf type theory without contexts are not without precedent. For example, Martin-Löf's own logical framework description of his type theory would qualify (but for not being described in categorical terms). A few other recent works have compared contextual and non-contextual notions of model:

1. Gratzer and Sterling[1] showed that the Martin-Löf algebras in the present sense are conservative over natural models in syntactic natural model of Martin-Löf type theory embeds fully faithfully into the free LCCC with a Martin-Löf algebra by means of a universal finitely continuous functor preserving pushforwards along representable maps. (In fact, they do this for any theory that can be described in the language of representable maps.)

2. Bocquet, Kaposi, and Sattler[2] compared what they called "first-order models" (models with contexts) with "higher-order models" (models without contexts, like ML-algebras) and use a comma category to turn a displayed higher-order model into a displayed first-order model. Unravelling the "displayed" language a bit, the goal seems to have been to use the universal property of the universal natural model to prove something that is most naturally expressed at the level of ML-algebras. This is usually the final step in semantic normalisation arguments in which some complicated glueing construction carried out in terms of ML-algebras must be turned into an honest model (if one does not wish to resort to the conservativity result of Gratzer and Sterling).

In Section 4, something very new and interesting is happening. Prior work of Awodey and Newstead situated the algebraic description of Martin-Löf type theory in the tricategory of polynomials, but the pertinent notions described there are taking place in the sub-tricategory whose 2-cells are cartesian. The

[1] Gratzer, Sterling (2020). *Syntactic categories for dependent type theory: sketching and adequacy.* Unpublished manuscript. `http://arxiv.org/abs/2012.10783`.

[2] Bocquet, Kaposi, and Sattler (2023). *For the Metatheory of Type Theory, Internal Sconing is Enough.* FSCD 2023. `http://drops.dagstuhl.de/entities/document/10.4230/LIPIcs.FSCD.2023.18`.

algebraic formulation of type theory is then phrased in terms of a pseudo-monad with a pseudo-algebra,[3] but what was a little disconcerting is that a there is always an essentially unique 3-cell connecting any two cartesian morphisms of polynomials, so the pseudomonad laws held automatically.

In *this* manuscript Awodey proposes a new way to de-trivialise the polynomial interpretation, inspired by intensional type theory. In particular, the polynomials in question are required to be classified by a ML-algebra, and this gives a natural notion of higher cell that incorporates automorphisms. The description here is a little complicated by the need to get an honest bicategory to replace $\mathbf{Poly}_{\mathcal{E}}(\mathbf{1},\mathbf{1})_{cart}$ in which composition of higher cells is strictly associative. An alternative description that can be expressed in the language of weak higher categories (*e.g.* $(\infty,1)$-categories), suggested by Mathieu Anel, is foreshadowed as future work.

I wonder if there is a sneaky way to bypass this difficulty. The root problem is that the groupoid of automorphisms is weak because composition of equivalences in intensional type theory is not definitionally associative. Aberlé and Spivak proposed to use identity types instead of equivalence types (in the presence of univalence), and that doesn't solve this particular problem, but it seems to me that you could combine it with Martín Escardó's strictified identity type[4]

$$x \sim_A y :\equiv \prod_{(z:A)} z =_A x \to z =_A y$$

which is strictly associative, and even strictly unital relative to

$$\begin{aligned} &\mathsf{refl}_A : \prod_{(x:A)} x \sim_A x \\ &\mathsf{refl}^{\sim}_A(x) :\equiv \lambda z : A.\, \lambda p : x =_A z.\, p \end{aligned}$$

In the presence of function extensionality, this is equivalent to the Martin-Löf identity type, and in the presence of univalence, it is therefore equivalent to the type of equivalences. Perhaps this could be a viable way to combine the proposal of this manuscript with the noted higher-categorical suggestion of Anel to get a version of the latter that still works in ordinary category theory.

[3]To my understanding, there is something a little bit subtle about the needful interpretation of "pseudo-algebra" here; this subtlety was recently revisited by Aberlé and Spivak in the cited paper *Polynomial Universes in Homotopy Type Theory* (MFPS 2025), where they seem to have found a way to express the appropriate notion of algebra in a way that is intrinsic to **Poly**.

[4]Thread entitled *Definitions of equivalence satisfying judgemental/strict groupoid laws?* in the `homotopytypetheory` mailing list, 2019.

2 Assessment

This manuscript was a joy to read, and the dedication to the memory of Phil Scott is fitting.

I did not notice any significant typographical errors. One slight annoyance is the frequent pronominal use of bracketed citations, but this is a matter of personal style that I won't interfere with.

I do think the manuscript can be published as-is, though perhaps some of the suggestions I have offered can be of help to the author.