

A note on Hofmann-Streicher universes

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Let \mathbb{C} be a small category and $\widehat{\mathbb{C}} = \mathbf{Set}^{\mathbb{C}^{\text{op}}}$ the category of presheaves on \mathbb{C} .

1. The Hofmann-Streicher universe

In [HS97] the authors define a (type-theoretic) *universe*

$$\mathbf{El} \rightarrow U \tag{1}$$

in $\widehat{\mathbb{C}}$ as follows. For $I \in \mathbb{C}$, set

$$U(I) = \mathbf{ob}(\widehat{\mathbb{C}/I}), \tag{2}$$

$$\mathbf{El}(\langle I, A \rangle) = A(id_I), \tag{3}$$

with an evident associated action on morphisms, which need not concern us for the moment. A few comments are required:

1. Since $U : \mathbb{C}^{\text{op}} \rightarrow \mathbf{Set}$, we have taken the underlying *set* $\mathbf{ob}(\widehat{\mathbb{C}/I})$ of objects of the category $\widehat{\mathbb{C}/I}$ in (2).
2. In (3), and throughout, the authors steadfastly adopt the “categories with families” point of view in describing the morphism $\mathbf{El} \rightarrow U$ in $\widehat{\mathbb{C}}$ as an object in

$$\widehat{\int_{\mathbb{C}} U} \simeq \widehat{\mathbb{C}}/U, \tag{4}$$

and thus as a presheaf on the *category of elements* $\int_{\mathbb{C}} U$ (rather than specifying the object \mathbf{El} in $\widehat{\mathbb{C}}$). Thus the argument $\langle I, A \rangle \in \int_{\mathbb{C}} U$ in (3) consists of an object $I \in \mathbb{C}$ and an element $A \in U(I)$.

3. In order to account for size issues, the authors assume a Grothendieck universe \mathcal{U} in \mathbf{Set} , the elements of which are called *small*. The category \mathbb{C} is then assumed to be small, as are the values of the presheaves (unless otherwise stated).

The presheaf U , which is not small, is regarded as the Grothendieck universe \mathcal{U} “lifted” from \mathbf{Set} to $\mathbf{Set}^{\mathbf{C}^{\text{op}}}$. We will analyse the construction of (1) from a slightly different perspective in order to arrive at its basic property as a classifier for small families in $\widehat{\mathbb{C}}$.

2. An unused adjunction

For a presheaf X on \mathbb{C} , recall that the category of elements is the comma category,

$$\int_{\mathbb{C}} X = y_{\mathbb{C}}/X,$$

where $y_{\mathbb{C}} : \mathbb{C} \rightarrow \mathbf{Set}^{\mathbf{C}^{\text{op}}}$ is the Yoneda embedding, which we may suppress and write simply $\mathbb{C}/_X$. While the category of elements $\int_{\mathbb{C}} X$ is used in the specification of the Hofmann-Streicher universe $\mathbf{El} \rightarrow U$ at the point (4), the authors seem to have missed a trick, which can be used to simplify things:

Proposition 1. *The category of elements functor $\int_{\mathbb{C}} : \widehat{\mathbb{C}} \rightarrow \mathbf{Cat}$ has a right adjoint, which we denote*

$$\nu_{\mathbb{C}} : \mathbf{Cat} \rightarrow \widehat{\mathbb{C}}.$$

For a small category \mathbb{A} , we call the presheaf $\nu_{\mathbb{C}}(\mathbb{A})$ the \mathbb{C} -nerve of \mathbb{A} .

Proof. For $\mathbb{A} \in \mathbf{Cat}$ and $c \in \mathbb{C}$ define $\nu_{\mathbb{C}}(\mathbb{A})(c)$ to be the Hom-set,

$$\nu_{\mathbb{C}}(\mathbb{A})(c) = \mathbf{Cat}(\mathbb{C}/_c, \mathbb{A}),$$

with contravariant action on $h : d \rightarrow c$ given by precomposing a functor $P : \mathbb{C}/_c \rightarrow \mathbb{A}$ with the postcomposition functor

$$\mathbb{C}/_h : \mathbb{C}/_d \rightarrow \mathbb{C}/_c.$$

For the adjunction, observe that the slice category $\mathbb{C}/_c$ is the category of elements of the representable functor y_c ,

$$\int_{\mathbb{C}} y_c \cong \mathbb{C}/_c.$$

Thus for all representables y_c , we have the required natural isomorphism

$$\widehat{\mathbb{C}}(y_c, \nu_{\mathbb{C}}(\mathbb{A})) \cong \nu_{\mathbb{C}}(\mathbb{A})(c) = \mathbf{Cat}(\mathbb{C}/_c, \mathbb{A}) \cong \mathbf{Cat}(\int_{\mathbb{C}} y_c, \mathbb{A}).$$

For arbitrary presheaves X , one uses the presentation of X as a colimit of representables over the index category $\int_{\mathbb{C}} X$, and the easy to prove fact that $\int_{\mathbb{C}}$ itself preserves colimits. Indeed, for any category \mathbb{D} , we have an isomorphism in \mathbf{Cat} ,

$$\lim_{\substack{\longrightarrow \\ d \in \mathbb{D}}} \mathbb{D}/_d \cong \mathbb{D}.$$

□

When \mathbb{C} is fixed, as here, we may omit the subscript from the notation $\int_{\mathbb{C}}$ and $y_{\mathbb{C}}$ and $\nu_{\mathbb{C}}$. The unit and counit maps of the adjunction $\int \dashv \nu$, vis.

$$\begin{aligned}\eta : X &\longrightarrow \nu \int X, \\ \epsilon : \int \nu \mathbb{A} &\longrightarrow \mathbb{A},\end{aligned}$$

are as follows. At $c \in \mathbb{C}$, for $x : y_c \rightarrow X$, the functor $(\eta_X)_c(x) : \mathbb{C}/_c \rightarrow \mathbb{C}/_X$ is just composition with x ,

$$(\eta_X)_c(x) = \mathbb{C}/_x : \mathbb{C}/_c \longrightarrow \mathbb{C}/_X. \quad (5)$$

For $\mathbb{A} \in \mathbf{Cat}$, the functor $\epsilon : \int \nu \mathbb{A} \rightarrow \mathbb{A}$ takes a pair $(c \in \mathbb{C}, f : \mathbb{C}/_c \rightarrow \mathbb{A})$ to the object $f(1_c) \in \mathbb{A}$,

$$\epsilon(c, f) = f(1_c).$$

Lemma 2. *For any $f : Y \rightarrow X$, the naturality square below is a pullback.*

$$\begin{array}{ccc} Y & \xrightarrow{\eta_Y} & \nu \int Y \\ f \downarrow & & \downarrow \nu f f \\ X & \xrightarrow{\eta_X} & \nu \int X. \end{array} \quad (6)$$

Proof. It suffices to prove it for the case $f : X \rightarrow 1$. Thus consider the square

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & \nu \int X \\ \downarrow & & \downarrow \\ 1 & \xrightarrow{\eta_1} & \nu \int 1. \end{array} \quad (7)$$

Evaluating at $c \in \mathbb{C}$ and applying (5) then gives the following square in \mathbf{Set} .

$$\begin{array}{ccc} Xc & \xrightarrow{\mathbb{C}/_-} & \mathbf{Cat}(\mathbb{C}/_c, \mathbb{C}/_X) \\ \downarrow & & \downarrow \\ 1c & \xrightarrow{\mathbb{C}/_-} & \mathbf{Cat}(\mathbb{C}/_c, \mathbb{C}/_1) \end{array} \quad (8)$$

The image of $* \in 1c$ along the bottom is the forgetful functor $U_c : \mathbb{C}/_c \rightarrow \mathbb{C}$, and its fiber under the map on the right is therefore the set of functors $F : \mathbb{C}/_c \rightarrow \mathbb{C}/_X$ such that $U_X \circ F = U_c$, where $U_X : \mathbb{C}/_X \rightarrow \mathbb{C}$ is also a forgetful functor. But any such F is easily seen to be uniquely of the form $\mathbb{C}/_x$ for $x = F(1_c) : y_c \rightarrow X$. \square

3. Classifying families

For the terminal presheaf $1 \in \widehat{\mathbb{C}}$, we have $\int 1 \cong \mathbb{C}$, so for every $X \in \widehat{\mathbb{C}}$ there is a canonical projection $\int X \rightarrow \mathbb{C}$, which is easily seen to be a discrete fibration. It follows that for any map $Y \rightarrow X$ of presheaves, the associated map $\int Y \rightarrow \int X$ is also a discrete fibration. Ignoring size issues for the moment, recall that discrete fibrations in \mathbf{Cat} are classified by the forgetful functor $\mathbf{Set}^{\text{op}} \rightarrow \mathbf{Set}^{\text{op}}$ from (the opposites of) the category of pointed sets to that of sets. For every presheaf $X \in \widehat{\mathbb{C}}$, we therefore have a pullback diagram in \mathbf{Cat} ,

$$\begin{array}{ccc} \int X & \longrightarrow & \mathbf{Set}^{\text{op}} \\ \downarrow & \lrcorner & \downarrow \\ \mathbb{C} & \xrightarrow[X]{} & \mathbf{Set}^{\text{op}}. \end{array} \quad (9)$$

Transposing by the adjunction $\int \dashv \nu$ then gives a commutative square in $\widehat{\mathbb{C}}$,

$$\begin{array}{ccc} X & \longrightarrow & \nu \mathbf{Set}^{\text{op}} \\ \downarrow & & \downarrow \\ 1 & \xrightarrow[\bar{X}]{} & \nu \mathbf{Set}^{\text{op}}. \end{array} \quad (10)$$

Lemma 3. *The square (10) is a pullback in $\widehat{\mathbb{C}}$. More generally, for any map $Y \rightarrow X$ in $\widehat{\mathbb{C}}$, there is a pullback square*

$$\begin{array}{ccc} Y & \longrightarrow & \nu \mathbf{Set}^{\text{op}} \\ \downarrow & \lrcorner & \downarrow \\ X & \longrightarrow & \nu \mathbf{Set}^{\text{op}}. \end{array} \quad (11)$$

Proof. Apply the right adjoint ν to the pullback square (9) and paste the naturality square (6) from Lemma 2 on the left, to obtain the transposed square (11) as a pasting of two pullbacks. \square

Let us write $\dot{\mathcal{V}} \rightarrow \mathcal{V}$ for the vertical map on the right in (11), so that

$$\begin{aligned} \dot{\mathcal{V}} &= \nu \mathbf{Set}^{\text{op}} \\ \mathcal{V} &= \nu \mathbf{Set}^{\text{op}}. \end{aligned} \quad (12)$$

We can summarize our results so far as follows.

Proposition 4. *The nerve $\dot{\mathcal{V}} \rightarrow \mathcal{V}$ of the classifier for discrete fibrations $\mathbf{Set}^{\text{op}} \rightarrow \mathbf{Set}^{\text{op}}$, as defined in (12), classifies natural transformations $Y \rightarrow X$ in $\widehat{\mathbb{C}}$, in the sense that there is always a pullback square,*

$$\begin{array}{ccc} Y & \longrightarrow & \dot{\mathcal{V}} \\ \downarrow & \lrcorner & \downarrow \\ X & \xrightarrow{\tilde{Y}} & \mathcal{V}. \end{array} \quad (13)$$

The classifying map $\tilde{Y} : X \rightarrow \mathcal{V}$ is determined by the adjunction $\int \dashv \mu$ as the transpose of the classifying map of the discrete fibration $\int X \rightarrow \int Y$.

Of course, $\dot{\mathcal{V}} \rightarrow \mathcal{V}$ itself cannot be a map in $\widehat{\mathbb{C}}$, for reasons of size.

4. Small maps

Let α be a cardinal number and call the sets that are strictly smaller α -small. Let $\mathbf{Set}_\alpha \hookrightarrow \mathbf{Set}$ be the full subcategory of α -small sets. Call a presheaf $X : \mathbb{C}^{\text{op}} \rightarrow \mathbf{Set}$ α -small if all of its values are α -small sets, and thus if, and only if, it factors through $\mathbf{Set}_\alpha \hookrightarrow \mathbf{Set}$. Call a map $f : Y \rightarrow X$ of presheaves α -small if all of the fibers $f_c^{-1}\{x\} \subseteq Yc$ are α -small sets (for all $c \in \mathbb{C}$ and $x \in Xc$). The latter condition is of course equivalent to saying that, in the pullback square over the element $x : yc \rightarrow X$,

$$\begin{array}{ccc} Y_x & \longrightarrow & Y \\ \downarrow & \lrcorner & \downarrow f \\ yc & \xrightarrow{x} & X, \end{array} \quad (14)$$

the presheaf Y_x is α -small.

Now let us restrict the specification (12) of $\dot{\mathcal{V}} \rightarrow \mathcal{V}$ to the α -small sets:

$$\begin{aligned} \dot{\mathcal{V}}_\alpha &= \nu \mathbf{Set}_\alpha^{\text{op}} \\ \mathcal{V}_\alpha &= \nu \mathbf{Set}_\alpha^{\text{op}}. \end{aligned} \quad (15)$$

Then the evident forgetful map $\dot{\mathcal{V}}_\alpha \rightarrow \mathcal{V}_\alpha$ is a map in the category $\widehat{\mathbb{C}}$ of presheaves, and it is in fact α -small. Moreover, it has the following basic property, which is just a restriction of the basic property of $\dot{\mathcal{V}} \rightarrow \mathcal{V}$ stated in Proposition 4.

Proposition 5. *The map $\dot{\mathcal{V}}_\alpha \rightarrow \mathcal{V}_\alpha$ classifies α -small maps $f : Y \rightarrow X$ in $\widehat{\mathbb{C}}$, in the sense that there is always a pullback square,*

$$\begin{array}{ccc} Y & \longrightarrow & \dot{\mathcal{V}}_\alpha \\ \downarrow & \lrcorner & \downarrow \\ X & \xrightarrow{\tilde{Y}} & \mathcal{V}_\alpha. \end{array} \quad (16)$$

The classifying map $\tilde{Y} : X \rightarrow \mathcal{V}_\alpha$ is determined by the adjunction $\int \dashv \mu$ as (the factorization of) the transpose of the classifying map of the discrete fibration $\int X \rightarrow \int Y$.

Proof. If $Y \rightarrow X$ is small, its classifying map $\tilde{Y} : X \rightarrow \mathcal{V}$ factors through $\mathcal{V}_\alpha \hookrightarrow \mathcal{V}$, as indicated below,

$$\begin{array}{ccccc} Y & \xrightarrow{\quad} & \nu \dot{\text{Set}}_\alpha^{\text{op}} & \hookrightarrow & \nu \dot{\text{Set}}^{\text{op}} \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{\quad} & \nu \text{Set}_\alpha^{\text{op}} & \hookrightarrow & \nu \text{Set}^{\text{op}}, \\ & & & & \uparrow \tilde{Y} \end{array} \quad (17)$$

in virtue of the following adjoint transposition,

$$\begin{array}{ccccc} \int Y & \xrightarrow{\quad} & \text{Set}_\alpha^{\text{op}} & \hookrightarrow & \text{Set}^{\text{op}} \\ \downarrow & & \downarrow & & \downarrow \\ \int X & \xrightarrow{\quad} & \text{Set}_\alpha^{\text{op}} & \hookrightarrow & \text{Set}^{\text{op}}. \\ & & & & \uparrow \end{array} \quad (18)$$

Note that the square on the right is evidently a pullback, and the one on the left therefore is, too, because the outer rectangle is the classifying pullback of the discrete fibration $\int Y \rightarrow \int X$, as stated. \square

5. Examples

1. Let $\alpha = \kappa$ be a strongly inaccessible cardinal, so that $\mathbf{ob}(\text{Set}_\kappa)$ is a Grothendieck universe. Then the Hofmann-Streicher universe of (1) is recovered in the present setting as the κ -small map classifier

$$\dot{\mathcal{V}}_\kappa \longrightarrow \mathcal{V}_\kappa$$

in the sense of Proposition 5. Indeed, for $c \in \mathbb{C}$, we have

$$\begin{aligned}\mathcal{V}_\kappa c &= \nu(\mathbf{Set}_\kappa^{\text{op}})(c) = \mathbf{Cat}(\mathbb{C}/_c, \mathbf{Set}_\kappa^{\text{op}}) = \mathbf{ob}(\widehat{\mathbb{C}/_c}) = \mathcal{U}c \\ \dot{\mathcal{V}}_\kappa c &= \nu(\dot{\mathbf{Set}}_\kappa^{\text{op}})(c) = \mathbf{Cat}(\mathbb{C}/_c, \dot{\mathbf{Set}}_\kappa^{\text{op}}) = \int_{\mathbb{C}/_c} \mathbf{El}(\langle c, A \rangle). \end{aligned} \quad (19)$$

2. By functoriality of $\nu : \mathbf{Cat} \rightarrow \widehat{\mathbb{C}}$, a sequence of Grothendieck universes

$$\mathcal{U} \subseteq \mathcal{U}' \subseteq \dots$$

in \mathbf{Set} gives rise to a (cumulative) sequence of such type-theoretic universes

$$\mathcal{V} \rightarrow \mathcal{V}' \rightarrow \dots$$

in $\widehat{\mathbb{C}}$. More precisely, there is a sequence of cartesian squares,

$$\begin{array}{ccccc} \dot{\mathcal{V}} & \rightarrow & \dot{\mathcal{V}}' & \rightarrow & \dots \\ \downarrow \lrcorner & & \downarrow \lrcorner & & \\ \mathcal{V} & \rightarrow & \mathcal{V}' & \rightarrow & \dots, \end{array} \quad (20)$$

in the image of $\nu : \mathbf{Cat} \rightarrow \widehat{\mathbb{C}}$, classifying small maps in $\widehat{\mathbb{C}}$ of increasing size, in the sense of Proposition 5.

3. Let $\alpha = 2$ so that $1 \rightarrow 2$ is the subobject classifier of \mathbf{Set} , and

$$1 = \dot{\mathbf{Set}}_2^{\text{op}} \rightarrow \mathbf{Set}_2^{\text{op}} = 2$$

is then a classifier in \mathbf{Cat} for full subcategories $\mathbb{S} \hookrightarrow \mathbb{A}$ that are closed under the domains of arrows $a \rightarrow s$ for $s \in \mathbb{S}$ (“total sieves”). The “lifted universe” $\dot{\mathcal{V}}_2 \rightarrow \mathcal{V}_2$ is then the subobject classifier $1 \rightarrow \Omega$ of $\widehat{\mathbb{C}}$,

$$1 = \nu 1 = \dot{\mathcal{V}}_2 \rightarrow \mathcal{V}_2 = \nu 2 = \Omega.$$

4. Let $i : 2 \hookrightarrow \mathbf{Set}_\kappa$ and $p : \mathbf{Set}_\kappa \rightarrow 2$ be the embedding-retraction pair with $i : 2 \hookrightarrow \mathbf{Set}_\kappa$ the inclusion of the full subcategory on the sets $\{0, 1\}$ and $p : \mathbf{Set}_\kappa \rightarrow 2$ the retraction that takes $0 = \emptyset$ to itself, and everything else (i.e. the non-empty sets) to $1 = \{\emptyset\}$. There is a retraction (of arrows) in \mathbf{Cat} ,

$$\begin{array}{ccccc} 1 & \hookrightarrow & \dot{\mathbf{Set}}_\kappa & \longrightarrow & 1 \\ \downarrow \lrcorner & & \downarrow & & \downarrow \\ 2 & \xrightarrow{i} & \mathbf{Set}_\kappa & \xrightarrow{p} & 2 \end{array} \quad (21)$$

where the left square is a pullback.

By the functoriality of $\nu : \mathbf{Cat} \rightarrow \widehat{\mathbb{C}}$ we then have a retract diagram in $\widehat{\mathbb{C}}$, again with a pullback on the left,

$$\begin{array}{ccccc}
 1 & \hookrightarrow & \dot{\mathcal{V}}_\kappa & \longrightarrow & 1 \\
 \downarrow & \lrcorner & \downarrow & & \downarrow \\
 \Omega & \xrightarrow{\{-\}} & \mathcal{V}_\kappa & \xrightarrow{[-]} & \Omega
 \end{array} \tag{22}$$

where for any $\phi : X \rightarrow \Omega$ the subobject $\{\phi\} \rightarrowtail X$ is classified as a small map by the composite $\{\phi\} : X \rightarrow \mathcal{V}_\kappa$, and for any small map $A \rightarrow X$, the image $[A] \rightarrowtail X$ is classified as a subobject by the composite $[\alpha] : X \rightarrow \mathcal{V}_\kappa \rightarrow \Omega$, where $\alpha : X \rightarrow \mathcal{V}_\kappa$ classifies $A \rightarrow X$. The idempotent composite

$$\| - \| = [\{-\}] : \mathcal{V}_\kappa \longrightarrow \mathcal{V}_\kappa$$

is the *propositional truncation modality* in the natural model of type theory given by $\dot{\mathcal{V}}_\kappa \rightarrow \mathcal{V}_\kappa$ (see [AGH21]).

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References

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