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Higher Lawvere theories

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A B S T R A C T

We survey Lawvere theories at the level of ∞-categories, as an alternative framework for higher algebra (rather than ∞-operads). From a pedagogical perspective, they make many key definitions and constructions less technical. They also play a prominent role in equivariant homotopy theory and its relatives.

Our main result establishes a universal property for the ∞-category of Lawvere theories, which completely characterizes the relationship between a Lawvere theory and its ∞-category of models. Many familiar properties of Lawvere theories follow directly.

As a consequence, we establish a correspondence between enriched and module Lawvere theories, which implies that the Burnside category is a classifying object for additive categories. This completes a proof from our earlier paper on the commutative algebra of categories.

1. Introduction

The primary goal of this paper is to lay the foundations for a Lawvere theoretic approach to higher algebra. As such, we organize the paper as a survey, with new results and new proofs throughout. In this introduction, we summarize the new results; however, some readers less familiar with the subject may prefer to begin at section 2.

A Lawvere theory $\mathcal{L}$ encodes a particular type of algebraic theory (for example, commutative monoids). A *model* of $\mathcal{L}$ is an instance of that algebraic structure. When we work with *higher* (or homotopical) Lawvere theories, we take models in an ∞-category, by default the ∞-category $\mathcal{S}$ of spaces (or homotopy types).

Specifically, $\mathcal{L}$ is an ∞-category with finite products, generated by a single object, and models are functors $\mathcal{L} \to \mathcal{S}$ which preserve finite products.

We write $\mathrm{Mdl}_{\mathcal{L}} = \mathrm{Fun}^{\times}(\mathcal{L}, \mathcal{S})$ for the ∞-category of models. Given a map of Lawvere theories $F : \mathcal{L} \to \mathcal{L}'$, restriction along F induces $F^* : \mathrm{Mdl}_{\mathcal{L}'} \to \mathrm{Mdl}_{\mathcal{L}}$, a right adjoint functor. Therefore, the assignment from $\mathcal{L}$ to the ∞-category of models is functorial

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$$\mathrm{Mdl} : \mathrm{Lwv} \to (\mathrm{Pr}^R)^{\mathrm{op}} \cong \mathrm{Pr}^L,$$

where an object of Pr^R (respectively Pr^L) is a presentable ∞-category, and a morphism is a right adjoint (respectively left adjoint) functor.

In fact, since any Lawvere theory $\mathcal{L}$ has a distinguished object 1, $\mathrm{Mdl}_{\mathcal{L}}$ also has a distinguished object, namely the model $\mathrm{Map}_{\mathcal{L}}(1, -)$, and these are preserved by the left adjoint functors. Therefore, Mdl may be promoted to a functor $\mathrm{Lwv} \to \mathrm{Pr}^L_*$ to *pointed* presentable ∞-categories.

Theorem *(Theorems 2.10, 3.4). The functor $Mdl : Lwv \to Pr^L_*$ is fully faithful and symmetric monoidal, and it has a right adjoint which sends $\mathcal{C} \in Pr^L_*$ to $\mathcal{C}^{op}_{fgf}$, where $\mathcal{C}_{fgf} \subseteq \mathcal{C}$ is the full subcategory of* finitely generated free *objects.*

In this way, Lwv is a *symmetric monoidal colocalization* of the better understood ∞-category Pr^L_*. This colocalization allows us to study Lawvere theories using familiar tools like the adjoint functor theorem and Lurie's commutative algebra of categories ([19] 4.8).

The theorem encapsulates the following known results about Lawvere theories in novel packaging:

- (the adjunction) $\mathrm{Mdl}_{\mathcal{L}}$ is the *free cocompletion* of $\mathcal{L}^{\mathrm{op}}$, regarding the latter as an ∞-category with finite coproducts ([20] 5.3.6.10);
- (Mdl is fully faithful) $\mathcal{L}$ can be recovered from $\mathrm{Mdl}_{\mathcal{L}}$ as the full subcategory of finitely generated free objects (Proposition 2.7);

The fact that Mdl is symmetric monoidal may be less familiar, but it is a consequence of Lurie's 'tensor products of categories' machinery in [19] 4.8.

In Section 3, we explore two direct corollaries of this monoidality:

- (Corollary 3.5) If a Lawvere theory $\mathcal{L}$ admits a symmetric monoidal structure compatible with finite products, then $\mathrm{Mdl}_{\mathcal{L}}$ inherits a closed symmetric monoidal structure called *Day convolution*;
- (Corollary 3.8) If $\mathcal{L}$ is as above, then $\mathcal{L}$ is a semiring ∞-category. Any module over $\mathcal{L}$ is naturally enriched in $\mathrm{Mdl}_{\mathcal{L}}$ with its Day convolution.

The first of these is standard, but the second we believe is new. It also admits a partial converse:

Theorem *(Theorem 3.12). If $\mathcal{L}$ admits a symmetric monoidal structure compatible with finite products, and another Lawvere theory $\mathcal{K}$ is enriched in $Mdl_{\mathcal{L}}$, then $\mathcal{K}$ is naturally tensored over $\mathcal{L}$ via a map of Lawvere theories*

$$\mathcal{L} \otimes \mathcal{K} \to \mathcal{K},$$

which behaves 'as expected' on models (see Theorem 3.12).

This theorem suggests that there may be a strong converse to Corollary 3.8:

Conjecture *(Conjecture 3.13). If $\mathcal{L}$ is as above, the ∞-categories of $Mdl_{\mathcal{L}}$-enriched Lawvere theories and $\mathcal{L}$-module Lawvere theories are equivalent.*

Sections 4 and 5 are devoted to examples and applications, as follows:

For any $\mathbb{E}_1$-semiring space R, there is a Lawvere theory Burn_R whose models are R-modules. In Section 4, we show that these are the only semiadditive Lawvere theories; that is, semiadditive Lawvere theories can be identified with semiring spaces. This is an easy result for classical Lawvere theories. We record it here because it is slightly more subtle for ∞-categories. It also suggests a philosophy that we are fond of: Lawvere theories may be regarded as generalized (that is, non-additive) rings.

For example, when $R = \Omega^\infty\mathbb{S}$ is the infinite loop space underlying the sphere spectrum (the commutative semiring space analogous to the commutative ring $\mathbb{Z}$), then $\mathrm{Burn}_{\Omega^\infty\mathbb{S}}$ is the Lawvere theory for $\Omega^\infty\mathbb{S}$-modules, or connective spectra.

We also write just $\mathrm{Burn} = \mathrm{Burn}_{\Omega^\infty\mathbb{S}}$ for this *Burnside ∞-category*. The name 'Burnside' is traditionally applied here because Burn admits another, combinatorial construction: an object is a finite set, and a morphism from X to Y is a virtual span of finite sets $X \leftarrow T \rightarrow Y$. This construction will not be important herein, but see [3] for details.

In Section 5.1, we use our ideas relating to Conjecture 3.13, proving a result promised in the author's earlier paper [5]:

Theorem *(Theorem 5.2). Burn is a commutative semiring ∞-category, and there is a canonical equivalence*

$$Mod_{Burn} \cong AddCat_\infty,$$

where $AddCat_\infty$ denotes the ∞-category of additive ∞-categories.

Remark. Remembering that Burn is the Lawvere theory for connective spectra $\mathrm{Sp}_{\geq 0}$, compare Theorem 5.2 to the following theorem of Gepner-Groth-Nikolaus ([9] 4.6): Regarding $\mathrm{Sp}_{\geq 0}$ as a commutative algebra in Pr^L, a presentable ∞-category is a $\mathrm{Sp}_{\geq 0}$-module if and only if it is additive.

Finally, in Section 5.2, we describe an application to equivariant homotopy theory. There are no results, but our discussion is meant to motivate both our study of Lawvere theories (with an eye towards future applications), as well as recent progress in *equivariant higher category theory*, like [2] and [15].

1.1. Acknowledgment

This paper is largely drawn from the author's thesis [4], of which it is the second part. It has been in the works for years, and benefited from conversations with many others, including Ben Antieau, Clark Barwick, Saul Glasman, Rune Haugseng, Mike Hill, Bogdan Krstic, and others.

2. Fundamentals of Lawvere theories

2.1. Lawvere theories and their models

Classical Lawvere theories are one of the earliest formulations of algebraic theory, dating to Lawvere's 1963 thesis [18], and they have been thoroughly studied since then; see [1]. In the setting of ∞-categories, Lawvere theories have been studied by just a few authors, notably by Cranch [6] and Gepner-Groth-Nikolaus [9], and in the prequel to this paper [5].

The literature in this area is sparse in part because Lurie's book Higher Algebra [19] founds the subject on operads, instead. That approach is now well-developed, and it has many benefits: Operads are restrictive enough to have excellent formal properties, but general enough that we can study $\mathbb{E}_n$-algebras, which are common in homotopy theory.

Lawvere theories are more general than operads. That is, every operad gives rise to an associated Lawvere theory, but Lawvere theories can also encode multiple operations at once, such as ring and module structures. Notably, we will see that Lawvere theories still have good formal properties.

We will present ∞-categorical Lawvere theories in a model-independent way. That is, our definitions will be essentially the same as familiar definitions from ordinary category theory. Contrast this with the approach to operads in Higher Algebra, which depends heavily on properties of fibrations of simplicial sets. Hence, the author believes there is expository value in developing this theory, as higher category theory has become increasingly applicable outside of homotopy theory.

We would like to see an intuitive approach to higher algebra that develops the subject 'from scratch' using Lawvere theories; however, that is not the purpose of this article, which draws heavily on results in Higher Algebra.

Definition 2.1. A *Lawvere theory* is an ∞-category $\mathcal{L}$ which is closed under finite products and generated by a distinguished object 1. That is, every object is equivalent to $1^{\times n}$ for some $n \geq 0$. An algebra or *model* of $\mathcal{L}$ in $\mathcal{C}$ is a functor $\mathcal{L} \to \mathcal{C}$ which preserves finite products.

Typically, we want to take $\mathcal{C}$ to be an ∞-category which is presentable and cartesian closed, such as Set, $\mathcal{S}$, or Cat_∞, but there is no such requirement. By default, we take $\mathcal{C}$ to be the ∞-category $\mathcal{S}$ of CW complexes, or homotopy types, which is the initial object among ∞-categories which are presentable and closed symmetric monoidal. We write

$$\mathrm{Mdl}_{\mathcal{L}} = \mathrm{Fun}^{\times}(\mathcal{L}, \mathcal{S}).$$

Example 2.2. Let Fin denote the category of finite sets. Then $\mathrm{Fin}^{\mathrm{op}}$ is a Lawvere theory, with product given by disjoint union of sets, and evaluation at the singleton

$$\mathrm{Mdl}_{\mathrm{Fin}^{\mathrm{op}}} \to \mathcal{S}$$

is an equivalence of ∞-categories. We say $\mathrm{Fin}^{\mathrm{op}}$ is the *trivial Lawvere theory.*

Example 2.3. Let $\mathrm{Burn}^{\mathrm{eff}}$ denote the effective Burnside 2-category, whose objects are finite sets. For finite sets X, Y, the groupoid of morphisms from X to Y is the groupoid of span diagrams $X \leftarrow T \to Y$, and composition is via pullback. Then $\mathrm{Burn}^{\mathrm{eff}}$ is a Lawvere theory, with product given by disjoint union of sets.

If $f : \mathrm{Burn}^{\mathrm{eff}} \to \mathcal{S}$ is a model, then the spans $0 \overset{=}{\leftarrow} 0 \to 1$, respectively $2 \overset{=}{\leftarrow} 2 \to 1$ endow $f(1)$ with a distinguished point, respectively a binary operation $f(1) \times f(1) \to f(1)$. Composition in $\mathrm{Burn}^{\mathrm{eff}}$ precisely enforces the structure of a commutative (or $\mathbb{E}_\infty$) monoid on $f(1)$, and evaluation at 1

$$\mathrm{Mdl}_{\mathrm{Burn}^{\mathrm{eff}}} \to \mathrm{CMon}_\infty$$

is an equivalence of ∞-categories. This is due to Cranch ([6] Section 4) and Glasman ([12] Appendix A).

We say $\mathrm{Burn}^{\mathrm{eff}}$ is the commutative (or $\mathbb{E}_\infty$) Lawvere theory.

In fact, for any cartesian monoidal $\mathcal{C}^{\times}$, commutative monoids in $\mathcal{C}$ are equivalent to models of $\mathrm{Burn}^{\mathrm{eff}}$ in $\mathcal{C}$ ([5] 3.6). Because a symmetric monoidal ∞-category is a commutative monoid in Cat_∞ ([19] 2.4.2.4), we have:

Example 2.4. A symmetric monoidal ∞-category may be regarded as a functor $\mathrm{Burn}^{\mathrm{eff}} \to \mathrm{Cat}_\infty$ which preserves finite products.

More generally, given any ∞-operad $\mathcal{O}$, there is a Lawvere theory $\mathcal{L}$ such that

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}^{\times}) \cong \mathrm{Mdl}_{\mathcal{L}}(\mathcal{C}^{\times})$$

for any cartesian monoidal $\mathcal{C}^\times$. The Lawvere theory can even be more-or-less explicitly described in terms of $\mathcal{O}$ ([5] 3.16).

It may appear that Lawvere theories are less general than operads because they apply only to cartesian monoidal ∞-categories. This is apparently a significant obstacle: a major application of operads is to understanding multiplicative structure on rings. For example, a ring spectrum is an algebra in spectra under *smash product* (which is not cartesian monoidal). This problem can be overcome, as long as we restrict attention to *connective* ring spectra:

Example 2.5. For any connective ring spectrum R, there are Lawvere theories whose models are equivalent to $\mathrm{Mod}_R^{\geq 0}$, $\mathrm{Alg}_R^{\geq 0}$, and $\mathrm{CAlg}_R^{\geq 0}$.

This example and others like it follow from a result of Gepner, Groth, and Nikolaus [9] which describes *exactly* which ∞-categories are equivalent to $\mathrm{Mdl}_\mathcal{L}$ for some Lawvere theory $\mathcal{L}$ (Theorem 2.6 below).

Before stating their result, we recall the theory of presentable ∞-categories. By the adjoint functor theorem ([20] 5.5.2.9), a functor $\mathcal{C} \to \mathcal{D}$ between presentable ∞-categories preserves small colimits if and only if it has a right adjoint. We write Pr^L for the ∞-category of presentable ∞-categories along with these *left adjoint* functors.

If $\mathcal{C} \in \mathrm{Pr}^L$, then since $\mathcal{S}$ is freely generated by one object under colimits, the following data are equivalent:

- an object $X \in \mathcal{C}$ (we say $\mathcal{C}$ is *pointed*);
- a left adjoint functor $L = - \otimes X : \mathcal{S} \to \mathcal{C}$ (we say $L(S)$ is the *free* object on S);
- a right adjoint functor $R = \mathrm{Map}(X, -) : \mathcal{C} \to \mathcal{S}$ (we say $R(Y)$ is the *underlying space* of Y).

We will denote by Pr^L_* the ∞-category of these pointed presentable ∞-categories, along with left adjoint basepoint-preserving functors. (Formally, Pr^L_* is defined to be the undercategory $\mathrm{Pr}^L_{\mathcal{S}/}$.)

If $\mathcal{L}$ is a Lawvere theory, generated by the distinguished object 1, then we regard $\mathrm{Mdl}_\mathcal{L}$ as canonically pointed by the right adjoint forgetful functor

$$\text{evaluate at } 1 : \mathrm{Mdl}_\mathcal{L} \to \mathcal{S}.$$

By the Yoneda lemma, the corresponding basepoint is the model

$$\mathrm{Map}(1, -) : \mathcal{L} \to \mathcal{S}.$$

Theorem 2.6 *(Gepner-Groth-Nikolaus [9] Theorem B.7). A pointed presentable ∞-category $\mathcal{C}$ is equivalent to $\mathrm{Mdl}_\mathcal{L}$ for some Lawvere theory $\mathcal{L}$ if and only if the forgetful functor $\mathcal{C} \to \mathcal{S}$ is conservative and preserves sifted colimits.*

2.2. Reconstructing Lawvere theories from their models

We have just seen that many ∞-categories $\mathcal{M}$ can be described as models over a Lawvere theory (roughly, those which are presentable and *algebraic* in nature). We may now ask: is that Lawvere theory unique, and to what extent can it be recovered from $\mathcal{M}$?

If $\mathcal{L}$ admits finite products, then $\mathrm{Mdl}_\mathcal{L} = \mathrm{Fun}^\times(\mathcal{L}, \mathcal{S})$ is a full subcategory, by definition, of the ∞-category of presheaves, $\mathcal{P}(\mathcal{L}^{\mathrm{op}}) = \mathrm{Fun}(\mathcal{L}, \mathcal{S})$. Moreover, every representable presheaf $\mathrm{Map}(X, -) : \mathcal{L} \to \mathcal{S}$ preserves any limits that exist in $\mathcal{L}$, and therefore is in $\mathrm{Mdl}_\mathcal{L}$.

By the Yoneda lemma, then $\mathcal{L}^{\mathrm{op}}$ is a full subcategory of $\mathrm{Mdl}_\mathcal{L}$. (We called this a *cartesian monoidal Yoneda lemma* in [5] 3.7.)

If $\mathcal{L}$ is a Lawvere theory, we can explicitly identify $\mathcal{L}^{\mathrm{op}}$ as a subcategory of $\mathrm{Mdl}_{\mathcal{L}}$: The embedding $\mathcal{L}^{\mathrm{op}} \subseteq \mathrm{Mdl}_{\mathcal{L}}$ identifies $1^{\amalg n} \in \mathcal{L}^{\mathrm{op}}$ with $\mathbb{I}^{\amalg n} \in \mathrm{Mdl}_{\mathcal{L}}$. Here $\mathbb{I}$ is the distinguished object of $\mathrm{Mdl}_{\mathcal{L}}$, so that $\mathbb{I}^{\amalg n}$ can also be regarded as the free model on n generators. In conclusion:

Proposition 2.7. *Suppose $\mathcal{M}$ is a pointed, presentable ∞-category with distinguished object $\mathbb{I}$. Let $\mathcal{M}_{fgf}$ be the full subcategory of finitely generated free objects; that is, those of the form $\mathbb{I}^{\amalg n}$ for integers $n \geq 0$. If $\mathcal{M} \cong Mdl_{\mathcal{L}}$ for some Lawvere theory $\mathcal{L}$, then $\mathcal{L} \cong \mathcal{M}^{op}_{fgf}$.*

Theoretically, this proposition combined with Theorem 2.6 allow us to describe the Lawvere theories associated to any operad, modules over a ring, algebras over a ring, etc. – *provided we already understand the ∞-category of models.*

However, we may instead seek to describe the Lawvere theory first, and use it to construct some new ∞-category of models. (For example, we might want to do this for pedagogical purposes as in Example 2.4.) Although we can't say anything generally, this is often possible:

Principle 2.8. Lawvere theories often have combinatorial descriptions, in which their objects are finite sets, morphisms are given by diagrams of finite sets, and products are given by disjoint union.

Example 2.9. The commutative Lawvere theory $\mathrm{Burn}^{\mathrm{eff}}$ is equivalent to the 2-category of spans of finite sets.

We may revisit this principle in a future paper on *combinatorial Lawvere theories*; for now, we will not emphasize it.

2.3. Main theorem

We have described how to pass back and forth between a Lawvere theory and its ∞-category of models. We will show this relation is exceptionally robust.

Let Lwv denote the ∞-category whose objects are Lawvere theories and morphisms are functors which preserve finite products and the distinguished object.[2]

Theorem 2.10. *There is an adjunction*

$$Mdl : Lwv \rightleftarrows Pr^L_* : (-)^{op}_{fgf},$$

and the left adjoint Mdl is fully faithful.

In other words, Lwv is a *colocalization* of Pr^L_*. Theorem 2.6 described explicitly *which* colocalization by providing a testable criterion to determine the essential image of Lwv in Pr^L_*.

Proof. We know $\mathrm{Mdl} \cong \mathrm{Fun}^{\times}(-, \mathcal{S})$ is functorial $\mathrm{Lwv}^{\mathrm{op}} \to \mathrm{Cat}_{\infty}$, and lands in presentable ∞-categories and right adjoint functors. As $\mathrm{Pr}^L \cong (\mathrm{Pr}^R)^{\mathrm{op}}$, there is an induced functor $\mathrm{Mdl} : \mathrm{Lwv} \to \mathrm{Pr}^L$. If $F : \mathcal{L} \to \mathcal{L}'$ preserves finite products, Mdl_F is left adjoint to restriction along F.

Call $L = \mathrm{Mdl}$ and $R = (-)^{\mathrm{op}}_{\mathrm{fgf}}$, suggestive of left and right adjoints. The composite RL is equivalent to the identity by Proposition 2.7. Therefore, applying R induces a map of spaces

$$\mathrm{Fun}^L_*(L(\mathcal{L}), \mathcal{C})^{\mathrm{iso}} \xrightarrow{R_*} \mathrm{Fun}^{\times}_*(\mathcal{L}, R(\mathcal{C}))^{\mathrm{iso}},$$

[2] Formally, Lwv is a full subcategory of pointed cartesian monoidal ∞-categories.

where Fun^L_* and $\mathrm{Fun}^\times_*$ refer to left adjoint or finite-product-preserving functors preserving the basepoint, and $(-)^{\mathrm{iso}}$ is the maximal subgroupoid. (That is, $\mathrm{Fun}^L_*(-,-)^{\mathrm{iso}}$ and $\mathrm{Fun}^\times_*(-,-)^{\mathrm{iso}}$ are the mapping spaces in Pr^L_* and Lwv, respectively.)

Moreover, this R_* is natural in both $\mathcal{L} \in \mathrm{Lwv}$ and $\mathcal{C} \in \mathrm{Pr}^L_*$, and R_* is a natural isomorphism by [20] 5.3.6.10 (which asserts that $\mathrm{Mdl}_\mathcal{L}$ is the *free cocompletion* of $\mathcal{L}^{\mathrm{op}}$, regarding the latter as an ∞-category which already has finite coproducts). Therefore, L and R are adjoint.

For $\mathcal{L}, \mathcal{K} \in \mathrm{Lwv}$, applying L induces a map of spaces

$$\mathrm{Fun}^\times_*(\mathcal{L}, \mathcal{K})^{\mathrm{iso}} \xrightarrow{L_*} \mathrm{Fun}^L_*(L(\mathcal{L}), L(\mathcal{K}))^{\mathrm{iso}}.$$

Since $RL \cong \mathrm{Id}$, $R_* L_*$ is equivalent to the identity, and because R_* is an equivalence (as above), so is L_*. Therefore, L is fully faithful. □

Remark 2.11. As seen in the proof, Theorem 2.10 combines two facts in one: the adjunction asserts [20] 5.3.6.10, that $\mathrm{Mdl}_\mathcal{L}$ is the *free cocompletion* of $\mathcal{L}^{\mathrm{op}}$.

That the left adjoint is fully faithful asserts Proposition 2.7, that $\mathcal{L}^{\mathrm{op}}$ is a full subcategory of $\mathrm{Mdl}_\mathcal{L}$.

Remark 2.12. More generally, if $\mathrm{CartMonCat}_\infty$ denotes the ∞-category of ∞-categories which admit finite products (and functors which preserve them), then the same proof shows

$$\mathrm{Mdl} : \mathrm{CartMonCat}_\infty \to \mathrm{Pr}^L$$

is fully faithful. It also comes very close to being a left adjoint to the functor

$$(-)^{\mathrm{op}} : \mathrm{Pr}^L \to \widehat{\mathrm{CartMonCat}}_\infty,$$

with the following fatal obstacle: The domain of Mdl consists of *small* cartesian monoidal ∞-categories, while the codomain of $(-)^{\mathrm{op}}$ consists of *large* cartesian monoidal ∞-categories.

This set-theoretic problem arises because of the definition of presentable ∞-categories: they are required to have a small set of generating objects, but these objects are not remembered as part of the data. Theorem 2.10 solves the problem by introducing a *framing*; that is, by remembering these objects (or in this case, a single object).

3. Algebra of Lawvere theories

In Section 3.1, we show that the functor $\mathrm{Mdl}_\mathcal{L} : \mathrm{Lwv} \to \mathrm{Pr}^L_*$ is symmetric monoidal. Then we study its behavior on commutative algebras in 3.2, constructing Day convolution products of models. Finally, we study its behavior on modules in Section 3.3, showing that many Lawvere theories have canonical enrichments. We offer evidence that $\mathcal{L}$-module Lawvere theories can be identified with $\mathrm{Mdl}_\mathcal{L}$-enriched Lawvere theories (Conjecture 3.13).

Remark 3.1. To study algebraic properties of *pointed* categories like Lwv and Pr^L_*, we recall from [19] 2.1.3.10: If $\mathcal{C}$ is a symmetric monoidal ∞-category with unit 1, then there is an equivalence

$$\mathcal{C}_{1/} \cong \mathrm{Alg}_{\mathbb{E}_0}(\mathcal{C})$$

between ∞-categories of pointed objects and $\mathbb{E}_0$-algebras.

Unpacking the construction of the $\mathbb{E}_0$-operad [19] 2.1.1.19 and symmetric monoidal envelope [19] 2.2.4, the symmetric monoidal envelope of $\mathbb{E}_0$ is the category $\mathrm{Fin}^{\mathrm{inj}}$ of finite sets and injections (under disjoint union). The universal property of the symmetric monoidal envelope [19] 2.2.4.9 then states

$$\mathcal{C}_{1/} \cong \mathrm{Alg}_{\mathbb{E}_0}(\mathcal{C}) \cong \mathrm{Fun}^{\otimes}(\mathrm{Fin}^{\mathrm{inj}}, \mathcal{C}).$$

3.1. Kronecker products of Lawvere theories

The primary technical contribution of this paper is first to cast the relationship between Lawvere theories and their models as a colocalization (Theorem 2.10) and second that this colocalization is compatible with symmetric monoidal structures on Lwv (the Kronecker product) and Pr^L_* (Lurie's tensor product). We review each of these symmetric monoidal operations:

Remark 3.2 *(Lurie's tensor product of presentable ∞-categories).* There is a closed symmetric monoidal tensor product on Pr^L with the following universal property: left adjoint functors $\mathcal{C}\otimes\mathcal{D} \to \mathcal{E}$ can be identified with functors $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ which preserve small colimits in each variable separately. This is constructed by Lurie in [19] 4.8.1, and the unit is $\mathcal{S}$.

If $\mathcal{V}$ is presentable, to endow $\mathcal{V}$ with the structure of a commutative algebra in $\mathrm{Pr}^{L,\otimes}$ is precisely to endow $\mathcal{V}$ with its own closed symmetric monoidal structure.[3]

If $\mathcal{C}, \mathcal{D}$ are two *pointed* presentable ∞-categories, $\mathcal{C} \otimes \mathcal{D}$ is also canonically pointed (for example, by the free functor $\mathcal{S} \cong \mathcal{S} \otimes \mathcal{S} \to \mathcal{C} \otimes \mathcal{D}$), so that Pr^L_* is also symmetric monoidal via Lurie's tensor product.

A commutative algebra object in Pr^L_* is a presentable ∞-category with a closed symmetric monoidal structure $\otimes$, pointed by the unit of $\otimes$ (because of Remark 3.1 and the identity $\mathbb{E}_0 \otimes \mathbb{E}_\infty \cong \mathbb{E}_\infty$.)

Remark 3.3 *(Kronecker tensor product of Lawvere theories).* There is a closed symmetric monoidal tensor product of cartesian monoidal ∞-categories with the following universal property: functors $\mathcal{C}\otimes\mathcal{D} \to \mathcal{E}$ which preserve finite products can be identified with functors $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ which preserve finite products in each variable separately. This can be made precise in two equivalent ways:

- by Lurie's general framework of tensor products of categories ([19] 4.8.1);
- as in [5]: cartesian monoidal ∞-categories can be identified with modules over the commutative semiring category $\mathrm{Fin}^{\mathrm{op}}$ ([5] 3.1), which admit a relative tensor product $\otimes_{\mathrm{Fin}^{\mathrm{op}}}$.

If $\mathcal{C}, \mathcal{D}$ are Lawvere theories (that is, generated by a single object under $\times$), then $\mathcal{C} \otimes \mathcal{D}$ is also a Lawvere theory ([5] 2.26), so that Lwv inherits a symmetric monoidal operation $\otimes$ called *Kronecker product*, and the unit is $\mathrm{Fin}^{\mathrm{op}}$.

For classical Lawvere theories, the Kronecker product goes back to Freyd [8], and it is also compatible with the Boardman-Vogt tensor product ([19] 2.2.5) of operads. That is, if $\mathcal{L}_{\mathcal{O}}, \mathcal{L}_{\mathcal{O}'}$ are Lawvere theories associated to operads $\mathcal{O}, \mathcal{O}'$, then

$$\mathrm{Mdl}_{\mathcal{L}_{\mathcal{O}} \otimes \mathcal{L}_{\mathcal{O}'}} \cong \mathrm{Fun}^{\times}(\mathcal{L}_{\mathcal{O}} \otimes \mathcal{L}_{\mathcal{O}'}, \mathcal{S}) \cong \mathrm{Fun}^{\times}(\mathcal{L}_{\mathcal{O}}, \mathrm{Fun}^{\times}(\mathcal{L}_{\mathcal{O}'}, \mathcal{S}))$$

which is equivalent to

$$\mathrm{Alg}_{\mathcal{O}}(\mathrm{Alg}_{\mathcal{O}'}(\mathcal{S})) \cong \mathrm{Alg}_{\mathcal{O}\otimes\mathcal{O}'}(\mathcal{S}) \cong \mathrm{Mdl}_{\mathcal{L}_{\mathcal{O}\otimes\mathcal{O}'}},$$

so that $\mathcal{L}_{\mathcal{O}} \otimes \mathcal{L}_{\mathcal{O}'} \cong \mathcal{L}_{\mathcal{O}\otimes\mathcal{O}'}$.

[3] A consequence of the adjoint functor theorem.

Theorem 3.4. *The functors*

$$Mdl\colon CartMonCat_\infty \to Pr^L$$
$$Mdl\colon Lwv \to Pr^L_*$$

are compatible with the symmetric monoidal structures of the last two remarks.

That is, Lwv is a *symmetric monoidal colocalization* of Pr^L_*.

In the next two sections, we will explore the consequences of this theorem when applied to (first) commutative algebras and (second) modules with respect to the two symmetric monoidal structures.

Proof. By [19] 4.8.1.8, the functor

$$\mathrm{Fun}^\times((-)^{\mathrm{op}}, \mathcal{S}) : \mathrm{CocartMonCat}_\infty \to \mathrm{Pr}^L$$

is symmetric monoidal. Passing via the symmetric monoidal equivalence $(-)^{\mathrm{op}} : \mathrm{CartMonCat}_\infty \to \mathrm{CocartMonCat}_\infty$, we have the first part.

Taking pointed objects on each side, $\mathrm{Mdl} : \mathrm{CartMonCat}_* \to \mathrm{Pr}^L_*$ is equivalent by Remark 3.1 to the functor 'postcompose by $\mathrm{Fun}^\times(-, \mathcal{S})$'

$$\mathrm{Fun}^\otimes(\mathrm{Fin}^{\mathrm{inj}}, \mathrm{CartMonCat}) \to \mathrm{Fun}^\otimes(\mathrm{Fin}^{\mathrm{inj}}, \mathrm{Pr}^L),$$

which is then symmetric monoidal because $\mathrm{Fun}^\times(-, \mathcal{S})$ is. □

3.2. Algebra Lawvere theories and Day convolution

For suitable Lawvere theories $\mathcal{L}$, we can use Theorem 3.4 to construct tensor products of $\mathcal{L}$-models. A commutative algebra structure on $\mathcal{L} \in \mathrm{Lwv}^\otimes$ amounts to a symmetric monoidal structure on $\mathcal{L}$ which preserves finite products independently in each variable, and such that the unit is the distinguished object 1.

Such Lawvere theories are sometimes called *commutative algebraic theories* in the classical literature [17].

Corollary 3.5. *If $\mathcal{L} \in CAlg(Lwv^\otimes)$, then $Mdl_\mathcal{L}$ inherits a closed symmetric monoidal structure called* Day convolution, *with unit* $\mathbb{I}$.[4]

Conversely, if $Mdl_\mathcal{L}$ has a closed symmetric monoidal structure with unit $\mathbb{I}$, then $\mathcal{L}$ inherits a commutative algebra structure in Lwv.

Proof. Since Mdl is symmetric monoidal, it takes commutative algebras to commutative algebras. Therefore, if $\mathcal{L} \in \mathrm{CAlg}(\mathrm{Lwv}^\otimes)$, then $\mathrm{Mdl}_\mathcal{L}$ has a commutative algebra structure in Pr^L_*, which is to say a closed symmetric monoidal structure with unit $\mathbb{I}$.

Conversely, the right adjoint to a symmetric monoidal functor is lax symmetric monoidal [10]. If $\mathrm{Mdl}_\mathcal{L}$ has a closed symmetric monoidal structure with unit $\mathbb{I}$, it is a commutative algebra in Pr^L_*, so $\mathcal{L} \in \mathrm{CAlg}(\mathrm{Lwv}^\otimes)$. □

Example 3.6. The effective Burnside 2-category is symmetric monoidal under cartesian product, which makes it a commutative algebraic theory. Therefore, Day convolution provides a closed symmetric monoidal *smash product* for $\mathbb{E}_\infty$-spaces.

[4] This is the Day convolution of Lurie [19] and Glasman [11].

Remark 3.7 *(Models in other ∞-categories).* More generally, suppose $\mathcal{L}$ is a Lawvere theory, and $\mathcal{V}$ is a presentable ∞-category. By general theory of presentable ∞-categories,

$$\mathrm{Fun}^{\times}(\mathcal{L},\mathcal{V}) \cong \mathrm{Fun}^{\amalg}(\mathcal{L}^{\mathrm{op}},\mathcal{V}^{\mathrm{op}})^{\mathrm{op}} \cong \mathrm{Fun}^{L}(\mathrm{Mdl}_{\mathcal{L}},\mathcal{V}^{\mathrm{op}})^{\mathrm{op}} \cong \mathrm{Fun}^{R}(\mathcal{V}^{\mathrm{op}},\mathrm{Mdl}_{\mathcal{L}}).$$

Lurie proves ([19] 4.8.1.17) for two presentable ∞-categories $\mathcal{C}$ and $\mathcal{D}$, that $\mathcal{C}\otimes\mathcal{D} \cong \mathrm{Fun}^{R}(\mathcal{C}^{\mathrm{op}},\mathcal{D})$. Therefore, models of $\mathcal{L}$ in $\mathcal{V}$ can be identified with the tensor product

$$\mathrm{Mdl}_{\mathcal{L}}(\mathcal{V}) \cong \mathrm{Mdl}_{\mathcal{L}} \otimes \mathcal{V}.$$

This equivalence is due to [9] Proposition B.3.

In particular, if $\mathcal{L} \in \mathrm{CAlg}(\mathrm{Lwv}^{\otimes})$ and $\mathcal{V}$ has a closed symmetric monoidal structure, then $\mathrm{Mdl}_{\mathcal{L}}(\mathcal{V})$ also has a closed symmetric monoidal structure[5] (Day convolution).

3.3. Module Lawvere theories and enrichment

Let $\mathcal{V}$ be presentable and closed symmetric monoidal; i.e., $\mathcal{V} \in \mathrm{CAlg}(\mathrm{Pr}^{L,\otimes})$. If $\mathcal{M} \in \mathrm{Pr}^{L}$ is a $\mathcal{V}$-module, and $X \in \mathcal{M}$, then $-\otimes X : \mathcal{V} \to \mathcal{M}$ has a right adjoint $\mathrm{Map}(X,-) : \mathcal{M} \to \mathcal{V}$. This promotes the mapping spaces in $\mathcal{M}$ to objects of $\mathcal{V}$, and therefore makes $\mathcal{M}$ naturally $\mathcal{V}$-enriched. Gepner and Haugseng have made this precise ([10] 7.4.13).

Conversely, we may think of $\mathcal{V}$-modules in Pr^{L} as precisely those $\mathcal{V}$-enriched categories which are *presentable in an enriched sense*. As far as the author is aware, the notion of 'presentable in an enriched sense' has not yet been made rigorous for ∞-categories, but this is a philosophy already familiar to experts.

We have a second corollary of Theorem 3.4:

Corollary 3.8. *If $\mathcal{L}$ is a commutative semiring ∞-category whose additive structure is cartesian monoidal, and $\mathcal{M}$ is an $\mathcal{L}$-module, then $\mathcal{M}$ is naturally enriched in $\mathrm{Mdl}_{\mathcal{L}}$.*

Proof. If $\mathcal{M}$ is an $\mathcal{L}$-module in $\mathrm{CartMonCat}_{\infty}$ (or Lwv), then $\mathrm{Mdl}_{\mathcal{M}}$ is a $\mathrm{Mdl}_{\mathcal{L}}$-model in Pr^{L}. As above, $\mathrm{Mdl}_{\mathcal{M}}$ inherits a canonical $\mathrm{Mdl}_{\mathcal{L}}$-enrichment, which restricts to a $\mathrm{Mdl}_{\mathcal{L}}$-enrichment on the full subcategory $\mathcal{M} \subseteq \mathrm{Mdl}_{\mathcal{M}}^{\mathrm{op}}$. $\square$

Example 3.9. If $\mathcal{L} = \mathrm{Burn}^{\mathrm{eff}}$ is the Lawvere theory for $\mathbb{E}_{\infty}$-spaces, then $\mathrm{Burn}^{\mathrm{eff}}$-modules can be identified with semiadditive ∞-categories ([5] Theorem 1.2). By Corollary 3.8, any semiadditive ∞-category is naturally enriched in $\mathbb{E}_{\infty}$-spaces.

This is the homotopical analogue of a classical fact: semiadditive categories are naturally enriched in commutative monoids.

Conversely, if $\mathcal{L} \in \mathrm{CAlg}(\mathrm{Lwv}^{\otimes})$ and $\mathcal{K}$ is a Lawvere theory which is enriched in $\mathrm{Mdl}_{\mathcal{L}}$, we might ask whether $\mathcal{K}$ is an $\mathcal{L}$-module. We conjecture that some statement of this form is true.

As evidence, we will prove the (presumably weaker statement) that any Lawvere theory with $\mathcal{L}$-algebra structures on all its mapping spaces is tensored over $\mathcal{L}$ in a universal way.

Definition 3.10. If $\mathcal{V}$ is presentable and closed symmetric monoidal, there is a unique symmetric monoidal functor $\mathcal{S} \to \mathcal{V}$ which preserves colimits. Denote its right adjoint by U for *underlying space*. A Lawvere theory (or any ∞-category) $\mathcal{K}$ is *weakly $\mathcal{V}$-enriched* if there is a functor

[5] Because tensor products of commutative algebras are commutative algebras.

$$\mathrm{Map}^{\mathrm{enr}}_{\mathcal{K}}(-,-) : \mathcal{K}^{\mathrm{op}} \times \mathcal{K} \to \mathcal{V}$$

such that $U(\mathrm{Map}^{\mathrm{enr}}_{\mathcal{K}}(-,-)) \cong \mathrm{Map}_{\mathcal{K}}(-,-)$.

Remark 3.11. We believe it is true that any $\mathcal{V}$-enriched category is weakly $\mathcal{V}$-enriched in the sense of Definition 3.10. This statement should follow from the Yoneda lemma for enriched higher categories. (The only known proof, due to Hinich [16], is highly nontrivial.)

Specifically, Hinich proves there is a Yoneda embedding $Y : \mathcal{C} \to \mathcal{P}_{\mathcal{V}}(\mathcal{C})$, where $\mathcal{P}_{\mathcal{V}}(\mathcal{C})$ is an enriched category of enriched functors $\mathcal{C}^{\mathrm{op}} \to \mathcal{V}$. Forgetting all enrichment, there is certainly a forgetful functor $\mathcal{P}_{\mathcal{V}}(\mathcal{C}) \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{V})$. By adjunction, therefore Y induces the functor

$$\mathrm{Map}^{\mathrm{enr}}_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{V}$$

of Definition 3.10. The equivalence $U \circ \mathrm{Map}^{\mathrm{enr}}_{\mathcal{C}} \cong \mathrm{Map}_{\mathcal{C}}$ should follow from naturality of the Yoneda embedding, but there may not be enough theory to make this precise at this time.

Theorem 3.12. *Suppose $\mathcal{L} \in CAlg(Lwv^{\otimes})$, and $\mathcal{K}$ is a Lawvere theory weakly enriched in $Mdl_{\mathcal{L}}$. Then there is a morphism of Lawvere theories $\mathcal{L} \otimes \mathcal{K} \xrightarrow{\otimes} \mathcal{K}$ with the following universal property: For any $X \in \mathcal{L}$, $K \in \mathcal{K}$, there is a natural isomorphism*

$$Map^{enr}(-, K)(X) \cong Map(-, X \otimes K).$$

Proof of Theorem 3.12. By assumption, there is a functor

$$\mathrm{Map}^{\mathrm{enr}}_{\mathcal{K}}(-,-) : \mathcal{K}^{\mathrm{op}} \times \mathcal{K} \to \mathrm{Mdl}_{\mathcal{L}},$$

and the composite with the forgetful functor $\mathrm{ev}_1 : \mathrm{Mdl}_{\mathcal{L}} \to \mathcal{S}$ is the ordinary mapping space $\mathrm{Map}_{\mathcal{K}}$. In particular, that composite preserves finite products in the second variable.

By Theorem 2.6, the forgetful functor ev_1 (evaluation at 1) is conservative. It also preserves finite products because it is a right adjoint. Therefore, the functor $\mathrm{Map}^{\mathrm{enr}}_{\mathcal{K}}(-,-)$ preserves finite products in the second variable.

It follows that the adjoint $\mathcal{L} \times \mathcal{K} \to \mathrm{Fun}(\mathcal{K}^{\mathrm{op}}, \mathcal{S})$ preserves finite products independently in each variable, thereby inducing a functor

$$\phi : \mathcal{L} \otimes \mathcal{K} \to \mathrm{Fun}(\mathcal{K}^{\mathrm{op}}, \mathcal{S})$$

which preserves finite products. By construction, $\phi(X \otimes K) \cong \mathrm{Map}^{\mathrm{enr}}(-, K)(X)$. In particular, if n denotes $1^{\times n}$ in a Lawvere theory,

$$\phi(n) \cong \mathrm{Map}^{\mathrm{enr}}_{\mathcal{K}}(-, n)(1) \cong \mathrm{ev}_1 \circ \mathrm{Map}^{\mathrm{enr}}_{\mathcal{K}}(-, n) \cong \mathrm{Map}_{\mathcal{K}}(-, n),$$

so that ϕ factors through the full subcategory $\mathcal{K} \subseteq \mathrm{Fun}(\mathcal{K}^{\mathrm{op}}, \mathcal{S})$. This completes the proof. □

Together, Corollary 3.8 and Theorem 3.12 are suggestive of:

Conjecture 3.13. *If $\mathcal{L} \in CAlg(Lwv^{\otimes})$, the ∞-categories of $Mdl_{\mathcal{L}}$-enriched Lawvere theories and $\mathcal{L}$-module Lawvere theories are equivalent.*

4. Additive Lawvere theories

Suppose that $\mathcal{L}$ is semiadditive as an ∞-category; essentially, finite products are also finite coproducts. By Example 3.9, $\mathcal{L}$ is enriched in $\mathbb{E}_\infty$-spaces.

Therefore, End(1) is an $\mathbb{E}_1$-semiring space, where 1 is the distinguished object of $\mathcal{L}$, and there is a functor

$$\mathrm{End}(1) : \mathrm{SemiaddLwv} \to \mathbb{E}_1\mathrm{Semiring}.$$

Proposition 4.1. *This functor is an equivalence of symmetric monoidal ∞-categories, identifying a semiring R with the Lawvere theory modeling Mod_R.*

The proposition should not be surprising; if $\mathcal{L}$ is semiadditive, then we know

$$\mathrm{Map}(1^{\times m}, 1^{\times n}) \cong \mathrm{Map}(1,1)^{\times mn},$$

which depends only on the semiring End(1). Hence, we expect End(1) to encode all the data of the Lawvere theory.

Lemma 4.2. *As symmetric monoidal ∞-categories, $\mathbb{E}_1$Semiring (the ∞-category of $\mathbb{E}_1$-semiring spaces under $\otimes$) is equivalent to the full subcategory of Pr^L_* spanned by ∞-categories Mod_R, also under $\otimes$.*

Proof. The lemma is a special case of [19] 4.8.5.5 and (for the symmetric monoidal part) 4.8.5.16. Because the statements of those results are technical, we unpack them below.

Following Lurie, let $\widehat{\mathrm{Cat}}_\infty^{\mathrm{Alg}}(\mathcal{K})$ denote the ∞-category of pairs $(\mathcal{C}, A)$ where $\mathcal{C}$ is a large ∞-category which admits all small colimits and has a monoidal structure which respects colimits in each variable separately, and A is an algebra in $\mathcal{C}$. Let $\widehat{\mathrm{Cat}}_\infty^{\mathrm{Mod}}(\mathcal{K})_*$ denote the ∞-category of triples $(\mathcal{C}, \mathcal{M}, X)$ where $\mathcal{C}$ is as before, and $\mathcal{M}$ is a left $\mathcal{C}$-module, pointed by $X \in \mathcal{M}$.

Then [19] 4.8.5.5 asserts that there is a fully faithful functor

$$\Theta_* : \widehat{\mathrm{Cat}}_\infty^{\mathrm{Alg}}(\mathcal{K}) \to \widehat{\mathrm{Cat}}_\infty^{\mathrm{Mod}}(\mathcal{K})_*$$

which sends $(\mathcal{C}, A)$ to $(\mathcal{C}, \mathrm{Mod}_A, A)$.

Let CMon_∞ denote the ∞-category of $\mathbb{E}_\infty$-algebras, and restrict Θ_* to the subcategories of $\widehat{\mathrm{Cat}}_\infty^{\mathrm{Alg}}(\mathcal{K})$ and $\widehat{\mathrm{Cat}}_\infty^{\mathrm{Mod}}(\mathcal{K})_*$ spanned by pairs $(\mathrm{CMon}_\infty, A)$ and triples $(\mathrm{CMon}_\infty, \mathcal{M}, X)$ for which $\mathcal{M}$ is presentable, and morphisms which are the identity on CMon_∞. Unpacking the definitions in [19] 4.8.1, then we have

$$\Theta_* : \mathbb{E}_1\mathrm{Semiring} \to \mathrm{Mod}_{\mathrm{CMon}_\infty}(\mathrm{Pr}^L)_*$$

is fully faithful. Moreover, $\mathrm{Mod}_{\mathrm{CMon}_\infty}(\mathrm{Pr}^L) \cong \mathrm{SemiaddPr}^L$, which is a full subcategory of Pr^L ([9] 4.10). Hence there are full subcategory inclusions

$$\mathbb{E}_1\mathrm{Semiring} \subseteq \mathrm{Mod}_{\mathrm{CMon}_\infty}(\mathrm{Pr}^L)_* \subseteq \mathrm{Pr}^L_*.$$

Moreover, by [19] 4.8.5.16, Θ_* is symmetric monoidal (with respect to the expected symmetric monoidal structures, described in [19] 4.8.5.14). Since $\mathrm{CMon}_\infty \otimes \mathrm{CMon}_\infty \cong \mathrm{CMon}_\infty$ as presentable ∞-categories, the subcategories we consider inherit their usual symmetric monoidal structures from $\widehat{\mathrm{Cat}}_\infty^{\mathrm{Alg}}(\mathcal{K})$ and $\widehat{\mathrm{Cat}}_\infty^{\mathrm{Mod}}(\mathcal{K})_*$. Therefore, the restriction

$$\Theta_* : \mathbb{E}_1\mathrm{Semiring} \to \mathrm{Pr}^L_*$$

is not only fully faithful but compatible with the symmetric monoidal structures. □

Proof of Proposition 4.1. By the lemma, $\mathbb{E}_1$Semiring is equivalent to the symmetric monoidal full subcategory of Pr^L_* spanned by Mod_R (R ranges over $\mathbb{E}_1$-semirings). Meanwhile, by Theorems 2.10 and 3.4, SemiaddLwv $\subseteq$ Lwv $\subseteq \mathrm{Pr}^L_*$ is also a symmetric monoidal full subcategory of Pr^L_*.

If R is an $\mathbb{E}_1$-semiring space, let Burn_R denote the Lawvere theory whose models are Mod_R, which exists by Theorem 2.6. In particular, there is a chain of full subcategory inclusions

$$\mathbb{E}_1\mathrm{Semiring} \subseteq \mathrm{SemiaddLwv} \subseteq \mathrm{Pr}^L_*.$$

To conclude $\mathbb{E}_1\mathrm{Semiring} \cong \mathrm{SemiaddLwv}$, we need only show that every semiadditive Lawvere theory is equivalent to Burn_R for some R; namely, $R = \mathrm{End}(1)$.

Choose a semiadditive Lawvere theory $\mathcal{L}$, and let $B\mathrm{End}(1)$ denote the full subcategory spanned by the object 1. Denote by α the composite

$$\mathcal{L} \xrightarrow{Y} \mathrm{Fun}^{\mathrm{CMon}}(\mathcal{L}^{\mathrm{op}}, \mathrm{CMon}) \to \mathrm{Fun}^{\mathrm{CMon}_\infty}(B\mathrm{End}(1)^{\mathrm{op}}, \mathrm{CMon}_\infty) \cong \mathrm{Mod}_{\mathrm{End}(1)},$$

where Y is the Yoneda embedding constructed by Hinich [16]. Explicitly, $\alpha(X) = \mathrm{Map}_{\mathcal{L}}(1, X)$, as an End(1)-module.

By construction, $\alpha(1) \cong \mathrm{End}(1)$, and the composite of α with the forgetful functor $U : \mathrm{Mod}_{\mathrm{End}(1)} \to \mathcal{S}$ is $U\alpha(-) \cong \mathrm{Map}(1, -)$, the usual mapping space. Since U and $U\alpha$ preserve finite products, and U is conservative, then α preserves finite products. Thus α restricts to a map of Lawvere theories

$$\mathcal{L} \xrightarrow{\alpha} \mathrm{Burn}_{\mathrm{End}(1)} \subseteq \mathrm{Mod}_{\mathrm{End}(1)}.$$

To conclude, we will prove that this α is an equivalence. Like any map of Lawvere theories, it is essentially surjective, so we need only show it is fully faithful. Given objects $m = 1^{\amalg m}$ and $n = 1^{\amalg n}$ in $\mathcal{L}$, we wish to prove that

$$\mathrm{Map}_{\mathcal{L}}(m, n) \xrightarrow{\alpha_*} \mathrm{Map}_{\mathrm{Burn}_R}(m, n)$$

is an equivalence. When $m = n = 1$, this is true by construction. Otherwise, since $\mathcal{L}$ and Burn_R are semiadditive, we know on both sides that $\mathrm{Map}(m, n) \cong R^{mn}$, so α_* is an equivalence.

Hence, if $\mathcal{M} \subseteq \mathrm{Pr}^L_*$ is the full subcategory spanned by the ∞-categories Mod_R, then

$$\mathrm{Mdl} : \mathrm{SemiaddLwv} \to \mathcal{M}$$

$$\mathrm{End}(1) : \mathcal{M} \to \mathbb{E}_1\mathrm{Semiring}$$

are equivalences, so their composite $\mathrm{End}(1) : \mathrm{SemiaddLwv} \to \mathbb{E}_1\mathrm{Semiring}$ is an equivalence, as desired. □

Corollary 4.3. *If R is an $\mathbb{E}_1$-semiring space and R^{op} denotes the same space with the opposite multiplication, then $Burn_R^{op} \cong Burn_{R^{op}}$.*

In particular, if R is an $\mathbb{E}_\infty$-semiring space, then $Burn_R^{op} \cong Burn_R$.

Proof. Since Burn_R is semiadditive, $\mathrm{Burn}_R^{\mathrm{op}}$ remains semiadditive. It is still generated by the same object 1, so it is a Lawvere theory. By Proposition 4.1, $\mathrm{Burn}_R^{\mathrm{op}} \cong \mathrm{Burn}_S$ where $S = \mathrm{End}(1) \cong R^{\mathrm{op}}$. □

From the proposition, we deduce an important philosophy: Lawvere theories are more like *algebraic* objects than *categorical* objects. We might even regard them as generalized (non-additive) rings.

Finally, we apply Theorem 2.6 to deduce:

Corollary 4.4. *If $\mathcal{M}$ is presentable and semiadditive, and $\mathcal{M} \to \mathcal{S}$ is a right adjoint functor which is conservative and preserves geometric realizations, then $\mathcal{M} \cong Mod_R$ for some $\mathbb{E}_1$-semiring space R, compatibly with the forgetful functor $Mod_R \to \mathcal{S}$.*

5. Applications of Lawvere theories

We will end with two applications. The first is to the higher algebra of semiring ∞-categories, and the second to equivariant homotopy theory.

5.1. Commutative algebra of categories

Let $\mathbb{S}$ denote the sphere spectrum and $\Omega^\infty\mathbb{S}$ the underlying infinite loop space, which is an $\mathbb{E}_\infty$-semiring. Then $\Omega^\infty\mathbb{S}$-modules can be described equivalently as infinite loop spaces, grouplike $\mathbb{E}_\infty$-spaces, or connective spectra:

$$\mathrm{Mod}_{\Omega^\infty\mathbb{S}} \cong \mathrm{Ab}_\infty \cong \mathrm{Sp}_{\geq 0}.$$

We call the associated Lawvere theory $\mathrm{Burn} = \mathrm{Burn}_{\Omega^\infty\mathbb{S}}$, for the *Burnside ∞-category*. Notice by Corollary 4.3 that $\mathrm{Burn}^{\mathrm{op}} \cong \mathrm{Burn}$.

As a Lawvere theory, we may describe Burn as the full subcategory

$$\mathrm{Burn} \cong \mathrm{Burn}^{\mathrm{op}} \subseteq \mathrm{Sp}_{\geq 0}$$

of spectra spanned by wedge powers $\mathbb{S}^{\vee n}$. From the wedge product and closed symmetric monoidal smash product on spectra, Burn inherits its (semiadditive) direct sum and a second symmetric monoidal operation $\otimes$ which makes it a commutative algebraic theory (Corollary 3.5). That is, $\mathrm{Burn} \in \mathrm{CAlg}(\mathrm{Lwv}^\otimes)$, and the embedding $\mathrm{Burn} \cong \mathrm{Burn}^{\mathrm{op}} \subseteq \mathrm{Sp}_{\geq 0}$ is a functor of commutative semiring ∞-categories.

In [5], we asserted that additive ∞-categories are precisely modules over the commutative semiring ∞-category Burn, and we promised to prove this in a sequel. We will now do so, and we will find that all of the work was already done in developing the machinery of Lawvere theories.

Definition 5.1. An ∞-category is *additive* if it is semiadditive and each mapping $\mathbb{E}_\infty$-space is grouplike (that is, π_0 is not just a monoid but a group).

In other words, an additive ∞-category is semiadditive and enriched in $\mathrm{Ab}_\infty \cong \mathrm{Sp}_{\geq 0}$ (grouplike $\mathbb{E}_\infty$-spaces, or connective spectra).

Theorem 5.2. *There is an equivalence $Mod_{Burn} \cong AddCat_\infty$, compatible with the two forgetful functors to $SymMonCat_\infty$.*

Remark 5.3. Note that Theorem 5.2 implies Conjecture 3.13 for the specific Lawvere theory Burn. Indeed, $\mathrm{Mdl}_{\mathrm{Burn}} \cong \mathrm{Sp}_{\geq 0}$-enriched Lawvere theories are additive Lawvere theories, so the conjecture asserts that a Lawvere theory is a Burn-module if and only if it is additive.

Lemma 5.4. *If $\mathcal{C}$ is an additive ∞-category, then $Mdl_{\mathcal{C}} = Fun^\times(\mathcal{C}, \mathcal{S})$ is also additive.*

Proof. Since $\mathcal{C}$ is semiadditive, it is (equivalently) a $\mathrm{Burn}^{\mathrm{eff}}$-module. Therefore, $\mathrm{Mdl}_{\mathcal{C}}$ is a $\mathrm{Mdl}_{\mathrm{Burn}^{\mathrm{eff}}} = \mathrm{CMon}_{\infty}$-module, which is to say semiadditive [9]. To prove $\mathrm{Mdl}_{\mathcal{C}}$ is additive, we need only construct, for any morphism $f : X \to Y$ in $\mathrm{Mdl}_{\mathcal{C}}$, a morphism $-f : X \to Y$ such that $f + (-f) \cong 0$.

Of course, X, Y are functors $\mathcal{C} \to \mathcal{S}$. Since $\mathcal{C}$ is additive, there is a natural transformation $-1 : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ such that $1 + (-1) \cong 0$. Composing with X, there is also a morphism $-1 : X \to X$ in $\mathrm{Mdl}_{\mathcal{C}}$ such that $1 + (-1) = 0$. We can choose $-f = f \circ (-1)$, so that $f + (-f) \cong 0$ as desired. □

Proof of Theorem 5.2. As in [9], the product map $\mathrm{Sp}_{\geq 0} \otimes \mathrm{Sp}_{\geq 0} \to \mathrm{Sp}_{\geq 0}$ is an equivalence. Identifying $\mathrm{Sp}_{\geq 0}$ with $\mathrm{Mdl}_{\mathrm{Burn}}$ and noting that Mdl is symmetric monoidal, we find that $\mathrm{Burn} \otimes \mathrm{Burn} \to \mathrm{Burn}$ is also an equivalence.

Any time $R \in \mathrm{CAlg}(\mathcal{C})$ is idempotent in this way, then Mod_R is a full subcategory of $\mathcal{C}$ (see [9] Section 3 or [5] 2.11). Therefore, the forgetful functor $\mathrm{Mod}_{\mathrm{Burn}} \to \mathrm{SymMonCat}_{\infty}$ is fully faithful. We need to show that a symmetric monoidal ∞-category admits the structure of a Burn-module if and only if it is additive.

First, if $\mathcal{C}$ is a Burn-module, it is a $\mathrm{Burn}^{\mathrm{eff}}$-module and therefore semiadditive (Example 3.9), but it is also $\mathrm{Mdl}_{\mathrm{Burn}}$-enriched by Corollary 3.8. By definition, it is therefore additive.

Conversely, suppose $\mathcal{C}$ is additive. By the lemma,

$$\mathrm{Mdl}_{\mathcal{C}^{\mathrm{op}}} = \mathrm{Fun}^{\times}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$$

is additive; it is also presentable because $\mathcal{S}$ is. Since presentable ∞-categories are additive if and only if they are $\mathrm{Sp}_{\geq 0}$-modules [9], $\mathrm{Fun}^{\times}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ is a $\mathrm{Sp}_{\geq 0}$-module in Pr^L. Moreover, the forgetful functor $\mathrm{Pr}^L \to \mathrm{SymMonCat}_{\infty}$ (which remembers only the cocartesian monoidal structure) is lax symmetric monoidal, so $\mathrm{Fun}^{\times}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ is a $\mathrm{Sp}_{\geq 0}$-module in $\mathrm{SymMonCat}_{\infty}$.

As above, $\mathrm{Burn} \subseteq \mathrm{Mdl}_{\mathrm{Burn}}$ is an inclusion of commutative semiring ∞-categories, so $\mathrm{Fun}^{\times}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$. Moreover, the Yoneda embedding $\mathcal{C} \subseteq \mathrm{Fun}^{\times}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ is product-preserving (a quick check, but see [5] Lemma 3.7), so the action of Burn restricts to $\mathcal{C}$, and $\mathcal{C}$ admits a Burn-module structure. This completes the proof. □

5.2. *Equivariant homotopy theory*

Throughout this section, G is a finite group. We write Fin_G for the category of finite G-sets, and Burn_G for the associated ∞-category of virtual spans, often referred to as the *Burnside ∞-category* without mention of the particular group. See [3] for more on this.

All group actions will be on the right.

There are two classical model categories of equivariant G-spaces: the 'naive' model structure has weak equivalences those maps which are weak equivalences of the underlying space. The corresponding ∞-category is $\mathrm{Fun}(BG, \mathcal{S})$, because equivalences in a functor ∞-category are likewise checked objectwise, and BG has only one object (up to equivalence).

On the other hand, the 'genuine' model structure has weak equivalences those maps which have inverses up to homotopy. This model category corresponds to an ∞-category $\mathcal{S}_G$ which is certainly not equivalent to $\mathrm{Fun}(BG, \mathcal{S})$! For example, the map $EG \to *$ is an equivalence in the former but not in the latter model structure.

For spectra as well, there is a distinction between $\mathrm{Fun}(BG, \mathrm{Sp})$ and the ∞-category Sp_G of genuine equivariant spectra. Consult [13] for a classical survey.

We might ask how to describe $\mathcal{S}_G$ and Sp_G in higher categorical terms. For this, we have the two theorems:

- (Elmendorf's Theorem: [7] Theorem 1) $\mathcal{S}_G \cong \mathrm{Mdl}(\mathrm{Fin}_G^{\mathrm{op}})$;
- (Guillou-May's Theorem: [14] Theorem 0.1, [3] Example B.6) $\mathrm{Sp}_G^{\geq 0} \cong \mathrm{Mdl}(\mathrm{Burn}_G)$.

Recall that we have used the notation $\mathrm{Mdl}(\mathcal{L}) = \mathrm{Fun}^{\times}(\mathcal{L}, \mathcal{S})$ whenever $\mathcal{L}$ admits finite products, even if it is not a Lawvere theory. However, Fin_G and $\mathrm{Burn}_G^{\mathrm{eff}}$ are not far from being Lawvere theories: although they do not have single generating objects, they are generated freely by the set of orbits G/H, as H ranges over subgroups of G.

We call them *colored Lawvere theories*, with set of colors $\{G/H\}$, or *equivariant Lawvere theories*, because they admit essentially surjective, product-preserving maps from the groupoid of finite G-sets, $\mathrm{Fin}_G^{\mathrm{iso}}$.

Remark 5.5. The word 'genuine', used to describe equivariant spaces and spectra, can be misleading. Frequently, group actions on spectra arise via abstract homotopy-theoretic means, such as when the spectra themselves are algebraic in nature (as in chromatic homotopy theory). In these cases, we typically do not expect 'genuine' equivariant structures.

However, when our spaces or spectra arise geometrically out of point-set constructions, group actions will be 'genuine'. This is because we can pass through the model category of genuine equivariant objects, on our way to the abstract ∞-categories $\mathcal{S}_G$ and Sp_G.

It would almost be better to regard the 'genuine' actions as 'geometric', and the 'naive' actions as 'homotopical'.

The theorems of Elmendorf and Guillou-May may be combined with Corollary 3.8 as follows:

Corollary 5.6. *Regarding Fin_G and $\mathrm{Burn}_G^{\mathrm{eff}}$ as commutative semiring ∞-categories, any Fin_G-module is naturally enriched in genuine G-spaces, and any $\mathrm{Burn}_G^{\mathrm{eff}}$-module is naturally enriched in (connective) genuine G-spectra.*

More generally, suppose we have some algebraic structure, whose homotopical instances form an ∞-category $\mathcal{C}$. For example, $\mathcal{C} = \mathrm{Sp}_{\geq 0}$ corresponds to the structure 'abelian group'. We might ask: what kind of structure does a *genuine* equivariant G-object of $\mathcal{C}$ have?

This is a question which is not entirely idle. Following Remark 5.5, if an object of $\mathcal{C}$ has an action of G at some sufficiently concrete point-set level, we might expect *additional structure* to carry over to the ∞-category $\mathcal{C}$, beyond a naive G-action.

By analogy with the theorems of Elmendorf and Guillou-May, we propose addressing this question via a 3-step procedure:

1. check whether $\mathcal{C}$ is of the form $\mathrm{Mdl}_{\mathcal{L}}$ for some Lawvere theory $\mathcal{L}$ (possibly by means of Theorem 2.6);
2. check whether $\mathcal{L}$ can be described combinatorially, by applying some construction $\mathcal{M}$ to Fin (as in Principle 2.8);
3. $\mathrm{Fun}^{\times}(\mathcal{M}(\mathrm{Fin}_G), \mathcal{C})$ is a candidate for genuine G-objects of $\mathcal{C}$.

Example 5.7. When $\mathcal{C} = \mathcal{S}$, $\mathcal{L} = \mathrm{Fin}^{\mathrm{op}}$ and the combinatorial construction $\mathcal{M}$ is the opposite category construction, so that (3) is Elmendorf's Theorem.

When $\mathcal{C} = \mathrm{Sp}_{\geq 0}$, $\mathcal{L} = \mathrm{Burn}$ and the combinatorial construction $\mathcal{M}$ is the virtual span construction, so that (3) is Guillou-May's Theorem.

One goal is to use this strategy to understand equivariant $\mathbb{E}_{\infty}$-ring spectra via the Lawvere theory of *bispans* of finite G-sets, by analogy with the construction of Tambara functors [21].

We hope to address these problems in a sequel, in which we will discuss combinatorial constructions of Lawvere theories (as in Principle 2.8).

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