

Notes on cubical models of type theory

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Roughly following the paper of Bezem, Coquand, and Huber [?] and reformulating things in functorial style.

1 Some cube categories

We consider three different cube categories, to be used as index categories for cubical sets:

1. \mathbb{C} the (classical) cube category: the *free monoidal category on an interval*.
2. \mathbb{C}_s the symmetric cube category: the *free symmetric monoidal category on an interval*.
3. \mathbb{C}_c the cartesian cube category: the *free finite product category on an interval*.

1.1 The classical cube category \mathbb{C}

(Cf. Jardine [?, ?].) The *objects* are the sets of binary n -tuples:

$$I^n = \{\langle d_1, \dots, d_n \rangle \mid d_i = 0, 1\}$$

where

$$I = \{0, 1\}$$

and we let $I^0 = \{*\}$.

The *arrows*

$$f : I^n \longrightarrow I^m$$

are those functions generated by compositions of the following primitive ones:

- *face maps* $\alpha_i^d : I^n \longrightarrow I^{n+1}$, taking $\langle d_1, \dots, d_n \rangle$ to $\langle d_1, \dots, d_{(i)}, \dots, d_n \rangle$, with a new digit $d = 0, 1$ inserted as the i^{th} coordinate. There are $2(n+1)$ such maps.
- *degeneracies* $\beta_i : I^n \longrightarrow I^{n-1}$, taking $\langle d_1, \dots, d_n \rangle$ to $\langle d_1, \dots, \hat{d}_i, \dots, d_n \rangle$, omitting the i^{th} coordinate. There are n such maps.

Note that the order of the d_i 's does not change.

Remarks

1. It can be shown that every map factors as:

$$\begin{array}{ccc} I^n & \xrightarrow{f} & I^m \\ & \searrow \beta & \nearrow \alpha \\ & I^k & \end{array}$$

where $\alpha : I^k \rightarrow I^m$ is a composite of faces, and $\beta : I^n \rightarrow I^k$ is a composite of degeneracies. Using this, it can be shown that \mathbb{C} is the free monoidal category on an *interval*: an object I equipped with maps:

$$1 \begin{array}{c} \top \\ \rightrightarrows \\ \perp \end{array} I \xrightarrow{!} 1$$

satisfying $! \circ \top = \text{id}_1 = ! \circ \perp$, where 1 is the monoidal unit.

2. The presheaf category $\mathbf{cSet} = \mathbf{Set}^{\mathbb{C}^{\text{op}}}$ of *cubical sets* has the same homotopy theory as the classical simplicial sets $\mathbf{sSet} = \mathbf{Set}^{\Delta^{\text{op}}}$, in the sense that the two are Quillen equivalent.
3. The objects I^n are *not* the n -fold cartesian products of the interval I , either in the site \mathbb{C} or as presheaves. Rather, there is a monoidal product \otimes on \mathbf{cSet} extending that on \mathbb{C} , such that $I^m \otimes I^n \cong I^{m+n}$. Similarly, the geometric realization functor to topological spaces

$$R : \mathbf{cSet} \longrightarrow \mathbf{Top}$$

does not in general preserve cartesian products, but instead takes tensor products in \mathbf{cSet} to cartesian ones in \mathbf{Top} ,

$$R(X \otimes Y) \cong R(X) \times R(Y).$$

1.2 The symmetric cube category \mathbb{C}_s

(Cf. Grandis [?].) As before, the *objects* are the sets of binary n -tuples:

$$1 = I^0, I, \dots, I^n$$

The *arrows*

$$f : I^n \longrightarrow I^m$$

are still functions generated by compositions of primitive ones, including the faces and degeneracies as before, but now also including the primitive:

- *permutations* $\sigma_i : I^n \longrightarrow I^n$, swapping d_i and d_{i+1} .

For each I^n there are $n - 1$ such maps. Of course, for any permutation $\sigma \in S_n$ one can define a corresponding $\sigma : I^n \longrightarrow I^n$ taking $\langle d_1, \dots, d_n \rangle$ to $\langle d_{\sigma(1)}, \dots, d_{\sigma(n)} \rangle$ as a suitable composite of σ_i 's.

Remarks

1. It can be shown that now every map factors as:

$$\begin{array}{ccc} I^n & \xrightarrow{f} & I^m \\ \beta \downarrow & & \uparrow \alpha \\ I^k & \xrightarrow[\sigma]{\sim} & I^k \end{array}$$

where $\alpha : I^k \hookrightarrow I^m$ is a (composite) face, $\sigma : I^k \xrightarrow{\sim} I^k$ is a (composite) permutation, and $\beta : I^n \twoheadrightarrow I^k$ is a (composite) degeneracy. Using this, it can be shown that \mathbb{C}_s is the free *symmetric* monoidal category on an interval.

2. The presheaf category $\mathbf{csSet} = \mathbf{Set}^{\mathbb{C}_s^{\text{op}}}$ of *symmetric cubical sets* again has the same homotopy theory as simplicial sets.
3. The objects I^n are again n -fold *tensor* products of the interval I , but not *cartesian* products, either in the site \mathbb{C}_s or in \mathbf{csSet} . And again, the geometric realization functor from \mathbf{csSet} to topological spaces does not preserve cartesian products, but instead takes tensor products to cartesian ones. Relatedly, there is a functor $\mathbf{Hom}(X, -)$, right adjoint to the tensor $X \otimes (-)$, which is not an exponential.

Covariant presentation

(Cf. Bezem, Coquand, and Huber [?], Pitts [?].) There is a dual presentation of the symmetric site \mathbb{C}_s . Let the category \mathcal{C} have as *objects* the finite sets

$$[n] = \{1, \dots, n\}$$

and write

$$[n]^+ = [n] \cup \{\top, \perp\} = \{\top, 1, \dots, n, \perp\}.$$

The *arrows*

$$f : [n] \longrightarrow [m]$$

in \mathcal{C} are all functions $f : [n] \longrightarrow [m]^+$ satisfying the following *partial injectivity condition*:

$$f(i) = f(j) \implies (i = j \text{ or } f(i) = \top = f(j) \text{ or } f(i) = \perp = f(j))$$

In other words, f is injective on the preimage of $[m] \subseteq [m]^+$,

$$\begin{array}{ccc} & \xrightarrow{\quad} & [m] \\ \downarrow \lrcorner & & \downarrow \\ [n] & \xrightarrow{\quad f \quad} & [m]^+ \end{array}$$

Identity and composition are just as in the Kleisli-category of the monad $X \mapsto X^+$. Specifically, $\text{id} : [n] \longrightarrow [n]$ is the inclusion $[n] \hookrightarrow [n]^+$, and $g \circ f : [n] \longrightarrow [m] \longrightarrow [k]$ is $\bar{g} \circ f$, where

$$\bar{g} : [m]^+ \longrightarrow [k]^+$$

is the unique (\top, \perp) -preserving extension of g , as indicated in the following.

$$\begin{array}{ccccc} [n] & & [m] & & [k] \\ & \searrow f & \downarrow & \searrow g & \downarrow \\ & & [m]^+ & \xrightarrow{\bar{g}} & [k]^+ \end{array}$$

One can show easily that this category \mathcal{C} (called the category of “names and substitutions” in [?]) is dual to the category of symmetric cubes,

$$\mathcal{C} \cong \mathbb{C}_s^{\text{op}}$$

and so we have an alternate presentation of the symmetric cubical sets as *covariant* functors,

$$\text{scSet} = \text{Set}^{\mathbb{C}_s^{\text{op}}} \cong \text{Set}^{\mathcal{C}}.$$

1.3 The cartesian cube category \mathbb{C}_c

As a modification of the foregoing, we consider a notion of *cartesian cubical sets*. The *objects* of \mathbb{C}_c are again the sets of binary n -tuples:

$$1 = I^0, I, \dots, I^n$$

The *arrows* of \mathbb{C}_c ,

$$f : I^n \longrightarrow I^m$$

are still functions generated by compositions of primitive ones, including the faces, degeneracies, and permutations, but now also including the primitive

- *diagonal maps* $\delta_i : I^n \longrightarrow I^{n+1}$, which double the i^{th} coordinate:

$$\langle d_1, \dots, d_n \rangle \mapsto \langle d_1, \dots, d_i, d_i, \dots, d_n \rangle.$$

Proposition 1. \mathbb{C}_c is the free category with finite products and an interval,

$$1 \begin{array}{c} \top \\ \rightrightarrows \\ \perp \end{array} I \xrightarrow{!} 1.$$

Proof. The free category with finite products and an interval is the classifying category for the algebraic theory consisting of the two constants $\{\top, \perp\}$, which can be described as follows (see [?]):

objects: finite lists $[x_1, \dots, x_n]$ of distinct variables,

arrows: $f : [x_1, \dots, x_n] \longrightarrow [x_1, \dots, x_m]$ are (equivalence classes of) m -tuples

$$f = \langle f_1, \dots, f_m \rangle$$

of terms in context,

$$x_1, \dots, x_n \vdash f_i.$$

But in this simple theory, the only such terms are the variables x_1, \dots, x_n themselves and the constants $\{\top, \perp\}$, and the equivalence relation is trivial, since there are no equations. Thus an arrow is just an m -tuple of arbitrary elements taken from the set $\{x_1, \dots, x_n, \top, \perp\}$. The identity arrow is the list of variables $\langle x_1, \dots, x_n \rangle$, and composition is by the usual substitution of terms for variables. But this is evidently just another description of the category \mathbb{C}_c .

In more detail, each of the primitive kinds of maps $\alpha_i^d, \beta_i, \sigma_i, \delta_i$ can clearly be presented in this form, e.g. $\alpha_i^d = \langle x_1, \dots, d'_{(i)}, \dots, x_n \rangle$, where $d' = \top, \perp$, respectively, when $d = 1, 0$. Conversely, an m -tuple (e_1, \dots, e_m) of elements

from the set $\{x_1, \dots, x_n, \top, \perp\}$ determines a map $\epsilon : I^n \longrightarrow I^m$ in \mathbb{C}_c as follows: beginning with a binary n -tuple (d_1, \dots, d_n) , first apply degeneracies β_i corresponding to each x_i not occurring in (e_1, \dots, e_m) ; next apply a permutation σ that reorders the terms d_j in accordance with the order of the non-constant terms e_j appearing in (e_1, \dots, e_m) ; apply suitable δ 's to duplicate coordinates appearing more than once; and finally use α 's to insert the required constants. \square

Corollary 2. *The cartesian cube category \mathbb{C}_c is equivalent to a non-full subcategory of \mathbf{Cat} (respectively \mathbf{Pos}) on the objects $I^n = I \times \dots \times I$, where $I = (0 \leq 1)$ is the 2-element poset.*

Proof. Each of the maps $\alpha_i^d, \beta_i, \sigma_i, \delta_i$ is monotone, and these are all distinct as monotone maps. To see that this is not full, observe that every monotone $f : I^n \longrightarrow I^m$ is an m -tuple of monotone $f_i : I^n \longrightarrow I$, each of which coming from \mathbb{C}_c is either a projection or a constant. But the map $f : I^2 \longrightarrow I$ with $f(1, 1) = 1$, and $f(d, d') = 0$ otherwise, is neither. \square

Note that the non-monotone “negation” map $n : I \longrightarrow I$, with $n(0) = 1$ and $n(1) = 0$, is also not in \mathbb{C}_c .

Covariant presentation

As a classifying category for an algebraic theory, the category \mathbb{C}_c of cartesian cubes also has a covariant presentation by Lawvere duality, namely as the opposite of the full subcategory of finitely-generated, free algebras $\mathbf{Alg}_{\mathbf{fg}}$. In this case, the algebras are simply *bipointed sets* (A, a_0, a_1) , and the free ones are the *strictly* bipointed sets $a_0 \neq a_1$. Thus $\mathbf{Alg}_{\mathbf{fg}}$ consists of the finite, strictly bipointed sets and all bipointed maps between them. Specifically, let the objects of \mathbb{B} be the sets $[n] = \{1, \dots, n\}$, and the arrows,

$$f : [m] \longrightarrow [n],$$

be arbitrary, $\{\top, \perp\}$ -preserving maps $[m]^+ \longrightarrow [n]^+$, where as before $[n]^+ = [n] \cup \{\top, \perp\}$. Then clearly $\mathbb{B} = \mathbf{Alg}_{\mathbf{fg}}$, and we know by Lawvere duality that

$$\mathbb{C}_c \cong \mathbb{B}^{\text{op}},$$

as can be read off from the foregoing description of the arrows in \mathbb{C}_c as “ m -tuples of arbitrary elements taken from the set $\{x_1, \dots, x_n, \top, \perp\}$ ”.

As a full subcategory of free algebras, the category \mathbb{B} can also be described as the Kleisli category of the monad $[n] \mapsto [n]^+$. Thus we arrive at the covariant description \mathcal{C} of the symmetric cubes, but without the partial injectivity condition, which is violated by (the duals of) the diagonal maps.

2 Hypercubical sets

Definition 3. We may refer to the objects of the cartesian cube category \mathbb{C}_c as *hypercubes* and write $\mathbb{H} = \mathbb{C}_c$ for the *category of hypercubes*. Again, the objects may be taken to be finite sets of the form

$$[n] = \{x_1, \dots, x_n\},$$

and the arrows,

$$f : [n] \longrightarrow [m],$$

are then taken to be m -tuples of elements drawn from the set

$$[n]^+ = \{\top, x_1, \dots, x_n, \perp\},$$

regarded as terms (alternately, bipointed maps $[m]^+ \longrightarrow [n]^+$).

The category \mathcal{H} of *hypercubical sets* is then the category of presheaves on \mathbb{H} ,

$$\mathcal{H} = \mathbf{Set}^{\mathbb{H}^{\text{op}}}.$$

It is generated by the representable presheaves $y([n])$, which will be written

$$\mathbf{I}^n = y([n])$$

and called the *standard n -cubes*. In particular, the standard 1-cube is $\mathbf{I} = y([1])$, and the standard 0-cube is $\mathbf{I}^0 = y([0]) = 1$. For any hypercubical set $X : \mathbb{H}^{\text{op}} \longrightarrow \mathbf{Set}$, we shall write $X_n = X([n])$ and call this the *set of n -cubes in X* . In particular $\mathbf{I}_m^n = \mathbb{H}([m], [n])$ is the set of m -cubes in the standard n -cube.

Proposition 4. *We now have $\mathbf{I}^n \times \mathbf{I}^m \cong \mathbf{I}^{n+m}$, in virtue of the preservation of products by the Yoneda embedding.*

Proposition 5. *The category \mathcal{H} of hypercubical sets is the classifying topos for bipointed objects.*

Proposition 6. *The geometric realization functor to topological spaces*

$$R : \mathcal{H} \longrightarrow \mathbf{Top}$$

preserves cartesian products, $R(X \times Y) \cong R(X) \times R(Y)$.

Proposition 7. *Since $\mathbb{H} \hookrightarrow \mathbf{Cat}$ is a subcategory, the nerve functor*

$$N : \mathbf{Cat} \longrightarrow \mathcal{H}$$

can be defined as usual by:

$$N(\mathbb{C})_n = \mathbf{Cat}(I^n, \mathbb{C}).$$

However, we do not expect the nerve to be full and faithful.

Proposition 8. *For any hypercubical set X , the exponential X^I can be calculated as:*

$$X^I(n) \cong X(n+1).$$

Proof.

$$\begin{aligned} X^I(n) &\cong \mathbf{hom}(y[n], X^I) \cong \mathbf{hom}(I^n, X^I) \cong \mathbf{hom}(I^n \times I, X) \\ &\cong \mathbf{hom}(I^{n+1}, X) \cong \mathbf{hom}(y[n+1], X) \cong X(n+1). \end{aligned}$$

□

Proposition 9. $I^I \cong I + 1$.

Proposition 10. *The functor $X \mapsto X^I$ has a right adjoint.*

Questions

1. According to Grothendieck [?], the category \mathbb{H} is a test category, and so the category $\mathcal{H} = \mathbf{Set}^{\mathbb{H}^{\text{op}}}$ has the same homotopy theory as simplicial sets. Prove this.
2. Want to know what a “hypercubical ω -groupoid” (i.e. a fibrant object) should be. Are the usual box-filling conditions sufficient to define this? Is there another characterization involving the new diagonal maps?
3. The hypercubical sets \mathcal{H} is perhaps a good setting in which to compare the globular, simplicial, and type-theoretic notions of ω -groupoid.
4. What is a hypercubical $(\infty, 1)$ -category (in analogy to the simplicial notion of quasicategory)? Does the type theory give rise to one?

References

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