

On Hofmann-Streicher universes

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to the memory of Erik Palmgren

Abstract

We have another look at the construction by Hofmann and Streicher of a universe (U, El) for the interpretation of Martin-Löf type theory in a presheaf category $[\mathbb{C}^{\text{op}}, \mathbf{Set}]$. It turns out that (U, El) can be described as the *categorical nerve* of the classifier $\mathbf{Set}^{\text{op}} \rightarrow \mathbf{Set}^{\text{op}}$ for discrete fibrations in \mathbf{Cat} , where the nerve functor is right adjoint to the so-called “Grothendieck construction” taking a presheaf $P : \mathbb{C}^{\text{op}} \rightarrow \mathbf{Set}$ to its category of elements $\int_{\mathbb{C}} P$.

Let $\widehat{\mathbb{C}} = [\mathbb{C}^{\text{op}}, \mathbf{Set}]$ be the category of presheaves on the small category \mathbb{C} .

1. The Hofmann-Streicher universe

In [HS97] the authors define a (type-theoretic) *universe* (U, El) with $U \in \widehat{\mathbb{C}}$ and $El \in \widehat{\int_{\mathbb{C}} U}$ as follows. For $I \in \mathbb{C}$, set

$$U(I) = \mathbf{Cat}(\mathbb{C}/I^{\text{op}}, \mathbf{Set}) \quad (1)$$

$$El(I, A) = A(id_I) \quad (2)$$

with an evident associated action on morphisms. A few comments are required:

1. In (1), we have taken the underlying set of objects of the category $\widehat{\mathbb{C}/I} = [\mathbb{C}/I^{\text{op}}, \mathbf{Set}]$ (in contrast to the specification in [HS97]).
2. In (2), and throughout, the authors steadfastly adopt a “categories with families” point of view in describing a morphism

$$E \longrightarrow U \quad (3)$$

in $\widehat{\mathbb{C}}$ equivalently as an object in

$$\widehat{\mathbb{C}}/U \simeq \widehat{\int_{\mathbb{C}} U}, \quad (4)$$

that is, as a presheaf on the *category of elements* $\int_{\mathbb{C}} U$, where:

$$E(I) = \coprod_{A \in U(I)} \mathbf{El}(I, A).$$

Thus the argument $(I, A) \in \int_{\mathbb{C}} U$ in (2) consists of an object $I \in \mathbb{C}$ and an element $A \in U(I)$.

3. In order to account for size issues, the authors assume a Grothendieck universe \mathcal{U} in **Set**, the elements of which are called *small*. The category \mathbb{C} is then assumed to be small, as are the values of the presheaves, unless otherwise stated.

The presheaf U , which is not small, is regarded as the Grothendieck universe \mathcal{U} “lifted” from **Set** to $[\mathbb{C}^{\text{op}}, \mathbf{Set}]$. We will analyse the construction of (U, \mathbf{El}) from a slightly different perspective in order to arrive at its basic property as a classifier for small families in $\widehat{\mathbb{C}}$.

2. An unused adjunction

For any presheaf X on \mathbb{C} , recall that the category of elements is the comma category,

$$\int_{\mathbb{C}} X = \mathbf{y}_{\mathbb{C}}/X,$$

where $\mathbf{y}_{\mathbb{C}} : \mathbb{C} \rightarrow [\mathbb{C}^{\text{op}}, \mathbf{Set}]$ is the Yoneda embedding, which we may suppress and write simply \mathbb{C}/X . While the category of elements $\int_{\mathbb{C}} X$ is used in the specification of the Hofmann-Streicher universe (U, \mathbf{El}) at the point (4), the authors seem to have missed a trick which would have simplified things:

Proposition 1 ([Gro83], §28). *The category of elements functor $\int_{\mathbb{C}} : \widehat{\mathbb{C}} \rightarrow \mathbf{Cat}$ has a right adjoint, which we denote*

$$\nu_{\mathbb{C}} : \mathbf{Cat} \rightarrow \widehat{\mathbb{C}}.$$

For a small category \mathbb{A} , we call the presheaf $\nu_{\mathbb{C}}(\mathbb{A})$ the \mathbb{C} -nerve of \mathbb{A} .

Proof. As suggested by the name, the adjunction $\int_{\mathbb{C}} \dashv \nu_{\mathbb{C}}$ can be seen as the familiar “realization \dashv nerve” construction with respect to the covariant

functor $\mathbb{C}/- : \mathbb{C} \rightarrow \mathbf{Cat}$, as indicated below.

$$\begin{array}{ccc}
 \widehat{\mathbb{C}} & \begin{array}{c} \xleftarrow{\nu_{\mathbb{C}}} \\ \xrightarrow{f_{\mathbb{C}}} \end{array} & \mathbf{Cat} \\
 \uparrow y & \nearrow \mathbb{C}/- & \\
 \mathbb{C} & &
 \end{array} \tag{5}$$

In detail, for $\mathbb{A} \in \mathbf{Cat}$ and $c \in \mathbb{C}$, let $\nu_{\mathbb{C}}(\mathbb{A})(c)$ be the Hom-set of functors,

$$\nu_{\mathbb{C}}(\mathbb{A})(c) = \mathbf{Cat}(\mathbb{C}/_c, \mathbb{A}),$$

with contravariant action on $h : d \rightarrow c$ given by pre-composing a functor $P : \mathbb{C}/_c \rightarrow \mathbb{A}$ with the post-composition functor

$$\mathbb{C}/_h : \mathbb{C}/_d \longrightarrow \mathbb{C}/_c.$$

For the adjunction, observe that the slice category $\mathbb{C}/_c$ is the category of elements of the representable functor y_c ,

$$\int_{\mathbb{C}} y_c \cong \mathbb{C}/_c.$$

Thus for representables y_c , we have the required natural isomorphism

$$\widehat{\mathbb{C}}(y_c, \nu_{\mathbb{C}}(\mathbb{A})) \cong \nu_{\mathbb{C}}(\mathbb{A})(c) = \mathbf{Cat}(\mathbb{C}/_c, \mathbb{A}) \cong \mathbf{Cat}(\int_{\mathbb{C}} y_c, \mathbb{A}).$$

For arbitrary presheaves X , one uses the presentation of X as a colimit of representables over the index category $\int_{\mathbb{C}} X$, and the easy to prove fact that $\int_{\mathbb{C}}$ itself preserves colimits. Indeed, for any category \mathbb{D} , we have an isomorphism in \mathbf{Cat} ,

$$\varinjlim_{d \in \mathbb{D}} \mathbb{D}/_d \cong \mathbb{D}.$$

□

When \mathbb{C} is fixed, as here, we may omit the subscript from the notation $y_{\mathbb{C}}$ and $\int_{\mathbb{C}}$ and $\nu_{\mathbb{C}}$. The unit and counit maps of the adjunction $\int \dashv \nu$,

$$\begin{aligned}
 \eta : X &\longrightarrow \nu \int X, \\
 \epsilon : \int \nu \mathbb{A} &\longrightarrow \mathbb{A},
 \end{aligned}$$

are as follows. At $c \in \mathbb{C}$, for $x : y_c \rightarrow X$, the functor $(\eta_X)_c(x) : \mathbb{C}/_c \rightarrow \mathbb{C}/_X$ is just composition with x ,

$$(\eta_X)_c(x) = \mathbb{C}/_x : \mathbb{C}/_c \longrightarrow \mathbb{C}/_X. \tag{6}$$

For $\mathbb{A} \in \mathbf{Cat}$, the functor $\epsilon : \int \nu \mathbb{A} \rightarrow \mathbb{A}$ takes a pair $(c \in \mathbb{C}, f : \mathbb{C}/_c \rightarrow \mathbb{A})$ to the object $f(1_c) \in \mathbb{A}$,

$$\epsilon(c, f) = f(1_c).$$

Lemma 2. *For any $f : Y \rightarrow X$, the naturality square below is a pullback.*

$$\begin{array}{ccc} Y & \xrightarrow{\eta_Y} & \nu \int Y \\ f \downarrow & & \downarrow \nu f \\ X & \xrightarrow{\eta_X} & \nu \int X. \end{array} \quad (7)$$

Proof. It suffices to prove this for the case $f : X \rightarrow 1$. Thus consider the square

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & \nu \int X \\ \downarrow & & \downarrow \\ 1 & \xrightarrow{\eta_1} & \nu \int 1. \end{array} \quad (8)$$

Evaluating at $c \in \mathbb{C}$ and applying (6) then gives the following square in \mathbf{Set} .

$$\begin{array}{ccc} Xc & \xrightarrow{\mathbb{C}/_-} & \mathbf{Cat}(\mathbb{C}/_c, \mathbb{C}/_X) \\ \downarrow & & \downarrow \\ 1c & \xrightarrow{\mathbb{C}/_-} & \mathbf{Cat}(\mathbb{C}/_c, \mathbb{C}/_1) \end{array} \quad (9)$$

The image of $* \in 1c$ along the bottom is the forgetful functor $U_c : \mathbb{C}/_c \rightarrow \mathbb{C}$, and its fiber under the map on the right is therefore the set of functors $F : \mathbb{C}/_c \rightarrow \mathbb{C}/_X$ such that $U_X \circ F = U_c$, where $U_X : \mathbb{C}/_X \rightarrow \mathbb{C}$ is also a forgetful functor. But any such F is easily seen to be uniquely of the form $\mathbb{C}/_x$ for $x = F(1_c) : yc \rightarrow X$. \square

3. Classifying families

For the terminal presheaf $1 \in \widehat{\mathbb{C}}$, we have an iso $\int 1 \cong \mathbb{C}$. So for every $X \in \widehat{\mathbb{C}}$ there is a canonical projection $\int X \rightarrow \mathbb{C}$, which is easily seen to be a discrete fibration. It follows that for any map $Y \rightarrow X$ of presheaves, the associated map $\int Y \rightarrow \int X$ is also a discrete fibration. Ignoring size issues for the moment, recall that discrete fibrations in \mathbf{Cat} are classified by the forgetful functor $\mathbf{Set}^{\text{op}} \rightarrow \mathbf{Set}^{\text{op}}$ from (the opposites of) the category of pointed sets

to that of sets (cf. [Web07]). For every presheaf $X \in \widehat{\mathbb{C}}$, we therefore have a pullback diagram in \mathbf{Cat} ,

$$\begin{array}{ccc} \int X & \longrightarrow & \mathbf{Set}^{\mathbf{op}} \\ \downarrow & \lrcorner & \downarrow \\ \mathbb{C} & \xrightarrow{X} & \mathbf{Set}^{\mathbf{op}}. \end{array} \quad (10)$$

Using $\int 1 \cong \mathbb{C}$ and transposing by the adjunction $\int \dashv \nu$ then gives a commutative square in $\widehat{\mathbb{C}}$,

$$\begin{array}{ccc} X & \longrightarrow & \nu \mathbf{Set}^{\mathbf{op}} \\ \downarrow & & \downarrow \\ 1 & \xrightarrow{\bar{X}} & \nu \mathbf{Set}^{\mathbf{op}}. \end{array} \quad (11)$$

Lemma 3. *The square (11) is a pullback in $\widehat{\mathbb{C}}$. More generally, for any map $Y \rightarrow X$ in $\widehat{\mathbb{C}}$, there is a pullback square*

$$\begin{array}{ccc} Y & \longrightarrow & \nu \mathbf{Set}^{\mathbf{op}} \\ \downarrow & \lrcorner & \downarrow \\ X & \longrightarrow & \nu \mathbf{Set}^{\mathbf{op}}. \end{array} \quad (12)$$

Proof. Apply the right adjoint ν to the pullback square (10) and paste the naturality square (7) from Lemma 2 on the left, to obtain the transposed square (12) as a pasting of two pullbacks. \square

Let us write $\dot{\mathcal{V}} \rightarrow \mathcal{V}$ for the vertical map on the right in (12); that is, let

$$\begin{aligned} \dot{\mathcal{V}} &= \nu \mathbf{Set}^{\mathbf{op}} \\ \mathcal{V} &= \nu \mathbf{Set}^{\mathbf{op}}. \end{aligned} \quad (13)$$

We can then summarize our results so far as follows.

Proposition 4. *The nerve $\dot{\mathcal{V}} \rightarrow \mathcal{V}$ of the classifier for discrete fibrations $\mathbf{Set}^{\mathbf{op}} \rightarrow \mathbf{Set}^{\mathbf{op}}$, as defined in (13), classifies natural transformations $Y \rightarrow X$ in $\widehat{\mathbb{C}}$, in the sense that there is always a pullback square,*

$$\begin{array}{ccc} Y & \longrightarrow & \dot{\mathcal{V}} \\ \downarrow & \lrcorner & \downarrow \\ X & \xrightarrow{\tilde{Y}} & \mathcal{V}. \end{array} \quad (14)$$

The classifying map $\tilde{Y} : X \rightarrow \mathcal{V}$ is determined by the adjunction $\int \dashv \nu$ as the transpose of the classifying map of the discrete fibration $\int Y \rightarrow \int X$.

The classifying map $\tilde{Y} : X \rightarrow \mathcal{V}$ of a given a natural transformation $Y \rightarrow X$ is, of course, not in general unique. Nonetheless, we can make use of the construction of $\dot{\mathcal{V}} \rightarrow \mathcal{V}$ as the nerve of the discrete fibration classifier $\mathbf{Set}^{\text{op}} \rightarrow \mathbf{Set}^{\text{op}}$, for which classifying functors $\mathbb{C} \rightarrow \mathbf{Set}^{\text{op}}$ are unique up to natural isomorphism, to infer the following proposition, which plays a role in [Shu15, GSS22] and elsewhere.

Proposition 5 (Realignment). *Given a monomorphism $c : C \hookrightarrow X$ and a family $Y \rightarrow X$, let $y_c : C \rightarrow \mathcal{V}$ classify the pullback $c^*Y \rightarrow C$. Then there is a classifying map $y : X \rightarrow \mathcal{V}$ for $Y \rightarrow X$ with $y \circ c = y_c$.*

$$\begin{array}{ccc}
 c^*Y & \xrightarrow{\quad} & \dot{\mathcal{V}} \\
 \downarrow & \searrow & \downarrow \\
 & Y & \\
 \downarrow & & \downarrow \\
 C & \xrightarrow{y_c} & \mathcal{V} \\
 \downarrow & \searrow & \downarrow \\
 & X & \\
 & \swarrow c & \nearrow y
 \end{array}
 \tag{15}$$

Proof. Transposing the realignment problem (15) for presheaves across the adjunction $\int \dashv \nu$ results in the following realignment problem for discrete fibrations.

$$\begin{array}{ccc}
 \int c^*Y & \xrightarrow{\quad} & \mathbf{Set}^{\text{op}} \\
 \downarrow & \searrow & \downarrow \\
 & \int Y & \\
 \downarrow & & \downarrow \\
 \int C & \xrightarrow{\tilde{y}_c} & \mathbf{Set}^{\text{op}} \\
 \downarrow & \searrow & \downarrow \\
 & \int X & \\
 & \swarrow \int c & \nearrow \tilde{y}
 \end{array}
 \tag{16}$$

The category of elements functor \int is easily seen to preserve pullbacks, hence monos; thus let us consider the general case of a functor $C : \mathbb{C} \rightarrow \mathbb{D}$ which is monic in \mathbf{Cat} , a pullback of discrete fibrations as on the left below, and a

presheaf $E : \mathbb{C} \rightarrow \mathbf{Set}^{\text{op}}$ with $\int E \cong \mathbb{E}$ over \mathbb{C} .

$$\begin{array}{ccc}
 \mathbb{E} & \xrightarrow{\quad} & \mathbf{Set}^{\text{op}} \\
 \downarrow & \searrow & \uparrow \text{dotted} \\
 & \mathbb{F} & \\
 \downarrow & \downarrow & \downarrow \\
 \mathbb{C} & \xrightarrow{E} & \mathbf{Set}^{\text{op}} \\
 \downarrow C & & \uparrow \text{dotted} F \\
 & \mathbb{D} &
 \end{array}
 \tag{17}$$

We seek $F : \mathbb{D} \rightarrow \mathbf{Set}^{\text{op}}$ with $\int F \cong \mathbb{F}$ over \mathbb{D} and $F \circ C = E$. Let $F_0 : \mathbb{D} \rightarrow \mathbf{Set}^{\text{op}}$ with $\int F_0 \cong \mathbb{F}$ over \mathbb{D} . Since $F_0 \circ C$ and E both classify \mathbb{E} , there is a natural iso $e : F_0 \circ C \cong E$. Consider the following diagram

$$\begin{array}{ccccc}
 \mathbb{C} & \xrightarrow{e} & (\mathbf{Set}^{\cong})^{\text{op}} & \xrightarrow{p_2} & \mathbf{Set}^{\text{op}} \\
 \downarrow C & & \uparrow \text{dotted } f & \downarrow p_1 & \\
 \mathbb{D} & \xrightarrow{F_0} & \mathbf{Set}^{\text{op}} & &
 \end{array}
 \tag{18}$$

where \mathbf{Set}^{\cong} is the category of isos in \mathbf{Set} , with p_1, p_2 the (opposites of the) domain and codomain projections. There is a well-known weak factorization system on \mathbf{Cat} (part of the “canonical model structure”) with injective-on-objects functors on the left and isofibrations on the right. Thus there is a diagonal filler f as indicated. The functor $F := p_2 f : \mathbb{D} \rightarrow \mathbf{Set}^{\text{op}}$ is then the one we seek. \square

Of course, as defined in (13), the classifier $\dot{\mathcal{V}} \rightarrow \mathcal{V}$ cannot be a map in $\widehat{\mathbb{C}}$, for reasons of size; we now address this.

4. Small maps

Let α be a cardinal number, and call the sets that are strictly smaller than it α -small. Let $\mathbf{Set}_\alpha \hookrightarrow \mathbf{Set}$ be the full subcategory of α -small sets. Call a presheaf $X : \mathbb{C}^{\text{op}} \rightarrow \mathbf{Set}$ α -small if all of its values are α -small sets, and thus if, and only if, it factors through $\mathbf{Set}_\alpha \hookrightarrow \mathbf{Set}$. Call a map $f : Y \rightarrow X$ of presheaves α -small if all of the fibers $f_c^{-1}\{x\} \subseteq Yc$ are α -small sets (for all

$c \in \mathbb{C}$ and $x \in Xc$). The latter condition is of course equivalent to saying that, in the pullback square over the element $x : yc \rightarrow X$,

$$\begin{array}{ccc} Y_x & \longrightarrow & Y \\ \downarrow & \lrcorner & \downarrow f \\ yc & \xrightarrow{x} & X, \end{array} \quad (19)$$

the presheaf Y_x is α -small.

Now let us restrict the specification (13) of $\dot{\mathcal{V}} \rightarrow \mathcal{V}$ to the α -small sets:

$$\begin{aligned} \dot{\mathcal{V}}_\alpha &= \nu \dot{\text{Set}}_\alpha^{\text{op}} \\ \mathcal{V}_\alpha &= \nu \text{Set}_\alpha^{\text{op}}. \end{aligned} \quad (20)$$

Then the evident forgetful map $\dot{\mathcal{V}}_\alpha \rightarrow \mathcal{V}_\alpha$ is a map in the category $\widehat{\mathbb{C}}$ of presheaves, and it is in fact α -small. Moreover, it has the following basic property, which is just a restriction of the basic property of $\dot{\mathcal{V}} \rightarrow \mathcal{V}$ stated in Proposition 4.

Proposition 6. *The map $\dot{\mathcal{V}}_\alpha \rightarrow \mathcal{V}_\alpha$ classifies α -small maps $f : Y \rightarrow X$ in $\widehat{\mathbb{C}}$, in the sense that there is always a pullback square,*

$$\begin{array}{ccc} Y & \longrightarrow & \dot{\mathcal{V}}_\alpha \\ \downarrow & \lrcorner & \downarrow \\ X & \xrightarrow{\tilde{Y}} & \mathcal{V}_\alpha. \end{array} \quad (21)$$

The classifying map $\tilde{Y} : X \rightarrow \mathcal{V}_\alpha$ is determined by the adjunction $\int \dashv \nu$ as (the factorization of) the transpose of the classifying map of the discrete fibration $\int X \rightarrow \int Y$.

Proof. If $Y \rightarrow X$ is α -small, its classifying map $\tilde{Y} : X \rightarrow \mathcal{V}$ factors through $\mathcal{V}_\alpha \hookrightarrow \mathcal{V}$, as indicated below,

$$\begin{array}{ccccc} & & \curvearrowright & & \\ Y & \longrightarrow & \nu \dot{\text{Set}}_\alpha^{\text{op}} & \hookrightarrow & \nu \dot{\text{Set}}^{\text{op}} \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & \nu \text{Set}_\alpha^{\text{op}} & \hookrightarrow & \nu \text{Set}^{\text{op}}, \\ & & \curvearrowleft & & \\ & & \tilde{Y} & & \end{array} \quad (22)$$

in virtue of the following adjoint transposition,

$$\begin{array}{ccccc}
 \int Y & \xrightarrow{\quad} & \mathsf{Set}_\alpha^{\mathsf{op}} & \hookrightarrow & \mathsf{Set}^{\mathsf{op}} \\
 \downarrow & & \downarrow & & \downarrow \\
 \int X & \xrightarrow{\quad} & \mathsf{Set}_\alpha^{\mathsf{op}} & \hookrightarrow & \mathsf{Set}^{\mathsf{op}}.
 \end{array}
 \quad (23)$$

Note that the square on the right is evidently a pullback, and the one on the left therefore is, too, because the outer rectangle is the classifying pullback of the discrete fibration $\int Y \rightarrow \int X$, as stated. Thus the left square in (22) is a pullback. \square

5. Examples

1. Let $\alpha = \kappa$ a strongly inaccessible cardinal, so that $\mathbf{ob}(\mathsf{Set}_\kappa)$ is a Grothendieck universe. Then the Hofmann-Streicher universe of (3) is recovered in the present setting as the κ -small map classifier

$$E \cong \dot{\mathcal{V}}_\kappa \longrightarrow \mathcal{V}_\kappa \cong U$$

in the sense of Proposition 6. Indeed, for $c \in \mathbb{C}$, we have

$$\mathcal{V}_\kappa c = \nu(\mathsf{Set}_\kappa^{\mathsf{op}})(c) = \mathsf{Cat}(\mathbb{C}/_c, \mathsf{Set}_\kappa^{\mathsf{op}}) = \mathbf{ob}(\widehat{\mathbb{C}/_c}) = U c. \quad (24)$$

For $\dot{\mathcal{V}}_\kappa$ we then have,

$$\begin{aligned}
 \dot{\mathcal{V}}_\kappa c &= \nu(\dot{\mathsf{Set}}_\kappa^{\mathsf{op}})(c) = \mathsf{Cat}(\mathbb{C}/_c, \dot{\mathsf{Set}}_\kappa^{\mathsf{op}}) \\
 &\cong \coprod_{A \in \mathcal{V}_\kappa c} \mathsf{Cat}_{\mathbb{C}/_c}(\mathbb{C}/_c, A^* \mathsf{Set}_\kappa^{\mathsf{op}})
 \end{aligned}
 \quad (25)$$

where the A -summand in (25) is defined by taking sections of the pullback indicated below.

$$\begin{array}{ccc}
 A^* \mathsf{Set}_\kappa^{\mathsf{op}} & \longrightarrow & \dot{\mathsf{Set}}_\kappa^{\mathsf{op}} \\
 \downarrow \lrcorner & \nearrow & \downarrow \\
 \mathbb{C}/_c & \xrightarrow{A} & \mathsf{Set}_\kappa^{\mathsf{op}}
 \end{array}
 \quad (26)$$

But $A^* \mathsf{Set}_\kappa^{\mathsf{op}} \cong \int_{\mathbb{C}/_c} A$ over $\mathbb{C}/_c$, and sections of this discrete fibration in Cat correspond uniquely to natural maps $1 \rightarrow A$ in $\widehat{\mathbb{C}/_c}$. Since 1 is

representable in $\widehat{\mathbb{C}/c}$ we can continue (25) by

$$\begin{aligned}
\dot{\mathcal{V}}_{\kappa}c &\cong \coprod_{A \in \mathcal{V}_{\kappa}c} \mathbf{Cat}_{\mathbb{C}/c}(\mathbb{C}/c, A^* \mathbf{Set}_{\kappa}^{\text{op}}) \\
&\cong \coprod_{A \in \mathcal{V}_{\kappa}c} \widehat{\mathbb{C}/c}(1, A) \\
&\cong \coprod_{A \in \mathcal{V}_{\kappa}c} A(1_c) \\
&= \coprod_{A \in \mathcal{V}_{\kappa}c} \mathbf{El}(\langle c, A \rangle) \\
&= E_c.
\end{aligned}$$

2. By functoriality of the nerve $\nu : \mathbf{Cat} \rightarrow \widehat{\mathbb{C}}$, a sequence of Grothendieck universes

$$\mathcal{U} \subseteq \mathcal{U}' \subseteq \dots$$

in \mathbf{Set} gives rise to a (cumulative) sequence of type-theoretic universes

$$\mathcal{V} \rightarrow \mathcal{V}' \rightarrow \dots$$

in $\widehat{\mathbb{C}}$. More precisely, there is a sequence of cartesian squares,

$$\begin{array}{ccccc}
\dot{\mathcal{V}} & \rightarrow & \dot{\mathcal{V}}' & \rightarrow & \dots \\
\downarrow \lrcorner & & \downarrow \lrcorner & & \\
\mathcal{V} & \rightarrow & \mathcal{V}' & \rightarrow & \dots,
\end{array} \tag{27}$$

in the image of $\nu : \mathbf{Cat} \rightarrow \widehat{\mathbb{C}}$, classifying small maps in $\widehat{\mathbb{C}}$ of increasing size, in the sense of Proposition 6.

3. Let $\alpha = 2$ so that $1 \rightarrow 2$ is the subobject classifier of \mathbf{Set} , and

$$\mathbb{1} = \dot{\mathbf{Set}}_2^{\text{op}} \rightarrow \mathbf{Set}_2^{\text{op}} = \mathbb{2}$$

is then a classifier in \mathbf{Cat} for *sieves*, i.e. full subcategories $\mathbb{S} \hookrightarrow \mathbb{A}$ closed under the domains of arrows $a \rightarrow s$ for $s \in \mathbb{S}$. The nerve $\dot{\mathcal{V}}_2 \rightarrow \mathcal{V}_2$ is then exactly the subobject classifier $1 \rightarrow \Omega$ of $\widehat{\mathbb{C}}$,

$$1 = \nu \mathbb{1} = \dot{\mathcal{V}}_2 \rightarrow \mathcal{V}_2 = \nu \mathbb{2} = \Omega.$$

4. Let $i : \mathbb{2} \hookrightarrow \mathbf{Set}_{\kappa}$ and $p : \mathbf{Set}_{\kappa} \rightarrow \mathbb{2}$ be the embedding-retraction pair with $i : \mathbb{2} \hookrightarrow \mathbf{Set}_{\kappa}$ the inclusion of the full subcategory on the sets $\{0, 1\}$ and $p : \mathbf{Set}_{\kappa} \rightarrow \mathbb{2}$ the retraction that takes $0 = \emptyset$ to itself,

and everything else (i.e. the non-empty sets) to $1 = \{\emptyset\}$. There is a retraction (of arrows) in \mathbf{Cat} ,

$$\begin{array}{ccccc} 1 & \hookrightarrow & \mathbf{Set}_\kappa & \longrightarrow & 1 \\ \downarrow & \lrcorner & \downarrow & & \downarrow \\ 2 & \xhookrightarrow{i} & \mathbf{Set}_\kappa & \twoheadrightarrow[p] & 2 \end{array} \quad (28)$$

where the left square is a pullback.

By the functoriality of $(-)^{\text{op}}$ and $\nu : \mathbf{Cat} \rightarrow \widehat{\mathbf{C}}$ we then have a retract diagram in $\widehat{\mathbf{C}}$, again with a pullback on the left,

$$\begin{array}{ccccc} 1 & \hookrightarrow & \dot{\mathcal{V}}_\kappa & \longrightarrow & 1 \\ \downarrow & \lrcorner & \downarrow & & \downarrow \\ \Omega & \xhookrightarrow{\{-\}} & \mathcal{V}_\kappa & \twoheadrightarrow[-] & \Omega \end{array} \quad (29)$$

where for any $\phi : X \rightarrow \Omega$ the subobject $\{\phi\} \rightarrowtail X$ is classified as a small map by the composite $\{\phi\} : X \rightarrow \mathcal{V}_\kappa$, and for any small map $A \rightarrow X$, the image $[A] \rightarrowtail X$ is classified as a subobject by the composite $[\alpha] : X \rightarrow \mathcal{V}_\kappa \rightarrow \Omega$, where $\alpha : X \rightarrow \mathcal{V}_\kappa$ classifies $A \rightarrow X$. The idempotent composite

$$\|-\| = \{[-]\} : \mathcal{V}_\kappa \longrightarrow \mathcal{V}_\kappa$$

is the *propositional truncation modality* in the natural model of type theory given by $\dot{\mathcal{V}}_\kappa \rightarrow \mathcal{V}_\kappa$ (see [AGH21]).

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