Notes on Algebraic Type Theory

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December 9, 2023

1 Natural models of type theory

We write $\widehat{\mathbb{C}} = [\mathbb{C}^{\mathsf{op}}, \mathsf{Set}]$ for the category of presheaves on a small category $\mathbb{C}$. In (**??**), a *natural model* of type theory is defined to be a representable natural tranformation $\mathrm{t} : \dot{\mathrm{T}} \to \mathrm{T}$ of presheaves on a small category $\mathbb{C}$. Theorem XX shows that such a map is equivalent to the notion of a *category with families* in the sense of [**?**].

A natural model $\mathrm{t} : \dot{\mathrm{T}} \to \mathrm{T}$ will be said to *model* the type-constructors $1, \Sigma, \Pi$ if there are pullback squares in $\mathbb{C}$ of the following form,

$$
\begin{tikzcd}
1 \arrow[r, "*"] \arrow[d, "!"'] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathrm{T}} \arrow[d, "\mathrm{t}"] \\
1 \arrow[r, "1"'] & \mathrm{T}
\end{tikzcd}
\qquad
\begin{tikzcd}
\dot{\mathrm{T}}_2 \arrow[r, "\sigma"] \arrow[d, "\mathrm{t}\cdot\mathrm{t}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathrm{T}} \arrow[d, "\mathrm{t}"] \\
\mathrm{T}_2 \arrow[r, "\Sigma"'] & \mathrm{T}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{P}_{\mathrm{t}}(\dot{\mathrm{T}}) \arrow[r, "\lambda"] \arrow[d, "\mathsf{P}_{\mathrm{t}}(\mathrm{t})"'] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathrm{T}} \arrow[d, "\mathrm{t}"] \\
\mathsf{P}_{\mathrm{t}}(\mathrm{T}) \arrow[r, "\Pi"'] & \mathrm{T}
\end{tikzcd}
\tag{1}
$$

where

1. $\mathsf{P}_{\mathrm{t}} = \mathrm{T}_! \circ \mathrm{t}_* \circ \dot{\mathrm{T}}^* : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is the polynomial endofunctor determined uniquely by $\mathrm{t} : \dot{\mathrm{T}} \to \mathrm{T}$ (its *signature*, cf. [**?**]),

2. $\mathrm{t} \cdot \mathrm{t} : \dot{\mathrm{T}}_2 \to \mathrm{T}_2$ is defined by $\mathsf{P}_{\mathrm{t}\cdot\mathrm{t}} = \mathsf{P}_{\mathrm{t}} \circ \mathsf{P}_{\mathrm{t}}$.

The terminology is justified by the following result, also from [**?**].

Theorem 1. *Let* $\mathrm{t} : \dot{\mathrm{T}} \to \mathrm{T}$ *be a natural model. The associated category with families satisfies the usual rules for the type-constructors* $1, \Sigma, \Pi$ *just if* $\mathrm{t} : \dot{\mathrm{T}} \to \mathrm{T}$ *models them, in the sense of the diagrams* (1).

1.1 Display maps and clans

[*define* $\mathcal{D}_{\mathrm{t}}$ *and show that it is a display map category with ...*]

[*given a display map category* $\mathcal{D}$ *with ... define* d *and show that it is ...*]

[*the constructions are not mutually inverse but rather adjoint inverse ...*]

In these terms, we have the following description of the type-constructors $\Pi, \Sigma, \mathbf{1}$: Given a natural model

Theorem 2 ([?], XX)**.** *Let* $\mathrm{t} : \dot{\mathrm{T}} \to \mathrm{T}$ *be a natural model of type theory. Then*

2. Martin-Löf algebras

Now let $\mathcal{E}$ be a locally cartesian closed category and $\mathrm{t} : \dot{\mathrm{T}} \to \mathrm{T}$ any map in $\mathcal{E}$. As in the representable case, t gives rise to a polynomial endofunctor $\mathsf{P}_{\mathrm{t}} : \mathcal{E} \to \mathcal{E}$, in terms of which we define the following abstraction of the notion of a natural model.

Definition 3. A *Martin-Löf algebra* in $\mathcal{E}$ is a map $\mathrm{t} : \dot{\mathrm{T}} \to \mathrm{T}$ equipped with structure maps $(*, 1, \sigma, \Sigma, \lambda, \Pi)$ making pullback squares

$$\begin{tikzcd} 1 \arrow[r, "*"] \arrow[d, "!"'] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathrm{T}} \arrow[d, "\mathrm{t}"] \\ 1 \arrow[r, "1"'] & \mathrm{T} \end{tikzcd} \qquad \begin{tikzcd} \dot{\mathrm{T}}^2 \arrow[r, "\sigma"] \arrow[d, "\mathrm{t}^2"'] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathrm{T}} \arrow[d, "\mathrm{t}"] \\ \mathrm{T}^2 \arrow[r, "\Sigma"'] & \mathrm{T} \end{tikzcd} \qquad \begin{tikzcd} \dot{\mathrm{T}}' \arrow[r, "\lambda"] \arrow[d, "\mathrm{t}'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathrm{T}} \arrow[d, "\mathrm{t}"] \\ \mathrm{T}' \arrow[r, "\Pi"'] & \mathrm{T} \end{tikzcd} \tag{2}$$

where the maps t^2 and t' are defined in terms of the polynomial endofunctor

$$\mathsf{P}_{\mathrm{t}} = \mathrm{T}_! \circ \mathrm{t}_* \circ \dot{\mathrm{T}}^* : \mathcal{E} \longrightarrow \mathcal{E}$$

as in the foregoing section.

In place of representability, we may require that $\mathrm{t} : \dot{\mathrm{T}} \to \mathrm{T}$ be *tiny*.

Definition 4. A map $f : A \to B$ in a locally cartesian closed category $\mathcal{E}$ is *tiny* if it is so as an object in $\mathcal{E}/_B$, in the sense that exponentiation by f has a right adjoint $(-)^f \dashv (-)_f$.

Lemma 5. *A map* $f : A \to B$ *is tiny just if the* pushforward $f_* : \mathcal{E}/_A \to \mathcal{E}/_B$ *has a right adjoint,*

$$f_* \dashv f^! : \mathcal{E}/_B \longrightarrow \mathcal{E}/_A \,.$$

Proposition 6. *Any representable natural transformation $f : A \to B$ between presheaves on a small category $\mathbb{C}$ is a tiny map in $\hat{\mathbb{C}}$.*

Proof. See [**?**]. Briefly, ... $\square$