

Cartesian cubical model categories

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Background

- There has recently been work on **cubical** homotopy theory.
- It is related to **homotopy type theory** which is being used for computerized proof checking.
- The cubes used for this are **closed under finite products**.
- This model of homotopy was also proposed by Lawvere who stressed the **tinyness of the geometric interval \mathbb{I}** .
- The tinyness of \mathbb{I} is also used in the current theory.

Cartesian cubical sets

The **Cartesian cube category** \square is the opposite of the category \mathbb{B} of finite, strictly bipointed sets,

$$\square := \mathbb{B}^{\text{op}}.$$

Thus \square is the **Lawvere theory of bipointed objects**: the free finite product category with a bipointed object $[0] \rightrightarrows [1]$.

The **Cartesian cubical sets** is the category of presheaves on \square ,

$$\text{cSet} = \text{Set}^{\square^{\text{op}}}.$$

Thus cSet consists of all **covariant** functors $\mathbb{B} \rightarrow \text{cSet}$.

The tiny interval \mathbb{I}

The 1-cube $[1]$ represents the cubical set that **forgets the points**,

$$\mathbb{I} := \mathbb{B}([1], -) : \mathbb{B} \longrightarrow \mathbf{cSet}.$$

It **generates** \mathbf{cSet} under finite products and colimits.

The two points $1 \rightrightarrows \mathbb{I}$ have a trivial intersection.

$$\begin{array}{ccc} 0 & \longrightarrow & 1 \\ \downarrow & \lrcorner & \downarrow \\ 1 & \longrightarrow & \mathbb{I} \end{array}$$

This is the universal **interval** in a topos.

It provides a **good cylinder** $X + X \rightarrowtail \mathbb{I} \times X$ for every object X ,
and a **good path object** $X^{\mathbb{I}} \twoheadrightarrow X \times X$ for every **fibrant** object X .

The main result

Theorem (A. 2023)

There is a Quillen model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ on \mathbf{cSet} where:

- the **cofibrations** \mathcal{C} are an axiomatized class of monos,
- the **fibrations** \mathcal{F} are those $f : X \rightarrow Y$ for which

$$(f^{\mathbb{I}} \times \mathbb{I}, \text{eval}) : X^{\mathbb{I}} \times \mathbb{I} \longrightarrow (Y^{\mathbb{I}} \times \mathbb{I}) \times_Y X$$

lifts on the right against all cofibrations,

- the **weak equivalences** \mathcal{W} are those $f : X \rightarrow Y$ for which $K^f : K^Y \longrightarrow K^X$ is bijective under π_0 whenever K is fibrant.

The construction of $(\mathcal{C}, \mathcal{W}, \mathcal{F})$

The **proof** of the theorem

- uses ideas from **type theory**,
- including the **univalence axiom** of Voevodsky,
- is **axiomatized** in terms of:
 1. a classifier $\Phi \hookrightarrow \Omega$ for the cofibrations,
 2. a tiny interval $1 \rightrightarrows \mathbb{I}$,
 3. a universal small map $\dot{V} \rightarrow V$,
- applies in several different cases.

$(\mathcal{C}, \mathcal{W}, \mathcal{F})$ from $(\Phi, \mathbb{I}, \mathbb{V})$

The model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ is constructed in 3 steps:

1. Φ is used to determine a wfs $(\mathcal{C}, \text{TFib})$,
2. \mathbb{I} is used to determine a wfs $(\text{TCof}, \mathcal{F})$ with $\text{TFib} \subseteq \mathcal{F}$,
3. \mathbb{V} is used to show 3-for-2 for $\mathcal{W} := \text{TFib} \circ \text{TCof}$.

1. The cofibration wfs $(\mathcal{C}, \mathbf{TFib})$

The **cofibrations** \mathcal{C} are the monos $C' \rightarrowtail C$ classified by $t : 1 \rightarrowtail \Phi$.

$$\begin{array}{ccccc}
 C' & \longrightarrow & 1 & \longrightarrow & 1 \\
 \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \top \\
 C & \longrightarrow & \Phi & \longrightarrow & \Omega
 \end{array}$$

t

The **trivial fibrations** \mathbf{TFib} are the maps $T \twoheadrightarrow X$ that lift against the cofibrations.

$$\mathcal{C}^{\pitchfork} =: \mathbf{TFib}$$

$$\begin{array}{ccc}
 C' & \longrightarrow & T \\
 \downarrow & \nearrow & \downarrow \\
 C & \longrightarrow & X
 \end{array}$$

1. The cofibration wfs $(\mathcal{C}, \mathbf{TFib})$

Proposition

$(\mathcal{C}, \mathbf{TFib})$ is an algebraic weak factorization system.

Proof.

The classifier $t : 1 \rightarrow \Phi$ determines a fibrewise polynomial monad

$$P_t = \Phi_! t_* : \mathbf{cSet} \longrightarrow \mathbf{cSet}$$

the algebras for which in \mathbf{cSet}/X are the trivial fibrations. □

2. The fibration wfs $(\mathbf{TCof}, \mathcal{F})$

The **fibrations** \mathcal{F} are defined in terms of the trivial fibrations by

$$(f : F \rightarrow X) \in \mathcal{F} \quad \text{iff} \quad (\delta \Rightarrow f) \in \mathbf{TFib}$$

where $\delta \Rightarrow f$ is the **gap map** with $\delta : 1 \longrightarrow \mathbb{I}$ in \mathbf{cSet}/\mathbb{I} .

$$\begin{array}{ccccc}
 F^{\mathbb{I}} & \xrightarrow{\quad} & F & & \\
 \downarrow & \searrow \delta \Rightarrow f & & & \parallel \\
 & & \cdot & \xrightarrow{\quad} & F \\
 & & \downarrow \lrcorner & & \downarrow f \\
 X^{\mathbb{I}} & \xlongequal{\quad} & X^{\mathbb{I}} & \xrightarrow{\quad} & X
 \end{array}$$

The **trivial cofibrations** \mathbf{TCof} are the maps that lift against \mathcal{F} .

$$\mathbf{TCof} := {}^{\pitchfork} \mathcal{F}$$

3. The weak equivalences \mathcal{W}

Let $\mathcal{W} := \text{TFib} \circ \text{TCof}$.

Proposition

$(\mathcal{C}, \text{TFib})$ and $(\text{TCof}, \mathcal{F})$ form a Barton **premodel structure**.

$$\text{TCof} = \mathcal{W} \cap \mathcal{C}$$

$$\text{TFib} = \mathcal{W} \cap \mathcal{F}$$

Corollary

If \mathcal{W} satisfies 3-for-2, then $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ is a QMS.

3. The weak equivalences \mathcal{W}

We use a **universal fibration** $\dot{U} \twoheadrightarrow U$ to show 3-for-2 for \mathcal{W} .

- (i) there is a **universal small map** $\dot{V} \rightarrow V$
- (ii) U is the **classifying type** for fibration structures on $\dot{V} \rightarrow V$
- (iii) $\dot{U} \twoheadrightarrow U$ is **univalent**
- (iv) U is **fibrant**
- (v) fibrant U implies **3-for-2** for \mathcal{W}

The idea of getting a QMS **from** univalence is due to Sattler.

3(i). The universal small map $\dot{V} \rightarrow V$

The **category of elements** functor $\int_{\mathbb{C}}$

$$\int_{\mathbb{C}} : \widehat{\mathbb{C}} \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \text{Cat} : \nu_{\mathbb{C}}$$

always has a right adjoint **nerve** functor $\nu_{\mathbb{C}}$.

Proposition

For any small map $Y \rightarrow X$ in $\widehat{\mathbb{C}}$ there is a canonical pullback

$$\begin{array}{ccc} Y & \xrightarrow{\quad} & \nu_{\mathbb{C}} \text{set}^{\text{op}} \\ \downarrow \lrcorner & & \downarrow \\ X & \xrightarrow{\quad} & \nu_{\mathbb{C}} \text{set}^{\text{op}} \end{array}$$

since $\text{set}^{\text{op}} \rightarrow \text{set}^{\text{op}}$ classifies small discrete fibrations in Cat .

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Proposition

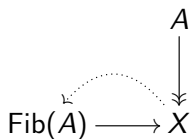
For any small map $Y \rightarrow X$ in $\hat{\mathbb{C}}$ there is a canonical pullback

$$\begin{array}{ccccc} Y & \longrightarrow & \nu_{\mathbb{C}} \dot{\text{set}}^{\text{op}} & \equiv & \dot{V} \\ \downarrow \lrcorner & & \downarrow & & \downarrow \\ X & \longrightarrow & \nu_{\mathbb{C}} \text{set}^{\text{op}} & \equiv & V \end{array}$$

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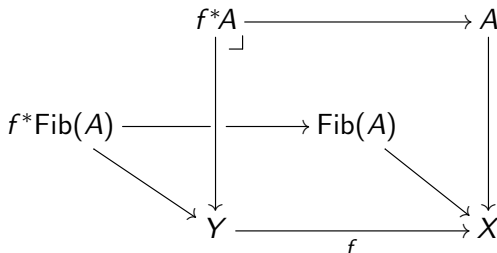
3(ii). The universal fibration $\dot{U} \rightarrow U$

For any $A \rightarrow X$ in \mathbf{cSet} there is a **classifying type** $\mathbf{Fib}(A) \rightarrow X$, the sections of which correspond to fibration structures.



3(ii). The universal fibration $\dot{U} \rightarrow U$

The construction of $\text{Fib}(A) \rightarrow X$ is stable under pullback.



$$f^*\text{Fib}(A) \cong \text{Fib}(f^*A)$$

This uses the **root** functor $(-)^{\mathbb{I}} \dashv (-)_{\mathbb{I}}$.

3(ii). The universal fibration $\dot{U} \rightarrow U$

Let U be the type of fibration structures on $\dot{V} \rightarrow V$

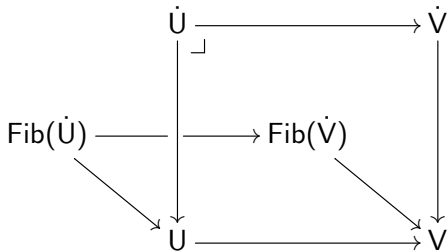
$$\begin{array}{ccc}
 & & \dot{V} \\
 & & \downarrow \\
 U := \text{Fib}(\dot{V}) & \longrightarrow & \dot{V}
 \end{array}$$

then define $\dot{U} \rightarrow U$ by pulling back.

$$\begin{array}{ccc}
 \dot{U} & \xrightarrow{\quad} & \dot{V} \\
 \downarrow \lrcorner & & \downarrow \\
 U & \longrightarrow & V
 \end{array}$$

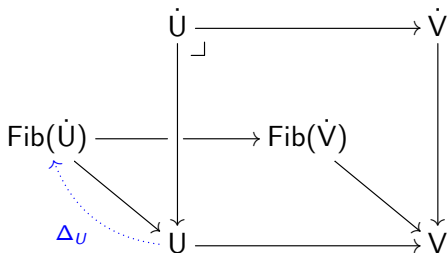
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Since $\text{Fib}(-)$ is stable, the lower square is a pullback.



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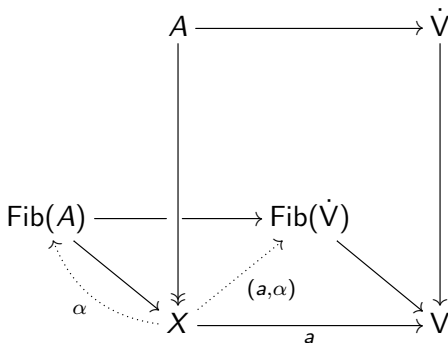
Since $\text{Fib}(-)$ is stable the lower square is also a pullback.



But since $U = \text{Fib}(\dot{V})$ there is a section of $\text{Fib}(\dot{U})$.
So $\dot{U} \rightarrow U$ is a fibration.

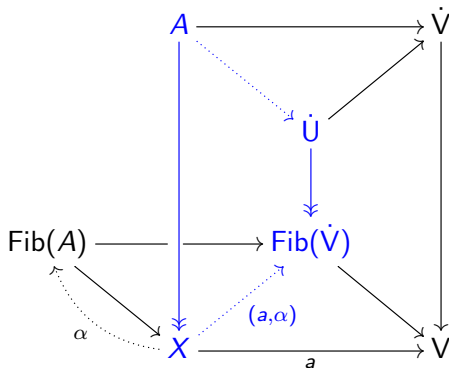
3(ii). The universal fibration $\dot{U} \rightarrow U$

A fibration structure α on a small map $A \rightarrow X$ determines a factorization (a, α) of its classifying map $a : X \rightarrow V$.



3(ii). The universal fibration $\dot{U} \twoheadrightarrow U$

A fibration structure α on a small map $A \rightarrow X$ determines a factorization (a, α) of its classifying map $a : X \rightarrow V$,



which classifies $A \twoheadrightarrow X$ as a **fibration** since $\text{Fib}(\dot{V}) = U$.

3(iii). $\dot{U} \rightarrow U$ is univalent

The universal fibration $\dot{U} \rightarrow U$ is **univalent** if the type

$$\text{Eq}_B = \Sigma_B \text{Eq}(-, B) \longrightarrow U$$

of **based equivalences** is always a trivial fibration.

$$\begin{array}{ccc} C' & \longrightarrow & \text{Eq}_B \\ \downarrow & & \downarrow \\ C & \xrightarrow{A} & U \end{array} \quad \begin{array}{c} \nearrow \\ A \simeq B \end{array}$$

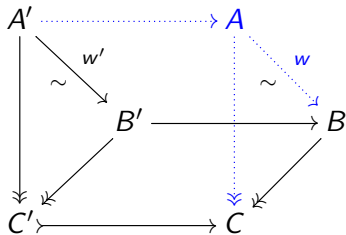
(*)

Remark

In HoTT this implies $(A = B) \simeq (A \simeq B)$.

3(iii). $\dot{U} \twoheadrightarrow U$ is univalent

Unwinding (*) gives the **equivalence extension property**:
weak equivalences extend along cofibrations $C' \rightarrowtail C$.



3(iii). $\dot{U} \twoheadrightarrow U$ is univalent

Proposition

The universal fibration $\dot{U} \twoheadrightarrow U$ is univalent.

Voevodsky proved this **classically** for Kan fibrations in \mathbf{sSet} .

Coquand gave a constructive proof in **type theory** for \mathbf{cSet} .

We have generalized Coquand's proof to cartesian cubical sets.

3(iv). \mathcal{U} is fibrant

Univalence of $\dot{\mathcal{U}} \rightarrow \mathcal{U}$ implies that \mathcal{U} is fibrant.

Proposition

The universe \mathcal{U} is fibrant.

Voevodsky proved this for Kan sSets using **minimal fibrations**.

Shulman proved it using **3-for-2** for \mathcal{W} .

Coquand proved it from univalence without 3-for-2 using **Kan composition** for cSets in type theory.

We give a general proof from univalence without using 3-for-2.

3(v). From fibrant U to 3-for-2

Finally, we can apply the following.

Proposition (Sattler)

\mathcal{W} satisfies 3-for-2 if fibrations extend along trivial cofibrations.

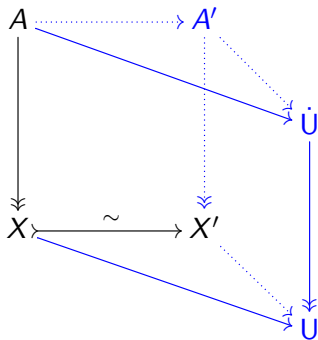
$$\begin{array}{ccc} A & \xrightarrow{\quad \cdots \quad} & A' \\ \downarrow \lrcorner & & \downarrow \ddots \\ X & \xrightarrow[\sim]{} & X' \end{array}$$

This is called the **fibration extension property**.

3(v). From fibrant U to 3-for-2 for \mathcal{W}

Lemma

Given a universal fibration $\dot{U} \twoheadrightarrow U$ the FEP holds if U is fibrant.



References

- S. Awodey, Cartesian cubical model categories, 2023.
- C. Cohen, et al., Cubical type theory: A constructive interpretation of the univalence axiom, 2016.
- C. Kapulkin and P. LeFanu Lumsdaine, The simplicial model of univalent foundations (after Voevodsky), 2021.
- C. Sattler, The equivalence extension property and model structures, 2017.
- M. Shulman, All $(\infty, 1)$ -toposes have strict univalent universes, 2019.

Appendix: U is fibrant (sketch)

It suffices to show the following.

Proposition

Evaluation at the **generic point** $U^{\mathbb{I}} \longrightarrow U$ is a trivial fibration.

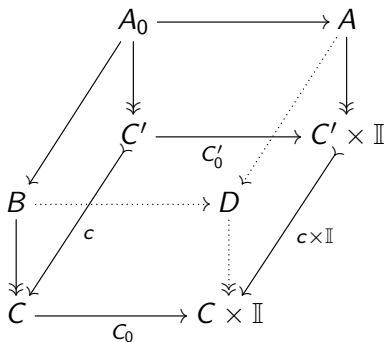
Proof.

We need a diagonal filler for any cofibration c .

$$\begin{array}{ccc} C' & \xrightarrow{a} & U^{\mathbb{I}} \\ c \downarrow & \nearrow \text{dotted} & \downarrow U^{\delta} \\ C & \xrightarrow{b} & U \end{array}$$

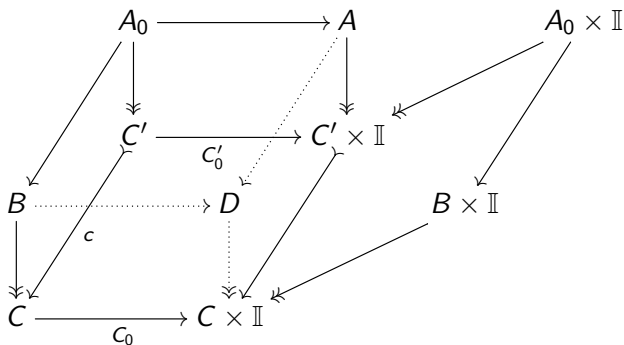
Appendix: \mathcal{U} is fibrant (sketch)

Transposing by \mathbb{I} and using the classifying property of \mathcal{U} gives the following equivalent problem.



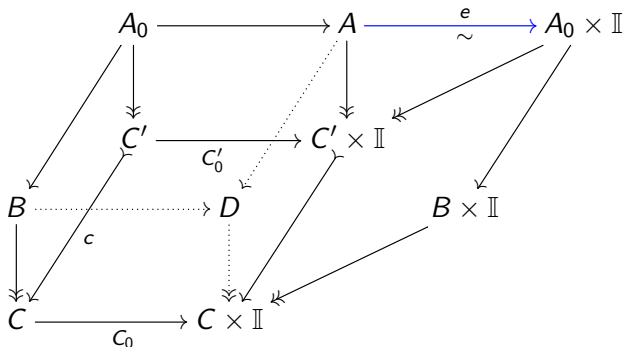
Appendix: U is fibrant (sketch)

Apply the functor $(-) \times \mathbb{I}$ to the left face to get:



Appendix: U is fibrant (sketch)

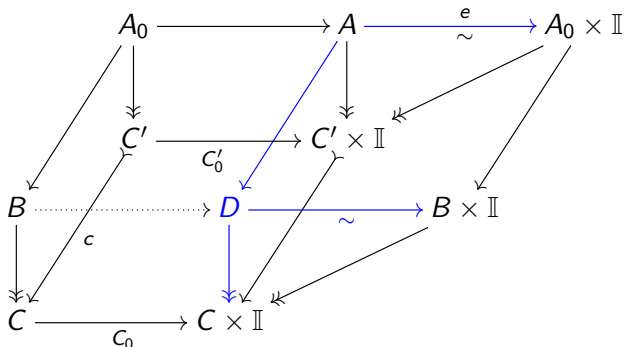
Apply the functor $(-) \times \mathbb{I}$ to the left face to get:



There is a weak equivalence $e : A \xrightarrow{\sim} A_0 \times \mathbb{I}$ to which we can apply the EEP.

Appendix: U is fibrant (sketch)

Apply the functor $(-) \times \mathbb{I}$ to the left face to get:



There is a weak equivalence $e : A \simeq A_0 \times \mathbb{I}$ to which we can apply the EEP. This produces the required fibration $D \twoheadrightarrow Z \times \mathbb{I}$. \square