# CARTESIAN CUBICAL MODEL CATEGORIES

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Abstract. The category of Cartesian cubical sets is introduced and endowed with a Quillen model structure, using ideas coming from recent constructions of cubical systems of univalent type theory.

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## Introduction

Recent years have seen renewed interest in the cubical approach to abstract homotopy theory. This contrasts with the more familiar and widespread simplicial approach, using which many sophisticated and powerful tools have been developed, such as simplicial model categories [DK80], quasi-categories [Joy08], and higher toposes [Lur09]. Of course, some early work like the original papers of D. Kan [Kan55, Kan56] employed cubical sets, and some researchers such as R. Brown [Bro18] and R. Jardine [Jar02] have developed such methods further in a more modern style, but they are swimming against the tide.

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The current interest in the cubical approach arises from connections with the formal system of *type theory* for the purpose of computerized proof checking [AC13]. Unlike previous cubical models of homotopy theory, however, the cubes being used for this purpose are generally assumed to be closed under finite products; we call such cube categories *Cartesian*. This is a natural enough assumption to make for cubes, but one that has somehow escaped serious consideration—but for two notable exceptions: in A. Grothendieck's famous letter to D. Quillen, and the accompanying 600 page manuscript *Pursuing stacks* [Gro83], such cubical sets make an appearance as *test categories*, which model the homotopy category of spaces in a particular way. In fact, the Cartesian cubes studied here are *strict* test categories in the terminology of *op.cit.*, meaning that the geometric realization functor preserves finite products ([BM17]). The more familiar category of "monoidal" cubical sets used since Kan is also a strict test category provided one includes *connections* [Mal09], but this is not necessarily Cartesian. The second source for Cartesian cubical sets is F.W. Lawvere, who proposed them as a model for homotopy theory in lectures, and in public and private correspondence, but never (to my knowledge) published anything on the subject. Among their advantages, he stressed the tinyness of the 1-cube, or "interval" I, which indeed plays a role in the current theory—although perhaps not the one envisioned by him.

We can define *the* Cartesian cube category $\Box$ to be the Lawvere theory of bipointed objects, the opposite of which is therefore the category of finite, strictly bipointed sets $\mathbb{B} = \Box^{\mathrm{op}}$. Thus $\Box$ is the free finite product category with a bipointed object $[0] \rightrightarrows [1]$. Our homotopy theory will be based on the category of *Cartesian cubical sets*, which is the category of presheaves on $\Box$,

$$\mathsf{cSet} = \mathsf{Set}^{\Box^{\mathrm{op}}}$$

and thus consists of all *covariant* functors $\mathbb{B} \to \mathsf{Set}$. Among these, there is an evident distinguished one, namely that which "forgets the points", and it is represented by the generating 1-cube $[1]$,

$$\mathrm{I} = \Box(-,[1]) : \mathbb{B} \longrightarrow \mathsf{Set}\,.$$

In cubical sets, the bipointed object $1 \rightrightarrows \mathrm{I}$ turns out to have the (non-algebraic) property that its two points have a trivial intersection.

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d] \\ 1 \arrow[r] & \mathrm{I} \end{tikzcd}$$

We call such an object in a topos an *interval*, and in a sense to be made precise, this is the universal one. Other categories of Cartesian cubical sets have a canonical comparison to this one, relating their respective homotopy theories.

For the purpose of homotopy theory, namely, this interval provides a good cylinder $X+X \rightarrowtail \mathrm{I}\times X$ for every object $X$, as well as a good path object $X^{\mathrm{I}} \twoheadrightarrow X \times X$ for every *fibrant* object $X$. The notion of fibrancy here is determined by the interval I in terms of *paths* $\mathrm{I} \to X$, and is a generalization of the *path-lifting* condition from classical homotopy theory, suitably modified for this setting. We formulate it using the now-standard notion of a Quillen model structure:

**Definition 1** (cf.[Qui67])**.** A *Quillen model structure* on a (bicomplete) category $\mathcal{E}$ consists of three classes of maps $\mathcal{C}, \mathcal{W}, \mathcal{F}$ satisfying the conditions:

(1) $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are weak factorization systems,
(2) $\mathcal{W}$ has the 3-for-2 property: if any two sides of the triangle $e = f \circ g$ are in $\mathcal{W}$, so is the third.

For the interval $1 \rightrightarrows \mathrm{I}$, we have the mono $\partial : 1 + 1 \rightarrowtail \mathrm{I}$ as one of two basic cofibrations $\mathcal{C}$ giving rise to all the others, in a certain sense. The other basic one is the diagonal $\delta : \mathrm{I} \rightarrowtail \mathrm{I} \times \mathrm{I}$, which is a special cofibration that, together with $\partial$, determines both $\mathcal{F}$ and $\mathcal{W}$ just from the conditions (1) and (2) in the definition (which we have restated in a form due to [JT99]). Condition (1) has recently been termed a *premodel structure* by R. Barton [Bar19], and its verification in our setting is fairly routine, occupying less than the first half of the paper. Condition (2) is where all the work is, and where our treatment is most likely to be of interest to the expert. We shall summarize those aspects below, but let us say now that the model structure is not the one determined by the method of Cisinski [Cis06], nor is it Reedy [Ree74], although the Cartesian cube category $\Box$ is "generalized Reedy" in the sense of [BM08].

Having identified $\Box$ as a strict test category, why not simply use standard tools to determine the test model structure making it equivalent to the standard homotopy theory of spaces? Because *we are mainly interested in how the model structure relates to the interpretation of type theory.* Our final result could no doubt be arrived at more directly with the help of more sophisticated tools from homotopy theory (although not by this author!). But like helicoptering to the top of a mountain, the view might be just as good, but something would be lost along the way. Our goal is not merely to arrive at the "theorem" that there is such-and-such a model structure on such-and-such a category, but to investigate the relationship between the individual ingredients of a Quillen model structure and certain standard constructions in type theory, in order to better understand the somewhat mysterious connection between the two.

The first models of homotopy type theory used the standard Kan-Quillen model structure on simplicial sets [AW09, CK21]. Much subsequent work has also relied on classical methods, including M. Shulman's *tour de force* result that every Grothendieck $\infty$-topos admits a model of HoTT with a univalent universe [Shu19]. This means that all of the results in the Homotopy Type Theory book [Uni13] hold not only in the standard model in "spaces," *i.e.* simplicial sets, but also in any such higher topos. In particular, the univalence *axiom* of V. Voevodsky is actually *true* in all such models. There is, however, a mismatch between such models of the univalence axiom and the design and implementation of computer systems based on type theory. Taken as an axiom, univalence blocks the normalization algorithm which forms the basis of type theoretic computation. Voevodsky recognized this, and conjectured (roughly) that the system with the univalence axiom admitted an interpretation into the system without it, in a way that would restore effective computation.

A version of this "homotopy canonicity conjecture" was finally verified a decade later by T. Coquand and collaborators [BCH14, CCHM18]. One key insight that apparently led to their success was the "change of shape" from simplicial to cubical sets.[1] Some aspects of Coquand's work were undoubtedly informed by homotopy theory, but much of it was driven by type-theoretic considerations: normalization, canonicity, constructivity, etc. Subsequent work on computational systems of univalent type theory (such as [OP18, LOPS18, ABC$^+$22]) also used intuitions from basic homotopy theory (and some of the jargon), but without bothering to verify the model category axioms. Of course, this research had a very different aim, namely the provision of a constructive system of type theory with univalence, which would facilitate its implementation in a computer proof system. Once that was accomplished, there was no need to determine whether a Quillen model structure was also lurking in the background; it simply remained a mystery that the ingredients required for a computational system of univalent type

[1]As suggested by [BC15]. Whether this alone is essential is still a matter of debate; arguably, it was rather the algebraic aspect underlying the "uniform Kan filling" condition that made the break-through possible. Whether the cubical shape is essential to *that* will perhaps be determined by recent work on an algebraic simplicial approach by [GH19].

theory seemed to align with the basic concepts of abstract homotopy theory.

It was C. Sattler who first recognized that a computational implementation of univalent type theory contains everything required to determine a Quillen model structure [Sat17]. An earlier result in this direction had been given by [GG08], who showed that the basic system of type theory with identity types not only interpreted into a weak factorization system (as had been shown by [AW09]), but that it actually *required* such a structure for its sound interpretation—essentially by constructing a weak factorization system from the system of type theory itself (P. LeFanu Lumsdaine subsequently used higher inductive types to construct a second weak factorization system within homotopy type theory [Lum11], making another step toward a full model structure). The relationship between the full system of univalent type theory and a full Quillen model structure is somewhat more subtle—and part of the present investigation—but the mystery of *why* the tools of model category theory seemed to work so well for constructing systems of univalent type theory is at least partially resolved by the insight that the type theory is apparently describing the same kind of structure as do certain model categories; namely, that of a higher topos.[2] So while there was no reason to expect *a priori* that the work on computer proof systems would have any relevance to homotopy theory, the methods developed for those purposes have now acquired such relevance nonetheless.

These new methods include various species of cubical sets with different combinatorial and homotopical properties [BM17], some still unknown, as well as various composition, filling, and uniformity conditions with as yet unclear relationships to homotopical algebra [OP18, CCHM18, BCH14, ABC+22, CMS20]. It is worth noting, for those not familiar with both, that translating between the language of type theory and that of model categories is by no means routine, nor is the converse anything like typesetting a commutative diagram in LaTeX. (Indeed, the limits of such translation are a matter of current investigation, with the question of how to handle the coherences arising in higher category theory in type theory at the very forefront of current research.)

The particular category of Cartesian cubes considered here has been studied by the author, in lectures and papers, since 2013 [Awo18,

[2] Logicians, who have always been puzzled by homotopy theorists' apparently backwards use of the term "model", can finally answer the question of what it is that is modeled by a Quillen model category: it is type theory.

Awo19b], with various different box-filling conditions. The condition explored in the present work, which we call *unbiased partial box-filling*, was apparently considered early on by Coquand [Coq14] but abandoned in favor of a monoidal one in [BCH14], and later modified to one depending on the presence of connections in [CCHM18]. The unbiased approach was resurrected and studied intensely in type theory by R. Harper and his students [BL14, AHW17, AHH18, Ang19], culminating in [ABC+22]. These type theoretic constructions are analyzed in terms of model categories here for the first time, doing for the system of Cartesian cubical type theory roughly what Gambino and Sattler [GS17a, Sat17] did for the system in [CCHM18].

Specifically, we ultimately show that the category of Cartesian cubical sets admits a Quillen model structure $\mathcal{C}, \mathcal{W}, \mathcal{F}$ with the unbiased fibrations as the class $\mathcal{F}$ and the cofibrations $\mathcal{C}$ axiomatized to allow for variations, including additional structure on the basic cube category $\Box$ and adjustments in the filling conditions. Since our proofs are given in elementary diagrammatic form, they will also hold in other categories of Cartesian cubical sets, including those with connections, reversals, etc. Indeed, part of our motivation is to apply the results obtained here, *mutatis mutandis*, in two other settings: realizability, and equivariant filling. The former (underway in [AAFS23]) imposes a strict condition of constructivity, about which we will say a bit more shortly. The latter (underway in [ACC+23]) is based on an unpublished result due to Sattler showing that an additional equivariance condition on the unbiased fibrations suffices to turn this model structure into the test model structure already mentioned.

The possibility of an entirely constructive verification of the Quillen model category axioms is a consequence of the constructive interpretation of univalent type theory labored over by Coquand and his collaborators, and it has applications for the homotopy theory of presheaves and sheaves that stand to be explored further (but cf. [CMR17]). The important uniformity condition on the Kan filling operations is closely related to E. Riehl's *algebraic model structures* [Rie11] and gives rise to a notion of structured fibration that admits classification, in the sense of classifying spaces, by means of what we here call *classifying types*. These classifiers are used to construct universal objects of various kinds: families, cofibrations, (trivial) fibrations, and ultimately a universal fibration $\dot{\mathcal{U}} \twoheadrightarrow \mathcal{U}$, which acts like an object classifier in higher topos theory, but with a stricter universal property. Our work shows that having such classifying types can be useful, *e.g.* when "changing the base" from one slice category $\mathcal{E}/_X$ to another $\mathcal{E}/_Y$ along a map $f : Y \to X$, or along a more general geometric morphism $f^* \dashv f_* : \mathcal{F} \to \mathcal{E}$.

Another application of the constructivity of the model structure is the computation of homotopy invariants as a result of a constructive proof. This was only a theoretical possibility until quite recently, when a breakthrough by A. Ljungström [Lju22], a student in Stockholm, finally allowed the computer system Cubical Agda [VMA19] (which is based on the results just mentioned of Coquand *et al.*) to compute the value of $k \in \mathbb{Z}$ from a proof in homotopy type theory that $\pi_4(S^3) \cong \mathbb{Z}/_{k\mathbb{Z}}$, which had been done by hand 10 years earlier at the IAS by Guillaume Brunerie [Bru13]. Realizability models of type theory based on constructively proven model structures should also have applications in computational homotopy theory.

One way to verify that our model structure is entirely constructive would be to formalize the proofs below in a proof assistant such as Agda. While this could be of interest for the practice of translating model category proofs into type theory, in principle one would learn very little that is not already known, since the model structure given here is in a certain sense "reverse-engineered" from a computational interpretation of type theory that has already been fully formalized and verified (namely, that in [ABC+22]). Although our definitions and proofs do not parallel those in *ibid.* in the way that a proper formalization would, the interpretation of type theory underlying them will be plainly visible to the experts—since the Quillen model category defined here was already found lurking, as it were, behind that system.

Let us now make this more explicit as we outline the contents of the paper (references to the literature occur at the corresponding points of the main text). After defining the Cartesian cubical sets and establishing some basic facts about them in Section 1, Section 2 specifies the *cofibrations* axiomatically, as a class of monomorphisms classified by a universal one $t : 1 \rightarrowtail \Phi$. This permits using the associated polynomial endofunctor $P_t : \mathsf{cSet} \to \mathsf{cSet}$ (which is shown to be a monad by the axioms for cofibrations), to give an algebraic weak factorization system with the cofibrations as the left maps and the (retracts of) $P_t$-algebras as the maps on the right, which we define to be the *trivial fibrations.* Since the monad is fibered, the factorization system is stable under change of base, which we use to derive the familiar diagonal filling characterization of the trivial fibrations in algebraic form, and relate this to the uniform filling condition from type theory. The polynomial monad $P_t : \mathsf{cSet} \to \mathsf{cSet}$ is related to the type theoretic partiality- or lifting-monad, and generalizes the partial map classifier from the early days of topos theory.

In Section 3, the *fibrations* are defined in terms of the cofibrations via the Joyal-Tierney calculus of pushout-products and pullback-homs. A "biased" version using the two endpoints $\delta_0, \delta_1 : 1 \rightrightarrows \mathrm{I}$ is given first, before specifying the "unbiased" version in terms of the generic point $\delta : 1 \to \mathrm{I}^*\mathrm{I}$ in the slice category $\mathsf{cSet}/_{\mathrm{I}}$, namely the diagonal $\mathrm{I} \to \mathrm{I} \times \mathrm{I}$. Specifically, a map $f : X \to Y$ in $\mathsf{cSet}$ is defined to be an unbiased fibration if its pullback to $\mathsf{cSet}/_{\mathrm{I}}$ has the right lifting property against all maps of the form $c \otimes_I \delta$ where $c : C \rightarrowtail Z$ is a cofibration over I and the pushout-product with $\delta$ is formed in $\mathsf{cSet}/_{\mathrm{I}}$.

$$\begin{tikzcd}
Z +_C (C \times \mathrm{I}^*\mathrm{I}) \arrow[r] \arrow[d, tail, "c \otimes_I \delta"'] & \mathrm{I}^*Y \arrow[d, "\mathrm{I}^* f"] \\
Z \times \mathrm{I}^*\mathrm{I} \arrow[r] \arrow[ur, dotted] & \mathrm{I}^*X
\end{tikzcd}$$

The two weak factorization systems of cofibrations and trivial fibrations, and trivial cofibrations and fibrations, are assembled formally into a Barton premodel structure in Section 4, where the *weak equivalences* are determined and related to *weak homotopy equivalences*: maps that induce isomorphisms in the homotopy category by precomposition. The 3-for-2 axiom is reduced to a technical condition dubbed the *fibration extension property*, the proof of which is deferred. This concludes Part 1, and attention shifts to establishing the fibration extension property.

Part 2, consisting of sections 5-8, is essentially a 60 page proof of a lemma. It seems entirely likely that a more direct proof could be given, dispatching the entire second part of the paper. Even in that event, however, the work done in Part 2 would remain worthwhile, for this is where an implicit construction of a model of (homotopy) type theory occurs: The Frobenius property in Section 5 establishes the interpretation of $\Pi$-types of fibrations along fibrations, and thus the right properness of the model structure, by an entirely new diagrammatic argument derived from one originally given in type theory. In section 6 we construct the classifying types for fibration structure and use them to give a new construction of a universal fibration $\dot{\mathcal{U}} \twoheadrightarrow \mathcal{U}$. This is where the tinyness of the interval I plays an unexpected role, and a related axiom on the cofibrations is discovered.

Sections 7 and 8 make implicit use of the model of type theory emerging in the background, and contribute new diagrammatic proofs of two fundamental facts about it. In section 7 an equivalence extension property is established which is closely related to the univalence of the universal fibration $\dot{\mathcal{U}} \twoheadrightarrow \mathcal{U}$, and in section 8 that property is used to finally establish the fibration extension property, which is seen to be equivalent to the statement that the base object $\mathcal{U}$ is fibrant. In sum, then, the missing 3-for-2 property of the premodel structure from Part 1 is proven in Part 2 by constructing a fibrant, univalent universe of fibrant objects.

One thing that we learn from the exercise is that one can get quite far in constructing a model of type theory in a *premodel* category, without assuming a fibrant universe, its univalence, or even the presence of a universe at all! Conversely, our results suggest that the presence of a fibrant, univalent universe in such homotopical semantics in a premodel structure is not just necessary for a full model of univalent type theory, but actually suffices for a full Quillen model structure. In this sense, a model of HoTT is *equivalent* to a Quillen model structure of a certain kind—namely, one that presents a higher topos.

**Acknowledgements.**

## 1. Cartesian cubical sets

There are many different categories of cubes $\square$ that could be taken as a site for homotopy theory [GM03, BM17], and indeed several different ones have recently been explored in connection with cubical systems of (homotopy) type theory, including [BCH14, OP18, CCHM18, ABC+22, CMS20], to name a few. The model structure developed here is intended to work with any of these, insofar as they are *cartesian*, in the sense that the indexing cubes $[n] \in \square$ are closed under finite products $[m] \times [n] = [m+n]$. Rather than working axiomatically, though, we shall work in the initial such category, which we call *the Cartesian cube category* $\square$, defined as the free finite product category on an interval $\delta_0, \delta_1 : 1 \rightrightarrows \mathrm{I}$.

**Definition 2.** The objects $[n]$ of the *Cartesian cube category* $\square$, called $n$-cubes, are finite sets of the form

$$[n] = \{0, x_1, ..., x_n, 1\}\,,$$

where the $x_1, ..., x_n$, are arbitrary but distinct elements, and $0, 1$ are further distinct, distinguished elements. The arrows,

$$f : [m] \to [n]\,,$$

are arbitrary bipointed maps $f' : [n] \to [m]$ (note the variance!). Thus $\mathbb{B} = \square^{\mathrm{op}}$ is the category if finite, strictly bipointed sets.

As a Lawvere theory, the arrows $f : [m] \to [n]$ in $\square$ may also be regarded as $n$-tuples of elements from the set $\{0, x_1, ..., x_m, 1\}$. These can be generated under composition by faces, degeneracies, permutations, and diagonals (see [Par15] for further details).

**Definition 3.** The category $\mathsf{cSet}$ of *Cartesian cubical sets* is the category of presheaves on the Cartesian cube category $\square$,

$$\mathsf{cSet} \;=\; \mathsf{Set}^{\square^{\mathrm{op}}}.$$

It is of course generated by the representable presheaves $\mathsf{y}[n]$, to be written

$$\mathrm{I}^n = \mathsf{y}[n]$$

and called the *geometric $n$-cubes*.

Note that the representables $\mathrm{I}^n$ are also closed under finite products, $\mathrm{I}^m \times \mathrm{I}^n = \mathrm{I}^{m+n}$. We write $\mathrm{I}$ for $\mathrm{I}^1$ and $1$ for $\mathrm{I}^0$, which is terminal. We will need the following basic fact about the cubes $\mathrm{I}^n$ in $\mathsf{cSet}$.

**Proposition 4** (Lawvere)**.** *The $n$-cubes $\mathrm{I}^n$ are* tiny*, in the sense that the endofunctor $X \mapsto X^{\mathrm{I}^n}$ is a left adjoint.*

(See [Law04] on such "amazing right adjoints".)

*Proof.* It clearly suffices to prove the claim for $n = 1$. For any cubical set $X$, the exponential $X^{\mathrm{I}}$ is a "shift by one dimension",

$$X^{\mathrm{I}}(n) \cong \mathrm{Hom}(\mathrm{I}^n, X^{\mathrm{I}}) \;\cong\; \mathrm{Hom}(\mathrm{I}^{n+1}, X) \cong\; X(n+1).$$

Thus $X^{\mathrm{I}}$ is given by precomposition with the "successor" functor $\square \to \square$ with $[n] \mapsto [n+1]$. Precomposition always has a right adjoint, which in this case we shall write as

$$(-)^{\mathrm{I}} \dashv (-)_{\mathrm{I}}$$

and call $X_{\mathrm{I}}$ the $\mathrm{I}^{th}$-*root* of $X$. See Appendix B below for a calculation of the root $X_{\mathrm{I}}$. $\square$

The exponential $X^{\mathrm{I}}$ will be called the *pathobject* of $X$, and plays a special role. As we have just seen, it classifies "paths" in $X$; so the 0-cubes $p \in (X^I)_0$ in the pathobject correspond to 1-cubes $p \in X_1$, the "endpoints" of which $p_0, p_1 \in X_0$ are given by composing with the evaluation maps

$$\epsilon_0, \epsilon_1 : X^{\mathrm{I}} \rightrightarrows X$$

at the points $\delta_0, \delta_1 : 1 \rightrightarrows \mathrm{I}$. More generally, higher cubes $c \in X_{n+1}$ correspond to maps $c : \mathrm{I}^{n+1} \to X$, which are thus paths between the $n$-cubes $c_0, c_1 : \mathrm{I}^n \to X$, corresponding to $c_0, c_1 \in X_n$.

We mention two facts that will be needed below, concerning the base change functors

$$f_! \dashv f^* \dashv f_* : \mathsf{cSet}/_X \longrightarrow \mathsf{cSet}/_Y$$

associated to a map $f : X \to Y$ in $\mathsf{cSet}$.

**Lemma 5.** *The pulled-back interval* $\mathrm{I}^*\mathrm{I} = \mathrm{I}\times\mathrm{I} \to \mathrm{I}$ *in* $\mathsf{cSet}/_{\mathrm{I}}$ *is also tiny.*

*Proof.* Since the interval $\mathrm{I} = \mathsf{y}[1]$ is representable, the slice category $\mathsf{cSet}/_{\mathrm{I}}$ is also a category of presheaves, namely over the sliced cube category $\Box/_{[1]}$ ,

$$\mathsf{cSet}/_{\mathrm{I}} \;=\; \mathsf{Set}^{\Box^{\mathrm{op}}}/_{\mathsf{y}[1]} \;\cong\; \mathsf{Set}^{(\Box/_{[1]})^{\mathrm{op}}} .$$

However, since $\Box$ does not have all finite limits, the sliced index category does not have all finite products, and so we cannot simply repeat the proof from Proposition 4. But as in that proof, we do have a "successor" functor

$$s_{[1]} : \Box/_{[1]} \to \Box/_{[1]} ,$$

resulting from the "predecessor" natural transformation $s \Rightarrow 1_{\Box}$ given by the projection $\mathrm{I}\times X \to X$. Evaluating $s$ at each object $f : [n] \to [1]$ in $\Box/_{[1]}$, we obtain a commutative diagram:

$$\begin{tikzcd}
s[n] \arrow[r, "\cong"] \arrow[d, "sf"'] & {[1]\times[n]} \arrow[r, "p_n"] & {[n]} \arrow[d, "f"] \\
s[1] \arrow[r, "\cong"'] & {[1]\times[1]} \arrow[r, "p_1"'] & {[1]}
\end{tikzcd} \tag{1}$$

We can then set $s_{[1]}(f) = p_1 \circ sf = f \circ p_n$. As in the foregoing proof, we can then calculate the values of the adjoints on presheaves, associated to $s_{[1]}$,

$$s_{[1]_!} \dashv s_{[1]}{}^* : \widehat{\Box/_{[1]}} \longrightarrow \widehat{\Box/_{[1]}}$$

to be, successively,

$$s_{[1]_!}(X) = \mathrm{I}^*\mathrm{I} \times X ,$$
$$s_{[1]}{}^*(X) = X^{\mathrm{I}^*\mathrm{I}} .$$

The first equation follows from the observation that the diagram (1) is a pullback, and so the object $s_{[1]}(f) : s[n] \to [1]$ of $\widehat{\Box/_{[1]}}$ given by the evident composite is just $\mathrm{I}^*\mathrm{I}\times f$, and the diagram itself represents the counit map $(\mathrm{I}^*\mathrm{I}\times f) \to f$ over I. The second line then follows by adjointness, as does the fact that we have a further right adjoint, namely, the $\mathrm{I}^*\mathrm{I}^{th}$-*root*:

$$s_{[1]_*}(X) =: X_{\mathrm{I}^*\mathrm{I}} .$$

∎

**Lemma 6.** *The pushforward functor along any map* $f : X \to Y$ *preserves pathobjects; for any object* $A \to X$ *over* $X$*, the pathobject of the* *pushforward $f_*A$ is (canonically isomorphic to) the pushforward of the pathobject,*

$$(f_*A)^{\mathrm{I}} \cong f_*(A^{\mathrm{I}})$$

*over $Y$.*

*Proof.* This is true for any constant family $X^*C = X \times C \to X$ with $C$ in place of I, as the reader can easily verify using the Beck-Chevalley condition. $\square$

## 2. The cofibration weak factorization system

To build a model structure on the presheaf category of cubical sets, one can simply take as the cofibrations *all* of the monomorphisms in $\mathsf{cSet}$; but for some purposes, it is convenient to know what is actually required of them (see *e.g.* Appendix A). Thus, to begin, the following axioms are assumed.

**Definition 7** (Cofibrations)**.** The *cofibrations* are a class $\mathcal{C}$ of monomorphisms satisfying the following conditions:

(C0) The map $0 \to C$ is always a cofibration.
(C1) All isomorphisms are cofibrations.
(C2) The composite of two cofibrations is a cofibration.
(C3) Any pullback of a cofibration is a cofibration.

We also require the cofibrations to be classified by a subobject $\Phi \hookrightarrow \Omega$ of the standard subobject classifier $\top : 1 \to \Omega$ of $\mathsf{cSet}$:

(C4) There is a terminal object $t : 1 \to \Phi$ in the category of cofibrations and cartesian squares.

Further axioms for cofibrations will be added later as needed: two early in Section 3, one later in Section 3, and a final one in Section 6 (see Appendix A for a summary). Cofibrations will be written

$$c : A \rightarrowtail B\,.$$

**The cofibrant partial map classifier.** Consider the polynomial endofunctor $P_t : \mathsf{cSet} \to \mathsf{cSet}$ determined by the cofibration classifier $t : 1 \rightarrowtail \Phi$ (see [GK13]). We will write the value of this functor at an object $X$ as

$$X^+ \;:=\; \Phi_!\, t_*(X) \;=\; \sum_{\varphi:\Phi} X^{[\varphi]}\,. \tag{2}$$

The reader familiar with type theory will recognize the similarity to the "partiality" or "lifting" monad [Mog91]. When all monos are cofibrations, so that $\Phi = \Omega$, the object $X^+$ agrees with the partial map classifier $\widetilde{X}$ from topos theory [Joh77]. We may therefore regard $X^+$ as the object of *cofibrant partial elements* of $X$, as we now explain.

Since $t : 1 \rightarrowtail \Phi$ is monic, $t^*t_* \cong 1$, so $X^+$ fits into the pullback square

$$\begin{tikzcd} X \arrow[r, rightarrowtail] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X^+ \arrow[d, "t_*X"] \\ 1 \arrow[r, rightarrowtail, "t"'] & \Phi. \end{tikzcd} \tag{3}$$

Let $\eta : X \rightarrowtail X^+$ be the indicated top horizontal map; we call this the *cofibrant partial map classifier* of $X$. By a *cofibrant partial map* (from an object $Z$) into $X$ we mean a span $(c, x) : Z \leftarrowtail C \to X$ with a cofibration on the left. The object $X^+$ is a *classifying type* for such cofibrant partial maps, in that it has the following universal property.

**Proposition 8.** *Let $\eta : X \rightarrowtail X^+$ be as defined in (3).*

1. *The map $\eta : X \rightarrowtail X^+$ is a cofibration.*
2. *For any object $Z$ and any partial map $(c, x) : Z \leftarrowtail C \to X$, with $c : C \rightarrowtail Z$ a cofibration, there is a unique $\chi : Z \to X^+$ fitting into a pullback square as follows.*

$$\begin{tikzcd} C \arrow[r, "x"] \arrow[d, rightarrowtail, "c"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, rightarrowtail, "\eta"] \\ Z \arrow[r, "\chi"'] & X^+ \end{tikzcd}$$

*The map $\chi : Z \to X^+$ is said to* classify *the partial map*

$$(c, x) : Z \leftarrowtail C \to X\,.$$

*Proof.* The map $\eta : X \rightarrowtail X^+$ is a cofibration since it is a pullback of the universal cofibration $t : 1 \rightarrowtail \Phi$. Observe that $(\eta, 1_X) : X^+ \leftarrowtail X \to X$ is therefore a cofibrant partial map into $X$. The second statement is just the universal property of $X^+$ as a polynomial (see [Awo16], prop. 7). □

**Proposition 9.** *The pointed endofunctor $\eta_X : X \rightarrowtail X^+$ has a natural multiplication $\mu_X : X^{++} \to X^+$ making it a monad.*

*Proof.* Since the cofibrations are closed under composition, the monad structure on $X^+$ follows as in [Awo16], Lemma 5. Explicitly, $\mu_X$ is determined by proposition 8 as the unique map making the following a pullback diagram.

$$\begin{tikzcd}
X \arrow[r, "="] \arrow[d, tail, "\eta_X"'] & X \arrow[dd, tail, "\eta"] \\
X^+ \arrow[d, tail, "\eta_{X^+}"'] & \\
X^{++} \arrow[r, dotted, "\mu"'] & X^+
\end{tikzcd}$$

□

**Relative partial map classifier.** For any object $X \in \mathsf{cSet}$ the pullback functor

$$X^* : \mathsf{cSet} \to \mathsf{cSet}/_X\ ,$$

taking any $A$ to the (say) first projection $X \times A \to X$, not only preserves the subobject classifier $\Omega$, but also the cofibration classifier $\Phi \hookrightarrow \Omega$, where a map in $\mathsf{cSet}/_X$ is defined to be a cofibration if it is one in $\mathsf{cSet}$ (under the forgetful functor $\mathsf{cSet}/_X \to \mathsf{cSet}$). Thus in $\mathsf{cSet}/_X$ we can define the *(relative) cofibration classifier* to be the map

$$X^*t : X^*1 \longrightarrow X^*\Phi \quad \text{over } X\ ,$$

which we may also write $t_X : 1_X \to \Phi_X$. Like $t : 1 \to \Phi$, this map determines a polynomial endofunctor

$$+_X : \mathsf{cSet}/_X \to \mathsf{cSet}/_X\ ,$$

which commutes (up to natural isomorphism) with $+ : \mathsf{cSet} \to \mathsf{cSet}$ and $X^* : \mathsf{cSet} \to \mathsf{cSet}/_X$ in the expected way, namely:

$$\begin{tikzcd}
\mathsf{cSet}/_X \arrow[r, "+_X"] & \mathsf{cSet}/_X \\
\mathsf{cSet} \arrow[u, "X^*"] \arrow[r, "+"'] & \mathsf{cSet} \arrow[u, "X^*"']
\end{tikzcd} \tag{4}$$

The endofunctor $+_X$ is also pointed $\eta_Y : Y \to Y^+$ and has a natural monad multiplication $\mu_Y : Y^{++} \to Y^+$, for any $Y \to X$, for the same reason that $+$ has this structure. Summarizing, we may say:

**Proposition 10.** *The polynomial monad* $+ : \mathsf{cSet} \to \mathsf{cSet}$ *of* cofibrant partial elements *is indexed (or fibered) over* $\mathsf{cSet}$.

**Definition 11.** A $+$-*algebra* in $\mathsf{cSet}$ is an algebra for the pointed endofunctor $+ : \mathsf{cSet} \to \mathsf{cSet}$. Explicitly, a $+$-algebra is a cubical set $A$ together with a retraction $\alpha : A^+ \to A$ of the unit $\eta_A : A \to A^+$. Algebras for the monad $(+, \eta, \mu)$ will be referred to explicitly as $(+, \eta, \mu)$-*algebras*, or $+$-*monad algebras*.

A *relative $+$-algebra* in $\mathsf{cSet}$ is a map $A \to X$, together with an algebra structure over the codomain $X$ for the pointed endofunctor

$$+_X : \mathsf{cSet}/_X \longrightarrow \mathsf{cSet}/_X \,.$$

**The cofibration weak factorization system.** The following proposition generalizes one in [BG16].

**Proposition 12.** *There is an (algebraic) weak factoriation system on* $\mathsf{cSet}$ *with the cofibrations as the left class, and as the right class, the maps underlying relative $+$-algebras. Thus a right map is one $p : A \to X$ for which there is a retract $\alpha : A' \to A$ over $X$ of the canonical map $\eta : A \to A'$,*

$$\begin{tikzcd} A \arrow[r, "\eta"] \arrow[rr, bend left, "="] \arrow[dr, "p"'] & A' \arrow[r, "\alpha"] \arrow[d, "p^+"] & A \arrow[dl, "p"] \\ & X. & \end{tikzcd}$$

(Note that the domain of $p^+ : A' \to X$ is not $A^+$, unless of course $X = 1$.)

*Proof.* The factorization of a map $f : Y \to X$ is given by applying the relative $+$-functor over the codomain,

$$\begin{tikzcd} Y \arrow[r, tail, "\eta_f"] \arrow[dr, "f"'] & Y' \arrow[d, "f^+"] \\ & X. \end{tikzcd}$$

We know by proposition 8 that the unit $\eta_f$ is always a cofibration, and since $f^+$ is the free algebra for the relative $+$-monad, it is in particular a $+$-algebra.

For the lifting condition, consider a cofibration $c : B \rightarrowtail C$, and a right map $p : A \to X$ with $+$-algebra structure map $\alpha : A' \to A$ over $X$, and a commutative square as indicated below.

$$\begin{tikzcd} B \arrow[r, "a"] \arrow[dd, tail, "c"'] & A \arrow[dd, "p"'] \arrow[dr, "\eta"] & \\ & & A' \arrow[ul, bend right, "\alpha"'] \arrow[dl, "p^+"] \\ C \arrow[r, "x"'] & X & \end{tikzcd}$$

Thus in the slice category over $X$, we have

$$\begin{tikzcd} B \arrow[r, "a"] \arrow[d, tail, "c"'] & A \arrow[d, "\eta"] \\ C \arrow[ur, dotted, "d"'] & A^+ \arrow[u, bend right, "\alpha"'] \end{tikzcd}$$

and we seek a diagonal filler $d$ as indicated. (Note that we are writing $A^+$ for the map $p^+ : A' \to A$ regarded as an object over $X$, and similarly $C$ for $x : C \to X$ and $B$ for $xc : B \to X$ and $A$ for $p : A \to X$.) Since $(c, a) : B \leftarrowtail C \to A$ is a cofibrant partial map into $A$, by the universal property of $\eta : A \rightarrowtail A^+$ (Proposition 8) there is a unique classifying map $\varphi : C \to A^+$ (over X) making a pullback square,

$$\begin{tikzcd} B \arrow[r, "a"] \arrow[d, tail, "c"'] & A \arrow[d, "\eta"] \\ C \arrow[r, dotted, "\varphi"'] & A^+. \arrow[u, bend right, "\alpha"'] \end{tikzcd}$$

We can set $d := \alpha \circ \varphi : C \to A$ to obtain the required diagonal filler, since $dc = \alpha\varphi c = \alpha\eta a = a$, because $\alpha$ is a retract of $\eta$.

The closure of the cofibrations under retracts follows from their classification by a universal object $t : 1 \rightarrowtail \Phi$, and the closure of the right maps under retracts follows from their being the algebras for a pointed endofunctor underlying a monad (cf. [Rie14]). Algebraicity of this weak factorization system is immediate, since $+$ is a monad. $\square$

Summarizing, we have an algebraic weak factorization system $(\mathcal{C}, \mathcal{C}^{\pitchfork})$ on the category $\mathsf{cSet}$ of cubical sets, where:

$$\begin{aligned} \mathcal{C} &= \text{the cofibrations} \\ \mathcal{C}^{\pitchfork} &= \text{the maps underlying relative +-algebras} \end{aligned}$$

We shall call this the *cofibration weak factorization system*. The right maps will be called *trivial fibrations*, and the class of all such denoted

$$\mathsf{TFib} = \mathcal{C}^{\pitchfork}.$$

The cofibration algebraic weak factorization system is a generalization of one defined in [BG16] and mentioned in [GS17b].

**Uniform filling structure.** It will be useful to relate relative +-algebra structure to the more familiar diagonal filling condition of cofibrantly generated weak factorization systems, and specifically the special ones occuring in [CCHM18] under the name *uniform filling structure* (this notion is also closely related to that of an *algebraic weak factorization system*, *cf.* [Gar09, Rie11]).

Consider a generating sub*set* of cofibrations consisting of those with representable codomain $c : C \rightarrowtail \mathrm{I}^n$, and call these the *basic cofibrations.*

$$\mathsf{BCof} = \{c : C \rightarrowtail \mathrm{I}^n \mid c \in \mathcal{C}, n \geq 0\}. \tag{5}$$

**Proposition 13.** *For any object $X$ in* $\mathsf{cSet}$ *the following are equivalent:*

(1) *$X$ admits a* $+$-algebra structure: *a retraction $\alpha : X^+ \to X$ of the unit $\eta : X \to X^+$.*
(2) *$X \to 1$ is a* trivial fibration: *it has the right lifting property with respect to all cofibrations,*

$$\mathcal{C} \pitchfork X.$$

(3) *$X$ admits a* uniform filling structure: *for each basic cofibration $c : C \rightarrowtail \mathrm{I}^n$ and map $x : C \to X$ there is given an extension $j(c,x)$,*

$$\begin{tikzcd} C \arrow[r, "x"] \arrow[d, tail, "c"'] & X, \\ \mathrm{I}^n \arrow[ur, dotted, "j(c,x)"'] & \end{tikzcd} \tag{6}$$

*and the choice is* uniform in $\mathrm{I}^n$ *in the following sense.*

*Given any cubical map $u : \mathrm{I}^m \to \mathrm{I}^n$, the pullback $u^*c : u^*C \rightarrowtail \mathrm{I}^m$, which is again a basic cofibration, fits into a commutative diagram of the form*

$$\begin{tikzcd} u^*C \arrow[r, "c^*u"] \arrow[d, tail, "u^*c"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[r, "x"] \arrow[d, tail, "c"'] & X. \\ \mathrm{I}^m \arrow[r, "u"'] & \mathrm{I}^n \arrow[ur, dotted, "j(c,x)"'] & \end{tikzcd} \tag{7}$$

*For the pair $(u^*c,\ x \circ c^*u)$ in (7), the chosen extension $j(u^*c, x \circ c^*u) : \mathrm{I}^m \to X$, is required to be equal to $j(c,x) \circ u$,*

$$j(u^*c, x \circ c^*u) = j(c,x) \circ u. \tag{8}$$

*Proof.* Let $(X, \alpha)$ be a $+$-*algebra* and suppose given the span $(c, x)$ as below, with $c$ a cofibration.

$$\begin{tikzcd} C \arrow[r, "x"] \arrow[d, tail, "c"'] & X \\ Z & \end{tikzcd}$$

Let $\chi(c,x) : Z \to X^+$ be the classifying map of the cofibrant partial map $(c,x) : Z \leftarrowtail C \to X$, so that we have a pullback square as follows.

$$\begin{tikzcd} C \arrow[r, "x"] \arrow[d, tail, "c"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\eta"] \\ Z \arrow[r, "\chi(c,x)"'] & X^+ \end{tikzcd} \tag{9}$$

Then set

$$j = \alpha \circ \chi(c,x) : Z \to X \tag{10}$$

to get a filler,

$$\begin{tikzcd} C \arrow[r, "x"] \arrow[d, tail, "c"'] & X \arrow[d, "\eta"] \\ Z \arrow[r, "\chi(c,x)"'] \arrow[ur, dotted, "j"] & X^+ \arrow[u, bend right, "\alpha"'] \end{tikzcd} \tag{11}$$

since

$$j \circ c = \alpha \circ \chi(c,x) \circ c = \alpha \circ \eta \circ x = x.$$

Thus (1) implies (2). To see that it also implies (3), observe that in the case where $Z = I^n$ and we specify, in (10), that

$$j(c,x) = \alpha \circ \chi(c,x) : \mathrm{I}^n \to X, \tag{12}$$

then the assignment is natural in $\mathrm{I}^n$. Indeed, given any $u : \mathrm{I}^m \to \mathrm{I}^n$, we have

$$j(c', xu') = \alpha \circ \chi(c', xu') = \alpha \circ \chi(c,x) \circ u = j(c,x)u, \tag{13}$$

by the uniqueness of the classifying maps.

It is clear that (2) implies (1), since if $\mathcal{C} \pitchfork X$ then we can take as an algebra structure $\alpha : X^+ \to X$ any filler for the universal span

$$\begin{tikzcd} X \arrow[r, "="] \arrow[d, tail, "\eta"'] & X. \\ X^+ \arrow[ur, dotted, "\alpha"'] & \end{tikzcd}$$

To see that (3) implies (1), suppose that $X$ has a uniform filling structure $j$ and we want to define an algebra structure $\alpha : X^+ \to X$. By Yoneda, for every $y : \mathrm{I}^n \to X^+$ we need a map $\alpha(y) : \mathrm{I}^n \to X$, naturally in $\mathrm{I}^n$, in the sense that for any $u : \mathrm{I}^m \to \mathrm{I}^n$, we have

$$\alpha(yu) = \alpha(y)u. \tag{14}$$

Moreover, to ensure that $\alpha\eta = 1_X$, for any $x : \mathrm{I}^n \to X$ we must have $\alpha(\eta \circ x) = x$. So take $y : \mathrm{I}^n \to X^+$ and let

$$\alpha(y) = j(y^*\eta, y'),$$

as indicated on the right below.

$$\begin{tikzcd} u^*C \arrow[r, "u'"] \arrow[d, tail, "u^*y^*\eta"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[r, "y'"] \arrow[d, tail, "y^*\eta"'] \arrow[dr, phantom, "\lrcorner", very near start] & X. \arrow[d, "\eta"] \\ \mathrm{I}^m \arrow[r, "u"'] & \mathrm{I}^n \arrow[r, "y"'] \arrow[ur, dotted, "j(y^*\eta, y')"'] & X^+ \end{tikzcd} \tag{15}$$

Then for any $u : \mathrm{I}^m \to \mathrm{I}^n$, we indeed have

$$\alpha(yu) = j\big((yu)^*\eta, y'u'\big) = j(y^*\eta, y') \circ u = \alpha(y)u,$$

by the uniformity of $j$. Finally, if $y = \eta \circ x$ for some $x : \mathrm{I}^n \to X$ then

$$\alpha(\eta x) = j\big((\eta x)^*\eta, (\eta x)'\big) = j(1_X, x) = x,$$

because the defining diagram for $\alpha(\eta x)$, i.e. the one on the right in (15), then factors as

$$\begin{tikzcd} \mathrm{I}^n \arrow[r, "x"] \arrow[d, tail, "="'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[r, "="] \arrow[d, tail, "="'] \arrow[dr, phantom, "\lrcorner", very near start] & X, \arrow[d, "\eta"] \\ \mathrm{I}^n \arrow[r, "x"'] & X \arrow[r, "\eta"'] & X^+ \end{tikzcd} \tag{16}$$

and the only possible extension $j(1_X, x)$ for the span $(1_{\mathrm{I}^n}, x)$ is $x$ itself. □

*Remark* 14. Observe that the uniformilty condition (3) can be extended to the *class of all* cofibrations, in the form:

4. $X$ admits a *(large) uniform filling structure:* for each cofibration $c : C \rightarrowtail Z$ and map $x : C \to X$ there is given an extension $j(c, x)$,

$$\begin{tikzcd} C \arrow[r, "x"] \arrow[d, tail, "c"'] & X, \\ Z \arrow[ur, dotted, "j(c,x)"'] & \end{tikzcd} \tag{17}$$

and the choice is *uniform in $Z$* in the following sense: Given any map $u : Y \to Z$, the pullback $u^*c : u^*C \rightarrowtail Y$, which is again a cofibration, fits into a commutative diagram of the form

$$\begin{tikzcd} u^*C \arrow[r, "c^*u"] \arrow[d, tail, "u^*c"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[r, "x"] \arrow[d, tail, "c"'] & X. \\ Y \arrow[r, "u"'] & z \arrow[ur, dotted, "j(c,x)"'] & \end{tikzcd} \tag{18}$$

For the pair $(u^*c,\ x \circ c^*u)$ in (18), the chosen extension $j(u^*c, x \circ c^*u) : \mathrm{I}^m \to X$, is required to be equal to $j(c, x) \circ u$,

$$j(u^*c, x \circ c^*u) = j(c, x) \circ u. \tag{19}$$

Indeed, the proof that (1) implies (2) and (3) works just as well to infer (4), which in turn implies (2) and (3) as special cases.

The relative version of the foregoing is entirely analogous, since the $+$-functor is fibered over $\mathsf{cSet}$ in the sense of diagram (4). We can therefore omit the entirely analogous proof of the following.

**Proposition 15.** *For any map $f : Y \to X$ in $\mathsf{cSet}$ the following are equivalent:*

(1) *$f : Y \to X$ admits a* relative $+$-algebra structure over *$X$, i.e. there is a retraction $\alpha : Y' \to Y$ over $X$ of the unit $\eta : Y \to Y'$, where $f^+ : Y' \to X$ is the result of the relative $+$-functor applied to $f$, as in definition 11.*

(2) *$f : Y \to X$ is a* trivial fibration,

$$\mathcal{C} \pitchfork f.$$

(3) *$f : Y \to X$ admits a (small)* uniform filling structure*: for each basic cofibration $c : C \rightarrowtail \mathrm{I}^n$ and maps $x : C \to X$ and $y : \mathrm{I}^n \to Y$ making the square below commute, there is given a diagonal filler $j(c, x, y)$,*

$$\begin{tikzcd} C \arrow[r, "x"] \arrow[d, tail, "c"'] & X \arrow[d, "f"] \\ \mathrm{I}^n \arrow[r, "y"'] \arrow[ur, dotted, "j(c,x,y)"] & Y, \end{tikzcd} \tag{20}$$

*and the choice is* uniform in $\mathrm{I}^n$ *in the following sense: given any cubical map $u : \mathrm{I}^m \to \mathrm{I}^n$, the pullback $u^*c : u^*C \rightarrowtail \mathrm{I}^m$ is again a basic cofibration and fits into a commutative diagram of the form*

$$\begin{tikzcd} u^*C \arrow[r, "c^*u"] \arrow[d, tail, "u^*c"'] \arrow[dr, phantom, very near start, "\lrcorner"] & C \arrow[r, "x"] \arrow[d, tail, "c"] & X \arrow[d, "f"] \\ \mathrm{I}^m \arrow[r, "u"'] & \mathrm{I}^n \arrow[r, "y"'] \arrow[ur, dotted, "j(c,x,y)"] & Y. \end{tikzcd} \tag{21}$$

*For the evident triple $(u^*c,\, x \circ c^*u, y \circ u)$ in (21) the chosen diagonal filler*

$$j(u^*c, x \circ c^*u, y \circ u) : \mathrm{I}^m \to X$$

*is equal to $j(c, x, y) \circ u$,*

$$j(u^*c, x \circ c^*u, y \circ u) = j(c, x, y) \circ u. \tag{22}$$

*And again, a large version of (3) with arbitrary cofibrations $c : C \rightarrowtail Z$ is again equivalent to (1)-(3).*

We next collect some basic facts about trivial fibrations that will be needed later: they have sections, they are closed under composition and retracts, and they are closed under pullback and pushforward along all maps.

**Corollary 16.** (1) *Every trivial fibration $A \to X$ has a section $s : X \to A$.*
(2) *If $a : A \to X$ is a trivial fibration and $b : B \to A$ is a trivial fibration, then $a \circ b : B \to X$ is a trivial fibration.*
(3) *If $a : A \to X$ is a trivial fibration and $a' : A' \to X'$ is a retract of $a$ in the arrow category, then $a'$ is a trivial fibration.*
(4) *For any map $f : X \to Y$ and any trivial fibration $B \to Y$, the pullback $f^*B \to X$ is a trivial fibration.*
(5) *For any map $f : X \to Y$ and any trivial fibration $A \to X$, the pushforward $f_*A \to Y$ is a trivial fibration.*

*Proof.* (1) holds because all objects are cofibrant by (C0). (5) is a consequence of (C3), stability of cofibrations under pullback, by a standard argument using the adjunction $f^* \dashv f_*$. The rest hold for the right maps in any weak factorization system. $\square$

*Remark* 17. The structured notion of trivial fibration, *vis.* relative +-algebra, can also be shown algebraically (i.e. not using Proposition 15) to be closed under composition and retracts and preserved by pullback and pushforward. We do just the case of pushforward as an example. Thus consider the following situation with $A \to X$ a +-algebra with structure $\alpha$, as indicated.

$$\begin{tikzcd}
A \arrow[dr, "\eta_A"] \arrow[dd] & & f_*A \arrow[dr, "\eta_{f_*A}"] \arrow[dd] & \\
& A^+ \arrow[dl] \arrow[ul, bend right, "\alpha"'] & & (f_*A)^+ \arrow[dl] \arrow[ul, dotted, bend right, "\beta"'] \\
X \arrow[rr, "f"'] & & Y &
\end{tikzcd} \tag{23}$$

A +-algebra structure for $f_*A \to Y$ would be a retract $\beta : (f_*A)^+ \to f_*A$ of $\eta_{f_*A} : f_*A \to (f_*A)^+$ over $Y$, which corresponds under $f^* \dashv f_*$ to a map $\tilde{\beta} : f^*((f_*A)^+) \to A$ over $X$ with

$$\tilde{\beta} \circ f^*\eta_{f_*A} = \epsilon_A$$

as indicated below.

$$\begin{tikzcd}
f^* f_* A \arrow[r, "f^*\eta_{f_*A}"'] \arrow[dr, "\epsilon_A"'] & f^*((f_*A)^+) \arrow[d, dotted, "\tilde\beta"'] & \\
& A \arrow[r, "\eta_A"'] & A^+ . \arrow[l, bend right, "\alpha"']
\end{tikzcd} \tag{24}$$

But since pullback $f^*$ commutes with $+$, there is a canonical iso $c : f^*((f_*A)^+) \cong (f^*f_*A)^+$ with $c \circ f^*\eta_{f_*A} = \eta_{f^*f_*A}$. So we can set $\tilde\beta := \alpha \circ (\epsilon_A)^+ \circ c$.

$$\begin{tikzcd}
f^* f_* A \arrow[r, "f^*\eta_{f_*A}"'] \arrow[dr, "\epsilon_A"'] \arrow[rr, bend left, "\eta_{f^*f_*A}"] & f^*((f_*A)^+) \arrow[d, dotted, "\tilde\beta"'] \arrow[r, "\sim", "c"'] & (f^*f_*A)^+ \arrow[d, "(\epsilon_A)^+"] \\
& A \arrow[r, "\eta_A"'] & A^+ \arrow[l, bend right, "\alpha"']
\end{tikzcd} \tag{25}$$

## 3. The fibration weak factorization system

We now specify a second weak factorization system, with a restricted class of "trivial" cofibrations on the left, and an expanded class of right maps, the *fibrations*. As explained in the introduction, we first recall from [GS17a] what we shall call the "biased" notion of fibration, before giving the "unbiased" one appropriate to our more general setting. The two versions are equivalent in the presence of *connections*

$$\vee, \wedge : \mathrm{I} \times \mathrm{I} \longrightarrow \mathrm{I}$$

on the cubes, which are used in [Sat17] to determine a model structure with biased fibrations. In [AGH21] it is shown that the biased fibrations of *op.cit.* agree with those specified in the "logical style" of [CCHM18, OP18]. Note that we do *not* assume connections in the category $\square$ of Cartesian cubical sets.

**Partial box filling (biased version).** The *generating biased trivial cofibrations* are all maps of the form

$$c \otimes \delta_\epsilon : D \rightarrowtail Z \times \mathrm{I}\,, \tag{26}$$

where:

(1) $c : C \rightarrowtail Z$ is an arbitrary cofibration,

(2) $\delta_\epsilon : 1 \to \mathrm{I}$ is one of the two *endpoint inclusions*, for $\epsilon = 0, 1$.
(3) $c \otimes \delta_\epsilon$ is the *pushout-product* indicated in the following diagram.

$$\begin{tikzcd}
C \times 1 \arrow[r, "C\times\delta_\epsilon"] \arrow[d, "C\times 1"'] & C \times \mathrm{I} \arrow[d] \arrow[ddr, bend left, "c\times\mathrm{I}"] & \\
Z \times 1 \arrow[r] \arrow[drr, bend right, "Z\times\delta_\epsilon"'] & Z +_C (C \times \mathrm{I}) \arrow[dr, dotted, "c\otimes\delta_\epsilon"'] & \\
 & & Z \times \mathrm{I}
\end{tikzcd} \tag{27}$$

(4) $D = Z +_C (C \times \mathrm{I})$ is the indicated domain of the map $c \otimes \delta_\epsilon$.

In order to ensure that such maps are indeed cofibrations, we assume two further axioms in addition to (C1)–(C4) from Definition 7:

(C5) The endpoint inclusions $\delta_\epsilon : 1 \to \mathrm{I}$ are cofibrations, for $\epsilon = 0, 1$.
(C6) The cofibrations are closed under joins $A \vee B \rightarrowtail C$ of subobjects $A, B \rightarrowtail C$ of any object $C$.

*Remark* 18. Note that since $\delta_0 : 1 \to \mathrm{I}$ and $\delta_1 : 1 \to \mathrm{I}$ are disjoint, by (C5) and stability under pullbacks we have that $0 \to 1$ is a cofibration, so by stability again $0 \to A$ is always a cofibration. Thus (C0) is no longer required. From (C6) it follows that cofibrations are closed under pushout-products $a \otimes b$ in the arrow category. It also then follows from (C5) that the boundary $\partial : 1 + 1 \to \mathrm{I}$ is a cofibration.

**Fibrations (biased version).** Now let

$$\mathcal{C} \otimes \delta_\epsilon \;=\; \{c \otimes \delta_\epsilon : D \rightarrowtail Z \times \mathrm{I} \mid c \in \mathcal{C},\ \epsilon = 0, 1\}$$

be the class of all generating biased trivial cofibrations. The *biased fibrations* are defined to be the right class of these maps,

$$(\mathcal{C} \otimes \delta_\epsilon)^{\pitchfork} \;=\; \mathcal{F}\,.$$

Thus a map $f : Y \to X$ is a biased fibration just if for every commutative square of the form

$$\begin{tikzcd}
Z +_C (C \times \mathrm{I}) \arrow[r] \arrow[d, tail, "c\otimes\delta_\epsilon"'] & Y \arrow[d, "f"] \\
Z \times \mathrm{I} \arrow[r] \arrow[ur, dotted, "j"'] & X
\end{tikzcd} \tag{28}$$

with a generating biased trivial cofibration on the left, there is a diagonal filler $j$ as indicated.

To relate this notion of fibration to the cofibration weak factorization system, fix any map $u : A \to B$, and recall (*e.g.* from [JT08, Rie14]) that the pushout-product with $u$ is a functor on the arrow category

$$(-)\otimes u : \mathsf{cSet}^{\mathbb{2}} \to \mathsf{cSet}^{\mathbb{2}}\,.$$

This functor has a right adjoint, the *pullback-hom*, which for a map $f : X \to Y$ we shall write as

$$(u \Rightarrow f) : Y^B \longrightarrow (X^B \times_{X^A} Y^A)\,.$$

The pullback-hom is determined as indicated in the following diagram.

$$\begin{tikzcd}
Y^B \arrow[drr, bend left, "Y^u"] \arrow[dr, dotted, "u\Rightarrow f"] \arrow[ddr, bend right, "f^B"'] & & \\
& X^B \times_{X^A} Y^A \arrow[r] \arrow[d] & Y^A \arrow[d, "f^A"] \\
& X^B \arrow[r, "X^u"'] & X^A
\end{tikzcd} \tag{29}$$

The $\otimes \dashv \Rightarrow$ adjunction on the arrow category has the following useful relation to weak factorization systems (cf. [GS17a, Rie14, JT08]), where, as usual, for any maps $a : A \to B$ and $f : X \to Y$ we write

$$a \pitchfork f$$

to mean that for every solid square of the form

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "a"'] & X \arrow[d, "f"] \\
B \arrow[r] \arrow[ur, dotted, "j"'] & Y
\end{tikzcd} \tag{30}$$

there exists a diagonal filler $j$ as indicated.

**Lemma 19.** *For any maps $a : A_0 \to A_1, b : B_0 \to B_1, c : C_0 \to C_1$ in* $\mathsf{cSet}$*,*

$$(a \otimes b) \pitchfork c \quad \textit{iff} \quad a \pitchfork (b \Rightarrow c)\,.$$

The following is now a direct corollary.

**Proposition 20.** *An object $X$ is fibrant if and only if both of the endpoint projections $X^{\mathrm{I}} \to X$ from the pathspace are trivial fibrations. More generally, a map $f : Y \to X$ is a fibration iff both of the maps*

$$(\delta_\epsilon \Rightarrow f) : Y^I \to X^I \times_X Y$$

*are trivial fibrations (for $\epsilon = 0, 1$).*

**Fibration structure (biased version).** The $\otimes \dashv \Rightarrow$ adjunction determines the fibrations in terms of the trivial fibrations, which in turn can be determined by *uniform* lifting against a *small category* consisting of basic cofibrations and pullback squares between them, by proposition 15. The fibrations are similarly determined by *uniform* lifting against the *small category* of basic, biased trivial cofibrations, consisting of all those $c \otimes \delta_\epsilon$ in $\mathcal{C} \otimes \delta_\epsilon$ where $c : C \rightarrowtail \mathrm{I}^n$ is a *basic* cofibration, *i.e.* one with representable codomain. Thus the set of *basic biased trivial cofibrations* is

$$\mathsf{BCof} \otimes \delta_\epsilon = \{c \otimes \delta_\epsilon : B \rightarrowtail \mathrm{I}^{n+1} \mid c : C \rightarrowtail \mathrm{I}^n,\ \epsilon = 0, 1,\ n \geq 0\}, \tag{31}$$

where the pushout-product $c \otimes \delta_\epsilon$ now takes the simpler form

$$\begin{tikzcd}
C \arrow[r] \arrow[d, tail] & C \times \mathrm{I} \arrow[d] \arrow[ddr, bend left] & \\
\mathrm{I}^n \arrow[r] \arrow[drr, bend right] & \mathrm{I}^n +_C (C \times \mathrm{I}) \arrow[dr, tail, "c \otimes \delta_\epsilon"'] & \\
& & \mathrm{I}^n \times \mathrm{I}
\end{tikzcd} \tag{32}$$

for a basic cofibration $c : C \rightarrowtail \mathrm{I}^n$, an endpoint $\delta_\epsilon : 1 \to \mathrm{I}$, and with domain $B = \big(\mathrm{I}^n +_C (C \times \mathrm{I})\big)$. These subobjects $B \rightarrowtail \mathrm{I}^{n+1}$ can be seen geometrically as generalized open box inclusions.

For any map $f : Y \to X$ a *uniform, biased fibration structure* on $f$ is a choice of diagonal fillers $j_\epsilon(c, x, y)$,

$$\begin{tikzcd}
\mathrm{I}^n +_C (C \times \mathrm{I}) \arrow[r, "x"] \arrow[d, "c \otimes \delta_\epsilon"'] & X \arrow[d, "f"] \\
\mathrm{I}^n \times \mathrm{I} \arrow[r, "y"'] \arrow[ur, dotted, "j_\epsilon(c,x,y)"'] & Y,
\end{tikzcd} \tag{33}$$

for each basic biased trivial cofibration $c \otimes \delta_\epsilon : B = (\mathrm{I}^n +_C (C \times \mathrm{I})) \rightarrowtail \mathrm{I}^{n+1}$ and maps $x : B \to X$ and $y : \mathrm{I}^{n+1} \to Y$, which is *uniform in* $\mathrm{I}^n$ in the following sense: Given any cubical map $u : \mathrm{I}^m \to \mathrm{I}^n$, the pullback $u^*c : u^*C \rightarrowtail \mathrm{I}^m$ of $c : C \rightarrowtail \mathrm{I}^n$ along $u$ determines another basic biased trivial cofibration

$$u^*c \otimes \delta_\epsilon : B' = (\mathrm{I}^m +_{u^*C} (u^*C \times \mathrm{I})) \rightarrowtail \mathrm{I}^{m+1},$$

which fits into a commutative diagram of the form

$$\begin{tikzcd} \mathrm{I}^m +_{u^*C} (u^*C \times \mathrm{I}) \arrow[r, "(u\times\mathrm{I})'"] \arrow[d, "u^*c\otimes\delta_\epsilon"'] & \mathrm{I}^n +_C (C \times \mathrm{I}) \arrow[r, "x"] \arrow[d, "c\otimes\delta_\epsilon"'] & X \arrow[d, "f"] \\ \mathrm{I}^m \times \mathrm{I} \arrow[r, "u\times\mathrm{I}"'] & \mathrm{I}^n \times \mathrm{I} \arrow[r, "y"'] \arrow[ur, dotted, "j_\epsilon(c,x,y)"'] & Y, \end{tikzcd} \tag{34}$$

by applying the functor $(-) \otimes \delta_\epsilon$ to the pullback square relating $u^*c$ to $c$. For the outer rectangle in (34) there is then a chosen diagonal filler

$$j_\epsilon(u^*c, x \circ (u \times \mathrm{I})', y \circ (u \times \mathrm{I})) : \mathrm{I}^m \times \mathrm{I} \to X\,,$$

and for this map we require that

$$j_\epsilon(u^*c, x \circ (u \times \mathrm{I})', y \circ (u \times \mathrm{I})) = j_\epsilon(c, x, y) \circ (u \times \mathrm{I}). \tag{35}$$

This can be seen to be a reformulation of the logical specification given in [CCHM18] (see [AGH21]).

**Definition 21.** A *uniform, biased fibration structure* on a map $f : Y \to X$ is a choice of fillers $j_\epsilon(c, x, y)$ as in (33) satisfying (35) for all maps $u : \mathrm{I}^m \to \mathrm{I}^n$.

Finally, we have the analogue of proposition 13 for fibrant objects. The analogous statement of proposition 15 for fibrations is omitted, as is the entirely analogous proof.

**Corollary 22.** *For any object $X$ in* $\mathsf{cSet}$ *the following are equivalent:*

(1) *$X$ is* biased fibrant, *in the sense that every map $D \to X$ from the domain of a generating biased trivial cofibration $D \rightarrowtail Z \times \mathrm{I}$ extends to a total map $Z \times \mathrm{I} \to X$,*

$$\mathcal{C} \otimes \delta_\epsilon \ \pitchfork \ X\,.$$

(2) *The canonical maps $(\delta_\epsilon \Rightarrow X) : X^I \to X$ are trivial fibrations.*

(3) *$X \to 1$ admits a* uniform biased fibration structure. *Explicitly, for each basic biased trivial cofibration $c \otimes \delta_\epsilon : B \rightarrowtail \mathrm{I}^{n+1}$ and map $x : B \to X$, there is given an extension $j_\epsilon(c, x)$,*

$$\begin{tikzcd} B \arrow[r, "x"] \arrow[d, tail, "c\otimes\delta_\epsilon"'] & X, \\ \mathrm{I}^{n+1} \arrow[ur, dotted, "j_\epsilon(c,x)"'] & \end{tikzcd} \tag{36}$$

*and, moreover, the choice is* uniform in $\mathrm{I}^n$ *in the following sense: Given any cubical map $u : \mathrm{I}^m \to \mathrm{I}^n$, the pullback $u^*c\otimes\delta_\epsilon$ :*

*$B' \rightarrowtail \mathrm{I}^m \times \mathrm{I}$ fits into a commutative diagram of the form*

$$\begin{tikzcd} B' \arrow[r, "(u\times \mathrm{I})'"] \arrow[d, tail, "u^*c\otimes\delta_\epsilon"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[r, "x"] \arrow[d, tail, "c\otimes\delta_\epsilon"'] & X. \\ \mathrm{I}^m \times \mathrm{I} \arrow[r, "u\times \mathrm{I}"'] & \mathrm{I}^n \times \mathrm{I} \arrow[ur, dotted, "j(c,x)"'] & \end{tikzcd} \tag{37}$$

*For the pair $(u^*c \otimes \delta_\epsilon,\ x \circ (u \times \mathrm{I})')$ in (37) the chosen extension*

$$j(u^*c \otimes \delta_\epsilon, x \circ (u \times \mathrm{I})') : \mathrm{I}^m \times \mathrm{I} \to X$$

*is equal to $j(c, x) \circ (u \times \mathrm{I})$,*

$$j(u^*c \otimes \delta_\epsilon, x \circ (u \times \mathrm{I})') = j(c, x)(u \times \mathrm{I}). \tag{38}$$

**Partial box filling (unbiased version).** Rather than building a weak factorization system based on the foregoing notion of biased fibration (as is done in [GS17a]), we shall first eliminate the "bias" with respect to the endpoints $\delta_\epsilon : 1 \to \mathrm{I}$, for $\epsilon = 0, 1$. This will have the effect of adding more trivial cofibrations, and thus more weak equivalences, to our model structure. Consider first the simple path-lifting condition for a map $f : Y \to X$, which is a special case of (28) with $c = ! : 0 \rightarrowtail 1$, so that $! \otimes \delta_\epsilon = \delta_\epsilon$.

$$\begin{tikzcd} 1 \arrow[r] \arrow[d, tail, "\delta_\epsilon"'] & Y \arrow[d, "f"] \\ \mathrm{I} \arrow[r] \arrow[ur, dotted, "j_\epsilon"'] & X \end{tikzcd}$$

In topological spaces, for instance, rather than requiring lifts $j_\epsilon$ for each of the endpoints $\epsilon = 0, 1$ of the real interval $\mathrm{I} = [0, 1]$, one could equivalently require there to be a lift $j_i$ *for each point* $i : 1 \to \mathrm{I}$. Such "unbiased path-lifting" can be formulated in $\mathsf{cSet}$ by introducing a "generic point" $\delta : 1 \to \mathrm{I}$ by passing to $\mathsf{cSet}/_{\mathrm{I}}$ via the pullback functor $\mathrm{I}^* : \mathsf{cSet} \to \mathsf{cSet}/_{\mathrm{I}}$, and then requiring path-lifting for $\mathrm{I}^*f$ with respect to $\delta : \mathrm{I} \to \mathrm{I} \times \mathrm{I}$, regarded as a map $\delta : 1 \to \mathrm{I}^*\mathrm{I}$ in $\mathsf{cSet}/_{\mathrm{I}}$. We shall therefore define $f$ to be an unbiased fibration just if $\mathrm{I}^*f$ is a $\delta$-biased fibration for the generic point $\delta$. The following specification implements that idea, while also adding cofibrant partiality, as in the biased case.

We first replace axiom (C5) with the following stronger assumption.

(C7) The diagonal map $\delta : \mathrm{I} \to \mathrm{I} \times \mathrm{I}$ of the interval I is a cofibration.

The unbiased notion of a fibration for $\mathsf{cSet}$ is now as follows.

**Definition 23** (unbiased fibration)**.** Let $\delta : \mathrm{I} \to \mathrm{I} \times \mathrm{I}$ be the diagonal map.

(1) An object $X$ is *unbiased fibrant* if the map
$$(\delta \Rightarrow X) = \langle \mathsf{eval}, p_2 \rangle : X^{\mathrm{I}} \times \mathrm{I} \to X \times \mathrm{I}$$
is a trivial fibration.

(2) A map $f : Y \to X$ is an *unbiased fibration* if the map
$$(\delta \Rightarrow f) = \langle f^{\mathrm{I}} \times \mathrm{I}, \langle \mathsf{eval}, p_2 \rangle \rangle : Y^{\mathrm{I}} \times \mathrm{I} \to (X^{\mathrm{I}} \times \mathrm{I}) \times_{(X \times \mathrm{I})} (Y \times \mathrm{I})$$
is a trivial fibration.

Let us (temporarily) write $\mathbb{I} = \mathrm{I}^*\mathrm{I}$ for the pulled-back interval in the slice category $\mathsf{cSet}/_{\mathrm{I}}$, so that the generic point is written $\delta : 1 \to \mathbb{I}$. Condition (1) above (which of course is a special case of (2)) then says that evaluation at the generic point $\delta : 1 \to \mathbb{I}$, the map $(\mathrm{I}^*X)^\delta : (\mathrm{I}^*X)^{\mathbb{I}} \to \mathrm{I}^*X$, constructed in the slice category $\mathsf{cSet}/_{\mathrm{I}}$, is a trivial fibration. Condition (2) says that the pullback-hom of the generic point $\delta : 1 \to \mathbb{I}$ with $\mathrm{I}^*f$, constructed in the slice category $\mathsf{cSet}/_{\mathrm{I}}$, is a trivial fibration. Thus a map $f : Y \to X$ is an *unbiased* fibration just if its base change $\mathrm{I}^*f$ is a $\delta$-*biased* fibration in the slice category $\mathsf{cSet}/_{\mathrm{I}}$. The latter condition can also be reformulated as follows.

**Proposition 24.** *A map $f : Y \to X$ is an unbiased fibration if and only if the canonical map $u$ to the pullback, in the following diagram in* $\mathsf{cSet}$*, is a trivial fibration.*

$$\begin{tikzcd}
Y^{\mathrm{I}} \times \mathrm{I} \arrow[dr, dotted, "u"] \arrow[drr, bend left, "\mathsf{eval}"] \arrow[ddr, bend right, "f^{\mathrm{I}} \times \mathrm{I}"'] & & \\
& Y_{\mathsf{eval}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "f"] \\
& X^{\mathrm{I}} \times \mathrm{I} \arrow[r, "\mathsf{eval}"'] & X.
\end{tikzcd} \tag{39}$$

*Proof.* We interpolate another pullback into the rectangle in (39) to obtain

$$\begin{tikzcd}
Y_{\mathsf{eval}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \times \mathrm{I} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "f"] \\
X^{\mathrm{I}} \times \mathrm{I} \arrow[r] & X \times \mathrm{I} \arrow[r] & X
\end{tikzcd} \tag{40}$$

with the evident maps. The left hand square is therefore a pullback, so we indeed have that
$$Y_{\mathsf{eval}} \cong (X^{\mathrm{I}} \times \mathrm{I}) \times_{(X \times \mathrm{I})} (Y \times \mathrm{I}) \cong (X^{\mathrm{I}} \times \mathrm{I}) \times_X Y$$
and $u = (\delta \Rightarrow f)$. □

As a special case, we have:

**Corollary 25.** *An object $X$ is unbiased fibrant if and only if the canonical map $u$ to the pullback, in the following diagram in* $\mathsf{cSet}$*, is a trivial fibration.*

$$\begin{tikzcd}
X^{\mathrm{I}} \times \mathrm{I} \arrow[dr, dotted, "u"'] \arrow[drr, bend left, "\mathsf{eval}"] \arrow[ddr, bend right, "p_2"'] & & \\
& \mathrm{I} \times X \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & X \arrow[d] \\
& \mathrm{I} \arrow[r] & 1.
\end{tikzcd} \tag{41}$$

Now we can run the proof of Proposition 20 backwards in order to determine a class of generating trivial cofibrations for the unbiased case. Consider pairs of maps $c : C \rightarrowtail Z$ and $i : Z \to \mathrm{I}$, where the former is a cofibration and the latter is regarded as an "I-indexing", so that

$$\begin{tikzcd}
C \arrow[r, tail, "c"] \arrow[dr] & Z \arrow[d, "i"] \\
& \mathrm{I}
\end{tikzcd}$$

is regarded as an "I-indexed family of cofibrations $c_i : C_i \rightarrowtail Z_i$". We shall use the notation

$$\langle i \rangle := \langle 1_Z, i \rangle : Z \longrightarrow Z \times \mathrm{I}\,, \tag{42}$$

for the graph of the indexing map $i : Z \to \mathrm{I}$. Then write

$$c \otimes_i \delta := [\langle i \rangle, c \times \mathrm{I}] : Z +_C (C \times \mathrm{I}) \longrightarrow Z \times \mathrm{I}\,,$$

which is easily seen to be well-defined on the indicated pushout below.

$$\begin{tikzcd}
C \arrow[r, "\langle ic \rangle"] \arrow[d, tail, "c"'] & C \times \mathrm{I} \arrow[d] \arrow[ddr, bend left, "c \times \mathrm{I}"] & \\
Z \arrow[r] \arrow[drr, bend right, "\langle i \rangle"'] & Z +_C (C \times \mathrm{I}) \arrow[dr, dotted, "c \otimes_i \delta"] & \\
& & Z \times \mathrm{I}\,.
\end{tikzcd} \tag{43}$$

*Remark* 26. The specification (43) differs from the similar (27) by using the graph $\langle i \rangle : Z \rightarrowtail Z \times \mathrm{I}$ for the inclusion of $Z$ into the cylinder over $Z$, rather than one of the two "ends",

$$\langle 1_Z, \delta_\epsilon ! \rangle : Z \cong Z \times 1 \xrightarrow{Z \times \delta_\epsilon} Z \times \mathrm{I} \tag{44}$$

arising from the endpoint inclusions $\delta_\epsilon : 1 \to \mathrm{I}$, for $\epsilon = 0, 1$. As an arrow over I, the graph $\langle i \rangle : Z \to Z \times \mathrm{I}$ also takes the form (44), namely

$$\langle i \rangle = \langle 1_Z, \delta ! \rangle : Z \to Z \times \mathrm{I}\,.$$

If we also regard $c : C \to Z$ as an arrow over I via $i : Z \to \mathrm{I}$, and use the generic point $\delta : 1 \to \mathbb{I}$ over I in place of $\delta_\epsilon : 1 \to \mathrm{I}$, then (43) agrees with (27), up to those changes. Thus the indicated map $c \otimes_i \delta$ in (43) *is the pushout-product constructed over* I of the generic point $\delta$ with the map $c$ regarded as an I-indexed family of cofibrations via the indexing $i : Z \to \mathrm{I}$.

Observe that for any map $i : Z \to \mathrm{I}$, the graph $\langle i \rangle = \langle 1_Z, i \rangle : Z \to Z \times \mathrm{I}$ is a cofibration, since it is a pullback of the diagonal of I along $i \times \mathrm{I}$. The subobject

$$c \otimes_i \delta \rightarrowtail Z \times \mathrm{I}$$

constructed in (43) is therefore a cofibration, since it is the join in the lattice $\mathsf{Sub}(Z \times \mathrm{I})$ of the cofibrant subobjects $\langle i \rangle \rightarrowtail Z \times \mathrm{I}$ and $C \times \mathrm{I} \rightarrowtail Z \times \mathrm{I}$, where the latter is the "cylinder over $C \rightarrowtail Z$".

**Definition 27.** The maps of the form $c \otimes_i \delta : Z +_C (C \times \mathrm{I}) \rightarrowtail Z \times \mathrm{I}$ now form the class of *generating unbiased trivial cofibrations*,

$$\mathcal{C} \otimes \delta \;=\; \{c \otimes_i \delta : D \rightarrowtail Z \times \mathrm{I} \mid c : C \rightarrowtail Z, i : Z \to \mathrm{I}\}\,. \tag{45}$$

We can then show that the unbiased fibrations are exactly the right class of these maps,

$$(\mathcal{C} \otimes \delta)^{\pitchfork} = \mathcal{F}.$$

**Proposition 28.** *A map $f : Y \to X$ is an unbiased fibration iff for every pair of maps $c : C \rightarrowtail Z$ and $i : Z \to \mathrm{I}$, where the former is a cofibration, every commutative square of the following form has a diagonal filler, as indicated in the following.*

$$\begin{tikzcd} Z +_C (C \times \mathrm{I}) \arrow[r] \arrow[d, tail, "c \otimes_i \delta"'] & Y \arrow[d, "f"] \\ Z \times \mathrm{I} \arrow[r] \arrow[ur, dotted, "j"'] & X. \end{tikzcd} \tag{46}$$

*Proof.* Suppose that for all $c : C \rightarrowtail Z$ and $i : Z \to \mathrm{I}$, we have $(c \otimes_i \delta) \pitchfork f$ in $\mathsf{cSet}$. Pulling $f$ back over I, this is equivalent to the condition $c \otimes \delta \pitchfork \mathrm{I}^* f$ in $\mathsf{cSet}/_{\mathrm{I}}$, for all cofibrations $c : C \rightarrowtail Z$ over I, which is equivalent to $c \pitchfork (\delta \Rightarrow \mathrm{I}^* f)$ in $\mathsf{cSet}/_{\mathrm{I}}$ for all cofibrations $c : C \rightarrowtail Z$. But this in turn means that $\delta \Rightarrow \mathrm{I}^* f$ is a trivial fibration, which by definition means that $f$ is an unbiased fibration. $\square$

*Remark* 29. Note that the endpoints $\delta_\epsilon : 1 \to \mathrm{I}$, in particular, are of the form $c \otimes_i \delta$ by taking $Z = 1$ and $i = \delta_\epsilon$ and $c = \,! : 0 \to 1$, so that the case of biased filling is subsumed. Moreover, for any $i : Z \to \mathrm{I}$ the graph $\langle i \rangle : Z \rightarrowtail Z \times \mathrm{I}$ is itself of the form $0 \otimes_i \delta$ for the cofibration $0 \rightarrowtail Z$, so the graph of any "I-indexing" map $i : Z \to \mathrm{I}$ is also a trivial cofibration.

The following sanity check will be needed later.

**Proposition 30.** *Let $f : F \twoheadrightarrow X$ be an unbiased fibration in* $\mathsf{cSet}$*. Then for the endpoints $\delta_0, \delta_1 : 1 \to \mathrm{I}$, the associated pullback-homs,*

$$\delta_\epsilon \Rightarrow f : F^{\mathrm{I}} \to X^{\mathrm{I}} \times_X F \qquad (\epsilon = 0, 1) \tag{47}$$

*are also trivial fibrations. Thus unbiased fibrations are also $\delta_\epsilon$-biased fibrations, for $\epsilon = 0, 1$.*

*Proof.* This follows from Remark 29 and the $\otimes \dashv \Rightarrow$ adjunction, but we give a different proof. Consider the case $X = 1$, the general one $f : F \to X$ being analogous. Thus let $F$ be an unbiased fibrant object in $\mathsf{cSet}$. So by definition $(\mathrm{I}^*F)^\delta : (\mathrm{I}^*F)^{\mathbb{I}} \longrightarrow \mathrm{I}^*F$ in $\mathsf{cSet}/_{\mathrm{I}}$ is a trivial fibration. Pulling back $\delta : 1 \to \mathbb{I}$ in $\mathsf{cSet}/_{\mathrm{I}}$ along the base change $\delta_\epsilon : 1 \to \mathrm{I}$ takes it to $\delta_\epsilon : 1 \to \mathrm{I}$ in $\mathsf{cSet}$, by the universal property of the generic point $\delta : 1 \to \mathbb{I}$; that is $\delta_\epsilon^*(\delta) = \delta_\epsilon : 1 \to \mathrm{I}$. So $(\mathrm{I}^*F)^\delta : (\mathrm{I}^*F)^{\mathbb{I}} \longrightarrow \mathrm{I}^*F$ is taken by $\delta_\epsilon^*$ to

$$\delta_\epsilon^*\big((\mathrm{I}^*F)^\delta\big) = (\delta_\epsilon^*\mathrm{I}^*F)^{\delta_\epsilon^*\delta} = F^{\delta_\epsilon} : F^{\mathrm{I}} \longrightarrow F\,,$$

as shown in the following.

$$\begin{tikzcd}
F^{\mathrm{I}} \arrow[r] \arrow[d, "F^{\delta_\epsilon}"'] \arrow[dr, phantom, "\lrcorner", very near start] & (\mathrm{I}^*F)^{\mathbb{I}} \arrow[d, "(\mathrm{I}^*F)^\delta"] & \\
F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{I}^*F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F \arrow[d] \\
1 \arrow[r, "\delta_\epsilon"'] & \mathrm{I} \arrow[r] & 1
\end{tikzcd} \tag{48}$$

And pullback preserves trivial fibrations. $\square$

**Unbiased fibration structure.** As in the biased case, the fibrations can be determined by *uniform* right-lifting against a *small category* of unbiased trivial cofibrations, now consisting of all those $c \otimes_i \delta$ in $\mathcal{C} \otimes \delta$ for which $c : C \rightarrowtail \mathrm{I}^n$ is basic, *i.e.* has representable codomain. Call these maps the *basic unbiased trivial cofibrations*, and let

$$\mathsf{BCof} \otimes \delta = \{c \otimes_i \delta : B \rightarrowtail \mathrm{I}^{n+1} \mid c : C \rightarrowtail \mathrm{I}^n,\ i : \mathrm{I}^n \to \mathrm{I},\ n \geq 0\}\,, \tag{49}$$

where the pushout-product $c \otimes_i \delta$ now has the form

$$\begin{tikzcd} C \arrow[r, "\langle ic \rangle"] \arrow[d, tail, "c"'] & C \times \mathrm{I} \arrow[d] \arrow[ddr, bend left, "c \times \mathrm{I}"] & \\ \mathrm{I}^n \arrow[r] \arrow[drr, bend right, "\langle i \rangle"'] & \mathrm{I}^n +_C (C \times \mathrm{I}) \arrow[dr, dotted, tail, "c \otimes_i \delta"'] & \\ & & \mathrm{I}^n \times \mathrm{I} \end{tikzcd} \tag{50}$$

for a basic cofibration $c : C \rightarrowtail \mathrm{I}^n$ and an indexing map $i : \mathrm{I}^n \to \mathrm{I}$, and with domain $B = \big(\mathrm{I}^n +_C (C \times \mathrm{I})\big)$. These subobjects $B \rightarrowtail \mathrm{I}^{n+1}$ can again be seen geometrically as "generalized open box inclusions", but now the floor and lid of the open box are generalized to the graph of an arbitrary map $i : \mathrm{I}^n \to \mathrm{I}$.

For any map $f : Y \to X$ a uniform, unbiased fibration structure on $f$ is then a choice of diagonal fillers $j(c, i, x, y)$,

$$\begin{tikzcd} B \arrow[r, "x"] \arrow[d, tail, "c \otimes_i \delta"'] & X \arrow[d, "f"] \\ \mathrm{I}^n \times \mathrm{I} \arrow[r, "y"'] \arrow[ur, dotted, "j(c,i,x,y)"'] & Y, \end{tikzcd} \tag{51}$$

for each basic trivial cofibration $c \otimes_i \delta : B \rightarrowtail \mathrm{I}^{n+1}$, which is *uniform in* $\mathrm{I}^n$ in the following sense: Given any cubical map $u : \mathrm{I}^m \to \mathrm{I}^n$, the pullback $u^*c : u^*C \rightarrowtail \mathrm{I}^m$ and the reindexing $iu : \mathrm{I}^m \to \mathrm{I}^n \to \mathrm{I}$ determine another basic trivial cofibration $u^*c \otimes_{iu} \delta : B' = (\mathrm{I}^m +_{u^*C} (u^*C \times \mathrm{I})) \rightarrowtail \mathrm{I}^{m+1}$, which fits into a commutative diagram of the form

$$\begin{tikzcd} B' \arrow[r, "(u \times \mathrm{I})'"] \arrow[d, tail, "u^*c \otimes_{iu} \delta"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[r, "x"] \arrow[d, tail, "c \otimes_i \delta"'] & X \arrow[d, "f"] \\ \mathrm{I}^m \times \mathrm{I} \arrow[r, "u \times \mathrm{I}"'] & \mathrm{I}^n \times \mathrm{I} \arrow[r, "y"'] \arrow[ur, dotted, "j(c,i,x,y)"'] & Y. \end{tikzcd} \tag{52}$$

For the outer rectangle in (52) there is a chosen diagonal filler

$$j(u^*c, iu, x(u \times \mathrm{I})', y(u \times \mathrm{I})) : \mathrm{I}^m \times \mathrm{I} \to X,$$

and for this map we require that

$$j(u^*c, iu, x(u \times \mathrm{I})', y(u \times \mathrm{I})) = j(c, i, x, y) \circ (u \times \mathrm{I}). \tag{53}$$

**Definition 31.** A *uniform, unbiased fibration structure* on a map

$$f : Y \to X$$

is a choice of fillers $j(c, i, x, y)$ as in (51) satisfying (53) for all cubical maps $u : \mathrm{I}^m \to \mathrm{I}^n$.

In these terms, we have the following analogue of corollary 22.

**Proposition 32.** *For any object $X$ in* $\mathsf{cSet}$ *the following are equivalent:*

(1) *$X$ is an unbiased fibrant object in the sense of Definition 23: the canonical map $\delta \Rightarrow X : X^{\mathrm{I}} \times \mathrm{I} \to X \times \mathrm{I}$ is a trivial fibration.*
(2) *$X$ has the right lifting property with respect to all generating unbiased trivial cofibrations,*

$$(\mathcal{C} \otimes \delta) \pitchfork X.$$

(3) *$X$ has a uniform, unbiased fibration structure in the sense of Definition 31.*

*Proof.* The equivalence between (1) and (2) is proposition 28. So assume (1). Then in $\mathsf{cSet}/_{\mathrm{I}}$, the evaluation at $\delta : 1 \to \mathbb{I}$,

$$(\mathrm{I}^*X)^{\delta} : (\mathrm{I}^*X)^{\mathbb{I}} \longrightarrow X$$

is a trivial fibration. By Proposition 15 it therefore has a uniform filling structure with respect to all basic cofibrations $c : C \rightarrowtail \mathrm{I}^n$ over I. Transposing by the $\otimes \dashv \Rightarrow$ adjunction and unwinding then gives exactly a uniform fibration structure on $X$. $\square$

A statement analogous to the foregoing also holds for maps $f : Y \to X$ in place of objects $X$. Indeed, as before, we have the following sharper formulation.

**Corollary 33.** *Uniform, unbiased fibration structures on a map $f : Y \to X$ correspond uniquely to relative $+$-algebra structures on the map $(\delta \Rightarrow f)$ (cf. definition 23),*

$$(\delta \Rightarrow f) : Y^I \times \mathrm{I} \longrightarrow (X^I \times \mathrm{I}) \times_{(X \times \mathrm{I})} (Y \times \mathrm{I})\,.$$

**Factorization.**

**Definition 34.** Summarizing the foregoing definitions, we have the following classes of maps:

- The *generating unbiased trivial cofibrations* were determined in (45) as

$$\mathcal{C} \otimes \delta = \{c \otimes_i \delta : D \rightarrowtail Z \times \mathrm{I} \mid c : C \rightarrowtail Z,\, i : Z \to \mathrm{I}\}\,, \qquad (54)$$

where $D = \big(Z +_C (C \times \mathrm{I})\big)$ and the pushout-product $c \otimes_i \delta$ has the form

$$
\begin{tikzcd}
C \arrow[r, "\langle ic \rangle"] \arrow[d, tail, "c"'] & C \times \mathrm{I} \arrow[d] \arrow[ddr, bend left, "c \times \mathrm{I}"] & \\
Z \arrow[r] \arrow[drr, bend right, "\langle ic \rangle"'] & Z +_C (C \times \mathrm{I}) \arrow[dr, dotted, "c \otimes_i \delta"'] & \\
& & Z \times \mathrm{I}
\end{tikzcd}
\tag{55}
$$

for any cofibration $c : C \rightarrowtail Z$ and indexing map $i : Z \to \mathrm{I}$.

- The class $\mathcal{F}$ of *unbiased fibrations*, which can be characterized as the right-lifting class of the generating unbiased trivial cofibrations,
$$(\mathcal{C} \otimes \delta)^{\pitchfork} \;=\; \mathcal{F}.$$
- The class of *unbiased trivial cofibrations* is then defined to be left-lifting class of the fibrations,
$$\mathsf{TCof} \;=\; {}^{\pitchfork}\mathcal{F}.$$

It follows that the classes $\mathsf{TCof}$ and $\mathcal{F}$ are closed under retracts and are mutually weakly orthogonal, $\mathsf{TCof} \pitchfork \mathcal{F}$. Thus in order to have a weak factorization system $(\mathsf{TCof}, \mathcal{F})$ it just remains to show the following.

**Lemma 35.** *Every map $f : X \to Y$ in $\mathsf{cSet}$ can be factored as $f = p \circ i$,*

$$
\begin{tikzcd}
X \arrow[r, tail, "i"] \arrow[dr, "f"'] & X' \arrow[d, two heads, "p"] \\
& Y
\end{tikzcd}
\tag{56}
$$

*with $i : X \rightarrowtail X'$ an unbiased trivial cofibration and $p : X' \twoheadrightarrow Y$ an unbiased fibration.*

*Proof.* We can use a standard argument (the “algebraic small object argument”, cf. [Gar09, Rie11]), which can be further simplified using the fact that the codomains of the basic trivial cofibrations $c \otimes_i \delta : B \rightarrowtail \mathrm{I}^{n+1}$ are not just representable, but *tiny* in the sense of Proposition 4, and the domains are not merely “small”, but *finitely presented*. The reader is referred to [Awo18] for details in a similar case. □

*Remark* 36. The proof in *ibid.* actually produces a stronger result than we need, namely an *algebraic* weak factorization system. This follows from the small generating *category* $\mathsf{BCof} \otimes \delta$ of basic unbiased trivial cofibrations (and pullback squares of the form on the left in (52)). The relationship between this stronger condition and the *classifying types* used in Section 6 is studied in [Swa18], which also gives an even more "constructive" proof of the factorization Lemma 35, not requiring quotients, exactness, or impredicativity. With this modification, the present approach can also be used in a *quasi*topos, as occurs in *e.g.* realizability and sheaves.

**Proposition 37.** *There is a weak factorization system on the category* $\mathsf{cSet}$ *in which the right maps are the unbiased fibrations and the left maps are the unbiased trivial cofibrations, both as specified in definition 34. This will be called the* (unbiased) fibration weak factorization system*.*

Hereafter, unless otherwise stated, all fibrations in $\mathsf{cSet}$ are assumed to be unbiased.

## 4. The weak equivalences

Our approach to proving that the classes $\mathcal{C}$ and $\mathcal{F}$ of cofibrations and fibrations, from Sections 2 and 3, determine a model structure will be to first identify a *premodel structure* in the sense of [Bar19], and then turn to the question of the 3-for-2 property for the resulting weak equivalences.

**Definition 38** (Weak equivalence)**.** A map $f : X \to Y$ in $\mathsf{cSet}$ is a *weak equivalence* if it can be factored as $f = g \circ h$,

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[dr, "f"'] & W \arrow[d, "g"] \\ & Y \end{tikzcd}$$

with $h \pitchfork \mathcal{F}$ and $\mathcal{C} \pitchfork g$. Accordingly, let

$$\begin{aligned} \mathcal{W} &= \mathsf{TFib} \circ \mathsf{TCof} \\ &= \{ f : X \to Y \mid f = g \circ h \text{ for some } g \in \mathsf{TFib} \text{ and } h \in \mathsf{TCof} \} \end{aligned}$$

be the class of weak equivalences.

Observe first that every trivial fibration $f \in \mathsf{TFib} = \mathcal{C}^{\pitchfork}$ is indeed a fibration, because the generating trivial cofibrations $c \otimes_i \delta$ are cofibrations. Moreover, every trivial fibration $f : X \to Y$ is also a weak equivalence $f = f \circ 1_X$, since the identity map $1_X$ is (trivially) a trivial cofibration $\mathsf{TCof} = {}^{\pitchfork}\mathcal{F}$. Thus we have

$$\mathsf{TFib} \subseteq (\mathcal{F} \cap \mathcal{W}).$$

Similarly, because $\mathsf{TFib} \subseteq \mathcal{F}$, we have $\mathsf{TCof} \subseteq \mathcal{C}$. Moreover, since identity maps are also trivial fibrations we have $\mathsf{TCof} \subseteq \mathsf{TFib} \circ \mathsf{TCof} = \mathcal{W}$. Thus we also have

$$\mathsf{TCof} \subseteq (\mathcal{C} \cap \mathcal{W}).$$

**Lemma 39.** $(\mathcal{C} \cap \mathcal{W}) \subseteq \mathsf{TCof}$.

*Proof.* Let $c : A \rightarrowtail B$ be a cofibration with a factorization

$$c = tf \circ tc : A \to W \to B$$

where $tc \in \mathsf{TCof}$ and $tf \in \mathsf{TFib}$. Let $f : X \twoheadrightarrow Y$ be a fibration and consider a commutative diagram,

$$\begin{tikzcd}
A \arrow[r, "x"] \arrow[d, tail, "c"'] & X \arrow[d, two heads, "f"] \\
B \arrow[r, "y"'] & Y.
\end{tikzcd}$$

Inserting the factorization of $c$, from $tc \pitchfork f$ we obtain $j : W \to X$ as indicated, with $j \circ tc = x$ and $f \circ j = y \circ tf$.

$$\begin{tikzcd}
A \arrow[rr, "x"] \arrow[dd, tail, "c"'] \arrow[dr, "tc"'] & & X \arrow[dd, two heads, "f"] \\
& W \arrow[ur, dotted, "j"'] \arrow[dl, "tf"] & \\
B \arrow[rr, "y"'] & & Y.
\end{tikzcd}$$

Moreover, since $c \pitchfork tf$ there is an $i : B \to W$ as indicated, with $i \circ c = tc$ and $tf \circ i = 1_B$.

$$\begin{tikzcd}
A \arrow[rr, "x"] \arrow[dd, tail, "c"'] \arrow[dr, "tc"'] & & X \arrow[dd, two heads, "f"] \\
& W \arrow[ur, dotted, "j"'] \arrow[dl, "tf"] & \\
B \arrow[rr, "y"'] \arrow[ur, dotted, bend right, "i"'] & & Y.
\end{tikzcd}$$

Let $k = j \circ i$. Then $k \circ c = j \circ i \circ c = j \circ tc = x$, and $f \circ k = f \circ j \circ i = y \circ tf \circ i = y$. $\square$

The proof of the following is exactly dual.

**Lemma 40.** $(\mathcal{F} \cap \mathcal{W}) \subseteq \mathsf{TFib}$.

**Proposition 41.** *The three classes of maps $\mathcal{C}, \mathcal{W}, \mathcal{F}$ in* $\mathsf{cSet}$ *constitute a* premodel structure *in the sense of* [Bar19]*. In particular, we have*

$$\mathcal{F} \cap \mathcal{W} = \mathsf{TFib},$$
$$\mathcal{C} \cap \mathcal{W} = \mathsf{TCof},$$

*and therefore two interlocking weak factorization systems:*

$$(\mathcal{C},\, \mathcal{W} \cap \mathcal{F}) \quad , \quad (\mathcal{C} \cap \mathcal{W},\, \mathcal{F}).$$

It now "only" remains to show that the weak equivalences $\mathcal{W}$ satisfy the 3-for-2 axiom from Definition 1 in order to verify that $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ is a model structure. Perhaps surprisingly, this will occupy the remainder of these lectures! We shall follow roughly the approach of [JT08]: the weak equivalences between fibrant objects are shown to be the usual *homotopy equivalences*, which evidently satisfy 3-for-2. So we reduce to this case using the fact that $K^X$ is fibrant whenever $K$ is. It suffices, namely, to show that the weak equivalences are those maps $w : X \to Y$ that induce homotopy equivalences $K^w : K^Y \simeq K^X$ for fibrant $K$. Such maps are termed *weak homotopy equivalences* (Definition 48), and our task will therefore be to show that a map is a weak equivalence if and only if it is a weak homotopy equivalence.

**Weak homotopy equivalence.**

**Definition 42.** A *homotopy* $\vartheta : f \sim g$ between maps $f, g : X \rightrightarrows Y$ is a map,

$$\vartheta : \mathrm{I} \times X \longrightarrow Y,$$

such that $\vartheta \circ \iota_0 = f$ and $\vartheta \circ \iota_1 = g$,

$$\begin{tikzcd} X \arrow[r, "\iota_0"] \arrow[dr, "f"'] & \mathrm{I} \times X \arrow[d, "\vartheta"] & X, \arrow[l, "\iota_1"'] \arrow[dl, "g"] \\ & Y & \end{tikzcd} \tag{57}$$

where $\iota_0, \iota_1$ are the canonical inclusions into the ends of the cylinder,

$$\iota_\epsilon : X \cong 1 \times X \xrightarrow{\delta_\epsilon \times X} \mathrm{I} \times X\,, \qquad \epsilon = 0, 1.$$

Note that each of the inclusions $\iota_\epsilon : X \rightarrowtail \mathrm{I} \times X$ is a cofibration, as is their join $X + X \rightarrowtail \mathrm{I} \times X$, by Remark 18.

**Proposition 43.** *The relation of homotopy $f \sim g$ between maps $f, g : X \rightrightarrows Y$ is preserved by pre- and post-composition. If $Y$ is fibrant, then $f \sim g$ is an equivalence relation.*

*Proof.* Inspecting (140), preservation of $f \sim g$ under post-composing with any $h : Y \to Z$ is obvious: we have $h \circ \vartheta : h \circ f \sim h \circ g$. Now observe that a homotopy $f \overset{\vartheta}{\sim} g : X \times \mathrm{I} \to Y$ determines a (unique) path $\tilde{\vartheta} : \mathrm{I} \to Y^X$ in the function space, with endpoints $\vartheta_0 = \vartheta \circ \delta_0 = \tilde{f} : 1 \to Y^X$ and $\vartheta_1 = \vartheta \circ \delta_1 = \tilde{g}$. Precomposing maps $f, g : X \rightrightarrows Y$ with any $e : W \to X$ is induced by post-composing $\tilde{f}, \tilde{g} : 1 \to Y^X$ with the map $Y^e : Y^X \to Y^W$, which then also takes the path $\tilde{\vartheta} : \mathrm{I} \to Y^X$ to a path $\tilde{\varphi} = Y^e \circ \tilde{\vartheta} : \mathrm{I} \to Y^W$ corresponding to a (unique) homotopy $\varphi : f \circ e \sim g \circ e$.

Now note that $Y^X$ is fibrant if $Y$ is fibrant, since the generating trivial cofibrations $c \times_i \delta$ are preserved by the functor $X \times (-)$. So we can use "box-filling" in $Y^X$ to verify the claimed equivalence relation.

- Reflexivity $f \sim f$ is witnessed by the homotopy $\rho : \mathrm{I} \to 1 \xrightarrow{f} Y^X$.
- For symmetry $f \sim g \Rightarrow g \sim f$ take $\vartheta : \mathrm{I} \to Y^X$ with $\vartheta_0 = f$ and $\vartheta_1 = g$ and we want to build $\vartheta' : \mathrm{I} \to Y^X$ with $\vartheta'_0 = g$ and $\vartheta'_1 = f$. Take an open 2-box in $Y^X$ of the following form.

$$\begin{tikzcd} g & f \\ f \arrow[u, "\vartheta"] \arrow[r, "\rho"'] & f \arrow[u, "\rho"'] \end{tikzcd}$$

  This box is a map $b : \mathrm{I} +_1 \mathrm{I} +_1 \mathrm{I} \to Y^X$ with the indicated components, and it has a filler $c : \mathrm{I} \times \mathrm{I} \to Y^X$, i.e. an extension along the canonical map $\mathrm{I} +_1 \mathrm{I} +_1 \mathrm{I} \rightarrowtail \mathrm{I} \times \mathrm{I}$, which is a trivial cofibration of the form $\partial\mathrm{I} \otimes \delta_0$. Let $t : \mathrm{I} \to \mathrm{I} \times \mathrm{I}$ be the top face of the 2-cube (the bipointed map $\{0, x_1, x_2, 1\} \to \{0, x, 1\}$ that is constantly 1). We can set $\vartheta' = c \circ t : \mathrm{I} \to Y^X$ to get a homotopy $\vartheta' : \mathrm{I} \to Y^X$ with $\vartheta'_0 = g$ and $\vartheta'_1 = f$ as required.
- For transitivity, $f \overset{\vartheta}{\sim} g$, $g \overset{\varphi}{\sim} h \Rightarrow f \sim h$, an analogous construction will fill the open box:

$$\begin{tikzcd} f & h \\ f \arrow[u, "\rho"] \arrow[r, "\vartheta"'] & g \arrow[u, "\varphi"'] \end{tikzcd}$$

$\square$

We then have the usual definition of homotopy equivalence:

**Definition 44.** A *homotopy equivalence* is a map $f : X \to Y$ together with a map $g : Y \to X$ and homotopies $\vartheta : 1_X \sim g \circ f$ and $\varphi : 1_Y \sim f \circ g$. We call $g$ a *quasi-inverse* of $f$.

Since these maps clearly compose and come with quasi-inverses, the following is then immediate.

**Lemma 45.** *The homotopy equivalences satisfy the 3-for-2 condition.*

For the comparison with the weak equivalences we need the following.

**Definition 46** (Connected components)**.** The functor

$$\pi_0 : \mathsf{cSet} \to \mathsf{Set}$$

is defined on a cubical set $X$ as the coequalizer

$$X_1 \rightrightarrows X_0 \to \pi_0 X,$$

where the two parallel arrows are the maps $X_{\delta_0}, X_{\delta_1} : X_1 \rightrightarrows X_0$ for the endpoints $\delta_0, \delta_1 : 1 \rightrightarrows \mathrm{I}$. If $K$ is fibrant, then by the foregoing Proposition 43, for any $X$ we have

$$\pi_0(K^X) = \mathrm{Hom}(X, K)/\!\sim\ .$$

That is, $\pi_0(K^X)$ is the set $[X, K]$ of homotopy equivalence classes of maps $X \to K$.

*Remark* 47. One can show that in fact $\pi_0 X = \varinjlim X_n$ where the colimit is taken over *all* objects $[n]$ in the index category $\square^{\mathrm{op}} = \mathbb{B}$, rather than just the "last" two $[1] \rightrightarrows [0]$. Since the category $\mathbb{B}$ of finite strictly bipointed sets is sifted, the functor $\pi_0 : \mathsf{cSet} \to \mathsf{Set}$ preserves finite products.

**Definition 48** (Weak homotopy equivalence)**.** A map $f : X \to Y$ is called a *weak homotopy equivalence* if for every fibrant object $K$, the canonical map $K^f : K^Y \to K^X$ is bijective on connected components,

$$\pi_0(K^f) : \pi_0(K^Y) \cong \pi_0(K^X)\,.$$

**Lemma 49.** *Every homotopy equivalence is a weak homotopy equivalence.*

*Proof.* Let $f : X \to Y$ be a homotopy equivalence. Then $K^f : K^Y \to K^X$ is also a homotopy equivalence for any $K$, since homotopy respects (post-) composition by all maps. If $K$ is fibrant, then so is $K^X$ and $\pi_0$ is well defined on homotopy classes of maps, by Proposition 43. It clearly takes homotopy equivalences to isomorphisms of sets, since it identifies homotopic maps. $\square$

**Lemma 50.** *The weak homotopy equivalences also satisfy the 3-for-2 condition.*

*Proof.* This follows by applying the $\mathsf{Set}$-valued functors $\pi_0(K^{(-)})$, for all fibrant objects $K$, and the corresponding fact about bijections of sets. $\square$

In virtue of Lemma 50 it now suffices to show that a map is a weak equivalence if and only if it is a weak homotopy equivalence. The following characterization will be useful.

**Lemma 51.** *A map $f : X \to Y$ is a weak homotopy equivalence just if it satisfies the following two conditions.*

(1) *For every fibrant object $K$ and every map $x : X \to K$ there is a map $y : Y \to K$ such that $y \circ f \sim x$,*

$$\begin{tikzcd} X \arrow[r, "x"] \arrow[d, "f"'] \arrow[dr, phantom, "\sim"] & K. \\ Y \arrow[ur, dotted, "y"'] & \end{tikzcd}$$

*We say that $x$ "extends along $f$ up to homotopy".*

(2) *For every fibrant object $K$ and maps $y, y' : Y \to K$ such that $yf \sim y'f$, there is a homotopy $y \sim y'$,*

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & K^{\mathrm{I}} \arrow[d] \\ Y \arrow[r, "\langle y,y' \rangle"'] \arrow[ur, dotted] & K \times K. \end{tikzcd}$$

*Proof.* Condition (1) says exactly that the internal precomposition map $K^f : K^Y \to K^X$ is surjective under connected components $\pi_0$, while (2) says just that it is injective under $\pi_0$. $\square$

**Lemma 52.** (1) *Any trivial fibration is a homotopy equivalence.*
(2) *Any weak equivalence is a weak homotopy equivalence.*

*Proof.* For (1), any trivial fibration $f : X \twoheadrightarrow Y$ has a section $s : Y \to X$ by Corollary 16. Consider the following lifting problem:

$$\begin{tikzcd} X + X \arrow[r, "{[sf,1]}"] \arrow[d, tail, "{[\iota_0,\iota_1]}"'] & X \arrow[d, two heads, "f"] \\ \mathrm{I} \times X \arrow[r, "f\pi_2"'] & Y \end{tikzcd}$$

Since the map on the left is a cofibration, a diagonal filler provides a homotopy $\vartheta : sf \sim 1_X$. Thus $f$ is a homotopy equivalence.

For (2), by (1) and Lemma 49, a trivial fibration is also a weak homotopy equivalence. So it suffices to consider the trivial cofibrations, since weak homotopy equivalences are closed under composition, by Lemma 50. Thus let $f : X \rightarrowtail Y$ be a trivial cofibration, and apply Lemma 51: condition (1) is immediate, and (2) follows because $K^{\mathrm{I}} \twoheadrightarrow K \times K$ is a fibration when $K$ is fibrant, since $\partial : 1 + 1 \rightarrowtail \mathrm{I}$ is a cofibration (by Remark 18). $\square$

Our goal is to show the converse of Lemma 52(2), that a weak homotopy equivalence is a weak equivalence. We shall first restrict attention to maps $f : X \to K$ with a fibrant codomain $K$. By factoring such maps, we can split into the cases of a fibration and a cofibration.

**Lemma 53.** *If $K$ is fibrant, then any fibration $f : X \twoheadrightarrow K$ that is a homotopy equivalence is a weak equivalence.*

*Proof.* This is a standard argument, which we just sketch. It suffices to show that any diagram of the form

$$\begin{tikzcd} C \arrow[r, "x"] \arrow[d, tail, "c"'] & X \arrow[d, two heads, "f"] \\ K \arrow[r, "="'] & K, \end{tikzcd} \tag{58}$$

with $c : C \rightarrowtail X$ a cofibration, has a diagonal filler, for then $f$ is a trivial fibration. Since $f$ is a homotopy equivalence, it has a quasi-inverse $s : K \to X$ with $\vartheta : fs \sim 1_K$, which we claim can be corrected to a section $s' : K \to X$. Indeed, consider

$$\begin{tikzcd} K \arrow[r, "s"] \arrow[d, tail, "\iota_0"'] & X \arrow[d, two heads, "f"] \\ K \times \mathrm{I} \arrow[r, "\vartheta"'] \arrow[ur, dotted, "\vartheta'"] & K \\ K \arrow[u, tail, "\iota_1"] \arrow[ur, "="'] & \end{tikzcd}$$

where $\vartheta'$ results from $\iota_0 \pitchfork f$. Let $s' = \vartheta' \iota_1$, so that $\vartheta' : s \sim s'$ and $fs' = 1_K$.

Thus we can assume that $s = s' : K \to X$ is a section, which fills the diagram (58) up to a homotopy in the upper triangle.

$$\begin{tikzcd} C \arrow[r, "x"] \arrow[d, tail, "c"'] \arrow[dr, phantom, "\sim", pos=0.3] & X \arrow[d, two heads, "f"] \\ K \arrow[r, "="'] \arrow[ur, "s"'] & K \end{tikzcd}$$

Now we can correct $s : K \to X$ to a homotopic $t : K \to X$ over $f$ by using the homotopy $\varphi : sc \sim x$ to get a map $\varphi : C \to X^{\mathrm{I}}$ over $f$. Since $f$ is a fibration, the projections $p_0, p_1 : X^{\mathrm{I}} \to X$ over $f$ are trivial fibrations, and so there is a lift $\varphi' : K \to X^{\mathrm{I}}$ for which $t := p_1\varphi'$ has $tc = x$ and $ft = 1_K$, and so is a filler for (58). □

**Lemma 54.** *If $K$ is fibrant, then any fibration $f : X \twoheadrightarrow K$ that is a weak homotopy equivalence is a weak equivalence.*

*Proof.* Since $K$ is fibrant, so is $X$, and since $f$ is a weak homotopy equivalence, by lemma 51(1) there is then a map $s : K \to X$ and a homotopy $\theta : sf \sim 1_X$. Postcomposing with $f$ gives a homotopy $f\vartheta : fsf \sim f$, forming the outer commutative square in

$$\begin{tikzcd}
X \arrow[r, "f\vartheta"] \arrow[d, "f"'] & K^{\mathrm{I}} \arrow[d] \\
K \arrow[r, "\langle fs, 1_K \rangle"'] \arrow[ur, dotted, "\varphi"'] & K \times K.
\end{tikzcd}$$

By lemma 51(2) there is a diagonal filler $\varphi : fs \sim 1_K$, and so $f$ is a homotopy equivalence. Now apply lemma 53. □

We now have the following.

**Proposition 55.** *If $A$ and $K$ are both fibrant, then for any cofibration $c : A \rightarrowtail K$ the following are equivalent.*

(1) *$c : A \rightarrowtail K$ is a weak equivalence.*
(2) *$c : A \rightarrowtail K$ is a homotopy equivalence.*
(3) *$c : A \rightarrowtail K$ is a weak homotopy equivalence.*

*Proof.* Suppose (1), so $c : A \rightarrowtail K$ is a trivial cofibration. Then since $A$ is fibrant, it has a retraction $r : K \to A$.

$$\begin{tikzcd}
A \arrow[r, "="] \arrow[d, tail, "c"'] & A. \\
K \arrow[ur, "r"'] &
\end{tikzcd}$$

Since $K$ is fibrant, $K^{\mathrm{I}} \to K \times K$ is a fibration. So the following has a diagonal filler, which is a homotopy $1_K \sim cr$.

$$\begin{tikzcd}
A \arrow[r, "c"] \arrow[d, tail, "c"'] & K \arrow[r, "K^!"] & K^{\mathrm{I}} \arrow[d, "\langle K^{d_0}, K^{d_1} \rangle"] \\
K \arrow[rr, "\langle 1_K, cr \rangle"'] \arrow[urr, dotted, "\vartheta"'] & & K \times K
\end{tikzcd}$$

(2) ⇒ (3) is Lemma 49.

Suppose (3), that $c : A \rightarrowtail K$ is a weak homotopy equivalence. Factor $c = f \circ tc$ with a trivial cofibration $tc : A \rightarrowtail C$ followed by a fibration $f : C \twoheadrightarrow K$. By parts (1) and (2), $tc : A \rightarrowtail C$ is then a weak homotopy equivalence. By 3-for-2 for weak homotopy equivalences, Lemma 50, $f : C \twoheadrightarrow K$ is then also a weak homotopy equivalence. By Lemma 54, $f : C \twoheadrightarrow K$ is then a weak equivalence. □

**Proposition 56.** *For fibrations $f : X \twoheadrightarrow K$ with fibrant codomain $K$, all three concepts coincide: weak equivalences, weak homotopy equivalences, and homotopy equivalences.*

*Proof.* Let $K$ be fibrant and suppose that $f : X \twoheadrightarrow K$ is a weak homotopy equivalence. Then it is a weak equivalence by Lemma 54. By Lemma 52 any fibration weak equivalence is a homotopy equivalence, and by Lemma 49 any homotopy equivalence is a weak homotopy equivalence. □

**Corollary 57.** *For all maps $f : X \to Y$ between fibrant objects $X$ and $Y$, all three concepts coincide: weak equivalence, weak homotopy equivalence, and homotopy equivalence.*

*Proof.* Let $X$ and $Y$ be fibrant and factor $f = tf \circ tc$ with a trivial cofibration $tc : X \rightarrowtail F$ followed by a trivial fibration $tf : F \twoheadrightarrow Y$. Then by Proposition 55, $tc : X \rightarrowtail F$ is a homotopy equivalence, and by Proposition 56 so is $tf : F \twoheadrightarrow Y$, thus $f = tf \circ tc$ is a homotopy equivalence. Again by Lemma 49, any homotopy equivalence is a weak homotopy equivalence, and weak homotopy equivalence between fibrant objects is clearly a weak equivalence, by factoring and using the foregoing Propositions 55 and 56. □

**Lemma 58.** *If $K$ is fibrant, then any cofibration $c : A \rightarrowtail K$ that is a weak homotopy equivalence is a weak equivalence.*

*Proof.* Let $c : A \rightarrowtail K$ be a cofibration weak homotopy equivalence and factor it into a trivial cofibration $i : A \rightarrowtail Z$ followed by a fibration $p : Z \twoheadrightarrow K$. By lemma 51, any trivial cofibration is clearly a weak homotopy equivalence. So both $c$ and $i$ are weak homotopy equivalences, and therefore so is $p$ by 3-for-2 for weak homotopy equivalences. Since $K$ is fibrant, $p$ is a trivial fibration by lemma 54, and thus $c$ is a weak equivalence. □

It now follows that a weak homotopy equivalence $f : X \to K$ with a fibrant codomain is a weak equivalence. To eliminate the condition on the codomain we use the following lemma due to D.-C. Cisinski [Cis06].

**Lemma 59.** *A cofibration $c : A \rightarrowtail B$ weak homotopy equivalence lifts against any fibration $f : Y \twoheadrightarrow K$ with fibrant codomain.*

*Proof.* Let $c : A \rightarrowtail B$ be a cofibration weak homotopy equivalence and $f : Y \twoheadrightarrow K$ a fibration with fibrant codomain $K$, and consider a lifting problem

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, tail, "c"'] & Y \arrow[d, two heads, "f"] \\ B \arrow[r, "b"'] & K. \end{tikzcd}$$

Let $\eta : B \rightarrowtail B'$ be a fibrant replacement of $B$, since $K$ is fibrant, $b$ extends along $\eta$ to give $b' : B' \to K$ as shown below.

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, tail, "c"'] & Y \arrow[d, two heads, "f"] \\ B \arrow[r, "b"] \arrow[d, "\eta"'] & K \\ B' \arrow[ur, "b'"'] & \end{tikzcd}$$

Since $\eta$ is a trivial cofibration, it is a weak homotopy equivalence. So the composite $\eta c$ is also a weak homotopy equivalence. But since $B'$ is fibrant, $\eta c$ is then a trivial cofibration by lemma 58. Thus there is a lift $j : B' \to Y$, and therefore also one $k = j\eta : B \to Y$. $\square$

To complete the proof that a weak homotopy equivalence is a weak equivalence, we shall make use of the following *fibration extension property*, the proof of which is deferred to section 8.

**Definition 60** (Fibration extension property)**.** For any fibration $f : Y \twoheadrightarrow X$ and any trivial cofibration $\eta : X \to X'$, there is a fibration $f' : Y' \twoheadrightarrow X'$ that pulls back to $f$ along $\eta$, as shown below.

$$\begin{tikzcd} Y \arrow[r] \arrow[d, two heads, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y' \arrow[d, two heads, "f'"] \\ X \arrow[r, tail, "\eta"'] & X' \end{tikzcd} \tag{59}$$

**Lemma 61.** *Assuming the fibration extension property, a cofibration that lifts against every fibration $f : Y \twoheadrightarrow K$ with fibrant codomain is a weak equivalence.*

*Proof.* Let $c : A \rightarrowtail B$ be a cofibration and consider a lifting problem against an arbitrary fibration $f : Y \twoheadrightarrow X$,

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, tail, "c"'] & Y \arrow[d, two heads, "f"] \\ B \arrow[r, "b"'] & X. \end{tikzcd} \tag{60}$$

Let $\eta : X \to X'$ be a fibrant replacement, so $\eta$ is a trivial cofibration and $X'$ is fibrant. By the fibration extension property of definition 60, there is a fibration $f' : Y' \twoheadrightarrow X'$ such that $f$ is a pullback of $f'$ along $\eta$. So we can extend diagram (60) to obtain the following, in which the righthand square is a pullback.

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, tail, "c"'] & Y \arrow[r, "y"] \arrow[d, two heads, "f"] \arrow[dr, phantom, "\lrcorner", very near start] & Y' \arrow[d, two heads, "f'"] \\ B \arrow[r, "b"'] & X \arrow[r, "\eta"'] & X'. \end{tikzcd} \tag{61}$$

By assumption, there is a lift $j' : B \to Y'$ with $f'j' = \eta b$ and $j'c = yb$. Therefore, since $f$ is a pullback, there is a map $j : B \to Y$ with $fj = b$ and $yj = j'$.

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, tail, "c"'] & Y \arrow[r, "y"] \arrow[d, two heads, "f"] \arrow[dr, phantom, "\lrcorner", very near start] & Y' \arrow[d, two heads, "f'"] \\ B \arrow[r, "b"'] \arrow[ur, dotted, "j"] \arrow[urr, dotted, "j'"'] & X \arrow[r, "\eta"'] & X'. \end{tikzcd} \tag{62}$$

Thus $yjc = j'c = ya$. But as a trivial cofibration, $\eta$ is monic, and as a pullback of $\eta$, $y$ is also monic. So $jc = a$. □

**Corollary 62.** *Assuming the fibration extension property,*

(1) *a cofibration $c : A \rightarrowtail B$ weak homotopy equivalence is a weak equivalence,*
(2) *a fibration $f : Y \twoheadrightarrow X$ weak homotopy equivalence is a weak equivalence.*

*Proof.* (1) follows immediately by combining the previous lemmas 59 and 61.

For (2), factor $f : Y \twoheadrightarrow X$ into a cofibration $i : Y \rightarrowtail Z$ followed by a trivial fibration $p : Z \twoheadrightarrow X$. Then $f$ is itself a trivial fibration if $i \pitchfork f$, for then it is a retract of $p$. Since $p$ is a trivial fibration, it is a weak homotopy equivalence by Lemma 52. Since $f$ is also a weak homotopy equivalence, so is $i$ by Lemma 50. Thus $i$ is a trivial cofibration by (1). Since $f$ is a fibration, $i \pitchfork f$ as required. □

We have now shown:

**Proposition 63.** *Assuming the fibration extension property, a map $f : X \to Y$ is a weak homotopy equivalence if and only if it is a weak equivalence. The weak equivalences $\mathcal{W}$ therefore satisfy the 3-for-2 condition.*

The results of this section are summarized in the following.

**Theorem 64.** *Assume the fibration weak factorization system of Definition 34 satisfies the fibration extension property of Definition 60 (as will be shown in Corollary 115). Then the weak equivalences $\mathcal{W}$ have the 3-for-2 property, and so by Proposition 41, the classes $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ form a Quillen model structure. The weak equivalences $\mathcal{W}$ are the* weak homotopy equivalences*: those maps $f : X \to Y$ for which $K^f : K^Y \to K^X$ is bijective on connected components whenever $K$ is fibrant.*

The proof of the fibration extension property will be given in Section 8. It uses the equivalence extension property (Section 7), a universal fibration (Section 6), and the Frobenius condition (Section 5), to which we now turn.

## 5. The Frobenius condition

In this section, we show that the (unbiased) fibration weak factorization system from Section 3 satisfies what has been called the *Frobenius condition*: the left maps are stable under pullback along the right maps (see [BG12]). This will imply the *right properness* of our model structure: the weak equivalences are preserved by pullback along fibrations. In the present setting, it then follows that the entire model structure is stable under such a base change. The Frobenius condition will be used in the proof of the equivalence extension property in Section 7.

An proof of Frobenius in the related setting of cubical sets *with connections* was given in [GS17a] using conventional, functorial methods. By contrast, the type theoretic approach of [CCHM18] provides a proof that is much more direct, and can also be modified to work without connections (as in [ABC+22]). That approach proves the dual fact that the *pushforward* operation, which is right adjoint to pullback and always exists in a topos, preserves fibrations when applied along a fibration. This corresponds to the type-theoretic $\Pi$-formation rule, and the proof given in *op.cit.* is entirely in type theory. It also employs a reduction of box filling (in all dimensions) to an apparently weaker condition of *Kan composition* (in all dimensions), which merely "puts a lid on" the open box, rather than filling it. This aspect of the type theoretic proof can also be described functorially, but is not used in the proof given here, and will therefore not be discussed further (see [LOPS18] for a description of Kan composition with connections, and [Awo19a] for the same without connections).

Our proof takes the approach that was used to determine the unbiased fibrations, namely we first establish the result in the *biased but generic* setting, and then transfer it to the unbiased setting by pulling back along the base change $\mathsf{cSet} \to \mathsf{cSet}/_{\mathrm{I}}$. We first give the second step as a conditional statement.

**Proposition 65.** *Suppose the $\delta$-biased fibrations in $\mathsf{cSet}/_{\mathrm{I}}$ satisfy the Frobenius condition. Then the unbiased fibrations in $\mathsf{cSet}$ also satisfy the Frobenius condition.*

*Proof.* This follows almost immeditely from the fact that the pullback functor $\mathrm{I}^* : \mathsf{cSet} \to \mathsf{cSet}/_{\mathrm{I}}$ preserves the locally cartesian closed structure, takes unbiased fibrations to $\delta$-biased ones, and reflects $\delta$-biased fibrations to unbiased ones. In detail, let unbiased fibrations $B \twoheadrightarrow A$ and $A \twoheadrightarrow X$ in $\mathsf{cSet}$ be given, and we wish to find $C \twoheadrightarrow X$ and $e : A \times_X C \to B$ over $A$, universal in the way recalled in the diagram below.

$$\begin{tikzcd}
A \times_X C \arrow[r, dotted] \arrow[d, dotted, "e"] & C \arrow[dd, dotted, two heads] \\
B \arrow[d, two heads] & \\
A \arrow[r, two heads] & X
\end{tikzcd} \tag{63}$$

Take the pushforward $C := A_* B \to X$, and its associated map $e : A \times_X C \to B$, in the locally cartesian closed category $\mathsf{cSet}$. Since fibrations are stable under (all) pullbacks, it then suffices to show that $C \to X$ is a fibration.

By definition, $C \to X$ is an unbiased fibration in $\mathsf{cSet}$ just in case the base change $\mathrm{I}^* C \to \mathrm{I}^* X$ is a $\delta$-biased fibration in the slice category $\mathsf{cSet}/_{\mathrm{I}}$. Since the pullback functor $\mathrm{I}^* : \mathsf{cSet} \to \mathsf{cSet}/_{\mathrm{I}}$ preserves all lcc structure, over $\mathrm{I}^* X$ we have an iso,

$$\mathrm{I}^* C = \mathrm{I}^*(A_* B) \cong (\mathrm{I}^* A)_* \mathrm{I}^* B\,,$$

where the pushforward $(\mathrm{I}^* A)_* \mathrm{I}^* B$ is taken in the topos $\mathsf{cSet}/_{\mathrm{I}}$. But $\mathrm{I}^* B \to \mathrm{I}^* A$ and $\mathrm{I}^* A \to \mathrm{I}^* X$ are $\delta$-biased fibrations in $\mathsf{cSet}/_{\mathrm{I}}$ because $B \to A$ and $A \to X$ were assumed to be unbiased fibrations in $\mathsf{cSet}$. Since we are assuming the Frobenius condition for $\delta$-biased fibrations in $\mathsf{cSet}/_{\mathrm{I}}$, the pushforward $\mathrm{I}^* C \cong (\mathrm{I}^* A)_* \mathrm{I}^* B \to \mathrm{I}^* X$ is also a $\delta$-biased fibration, as required. ∎

**Frobenius for biased fibrations.** The results proved in this section will be applied to the slice category $\mathsf{cSet}/_{\mathrm{I}}$ and the generic point $\delta : 1 \to \mathrm{I} = \mathrm{I}^*\mathrm{I}$, but nothing depends on this particular case, and so we shall write simply $\delta : 1 \to \mathrm{I}$ for a chosen pointed object in an arbitrary topos $\mathcal{E}$. (Indeed, in this section $\mathcal{E}$ may even be just a locally cartesian closed category with a class of cofibrations in the sense of Appendix A.)

Recall from Definition 23 that a map $f : A \to X$ is a $\delta$-biased fibration just if the map $\delta \Rightarrow f$ admits a relative $+$-algebra structure, and is therefore a trivial fibration. The definition of the pullback-hom $\delta \Rightarrow f$ is recalled below.

$$\begin{tikzcd} A^{\mathrm{I}} \arrow[drr, bend left, "A^{\delta}"] \arrow[dr, dotted, "\delta\Rightarrow f"] \arrow[ddr, bend right, "f^{\mathrm{I}}"'] & & \\ & X^{\mathrm{I}} \times_X A \arrow[r] \arrow[d] & A \arrow[d, "f"] \\ & X^{\mathrm{I}} \arrow[r, "X^{\delta}"'] & X \end{tikzcd} \tag{64}$$

Let us write this condition schematically as follows:

$$\begin{tikzcd} A^{\mathrm{I}} \arrow[r, "+" description] & A_{\epsilon} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & A \arrow[d, "f"] \\ & X^{\mathrm{I}} \arrow[r, "\epsilon"'] & X \end{tikzcd} \tag{65}$$

where $\epsilon = X^{\delta}$ and $A_{\epsilon} = X^{\mathrm{I}} \times_X A$, and the struck-through arrow indicates that it admits a $+$-algebra structure.

**Lemma 66.** *Let $A \to X$ be a $\delta$-biased fibration and $t : Y \to X$ any map, then the pullback $t^*A \to Y$ is also a $\delta$-biased fibration.*

*Proof.* This is clear from the fact that the $\delta$-biased fibrations can be made into the right class of a weak factorization system (by reasoning analogous to that for Proposition 37), but it will be useful to see how the structure indicated in (64) is itself stable under pullback. Indeed, consider the following commutative diagram, in which the front face of the cube is the pullback in question, and the right and left sides are the respective versions of the construction in (64).

$$\begin{tikzcd}
(t^*A)^{\mathrm{I}} \arrow[rr] \arrow[d] & & A^{\mathrm{I}} \arrow[d] & \\
(t^*A)_\epsilon \arrow[rr, dotted] \arrow[dr] \arrow[dd] & & A_\epsilon \arrow[dr] \arrow[dd] & \\
& t^*A \arrow[rr] \arrow[dd] & & A \arrow[dd] \\
Y^{\mathrm{I}} \arrow[rr] \arrow[dr, "\epsilon"'] & & X^{\mathrm{I}} \arrow[dr, "\epsilon"'] & \\
& Y \arrow[rr, "t"'] & & X
\end{tikzcd} \tag{66}$$

The rear square of solid arrows is the image of the front face under the pathobject functor and is therefore also a pullback. The base commutes by the naturality of the maps $\epsilon$, as does a corresponding top square involving further such $\epsilon$'s not shown. Note that these naturality squares need not be pullbacks, but the vertical squares on the sides are, by construction. It follows that there is a dotted arrow as shown, making the resulting lower rear square commute. That lower square is then also a pullback, since the other vertical faces of the resulting cube are pullbacks, and thus finally, the upper rear square is also a pullback.

Now if $A \to X$ is a $\delta$-biased fibration, then $A^{\mathrm{I}} \to A_\epsilon$ is a trivial fibration, and then so is its pullback $(t^*A)^{\mathrm{I}} \to (t^*A)_\epsilon$ since relative +-algebras are stable under pullback. Therefore the pullback $t^*A \to Y$ is also a $\delta$-biased fibration. $\square$

*Remark* 67. In this way we can show algebraically that the pullback of a $\delta$-biased fibration is again one by pulling back the structure that makes it so. In Section 6, the pullback stability of the fibration structure will be used in the construction of a universal fibration via a closely related argument.

**Lemma 68.** *Let $\alpha : A \to X$ and $\beta : B \to A$ be $\delta$-biased fibrations, then the composite $\alpha \circ \beta : B \to X$ is also a $\delta$-biased fibration.*

*Proof.* Again for maps in the right class of a weak factorization system this is immediate. But let us see how the fibration structures also compose. We have the following diagram for the fibration structures on $B \to A$ and $A \to X$ (with obvious notation).

$$
\begin{tikzcd}
B^{\mathrm{I}} \arrow[r, "\shortmid" marking] & B_{\epsilon_A} \arrow[rr] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & & B \arrow[d] \\
& A^{\mathrm{I}} \arrow[r, "\shortmid" marking] & A_{\epsilon_X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d] \\
& & X^{\mathrm{I}} \arrow[r, "\epsilon_X"'] & X,
\end{tikzcd}
\tag{67}
$$

Pulling back $B \to A$ in two steps we therefore obtain the intermediate map $B_{\epsilon_X} \to A_{\epsilon_X}$ indicated in the following diagram.

$$
\begin{tikzcd}
B^{\mathrm{I}} \arrow[r, "\shortmid" marking] & B_{\epsilon_A} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B_{\epsilon_X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d] \\
& A^{\mathrm{I}} \arrow[r, "\shortmid" marking] & A_{\epsilon_X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d] \\
& & X^{\mathrm{I}} \arrow[r, "\epsilon_X"'] & X
\end{tikzcd}
\tag{68}
$$

Now use the fact that a trivial fibration structure (*i.e.* a $+$-algebra structure) has a canonical pullback along any map, and that two such structures have a canonical composition (cf. Remark 17), to obtain a trivial fibration structure for the indicated composite map $B^{\mathrm{I}} \to B_{\epsilon_X}$, which is then a fibration structure for the composite $B \to A \to X$. $\square$

**Proposition 69** ($\delta$-Biased Frobenius)**.** *If $\alpha : A \to X$ and $\beta : B \to A$ are $\delta$-biased fibrations, then the pushforward $\alpha_*\beta : \Pi_A B \to X$ is also a $\delta$-biased fibration.*

*Proof.* Given $\delta$-biased fibrations $\alpha : A \to X$ and $\beta : B \to A$, let $a : A^{\mathrm{I}} \to A_\epsilon$ and $b : B^{\mathrm{I}} \to a^* B_\epsilon$ be the associated trivial fibrations, so that we have the situation of diagram (68), with all three squares pullbacks.

$$
\begin{tikzcd}
B^{\mathrm{I}} \arrow[r, "b", "\shortmid" marking] \arrow[dr, "\beta^{\mathrm{I}}"'] & a^* B_\epsilon \arrow[r] \arrow[d] & B_\epsilon \arrow[r] \arrow[d] & B \arrow[d, "\beta"] \\
& A^{\mathrm{I}} \arrow[r, "a", "\shortmid" marking] \arrow[dr, "\alpha^{\mathrm{I}}"'] & A_\epsilon \arrow[r] \arrow[d] & A \arrow[d, "\alpha"] \\
& & X^{\mathrm{I}} \arrow[r, "\epsilon"'] & X.
\end{tikzcd}
\tag{69}
$$

Taking the pushforward of the righthand vertical column give a map,

$$\gamma := \alpha_* \beta : \Pi_A B \to X \, ,$$

and placing it underneath, along with the corresponding construction from (64), we then have the following commutative diagram.

$$\begin{tikzcd}
B^{\mathrm{I}} \arrow[r, "b", "\shortmid"{marking}] \arrow[dr, "\beta^{\mathrm{I}}"'] & a^* B_\epsilon \arrow[r] \arrow[d] & B_\epsilon \arrow[r] \arrow[d] & B \arrow[d, "\beta"] \\
& A^{\mathrm{I}} \arrow[r, "a", "\shortmid"{marking}] \arrow[dr, "\alpha^{\mathrm{I}}"'] & A_\epsilon \arrow[r] \arrow[d] & A \arrow[d, "\alpha"] \\
& & X^{\mathrm{I}} \arrow[r, "\epsilon"'] & X \\
& (\Pi_A B)^{\mathrm{I}} \arrow[r, "c"'] \arrow[ur, "\gamma^{\mathrm{I}}"] & (\Pi_A B)_\epsilon \arrow[r] \arrow[u] & \Pi_A B \arrow[u, "\gamma"']
\end{tikzcd} \tag{70}$$

We wish to show that the indicated map $c : (\Pi_A B)^{\mathrm{I}} \to (\Pi_A B)_\epsilon$ admits a $+$-algebra structure. This we will do by showing that it is a retract of a known $+$-algebra. Namely, we can apply the pushforward along the map $\alpha^{\mathrm{I}} : A^{\mathrm{I}} \to X^{\mathrm{I}}$ to the $+$-algebra $b : B^{\mathrm{I}} \to a^* B_\epsilon$ regarded as an arrow over $A^{\mathrm{I}}$. We obtain an arrow over $X^{\mathrm{I}}$ of the form

$$\Pi_{A^{\mathrm{I}}}\, b : \Pi_{A^{\mathrm{I}}}\, B^{\mathrm{I}} \longrightarrow \Pi_{A^{\mathrm{I}}}\, a^* B_\epsilon \tag{71}$$

which is indeed a $+$-algebra, since these are preserved under pushing forward, by Remark 17.

Next, observe that by the Beck-Chevalley condition for the central pullback, for the codomain of $c$ we have an isomorphism

$$(\Pi_A B)_\epsilon \;\cong\; \Pi_{A_\epsilon} B_\epsilon \qquad \text{over } X^{\mathrm{I}}.$$

And since $\Pi_{A^{\mathrm{I}}} \cong \Pi_{A_\epsilon} \circ a_*$, for the codomain of our $+$-algebra $\Pi_{A^{\mathrm{I}}}\, b$ from (71) we also have

$$\Pi_{A^{\mathrm{I}}}\, a^* B_\epsilon \;\cong\; \Pi_{A_\epsilon} a_* a^* B_\epsilon \, .$$

Thus the image of the unit $\eta : B_\epsilon \to a_* a^* B_\epsilon$ under $\Pi_{A_\epsilon}$ provides a map $\sigma := \Pi_{A_\epsilon}\eta$ over $X^{\mathrm{I}}$ of the form:

$$\begin{tikzcd}
 & X^{\mathrm{I}} \\
(\Pi_A B)^{\mathrm{I}} \arrow[ur] \arrow[r, "c"'] & \Pi_{A_\epsilon} B_\epsilon \arrow[u] \arrow[d, "\sigma"] \\
\Pi_{A^{\mathrm{I}}} B^{\mathrm{I}} \arrow[r, "\Pi_{A^{\mathrm{I}}} b"'] & \Pi_{A_\epsilon} a_* a^* B_\epsilon
\end{tikzcd} \tag{72}$$

Our goal is now to determine further arrows $\varphi, \psi, \tau$ as indicated below, exhibiting $c$ as a retract of $\Pi_{A^{\mathrm{I}}}\, b$ in the arrow category over $X^{\mathrm{I}}$.

$$\begin{tikzcd}
 & X^{\mathrm{I}} \\
(\Pi_A B)^{\mathrm{I}} \arrow[ur] \arrow[r, "c"'] \arrow[d, dotted, "\varphi"'] & \Pi_{A_\epsilon} B_\epsilon \arrow[u] \arrow[d, "\sigma"] \\
\Pi_{A^{\mathrm{I}}} B^{\mathrm{I}} \arrow[r, "\Pi_{A^{\mathrm{I}}} b"'] \arrow[d, dotted, "\psi"'] & \Pi_{A_\epsilon} a_* a^* B_\epsilon \arrow[d, dotted, "\tau"] \\
(\Pi_A B)^{\mathrm{I}} \arrow[r, "c"'] & \Pi_{A_\epsilon} B_\epsilon
\end{tikzcd} \tag{73}$$

- For $\varphi$, we require a map

$$\varphi : (\Pi_A B)^{\mathrm{I}} \to \Pi_{A^{\mathrm{I}}} B^{\mathrm{I}} \qquad \text{over } X^{\mathrm{I}}.$$

Consider the following diagram, which is based on (65).

$$\begin{tikzcd}
B^{\mathrm{I}} \arrow[rr, "b"'] \arrow[drr, "\beta^{\mathrm{I}}"'] & & a^* B_\epsilon \arrow[r] \arrow[d] & B_\epsilon \arrow[r] \arrow[d] & B \arrow[d, "\beta"] & \\
 & & A^{\mathrm{I}} \arrow[r, "a"'] \arrow[dr, "\alpha^{\mathrm{I}}"'] & A_\epsilon \arrow[r] \arrow[d] & A & \Pi_A B \times_X A \arrow[l] \arrow[d] \arrow[ul, dotted, "e"'] \\
(\Pi_A B \times_X A)^{\mathrm{I}} \arrow[uu, dotted, "e^{\mathrm{I}}"] \arrow[urr] \arrow[drr] & & & X^{\mathrm{I}} \arrow[r, "\epsilon"'] & X & \Pi_A B \arrow[l] \\
 & & (\Pi_A B)^{\mathrm{I}} \arrow[ur] \arrow[r, dotted, "\varphi"'] & \Pi_{A^{\mathrm{I}}} B^{\mathrm{I}} \arrow[u] & &
\end{tikzcd} \tag{74}$$

The map $e$ is the counit at $\beta : B \to A$ of the pullback-pushforward adjunction along $\alpha : A \to X$. The right-hand side of the diagram, including $e$ and the associated pullback square, reappears (mirrored) on the left under the functor $(-)^{\mathrm{I}}$, which preserves the pullback. Thus we can take $\varphi$ to be the transpose of $e^{\mathrm{I}}$ under the pullback-pushforward adjunction along $\alpha^{\mathrm{I}} : A^{I} \to X^{\mathrm{I}}$,

$$\varphi := \widetilde{e^{\mathrm{I}}}\,.$$

An easy diagram chase involving the pullback-pushforward adjunction along $A_\epsilon \to X^{\mathrm{I}}$ shows that the upper square in (73) then commutes.

• For $\tau$: referring to the diagram (65), since $a : A^{\mathrm{I}} \to A_\epsilon$ is a trivial fibration, it has a section $o : A_\epsilon \to A^{\mathrm{I}}$ by lemma 16. Pulling $a^*B_\epsilon \to A^{\mathrm{I}}$ back along $o$ results in an iso,

$$o^*a^*B_\epsilon \cong B_\epsilon \quad \text{over } A_\epsilon$$

and so by the adjunction $o^* \dashv o_*$ there is an associated map,

$$a^*B_\epsilon \to o_*B_\epsilon \quad \text{over } A^{\mathrm{I}}$$

to which we can apply $a_*$ to obtain a map,

$$t : a_*a^*B_\epsilon \to a_*o_*B_\epsilon \cong B_\epsilon \quad \text{over } A_\epsilon\,.$$

This map $t$ is evidently a retraction of the unit $\eta : B_\epsilon \to a_*a^*B_\epsilon$ over $A_\epsilon$. Applying the functor $\Pi_{A_\epsilon}$ therefore gives the desired retraction of $\sigma$,

$$\tau := \Pi_{A_\epsilon} t : \Pi_{A_\epsilon} a_*a^*B_\epsilon \to \Pi_{A_\epsilon} B_\epsilon\,.$$

• For $\psi$, we require a map

$$\psi : \Pi_{A^{\mathrm{I}}} B^{\mathrm{I}} \to (\Pi_A B)^{\mathrm{I}} \qquad \text{over } X^{\mathrm{I}}.$$

Consider the following diagram resulting from combining (65) and (73), in which all solid arrows are those already introduced. The dotted arrow labelled $p$ is the evident composite.

$$\begin{tikzcd}
 & X^{\mathrm{I}} \arrow[r, "\epsilon"] & X \\
(\Pi_A B)^{\mathrm{I}} \arrow[ur] \arrow[r] \arrow[d] & \Pi_{A_\epsilon} B_\epsilon \arrow[r] \arrow[u] \arrow[d] & \Pi_A B \arrow[u] \arrow[dd, "="] \\
\Pi_{A^{\mathrm{I}}} B^{\mathrm{I}} \arrow[r] \arrow[drr, dotted, "p"] & \Pi_{A_\epsilon} a_*a^*B_\epsilon \arrow[d] & \\
(\Pi_A B)^{\mathrm{I}} \arrow[r] & \Pi_{A_\epsilon} B_\epsilon \arrow[r] & \Pi_A B
\end{tikzcd} \tag{75}$$

The lower horizontal composite is the evaluation of the pathobject $(\Pi_A B)^{\mathrm{I}}$ at the point $\delta : 1 \to \mathrm{I}$,

$$\epsilon_{\Pi_A B} = (\Pi_A B)^{\delta} : (\Pi_A B)^{\mathrm{I}} \longrightarrow (\Pi_A B)^{1} \cong \Pi_A B\,.$$

This is constructed from the (cartesian closed) evaluation,

$$\mathsf{eval} : \mathrm{I} \times (\Pi_A B)^{\mathrm{I}} \longrightarrow \Pi_A B$$

which is the counit of $\mathrm{I} \times (-) \dashv (-)^{\mathrm{I}}$, as the composite shown below.

$$\begin{tikzcd}
(\Pi_A B)^{\mathrm{I}} \arrow[r, "\epsilon_{\Pi_A B}"] \arrow[d, "\cong"'] & \Pi_A B \\
1 \times (\Pi_A B)^{\mathrm{I}} \arrow[r, "\delta \times (\Pi_A B)^{\mathrm{I}}"'] & \mathrm{I} \times (\Pi_A B)^{\mathrm{I}} \arrow[u, "\mathsf{eval}"']
\end{tikzcd} \tag{76}$$

Let us analyse this evaluation at $\delta$ further, in terms of the *locally* cartesian closed structure associated to the base changes along the section $\delta : 1 \to \mathrm{I}$ and retraction $\mathrm{I} \to 1$ in $\mathcal{E}$. Since $\mathsf{id} \cong \delta^*\mathrm{I}^* : \mathcal{E} \to \mathcal{E}/_{\mathrm{I}} \to \mathcal{E}$, the map $\epsilon_{\Pi_A B}$ can be rewritten as follows.

$$\begin{tikzcd}
(\Pi_A B)^{\mathrm{I}} \arrow[r, "\epsilon_{\Pi_A B}"] \arrow[d, "\cong"'] & \Pi_A B \arrow[d, "\cong"] \\
\delta^*\mathrm{I}^*((\Pi_A B)^{\mathrm{I}}) \arrow[r, "\delta^*\mathrm{I}^*\epsilon_{\Pi_A B}"] \arrow[d, "\cong"'] & \delta^*\mathrm{I}^*\Pi_A B \arrow[d, "="] \\
\delta^*\mathrm{I}^*\mathrm{I}_*\mathrm{I}^*\Pi_A B \arrow[r, "\delta^*\varepsilon"'] & \delta^*\mathrm{I}^*\Pi_A B
\end{tikzcd} \tag{77}$$

where the map $\delta^*\varepsilon$ across the bottom is the counit of the adjunction $\mathrm{I}^* \dashv \mathrm{I}_*$, taken at $\mathrm{I}^*\Pi_A B$, and then pulled back along $\delta : 1 \to \mathrm{I}$. Before taking the pullback, we therefore have the following iso over I between that counit $\varepsilon_{\mathrm{I}^*}$ and the image under $\mathrm{I}^*$ of the previously considered evaluation $\epsilon : (\Pi_A B)^{\mathrm{I}} \to \Pi_A B$ from (76).

$$\begin{tikzcd}
\mathrm{I}^*((\Pi_A B)^{\mathrm{I}}) \arrow[r, "I^*\epsilon"] \arrow[d, "\cong"'] & \mathrm{I}^*\Pi_A B \arrow[d, "="] \\
\mathrm{I}^*\mathrm{I}_*\mathrm{I}^*\Pi_A B \arrow[r, "\varepsilon_{\mathrm{I}^*}"'] & \mathrm{I}^*\Pi_A B\,.
\end{tikzcd} \tag{78}$$

Now let us apply $\mathrm{I}^*$ to (75) to get the map $\mathrm{I}^*p$ in the diagram below, which therefore factors (up to (78)) through the counit $\varepsilon_{\mathrm{I}^*}$ as $\varepsilon_{\mathrm{I}^*} \circ$

$\mathrm{I}^*(\widetilde{\mathrm{I}^*p})$, where $\widetilde{\mathrm{I}^*p}$ is the adjoint transpose of $\mathrm{I}^*p$, as shown.

$$\begin{tikzcd}
\mathrm{I}^*\Pi_{A^\mathrm{I}}B^\mathrm{I} \arrow[r] \arrow[d, "\mathrm{I}^*(\widetilde{\mathrm{I}^*p})"'] \arrow[rrd, dotted, "\mathrm{I}^*p"] & \mathrm{I}^*\Pi_{A_\epsilon}a_*a^*B_\epsilon \arrow[d] & \\
\mathrm{I}^*\mathrm{I}_*\mathrm{I}^*\Pi_A B \arrow[r] \arrow[rr, bend right, "\varepsilon_{\mathrm{I}^*}"'] & \mathrm{I}^*\Pi_{A_\epsilon}B_\epsilon \arrow[r] & \mathrm{I}^*\Pi_A B
\end{tikzcd} \tag{79}$$

We can therefore set

$$\psi := \widetilde{\mathrm{I}^*p}\,,$$

and we obtain $\epsilon \circ \psi = p$, from which it follows that the square in (79) commutes by the definition of $\Pi_{A_\epsilon}B_\epsilon$ as a pullback. The same square without $\mathrm{I}^*$ then also commutes by applying the retraction $\delta^*$.

We have now defined all the maps indicated below, the squares involving $\varphi$ and $\psi$ commute, and the composite of $\sigma$ and $\tau$ is the identity.

$$\begin{tikzcd}
& X^\mathrm{I} \arrow[r] & X \\
(\Pi_A B)^\mathrm{I} \arrow[r] \arrow[ur] \arrow[d, "\varphi"'] & \Pi_{A_\epsilon}B_\epsilon \arrow[r] \arrow[u] \arrow[d, "\sigma"] & \Pi_A B \arrow[u] \arrow[dd, "="] \\
\Pi_{A^\mathrm{I}}B^\mathrm{I} \arrow[r] \arrow[d, "\psi"'] \arrow[rrd, dotted, "p"] & \Pi_{A_\epsilon}p_*p^*B_\epsilon \arrow[d, "\tau"] & \\
(\Pi_A B)^\mathrm{I} \arrow[r] \arrow[rr, bend right, "\epsilon"'] & \Pi_{A_\epsilon}B_\epsilon \arrow[r] & \Pi_A B
\end{tikzcd} \tag{80}$$

To see that $\psi \circ \varphi = 1$, an easy chase through the diagram (80) shows that

$$\epsilon \circ \psi \circ \varphi = p \circ \varphi = \epsilon\,.$$

Thus by applying $\mathrm{I}^*$ and using (78) we have $\varepsilon_{\mathrm{I}^*} \circ \mathrm{I}^*(\psi \circ \varphi) = \varepsilon_{\mathrm{I}^*}$, and so $\psi \circ \varphi = \widetilde{\varepsilon_{\mathrm{I}^*}} = 1$.

□

From Proposition 65 we then have:

**Corollary 70** (Unbiased Frobenius)**.** *The unbiased fibration weak factorization system on* $\mathsf{cSet}$ *satisfies the Frobenius condition.*

**Corollary 71.** *Unbiased fibrations are closed under pushforward along unbiased fibrations. Thus given unbiased fibrations $X \twoheadrightarrow Z$ and $Y \twoheadrightarrow Z$ over any base $Z$, the relative exponential $Y^X = X_*X^*Y \to Z$, formed in the slice over $Z$, is again an unbiased fibration.*

*Remark* 72. We note in passing that the proof just given for the $\delta$-biased case of Frobenius, Proposition 69, made no use of the fact that $\delta : 1 \to \mathrm{I}$ is generic, nor even that we were working in the slice category over I. Indeed the same algebraic argument works for $p$-biased fibrations for any point $p : 1 \to \mathrm{I}$ of any object I, in any (quasi-)topos $\mathcal{E}$.

## 6. A universal fibration

We shall construct a *universal small fibration* $\dot{\mathcal{U}} \to \mathcal{U}$, which is a classifier for small fibrations. It will be shown in Section 8 that the base object $\mathcal{U}$ is fibrant, using the fact to be proved in Section 7 that the map $\dot{\mathcal{U}} \to \mathcal{U}$ itself is *univalent*, in a sense to be made precise.

Our construction of $\dot{\mathcal{U}} \to \mathcal{U}$ makes use, first of all, of a new description of the well-known Hofmann-Streicher universe in a category $\widehat{\mathbb{C}} = [\mathbb{C}^{\mathrm{op}}, \mathsf{Set}]$ of presheaves on a small category $\mathbb{C}$, which was used in [HS97] to interpret dependent type theory. See [Awo22] for further details.

### Classifying families.

**Definition 73** ([HS97])**.** Let $\mathbb{C}$ be a small category. A (type-theoretic) *universe* $(U, \mathsf{E}l)$ consists of $U \in \widehat{\mathbb{C}}$ and $\mathsf{E}l \in \widehat{\int_{\mathbb{C}} U}$ with:

$$U(c) \;=\; \mathsf{Cat}\big(\mathbb{C}/_c{}^{\mathrm{op}}, \mathsf{Set}\big) \tag{81}$$

$$\mathsf{E}l(c, A) \;=\; A(id_c) \tag{82}$$

with the evident associated action on morphisms.

A few comments are required:

- In contrast to [HS97], in (81) we take the underlying set of objects of the functor category $\widehat{\mathbb{C}/_c} = [\mathbb{C}/_c{}^{\mathrm{op}}, \mathsf{Set}]$.
- As in [HS97], (82) adopts the "categories with families" point of view in describing an arrow $E \to U$ in $\widehat{\mathbb{C}}$ equivalently as a presheaf on the category of elements $\int_{\mathbb{C}} U$, using
$$\widehat{\mathbb{C}}/_U \;\simeq\; \widehat{\textstyle\int_{\mathbb{C}} U} \tag{83}$$
where
$$E(c) \;=\; \textstyle\coprod_{A \in U(c)} \mathsf{E}l(c, A).$$
The argument $(c, A) \in \int_{\mathbb{C}} U$ in (82) thus consists of an object $c \in \mathbb{C}$ and an element $A \in U(c)$.
- To account for size issues, the authors of [HS97] assume a Grothendieck universe $u$ in $\mathsf{Set}$, the elements of which are called *small*. The category $\mathbb{C}$ is assumed to be small, as are the values of the presheaves, unless otherwise stated.

The presheaf $U$, which is not small, is then regarded as the Grothendieck universe $u$ "lifted" from $\mathsf{Set}$ to $[\mathbb{C}^{\mathrm{op}}, \mathsf{Set}]$. We first analyse this specification of $(U, \mathsf{E}l)$ from a different perspective, in order to establish its basic property as a classifier for small families in $\widehat{\mathbb{C}}$.

*A realization-nerve adjunction.* For a presheaf $X$ on $\mathbb{C}$, recall that the category of elements is the comma category,

$$\textstyle\int_{\mathbb{C}} X \;=\; \mathsf{y}_{\mathbb{C}}/_{X}\,,$$

where $\mathsf{y}_{\mathbb{C}} : \mathbb{C} \to \mathsf{Set}^{\mathbb{C}^{\mathrm{op}}}$ is the Yoneda embedding, which we sometimes supress and write simply $\mathbb{C}/_X$ for $\mathsf{y}_{\mathbb{C}}/_X$.

**Proposition 74** ([Gro83],§28)**.** *The category of elements functor*

$$\textstyle\int_{\mathbb{C}} : \widehat{\mathbb{C}} \longrightarrow \mathsf{Cat}$$

*has a right adjoint,*

$$\nu_{\mathbb{C}} : \mathsf{Cat} \longrightarrow \widehat{\mathbb{C}}\,.$$

*For a small category $\mathbb{A}$, we shall call the presheaf $\nu_{\mathbb{C}}(\mathbb{A})$ the* ($\mathbb{C}$-)nerve *of $\mathbb{A}$.*

*Proof.* The adjunction $\int_{\mathbb{C}} \dashv \nu_{\mathbb{C}}$ is an instance of the usual "realization/nerve" adjunction, here with respect to the covariant slice category functor $\mathbb{C}/- : \mathbb{C} \to \mathsf{Cat}$, as indicated below.

$$\begin{tikzcd}
\widehat{\mathbb{C}} \arrow[r, "\int_{\mathbb{C}}"'] & \mathsf{Cat} \arrow[l, bend right, "\nu_{\mathbb{C}}"'] \\
\mathbb{C} \arrow[u, hook, "\mathsf{y}"] \arrow[ur, "\mathbb{C}/_-"'] &
\end{tikzcd} \tag{84}$$

In detail, for $\mathbb{A} \in \mathsf{Cat}$ and $c \in \mathbb{C}$, let $\nu_{\mathbb{C}}(\mathbb{A})(c)$ be the Hom-set of functors,

$$\nu_{\mathbb{C}}(\mathbb{A})(c) = \mathsf{Cat}\big(\mathbb{C}/_c\,,\, \mathbb{A}\big)\,,$$

with contravariant action on $h : d \to c$ given by pre-composing a functor $P : \mathbb{C}/_c \to \mathbb{A}$ with the post-composition functor

$$\mathbb{C}/_h : \mathbb{C}/_d \longrightarrow \mathbb{C}/_c\,.$$

For the adjunction, observe that the slice category $\mathbb{C}/_c$ is the category of elements of the representable functor $\mathsf{y}c$,

$$\textstyle\int_{\mathbb{C}} \mathsf{y}c \;\cong\; \mathbb{C}/_c\,.$$

Thus for representables $\mathsf{y}c$, we have the required natural isomorphism

$$\widehat{\mathbb{C}}\big(\mathsf{y}c\,,\, \nu_{\mathbb{C}}(\mathbb{A})\big) \;\cong\; \nu_{\mathbb{C}}(\mathbb{A})(c) \;=\; \mathsf{Cat}\big(\mathbb{C}/_c\,,\, \mathbb{A}\big) \;\cong\; \mathsf{Cat}\big(\textstyle\int_{\mathbb{C}} \mathsf{y}c\,,\, \mathbb{A}\big)\,.$$

For arbitrary presheaves $X$, one uses the presentation of $X$ as a colimit of representables over the index category $\int_{\mathbb{C}} X$, and the easy to prove fact that $\int_{\mathbb{C}}$ itself preserves colimits. Indeed, for any category $\mathbb{D}$, we have an isomorphism in $\mathsf{Cat}$,

$$\varinjlim_{d\in\mathbb{D}} \mathbb{D}/_d \;\cong\; \mathbb{D}\,.$$

$\square$

When $\mathbb{C}$ is fixed, we may omit the subscript in the notation $\mathsf{y}_{\mathbb{C}}$ and $\int_{\mathbb{C}}$ and $\nu_{\mathbb{C}}$. The unit and counit maps of the adjunction $\int \dashv \nu$,

$$\eta : X \longrightarrow \nu\textstyle\int X\,,$$
$$\epsilon :\ \textstyle\int \nu\mathbb{A} \longrightarrow \mathbb{A}\,,$$

are then as follows. At $c \in \mathbb{C}$, for $x : \mathsf{y}c \to X$, the functor $(\eta_X)_c(x) : \mathbb{C}/_c \to \mathbb{C}/_X$ is just composition with $x$,

$$(\eta_X)_c(x) = \mathbb{C}/_x : \mathbb{C}/_c \longrightarrow \mathbb{C}/_X\,. \tag{85}$$

For $\mathbb{A} \in \mathsf{Cat}$, the functor $\epsilon : \int \nu\mathbb{A} \to \mathbb{A}$ takes a pair $(c \in \mathbb{C}, f : \mathbb{C}/_c \to \mathbb{A})$ to the object $f(1_c) \in \mathbb{A}$,

$$\epsilon(c, f) = f(1_c).$$

**Lemma 75.** *For any $f : Y \to X$, the naturality square below is a pullback.*

$$\begin{tikzcd} Y \arrow[r, "\eta_Y"] \arrow[d, "f"'] & \nu\int Y \arrow[d, "\nu\int f"] \\ X \arrow[r, "\eta_X"'] & \nu\int X. \end{tikzcd} \tag{86}$$

*Proof.* It suffices to prove this for the case $f : X \to 1$. Thus consider the square

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d] & \nu\int X \arrow[d] \\ 1 \arrow[r, "\eta_1"'] & \nu\int 1. \end{tikzcd} \tag{87}$$

Evaluating at $c \in \mathbb{C}$ and applying (85) gives the following square in $\mathsf{Set}$.

$$\begin{tikzcd} Xc \arrow[r, "\mathbb{C}/_{-}"] \arrow[d] & \mathsf{Cat}\big(\mathbb{C}/_c\,, \mathbb{C}/_X\big) \arrow[d] \\ 1c \arrow[r, "\mathbb{C}/_{-}"'] & \mathsf{Cat}\big(\mathbb{C}/_c\,, \mathbb{C}/_1\big) \end{tikzcd} \tag{88}$$

The image of $* \in 1c$ along the bottom is the forgetful functor $U_c : \mathbb{C}/_c \to \mathbb{C}$, and its fiber under the map on the right is the set of functors $F : \mathbb{C}/_c \to \mathbb{C}/_X$ such that $U_X \circ F = U_c$, where $U_X : \mathbb{C}/_X \to \mathbb{C}$ is also a forgetful functor. But any such $F$ is uniquely of the form $\mathbb{C}/_x$ for $x = F(1_c) : \mathsf{y}c \to X$. $\square$

*A universal family.* For the terminal presheaf $1 \in \widehat{\mathbb{C}}$ we have an iso $\int 1 \cong \mathbb{C}$, so for every $X \in \widehat{\mathbb{C}}$ there is a canonical projection $\int X \to \mathbb{C}$, which is a discrete fibration. It follows that for any map $Y \to X$ of presheaves, the associated map $\int Y \to \int X$ is also a discrete fibration. Ignoring size issues temporarily, recall that discrete fibrations in $\mathsf{Cat}$ are classified by the forgetful functor $\dot{\mathsf{Set}}^{\mathrm{op}} \to \mathsf{Set}^{\mathrm{op}}$ from (the opposites of) the category of pointed sets to that of sets (cf. [Web07]). For every presheaf $X \in \widehat{\mathbb{C}}$, we therefore have a pullback diagram in $\mathsf{Cat}$,

$$\begin{tikzcd} \int X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathsf{Set}}^{\mathrm{op}} \arrow[d] \\ \mathbb{C} \arrow[r, "X"'] & \mathsf{Set}^{\mathrm{op}}. \end{tikzcd} \tag{89}$$

Using $\mathbb{C} \cong \int 1$ and transposing by the adjunction $\int \dashv \nu$ then gives a commutative square in $\widehat{\mathbb{C}}$ of the form:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \nu\dot{\mathsf{Set}}^{\mathrm{op}} \arrow[d] \\ 1 \arrow[r, "\tilde{X}"'] & \nu\mathsf{Set}^{\mathrm{op}}. \end{tikzcd} \tag{90}$$

**Lemma 76.** *The square (90) is a pullback in $\widehat{\mathbb{C}}$. More generally, for any map $Y \to X$ in $\widehat{\mathbb{C}}$, there is a canonical pullback square*

$$\begin{tikzcd} Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \nu\dot{\mathsf{Set}}^{\mathrm{op}} \arrow[d] \\ X \arrow[r] & \nu\mathsf{Set}^{\mathrm{op}}. \end{tikzcd} \tag{91}$$

*Proof.* Apply the right adjoint $\nu$ to the pullback square (89) and paste the naturality square (86) from Lemma 75 on the left, to obtain the transposed square (91) as a pasting of two pullbacks. $\square$

Let us write $\dot{\mathcal{V}} \to \mathcal{V}$ for the vertical map on the right in (91), setting

$$\begin{aligned} \dot{\mathcal{V}} &:= \nu\dot{\mathsf{Set}}^{\mathrm{op}} \\ \mathcal{V} &:= \nu\mathsf{Set}^{\mathrm{op}}. \end{aligned} \tag{92}$$

We summarize our results so far as follows.

**Proposition 77.** *The nerve $\dot{\mathcal{V}} \to \mathcal{V}$ of the classifier for discrete fibrations $\dot{\mathsf{Set}}^{\mathrm{op}} \to \mathsf{Set}^{\mathrm{op}}$, as defined in (92), classifies natural transformations $Y \to X$ in $\widehat{\mathbb{C}}$, in the sense that there is always a pullback square,*

$$\begin{tikzcd}
Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathcal{V}} \arrow[d] \\
X \arrow[r, "\tilde{Y}"'] & \mathcal{V}.
\end{tikzcd} \tag{93}$$

*The classifying map $\tilde{Y} : X \to \mathcal{V}$ is determined by the adjunction $\int \dashv \nu$ as the transpose of the classifying map of the discrete fibration $\int Y \to \int X$.*

Given a natural transformation $Y \to X$, the classifying map $\tilde{Y} : X \to \mathcal{V}$ is of course not in general unique. Nonetheless, we can use the construction of $\dot{\mathcal{V}} \to \mathcal{V}$ as the nerve of the discrete fibration classifier $\dot{\mathsf{Set}}^{\mathrm{op}} \to \mathsf{Set}^{\mathrm{op}}$, for which classifying functors $\mathbb{C} \to \mathsf{Set}^{\mathrm{op}}$ are unique up to natural isomorphism, to infer the following proposition, which will be required below (cf. [Shu15, GSS22]).

**Proposition 78** (Realignment for families)**.** *Given a monomorphism $c : C \rightarrowtail X$ and a family $Y \to X$, let $y_c : C \to \mathcal{V}$ classify the pullback $c^*Y \to C$. Then there is a classifying map $y : X \to \mathcal{V}$ for $Y \to X$ with $y \circ c = y_c$.*

$$\begin{tikzcd}
c^*Y \arrow[rr] \arrow[dr] \arrow[dd] & & \dot{\mathcal{V}} \arrow[dd] \\
& Y \arrow[ur, dotted] & \\
C \arrow[rr, "y_c"] \arrow[dr, tail, "c"'] & & \mathcal{V} \\
& X \arrow[ur, dotted, "y"'] \arrow[from=uu, crossing over] &
\end{tikzcd} \tag{94}$$

*Proof.* Transposing the realignment problem (94) for presheaves across the adjunction $\int \dashv \nu$ results in the following realignment problem for discrete fibrations.

$$
\begin{tikzcd}
\int c^*Y \arrow[rr] \arrow[dr] \arrow[dd] & & \dot{\mathsf{Set}}^{\mathrm{op}} \arrow[dd] \\
& \int Y \arrow[ur, dotted] \arrow[dd] & \\
\int C \arrow[rr, "\widetilde{y_c}"] \arrow[dr, tail, "\int c"'] & & \mathsf{Set}^{\mathrm{op}} \\
& \int X \arrow[ur, dotted, "\tilde{y}"'] &
\end{tikzcd}
\tag{95}
$$

The category of elements functor $\int$ is easily seen to preserve pullbacks, hence monos; thus let us consider the general case of a functor $C : \mathbb{C} \rightarrowtail \mathbb{D}$ which is monic in $\mathsf{Cat}$, a pullback of discrete fibrations as on the left below, and a presheaf $E : \mathbb{C} \to \mathsf{Set}^{\mathrm{op}}$ with $\int E \cong \mathbb{E}$ over $\mathbb{C}$.

$$
\begin{tikzcd}
\mathbb{E} \arrow[rr] \arrow[dr] \arrow[dd] & & \dot{\mathsf{Set}}^{\mathrm{op}} \arrow[dd] \\
& \mathbb{F} \arrow[ur, dotted] \arrow[dd] & \\
\mathbb{C} \arrow[rr, "E"] \arrow[dr, tail, "C"'] & & \mathsf{Set}^{\mathrm{op}} \\
& \mathbb{D} \arrow[ur, dotted, "F"'] &
\end{tikzcd}
\tag{96}
$$

We seek $F : \mathbb{D} \to \mathsf{Set}^{\mathrm{op}}$ with $\int F \cong \mathbb{F}$ over $\mathbb{D}$ and $F \circ C = E$. Let $F_0 : \mathbb{D} \to \mathsf{Set}^{\mathrm{op}}$ with $\int F_0 \cong \mathbb{F}$ over $\mathbb{D}$, which exists since $\mathbb{F} \to \mathbb{D}$ is a discrete fibration. Since $F_0 \circ C$ and $E$ both classify $\mathbb{E}$, there is a natural iso $e : F_0 \circ C \cong E$. Consider the following diagram

$$
\begin{tikzcd}
\mathbb{C} \arrow[r, "e"] \arrow[d, tail, "C"'] & (\mathsf{Set}^{\cong})^{\mathrm{op}} \arrow[r, "p_2"'] \arrow[d, "p_1"] & \mathsf{Set}^{\mathrm{op}} \\
\mathbb{D} \arrow[r, "F_0"'] \arrow[ur, dotted, "f"'] & \mathsf{Set}^{\mathrm{op}} &
\end{tikzcd}
\tag{97}
$$

where $\mathsf{Set}^{\cong}$ is the category of isos in $\mathsf{Set}$, with $p_1, p_2$ the (opposites of the) domain and codomain projections. There is a well-known weak factorization system on $\mathsf{Cat}$ (part of the "canonical model structure") with injective-on-objects functors on the left and isofibrations on the right. Thus there is a diagonal filler $f$ as indicated. The functor $F := p_2 \circ f : \mathbb{D} \to \mathsf{Set}^{\mathrm{op}}$ is then the one we seek. $\square$

*Small maps.* Of course, as defined in (92), the classifier $\dot{\mathcal{V}} \to \mathcal{V}$ cannot be a map in $\widehat{\mathbb{C}}$, for reasons of size; we now address this. Let $\alpha$ be a cardinal number, and call the sets strictly smaller than it $\alpha$-*small*. Let $\mathsf{Set}_\alpha \hookrightarrow \mathsf{Set}$ be the full subcategory of $\alpha$-small sets. Call a presheaf $X : \mathbb{C}^{\mathrm{op}} \to \mathsf{Set}$ $\alpha$-small if all of its values are $\alpha$-small sets, and thus if, and only if, it factors through $\mathsf{Set}_\alpha \hookrightarrow \mathsf{Set}$. Call a map $f : Y \to X$ of presheaves $\alpha$-small if all of the fibers $f_c^{-1}\{x\} \subseteq Yc$ are $\alpha$-small sets (for all $c \in \mathbb{C}$ and $x \in Xc$). The latter condition is of course equivalent to saying that, in the pullback square over the element $x : \mathsf{y}c \to X$,

$$\begin{tikzcd} Y_x \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "f"] \\ \mathsf{y}c \arrow[r, "x"'] & X, \end{tikzcd} \tag{98}$$

the presheaf $Y_x$ is $\alpha$-small.

Now let us restrict the specification (92) of $\dot{\mathcal{V}} \to \mathcal{V}$ to the $\alpha$-small sets:

$$\begin{aligned} \dot{\mathcal{V}}_\alpha &:= \nu\dot{\mathsf{Set}}_\alpha^{\mathsf{op}} \\ \mathcal{V}_\alpha &:= \nu\mathsf{Set}_\alpha^{\mathsf{op}}. \end{aligned} \tag{99}$$

Then the evident forgetful map $\dot{\mathcal{V}}_\alpha \to \mathcal{V}_\alpha$ *is* a map in the category $\widehat{\mathbb{C}}$ of presheaves, and it is in fact $\alpha$-small. Moreover, it has the following basic property, which is just a restriction of the basic property of $\dot{\mathcal{V}} \to \mathcal{V}$ stated in Proposition 77.

**Proposition 79.** *The map $\dot{\mathcal{V}}_\alpha \to \mathcal{V}_\alpha$ classifies $\alpha$-small maps $f : Y \to X$ in $\widehat{\mathbb{C}}$, in the sense that there is always a pullback square,*

$$\begin{tikzcd} Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathcal{V}}_\alpha \arrow[d] \\ X \arrow[r, "\tilde{Y}"'] & \mathcal{V}_\alpha. \end{tikzcd} \tag{100}$$

*The classifying map $\tilde{Y} : X \to \mathcal{V}_\alpha$ is determined by the adjunction $\int \dashv \nu$ as (the factorization of) the transpose of the classifiyng map of the discrete fibration $\int X \to \int Y$.*

*Proof.* If $Y \to X$ is $\alpha$-small, its classifying map $\tilde{Y} : X \to \mathcal{V}$ factors through $\mathcal{V}_\alpha \hookrightarrow \mathcal{V}$, as indicated below,

$$\begin{tikzcd}
Y \arrow[r] \arrow[d] \arrow[rr, bend left] & \nu\dot{\mathsf{Set}}_\alpha{}^{\mathrm{op}} \arrow[r, hook] \arrow[d] & \nu\dot{\mathsf{Set}}^{\mathrm{op}} \arrow[d] \\
X \arrow[r] \arrow[rr, bend right, "\tilde{Y}"'] & \nu\mathsf{Set}_\alpha{}^{\mathrm{op}} \arrow[r, hook] & \nu\mathsf{Set}^{\mathrm{op}},
\end{tikzcd} \tag{101}$$

in virtue of the following adjoint transposition,

$$\begin{tikzcd}
\int Y \arrow[r] \arrow[d] \arrow[rr, bend left] & \dot{\mathsf{Set}}_\alpha{}^{\mathrm{op}} \arrow[r, hook] \arrow[d] & \dot{\mathsf{Set}}^{\mathrm{op}} \arrow[d] \\
\int X \arrow[r] \arrow[rr, bend right] & \mathsf{Set}_\alpha{}^{\mathrm{op}} \arrow[r, hook] & \mathsf{Set}^{\mathrm{op}}.
\end{tikzcd} \tag{102}$$

Note that the square on the right is evidently a pullback, and so the one on the left is, too, because the outer rectangle is the classifying pullback of the discrete fibration $\int Y \to \int X$, as stated. Thus the left square in (101) is also a pullback. $\square$

*Examples of universal families* $\dot{\mathcal{V}}_\alpha \longrightarrow \mathcal{V}_\alpha$.

(1) Let $\alpha = \kappa$ a strongly inaccessible cadinal, so that $\mathsf{ob}(\mathsf{Set}_\kappa)$ is a Grothendieck universe. Then the Hofmann-Streicher universe of Definition 73 is recovered as the $\kappa$-small map classifier

$$E \cong \dot{\mathcal{V}}_\kappa \longrightarrow \mathcal{V}_\kappa \cong U$$

in the sense of Proposition 79. Indeed, for $c \in \mathbb{C}$, we have

$$\mathcal{V}_\kappa c \;=\; \nu(\mathsf{Set}_\kappa^{\mathsf{op}})(c) = \mathsf{Cat}\big(\mathbb{C}/_c\,,\ \mathsf{Set}_\kappa^{\mathsf{op}}\big) \;=\; \mathsf{ob}(\widehat{\mathbb{C}/_c}) \;=\; Uc\,. \tag{103}$$

For $\dot{\mathcal{V}}_\kappa$ we then have,

$$\begin{aligned}
\dot{\mathcal{V}}_\kappa c \;=\; \nu(\,\dot{\mathsf{Set}}_\kappa^{\mathsf{op}})(c) \;&=\; \mathsf{Cat}\big(\mathbb{C}/_c\,,\ \dot{\mathsf{Set}}_\kappa^{\mathsf{op}}\big) \\
&\cong\; \textstyle\coprod_{A \in \mathcal{V}_\kappa c} \mathsf{Cat}_{\mathbb{C}/_c}\big(\mathbb{C}/_c\,,\ A^*\mathsf{Set}_\kappa^{\mathsf{op}}\big)
\end{aligned} \tag{104}$$

where the $A$-summand in (104) is defined by taking sections of the pullback indicated below.

$$\begin{tikzcd}
A^*\mathsf{Set}_\kappa^{\mathsf{op}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathsf{Set}}_\kappa^{\mathsf{op}} \arrow[d] \\
\mathbb{C}/_c \arrow[r, "A"'] \arrow[u, dotted, bend left] \arrow[ur, dotted] & \mathsf{Set}_\kappa^{\mathsf{op}}
\end{tikzcd} \tag{105}$$

But $A^*\mathsf{Set}_\kappa^{\mathsf{op}} \cong \int_{\mathbb{C}/_c} A$ over $\mathbb{C}/_c$, and sections of this discrete fibration in $\mathsf{Cat}$ correspond uniquely to natural maps $1 \to A$ in $\widehat{\mathbb{C}/_c}$. Since 1 is representable in $\widehat{\mathbb{C}/_c}$ we can continue (104) by

$$\begin{aligned}
\dot{\mathcal{V}}_\kappa c &\cong \textstyle\coprod_{A \in \mathcal{V}_\kappa c} \mathsf{Cat}_{\mathbb{C}/_c}\big(\mathbb{C}/_c\,,\, A^*\mathsf{Set}_\kappa^{\mathsf{op}}\big) \\
&\cong \textstyle\coprod_{A \in \mathcal{V}_\kappa c} \widehat{\mathbb{C}/_c}(1, A) \\
&\cong \textstyle\coprod_{A \in \mathcal{V}_\kappa c} A(1_c) \\
&= \textstyle\coprod_{A \in \mathcal{V}_\kappa c} \mathsf{E}l(\langle c, A\rangle) \\
&= Ec\,.
\end{aligned}$$

(2) By functoriality of the nerve $\nu : \mathsf{Cat} \to \widehat{\mathbb{C}}$, a sequence of Grothendieck universes

$$\mathsf{Set}_\alpha \subseteq \mathsf{Set}_\beta \subseteq \ldots$$

in $\mathsf{Set}$ gives rise to a (cumulative) sequence of type-theoretic universes

$$\mathcal{V}_\alpha \rightarrowtail \mathcal{V}_\beta \rightarrowtail \ldots$$

in $\widehat{\mathbb{C}}$. More precisely, there is a sequence of cartesian squares,

$$\begin{tikzcd}
\dot{\mathcal{V}}_\alpha \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathcal{V}}_\beta \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \ldots \\
\mathcal{V}_\alpha \arrow[r, tail] & \mathcal{V}_\beta \arrow[r, tail] & \ldots\,,
\end{tikzcd} \tag{106}$$

in the image of $\nu : \mathsf{Cat} \longrightarrow \widehat{\mathbb{C}}$, classifying small maps in $\widehat{\mathbb{C}}$ of increasing size, in the sense of Proposition 79.

(3) Let $\alpha = 2$ so that $1 \to 2$ is the subobject classifier of $\mathsf{Set}$, and

$$\mathbb{1} = \dot{\mathsf{Set}}_2^{\mathsf{op}} \longrightarrow \mathsf{Set}_2^{\mathsf{op}} = \mathbb{2}$$

is then a classifier in $\mathsf{Cat}$ for *sieves*, i.e. full subcategories $\mathbb{S} \hookrightarrow \mathbb{A}$ closed under the domains of arrows $a \to s$ for $s \in \mathbb{S}$. The nerve

$\dot{\mathcal{V}}_2 \to \mathcal{V}_2$ is then the usual subobject classifier $1 \to \Omega$ of $\widehat{\mathbb{C}}$,

$$\begin{tikzcd} \dot{\mathcal{V}}_2 \arrow[r, equal] \arrow[d] & \nu\mathbb{1} \arrow[r, "\sim"] \arrow[d] & 1 \arrow[d] \\ \mathcal{V}_2 \arrow[r, equal] & \nu\mathbb{2} \arrow[r, "\sim"] & \Omega \end{tikzcd} \tag{107}$$

(4) For any $X \in \widehat{\mathbb{C}}$, we have an equivalence

$$\widehat{\mathbb{C}}/_X \;\simeq\; \widehat{\textstyle\int_{\mathbb{C}} X} \;\simeq\; \mathsf{dFib}/_{\int_{\mathbb{C}} X}$$

where, generally, $\mathsf{dFib}/_{\mathbb{D}}$ is the category of discrete fibrations over a category $\mathbb{D}$. This equivalence commutes with composition along discrete fibrations, in the sense that the forgetful functor

$$X_! : \widehat{\mathbb{C}}/_X \to \widehat{\mathbb{C}}$$

given by composition along $X \to 1$ agrees (up to canonical isomorphism) with the base change $(p_X)_! \dashv (p_X)^*$ of presheaves along the projection $p_X : \int_{\mathbb{C}} X \to \mathbb{C}$, and with composition along the discrete fibration $p_X$, as indicated in:

$$\begin{tikzcd} \widehat{\mathbb{C}}/_X \arrow[r, "\sim"] \arrow[d, "X_!"'] & \widehat{\int_{\mathbb{C}} X} \arrow[r, "\sim"] \arrow[d, "(p_X)_!"'] & \mathsf{dFib}/_{\int_{\mathbb{C}} X} \arrow[d, "p_X \circ (-)"] \\ \widehat{\mathbb{C}} \arrow[r, "\sim"'] & \widehat{\mathbb{C}} \arrow[r, "\sim"'] & \mathsf{dFib}/_{\mathbb{C}}. \end{tikzcd} \tag{108}$$

It follows that the pullback functor $X^* : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}/_X$ commutes with the corresponding right adjoints (one of which is the nerve), and therefore preserves the respective universes,

$$X^*\mathcal{V}_{\mathbb{C}} \;\cong\; (p_X)^*\nu_{\mathbb{C}}(\mathsf{Set}^{\mathsf{op}}) \;\cong\; \nu_{\int_{\mathbb{C}} X}(\mathsf{Set}^{\mathsf{op}}) \;\cong\; \mathcal{V}_{\int_{\mathbb{C}} X}\,.$$

**Corollary 80.** *Let $\dot{\mathcal{V}}_\alpha \to \mathcal{V}_\alpha$ classify $\alpha$-small maps in $\widehat{\mathbb{C}}$, as in Proposition 79. Then for any $X \in \mathbb{C}$, the pullback $X^*\dot{\mathcal{V}}_\alpha \to X^*\mathcal{V}_\alpha$ classifies $\alpha$-small maps in $\widehat{\mathbb{C}}/_X$*

**Classifying trivial fibrations.** Returning now to the presheaf category $\mathsf{cSet} = \mathsf{Set}^{\square^{\mathrm{op}}}$ of cubical sets, recall from section 2 that (uniform) trivial fibration structures on a map $A \to X$ correspond bijectively to relative $+$-algebra structures over $X$ (definition 11). A relative $+$-algebra structure on $A \to X$ is an algebra structure for the pointed endofunctor $+_X : \mathsf{cSet}/X \to \mathsf{cSet}/X$, where recall from (2),

$$A^+ = \sum_{\varphi:\Phi} A^{[\varphi]} \quad \text{over } X.$$

A $+$-algebra structure is then a retract $\alpha : A^+ \to A$ over $X$ of the canonical map $\eta_A : A \to A^+$,

$$
\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[rr, bend left, "="] \arrow[dr] & A^+ \arrow[r, "\alpha"] \arrow[d] & A \arrow[dl] \\
& X. &
\end{tikzcd}
\tag{109}
$$

In more detail, let us write $A \to X$ as a family $(A_x)_{x \in X}$, so that $A = \sum_{x:X} A_x \to X$. Since the $+$-functor acts fiberwise, the object $A^+$ in (109) is then the indexing projection

$$\sum_{x:X} A_x^+ \to X.$$

Working in the slice $\mathsf{cSet}/X$, the (relative) exponentials (internal Hom's) $[A^+, A]$ and $[A, A]$ and the "precomposition by $\eta_A$" map $[\eta_A, A]$, fit into the following pullback diagram

$$
\begin{tikzcd}
+\mathsf{Alg}(A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & {[A^+, A]} \arrow[d, "{[\eta_A, A]}"] \\
1 \arrow[r, "'id_A'"'] & {[A, A].}
\end{tikzcd}
\tag{110}
$$

The constructed object $+\mathsf{Alg}(A) \to X$ over $X$ is then the *object of $+$-algebra structures on* $A \to X$, in the sense that sections $X \to +\mathsf{Alg}(A)$ correspond uniquely to $+$-algebra structures on $A \to X$. Moreover, $+\mathsf{Alg}(A) \to X$ is stable under pullback, in the sense that for any $f : Y \to X$, we have two pullback squares,

$$
\begin{tikzcd}
f^*A \arrow[r] \arrow[d] & A \arrow[d] \\
Y \arrow[r, "f"'] & X \\
+\mathsf{Alg}(f^*A) \arrow[r] \arrow[u] & +\mathsf{Alg}(A) \arrow[u]
\end{tikzcd}
\tag{111}
$$

because the $+$-functor, exponentials and pullbacks occurring in the construction of $+\mathsf{Alg}(A) \to X$ are themselves all stable.

It then follows from Proposition 79 that, if $A \to X$ is small, then $+\mathsf{Alg}(A) \to X$ is itself a pullback of the analogous object $+\mathsf{Alg}(\dot{\mathcal{V}}) \to \mathcal{V}$ constructed from the universal small family $\dot{\mathcal{V}} \to \mathcal{V}$ of Proposition 79, so there are two pullback squares:

$$
\begin{tikzcd}
A \arrow[r] \arrow[d] & \dot{\mathcal{V}} \arrow[d] \\
X \arrow[r, "\chi_A"'] & \mathcal{V} \\
+\mathsf{Alg}(A) \arrow[r] \arrow[u] & +\mathsf{Alg}(\dot{\mathcal{V}}) \arrow[u]
\end{tikzcd}
\tag{112}
$$

**Proposition 81.** *There is a* universal small trivial fibration

$$\mathsf{T\dot{F}ib} \to \mathsf{TFib}.$$

*Every small trivial fibration $A \to X$ is a pullback of $\mathsf{T\dot{F}ib} \to \mathsf{TFib}$ along a canonically determined classifying map $X \to \mathsf{TFib}$.*

$$
\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{T\dot{F}ib} \arrow[d] \\
X \arrow[r] & \mathsf{TFib}
\end{tikzcd}
\tag{113}
$$

*Proof.* We can take

$$\mathsf{TFib} := +\mathsf{Alg}(\dot{\mathcal{V}}),$$

which comes with its projection $+\mathsf{Alg}(\dot{\mathcal{V}}) \to \mathcal{V}$ as in diagram (112). Now define $p_t : \mathsf{T\dot{F}ib} \to \mathsf{TFib}$ by pulling back the universal small family,

$$
\begin{tikzcd}
\mathsf{T\dot{F}ib} \arrow[r] \arrow[d, "p_t"'] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathcal{V}} \arrow[d, "p"] \\
\mathsf{TFib} \arrow[r] & \mathcal{V}.
\end{tikzcd}
$$

Consider the following diagram, in which all the squares (including the distorted ones) are pullbacks, with the outer one coming from proposition 79 and the lower one from (112).

$$
\begin{tikzcd}
& A \arrow[rr, "q_A"] \arrow[dr, dotted] \arrow[ddd] & & \dot{\mathcal{V}} \arrow[ddd, "p"] \\
& & \mathsf{T\dot{F}ib} \arrow[ur] \arrow[d, "p_t"'] & \\
\mathsf{TFib}(A) \arrow[rr, crossing over] \arrow[dr] & & \mathsf{TFib} \arrow[dr] & \\
& X \arrow[rr, "\chi_A"'] \arrow[ur, dotted, "\alpha'"] \arrow[ul, dotted, bend left, "\alpha"] & & \mathcal{V}.
\end{tikzcd}
\tag{114}
$$

A trivial fibration structure $\alpha$ on $A \to X$ is a section the object of $+$-algebra structures on $A$, occurring in the diagram as

$$\mathsf{TFib}(A) := +\mathsf{Alg}(A),$$

the pullback of $\mathsf{TFib} = +\mathsf{Alg}(\dot{\mathcal{V}})$ along the classifying map $\chi_A : X \to \mathcal{V}$ for the small family $A \to X$. Such sections correspond uniquely to factorizations $\alpha'$ of $\chi_A$ as indicated, which in turn induce pullback squares of the required kind (113).

Note that the map $p_t : \dot{\mathsf{TFib}} \to \mathsf{TFib}$ has a canonical trivial fibration structure. Indeed, consider the following diagram, in which both squares are pullbacks.

$$\begin{tikzcd}
\dot{\mathsf{TFib}} \arrow[r] \arrow[d, "p_t"'] & \dot{\mathcal{V}} \arrow[d] \\
\mathsf{TFib} \arrow[r] & \mathcal{V} \\
\mathsf{TFib}(\dot{\mathsf{TFib}}) \arrow[r] \arrow[u] & \mathsf{TFib}(\dot{\mathcal{V}}) \arrow[u]
\end{tikzcd} \tag{115}$$

$\mathsf{TFib}(\dot{\mathcal{V}})$ is the object of trivial fibration structures on $\dot{\mathcal{V}} \to \mathcal{V}$, and its pullback $\mathsf{TFib}(\dot{\mathsf{TFib}})$ is therefore the object of trivial fibration structures on $p_t : \dot{\mathsf{TFib}} \to \mathsf{TFib}$. Thus we seek a section of $\mathsf{TFib}(\dot{\mathsf{TFib}}) \to \mathsf{TFib}$. But recall that $\mathsf{TFib} = \mathsf{TFib}(\dot{\mathcal{V}})$ by definition, so the lower pullback square is the pullback of $\mathsf{TFib}(\dot{\mathcal{V}}) \to \mathcal{V}$ against itself, which does indeed have a distinguished section, namely the diagonal

$$\Delta : \mathsf{TFib}(\dot{\mathcal{V}}) \to \mathsf{TFib}(\dot{\mathcal{V}}) \times_{\mathcal{V}} \mathsf{TFib}(\dot{\mathcal{V}}).$$

□

We record the following notation and corresponding fact from the foregoing proof for future reference:

**Lemma 82.** *The classifying type* $\mathsf{TFib}(A) := +\mathsf{Alg}(A) \to X$ *for trivial fibration structures on a map* $A \to X$ *is stable under pullback, in the sense that for any* $f : Y \to X$*, we have two pullback squares,*

$$\begin{tikzcd}
f^*A \arrow[r] \arrow[d] & A \arrow[d] \\
Y \arrow[r, "f"'] & X \\
\mathsf{TFib}(f^*A) \arrow[r] \arrow[u] & \mathsf{TFib}(A) \arrow[u]
\end{tikzcd} \tag{116}$$

Since the universal small trivial fibration $\dot{\mathsf{TFib}} \to \mathsf{TFib}$ in $\mathsf{cSet}$ from Proposition 81 was constructed as $\mathsf{TFib} = \mathsf{TFib}(\dot{\mathcal{V}})$ for the universal small family $\dot{\mathcal{V}} \to \mathcal{V}$, which in turn is stable under pullback by Corollary 80, we also have:

**Corollary 83.** *The base change of the universal small trivial fibration*

$$\dot{\mathsf{TFib}} \to \mathsf{TFib}$$

*in* $\mathsf{cSet}$ *along* $\mathrm{I}^* : \mathsf{cSet} \to \mathsf{cSet}/_{\mathrm{I}}$ *is a universal small trivial fibration in* $\mathsf{cSet}/_{\mathrm{I}}$.

**Classifying fibrations.** In order to classify fibrations $A \twoheadrightarrow X$, we shall proceed as for trivial fibrations by constructing, for any map $A \to X$, an object $\mathsf{Fib}(A) \to X$ of fibration structures which, moreover, is stable under pullback. We then apply the construction to the universal small family $\dot{\mathcal{V}} \to \mathcal{V}$ of Proposition 79 to obtain a universal small fibration. Here we will of course need to distinguish between biased and unbiased fibrations. In Lemma 84, we first construct a stable classifying type $\mathsf{Fib}(A) \to X$ for $\delta$-biased fibration structures on any map $A \to X$ in $\mathsf{cSet}/_{\mathrm{I}}$ where $\delta$ is the generic point. In Lemma 87 we then transfer the construction along the base change $\mathrm{I}^* : \mathsf{cSet} \to \mathsf{cSet}/_{\mathrm{I}}$ to obtain a classifier $\mathsf{Fib}(A) \to X$ for unbiased fibration structures on any $A \to X$ in $\mathsf{cSet}$.

The construction of $\mathsf{Fib}(A) \to X$ for biased fibration structures with respect to a point $\delta : 1 \to \mathrm{I}$ is already a bit more involved than was that of $\mathsf{TFib}(A) \to X$. In particular, it requires the codomain I of $\delta$ to be *tiny*, which is indeed the case for the generic point $\delta : 1 \to \mathrm{I}^*\mathrm{I}$ in $\mathsf{cSet}/_{\mathrm{I}}$ by Lemma 5.

*The classifying type of biased fibration structures.* A classifying type $\mathsf{Fib}(A) \to X$ of (uniform, $\delta$-biased) fibration structures on a map $p : A \to X$, as defined in Section 3, can be constructed as follows.

(1) First form the pullback-hom $\delta \Rightarrow p : A^{\mathrm{I}} \to X^{\mathrm{I}} \times_X A$ with the point $\delta : 1 \to \mathrm{I}$, as indicated in the following diagram.

$$\begin{tikzcd}
A^{\mathrm{I}} \arrow[drr, bend left, "A^{\delta}"] \arrow[dr, dotted, "\delta\Rightarrow p"] \arrow[ddr, bend right, "(p_A)^{\mathrm{I}}"'] & & \\
& X^{\mathrm{I}} \times_X A \arrow[r] \arrow[d] & A \arrow[d, "p"] \\
& X^{\mathrm{I}} \arrow[r, "X^{\delta}"'] & X
\end{tikzcd} \tag{117}$$

(2) A fibration structure on $p : A \to X$ is then a relative $+$-algebra structure on $\delta \Rightarrow p$ in the slice category over its codomain $X^{\mathrm{I}} \times_X A$. To construct a classifier for such structures, let us first relabel the objects and arrows in diagram (117) as follows:

$$\begin{aligned} \epsilon &:= X^{\delta} : X^{\mathrm{I}} \to X \\ A_\epsilon &:= X^{\mathrm{I}} \times_X A \\ \epsilon_A &:= \delta \Rightarrow p \end{aligned}$$

so that the working part of (117) becomes:

$$\begin{tikzcd} A^{\mathrm{I}} \arrow[dr, "\epsilon_A"'] & & \\ & A_\epsilon \arrow[r] \arrow[d, "p_\epsilon"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "p"] \\ & X^{\mathrm{I}} \arrow[r, "\epsilon"'] & X \end{tikzcd} \tag{118}$$

(3) Now a relative $+$-algebra structure on $\epsilon_A$ (Definition 11) is a retract $\alpha$ over $A_\epsilon$ of the unit $\eta$, as indicated below, where $D$ is simply the domain of the map $(\epsilon_A)^+$ resulting from applying the relative $+$-functor in the slice category over $A_\epsilon$ to the object $\epsilon_A$.

$$\begin{tikzcd} A^{\mathrm{I}} \arrow[r, "\eta"'] \arrow[dr, "\epsilon_A"'] & D \arrow[l, bend right, dotted, "\alpha"'] \arrow[d, "(\epsilon_A)^+"] & \\ & A_\epsilon \arrow[r] \arrow[d, "p_\epsilon"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "p"] \\ & X^{\mathrm{I}} \arrow[r, "\epsilon"'] & X \end{tikzcd} \tag{119}$$

(4) As in the construction (110), there is an object $\mathsf{TFib}(\epsilon_A) = +\mathsf{Alg}(\epsilon_A)$ over $A_\epsilon$ of relative +-algebra structures on $\epsilon_A$, the sections of which correspond uniquely to relative +-algebra structures on $\epsilon_A$ (and thus to fibration structures on $A$).

$$\begin{tikzcd}
 & A^{\mathrm{I}} \arrow[r, "\eta"'] \arrow[d, "\epsilon_A"'] & D \arrow[l, dotted, bend right, "\alpha"'] \arrow[dl, "(\epsilon_A)^+"] \\
\mathsf{TFib}(\epsilon_A) \arrow[r] & A_\epsilon \arrow[r] \arrow[d, "p_\epsilon"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "p"] \\
 & X^{\mathrm{I}} \arrow[r, "\epsilon"'] & X
\end{tikzcd} \tag{120}$$

(5) Sections of $\mathsf{TFib}(\epsilon_A) \longrightarrow A_\epsilon$ then correspond to sections of its push-forward along $p_\epsilon$, which we shall call $F_A$:

$$F_A := (p_\epsilon)_* \mathsf{TFib}(\epsilon_A) \,.$$

$$\begin{tikzcd}
 & A^{\mathrm{I}} \arrow[r, "\eta"'] \arrow[d, "\epsilon_A"'] & D \arrow[l, dotted, bend right, "\alpha"'] \arrow[dl, "(\epsilon_A)^+"] \\
\mathsf{TFib}(\epsilon_A) \arrow[r] & A_\epsilon \arrow[r] \arrow[d, "p_\epsilon"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "p"] \\
F_A \arrow[r] & X^{\mathrm{I}} \arrow[r, "\epsilon"'] & X
\end{tikzcd} \tag{121}$$

(6) One might now try taking another pushforward of $F_A \to X^{\mathrm{I}}$ along $\epsilon : X^{\mathrm{I}} \to X$ to get the object $\mathsf{Fib}(A) \to X$ that we seek, but unfortunately, this would not be stable under pullback along arbitrary maps $Y \to X$, because the evaluation $\epsilon = X^\delta : X^{\mathrm{I}} \to X$ is not stable in that way. Instead we use the *root* functor, i.e. the right adjoint of the pathspace, $(-)^{\mathrm{I}} \dashv (-)_{\mathrm{I}}$ (Proposition 4).

Let $f : F_A \to X^{\mathrm{I}}$ be the map $(p_\epsilon)_* \mathsf{TFib}(\epsilon_A)$ indicated in (121), and let $\eta : X \to (X^{\mathrm{I}})_{\mathrm{I}}$ be the unit of the root adjunction at $X$. Then define $\mathsf{Fib}(A) \to X$ by

$$\mathsf{Fib}(A) := \eta^* f_{\mathrm{I}}$$

as indicated in the following pullback diagram.

$$\begin{tikzcd}
\mathsf{Fib}(A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (F_A)_{\mathrm{I}} \arrow[d, "f_{\mathrm{I}}"] \\
X \arrow[r, "\eta"'] & (X^{\mathrm{I}})_{\mathrm{I}}
\end{tikzcd} \tag{122}$$

By adjointness, sections of $\mathsf{Fib}(A) \to X$ then correspond bijectively to sections of $f : F_A \to X^{\mathrm{I}}$.

**Lemma 84.** *For any map $A \to X$ in $\mathsf{cSet}/_{\mathrm{I}}$, the map $\mathsf{Fib}(A) \to X$ in* (122) *is a* classifying type for $\delta$-biased fibration structures*: sections of $\mathsf{Fib}(A) \to X$ correspond bijectively to $\delta$-biased fibration structures on $A \to X$, and the construction is stable under pullback in the sense that for any $f : Y \to X$, we have two pullback squares,*

$$\begin{tikzcd}
f^*A \arrow[r] \arrow[d] & A \arrow[d] \\
Y \arrow[r, "f"'] & X \\
\mathsf{Fib}(f^*A) \arrow[r] \arrow[u] & \mathsf{Fib}(A) \arrow[u]
\end{tikzcd} \tag{123}$$

*Proof.* It is clear from the construction that fibration structures on $A \to X$ correspond bijectively to sections of $\mathsf{Fib}(A) \to X$. We show that $\mathsf{Fib}(A) \to X$ is also stable under pullback. To that end, the relevant steps of the construction are recalled schematically below.

$$\begin{tikzcd}
 & A^{\mathrm{I}} \arrow[d, "\epsilon_A"'] & & \\
\mathsf{TFib}(\epsilon_A) \arrow[r] & A_\epsilon \arrow[r] \arrow[d, "p_\epsilon"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "p"] & \\
F_A \arrow[r] & X^{\mathrm{I}} \arrow[r, "\epsilon"'] & X & \mathsf{Fib}(A) \arrow[l]
\end{tikzcd} \tag{124}$$

Now consider the following diagram, in which the right hand side consists of the data from (124), and the front, central square is a pullback.

$$
\begin{tikzcd}
 & B^{\mathrm{I}} \arrow[rrr] \arrow[d, "\epsilon_B"'] & & & A^{\mathrm{I}} \arrow[d, "\epsilon_A"] & \\
\mathsf{TFib}(\epsilon_B) \arrow[r] & B_\epsilon \arrow[rrr, dotted] \arrow[d] \arrow[dr] & & & A_\epsilon \arrow[d] \arrow[dl] & \mathsf{TFib}(\epsilon_A) \arrow[l] \\
F_B \arrow[r] & Y^{\mathrm{I}} \arrow[rrr, dotted, bend right=10] \arrow[dr] & B \arrow[r] \arrow[d] & A \arrow[d] & X^{\mathrm{I}} \arrow[dl] & F_A \arrow[l] \\
 & \mathsf{Fib}(B) \arrow[r] & Y \arrow[r, "f"'] & X & \mathsf{Fib}(A) \arrow[l] &
\end{tikzcd}
\tag{125}
$$

As in the proof of Lemma 66, on the left side we repeat the construction with $B \to Y$ in place of $A \to X$. The left face of the indicated (distorted) cube is then also a pullback, whence the back (dotted) face is a pullback, since the two-story square in back is the image of the front pullback square under the right adjoint $(-)^{\mathrm{I}}$. Finally, the top rectangle in the back is therefore also a pullback.

It follows that $\mathsf{TFib}(\epsilon_B)$ is a pullback of $\mathsf{TFib}(\epsilon_A)$ along the upper dotted arrow, as in Lemma 82, and so the pushforward $F_B$ is a pullback of the corresponding $F_A$, along the lower dotted arrow (which is $f^{\mathrm{I}}$), by the Beck-Chevalley condition for the dotted pullback square. Let us record this for later reference:

$$
F_B \cong (f^{\mathrm{I}})^* F_A. \tag{126}
$$

It remains to show that $\mathsf{Fib}(B)$ is a pullback of $\mathsf{Fib}(A)$ along $f : Y \to X$, and now it is good that we did not take these to be pushforwards of $F_B$ and $F_A$, because the floor of the cube need not be a pullback, and so the Beck-Chavalley condition would not apply. Instead, consider the following diagram.

$$\begin{tikzcd}
& \mathsf{Fib}(B) \arrow[r] \arrow[dl] \arrow[ddd] & \mathsf{Fib}(A) \arrow[dr] \arrow[ddd] & \\
(F_B)_{\mathrm{I}} \arrow[rrr] \arrow[dddd] & & & (F_A)_{\mathrm{I}} \arrow[dddd] \\
&&& \\
& Y \arrow[r, "f"'] \arrow[dl, "\eta"'] & X \arrow[dr, "\eta"] & \\
(Y^{\mathrm{I}})_{\mathrm{I}} \arrow[rrr, "(f^{\mathrm{I}})_{\mathrm{I}}"'] & & & (X^{\mathrm{I}})_{\mathrm{I}}
\end{tikzcd} \tag{127}$$

The sides of the cube are pullbacks by the construction of $\mathsf{Fib}(A)$ and $\mathsf{Fib}(B)$. The front face is the root of the pullback (126) and is thus also a pullback, since the root is a right adjoint. The base commutes by naturality of the unit of the adjunction, and so the back face is also a pullback, as required. $\square$

Now let us apply the foregoing construction of $\mathsf{Fib}(A)$ to the universal family $\dot{\mathcal{V}} \to \mathcal{V}$ to get $\mathsf{Fib}(\dot{\mathcal{V}}) \to \mathcal{V}$, and define the universal small ($\delta$-biased) fibration in $\mathsf{cSet}/_{\mathrm{I}}$ by setting $\mathsf{Fib} := \mathsf{Fib}(\dot{\mathcal{V}})$ and $\dot{\mathsf{Fib}} \twoheadrightarrow \mathsf{Fib}$ by pulling back the universal family,

$$\begin{tikzcd}
\dot{\mathsf{Fib}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathcal{V}} \arrow[d, "p"] \\
\mathsf{Fib} \arrow[r] & \mathcal{V}.
\end{tikzcd} \tag{128}$$

The proof of the following then proceeds just as that given for $\mathsf{T}\dot{\mathsf{Fib}} \to \mathsf{TFib}$ in Proposition 81.

**Proposition 85.** *The map* $\dot{\mathsf{Fib}} \to \mathsf{Fib}$ *constructed in* (128) *is a* universal small $\delta$-biased fibration *in* $\mathsf{cSet}/_{\mathrm{I}}$*: every small $\delta$-biased fibration* $A \twoheadrightarrow X$ *in* $\mathsf{cSet}/_{\mathrm{I}}$ *is a pullback of* $\dot{\mathsf{Fib}} \twoheadrightarrow \mathsf{Fib}$ *along a canonically determined classifying map* $X \to \mathsf{Fib}$.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathsf{Fib}} \arrow[d, two heads] \\
X \arrow[r] & \mathsf{Fib}
\end{tikzcd} \tag{129}$$

*Remark* 86. Proposition 85 made no use of the fact that we were working in the slice category $\mathsf{cSet}/_{\mathrm{I}}$ with $\delta : 1 \to \mathrm{I}$ the generic point. It holds equally for $\delta$-biased fibrations with respect to any point $\delta : 1 \to \mathrm{I}$ of a tiny object I. Thus *e.g.* it could be used (with obvious adjustment) to construct a classifier for the $\{\delta_0, \delta_1\}$-biased fibrations of Section 3 in (Cartesian, Dedekind, or other varieties of) cubical sets $\mathsf{cSet}$.

*The classifying type of unbiased fibration structures.* In order to classify *unbiased* fibration structures on maps $A \to X$ in $\mathsf{cSet}$, we first apply the pullback $\mathrm{I}^* : \mathsf{cSet} \to \mathsf{cSet}/_{\mathrm{I}}$ and take the classifier $\mathsf{Fib}(\mathrm{I}^*A) \to \mathrm{I}^*X$ for $\delta$-biased fibration structures, then apply the pushforward $\mathrm{I}_* : \mathsf{cSet}/_{\mathrm{I}} \to \mathsf{cSet}$ and pull the result $\mathrm{I}_*\mathsf{Fib}(\mathrm{I}^*A) \to \mathrm{I}_*\mathrm{I}^*X$ back along the unit $X \to \mathrm{I}_*\mathrm{I}^*X$.

To show that this indeed classifies unbiased fibration structures on $A \to X$, let us first rename the classifying type from Lemma 84, which was constructed over I, to $\mathsf{Fib}_i(\mathrm{I}^*A) \to \mathrm{I}^*X$, and then apply $\mathrm{I}_*$ to get the map,

$$\Pi_{i:\mathrm{I}}\mathsf{Fib}_i(\mathrm{I}^*A) := \mathrm{I}_*(\mathsf{Fib}_i(\mathrm{I}^*A)) \longrightarrow X^{\mathrm{I}}$$

in $\mathsf{cSet}$. Then, as just said, we define the desired map $\mathsf{Fib}(A) \to X$ as the pullback along the unit $\rho : X \to X^{\mathrm{I}}$ of $\mathrm{I}^* \dashv \mathrm{I}_*$ as indicated below.

$$\begin{tikzcd}
\mathsf{Fib}(A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Pi_{i:\mathrm{I}}\mathsf{Fib}_i(\mathrm{I}^*A) \arrow[d] \\
X \arrow[r, "\rho"'] & X^{\mathrm{I}}
\end{tikzcd} \tag{130}$$

It now follows immediately from the adjunction $\mathrm{I}^* \dashv \mathrm{I}_*$ that sections of $\mathsf{Fib}(A) \to X$ correspond bijectively to sections of $\mathsf{Fib}_i(\mathrm{I}^*A) \to \mathrm{I}^*X$ over I, and thus to *unbiased* fibration structures on $A \to X$.

**Lemma 87.** *For any map $A \to X$ in $\mathsf{cSet}$, the map $\mathsf{Fib}(A) \to X$ in (130) is a* classifying type for unbiased fibration structures*: sections of $\mathsf{Fib}(A) \to X$ correspond bijectively to unbiased fibration structures on $A \to X$, and the construction is stable under pullback in the expected sense (as in Lemma 84).*

*Proof.* It remains only to check the stability, but since both of the adjoints in $\mathrm{I}^* \dashv \mathrm{I}_* : \mathsf{cSet}/_{\mathrm{I}} \to \mathsf{cSet}$ preserve pullbacks, this follows easily from the fact that the classifying types $\mathsf{Fib}_i$ are stable under pullback by Lemma 84. ☐

Finally, we can again take $\mathsf{Fib} := \mathsf{Fib}(\dot{\mathcal{V}})$ to now obtain a universal small *unbiased* fibration $\dot{\mathsf{Fib}} \to \mathsf{Fib}$ in $\mathsf{cSet}$, as in (128), and the proof can conclude just as in that for Proposition 81.

**Proposition 88.** *The map $\dot{\mathsf{Fib}} \to \mathsf{Fib}$ just constructed is a* universal small unbiased fibration *in $\mathsf{cSet}$: every small unbiased fibration $A \twoheadrightarrow X$ is a pullback of $\dot{\mathsf{Fib}} \twoheadrightarrow \mathsf{Fib}$ along a canonically determined classifying* *map* $X \to \mathsf{Fib}$.

$$\begin{tikzcd} A \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & \dot{\mathsf{Fib}} \arrow[d, two heads] \\ X \arrow[r] & \mathsf{Fib} \end{tikzcd} \tag{131}$$

*Remark* 89. Recall from Proposition 80 that the universe in the slice category $\mathsf{cSet}/_{\mathrm{I}}$ is the pullback of the universe $\mathcal{V}$ from $\mathsf{cSet}$ along the base change $\mathrm{I}^* : \mathsf{cSet} \to \mathsf{cSet}/_{\mathrm{I}}$. Thus in the construction just given of the classifier $\dot{\mathsf{Fib}} \to \mathsf{Fib}$ for unbiased fibrations in $\mathsf{cSet}$ we are first building the classifying type

$$\mathsf{Fib}_i(\mathrm{I}^*\dot{\mathcal{V}}) \to \mathrm{I}^*\mathcal{V}$$

for $\delta$-biased fibration structures on the universal family in $\mathsf{cSet}/_{\mathrm{I}}$, and then taking a pushforward $\mathrm{I}_* : \mathsf{cSet}/_{\mathrm{I}} \to \mathsf{cSet}$ to obtain the (base of the) classifier for unbiased fibrations as the pullback along the unit:

$$\begin{tikzcd} \mathsf{Fib}(\dot{\mathcal{V}}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & \Pi_{i:\mathrm{I}}\mathsf{Fib}_i(\mathrm{I}^*\dot{\mathcal{V}}) \arrow[d] \\ \mathcal{V} \arrow[r, "\rho"'] & \mathcal{V}^{\mathrm{I}} \end{tikzcd} \tag{132}$$

We remark for later reference that this classifying type $\mathsf{Fib} = \mathsf{Fib}(\dot{\mathcal{V}}) \to \mathcal{V}$ for unbiased fibration structures can therefore be constructed as the pushforward of the classifier $\mathsf{Fib}_i(\mathrm{I}^*\dot{\mathcal{V}}) \to \mathrm{I}^*\mathcal{V}$ for $\delta$-biased fibration structures along the projection $q : \mathrm{I}^*\mathcal{V} = \mathrm{I} \times \mathcal{V} \to \mathcal{V}$ indicated below.

$$\begin{tikzcd} \mathsf{Fib}_i(\mathrm{I}^*\dot{\mathcal{V}}) \arrow[d] & \mathsf{Fib}(\dot{\mathcal{V}}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & \Pi_{i:\mathrm{I}}\mathsf{Fib}_i(\dot{\mathcal{V}}) \arrow[d] \\ \mathrm{I}^*\mathcal{V} \arrow[r, "q"'] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & \mathcal{V} \arrow[r, "\rho"'] \arrow[d] & \mathcal{V}^{\mathrm{I}} \\ \mathrm{I} \arrow[r] & 1 & \end{tikzcd} \tag{133}$$

We record this fact as:

*Corollary* 90. $\mathsf{Fib} = \Sigma_{\mathcal{V}}\, q_*\mathsf{Fib}_i(\mathrm{I}^*\dot{\mathcal{V}})$.

The reader may also find it illuminating to reconsider the construction of the universal small unbiased fibration in more type theoretic terms. It was defined to be $\dot{\mathsf{Fib}} \twoheadrightarrow \mathsf{Fib} = \mathsf{Fib}(\dot{\mathcal{V}})$, for the universal family $\dot{\mathcal{V}} \to \mathcal{V}$, with $\dot{\mathsf{Fib}}$ the pullback of $\dot{\mathcal{V}} \to \mathcal{V}$ along the canonical projection $\mathsf{Fib}(\dot{\mathcal{V}}) \to \mathcal{V}$. Since, type theoretically, we have $\dot{\mathcal{V}} = \Sigma_{A:\mathcal{V}}A$, by the stability of the classifying type $\mathsf{Fib}(-)$ we can write $\mathsf{Fib} = \Sigma_{A:\mathcal{V}}\mathsf{Fib}(A)$ so that:

$$\dot{\mathsf{Fib}} = \Sigma_{A:\mathcal{V}}\mathsf{Fib}(A) \times A \longrightarrow \Sigma_{A:\mathcal{V}}\mathsf{Fib}(A) = \mathsf{Fib}\,.$$

**Realignment for fibration structure.** The realignment for families of Proposition 78 will need to be extended to (structured) fibrations. Our approach makes use of the notion of a *weak proposition*. Informally, a map $P \to X$ may be said to be a weak proposition if it is "conditionally contractible", in the sense that it is contractible if it has a section (recall that a proposition may be defined as a fibration that is "contractible if inhabited"). More formally, we have the following.

**Definition 91.** A map $P \to X$ is said to be a *weak proposition* if the projection $P \times_X P \to P$ is a trivial fibration.

$$\begin{tikzcd} P^2 \arrow[r] \arrow[d, two heads, "\sim"'] \arrow[dr, phantom, "\lrcorner", very near start] & P \arrow[d] \\ P \arrow[r] & X. \end{tikzcd} \tag{134}$$

Note that if either projection is a trivial fibration, then both are.

As an object over the base, a weak proposition is thus one that "thinks it is contractible". The key fact needed for realignment is the following.

**Lemma 92.** *For any $A \to X$, the classifying type $\mathsf{TFib}(A) \to X$ is a weak proposition. Moreover, the same is true for $\mathsf{Fib}(A) \to X$ (both the biased and unbiased versions) if the cofibrations are closed under exponentiation by the interval* $\mathrm{I}$.

*Proof.* Let $A \to X$ and consider the following diagram, in which we have written $A' = \mathsf{TFib}(A) \times_X A$ and $\mathsf{TFib}(A)^2 = \mathsf{TFib}(A) \times_X \mathsf{TFib}(A)$.

$$\begin{tikzcd} A' \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & A \arrow[dd] & \\ & \mathsf{TFib}(A)^2 \arrow[rr] \arrow[dl] \arrow[dr, phantom, "\lrcorner", very near start] & & \mathsf{TFib}(A) \arrow[dl] \\ \mathsf{TFib}(A) \arrow[rr] & & X & \end{tikzcd} \tag{135}$$

Since $\mathsf{TFib}$ is stable under pullback (by Lemma 82), we have $\mathsf{TFib}(A)^2 \cong \mathsf{TFib}(A')$, and since $\mathsf{TFib}(A)^2$ has a canonical section, $A' \to \mathsf{TFib}(A)$ is therefore a trivial fibration. Inspecting the definition of $\mathsf{TFib}(A) = {+}\mathsf{Alg}(A)$ in (110), we see that if a map $A \to X$ is a trivial fibration, then so is $\mathsf{TFib}(A) \to X$ (since $\eta : A \to A^+$ is always a cofibration). Thus $\mathsf{TFib}(A)^2 \cong \mathsf{TFib}(A') \to \mathsf{TFib}(A)$ is also a trivial fibration.

For $\mathsf{Fib}(A) \to X$, with reference to the construction (124) we use the foregoing to infer that $\mathsf{TFib}(\epsilon_A) \to A_\epsilon$ is a weak proposition, and so therefore is its pushforward $F_A = (p_\epsilon)_*\mathsf{TFib}(\epsilon_A) \to X^{\mathrm{I}}$ along the projection $p_\epsilon : A_\epsilon = X^I \times_X A \to X^{\mathrm{I}}$, since pushforward clearly preserves weak propositions. Applying the root $(-)_{\mathrm{I}}$ preserves trivial fibrations, by the assumption that its left adjoint $(-)^{\mathrm{I}}$ preserves cofibrations, and so, as a right adjoint, it also preserves weak propositions. Therefore $(F_A)_{\mathrm{I}} \to (X^{\mathrm{I}})_{\mathrm{I}}$ is a weak proposition, but then so is its pullback along the unit $X \to (X^{\mathrm{I}})_{\mathrm{I}}$, which is $\mathsf{Fib}_i(A) \to X$, the classifier for $\delta$-biased fibration structures. The same reasoning shows that $\mathsf{Fib}(A) = \rho^*\Pi_{i:\mathrm{I}}\mathsf{Fib}_i(\mathrm{I}^*A)$ (as in (130)) is also a weak proposition. $\square$

In light of Lemma 92 we shall assume as a final axiom on cofibrations:

(C8) The pathobject functor preserves cofibrations: thus $c : A \rightarrowtail B$ implies $c^{\mathrm{I}} : A^{\mathrm{I}} \rightarrowtail B^{\mathrm{I}}$.

Now, by Propositions 85 and 88 we have universal small $\delta$-biased and unbiased fibrations, the former in $\mathsf{cSet}/_{\mathrm{I}}$, the latter in $\mathsf{cSet}$. The following remarks apply to both, which we refer to neutrally as $\dot{\mathcal{U}} \twoheadrightarrow \mathcal{U}$. The base object $\mathcal{U}$ is (the domain of) the classifying type $\mathsf{Fib}(\dot{\mathcal{V}}) \to \mathcal{V}$, where $\dot{\mathcal{V}} \to \mathcal{V}$ is the universal small family. Type theoretically, this object can be written as

$$\mathcal{U} = \Sigma_{E:\mathcal{V}}\mathsf{Fib}(E)\,,$$

which comes with the canonical projection

$$\mathcal{U} = \Sigma_{E:\mathcal{V}}\mathsf{Fib}(E) \longrightarrow \mathcal{V}\,.$$

In these terms, a fibration $E \twoheadrightarrow X$ is a pair $\langle E, e\rangle$, consisting of the underlying family $E \to X$, equipped with a fibration structure $e : \mathsf{Fib}(E)$. Lemma 92 then allows us to establish the following, which was first isolated in [Shu15] (as condition (2'), also see [GSS22]). It holds for both biased and unbiased fibrations, and will be used in the sequel to "correct" the fibration structure on certain maps.

**Lemma 93** (Realignment for fibrations)**.** *Given a fibration $F \twoheadrightarrow X$ and a cofibration $c : C \rightarrowtail X$, let $f_c : C \to \mathcal{U}$ classify the pullback $c^*F \twoheadrightarrow C$. Then there is a classifying map $f : X \to \mathcal{U}$ for $F$ with*

$f \circ c = f_c$.

$$
\begin{tikzcd}
c^*F \arrow[rr] \arrow[dr] \arrow[dd, two heads] & & \dot{\mathcal{U}} \arrow[dd, two heads] \\
& F \arrow[ur, dotted] \arrow[dd, two heads] & \\
C \arrow[rr, "f_c"] \arrow[dr, tail, "c"'] & & \mathcal{U} \\
& X \arrow[ur, dotted, "f"'] &
\end{tikzcd}
\tag{136}
$$

*Proof.* First, let $|f_c| : C \to \mathcal{V}$ be the composite of $f_c : C \to \mathcal{U}$ with the canonical projection $\mathcal{U} \to \mathcal{V}$, thus classifying the underlying family $c^*F \to C$. Next, let $f_0 : X \to \mathcal{V}$ classify the underlying family $F \to X$. We may assume that $f_0 \circ c = |f_c|$ by realignment for families, Proposition 78.

$$
\begin{tikzcd}
c^*F \arrow[rr] \arrow[dr] \arrow[dd, two heads] & & \dot{\mathcal{U}} \arrow[r] \arrow[dd, two heads] & \dot{\mathcal{V}} \arrow[dd] \\
& F \arrow[urr, dotted] \arrow[dd, two heads] & & \\
C \arrow[rr, "f_c"] \arrow[dr, tail, "c"'] & & \mathcal{U} \arrow[r] & \mathcal{V} \\
& X \arrow[urr, dotted, "f_0"'] & &
\end{tikzcd}
\tag{137}
$$

Since $F \twoheadrightarrow X$ is a fibration, there is a lift $f_1 : X \to \mathcal{U}$ of $f_0$ classifying the fibration structure. We thus have the following commutative diagram in the base of (137).

$$
\begin{tikzcd}
C \arrow[r, "f_c"'] \arrow[rr, bend left, "|f_c|"] \arrow[d, tail, "c"'] & \mathcal{U} \arrow[r] & \mathcal{V} \arrow[d, equal] \\
X \arrow[r, "f_1"] \arrow[rr, bend right, "f_0"'] & \mathcal{U} \arrow[r] & \mathcal{V}
\end{tikzcd}
\tag{138}
$$

Now pull $\mathcal{U} \to \mathcal{V}$ back against itself and rearrange the previous data to give (the solid part of) the following, which also commutes.

$$\begin{tikzcd}
C \arrow[r, "\langle f_1 c, f_c \rangle"'] \arrow[d, tail, "c"'] \arrow[rr, bend left, "f_c"] & \mathcal{U} \times_{\mathcal{V}} \mathcal{U} \arrow[r, "\pi_2"'] \arrow[d, two heads, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{U} \arrow[d] \\
X \arrow[r, "f_1"'] \arrow[ur, dotted, "f_2"] \arrow[rr, bend right, "f_0"'] & \mathcal{U} \arrow[r] & \mathcal{V}
\end{tikzcd} \tag{139}$$

Since $\mathcal{U} = \mathsf{Fib}(\dot{\mathcal{V}}) \to \mathcal{V}$ is a weak proposition by Lemma 92 and (C8), the projection $\pi_1 : \mathcal{U} \times_{\mathcal{V}} \mathcal{U} \twoheadrightarrow \mathcal{U}$ is a trivial fibration, so there is a diagonal filler $f_2 : X \to \mathcal{U} \times_{\mathcal{V}} \mathcal{U}$ as indicated. Taking $f := \pi_2 \circ f_2 : X \to \mathcal{U} \times_{\mathcal{V}} \mathcal{U} \to \mathcal{U}$ gives another classifying map for the fibration structure on $F \to X$, for which $f \circ c = f_c$ as required. $\square$

## 7. The equivalence extension property

The equivalence extension property is closely related to the *univalence* of the universal fibration $\dot{\mathcal{U}} \twoheadrightarrow \mathcal{U}$ constructed in section 6 (see [Shu15]). It will be used in section 8 to show that the base object $\mathcal{U}$ is fibrant. The proof of the equivalence extension property given here is a reformulation of a type-theoretic argument due to Coquand [CCHM18], which in turn is a modification of the original argument of Voevodsky [CK21]. See [Sat17] for another reformulation.

**The sliced premodel structure.** We begin by recalling some basic facts and making some simple observations that are well-known in general model categories, but need to be checked again here, because we do not yet have a full model structure. The reader is reminded that the word "fibration" unqualified always refers to *unbiased* fibrations as in Definition 23. First, for any object $Z \in \mathsf{cSet}$, the slice category $\mathsf{cSet}/_Z$ inherits the premodel structure of Proposition 41 from $\mathsf{cSet}$ via the forgetful functor

$$Z_! : \mathsf{cSet}/_Z \to \mathsf{cSet}\,.$$

In more detail:

**Definition 94.** A map $f : X \to Y$ over $Z$ is a *(trivial) cofibration or (trivial) fibration* over $Z$ just if it is one in $\mathsf{cSet}$ after forgetting the $Z$-indexing via $Z_! : \mathsf{cSet}/_Z \to \mathsf{cSet}$. This will be called the *sliced premodel structure* on $\mathsf{cSet}/_Z$. Accordingly, a map $f : X \to Y$ over $Z$ will be called a *weak equivalence* over $Z$ just if it factors over $Z$ as a trivial fibration over $Z$ after a trivial cofibration over $Z$, which therefore holds just if it is a weak equivalence in $\mathsf{cSet}$ after forgetting the $Z$-indexing.

That the specification in Definition 94 does determine a premodel structure is a consequence of Proposition 41, and the well-known fact that (pre)model structures are stable under slicing in this way [Hir03]. In more detail:

**Lemma 95.** *A map $f : X \to Y$ over $Z$ is a fibration (respectively, a trivial fibration) over $Z$ if, and only if, it lifts on the right in the slice category $\mathsf{cSet}/_Z$ against all trivial cofibrations (respectively, cofibrations) over $Z$.*

*Proof.* Let $X \xrightarrow{f} Y \xrightarrow{p_Y} Z$, regarded as a map in the slice category over $Z$, with $p_X = p_Y \circ f : X \to Z$. Then by definition $f$ is a fibration in $\mathsf{cSet}/_Z$ just if $f : X \to Y$ is a fibration in the total category $\mathsf{cSet}$, which holds just if $f$ lifts on the right against all trivial cofibrations $t : A \rightarrowtail B$ in $\mathsf{cSet}$. But every lifting problem of the form $t \pitchfork f$ in $\mathsf{cSet}$,

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, tail, "t"'] & X \arrow[d, "f"] \\ B \arrow[r, "y"'] \arrow[ur, dotted] & Y \arrow[d, "p_Y"] \\ & Z \end{tikzcd}$$

gives rise to a corresponding one over $Z$, just by composing everything with $p_Y : Y \to Z$. Moreover, the evident resulting map $A \overset{t}{\rightarrowtail} B \to Z$ is then a trivial cofibration over $Z$, and every such lifting problem for $f$ over $Z$ arises in this way. Finally, the diagonal fillers for the resulting lifting problem in $\mathsf{cSet}/_Z$ are exactly the diagonal fillers for the original one in $\mathsf{cSet}$. Thus the map $f$ over $Z$ is a fibration over $Z$ just in case it lifts on the right over $Z$ against all trivial cofibrations over $Z$, as claimed. The case of trivial fibrations and cofibrations is exactly analogous. $\square$

**Lemma 96.** *A map $f : X \to Y$ over $Z$ is a cofibration (respectively, a trivial cofibration) over $Z$ if, and only if, it lifts on the left in the slice category $\mathsf{cSet}/_Z$ against all trivial fibrations (respectively, fibrations) over $Z$.*

*Proof.* Let $X \xrightarrow{f} Y \xrightarrow{p_Y} Z$, regarded as a map in the slice category over $Z$, with $p_X = p_Y \circ f : X \to Z$. Then by definition $f$ is a cofibration in $\mathsf{cSet}/_Z$ just if $Z_!f : Z_!X \to Z_!Y$ is a cofibration in the total category

$\mathsf{cSet}$, which holds just if $Z_!f$ lifts on the left against all trivial fibrations $t : E \twoheadrightarrow F$ in $\mathsf{cSet}$. But every lifting problem of the form $Z_!f \pitchfork t$ in $\mathsf{cSet}$,

$$\begin{tikzcd}
X \arrow[r, "x"] \arrow[d, "f"'] & E \arrow[d, two heads, "t"] \\
Y \arrow[r, "y"'] \arrow[ur, dotted] \arrow[d, "p_Y"'] & F \\
Z &
\end{tikzcd}$$

gives rise to a corresponding one over $Z$ of the form $f \pitchfork Z^*t$, by pulling $t$ back along $Z \to 1$. Moreover, since trivial fibrations are stable under pullback in $\mathsf{cSet}$, the pullback $Z^*t$ is a trivial fibration, and so $Z^*t$ is a trivial fibration over $Z$. Thus $f$ is a cofibration in $\mathsf{cSet}/_Z$ if and only if $f \pitchfork Z^*t$ in $\mathsf{cSet}/_Z$ for all trivial fibrations $t : E \twoheadrightarrow F$ in $\mathsf{cSet}$. Now observe that for any map $A \xrightarrow{g} B \xrightarrow{p_B} Z$ over $Z$, with $p_A = p_B \circ g$, the following unit square is a pullback, as indicated below,

$$\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[d, "g"'] \arrow[dr, phantom, "\lrcorner", very near start] & Z^*Z_!AE \arrow[r, equal] \arrow[d, "Z^*Z_!g"] & Z \times A \arrow[d, "Z\times g"] \\
B \arrow[r, "\eta_B"'] \arrow[d] & Z^*Z_!AF \arrow[r, equal] \arrow[d] & Z \times B \arrow[d] \\
Z \arrow[r, equal] & Z \arrow[r, equal] & Z,
\end{tikzcd}$$

because $\eta_A = \langle p_A, 1_Z\rangle : A \to Z \times A$ is a pullback of $\Delta_A = \langle 1_Z, 1_Z\rangle : Z \to Z \times Z$ along $1_Z \times p_A : Z \times A \to Z \times Z$, and similarly for $\eta_B$. Thus in particular, every trivial fibration $A \overset{g}{\twoheadrightarrow} B \xrightarrow{p_B} Z$ over $Z$ is a pullback over $Z$ of one of the form $Z^*t : Z^*E \twoheadrightarrow Z^*F$ for a trivial fibration $t : E \twoheadrightarrow F$ in $\mathsf{cSet}$. Therefore $f$ is a cofibration in $\mathsf{cSet}/_Z$ if and only if $f \pitchfork g$ in $\mathsf{cSet}/_Z$ for all trivial fibrations $g : A \twoheadrightarrow B$ in $\mathsf{cSet}/_Z$, as claimed. The case of trivial cofibrations and fibrations is exactly analogous. $\square$

Since factoring a map in the slice category is evidently given simply by factoring it after forgetting the indexing, we now have:

**Proposition 97.** *The specification in Definition 94 determines a pre-model structure on* $\mathsf{cSet}/_Z$ *for any object* $Z \in \mathsf{cSet}$.

Note that the axioms (C1)–(C6) for cofibrations are also stable under slicing (the cofibration classifier in $\mathsf{cSet}/_Z$ is $Z^*t : Z^*1 \rightarrowtail Z^*\Phi$). The reader is warned that when $Z = \mathrm{I}$ there is a possibility of confusion with the $\delta$-biased fibrations in $\mathsf{cSet}/_{\mathrm{I}}$, which do not in general agree with the I-sliced (unbiased) fibrations.

[*Joyal-Tierney calculus in the slice, fibrations are $\delta_Z$-biased trivial fibrations in the slice over* $\mathrm{I}_Z$*.*]

In order to apply the results on weak equivalences from section 4 in arbitrary slice categories $\mathsf{cSet}/_Z$ we also require analogues of Lemmas 95 and 96 for the notions of homotopy equivalence and weak homotopy equivalence. Toward that end, let $Z^*1 \rightrightarrows Z^*\mathrm{I}$ in $\mathsf{cSet}/_Z$ be the result of pulling the interval $1 \rightrightarrows \mathrm{I}$ back along $Z \to 1$. Observe that $1_Z + 1_Z \cong Z^*1 + Z^*1 \rightarrowtail Z^*\mathrm{I}$ since the pullback functor $Z^* : \mathsf{cSet} \to \mathsf{cSet}/_Z$ preserves (co)limits and cofibrations, so $Z^*1 \rightrightarrows Z^*\mathrm{I}$ is an interval in $\mathsf{cSet}/_Z$, which we will also write as $\delta_0, \delta_1 : 1_Z \rightrightarrows \mathrm{I}_Z$. We can use this interval to define homotopy between maps over $Z$ in the expected way, namely:

**Definition 98.** For any object $Z$ and maps $f, g : X \rightrightarrows Y$ in $\mathsf{cSet}/_Z$, a *homotopy over* $Z$, written

$$\vartheta : f \sim_Z g\,,$$

is a map over $Z$,

$$\vartheta : \mathrm{I}_Z \times_Z X \longrightarrow Y,$$

such that $\vartheta \circ \iota_0 = f$ and $\vartheta \circ \iota_1 = g$,

$$\begin{tikzcd} X \arrow[r, "\iota_0"] \arrow[dr, "f"'] & \mathrm{I}_Z \times_Z X \arrow[d, "\vartheta"] & X, \arrow[l, "\iota_1"'] \arrow[dl, "g"] \\ & Y & \end{tikzcd} \tag{140}$$

where, as usual, $\iota_0, \iota_1$ are the canonical inclusions into the ends of the cylinder,

$$\iota_\epsilon : X \cong 1_Z \times_Z X \xrightarrow{\delta_\epsilon \times_Z X} \mathrm{I}_Z \times_Z X\,, \qquad \epsilon = 0, 1.$$

**Lemma 99.** *For any object $Z$ and maps $f, g : X \rightrightarrows Y$ in $\mathsf{cSet}/_Z$, a homotopy over $Z$ determines a homotopy of the underlying maps by applying the functor $Z_! : \mathsf{cSet}/_Z \to \mathsf{cSet}$ that forgets the $Z$-indexing,*

$$\vartheta : f \sim_Z g \qquad \mapsto \qquad Z_!\vartheta : Z_!f \sim Z_!g\,.$$

*Proof.* Consider the following diagram, which depicts a homotopy $\vartheta : f \sim_Z g$ over $Z$.

$$\begin{tikzcd}
& & X \arrow[r, shift left, "g"] \arrow[r, shift right, "f"'] \arrow[d, shift right, "\iota_0"'] \arrow[d, shift left, "\iota_1"] & Y \arrow[dd] \\
\mathrm{I} \arrow[d] & \mathrm{I} \times Z \arrow[l] \arrow[d] & \mathrm{I}_Z \times_Z X \arrow[l] \arrow[d] \arrow[ur, "\vartheta"'] & \\
1 & Z \arrow[l] & X \arrow[l] \arrow[r] & Z
\end{tikzcd}$$

Since the lower left two squares are pullbacks, $\mathrm{I}_Z \times_Z X \cong \mathrm{I} \times X$. So applying $Z_!$ to $\vartheta$ results in a homotopy $Z_!\vartheta : Z_!f \sim Z_!g$.

Note that an arbitrary homotopy $\varphi : f \sim g$ will not result in one over $Z$, however, unless $\varphi$ commutes with the indexing maps to $Z$. $\square$

**Proposition 100.** *For any object $Z$, the relation of homotopy over $Z$ between maps $f, g : X \rightrightarrows Y$ over $Z$ is preserved by pre- and post-composition. If $X \twoheadrightarrow Z$ and $Y \twoheadrightarrow Z$ are both fibrations, then the relation $f \sim_Z g$ of maps between them is an equivalence relation.*

The proof is essentially the same as the corresponding one for homotopy over 1, Proposition 43, with the exception that both $X$ and $Y$ are required to be fibrant objects over $Z$, so that the exponential $Y^X$ over $Z$ is also a fibration $Y^Z \twoheadrightarrow Z$ (by Corollary 71).

Next we define a *connected components* functor on the full subcategory $\mathsf{Fib}_Z \hookrightarrow \mathsf{cSet}/_Z$ of fibrations over $Z$,

$$(\pi_0)_Z : \mathsf{Fib}_Z \to \mathsf{Set}\,,$$

by taking the global sections of a fibration $F \twoheadrightarrow Z$, modulo the relation $\sim_Z$ of homotopy over $Z$. In more detail, for $F \twoheadrightarrow Z$ in $\mathsf{Fib}_Z$ let $(\pi_0)_Z(F)$ be the coequalizer,

$$\mathrm{Hom}_Z(\mathrm{I}_Z, F) \rightrightarrows \mathrm{Hom}_Z(1_Z, F) \to (\pi_0)_Z(F)\,, \tag{141}$$

where the two maps are given by precomposition with the interval $1_Z \rightrightarrows \mathrm{I}_Z$ over $Z$, and the Hom-sets are those in $\mathsf{cSet}/_Z$.

For fibrations $X \twoheadrightarrow Z$ and $F \twoheadrightarrow Z$ we then again have

$$(\pi_0)_Z(F^X) = \mathrm{Hom}_Z(X, F)/{\sim_Z}\,,$$

so $(\pi_0)_Z(F^X)$ is the set $[X, F]_Z$ of $Z$-homotopy equivalence classes of maps $X \to F$ over $Z$.

For maps over a base object $Z$, we can now define the notions of homotopy equivalence over $Z$ and, between fibrations, weak homotopy equivalence over $Z$ as before (cf. Section 4):

**Definition 101.** Let $Z$ be any object in $\mathsf{cSet}$, and let $X \to Z$ and $Y \to Z$ be regarded as objects over $Z$.

(1) A map $f : X \to Y$ over $Z$ is a *homotopy equivalence over* $Z$ if there is a map $g : Y \to X$ over $Z$ and two homotopies over $Z$,
$$\vartheta : g \circ f \sim_Z 1_X , \qquad \varphi : f \circ g \sim_Z 1_Y \,.$$

(2) For $X \twoheadrightarrow Z$ and $Y \twoheadrightarrow Z$ fibrations, a map $f : X \to Y$ over $Z$ is a *weak homotopy equivalence over* $Z$ if for every fibration $F \twoheadrightarrow Z$, the precomposition map over $Z$,
$$F^f : F^Y \to F^X \,,$$
is bijective on connected components,
$$(\pi_0)_Z(F^f) : (\pi_0)_Z(F^Y) \cong (\pi_0)_Z(F^X) \,,$$
where the indicated exponentials by $F$ are taken in the slice category $\mathsf{cSet}/_Z$.

**Lemma 102.** *Let $f : X \to Y$ be any map over $Z$.*

(1) *If $f : X \to Y$ is homotopy equivalence over $Z$, then $Z_! f : Z_! X \to Z_! Y$ is a homotopy equivalence.*

(2) *If $X \twoheadrightarrow Z$ and $Y \twoheadrightarrow Z$ are fibrations and $f : X \to Y$ is weak homotopy equivalence over $Z$, then $Z_! f : Z_! X \to Z_! Y$ is a weak homotopy equivalence.*

*Proof.* (1) is immediate from the fact that $Z_!$ preserves homotopies, Lemma 99.

For (2), let $f : X \to Y$ be a weak homotopy equivalence over $Z$ between fibrations $X \twoheadrightarrow Z$ and $Y \twoheadrightarrow Z$, and let $K$ be any fibrant object in $\mathsf{cSet}$. Consider the internal precomposition map,
$$K^{Z_! f} : K^{Z_! Y} \to K^{Z_! X} \,,$$
which we would like to show is a bijection under $\pi_0 : \mathsf{cSet} \to \mathsf{Set}$. Since $K \twoheadrightarrow 1$ is a fibration, so is its pullback $Z^* K \twoheadrightarrow Z$. Therefore, since $f : X \to Y$ is weak homotopy equivalence over $Z$, the precomposition map over $Z$,
$$(Z^* K)^f : (Z^* K)^Y \to (Z^* K)^X \,,$$
is bijective on connected components,
$$(\pi_0)_Z\big((Z^* K)^f\big) : (\pi_0)_Z\big((Z^* K)^Y\big) \cong (\pi_0)_Z\big((Z^* K)^X\big) \,.$$
But now observe that in the coequalizer (141) that defines $(\pi_0)_Z\big((Z^* K)^X\big)$, we have
$$\begin{aligned}\mathrm{Hom}_Z(1_Z, (Z^* K)^X) &\cong \mathrm{Hom}_Z(X, (Z^* K)) \\ &\cong \mathrm{Hom}(Z_! X, K) \cong \mathrm{Hom}(1, K^{Z_! X}) \,,\end{aligned}$$

and similarly

$$\begin{aligned}\mathrm{Hom}_Z(\mathrm{I}_Z, (Z^*K)^X) &\cong \mathrm{Hom}_Z(Z^*\mathrm{I} \times X, Z^*K)\\ &\cong \mathrm{Hom}_Z(\mathrm{I} \times Z_!X, K) \cong \mathrm{Hom}(\mathrm{I}, K^{Z_!X})\,.\end{aligned}$$

Thus $(\pi_0)_Z((Z^*K)^X) \cong (\pi_0)(K^{Z_!X})$, and the same is true with $Y$ in place of $X$. So $K^{Z_!f} : K^{Z_!Y} \to K^{Z_!X}$ is also bijective on connected components. □

**Proposition 103.** *Let $Z$ be any object in* $\mathsf{cSet}$ *and $X \twoheadrightarrow Z$ and $Y \twoheadrightarrow Z$ fibrations, and let $f : X \to Y$ be a map over $Z$. Then the following are equivalent.*

(1) *$f : X \to Y$ over $Z$ is a homotopy equivalence over $Z$.*
(2) *$f : X \to Y$ over $Z$ is a weak homotopy equivalence over $Z$.*
(3) *$f : X \to Y$ is a homotopy equivalence in* $\mathsf{cSet}$.
(4) *$f : X \to Y$ is a weak homotopy equivalence in* $\mathsf{cSet}$.

*Proof.* We shall prove these in the order $1 \Rightarrow 3 \Rightarrow 4 \Rightarrow 2 \Rightarrow 1$.
$1 \Rightarrow 3$ was Lemma 102(1).
$3 \Rightarrow 4$ was Lemma 49.
$4 \Rightarrow 2$: [*fill in!*]
$2 \Rightarrow 1$: was Lemma 54.

□

**Pathobject factorizations.** For any map $f : X \to Y$ in $\mathsf{cSet}$, recall the *pathobject factorization* $f = t \circ s$ indicated below.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "s"] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dd, bend right=60, "="'] & Y \arrow[d, "r"'] \arrow[dr, "="] & \\
P_f \arrow[r, "p_0^* f"] \arrow[d, "p_f"] \arrow[rr, dotted, bend right, "t"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y^{\mathrm{I}} \arrow[r, "p_1"] \arrow[d, "p_0"'] & Y \\
X \arrow[r, "f"'] & Y &
\end{tikzcd} \tag{142}$$

Here $p_0, p_1$ are the evaluations $Y^{\delta_0}, Y^{\delta_1} : Y^{\mathrm{I}} \to Y$ at the endpoints $\delta_0, \delta_1 : 1 \to \mathrm{I}$, and let $r := Y^!$ for $! : \mathrm{I} \to 1$, so that $p_0 r = p_1 r = 1_Y$. Then let $p_f := f^* p_0 : P_f \to Y$, the pullback of $p_0$ along $f$, and $s := f^* r : X \to P_f$ (as a map over $X$). Finally, let $t := p_1 \circ p_0^* f : P_f \to Y$ be the indicated horizontal composite.

We then have the following facts:

(1) The retraction $p_0 \circ r = 1_Y$ pulls back along $f$ to a retraction $p_f \circ s = 1_X$.
(2) If $Y$ is a fibrant object, then $p_0, p_1 : Y^{\mathrm{I}} \to Y$ are both trivial fibrations, by Proposition 30.
(3) If $X$ and $Y$ are both fibrant then $t = p_1 \circ p_0^* f : P_f \to Y$ is a fibration. This can be seen by factoring the maps $p_0, p_1 : Y^{\mathrm{I}} \rightrightarrows Y$ through the product projections as

$$\pi_0 \circ p,\ \ \pi_1 \circ p : Y^{\mathrm{I}} \to Y \times Y \rightrightarrows Y$$

where $p = (p_0, p_1)$, and then interpolating the pullback $(f, 1_Y) : X \times Y \to Y \times Y$ into (142) as indicated below.

$$\begin{tikzcd}
 & X \arrow[r, "f"] \arrow[d, "s"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "r"] & \\
 & P_f \arrow[r, "p_0^* f"] \arrow[d, "f^* p"] \arrow[dl, "t"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y^{\mathrm{I}} \arrow[d, "p"'] \arrow[dr, "p_1"] & \\
Y & X \times Y \arrow[l, "\pi_1'"] \arrow[r, "{(f,1_Y)}"'] \arrow[d, "\pi_0'"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \times Y \arrow[r, "\pi_1"'] \arrow[d, "\pi_0"] & Y \\
 & X \arrow[r, "f"'] & Y &
\end{tikzcd} \tag{143}$$

The second factor $t = p_1 \circ p_0^* f : P_f \to Y$ now appears also as $\pi_1 \circ (f, 1_Y) \circ f^* p$, which is equal to the pullback $f^* p : P_f \to X \times Y$ followed by the second projection $\pi_1' : X \times Y \to Y$ (which is not a pullback). But if $Y$ is fibrant, then $p : Y^{\mathrm{I}} \to Y \times Y$ is a fibration by the $\otimes \dashv \Rightarrow$ adjunction, since $p = \partial \Rightarrow Y$ (this is just as in Proposition 30, but with the cofibration $\partial : 1 + 1 \rightarrowtail \mathrm{I}$ in place of the trivial cofibration $\delta_\epsilon : 1 \to \mathrm{I}$). Therefore the pullback $f^* p$ is also a fibration. And if $X$ is fibrant, then the second projection $\pi_1' : X \times Y \to Y$ is a fibration. Thus in this case, $t = \pi_1' \circ f^* p : P_f \to Y$ is a fibration, as claimed.

Summarizing (1)-(3):

**Lemma 104.** *For any map $f : X \to Y$ there is a factorization $f = t \circ s : X \to P_f \to Y$ in which $s$ is a section of a trivial fibration $p_f$ if*

*$Y$ is fibrant, and $t$ is a fibration if both $X$ and $Y$ are fibrant.*

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[dr, "f"'] & P_f \arrow[d, "t"] \arrow[l, bend right, dotted, "p_f"'] \\ & Y \end{tikzcd} \tag{144}$$

*Note that the retraction $p_f : P_f \to X$ is not over $Y$.*

Next, if $f : X \to Y$ is a map over any base object $Z$ in $\mathsf{cSet}$, we can use the same factorization $f = t \circ s$ to get a factorization in the slice category $\mathsf{cSet}/_Z$,

$$\begin{tikzcd} & P_f \arrow[dr, "t"] & \\ X \arrow[ur, "s"] \arrow[rr, "f"] \arrow[dr, "p_X"'] & & Y \arrow[dl, "p_Y"] \\ & Z & \end{tikzcd} \tag{145}$$

with $p_Y \circ t : P_f \to Z$; however, the maps $s, t$ will no longer have the properties stated in Lemma 104, because *e.g.* $p_0 : Y^{\mathrm{I}} \to Y$ need not be a trivial fibration even when $p_Y : Y \to Z$ is a fibration, since the object $Y$ need not be fibrant if the base $Z$ is not fibrant.

To remedy this, we can instead build a *fiberwise pathobject factorization* by using the relative pathobject $Y^{\mathrm{I}} \to Z$, where the indicated exponential is taken in the slice over $Z$, and the interval object occurring in the exponent is the result of pulling I back from $\mathsf{cSet}$ along $Z \to 1$. The pathspace factorization is then constructed as in (142), but in the slice $\mathsf{cSet}/_Z$, using the pulled back interval $Z^*1 \rightrightarrows Z^*\mathrm{I}$. Moreover, the resulting factorization $f = t \circ s : X \to P_f \to Y$ is then stable under pullback along any map $g : Z' \to Z$, in the sense that $g^*(Y^{\mathrm{I}}) \cong g^*(Y)^{\mathrm{I}}$ and so $g^*P_f = P_{g^*f}$, where $g^*f : g^*X \to g^*Y$, and similarly for the factors $g^*s$ and $g^*t$.

In more detail, let us review the foregoing steps in the relative case, with reference to the following diagram.

$$
\begin{tikzcd}
X \arrow[rr, "f"] \arrow[d, "s"] \arrow[dd, bend right, "="'] & & Y \arrow[d, "r"'] \arrow[dr, "="] & \\
P_f \arrow[rr, "p_0^* f"] \arrow[d, "p_f"] \arrow[rrr, dotted, bend right, "t"'] & & Y^{\mathrm{I}} \arrow[r, "p_1"] \arrow[d, "p_0"'] & Y \\
X \arrow[rr, "f"'] \arrow[dr, "p_X"'] & & Y \arrow[dl, "p_Y"] & \\
& Z & &
\end{tikzcd}
\tag{146}
$$

(1) The exponential of $p_Y : Y \to Z$, taken in $\mathsf{cSet}/_Z$, by the *constant* maps $Z^*\delta_\epsilon : Z^*1 \to Z^*\mathrm{I}$, which we continue to write as $\delta_\epsilon : 1 \to \mathrm{I}$, are now maps $p_\epsilon := Y^{\delta_\epsilon} : Y^{\mathrm{I}} \to Y$ over $Z$, for $\epsilon = 0, 1$. The retraction $p_0 \circ r = 1_Y$ (with $r$ defined accordingly) is now also over $Z$, and it still pulls back along $f$ to a retraction $p_f \circ s = 1_X$, also over $Z$.

(2) If $p_Y : Y \twoheadrightarrow Z$ a fibration, then the maps $p_0, p_1 : Y^{\mathrm{I}} \to Y$ over $Z$ are trivial fibrations. Indeed, forgetting the $Z$ indexing and testing against an arbitrary cofibration $c : A \to C$,

[*to here*]

$Z$ via $z : C \to Z$ the map $p_\epsilon = Y^{\delta_\epsilon}$ transposes to a filling problem over $Z$ of the form $(c \otimes_z \delta_\epsilon) \pitchfork Y$, where the pushout-product $c \otimes_z \delta_\epsilon : D \to C \times_Z \mathrm{I}$ over $Z$ is easily seen to be equal to the (non-relative) pushout-product $c \otimes \delta_\epsilon : D \to C \times \mathrm{I}$, *because* $\delta_\epsilon : 1 \to \mathrm{I}$ *is constant over* $Z$. Since $p_Y$ is a fibration and $c \otimes \delta_\epsilon$ is a trivial fibration, we have $c \otimes \delta_\epsilon \pitchfork p_Y$, and so $(c \otimes_z \delta_\epsilon) \pitchfork Y$ over $Z$, and so $Y^{\delta_\epsilon} : Y^{\mathrm{I}} \to Y$ over $Z$ is indeed a trivial fibration.

(3) If $X \twoheadrightarrow Z$ and $Y \twoheadrightarrow Z$ are both fibrations, then by the same argument as before, $t = p_1 \circ p_0^* f : P_f \to Y$ is a fibration. Indeed, this can be seen by factoring the maps $p_0, p_1 : Y^{\mathrm{I}} \rightrightarrows Y$ through the product projections as

$$\pi_0 \circ p,\ \pi_1 \circ p : Y^{\mathrm{I}} \to Y \times Y \rightrightarrows Y$$

where $p = (p_0, p_1)$, and then interpolating the pullback $(f, 1_Y) : X \times Y \to Y \times Y$ into (142) as indicated below.

$$
\begin{tikzcd}
& X \arrow[r, "f"] \arrow[d, "s"] \arrow[dr, phantom, very near start, "\lrcorner"] & Y \arrow[d, "r"'] & \\
& P_f \arrow[r, "p_0^* f"] \arrow[d, "f^* p"] \arrow[dl, "t"'] \arrow[dr, phantom, very near start, "\lrcorner"] & Y^{\mathrm{I}} \arrow[d, "p"'] \arrow[dr, "p_1"] & \\
Y & X \times Y \arrow[l, "\pi_1'"] \arrow[r, "(f, 1_Y)"'] \arrow[d, "\pi_0'"'] \arrow[dr, phantom, very near start, "\lrcorner"] & Y \times Y \arrow[r, "\pi_1"'] \arrow[d, "\pi_0"] & Y \\
& X \arrow[r, "f"'] & Y &
\end{tikzcd}
\tag{147}
$$

The second factor $t = p_1 \circ p_0^* f : P_f \to Y$ now appears also as $\pi_1 \circ (f, 1_Y) \circ f^* p$, which is equal to the pullback $f^* p : P_f \to X \times Y$ followed by the second projection $\pi_1' : X \times Y \to Y$ (which is not a pullback). But if $Y$ is fibrant, then $p : Y^{\mathrm{I}} \to Y \times Y$ is a fibration, again by Proposition 30 applied to the fibration $p_Y : Y \twoheadrightarrow Z$ (by the same argument as for (3), but with the cofibration $\partial : 1 + 1 \rightarrowtail \mathrm{I}$ in place of the trivial cofibration $\delta_\epsilon : 1 \to \mathrm{I}$). Therefore the pullback $f^* p$ is also a fibration. And if $X$ is fibrant, then the second projection $\pi_1' : X \times Y \to Y$ is a fibration. Thus in this case, $t = \pi_1' \circ f^* p : P_f \to Y$ is a fibration, as claimed.

Summarizing (1)-(3), we have now shown the following.

**Lemma 105.** *For any map $f : X \to Y$ over any base $Z \in \mathsf{cSet}$, there is a stable factorization $f = t \circ s : X \to P_f \to Y$ over $Z$, in which $s$ is a section of a trivial fibration $p_f$ if $Y$ is fibrant, and $t$ is a fibration if both $X$ and $Y$ are fibrant.*

$$
\begin{tikzcd}
X \arrow[r, "s"] \arrow[dr, "f"'] & P_f \arrow[l, bend right, dotted, "p_f"'] \arrow[d, "t"] \\
& Y
\end{tikzcd}
\tag{148}
$$

*Note that the retraction $p_f : P_f \to X$ is over $Z$ but not over $Y$.*

**Corollary 106.** *Suppose $p_X : X \twoheadrightarrow Z$ and $p_Y : Y \twoheadrightarrow Z$ are fibrations and $h : X \to Y$ is a map over $Z$, so $p_X = p_Y \circ h$. If $h$ is a homotopy equivalence (not necessarily over $Z$), then it can be corrected to a homotopy equivalence over $Z$.*

*Proof.* Let $j : Y \to X$ and homotopies $\alpha : 1_X \sim j \circ h$ and $\beta : 1_Y \sim h \circ j$ be given.

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[dr, two heads, "p_X"'] & Y \arrow[r, "j"] \arrow[d, two heads, "p_Y"] & X \\ & Z & \end{tikzcd} \tag{149}$$

□

The following simple fact concerning just the cofibration weak factorization system will also be needed.

**Lemma 107.** *Let $p : E \twoheadrightarrow B$ be a trivial fibration and $c : C \rightarrowtail B$ a cofibration. Then the unit $\eta : E \to c_*c^*E$ of the base change $c^* \dashv c_*$ along $c$ is a trivial fibration.*

*Proof.* The unit map $\eta : E \to c_*c^*E$ is the pullback-hom $c \Rightarrow p$, as is easily checked. By lemma 19, for any map $a : A \to Z$ we have the equivalence of diagonal filling conditions,

$$a \pitchfork c \Rightarrow p \quad \text{iff} \quad a \otimes c \pitchfork p.$$

But since $c : C \rightarrowtail B$ is a cofibration, $a \otimes c$ is also a cofibration if $a : A \rightarrowtail Z$ is one, by axiom (C6), which says that cofibrations are closed under pushout-products. So $a \otimes c \pitchfork p$ indeed holds, since $p$ is a trivial fibration. □

**Proposition 108** (Equivalence extension property)**.** *Weak equivalences extended along cofibrations in the following sense: given a cofibration $c : C' \rightarrowtail C$ and fibrations $A' \twoheadrightarrow C'$ and $B \twoheadrightarrow C$, and a weak equivalence $w' : A' \simeq c^*B$ over $C'$,*

$$\begin{tikzcd} A' \arrow[rr, dotted] \arrow[dr, "w'", "\sim"'] \arrow[dd, two heads] & & A \arrow[dr, dotted, "w", "\sim"'] \arrow[dd, dotted, two heads] & \\ & c^*B \arrow[rr] \arrow[dl, two heads] & & B \arrow[dl, two heads] \\ C' \arrow[rr, tail, "c"'] & & C & \end{tikzcd} \tag{150}$$

*there is a fibration $A \twoheadrightarrow C$ and a weak equivalence $w : A \simeq B$ over $C$ that pulls back along $c : C' \rightarrowtail C$ to $w'$, so $c^*w = w'$.*

*Proof.* Call the given fibration $q : B \to C$ and let $b := q^*c : c^*B \to B$ be the indicated pullback, which is thus also a cofibration. Let $w := b_*w' : A \to B$ be the pushforward of $w'$ along $b$. Composing with $q$ gives the map $p := q \circ w : A \to C$. Since $b$ is monic, we indeed have $b^*w = w'$, thus filling in all the dotted arrows in (150). Note moreover that $c^*w = b^*w = w'$, as required. It remains to show that $p : A \to C$ is a fibration and $w : A \to B$ is a weak equivalence.

$$
\begin{tikzcd}
A' \arrow[rr, tail] \arrow[dd, two heads] \arrow[dr, "w'", "\sim"'] & & A \arrow[r, "s"] \arrow[dr, "w"] \arrow[dd, "p" near start] & P_w \arrow[l, dotted, bend right, two heads, "p_w"'] \arrow[d, "t"] \\
& c^*B \arrow[rr, tail, "b"' near start] \arrow[dl, two heads] & & B \arrow[dl, two heads, "q"] \\
C' \arrow[rr, tail, "c"'] & & C &
\end{tikzcd}
\tag{151}
$$

Let us rename $p' := c^*p : A' \to C'$ and $B' := c^*B$ and $q' := c^*q$. Now let $w = t \circ s$ be the pathspace factorization (**??**) of $w$, as a map over $C$. Since $q : B \to C$ is a fibration, by the foregoing remarks on pathspace factorizations, we know that $s : A \to P_w$ has a retraction $p_w : P_w \to A$ which is a trivial fibration. The retraction $p_w$ is a map over $C$.

The pathspace factorization $w = t \circ s : A \to P_w \to B$ is stable under pullback along $c$, providing a pathspace factorization $w' = t' \circ s' : A' \to P_{w'} \to B'$ over $C'$. Since both $p'$ and $q'$ are fibrations, the retraction $p_{w'} : P_{w'} \to A'$ is a trivial fibration, and now $t' : P_{w'} \to B'$ is a fibration.

$$
\begin{tikzcd}
A' \arrow[r, "s'"] \arrow[dr, "w'", "\sim"'] \arrow[dd, two heads, "p'"'] & P_{w'} \arrow[l, dotted, bend right, two heads, "p_{w'}"'] \arrow[d, two heads, "t'"] \\
& B' \arrow[dl, two heads, "q'"] \\
C' &
\end{tikzcd}
\tag{152}
$$

Thus the composite $q' \circ t' : P_{w'} \to B' \to C'$ is a fibration and therefore, by the retraction over $C'$ with the trivial fibration $p_{w'}$, we have that $s' : A' \to P_{w'}$ is a weak equivalence, by 3-for-2 for weak equivalences between fibrations. For the same reason, $t'$ is then a weak equivalence, and therefore a trivial fibration.

Since $t' = c^*t = b^*t$ is a trivial fibration, its pushforward $b_*b^*t$ along $b$ is also one by by Corollary 16. Moreover, $b_*b^*t : b_*b^*P_w \to B$ admits a unit $\eta : P_w \to b_*b^*P_w$ (over $B$).

$$
\begin{tikzcd}
A' \arrow[rr, tail] \arrow[dd, two heads] \arrow[dr, "w'", "\sim"'] & & A \arrow[r, "s"] \arrow[dr, "w"] \arrow[dd, "p" near start] & P_w \arrow[r, "\eta"] \arrow[d, "t"] \arrow[l, dotted, bend right, "p_w"'] & b_*b^*P_w \arrow[dl, "b_*b^*t"] \\
& B' \arrow[rr, tail, "b"' near start] \arrow[dl] & & B \arrow[dl, "q"] & \\
C' \arrow[rr, tail, "c"'] & & C & &
\end{tikzcd}
\tag{153}
$$

We now *claim* that $\eta : P_w \to b_*b^*P_w$ is a trivial fibration. Given that, the composite $t = b_*b^*t \circ \eta$ is also a trivial fibration, whence $q \circ t : P_w \to C$ is a fibration, and so its retract $p : A \to C$ is a fibration. Moreover, since $s$ is a section of the trivial fibration $p_w : P_w \to A$ between fibrations, as before it is also a weak equivalence. Thus $w = t \circ s$ is a weak equivalence, and we are finished.

To prove the remaining claim that $\eta : P_w \to b_*b^*P_w$ is a trivial fibration, we shall use lemma 107. It does not apply directly, however, since $t : P_w \to B$ is not yet known to be a trivial fibration. Instead, we show that $\eta$ is a pullback of the corresponding unit at the trivial fibration $p_1 : B^{\mathrm{I}} \to B$.

Consider the following cube (viewed with $b : B' \to B$ at the front).

$$
\begin{tikzcd}
P_{w'} \arrow[rr, tail, "\overline{a}"] \arrow[dr, "(p_0')^*w'"'] \arrow[dd, two heads, "p_{w'}"'] & & P_w \arrow[dr, "p_0^*w"] \arrow[dd, "p_w" near start] & \\
& B'^{\mathrm{I}} \arrow[rr, tail, "\overline{b}" near end] \arrow[dd, two heads, "p_0'" near start] & & B^{\mathrm{I}} \arrow[dd, two heads, "p_0"] \\
A' \arrow[rr, tail, "a"' near end] \arrow[dr, "w'"'] & & A \arrow[dr, "w"] & \\
& B' \arrow[rr, tail, "b"'] & & B
\end{tikzcd}
\tag{154}
$$

The right hand face is a pullback by definition, and the remainder results from pulling the right face back along $b$, by the stability of the pathspace factorization (148). Thus all faces are pullbacks. The base is also a pushforward, $b_*w' = w$, again by definition. Thus the top face is also a pushforward, $\overline{b}_*((p_0')^*w') = p_0^*w$. Indeed, since the front face is a pullback, the Beck-Chevalley condition applies, and so we have $\overline{b}_*(p_0')^*(w') = p_0^*\, b_*(w') = p_0^* w$.

Now consider the following, in which the top square remains the same as in (154), but $p_0$ has been relaced by $p_1 : B^{\mathrm{I}} \to B$, so the composite at right is by definition $t = p_1 \circ p_0^* w$.

$$
\begin{tikzcd}
P_{w'} \arrow[rr, tail, "\overline{a}"] \arrow[dr, "(p_0')^*w'"'] \arrow[ddr, two heads, "t'"'] & & P_w \arrow[dr, "p_0^*w"] \arrow[ddr, two heads, "t"] & \\
& B'^{\mathrm{I}} \arrow[rr, tail, "\overline{b}"] \arrow[d, two heads, "p_1'"] & & B^{\mathrm{I}} \arrow[d, two heads, "p_1"] \\
& B' \arrow[rr, tail, "b"'] & & B
\end{tikzcd}
\tag{155}
$$

The horizontal direction is still pullback along $b$; let us rename $p_0^* w =: u$ so that $(p_0')^*w' = b^*u$ and $t' = b^*t$ and $p_1' = b^*p_1$ to make this clear. We then add the pushforward along $b$ on the right, in order to obtain the two units $\eta$.

$$
\begin{tikzcd}
b^*P_w \arrow[rr, tail, "\overline{a}"] \arrow[dr, "b^*u"'] \arrow[ddr, two heads, "b^*t"'] & & P_w \arrow[rr, "\eta_t"] \arrow[dr, "u"] \arrow[ddr, two heads, "t"] & & b_*b^*P_w \arrow[dr, "b_*b^*u"] \arrow[ddr, "b_*b^*t"'] & \\
& b^*B^{\mathrm{I}} \arrow[rr, tail, "\overline{b}"] \arrow[d, two heads, "b^*p_1"] & & B^{\mathrm{I}} \arrow[rr, "\eta_{p_1}"] \arrow[d, two heads, "p_1"] & & b_*b^*B^{\mathrm{I}} \arrow[d, "b_*b^*p_1"] \\
& B' \arrow[rr, tail, "b"'] & & B \arrow[rr, "="'] & & B
\end{tikzcd}
\tag{156}
$$

By the usual calculation of pushforwards in slice categories, $\overline{b}_* \cong \eta_{p_1}^* \circ b_*$, and so for $b^*u$ we have $\overline{b}_* b^*u = \eta_{p_1}^* b_* b^* u$. But as we just determined in (154) the top left square is already a pushforward, and therefore $u = \eta_{p_1}^* b_* b^* u$, so the top right naturality square is a pullback.

To finish the proof as planned, $p_1 : B^{\mathrm{I}} \to B$ is a trivial fibration because $q : B \to C$ is a fibration, and $b : B' \rightarrowtail B$ is a cofibration because it is a pullback of $c : C' \rightarrowtail C$. Thus by lemma 107, we have that $\eta_{p_1} : B^{\mathrm{I}} \to b_*b^*B^{\mathrm{I}}$ is a trivial fibration, and so its pullback $\eta_t : P_w \to b_*b^*P_w$ is a trivial fibration, as claimed. $\square$

*Remark* 109. Note that $p : A \to C$ is small if $q : B \to C$ is small.

## 8. The fibration extension property

Given a universal fibration $\dot{\mathcal{U}} \twoheadrightarrow \mathcal{U}$, such as $\dot{\mathsf{Fib}} \twoheadrightarrow \mathsf{Fib}$ of Proposition 88, the fibration extension property (Definition 60) is closely related to the statement that the base object $\mathcal{U}$ is fibrant. For Kan simplicial sets, Voevodsky proved the latter directly, using the theory of minimal fibrations [CK21]. In a more general (but still simplicial) setting, Shulman [Shu15] gives a proof using univalence, in the form of the equivalence extension property of Section 7, but that proof also uses the 3-for-2 property for weak equivalences, which we do not yet have. For cubical sets, Coquand [CCHM18] uses the equivalence extension property to prove that $\mathcal{U}$ is fibrant without assuming 3-for-2 for weak equivalences, via a neat type theoretic argument reducing box-filling to an operation of *Kan-composition*. We shall prove that $\mathcal{U}$ is fibrant using the equivalence extension property, also without assuming 3-for-2 for weak equivalences, but via a different argument than that in [CCHM18], one not using (type theory or) Kan composition.

Returning to the relation between the fibration extension property and the fibrancy of the base object of the universal fibration $\dot{\mathcal{U}} \twoheadrightarrow \mathcal{U}$, it is easy to see that the latter implies the former. Indeed, let $t : X \rightarrowtail X'$ be a trivial cofibration and $Y \twoheadrightarrow X$ a fibration. To extend $Y$ along $t$, take a classifying map $y : X \to \mathcal{U}$, so that $Y \cong y^*\dot{\mathcal{U}}$ over $X$. If $\mathcal{U}$ is fibrant then we can extend $y$ along $t : X \rightarrowtail X'$ to get $y' : X' \to \mathcal{U}$ with $y = y' \circ t$. The pullback $Y' = (y')^*\dot{\mathcal{U}} \twoheadrightarrow X'$ is then a (small) fibration such that $t^*Y' \cong t^*(y')^*\dot{\mathcal{U}} \cong y^*\dot{\mathcal{U}} \cong Y$ over $X$.

$$\begin{tikzcd}
Y \arrow[rr] \arrow[dr] \arrow[dd, two heads] & & \dot{\mathcal{U}} \arrow[dd, two heads] \\
& Y' \arrow[ur, dotted] \arrow[dd, two heads] & \\
X \arrow[rr, "y"] \arrow[dr, tail, "t"'] & & \mathcal{U} \\
& X' \arrow[ur, dotted, "y'"'] &
\end{tikzcd}$$

Thus, for the record, we have:

**Proposition 110.** *If the base object $\mathcal{U}$ of the universal fibration $\dot{\mathcal{U}} \twoheadrightarrow \mathcal{U}$ is fibrant, then the fibration weak factorization system has the fibration extension property.*

Conversely, given the Realignment Lemma 93, the fibration extension property also implies the fibrancy of $\mathcal{U}$:

**Corollary 111.** *The fibration extension property implies that the base $\mathcal{U}$ of the universal fibration $\dot{\mathcal{U}} \twoheadrightarrow \mathcal{U}$ is fibrant: given any $y : X \to \mathcal{U}$ and trivial cofibration $t : X \rightarrowtail X'$, there is a map $y' : X' \to \mathcal{U}$ with $y' \circ t = y$.*

*Proof.* Take the pullback of $\dot{\mathcal{U}} \twoheadrightarrow \mathcal{U}$ along $y : X \to \mathcal{U}$ to get a (small) a fibration $Y \twoheadrightarrow X$, which extends along the (trivial) cofibration $t : X \rightarrowtail X'$ by the fibration extension property, to a (small) fibration $Y' \twoheadrightarrow X'$ with $Y \cong t^*Y'$ over $X$. By realignment there is a classifying map $y' : X' \to \mathcal{U}$ for $Y'$ with $y' \circ t = y$. □

Now let us show the following.

**Proposition 112.** *The base $\mathcal{U}$ of the universal fibration $\dot{\mathcal{U}} \twoheadrightarrow \mathcal{U}$ in* $\mathsf{cSet}$*, as constructed in section 6, is a fibrant object.*

*Proof.* By Corollary 25, $\mathcal{U}$ is an unbiased fibrant object if the canonical map $u = \langle \mathsf{p}_2, \mathsf{eval}\rangle$ in the following diagram in $\mathsf{cSet}$, is a trivial fibration.

$$\begin{tikzcd}
\mathcal{U}^{\mathrm{I}} \times \mathrm{I} \arrow[dr, dotted, "u"] \arrow[drr, bend left, "\mathsf{eval}"] \arrow[ddr, bend right, "\mathsf{p}_2"'] & & \\
& \mathrm{I} \times \mathcal{U} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{U} \arrow[d] \\
& \mathrm{I} \arrow[r] & 1
\end{tikzcd} \tag{157}$$

Thus consider a filling problem of the following form, with an arbitrary cofibration $c : C \rightarrowtail Z$.

$$\begin{tikzcd}
C \arrow[r] \arrow[d, tail, "c"'] & \mathcal{U}^{\mathrm{I}} \times \mathrm{I} \arrow[d, "\langle \mathsf{p}_2, \mathsf{eval}\rangle"] \\
Z \arrow[r] \arrow[ur, dotted] & \mathrm{I} \times \mathcal{U}
\end{tikzcd} \tag{158}$$

The horizontal maps may be written in the form $\langle i, b\rangle : Z \to \mathrm{I} \times \mathcal{U}$ and $\langle \tilde{a}, ic\rangle : C \to \mathcal{U}^{\mathrm{I}} \times \mathrm{I}$, regarding $i : Z \to \mathrm{I}$ as an I-indexing.

Transposing $\tilde{a}$ to $a : C \times \mathrm{I} \to \mathcal{U}$ we obtain the new problem

$$\begin{tikzcd}
C \arrow[r, "\langle ic\rangle"] \arrow[d, tail, "c"'] & C \times \mathrm{I} \arrow[d, "c \times \mathrm{I}"] \arrow[ddr, bend left, "a"] & \\
Z \arrow[r, "\langle i\rangle"'] \arrow[drr, bend right, "b"'] & Z \times \mathrm{I} \arrow[dr, dotted, "d"] & \\
& & \mathcal{U}
\end{tikzcd} \tag{159}$$

in which we recall from (42) the graph notation $\langle i\rangle = \langle 1_Z, i\rangle : Z \to Z \times \mathrm{I}$. Given a map $d$ as shown in (159), we can obtain the indicated diagonal filler in (158) as $\langle \tilde{d}, i\rangle : Z \to \mathcal{U}^{\mathrm{I}} \times \mathrm{I}$.

As a sanity check, note that $b \circ c = a \circ \langle ic\rangle$ turns the problem (159) into that of extending the copair $[b, a]$ along the unique map

$$Z +_C (C \times \mathrm{I}) \longrightarrow Z \times \mathrm{I}\,,$$

which is exactly the (trivial cofibration) pushout-product $c \otimes_i \delta$ from (43), recalled below for the reader's convenience.

$$\begin{tikzcd}
C \arrow[r, "\langle ic\rangle"] \arrow[d, tail, "c"'] & C \times \mathrm{I} \arrow[d] \arrow[ddr, bend left, "c\times\mathrm{I}"] & \\
Z \arrow[r] \arrow[drr, bend right, "\langle i\rangle"'] & Z +_C (C \times \mathrm{I}) \arrow[dr, dotted, "c \otimes_i \delta"] & \\
& & Z \times \mathrm{I}
\end{tikzcd} \tag{160}$$

Returning to (159), take pullbacks of $\dot{\mathcal{U}} \twoheadrightarrow \mathcal{U}$ along $a$ and $b$ to get fibrations $p_a : A \twoheadrightarrow C \times \mathrm{I}$ and $p_b : B \twoheadrightarrow Z$ respectively, and let

$$p_c := \langle ic\rangle^* p_a : A_c \longrightarrow C$$

be the corresponding "fiber of $A$ over the graph of $ic$". We then have $c^*B \cong A_c$ over $C$ by the commutativity of the outer square of (158).

$$\begin{tikzcd}
& A_c \arrow[rr] \arrow[d, two heads, "p_c"] \arrow[dl] \arrow[drr, phantom, very near start, "\lrcorner"] & & A \arrow[d, two heads, "p_a"] \\
B \arrow[dd, two heads, "p_b"'] & C \arrow[rr, "\langle ic\rangle"'] \arrow[ddl, "c"] & & C \times \mathrm{I} \arrow[ddl, "c\times\mathrm{I}"] \\
& & & \\
Z \arrow[rr, "\langle i\rangle"'] & & Z \times \mathrm{I} &
\end{tikzcd}$$

The diagonal filler sought in (158) corresponds, again by transposition and pullback of $\dot{\mathcal{U}} \twoheadrightarrow \mathcal{U}$, to a fibration $p_d : D \twoheadrightarrow Z \times \mathrm{I}$ with

$\langle i\rangle^* D \cong B$ over $Z$ and $(c\times \mathrm{I})^* D \cong A$ over $C \times \mathrm{I}$, as indicated below.

$$\begin{tikzcd}
& A_c \arrow[rr] \arrow[d, two heads, "p_c"] \arrow[ddl] & & A \arrow[d, two heads, "p_a"] \arrow[ddl, dotted] \\
& C \arrow[rr, "\langle ic\rangle"'] \arrow[ddl, "c"] & & C\times\mathrm{I} \arrow[ddl, "c\times\mathrm{I}"] \\
B \arrow[rr, dotted] \arrow[d, two heads, "p_b"'] & & D \arrow[d, dotted, two heads, "p_d"'] & \\
Z \arrow[rr, "\langle i\rangle"'] & & Z\times\mathrm{I} &
\end{tikzcd} \tag{161}$$

We shall construct $p_d : D \twoheadrightarrow Z \times \mathrm{I}$ using the equivalence extension property (Proposition 108) as follows. First apply the functor $(-)\times\mathrm{I}$ to the left vertical (pullback) face of the cube in (161) to get the following, with a new pullback square on the right with the indicated fibrations.

$$\begin{tikzcd}
& A_c \arrow[rr] \arrow[d, two heads, "p_c"] \arrow[ddl] & & A \arrow[d, two heads, "p_a"] \arrow[ddl, dotted] & & A_c\times\mathrm{I} \arrow[dll, two heads, "p_c\times\mathrm{I}"] \arrow[ddl] \\
& C \arrow[rr, "\langle ic\rangle"'] \arrow[ddl, "c"] & & C\times\mathrm{I} \arrow[ddl, "c\times\mathrm{I}"] & & \\
B \arrow[rr, dotted] \arrow[d, two heads, "p_b"'] & & D \arrow[d, dotted, two heads, "p_d"'] & & B\times\mathrm{I} \arrow[dll, two heads, "p_b\times\mathrm{I}"] & \\
Z \arrow[rr, "\langle i\rangle"'] & & Z\times\mathrm{I} & & &
\end{tikzcd} \tag{162}$$

We now *claim* that there is a weak equivalence $e : A \simeq A_c \times \mathrm{I}$ over $C \times \mathrm{I}$. From this it follows by the equivalence extension property (Proposition 108) that there are:

(i) a fibration $p_d : D \twoheadrightarrow Z \times \mathrm{I}$ with $(c \times \mathrm{I})^* D \cong A$ over $C \times \mathrm{I}$, and
(ii) a weak equivalence $f : D \simeq B \times \mathrm{I}$ over $Z \times \mathrm{I}$ with $(c \times \mathrm{I})^* f \cong e$ over $C \times \mathrm{I}$.

It then remains only to show that $B \cong \langle i\rangle^* D$ over $Z$ to complete the proof.

To obtain the claimed weak equivalence $e$, consider the following square,

$$\begin{tikzcd}
A_c \arrow[r, "\langle icp_c\rangle"] \arrow[d] & A_c\times\mathrm{I} \arrow[d, two heads, "p_c\times\mathrm{I}"] \\
A \arrow[r, two heads, "p_a"'] & C\times\mathrm{I}\,,
\end{tikzcd} \tag{163}$$

in which the top horizontal map is the graph of the composite,

$$A_c \overset{p_c}{\twoheadrightarrow} C \overset{c}{\rightarrowtail} Z \xrightarrow{i} \mathrm{I}\,,$$

and the others are the evident ones from (162). The square is easily seen to commute, and the top map is a trivial cofibration (by Remark 29), because it is the graph of a map into I. The left map is also a trivial cofibration by Frobenius (Proposition 69), because by its definition in (161) it is the pullback of another such graph $\langle ic\rangle$ along the fibration $p_a$. A simple lemma (Lemma 113 below) provides the claimed weak equivalence $e : A \simeq A_c \times \mathrm{I}$ over $C \times \mathrm{I}$.

To see that $B \cong \langle i\rangle^* D$ over $Z$, recall from the proof of the equivalence extension property that the map $f : D \cong B \times \mathrm{I}$ is the pushforward of $e : A \simeq A_c \times \mathrm{I}$ along the cofibration $b_c \times \mathrm{I} : A_c \times \mathrm{I} \rightarrowtail B \times \mathrm{I}$, where we are calling the evident map in (162) $b_c : A_c \rightarrowtail B$. Thus by construction $f = (b_c \times \mathrm{I})_* e$. We can then apply the Beck-Chevalley condition for the pushforward using the pullback square on the left below.

$$\begin{tikzcd}
A_c \arrow[r, tail, "\langle icp_c\rangle"] \arrow[d, tail, "b_c"'] \arrow[dr, phantom, "\lrcorner", very near start] & A_c \times \mathrm{I} \arrow[d, tail, "b_c \times \mathrm{I}"] & A \arrow[l, "e"'] \\
B \arrow[r, tail, "\langle ip_b\rangle"'] & B \times \mathrm{I} & D \arrow[l, "f"]
\end{tikzcd} \tag{164}$$

The pullback of $e$ along the top of the square is the identity on $A_c$, as can be seen by pulling back $e$ as a map over $C \times \mathrm{I}$ along $\langle ic\rangle : C \to C \times \mathrm{I}$. Thus the same is true up to isomorophism for the pullback of $f$ along the bottom.

An application of the Realignment Lemma 93 along the trivial cofibration $c \otimes_i \delta$ completes the proof. □

**Lemma 113.** *Suppose the following square commutes and the indicated cofibrations are trivial.*

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d, tail] & C \arrow[d, two heads] \\
B \arrow[r, two heads] & D
\end{tikzcd} \tag{165}$$

*Then there is a weak equivalence $e : B \simeq C$ over $D$ (and under $A$).*

*Proof.* Use the fact that any two diagonal fillers are homotopic to get a homotopy equivalence $e : B \simeq C$ filling the square. □

*Remark* 114. The foregoing proof also works, *mutatis mutandis*, for the case of *biased* fibrations, as in the setting of [CCHM18].

Applying proposition 110 now yields the following.

**Corollary 115.** *The fibration weak factorization system has the fibration extension property (definition 60).*

By Theorem 64, finally, we have the following.

**Theorem 116.** *There is a Quillen model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ on the category* $\mathsf{cSet}$ *of cubical sets for which:*

1. *the cofibrations $\mathcal{C}$ are any class of maps satisfying (C0)-(C8) (equivalently, the simplified axioms in the Appendix),*
2. *the fibrations $\mathcal{F}$ are the maps $f : Y \to X$ for which the canonical map*
$$(f^{\mathrm{I}} \times \mathrm{I}, \mathsf{eval}_Y) : Y^{\mathrm{I}} \times \mathrm{I} \longrightarrow (X^{\mathrm{I}} \times \mathrm{I}) \times_X Y$$
*lifts on the right against $\mathcal{C}$.*
3. *the weak equivalences $\mathcal{W}$ are the maps $w : X \to Y$ for which the internal precomposition $K^w : K^Y \to K^X$ is bijective on connected components for every fibrant object $K$.*

*Remark* 117. We note that in terms of the universal fibration $\dot{\mathcal{U}} \twoheadrightarrow \mathcal{U}$ constructed in Section 6 the equivalence extension property Proposition 108 says that the second projection from the classifying type of equivalences $A \simeq B$ between small families,
$$\pi_2 : \Sigma_{A,B}\mathsf{Eq}(A, B) \longrightarrow \mathcal{U}\,,$$
is a trivial fibration. From this, it follows that the canonical transport map
$$* : \mathcal{U}^{\mathrm{I}} \longrightarrow \Sigma_{A,B}\mathsf{Eq}(A, B) \tag{166}$$
is an equivalence over the base $\mathcal{U}$ via $p_2 : \mathcal{U}^{\mathrm{I}} \to \mathcal{U}$, which is also a trivial fibration, because $\mathcal{U}$ is fibrant (by Proposition 112). In type theory, the pathobject $\mathcal{U}^{\mathrm{I}}$ of course interprets the identity type $A = B$, so the equivalence (166) can be expressed as
$$(A = B) \simeq (A \simeq B)\,.$$

**Appendix A: Axioms for cartesian cofibrations.** A system of maps satisfying the axioms (C0)-(C8) above for the cofibrations in a cartesian cubical model category may be called *cartesian cofibrations.* The axioms can be restated equivalently as follows.

- (A0) All cofibrations are monomorphisms.
- (A1) All isomorphisms are cofibrations.
- (A2) The composite of two cofibrations is a cofibration.
- (A3) Any pullback of a cofibration is a cofibration.
- (A4) The category of cofibrations and cartesian squares has a terminal object.

(A5) The join of two cofibrant subobjects is a cofibration.
(A6) The diagonal map $\mathrm{I} \to \mathrm{I} \times \mathrm{I}$ is a cofibration.
(A7) Cofibrations are preserved by the pathobject functor $(-)^{\mathrm{I}}$.

**Proposition 118.** *In any topos, the locally decidable subobjects satisfy the axioms for cartesian cofibrations.*

*Proof.* We generalize (107) from the case of $\mathbb{2} \in \mathsf{Set}$ to an arbitrary base topos $\mathcal{E}$ and consider the presheaf category $\mathcal{E}^{\mathbb{C}}$ for a category $\mathbb{C}$ internal to $\mathcal{E}$. [*fill in!*] □

**Appendix B: Cartesian cubical sets classifies intervals.** Recall that the objects of the Cartesian cube category $\square$ may be taken concretely to be finite, strictly bipointed sets, written

$$[n] = \{0, x_1, ..., x_n, 1\},$$

and the arrows $f : [n] \to [m]$ to be all bipointed maps $[m] \to [n]$ (note the direction). The category of *(Cartesian) cubical sets* is the presheaf topos

$$\mathsf{cSet} = \mathsf{Set}^{\square^{\mathrm{op}}}.$$

It is generated by the representable presheaves $\mathrm{I}^n = \mathsf{y}[n]$, called the *$n$-cubes*. The 0-cube is $\mathrm{I}^0 = \mathsf{y}[0] = 1$; the 1-cube is $\mathrm{I} = \mathsf{y}[1]$; and $\mathrm{I}^n \times \mathrm{I}^m \cong \mathrm{I}^{n+m}$ in virtue of preservation of products by the Yoneda embedding $\mathsf{y} : \square^{\mathrm{op}} \hookrightarrow \mathsf{cSet}$. For a cubical set $X : \square^{\mathrm{op}} \to \mathsf{Set}$ we write $X_n = X[n]$ and call this the *set of $n$-cubes in $X$*, for which we have the usual Yoneda correspondence,

$$\{c \in X_n\} \;\cong\; \{c : \mathrm{I}^n \to X\}.$$

In particular, $\mathrm{I}^n_m = \mathsf{cSet}([m], [n])$ is the set of $m$-cubes in the $n$-cube.[3]

**Proposition 119.** *The category* $\mathsf{cSet}$ *of Cartesian cubical sets is the classifying topos for* intervals*: objects $\mathcal{I}$ with points $i, j : 1 \rightrightarrows \mathcal{I}$ the pullback of which is 0:*

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "j"] \\ 1 \arrow[r, "i"'] & \mathcal{I} \end{tikzcd}$$

[3]Note that the cardinality of $\mathrm{I}^n_m$ is therefore just $(m+2)^n$, in comparison to the *Dedekind* cubes, for which *e.g.* $\mathsf{cSet}([1], [n])$ the $n^{th}$ *Dedekind number*, the number of elements in the free distributive lattice on $n$ generators, which in general is a number so large that it is not known even for values of $n > 8$.

*Proof.* Consider the covariant presentation $\mathsf{cSet} = \mathsf{Set}^{\mathbb{B}}$ where $\mathbb{B}$ is the category of finite, strictly bipointed sets and bipointed maps. We can extend $\mathbb{B} \hookrightarrow \mathbb{B}_=$ by freely adjoining coequalizers, making $\mathbb{B}_=$ the free finite *colimit* category on a co-bipointed object. An concrete presentation of $\mathbb{B}_=$ is the finite bipointed sets, including those with $0 = 1$. Let us write $(n)$ for the bipointed set $\{x_1, ..., x_n, *\}$, with $n$ (non-constant) elements and a further element $0 = * = 1$. There is an evident coequalizer $[1] \rightrightarrows [n] \to (n)$, which just identifies the distinguished points, and every coqualizer has either the form $[m] \rightrightarrows [n] \to [k]$ or $[m] \rightrightarrows [n] \to (k)$, for a suitable choice of $k$. Note that there are no maps of the form $(m) \to [n]$, and that every map $[m] \to (n)$ factors uniquely as $[m] \to (m) \to (n)$ with $[m] \to (m)$ the canonical coequalizer of 0 and 1. The category $\mathbb{B}_=$ can therefore be decomposed into two "levels", the upper one of which is essentially $\mathbb{B}$, the lower one consisting of just the objects $(n)$ and thus essentially the finite *pointed* sets, and for each $n$, there is the canonical coequalizer $[n] \to (n)$ going from the upper level to the lower one.

$$\begin{tikzcd}
\dots \arrow[r] & {[m]} \arrow[r] \arrow[d, two heads] & {[n]} \arrow[r] \arrow[d, two heads] & \dots \\
\dots \arrow[r] & (m) \arrow[r] & (n) \arrow[r] & \dots
\end{tikzcd}$$

Write $u : \mathbb{B} \to \mathbb{B}_=$ for the upper inclusion, which is the classifying functor of generic co-bipointed object in $\mathbb{B}_=$ (which is strict).

Now consider the induced geometric morphism:

$$\begin{tikzcd}
\mathsf{Set}^{\mathbb{B}} \arrow[r, shift left=2, "u_*"] \arrow[r, shift right=2, "u_!"'] & \mathsf{Set}^{\mathbb{B}_=} \arrow[l, "u^*" description]
\end{tikzcd} \qquad u_! \dashv u^* \dashv u_*$$

Since $u^*$ is the restriction along $u$, the right adjoint $u_*$ must be "prolongation by 1",

$$\begin{aligned} u_*(P)[n] &= P[n], \\ u_*(P)(n) &= \{*\}, \end{aligned}$$

with the obvious maps,

$$\begin{tikzcd}
\dots \arrow[r] & P[m] \arrow[r] \arrow[d, two heads] & P[n] \arrow[r] \arrow[d, two heads] & \dots \\
\dots \arrow[r] & \{*\} \arrow[r] & \{*\} \arrow[r] & \dots
\end{tikzcd}$$

as is easily seen by considering maps in $\mathsf{Set}^{\mathbb{B}_=}$ of the form

$$\begin{tikzcd} Q[n] \arrow[r] \arrow[d, "\cdot"'] & P[n] \arrow[d, two heads, "\cdot"] \\ Q(n) \arrow[r] & \{*\}. \end{tikzcd}$$

Since $u_* : \mathsf{Set}^{\mathbb{B}} \to \mathsf{Set}^{\mathbb{B}_=}$ is evidently full and faithful, it is the inclusion part of a sheaf subtopos $\mathsf{sh}(\mathbb{B}_=^{\mathsf{op}}, j) \hookrightarrow \mathsf{Set}^{\mathbb{B}_=}$ for a suitable Grothendieck topology $j$ on $\mathbb{B}_=^{\mathsf{op}}$. We claim that $j$ is the closed complement topology of the subobject $[\bot = \top] \rightarrowtail 1$ represented by the coequalizer $[0] \to (0)$. Indeed, in $\mathsf{Set}^{\mathbb{B}_=}$ we have the representable functors:

$$\begin{aligned} \mathrm{I} &= y[1], \\ 1 &= y[0], \\ [0 = 1] &= y(0) \end{aligned}$$

fitting into an equalizer $[0 = 1] \to 1 \rightrightarrows \mathrm{I}$, which is the image under Yoneda of the canonical coequalizer $[1] \rightrightarrows [0] \to (0)$ in $\mathbb{B}_=$. The closed complement topology for $[0 = 1] \rightarrowtail 1$ is generated by the single cover $0 \to [0 = 1]$, which can be described logically as forcing the sequent $(0 = 1 \vdash 0)$ to hold. Recall from [Joh77], Proposition 3.53, the following simple characterization of the sheaves for a closed topology generated by an object $U \rightarrowtail 1$: an object $X$ is a sheaf iff $X \times U \cong U$. In the present case, it therefore suffices to show that for any $P : \mathbb{B}_= \to \mathsf{Set}$ we have:

$$P \times [0 = 1] \cong [0 = 1] \quad \text{iff} \quad P(n) = 1 \text{ for all } n.$$

For any object $b \in \mathbb{B}_=$, consider the map

$$\mathrm{Hom}(yb, P \times [0 = 1]) \cong \mathrm{Hom}(yb, P) \times \mathrm{Hom}(yb, [0 = 1]) \to \mathrm{Hom}(yb, [0 = 1]).$$

If $b = [k]$, then $\mathrm{Hom}(yb, [\bot = \top]) \cong \mathrm{Hom}_{\mathbb{B}_=}((0), [k]) \cong 0$, and so we always have an iso

$$\begin{aligned} \mathrm{Hom}(yb, P \times [\bot = \top]) &\cong \mathrm{Hom}(yb, P) \times \mathrm{Hom}(yb, [0 = 1]) \\ &\cong \mathrm{Hom}(yb, P) \times 0 \cong 0. \end{aligned}$$

If $b = (k)$, then $\mathrm{Hom}(y(k), [0 = 1]) \cong \mathrm{Hom}_{\mathbb{B}_=}((0), (k)) \cong 1$, and we have an iso

$$\begin{aligned} \mathrm{Hom}(y(k), P \times [\bot = \top]) &\cong \mathrm{Hom}(y(k), P) \times \mathrm{Hom}(y(k), [0 = 1]) \\ &\cong \mathrm{Hom}(y(k), P) \times 1 \cong \mathrm{Hom}(y(k), P) \cong P(k). \end{aligned}$$

Thus we will have an iso $P \times [0 = 1] \cong [0 = 1]$ iff $P(k) \cong 1$.

Thus the presheaf topos $\mathsf{Set}^{\mathbb{B}}$ is the closed complement of the open subtopos

$$\mathsf{Set}^{\mathbb{B}_=}/_{[0=1]} \hookrightarrow \mathsf{Set}^{\mathbb{B}_=}$$

given by forcing the proposition $0 \neq 1$. Since $\mathsf{Set}^{\mathbb{B}_=}$ is clearly the classifying topos for *arbitrary* bipointed objects $b_0, b_1 : 1 \to B$, the subtopos $\mathsf{Set}^{\mathbb{B}}$ then indeed classifies *strictly* bipointed objects, *i.e.* intervals, as claimed. □

**Corollary 120.** *The geometric realization functor to topological spaces*

$$R : \mathsf{cSet} \to \mathsf{Top}$$

*preserves cartesian products,* $R(X \times Y) \cong R(X) \times R(Y)$.[4]

*Proof.* This can of course be shown directly, but it follows immediately by composing the inverse image of the classifying geometric morphism $\mathsf{sSets} \to \mathsf{cSet}$ of the 1-simplex $\Delta^1$ with the standard geometric realization $\mathsf{sSets} \to \mathsf{Top}$, each of which preserves finite products. □

We conclude with a few general remarks about the category $\mathsf{cSet}$.

**Definition 121.** Let $\Box \to \mathsf{Cat}$ be the unique product-preserving functor taking the interval $[1]$ to the one arrow category $\mathbb{2} = (0 \leq 1)$. This functor then takes $[n]$ to $\mathbb{2}^n$, the $n$-fold product in $\mathsf{Cat}$, and maps $[m] \to [n]$ to the corresponding monotone functions of the posets $\mathbb{2}^n$. The *cubical nerve* functor

$$N : \mathsf{Cat} \to \mathsf{cSet}$$

is then defined by:

$$N(\mathbb{C})_n = \mathsf{Cat}(\mathbb{2}^n, \mathbb{C}).$$

Thus $N(\mathbb{C})_0$ is the set of objects of $\mathbb{C}$; $N(\mathbb{C})_1$ is the set of arrows; $N(\mathbb{C})_2$ consists of all commutative squares; $N(\mathbb{C})_3$ all commutative cubes, etc.

**Proposition 122.** *The nerve functor* $N : \mathsf{Cat} \to \mathsf{cSet}$ *is full and faithful.*

*Proof.* Given categories $\mathbb{C}$ and $\mathbb{D}$ and functors $F, G : \mathbb{C} \to \mathbb{D}$, suppose $F(f) \neq G(f)$ for some $f : A \to B$ in $\mathbb{C}$. Take $f^\sharp : \mathbb{2} \to \mathbb{C}$ with image $f$. Then $N(F)_1(f^\sharp) = F(f) \neq G(f) = N(G)_1(f^\sharp)$, and so $N(F) \neq N(G) : N(\mathbb{C}) \to N(\mathbb{D})$. So $N$ is faithful.

---

[4]This fact and the next one Proposition 122 are to be contrasted with the case of monoidal cubical sets, *e.g.* as studied by [Jar02]

For fullness, let $\varphi : N(\mathbb{C}) \to N(\mathbb{D})$ be a natural transformation, and define a proposed functor $F : \mathbb{C} \to \mathbb{D}$ by

$$F_0 = \varphi_0 : \mathbb{C}_0 = N(\mathbb{C})_0 \to N(\mathbb{D})_0 = \mathbb{D}_0$$
$$F_1 = \varphi_1 : \mathbb{C}_1 = N(\mathbb{C})_1 \to N(\mathbb{D})_1 = \mathbb{D}_1.$$

We just need to show that $F$ preserves identity arrows and composition. Consider the following diagram.

$$\begin{tikzcd}
\mathsf{Cat}(\mathbb{2}^1, \mathbb{C}) = N(\mathbb{C})_1 \arrow[r, "F_1"] & N(\mathbb{D})_1 = \mathsf{Cat}(\mathbb{2}^1, \mathbb{D}) \\
\mathsf{Cat}(\mathbb{2}^0, \mathbb{C}) = N(\mathbb{C})_0 \arrow[r, "F_0"'] \arrow[u, "!^*"] & N(\mathbb{D})_0 = \mathsf{Cat}(\mathbb{2}^0, \mathbb{D}). \arrow[u, "!^*"']
\end{tikzcd}$$

Here $!^* : \mathsf{Cat}(\mathbb{2}^0, \mathbb{C}) \to \mathsf{Cat}(\mathbb{2}, \mathbb{C})$ is precomposition with $! : \mathbb{2} = \mathbb{2}^1 \to \mathbb{2}^0 = \mathbb{1}$, so the diagram commutes. But since $! : \mathbb{2} \to \mathbb{1}$ is a functor,

$$\mathbb{C}_0 = \mathsf{Cat}(\mathbb{1}, \mathbb{C}) \xrightarrow{!^*} \mathsf{Cat}(\mathbb{2}, \mathbb{C}) = \mathbb{C}_1$$

takes objects in $\mathbb{C}$ to their identity arrows. Thus $F$ preserves identity arrows. Similarly, for composition, consider

$$\begin{tikzcd}
\mathsf{Cat}(\mathbb{2}^2, \mathbb{C}) = N(\mathbb{C})_2 \arrow[r, "\varphi_2"] \arrow[d, "d^*"'] & N(\mathbb{D})_2 = \mathsf{Cat}(\mathbb{2}^2, \mathbb{D}) \arrow[d, "d^*"] \\
\mathsf{Cat}(\mathbb{2}, \mathbb{C}) = N(\mathbb{C})_1 \arrow[r, "F_1"'] & N(\mathbb{D})_1 = \mathsf{Cat}(\mathbb{2}, \mathbb{D}).
\end{tikzcd}$$

where $\varphi_2 : N(\mathbb{C})_2 \to N(\mathbb{D})_2$ is the action of $\varphi$ on commutative squares of arrows, and $d^* : \mathsf{Cat}(\mathbb{2}^2, \mathbb{C}) \to \mathsf{Cat}(\mathbb{2}, \mathbb{C})$ is precomposition with the diagonal map $d : \mathbb{2} \to \mathbb{2}^2 = \mathbb{2} \times \mathbb{2}$, so the diagram commutes. For any composable pair of arrows $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathbb{C}$ there is a commutative square

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "f"'] & B \arrow[d, "g"] \\
B \arrow[r, "g"'] & C,
\end{tikzcd}$$

and the effect of $d^* : \mathsf{Cat}(\mathbb{2}^2, \mathbb{C}) \to \mathsf{Cat}(\mathbb{2}, \mathbb{C})$ on this square is exactly $g \circ f : A \to C$, and similarly for $d^* : \mathsf{Cat}(\mathbb{2}^2, \mathbb{D}) \to \mathsf{Cat}(\mathbb{2}, \mathbb{D})$. Thus the commutativity of the above diagram implies that $F$ preserves composition. Since clearly $N(F) = \varphi$, we indeed have that $N$ is also full. $\square$

**Proposition 123.** *For any cubical set $X$, the exponential $X^{\mathrm{I}}$ is calculated as the "shift by one dimension",*

$$X^{\mathrm{I}}(n) \cong X(n+1)\,.$$

*Proof.*

$$\begin{aligned} X^{\mathrm{I}}(n) \cong \mathrm{Hom}(y[n], X^{\mathrm{I}}) \cong \mathrm{Hom}(\mathrm{I}^n, X^{\mathrm{I}}) \ &\cong \ \mathrm{Hom}(\mathrm{I}^n \times \mathrm{I}, X) \\ \cong \ \mathrm{Hom}(\mathrm{I}^{n+1}, X) \cong \ \mathrm{Hom}(y[n+1], X) \ &\cong \ X(n+1). \end{aligned}$$

$\square$

**Corollary 124.** *The functor $X \mapsto X^{\mathrm{I}}$ has a* right *adjoint.*

*Proof.* The functor $X \mapsto X^{\mathrm{I}}$ is given by precomposition with the "successor" functor $S : \square \to \square$ with $S[n] = [n+1]$. Thus $X^{\mathrm{I}}([n]) = X(S[n]) = (S^*(X))([n])$. Precomposition always has a right adjoint $S^* \dashv S_*$, which can be calculated as:

$$S_*(X)(n) \cong \mathrm{Hom}(y[n], S_*X) \cong \mathrm{Hom}(S^*(y[n]), X) \ \cong \mathrm{Hom}(\square(S(-), [n]), X).$$

$\square$

We need the following fact in order to calculate the right adjoint further.

**Lemma 125.** *In* $\mathsf{cSet}$, *we have* $\mathrm{I}^{\mathrm{I}} \ \cong \ \mathrm{I} + 1$.

*Proof.* For any $[n] \in \square$ we have:

$$(\mathrm{I}^{\mathrm{I}})(n) \cong \mathrm{I}(n{+}1) \cong \mathrm{Hom}(\mathrm{I}^{(n+1)}, \mathrm{I}) \cong \square([n{+}1], [1]) \cong \mathbb{B}([1], [n{+}1]) \cong n{+}3.$$

On the other hand,

$$(\mathrm{I}+1)(n) \cong \mathrm{I}(n)+1(n) \cong \mathrm{Hom}(\mathrm{I}^n, \mathrm{I})+1 \cong \mathbb{B}([1], [n])+1 \cong (n+2)+1.$$

The isomorphism is natural in $n$. $\square$

We mention that a similar fact holds for the generic object in the object classifier topos, and in the Schanuel topos, and is used in the theory of "abstract higher-order syntax" [MF99].

**Definition 126.** Let us write

$$X_{\mathrm{I}} = S_*(X)$$

for the right adjoint of the path object functor $X^{\mathrm{I}} = S^*X$.

**Corollary 127.** *We have the following calculation for the right adjoint* $X_{\mathrm{I}}$*:*

$$\begin{aligned} X_{\mathrm{I}}(n) &\cong \mathrm{Hom}(\mathrm{I}^n, X_{\mathrm{I}}) \\ &\cong \mathrm{Hom}((\mathrm{I}^n)^{\mathrm{I}}, X) \\ &\cong \mathrm{Hom}((\mathrm{I}^{\mathrm{I}})^n, X) \\ &\cong \mathrm{Hom}((\mathrm{I}+1)^n, X) \\ &\cong \mathrm{Hom}(\mathrm{I}^n + C^n_{n-1}\mathrm{I}^{n-1} + \cdots + C^n_1\mathrm{I} + 1, X) \\ &\cong X(n) \times X(n-1)^{C^n_{n-1}} \times \cdots \times X(1)^{C^n_1} \times X(0), \end{aligned}$$

*where* $C^n_k$ *is the usual binomial coefficient* $\binom{n}{k}$*.*

**Corollary 128.** *There is a natural transformation* $X_{\mathrm{I}} \to X$*, given by the first projection from* $X_{\mathrm{I}}(n) \cong X(n) \times X(n-1)^{C^n_{n-1}} \times \cdots \times X(1)^{C^n_1} \times X(0)$*.*

Finally, we observe that that the path object functor $X^{\mathrm{I}}$ itself, as a left adjoint, preserves all *co*limits. This does not hold in general in type theory, but will be a special property of the cubical model. (Cf. Lawvere [Law04] on the notion of "tiny" objects and the "amazing right adjoint".)

*Example.* (P. Aczel) The cubical set $P$ of polynomials (over the integers, say), is defined by:

$$P_n = \{p(x_1, ..., x_n) \mid \text{polynomials in at most } x_1, ..., x_n\}$$

with the evident maps $P_m \to P_n$ for each function $[m] \to [n]$.

This is a ring object in the category of cubical sets, and the interval $\mathrm{I} = y[1]$ embeds into $P$. The same is true for any algebraic theory $\mathbb{T}$ with two constants, such as boolean algebras: there is a cubical $\mathbb{T}$-algebra $A$ and a monic $\mathrm{I} \rightarrowtail A$.

Let $\square[\mathrm{I}] = \square$ be the cube category, classifying intervals, and $\square[\mathbb{T}]$ the classifying category for $\mathbb{T}$-algebras. There is an interval $J$ in $\square_{\mathbb{T}}$ consisting of the generic $\mathbb{T}$-algebra and its two constants. This $J$ has a classifying functor $J : \square_{\mathrm{I}} \to \square_{\mathbb{T}}$, inducing functors on presheaves

$$J_! \dashv J^* \dashv J_* : \mathsf{Set}^{\square_{\mathrm{I}}^{\mathrm{op}}} \to \mathsf{Set}^{\square_{\mathbb{T}}^{\mathrm{op}}}$$

as usual, where $J_! \circ \mathsf{y}_{\square_{\mathrm{I}}} = \mathsf{y}_{\square_{\mathbb{T}}} \circ J$, with $\mathsf{y}$ the respective Yoneda embeddings.

We can calculate:

$$
\begin{aligned}
J^*J_!(\mathrm{I})([n]) &= J^*J_!(Y[1])([n]) \\
&= J^*Y(J[1])([n]) = Y(J[1])(J[n]) \\
&= \square_{\mathbb{T}}(J[n], J[1]) = \mathbb{T}\mathrm{-Alg}(J[1], J[n]) \\
&= \mathbb{T}\mathrm{-Alg}(F(1), F(n)) = |F(n)|,
\end{aligned}
\tag{167}
$$

where $F(n)$ is the free $\mathbb{T}$-algebra on $n$ generators. So in the case of polynomials we indeed have

$$P = J^*J_!(\mathrm{I}).$$

The unit of the adjunction $\mathrm{I} \to J^*J_!(\mathrm{I})$ is faithful, since $J$ itself is faithful and therefore the left adjoint $J_!$ is faithful. $P$ is a ring in $\mathsf{Set}^{\square_{\mathrm{I}}^{\mathrm{op}}}$ since $J_!(\mathrm{I})$ is a ring in $\mathsf{Set}^{\square_{\mathbb{T}}^{\mathrm{op}}}$ and $J^*$ is left exact.

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