Notes on cubical models of type theory

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Roughly following the paper of Bezem, Coquand, and Huber [**?**] ,and reformulating things in functorial style.

1 Some cube categories

We consider three different cube categories, to be used as index categories for cubical sets:

1. $\mathbb{C}$ the (classical) cube category: the *free monoidal category on an interval.*
2. $\mathbb{C}_s$ the symmetric cube category: the *free symmetric monoidal category on an interval.*
3. $\mathbb{C}_c$ the cartesian cube category: the *free finite product category on an interval.*

1.1 The classical cube category $\mathbb{C}$

(Cf. Jardine [**?**, **?**].) The *objects* are the sets of binary n-tuples:

$$I^n = \{\langle d_1, ..., d_n\rangle \mid d_i = 0, 1\}$$

where

$$I = \{0, 1\}$$

and we let $I^0 = \{*\}$.

The *arrows*

$$f : I^n \longrightarrow I^m$$

are those functions generated by compositions of the following primitive ones:

- *face maps* $\alpha_i^d : I^n \longrightarrow I^{n+1}$, taking $\langle d_1, ..., d_n \rangle$ to $\langle d_1, ..., d_{(i)}, ..., d_n \rangle$, with a new digit $d = 0, 1$ inserted as the i^{th} coordinate. There are $2(n+1)$ such maps.
- *degeneracies* $\beta_i : I^n \longrightarrow I^{n-1}$, taking $\langle d_1, ..., d_n \rangle$ to $\langle d_1, ..., \hat{d}_i, ..., d_n \rangle$, omitting the i^{th} coordinate. There are n such maps.

Note that the order of the d_i's does not change.

Remarks

1. It can be shown that every map factors as:

$$\begin{tikzcd} I^n \arrow[rr, "f"] \arrow[dr, two heads, "\beta"'] & & I^m \\ & I^k \arrow[ur, tail, "\alpha"'] & \end{tikzcd}$$

   where $\alpha : I^k \rightarrowtail I^m$ is a composite of faces, and $\beta : I^n \twoheadrightarrow I^k$ is a composite of degeneracies. Using this, it can be shown that $\mathbb{C}$ is the free monoidal category on an *interval*: an object I equipped with maps:

$$1 \underset{\bot}{\overset{\top}{\rightrightarrows}} I \xrightarrow{!} 1$$

   satisfying $! \circ \top = \mathrm{id}_1 = ! \circ \bot$, where 1 is the monoidal unit.

2. The presheaf category $\mathsf{cSet} = \mathsf{Set}^{\mathbb{C}^{\mathrm{op}}}$ of *cubical sets* has the same homotopy theory as the classical simplicial sets $\mathsf{sSet} = \mathsf{Set}^{\Delta^{\mathrm{op}}}$, in the sense that the two are Quillen equivalent.

3. The objects I^n are *not* the n-fold cartesian products of the interval I, either in the site $\mathbb{C}$ or as presheaves. Rather, there is a monoidal product $\otimes$ on cSet extending that on $\mathbb{C}$, such that $I^m \otimes I^n \cong I^{m+n}$. Similarly, the geometric realization functor to topological spaces

$$R : \mathsf{cSet} \longrightarrow \mathsf{Top}$$

   does not in general preserve cartesian products, but instead takes tensor products in cSet to cartesian ones in Top,

$$R(X \otimes Y) \cong R(X) \times R(Y).$$

1.2 The symmetric cube category $\mathbb{C}_s$

(Cf. Grandis [**?**].) As before, the *objects* are the sets of binary n-tuples:

$$1 = I^0,\ I,\ ..., I^n$$

The *arrows*

$$f : I^n \longrightarrow I^m$$

are still functions generated by compositions of primitive ones, including the faces and degeneracies as before, but now also including the primitive:

- *permutations* $\sigma_i : I^n \longrightarrow I^n$, swapping d_i and d_{i+1}.

For each I^n there are $n-1$ such maps. Of course, for any permutation $\sigma \in S_n$ one can define a corresponding $\sigma : I^n \longrightarrow I^n$ taking $\langle d_1, ..., d_n\rangle$ to $\langle d_{\sigma(1)}, ..., d_{\sigma(n)}\rangle$ as a suitable composite of σ_i's.

Remarks

1. It can be shown that now every map factors as:

$$\begin{tikzcd} I^n \arrow[r, "f"] \arrow[d, two heads, "\beta"'] & I^m \\ I^k \arrow[r, "\sim", "\sigma"'] & I^k \arrow[u, tail, "\alpha"'] \end{tikzcd}$$

   where $\alpha : I^k \rightarrowtail I^m$ is a (composite) face, $\sigma : I^k \xrightarrow{\sim} I^k$ is a (composite) permutation, and $\beta : I^n \twoheadrightarrow I^k$ is a (composite) degeneracy. Using this, it can be shown that $\mathbb{C}_s$ is the free *symmetric* monoidal category on an interval.

2. The presheaf category $\mathsf{csSet} = \mathsf{Set}^{\mathbb{C}_s^{\mathrm{op}}}$ of *symmetric cubical sets* again has the same homotopy theory as simplicial sets.

3. The objects I^n are again n-fold *tensor* products of the interval I, but not *cartesian* products, either in the site $\mathbb{C}_s$ or in csSet. And again, the geometric realization functor from csSet to topological spaces does not preserve cartesian products, but instead takes tensor products to cartesian ones. Relatedly, there is a functor $\mathsf{Hom}(X, -)$, right adjoint to the tensor $X \otimes (-)$, which is not an exponential.

Covariant presentation

(Cf. Bezem, Coquand, and Huber [**?**], Pitts [**?**].) There is a dual presentation of the symmetric site $\mathbb{C}_s$. Let the category $\mathcal{C}$ have as *objects* the finite sets

$$[n] = \{1, ..., n\}$$

and write

$$[n]^+ \; = \; [n] \cup \{\top, \bot\} \; = \; \{\top, 1, ..., n, \bot\}.$$

The *arrows*

$$f : [n] \longrightarrow [m]$$

in $\mathcal{C}$ are all functions $f : [n] \longrightarrow [m]^+$ satifying the following *partial injectivity condition*:

$$f(i) = f(j) \implies \big(i = j \textit{ or } f(i) = \top = f(j) \textit{ or } f(i) = \bot = f(j)\big)$$

In other words, f is injective on the preimage of $[m] \subseteq [m]^+$,

$$\begin{tikzcd} \cdot \arrow[r, tail] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & {[m]} \arrow[d, hook] \\ {[n]} \arrow[r, "f"'] & {[m]^+} \end{tikzcd}$$

Identity and composition are just as in the Kleisli-category of the monad $X \mapsto X^+$. Specifically, $\mathrm{id} : [n] \longrightarrow [n]$ is the inclusion $[n] \hookrightarrow [n]^+$, and $g \circ f : [n] \longrightarrow [m] \longrightarrow [k]$ is $\overline{g} \circ f$, where

$$\overline{g} : [m]^+ \longrightarrow [k]^+$$

is the unique $(\top, \bot)$-preserving extension of g, as indicated in the following.

$$\begin{tikzcd} {[n]} \arrow[dr, "f"] & {[m]} \arrow[d, hook] \arrow[dr, "g"] & {[k]} \arrow[d, hook] \\ & {[m]^+} \arrow[r, "\overline{g}"'] & {[k]^+} \end{tikzcd}$$

One can show easily that this category $\mathcal{C}$ (called the category of "names and substitutions" in [**?**]) is dual to the category of symmetric cubes,

$$\mathcal{C} \cong \mathbb{C}_s^{\mathrm{op}}$$

and so we have an alternate presentation of the symmetric cubical sets as *covariant* functors,

$$\mathsf{scSet} = \mathsf{Set}^{\mathbb{C}_s^{\mathrm{op}}} \cong \mathsf{Set}^{\mathcal{C}}.$$

1.3 The cartesian cube category $\mathbb{C}_c$

As a modification of the foregoing, we consider a notion of *cartesian cubical sets.* The *objects* of $\mathbb{C}_c$ are again the sets of binary n-tuples:

$$1 = I^0,\ I,\ ..., I^n$$

The *arrows* of $\mathbb{C}_c$,

$$f : I^n \longrightarrow I^m$$

are still functions generated by compositions of primitive ones, including the faces, degeneracies, and permutations, but now also including the primitive

- *diagonal maps* $\delta_i : I^n \longrightarrow I^{n+1}$, which double the i^{th} coordinate:

$$\langle d_1, ..., d_n\rangle \ \mapsto\ \langle d_1, ..., d_i, d_i, ..., d_n\rangle.$$

Proposition 1. *$\mathbb{C}_c$ is the free category with finite products and an interval,*

$$1 \underset{\bot}{\overset{\top}{\rightrightarrows}} I \xrightarrow{\ !\ } 1.$$

Proof. The free category with finite products and an interval is the classifying category for the algebraic theory consisting of the two constants $\{\top, \bot\}$, which can be described as follows (see [**?**]):

objects: finite lists $[x_1, ..., x_n]$ of distinct variables,

arrows: $f : [x_1, ..., x_n] \longrightarrow [x_1, ..., x_m]$ are (equivalence classes of) m-tuples

$$f = \langle f_1, ..., f_m\rangle$$

of terms in context,

$$x_1, ..., x_n \vdash f_i\,.$$

But in this simple theory, the only such terms are the variables $x_1, ..., x_n$ themselves and the constants $\{\top, \bot\}$, and the equivalence relation is trivial, since there are no equations. Thus an arrow is just an m-tuple of arbitrary elements taken from the set $\{x_1, ..., x_n, \top, \bot\}$. The identity arrow is the list of variables $\langle x_1, ..., x_n\rangle$, and composition is by the usual substitution of terms for variables. But this is evidently just another description of the category $\mathbb{C}_c$.

In more detail, each of the primitive kinds of maps $\alpha_i^d, \beta_i, \sigma_i, \delta_i$ can clearly be presented in this form, e.g. $\alpha_i^d = \langle x_1, ..., d'_{(i)}, ..., x_n\rangle$, where $d' = \top, \bot$, respectively, when $d = 1, 0$. Conversely, an m-tuple $(e_1, ..., e_m)$ of elements from the set $\{x_1, ..., x_n, \top, \bot\}$ determines a map $\epsilon : I^n \longrightarrow I^m$ in $\mathbb{C}_c$ as follows: beginning with a binary n-tuple $(d_1, ...d_n)$, first apply degeneracies β_i corresponding to each x_i not occurring in $(e_1, ..., e_m)$; next apply a permutation σ that reorders the terms d_j in accordance with the order of the non-constant terms e_j appearing in $(e_1, ..., e_m)$; apply suitable δ's to duplicate coordinates appearing more than once; and finally use α's to insert the required constants. □

Corollary 2. *The cartesian cube category $\mathbb{C}_c$ is equivalent to a* non-*full subcategory of* Cat *(respectively* Pos*) on the objects $I^n = I \times ... \times I$, where $I = (0 \leq 1)$ is the 2-element poset.*

Proof. Each of the maps $\alpha_i^d, \beta_i, \sigma_i, \delta_i$ is monotone, and these are all distinct as monotone maps. To see that this is not full, observe that every monotone $f : I^n \longrightarrow I^m$ is an m-tuple of monotone $f_i : I^n \longrightarrow I$, each of which coming from $\mathbb{C}_c$ is either a projection or a constant. But the map $f : I^2 \longrightarrow I$ with $f(1, 1) = 1$, and $f(d, d') = 0$ otherwise, is neither. □

Note that the non-monotone "negation" map $n : I \longrightarrow I$, with $n(0) = 1$ and $n(1) = 0$, is also not in $\mathbb{C}_c$.

Covariant presentation

As a classifying category for an algebraic theory, the category $\mathbb{C}_c$ of cartesian cubes also has a covariant presentation by Lawvere duality, namely as the opposite of the full subcategory of finitely-generated, free algebras $\mathsf{Alg}_{\mathrm{fg}}$. In this case, the algebras are simply *bipointed sets* (A, a_0, a_1), and the free ones are the *strictly* bipointed sets $a_0 \neq a_1$. Thus $\mathsf{Alg}_{\mathrm{fg}}$ consists of the finite, strictly bipointed sets and all bipointed maps between them. Specifically, let the objects of $\mathbb{B}$ be the sets $[n] = \{1, ..., n\}$, and the arrows,

$$f : [m] \longrightarrow [n]\,,$$

be arbitrary, $\{\top, \bot\}$-preserving maps $[m]^+ \longrightarrow [n]^+$, where as before $[n]^+ = [n] \cup \{\top, \bot\}$. Then clearly $\mathbb{B} = \mathsf{Alg}_{\mathrm{fg}}$, and we know by Lawvere duality that

$$\mathbb{C}_c \cong \mathbb{B}^{\mathrm{op}},$$

as can be read off from the foregoing descrition of the arrows in $\mathbb{C}_c$ as "m-tuples of arbitrary elements taken from the set $\{x_1, ..., x_n, \top, \bot\}$".

As a full subcategory of free algebras, the category $\mathbb{B}$ can also be described as the Kleisli category of the monad $[n] \mapsto [n]^+$. Thus we arrive at the covariant description $\mathcal{C}$ of the symmetric cubes, but without the partial injectivitity condition, which is violated by (the duals of) the diagonal maps.

2 Hypercubical sets

Definition 3. We may refer to the objects of the cartesian cube category $\mathbb{C}_c$ as *hypercubes* and write $\mathbb{H} = \mathbb{C}_c$ for the *category of hypercubes*. The objects may be taken to be finite sets of the form

$$[n] = \{x_1, ..., x_n\},$$

regarded as coordinate axes, and the arrows,

$$f : [n] \longrightarrow [m]\,,$$

are then taken to be m-tuples of elements drawn from the set

$$[n]^+ = \{0, x_1, ..., x_n, 1\},$$

regarded as formal terms representing composites of faces, degeneracies, permutations, and diagonals (alternately, the arrows are arbitrary bipointed maps $[m]^+ \longrightarrow [n]^+$).

The category $\mathcal{H}$ of *hypercubical sets* is the category of presheaves on $\mathbb{H}$,

$$\mathcal{H} \;=\; \mathsf{Set}^{\mathbb{H}^{\mathrm{op}}}.$$

It is generated by the representable presheaves $y([n])$, which will be written

$$\mathrm{I}^n \;=\; y([n])$$

and called the *standard n-cubes*. In particular, the standard 1-cube is $\mathrm{I} = y([1])$, and the standard 0-cube is $\mathrm{I}^0 = y([0]) = 1$. For any hypercubical set $X : \mathbb{H}^{\mathrm{op}} \longrightarrow \mathsf{Set}$, we shall write $X_n = X([n])$ and call this the *set of n-cubes in X*. For these, we have the usual Yoneda correspondence:

$$(c \in X_n) \;\cong\; (c : \mathrm{I}^n \longrightarrow X).$$

In particular $\mathrm{I}^n_m = \mathbb{H}([m], [n])$ is the set of m-cubes in the standard n-cube.

Proposition 4. *We now have $\mathrm{I}^n \times \mathrm{I}^m \cong \mathrm{I}^{n+m}$, in virtue of the preservation of products by the Yoneda embedding.*

Proposition 5. *The category $\mathcal{H}$ of hypercubical sets is the classifying topos for bipointed objects.*

Proposition 6. *The geometric realization functor to topological spaces*

$$R : \mathcal{H} \longrightarrow \mathsf{Top}$$

preserves cartesian products, $R(X \times Y) \cong R(X) \times R(Y)$.

Proposition 7. *Since* $\mathbb{H} \hookrightarrow \mathsf{Cat}$ *is a subcategory, the* nerve *functor*

$$N : \mathsf{Cat} \longrightarrow \mathcal{H}$$

can be defined as usual by:

$$N(\mathbb{C})_n = \mathsf{Cat}(I^n, \mathbb{C}).$$

However, we do not expect the nerve to be full and faithful.

Proposition 8. *For any hypercubical set* X*, the exponential* X^{I} *can be calculated as:*

$$X^{\mathrm{I}}(n) \cong X(n+1).$$

Proof.

$$\begin{aligned} X^{\mathrm{I}}(n) \cong \hom(y[n], X^{\mathrm{I}}) \cong \hom(\mathrm{I}^n, X^{\mathrm{I}}) \ &\cong \ \hom(\mathrm{I}^n \times \mathrm{I}, X) \\ \cong \ \hom(\mathrm{I}^{n+1}, X) \cong \ \hom(y[n+1], X) \ &\cong \ X(n+1). \end{aligned}$$

$\square$

Proposition 9. $\quad \mathrm{I}^{\mathrm{I}} \ \cong \ \mathrm{I} + 1.$

Proposition 10. *The functor* $X \mapsto X^{\mathrm{I}}$ *has a* right *adjoint.*

Example. The cubical set P of polynomials (over the integers, say), is defined by:

$$P_n = \{p(x_1, ..., x_n) \mid \text{polynomials in at most } x_1, , x_n\}$$

with the evident maps $P_m \longrightarrow P_n$ for each function $[m] \longrightarrow [n]$.

This is a ring object in the category of cubical sets, and the interval $\mathrm{I} \ = y[1]$ embeds into P. The same is true for any algebraic theory $\mathbb{T}$ with two constants, such as boolean algebras: there is a cubical $\mathbb{T}$-algebra A and a monic $\mathrm{I} \rightarrowtail A$.

Let $\mathbb{C}[\mathrm{I}] = \mathbb{H}$ be the cube category, classifying intervals, and $\mathbb{C}[\mathbb{T}]$ the classifying category for $\mathbb{T}$-algebras. There is an interval J in $\mathbb{C}_{\mathbb{T}}$ consisting of the generic $\mathbb{T}$-algebra and its two constants. This J has a classifying functor $J : \mathbb{C}_{\mathrm{I}} \longrightarrow \mathbb{C}_{\mathbb{T}}$, inducing functors on presheaves

$$J_! \dashv J^* \dashv J_* : \mathsf{Set}^{\mathbb{C}_{\mathrm{I}}^{\mathrm{op}}} \longrightarrow \mathsf{Set}^{\mathbb{C}_{\mathbb{T}}^{\mathrm{op}}}$$

as usual, where $J_! \circ \mathsf{y}_{\mathbb{C}_{\mathrm{I}}} = \ \mathsf{y}_{\mathbb{C}_{\mathbb{T}}} \circ J$, with y the respective Yoneda embeddings.

We can calculate:

$$
\begin{aligned}
J^*J_!(\mathrm{I})([n]) &= J^*J_!(Y[1])([n]) \\
&= J^*Y(J[1])([n]) = Y(J[1])(J[n]) \\
&= \mathbb{C}_{\mathbb{T}}(J[n], J[1]) = \mathbb{T}\mathrm{-Alg}(J[1], J[n]) \\
&= \mathbb{T}\mathrm{-Alg}(F(1), F(n)) = |F(n)|,
\end{aligned} \tag{1}
$$

where $F(n)$ is the free $\mathbb{T}$-algebra on n generators. So in the case of polynomials we indeed have

$$P = J^*J_!(\mathrm{I}).$$

The unit of the adjunction $\mathrm{I} \longrightarrow J^*J_!(\mathrm{I})$ is faithful, since J itself is faithful and therefore the left adjoint $J_!$ is faithful. P is a ring in $\mathsf{Set}^{\mathbb{C}_{\mathrm{I}}^{\mathrm{op}}}$ since $J_!(\mathrm{I})$ is a ring in $\mathsf{Set}^{\mathbb{C}_{\mathbb{T}}^{\mathrm{op}}}$ and J^* is left exact.

A closely related example is the cubical set of “boolean polynomials”,

$$B_n = \{\varphi(p_1, ..., p_n) \mid \text{propositional formulas in at most } p_1, ..., p_n\}$$

which is the free boolean algeba 2^n.

Questions

1. According to Grothedieck [**?**], the category $\mathbb{H}$ is a test category, and so the category $\mathcal{H} = \mathsf{Set}^{\mathbb{H}^{\mathrm{op}}}$ has the same homotopy theory as simplicial sets. Prove this.

2. Want to know what a “hypercubical ω-groupoid” (i.e. a fibrant object) should be. Are the usual box-filling conditions sufficient to define this? Is there another characterization involving the new diagonal maps?

3. The hypercubical sets $\mathcal{H}$ is perhaps a good setting in which to compare the globular, simplicial, and type-theoretic notions of ω-groupoid.

4. What is a hypercubical $(\infty, 1)$-category (in analogy to the simplicial notion of quasicategory)? Does the type theory give rise to one?

References

[1] S. Awodey and M.A. Warren. Homotopy theoretic models of identity types. *Math. Proc. Camb. Phil. Soc.*, 146, 45–55, 2009.

[2] B. van den Berg and R. Garner. Topological and Simplicial Models of Identity Types. *ACM Transactions on Computational Logic*, 13:1, 2012.

[3] P. Dybjer. “Internal Type Theory.” *LNCS* 1158, 120–134, 1996.

[4] N. Gambino and J. Kock, “Polynomial functors and polynomial monads” *Math. Proc. Cambridge Phil. Soc.* 154, 153–192, 2013.

[5]

[6] *Homotopy Type Theory: Univalent Foundations of Mathematics*, The Univalent Foundations Program, Institute for Advanced Study, 2013. `http://homotopytypetheory.org/book`

[7] C. Kapulkin, P. LeFanu Lumsdaine and V. Voevodsky, The Simplicial Model of Univalent Foundations. *In preparation*, 2013.