STACKS AND SHEAVES OF CATEGORIES AS FIBRANT OBJECTS

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ABSTRACT. We show that the category of categories fibred over a site is a generalized Quillen model category in which the weak equivalences are the local equivalences and the fibrant objects are the stacks, as they were defined by J. Giraud. The generalized model category restricts to one on the full subcategory whose objects are the categories fibred in groupoids. We show that the category of sheaves of categories is a model category that is Quillen equivalent to the generalized model category for stacks and to the model category for strong stacks due to A. Joyal and M. Tierney.

1. INTRODUCTION

The idea that stacks are the fibrant objects of a model category was developed by A. Joyal and M. Tierney in [19] and by S. Hollander in [15]. The former paper uses internal groupoids and categories in a Grothendieck topos instead of fibred categories, and the latter only considers categories fibred in groupoids. The fibrant objects of the Joyal-Tierney model category are called strong stacks (of groupoids or categories), and the fibrant objects of Hollander's model category are the stacks of groupoids. Using some elaborate results from the homotopy theory of simplicial presheaves on a site, Hollander shows that her model category is Quillen equivalent to the model category for strong stacks of groupoids.

The purpose of this paper is to extend Hollander's work to general stacks and to show that the category of internal categories in a Grothendieck topos admits another model category structure that is Quillen equivalent to the model category for strong stacks of categories. Our approach is different from both [15] and [19], and it was entirely inspired by J. Giraud's book [11]. In fact, the influence of Giraud's work on ours cannot be overestimated.

Concerning general stacks, we give a realization of the thought that parts of Giraud's presentation of the theory of stacks [11, Chapitre II §1, §2] hint at a connection with left Bousfield localizations of model categories as presented by P.S. Hirschhorn [14, Chapter 3]. In more detail, let E be a site, that is, a category E equipped with a Grothendieck topology and let $\mathrm{Fib}(E)$ be the category of fibred categories over E and cartesian functors between them. Let $\mathcal{C}$ be the class of maps $R \subset E_{/S}$ of $\mathrm{Fib}(E)$, where S ranges through the objects of E, $E_{/S}$ is the category of objects of E over S and R is a covering sieve (or, refinement) of S. Then Giraud's definition of stack resembles that of a $\mathcal{C}$-local object and his characterization of bicovering (*bicouvrant* in French) maps resembles the $\mathcal{C}$-local equivalences of [14,

Definition 3.1.4(1)]. The bicovering maps are better known under the name 'local equivalences'.

The realization goes as follows. In order to deal with the absence of all finite limits and colimits in $\mathrm{Fib}(E)$ we introduce, following a suggestion of A. Joyal, the notion of generalized model category (see Definition 8). Many concepts and results from the theory of model categories can be defined in the same way and have an exact analogue for generalized model categories. We disregard that E has a topology and we show that $\mathrm{Fib}(E)$ is naturally a generalized model category with the weak equivalences, cofibrations and fibrations defined on the underlying functors (see Theorem 13). Then we show that 'the left Bousfield localization of $\mathrm{Fib}(E)$ with respect to $\mathcal{C}$ exists', by which we mean that there is a generalized model category structure on $\mathrm{Fib}(E)$ having the bicovering maps as weak equivalences and the stacks over E as fibrant objects (see Theorem 29). We call this generalized model category the generalized model category for stacks over E and we denote it by $\mathrm{Champ}(E)$.

To construct $\mathrm{Champ}(E)$ we make essential use of the functorial construction of the stack associated to a fibred category (or, stack completion) and some of its consequences [11, Chapitre II §2], and of a special property of bicovering maps (see Lemma 33).

We adapt the method of proof of the existence of $\mathrm{Champ}(E)$ to show that $\mathrm{Fibg}(E)$, the full subcategory of $\mathrm{Fib}(E)$ whose objects are the categories fibred in groupoids, is a generalized model category in which the weak equivalences are the bicovering maps and the fibrant objects are the stacks of groupoids over E (see Theorem 46).

Concerning internal categories in a Grothendieck topos, let $\widetilde{E}$ be the category of sheaves on E. We show that the category $\mathrm{Cat}(\widetilde{E})$ of internal categories and internal functors in $\widetilde{E}$ (or, sheaves of categories) is a model category that is Quillen equivalent to $\mathrm{Champ}(E)$ (see Theorem 48). We denote this model category by $\mathrm{Stack}(\widetilde{E})_{proj}$. The fibrant objects of $\mathrm{Stack}(\widetilde{E})_{proj}$ are the sheaves of categories that are taken to stacks by the Grothendieck construction functor. To construct $\mathrm{Stack}(\widetilde{E})_{proj}$ we make essential use of the explicit way in which Giraud constructs the stack associated to a fibred category—a way that highlights the role of sheaves of categories, and of a variation of Quillen's path object argument (see Lemma 49). The model category $\mathrm{Stack}(\widetilde{E})_{proj}$ is also Quillen equivalent via the identity functors to the model category for strong stacks [19, Theorem 4] (see Proposition 52) and it behaves as expected with respect to morphisms of sites (see Proposition 53).

The paper contains a couple of other results, essentially easy consequences of some of the results we have proved so far: the bicovering maps and the natural fibrations make $\mathrm{Fib}(E)$ a category of fibrant objects [7] (see Proposition 40), and the 2-pullback (or, iso-comma object) of fibred categories is a model for the homotopy pullback in $\mathrm{Champ}(E)$ (see Lemma 42).

Appendix 1 is a review of Hollander's characterization of stacks of groupoids in terms of the homotopy sheaf condition [15, Theorem 1.1]. Appendix 2 studies the behaviour of left Bousfield localizations of model categories under change of cofibrations. The result contained in it is needed in Appendix 3, which is a review of the model category for strong stacks of categories [19, Theorem 4] made with the hope that it sheds some light on the nature of strong stacks.

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2. Fibred categories

In this section we recall, for completeness and to fix notations, some results from the theory of fibred categories.

We shall work in the setting of universes, as in [11], although we shall not mention the universe in which we shall be working. We shall also use the axiom of choice.

We denote by SET the category of sets and maps, by CAT the category of categories and functors and by $GRPD$ its full subcategory whose objects are groupoids.

Let E be a category. We denote by E^{op} the opposite category of E. We let $CAT_{/E}$ be the category of categories over E. Arrows of $CAT_{/E}$ will be called E-functors. If S is an object of E, $E_{/S}$ stands for the category of objects of E over S.

If A and B are two categories, we denote by $[A, B]$ the category of functors from A to B and natural transformations between them.

We denote by $*$ the terminal object of a category, when it exists. We denote by J the groupoid with two objects and one isomorphism between them.

2.1. **Isofibrations.** One says that a functor $A \to B$ is an *isofibration* (called *transportable* in [13, Exposé VI]) if it has the right lifting property with respect to one of the maps $* \to J$. A functor is both an isofibration and an equivalence of categories if and only if it is an equivalence which is surjective on objects (surjective equivalence, for short). Given a commutative diagram in CAT

$$
\begin{array}{ccc}
A & \longrightarrow & C \\
f \downarrow & & \downarrow g \\
B & \longrightarrow & D
\end{array}
$$

in which the horizontal arrows are surjective equivalences, if f is an isofibration then so is g.

2.2. **Fibrations and isofibrations.** Let E be a category.

Let $f \colon F \to E$ be a functor. We denote by F_S the fibre category over $S \in Ob(E)$. An E-functor $u \colon F \to G$ induces a functor $u_S \colon F_S \to G_S$ for every $S \in Ob(E)$.

Lemma 1. (1) *Let $u \colon F \to G$ be an E-functor with $F \to E$ an isofibration. Then the underlying functor of u is an isofibration if and only if for every $S \in Ob(E)$, the map $F_S \to G_S$ is an isofibration.*

(2) *Every fibration is an isofibration. Every surjective equivalence is a fibration.*

(3) *Let $u \colon F \to G$ be an E-functor such that the underlying functor of u is an equivalence. If F is a fibration then so is G.*

(4) *Let $u \colon F \to G$ be an E-functor such that the underlying functor of u is an equivalence. If G is a fibration and $F \to E$ is an isofibration then F is a fibration.*

(5) *Let F and G be two fibrations and $u \colon F \to G$ be an E-functor. If the underlying functor of u is full and faithful then u reflects cartesian arrows.*

(6) *Let*

$$
\begin{array}{ccc}
F & \longrightarrow & H \\
\downarrow u & & \downarrow v \\
G & \longrightarrow & K
\end{array}
$$

be a commutative diagram in $CAT_{/E}$ with F, G, H and K fibrations. If the underlying functors of the horizontal arrows are equivalences, then u is a cartesian functor if and only if v is cartesian.

Proof. (1) We prove sufficiency. Let $\beta \colon u(x) \to y$ be an isomorphism and let $S = g(y)$. Then $g(\beta) \colon f(x) \to S$ is an isomorphism therefore there is an isomorphism $\alpha \colon x \to x_0$ such that $f(\alpha) = g(\beta)$ since f is an isofibration. The composite $\beta u(\alpha^{-1}) \colon u(x_0) \to y$ lives in G_S hence there is an isomorphism $\alpha' \colon x_0 \to x_1$ such that $u(\alpha') = \beta u(\alpha^{-1})$ since u_S is an isofibration. One has $u(\alpha'\alpha) = \beta$.

(2) is straightforward. (3) and (4) are consequences of [13, Exposé VI Corollaire 4.4 et Proposition 6.2].

(5) Let $f \colon F \to E$ and $g \colon G \to E$ be the structure maps. Let $\alpha \colon x \to y$ be a map of F such that $u(\alpha)$ is cartesian. We can factorize α as $c\gamma$, where $c \colon z \to y$ is a cartesian map over $f(\alpha)$ and $\gamma \colon x \to z$ is a vertical map. Since $u(\alpha)$ is cartesian and $gu(\alpha) = gu(c)$, there is a unique $\epsilon \colon u(z) \to u(x)$ such that $u(\alpha)\epsilon = u(c)$. Then $\epsilon = u(\beta)$ since u is full, where $\beta \colon z \to x$. Hence $c = \alpha\beta$ since u is faithful. Since c is cartesian it follows that $\gamma\beta$ is the identity, and since $u(\alpha)$ is cartesian it follows that $\beta\gamma$ is the identity. Thus, γ is a cartesian map.

(6) is a consequence of (5) and [13, Exposé VI Corollaire 4.4 et Proposition 5.3(i)]. $\square$

A *sieve* of E is a collection R of objects of E such that for every arrow $X \to Y$ of E, $Y \in R$ implies $X \in R$. Let $F \to E$ be a fibration and R a sieve of F. The composite $R \subset F \to E$ is a fibration and $R \subset F$ is a cartesian functor.

A surjective equivalence takes sieves to sieves.

2.3. **The 2-categories $\mathscr{F}ib(E)$ and $\mathscr{F}ibg(E)$.** Let E be a category. We denote by $\mathrm{Fib}(E)$ the category whose objects are the categories fibred over E and whose arrows are the cartesian functors.

. Let F and G be two objects of $\mathrm{Fib}(E)$. The cartesian functors from F to G and the cartesian (sometimes called vertical) natural transformations between them form a category which we denote by $\mathbf{Cart}_E(F, G)$. This defines a functor

$$
\mathbf{Cart}_E(-, -) \colon \mathrm{Fib}(E)^{op} \times \mathrm{Fib}(E) \longrightarrow CAT
$$

so that the fibred categories over E, the cartesian functors and cartesian natural transformations between them form a 2-category which we denote by $\mathscr{F}ib(E)$.

The category $\mathrm{Fib}(E)$ has finite products. The product of two objects F and G is the pullback $F \times_E G$.

Let A be a category and $F \in \mathrm{Fib}(E)$. We denote by $A \times F$ the pullback of categories

$$
\begin{array}{ccc}
A \times F & \longrightarrow & F \\
\downarrow & & \downarrow \\
A \times E & \longrightarrow & E
\end{array}
$$

$A \times F$ is the product in $\mathrm{Fib}(E)$ of F and $A \times E$. The construction defines a functor

$$- \times -\colon CAT \times \mathrm{Fib}(E) \longrightarrow \mathrm{Fib}(E)$$

We denote by $F^{(A)}$ the pullback of categories

$$
\begin{array}{ccc}
F^{(A)} & \longrightarrow & [A, F] \\
\downarrow & & \downarrow \\
E & \longrightarrow & [A, E]
\end{array}
$$

so that $(F^{(A)})_S = [A, F_S]$. The functor $- \times F$ is left adjoint to $\mathbf{Cart}_E(F, -)$ and the functor $A \times -$ is left adjoint to $(-)^{(A)}$. These adjunctions are natural in F and A. There are isomorphisms that are natural in F and G

$$(2) \qquad \mathbf{Cart}_E(A \times F, G) \cong [A, \mathbf{Cart}_E(F, G)] \cong \mathbf{Cart}_E(F, G^{(A)})$$

so that $\mathscr{F}ib(E)$ is tensored and cotensored over the monoidal category CAT.

Let F and G be two objects of $\mathrm{Fib}(E)$. We denote by $\mathsf{CART}(F, G)$ the object of $\mathrm{Fib}(E)$ associated by the Grothendieck construction to the functor $E^{op} \to CAT$ which sends $S \in Ob(E)$ to $\mathbf{Cart}_E(E_{/S} \times F, G)$, so that

$$(3) \qquad \mathsf{CART}(F, G)_S = \mathbf{Cart}_E(E_{/S} \times F, G)$$

There is a natural equivalence of categories

$$(4) \qquad \mathbf{Cart}_E(F \times G, H) \simeq \mathbf{Cart}_E(F, \mathsf{CART}(G, H))$$

. The Grothendieck construction functor

$$[E^{op}, CAT] \xrightarrow{\ \Phi\ } \mathrm{Fib}(E)$$

has a right adjoint S given by $\mathsf{S}F(S) = \mathbf{Cart}_E(E_{/S}, F)$. Φ and S are 2-functors and the adjoint pair (Φ, S) extends to a 2-adjunction betweeen the 2-categories $[E^{op}, CAT]$ and $\mathscr{F}ib(E)$. $\mathsf{S}F$ is a split fibration and S sends maps in $\mathrm{Fib}(E)$ to split functors. The composite $\mathsf{S} = \Phi\mathsf{S}$ sends fibrations to split fibrations and maps in $\mathrm{Fib}(E)$ to split functors. The counit of the 2-adjunction (Φ, S) is a 2-natural transformation $\mathsf{v}\colon \mathsf{S} \to Id_{\mathscr{F}ib(E)}$. For every object F of $\mathrm{Fib}(E)$ and every $S \in Ob(E)$ the map

$$(5) \qquad (\mathsf{v}F)_S \colon \mathbf{Cart}_E(E_{/S}, F) \to F_S$$

is a surjective equivalence.

Φ has also a left adjoint L, constructed as follows. For any category A, the functor

$$- \times A\colon CAT \to \mathrm{Fib}(A)$$

has a left adjoint $\varinjlim(-/A)$ which takes F to the category obtained by inverting the cartesian morphisms of F. If $F \in \mathrm{Fib}(E)$,

$$\mathsf{L}F(S) = \varinjlim(E^{/S} \times_E F/E^{/S})$$

where $E^{/S}$ is the category of objects of E under S. We denote by I the unit of the adjoint pair (L, Φ). For every $S \in Ob(E)$, the map $(\mathsf{I}F)_S \colon F_S \to \mathsf{L}F(S)$ is an equivalence of categories. The adjoint pair (L, Φ) extends to a 2-adjunction betweeen the 2-categories $[E^{op}, CAT]$ and $\mathscr{F}ib(E)$.

. Let F be an object of $\mathrm{Fib}(E)$. We denote by F^{cart} the subcategory of F which has the same objects and whose arrows are the cartesian arrows. The composite $F^{cart} \subset F \to E$ is a fibration and $F^{cart} \subset F$ is a map in $\mathrm{Fib}(E)$. For each $S \in Ob(E)$, $(F^{cart})_S$ is the maximal groupoid associated to F_S. A map $u\colon F \to G$ of $\mathrm{Fib}(E)$ induces a map $u^{cart}\colon F^{cart} \to G^{cart}$ of $\mathrm{Fib}(E)$. In all, we obtain a functor $(-)^{cart}\colon \mathrm{Fib}(E) \to \mathrm{Fib}(E)$. One says that F is *fibred in groupoids* if the fibres of F are groupoids. This is equivalent to saying that $F^{cart} = F$ and, if $f\colon F \to E$ is the structure map of F, to saying that for every object x of F, the induced map $f_{/x}\colon F_{/x} \to E_{/f(x)}$ is a surjective equivalence.

We denote by $\mathrm{Fibg}(E)$ the full subcategory of $\mathrm{Fib}(E)$ consisting of categories fibred in groupoids. The inclusion functor $\mathrm{Fibg}(E) \subset \mathrm{Fib}(E)$ has $(-)^{cart}$ as right adjoint. $\mathrm{Fibg}(E)$ is a full subcategory of $CAT_{/E}$.

We denote by $\mathscr{F}ibg(E)$ the full sub-2-category of $\mathscr{F}ib(E)$ whose objects are the categories fibred in groupoids. $\mathscr{F}ibg(E)$ is a $GRPD$-category. If F and G are two objects of $\mathscr{F}ibg(E)$, we denote the $GRPD$-hom between F and G by $\mathbf{Cartg}_E(F, G)$. $\mathscr{F}ibg(E)$ is tensored and cotensored over $GRPD$ with tensor and cotensor defined by the same formulas as for $\mathrm{Fib}(E)$.

$\mathscr{F}ib(E)$ becomes a $GRPD$-category by change of base along the maximal groupoid functor $max\colon CAT \to GRPD$. Then the inclusion $\mathscr{F}ibg(E) \subset \mathscr{F}ib(E)$ becomes a $GRPD$-functor which has $(-)^{cart}$ as right $GRPD$-adjoint. In particular, we have a natural isomorphism

$$(6) \qquad\qquad \mathbf{Cartg}_E(F, G^{cart}) \cong max\mathbf{Cart}_E(F, G)$$

A change of base. Let $m\colon A \to B$ be a functor. There is a 2-functor

$$m_\bullet^{fib}\colon \mathscr{F}ib(B) \to \mathscr{F}ib(A)$$

given by $m_\bullet^{fib}(F) = F \times_B A$. If A is fibred in groupoids with structure map m, $m_\bullet^{fib}$ has a left 2-adjoint $m^\bullet$ that is given by composing with m.

Let P be a presheaf on A and $D\colon SET \to CAT$ be the discrete category functor. The functor D induces a functor $D\colon [E^{op}, SET] \to [E^{op}, CAT]$. We denote the category ΦDP by $A_{/P}$, often called the category of elements of P. As a consequence of the above 2-adjunction we have a natural isomorphism

$$(7) \qquad\qquad \mathbf{Cart}_A(A_{/P}, m_\bullet^{fib}(F)) \cong \mathbf{Cart}_B(A_{/P}, F)$$

Let E be a category and P a presheaf on E. Let m be the canonical map $E_{/P} \to E$. We denote $m_\bullet^{fib}(F)$ by $F_{/P}$. As a consequence of the above 2-adjunction we have a natural isomorphism

$$\mathbf{Cart}_{E_{/P}}(E_{/P}, F_{/P}) \cong \mathbf{Cart}_E(E_{/P}, F)$$

3. Generalized model categories

3.1. We shall need to work with a more general notion of (Quillen) model category than in the current literature (like [14]). In this section we shall introduce the notion of generalized model category. Many concepts and results from the theory of model categories can be defined in the same way and have an exact analogue for generalized model categories. We shall review below some of them.

Definition 8. A *generalized model category* is a category $\mathcal{M}$ together with three classes of maps W, C and F (called *weak equivalences*, *cofibrations* and *fibrations*) satisfying the following axioms:

A1: $\mathcal{M}$ has initial and terminal objects.

A2: The pushout of a cofibration along any map exists and the pullback of a fibration along any map exists.

A3: W has the two out of three property.

A4: The pairs $(C, F \cap W)$ and $(C \cap W, F)$ are weak factorization systems.

If follows from the definition that the classes C and $C \cap W$ are closed under pushout and that the classes F and $F \cap W$ are closed under pullback.

The opposite of the underlying category of a generalized model category is a generalized model category.

Let $\mathcal{M}$ be a generalized model category. A map of $\mathcal{M}$ is a *trivial fibration* if it is both a fibration and a weak equivalence, and it is a *trivial cofibration* if it is both a cofibration and a weak equivalence. An object of $\mathcal{M}$ is *cofibrant* if the map to it from the initial object is a cofibration, and it is *fibrant* if the map from it to the terminal object is a fibration. Let X be an object of $\mathcal{M}$. For every cofibrant object A of $\mathcal{M}$, the coproduct $A \sqcup X$ exists and the map $A \to A \sqcup X$ is a cofibration. Dually, for every fibrant object Z, the product $Z \times X$ exists and the map $Z \times X \to X$ is a fibration.

The class of weak equivalences of a generalized model category is closed under retracts [20, Proposition 7.8].

3.2.   Let $\mathcal{M}$ be a generalized model category with terminal object $*$. Let $f \colon X \to Y$ be a map of $\mathcal{M}$ between fibrant objects. We review the construction of the mapping path factorization of f [7]. Let

$$Y \xrightarrow{s} PathY \xrightarrow{p_0 \times p_1} Y \times Y$$

be a factorization of the diagonal map $Y \to Y \times Y$ into a weak equivalence s followed by a fibration $p_0 \times p_1$. Consider the following diagram

$$
\begin{array}{ccccc}
Pf & \xrightarrow{q} & X \times Y & \xrightarrow{p_X} & X \\
{\scriptstyle \pi_f} \downarrow & & {\scriptstyle f \times Y} \downarrow & & \downarrow {\scriptstyle f} \\
Y \xrightarrow{s} & PathY & \xrightarrow{p_0 \times p_1} Y \times Y & \xrightarrow{p_0} & Y \\
& & {\scriptstyle p_1} \downarrow & & \downarrow \\
& & Y & \longrightarrow & *
\end{array}
$$

in which all squares are pullbacks. The object Pf is fibrant. There is a unique map $j_f \colon X \to Pf$ such that $\pi_f j_f = sf$ and $p_X q j_f = 1_X$. The map $p_X q$ is a trivial fibration, hence the map j_f is a weak equivalence. Put $q_f = p_1(f \times Y)q$. Then q_f is a fibration and $f = q_f j_f$.

In a generalized model category, the pullback of a weak equivalence between fibrant objects along a fibration is a weak equivalence [7, Lemma 2 on page 428].

3.3.   Let $\mathcal{M}$ be a generalized model category. A *left Bousfield localization* of $\mathcal{M}$ is a generalized model category $L\mathcal{M}$ on the underlying category of $\mathcal{M}$ having the same class of cofibrations as $\mathcal{M}$ and a bigger class of weak equivalences.

Lemma 9. *Let $\mathcal{M}$ be a generalized model category with* W, C *and* F *as weak equivalences, cofibrations and fibrations. Let* W' *be a class of maps of $\mathcal{M}$ that contains*

W *and has the two out of three property. We define* F′ *to be the class of maps having the right lifting property with respect to every map of* C ∩ W′.

Then LM = (W′, C, F′) *is a left Bousfield localization of* M *if and only if the pair* (C ∩ W′, F′) *is a weak factorization system. Moreover,* (C ∩ W′, F′) *is a weak factorization system if and only if the class* C ∩ W′ *is closed under codomain retracts and every arrow of* M *factorizes as a map in* C ∩ W′ *followed by a map in* F′.

Proof. We prove the first statement. The necessity is clear. Conversely, since C ∩ W ⊂ C ∩ W′ it follows that F′ ⊂ F. This implies that the second part of Axiom A2 is satisfied. To complete the proof it suffices to show that F ∩ W = F′ ∩ W′. Since C ∩ W′ ⊂ C, it follows that F ∩ W ⊂ F′ and hence that F ∩ W ⊂ F′ ∩ W′. We show that F′ ∩ W′ ⊂ F ∩ W. Let $X \to Y$ be a map in F′ ∩ W′. We factorize it into a map $X \to Z$ in C followed by a map $Z \to Y$ in F ∩ W. Since W′ has the two out of three property, the map $X \to Z$ is in C ∩ W′. It follows that the commutative diagram

$$
\begin{array}{ccc}
X & = & X \\
\downarrow & & \downarrow \\
Z & \longrightarrow & Y
\end{array}
$$

has a diagonal filler, hence $X \to Y$ is a (domain) retract of $Z \to Y$. Thus, the map $X \to Y$ is in F ∩ W.

The second statement follows from a standard characterization of weak factorization systems.   □

Let LM be a left Bousfield localization of M. A map of M between fibrant objects in LM is a weak equivalence (fibration) in LM if and only if it is a weak equivalence (fibration) in M. Let $X \to Y$ be a weak equivalence in M between fibrant objects in M. Then X is fibrant in LM if and only if Y is fibrant in LM.

3.4.   A generalized model category is *left proper* if every pushout of a weak equivalence along a cofibration is a weak equivalence. Dually, a generalized model category is *right proper* if every pullback of a weak equivalence along a fibration is a weak equivalence. A generalized model category is *proper* if it is left and right proper. A left Bousfield localization of a left proper generalized model category is left proper.

Let M be a right proper generalized model category. Let

(10) $X \xrightarrow{\ g\ } Z \xleftarrow{\ f\ } Y$

be a diagram in M. We factorize f as a trivial cofibration $Y \xrightarrow{i_f} E(f)$ followed by a fibration $E(f) \xrightarrow{p_f} Z$. We factorize g as a trivial cofibration $X \xrightarrow{i_g} E(g)$ followed by a fibration $E(f) \xrightarrow{p_g} Z$. The *homotopy pullback* of diagram (10) is defined to be the pullback of the diagram

$$
E(g) \xrightarrow{\ p_g\ } Z \xleftarrow{\ p_f\ } E(f)
$$

The analogue of [14, Proposition 13.3.4] holds in this context. If X, Y and Z are fibrant, a model for the homotopy pullback is $X \times_Z Pf$, where Pf is the mapping path factorization of f described in Section 3.2.

Let $L\mathcal{M}$ be a left Bousfield localization of $\mathcal{M}$ that is right proper. We denote by $X \times_Z^h Y$ the homotopy pullback in $\mathcal{M}$ of diagram (10) and by $X \times_Z^{Lh} Y$ the homotopy pullback of the same diagram, but in $L\mathcal{M}$.

Proposition 11. (1) *Suppose that* X, Y *and* Z *are fibrant in* $L\mathcal{M}$. *Then* $X \times_Z^h Y$ *is weakly equivalent in* $\mathcal{M}$ *to* $X \times_Z^{Lh} Y$.

(2) *Suppose that the pullback of a map between fibrant objects in* $\mathcal{M}$ *that is both a fibration in* $\mathcal{M}$ *and a weak equivalence in* $L\mathcal{M}$ *is a weak equivalence in* $L\mathcal{M}$. *Suppose that* X, Y *and* Z *are fibrant in* $\mathcal{M}$. *Then* $X \times_Z^h Y$ *is weakly equivalent in* $L\mathcal{M}$ *to* $X \times_Z^{Lh} Y$.

Proof. (1) is a consequence of [14, Proposition 13.3.7]. To prove (2) we first factorize f in $L\mathcal{M}$ as a trivial cofibration $Y \to Y_0$ in followed by a fibration $Y_0 \to Z$. Then we factorize $Y \to Y_0$ in $\mathcal{M}$ as a trivial cofibration $Y \to Y'$ in followed by a fibration $Y' \to Y_0$. By assumption the map $X \times_Z Y' \to X \times_Z Y_0$ is a weak equivalence in $L\mathcal{M}$. $\square$

3.5. Let $\mathcal{M}$ and $\mathcal{N}$ be generalized model categories and $F \colon \mathcal{M} \to \mathcal{N}$ be a functor having a right adjoint G. The adjoint pair (F, G) is a *Quillen pair* if F preserves cofibrations and trivial cofibrations. Equivalently, if G preserves fibrations and trivial fibrations. If the classes of weak equivalences of $\mathcal{M}$ and $\mathcal{N}$ have the two out of six property, then (F, G) is a Quillen pair if and only if F preserves cofibrations between cofibrant objects and trivial cofibrations if and only if G preserves fibrations between fibrant objects and trivial fibrations (a result due to Joyal).

The adjoint pair (F, G) is a *Quillen equivalence* if (F, G) is a Quillen pair and if for every cofibrant object A in $\mathcal{M}$ and every fibrant object X in $\mathcal{N}$, a map $FA \to X$ is a weak equivalence in $\mathcal{N}$ if and only if its adjunct $A \to GX$ is a weak equivalence in $\mathcal{M}$.

4. The natural generalized model category on $\mathrm{Fib}(E)$

We recall [19] that CAT is a model category in which the weak equivalences are the equivalences of categories, the cofibrations are the functors that are injective on objects and the fibrations are the isofibrations. Therefore, for every category E, $CAT_{/E}$ is a model category in which a map is a weak equivalence, cofibration or fibration if it is one in CAT.

Let E be a category.

Definition 12. Let $u \colon F \to G$ be a map of $\mathrm{Fib}(E)$. We say that u is an *E-equivalence* (*isofibration*) if the underlying functor of u is an equivalence of categories (isofibration). We say that u is a *trivial fibration* if it is both an E-equivalence and an isofibration.

Theorem 13. *The category* $\mathrm{Fib}(E)$ *is a proper generalized model category with the* E-*equivalences as weak equivalences, the maps that are injective on objects as cofibrations and the isofibrations as fibrations.*

The proof of Theorem 13 will be given after some preparatory results.

Proposition 14. *Let* $u \colon F \to G$ *be a map of* $\mathrm{Fib}(E)$. *The following are equivalent:*
   (1) *u is an E-equivalence.*
   (2) *For every* $S \in Ob(E)$, *the map* $u_S \colon F_S \to G_S$ *is an equivalence of categories.*
   (3) *u is an equivalence in the 2-category* $\mathscr{F}ib(E)$.

(4) $\mathbf{Cart}_E(u, X)\colon \mathbf{Cart}_E(G, X) \to \mathbf{Cart}_E(F, X)$ *is an equivalence for all* $X \in$ $\mathrm{Fib}(E)$.

(5) $\mathbf{Cart}_E(X, u)\colon \mathbf{Cart}_E(X, F) \to \mathbf{Cart}_E(X, G)$ *is an equivalence for all* $X \in$ $\mathrm{Fib}(E)$.

Proof. All is contained in [13, Exposé VI]. $\qquad\square$

Corollary 15. *A map* $u\colon F \to G$ *of* $\mathrm{Fib}(E)$ *is a trivial fibration if and only if for every* $S \in Ob(E)$, $u_S\colon F_S \to G_S$ *is a surjective equivalence.*

Proof. This follows from Lemma 1((1) and (2)) and Proposition 14. $\qquad\square$

For part (2) of the next result, let $\mathscr{M}$ be a class of functors that is contained in the class of injective on objects functors. In our applications $\mathscr{M}$ will be the class of injective on objects functors or the set consisting of one of the inclusions $* \to J$. Let $\mathscr{M}^\perp$ be the class of functors that have the right lifting property with respect to every element of $\mathscr{M}$.

Proposition 16. *Let* $u\colon F \to G$ *be a map of* $\mathrm{Fib}(E)$.

(1) u *is an isofibration if and only if* u^{cart} *is an isofibration.*

(2) *If* u *has the right lifting property with respect to the maps* $f \times E_{/S}$, *where* $f \in \mathscr{M}$ *and* $S \in Ob(E)$, *then* $u_S \in \mathscr{M}^\perp$ *for every* $S \in Ob(E)$.

Proof. (1) This is a consequence of Lemma 1(1) and of the fact that for every $S \in$ $Ob(E)$ and every object F of $\mathrm{Fib}(E)$, $(F^{cart})_S$ is the maximal groupoid associated to F_S.

(2) Let $S \in Ob(E)$ and $A \to B$ be an element of $\mathscr{M}$. Consider the commutative solid arrow diagram

We recall that the category of arrows of CAT is a model category in which the weak equivalences and fibrations are defined objectwise. A functor is cofibrant in this model category if and only if it is injective on objects. If we regard the previous diagram as a diagram in the category of arrows of CAT, then it has by 2.3(5) and the assumption on $\mathscr{M}$ a diagonal filler, the two dotted arrows. From Section 2.3 and hypothesis this diagonal filler has itself a diagonal filler, hence the bottom face diagram has one. $\qquad\square$

Lemma 17. (1) *Let*

be a pullback diagram in $CAT_{/E}$. If F, G and H are fibrations, u and v are cartesian functors and u is an isofibration, then $F \times_H G$ is a fibration and the diagram is a pullback in $\mathrm{Fib}(E)$.

(2) *Let*

$$\begin{array}{ccc} F & \xrightarrow{\ u\ } & G \\ {\scriptstyle v}\downarrow & & \downarrow \\ H & \longrightarrow & G \sqcup_F H \end{array}$$

be a pushout diagram in $CAT_{/E}$. If F, G and H are fibrations, u and v are cartesian functors and u is injective on objects, then $G \sqcup_F H$ is a fibration and the diagram is a pushout in $\mathrm{Fib}(E)$.

Proof. (1) The objects of $F \times_H G$ are pairs (x, y) with $x \in Ob(F), y \in Ob(G)$ such that $u(x) = v(y)$. We shall briefly indicate how the composite map $F \times_H G \to F \xrightarrow{p} E$ is a fibration. Let $S \in Ob(E)$, $(x, y) \in F \times_H G$ and $f \colon S \to p(x)$. A cartesian lift of f is obtained as follows. Let $y^f \to y$ and $x^f \to x$ be cartesian lifts of f. Since H is a fibration, u and v are cartesian functors and u is an isofibration, there is $x_0^f \in Ob(F_S)$ such that $x^f \cong x_0^f$ and $u(x_0^f) = v(y^f)$. Then the obvious map $(x_0^f, y^f) \to (x, y)$ is a cartesian lift of f. The universal property of the pullback is easy to see.

(2) The set of objects of $G \sqcup_F H$ can be identified with $Ob(H) \sqcup (Ob(G) \setminus ImOb(u))$. Since the structure functors $G \to E$ and $H \to E$ are isofibrations, one can easily check that the canonical map $G \sqcup_F H \to E$ is an isofibration. We shall use Lemma 1(4) to show that it is a fibration. Consider the following cube in $CAT_{/E}$

$$\begin{array}{ccc} F & \xrightarrow{\ u\ } & G \\ & H & \longrightarrow & G \sqcup_F H \\ \Phi\mathsf{L}F & \longrightarrow & \Phi\mathsf{L}G \\ & \Phi\mathsf{L}H & \longrightarrow & \Phi\mathsf{L}G \sqcup_{\Phi\mathsf{L}F} \Phi\mathsf{L}H \end{array}$$

(see Section 2.3 for the functors Φ and L). The top and bottom faces are pushouts and the vertical arrows having sources F, G and H are weak equivalences. The map $\Phi\mathsf{L}u$ is a cofibration since u is one. By [14, Proposition 15.10.10(1)] the map

$$G \sqcup_F H \to \Phi\mathsf{L}G \sqcup_{\Phi\mathsf{L}F} \Phi\mathsf{L}H$$

is a weak equivalence. Since Φ is a left adjoint, the target of this map is in the image of Φ, hence it is a fibration. It follows from Lemma 1(4) that $G \sqcup_F H$ is a fibration. The canonical maps $H \to G \sqcup_F H$ and $G \to G \sqcup_F H$ are cartesian functors by Lemma 1(6) applied to the front and right faces of the above cube diagram. Finally, it remains to prove that if

$$\begin{array}{ccc} F & \xrightarrow{\ u\ } & G \\ {\scriptstyle v}\downarrow & & \downarrow \\ H & \longrightarrow & K \end{array}$$

is a commutative diagram in $\mathrm{Fib}(E)$, then the resulting functor $G \sqcup_F H \to K$ is cartesian. This follows from Lemma 1(6) applied to the diagram

$$
\begin{array}{ccc}
G \sqcup_F H & \longrightarrow & \Phi\mathsf{L}G \sqcup_{\Phi\mathsf{L}F} \Phi\mathsf{L}H \\
\downarrow & & \downarrow \\
K & \longrightarrow & \Phi\mathsf{L}K
\end{array}
$$

$\square$

Remark 18. A consequence of Lemma 17(1) is that the fibre category $(F \times_H G)_S$ is the pullback $F_S \times_{H_S} G_S$. Thus, if F, G and H are fibred in groupoids then so is $F \times_H G$. A consequence of Lemma 17(2) is that if F, G and H are fibred in groupoids then so is $G \sqcup_F H$.

Example 19. (1) Let $u \colon F \to G$ be a map of $\mathrm{Fib}(E)$ and H an object of $\mathrm{Fib}(E)$. Then the diagram

$$
\begin{array}{ccc}
F \times H & \xrightarrow{F \times u} & G \times H \\
\downarrow & & \downarrow \\
F & \xrightarrow{\ u\ } & G
\end{array}
$$

in a pullback in $\mathrm{Fib}(E)$.
  (2) Let

$$
E_/ \colon E \to CAT_{/E}
$$

be the functor which takes S to $E_{/S}$. The functor $E_/$ preserves all the limits that exist in E. Therefore, if

$$
\begin{array}{ccc}
U \times_S T & \longrightarrow & T \\
\downarrow & & \downarrow \\
U & \longrightarrow & S
\end{array}
$$

is a pullback diagram in E, then

$$
\begin{array}{ccc}
E_{/U \times_S T} & \longrightarrow & E_{/T} \\
\downarrow & & \downarrow \\
E_{/U} & \longrightarrow & E_{/S}
\end{array}
$$

is a pullback diagram in $\mathrm{Fib}(E)$.

Proof of Theorem 13. Axioms A1 and A3 from Definition 8 are clear. Axiom A2 was dealt with in Lemma 17. We prove Axiom 4. Any map $u \colon F \to G$ of $\mathrm{Fib}(E)$ admits a factorization $u = vi \colon F \to H \to G$ in $CAT_{/E}$, where i is injective on objects and the underlying functor of v is a surjective equivalence. By Lemma 1(2) H is an object of $\mathrm{Fib}(E)$. By Lemma 1(5) i is a cartesian functor. By [13, Exposé VI Proposition 5.3(i)] v is a cartesian functor. Any commutative diagram

$$
\begin{array}{ccc}
F & \longrightarrow & H \\
u\downarrow & & \downarrow v \\
G & \longrightarrow & K
\end{array}
$$

Fib(E) in which the underlying functor of u is injective on objects and v is a trivial fibration has a diagonal filler in $CAT_{/E}$. By Lemma 1(5) (or [13, Exposé VI Corollaire 5.4], for example) this diagonal filler is a cartesian functor. Thus, the first part of Axiom 4 is proved. The rest of the Axiom 4 is proved similarly, using Lemma 1((3) and (5)) and [13, Exposé VI Proposition 5.3(i)].

Properness is easy to see. $\square$

Remark 20. Let F be an object of Fib(E). Let $D2$ be the discrete category with two objects. By cotensoring the sequence $D2 \to J \to *$ with F we obtain a natural factorization of the diagonal $F \to F \times F$ as

$$F \longrightarrow F^{(J)} \longrightarrow F \times F$$

in which the map $F \to F^{(J)}$ is an E-equivalence and the map $F^{(J)} \to F \times F$ is an isofibration. We obtain the following model for the mapping path factorization (Section 3.2) of a map $u \colon F \to G$ of Fib(E). The objects of a fibre category $(Pu)_S$ are triples (x, y, θ) with $x \in Ob(F_S), y \in Ob(G_S)$ and $\theta \colon y \to u(x)$ an isomorphism in G_S. The arrows are pairs of arrows making the obvious diagram commute.

Proposition 21 (Compatibility with the 2-category structure). *Let $u \colon F \to G$ be a cofibration and $v \colon H \to K$ an isofibration. Then the canonical map*

$$\mathbf{Cart}_E(G, H) \longrightarrow \mathbf{Cart}_E(G, K) \times_{\mathbf{Cart}_E(F,K)} \mathbf{Cart}_E(F, H)$$

is an isofibration that is a surjective equivalence if either u or v is an E-equivalence.

Proof. By Section 2.3 the diagram

$$\begin{array}{ccc} * & \longrightarrow & \mathbf{Cart}_E(G, H) \\ \downarrow & & \downarrow \\ J & \longrightarrow & \mathbf{Cart}_E(G, K) \times_{\mathbf{Cart}_E(F,K)} \mathbf{Cart}_E(F, H) \end{array}$$

has a diagonal filler if and only if the diagram

$$\begin{array}{ccc} F & \longrightarrow & H^{(J)} \\ \downarrow & & \downarrow \\ G & \longrightarrow & K^{(J)} \times_K H \end{array}$$

has one (the pullback exists by Lemma 17(1)). The latter is true since the map $H^{(J)} \to K^{(J)} \times_K H$ is a trivial fibration using Corollary 15. Suppose that u is an E-equivalence. By Proposition 14, the functors $\mathbf{Cart}_E(u, H)$ and $\mathbf{Cart}_E(u, K)$ are surjective equivalences. Since surjective equivalences are stable under pullback, the functor

$$\mathbf{Cart}_E(G, H) \to \mathbf{Cart}_E(G, K) \times_{\mathbf{Cart}_E(F,K)} \mathbf{Cart}_E(F, H)$$

is an equivalence by the two out of three property of equivalences. Suppose that v is an E-equivalence. Then $\mathbf{Cart}_E(F, v)$ and $\mathbf{Cart}_E(G, v)$ are equivalences the functor

$$\mathbf{Cart}_E(G, K) \times_{\mathbf{Cart}_E(F,K)} \mathbf{Cart}_E(F, H) \longrightarrow \mathbf{Cart}_E(G, K)$$

is an equivalence being the pullback of an equivalence along an isofibration. Therefore the canonical map is an equivalence. $\square$

Corollary 22. *A map* $u\colon F \to G$ *is an isofibration if and only if for every object* X *of* $\mathrm{Fib}(E)$, *the map*

$$\mathbf{Cart}_E(X,u)\colon \mathbf{Cart}_E(X,F) \to \mathbf{Cart}_E(X,G)$$

is an isofibration.

Proof. One half is a consequence of Proposition 21. The other half follows by putting $X = E_{/S}$, where $S \in Ob(E)$, and using 2.3(5), Lemma 1(1) and Section 2.1. $\qquad\square$

Corollary 23 (Compatibility with the 'internal hom')**.** *Let* $u : F \to G$ *be a cofibration and* $v : H \to K$ *an isofibration. Then the canonical map*

$$\mathsf{CART}(G,H) \longrightarrow \mathsf{CART}(G,K) \times_{\mathsf{CART}(F,K)} \mathsf{CART}(F,H)$$

is an isofibration that is a trivial fibration if either u *or* v *is an* E-*equivalence.*

Proof. The map $\mathsf{CART}(u,K)$ is an isofibration by 2.3(3) and Proposition 21, therefore the pullback in the displayed arrow exists by Lemma 17(1). The result follows from Remark 18, 2.3(3) and Proposition 21 applied to v and the cofibration $E_{/S} \times u$, $S \in Ob(E)$. $\qquad\square$

We recall [19, Theorem 4] that the category $[E^{op}, CAT]$ is a model category in which a map is a weak equivalence or cofibration if it is objectwise an equivalence of categories or objectwise injective on objects. We denote this model category by $[E^{op}, CAT]_{inj}$. We recall that the category $[E^{op}, CAT]$ is a model category in which a map is a weak equivalence or fibration if it is objectwise an equivalence of categories or objectwise an isofibration. We denote this model category by $[E^{op}, CAT]_{proj}$. The identity functors form a Quillen equivalence between $[E^{op}, CAT]_{proj}$ and $[E^{op}, CAT]_{inj}$.

Recall from Section 2.3 the adjoint pairs $(\Phi, \mathcal{S})$ and (L, Φ).

Proposition 24. *The adjoint pair* $(\Phi, \mathcal{S})$ *is a Quillen equivalence between* $\mathrm{Fib}(E)$ *and* $[E^{op}, CAT]_{inj}$. *The adjoint pair* (L, Φ) *is a Quillen equivalence between* $\mathrm{Fib}(E)$ *and* $[E^{op}, CAT]_{proj}$.

Proof. The functor $\Phi\colon [E^{op}, CAT]_{inj} \to \mathrm{Fib}(E)$ preserves and reflects weak equivalences and preserves cofibrations. Since the map $\mathsf{v}F$ is a weak equivalence (2.3(5)), the pair $(\Phi, \mathcal{S})$ is a Quillen equivalence.

The functor $\Phi\colon [E^{op}, CAT]_{proj} \to \mathrm{Fib}(E)$ preserves fibrations. Since the map $\mathsf{l}F$ is a weak equivalence, the pair (L, Φ) is a Quillen equivalence. $\qquad\square$

Let $m\colon A \to B$ be a category fibred in groupoids. Recall from Section 2.3 that the functor $m_\bullet^{fib}\colon \mathrm{Fib}(B) \to \mathrm{Fib}(A)$ has a left adjoint $m^\bullet$. The proof of the next result is straightforward.

Proposition 25. *Let* $m\colon A \to B$ *be a category fibred in groupoids. The adjoint pair* $(m^\bullet, m_\bullet^{fib})$ *is a Quillen pair.*

Let $f\colon T \to S$ be a map of E. The functor $f_\bullet^{fib}\colon \mathrm{Fib}(E_{/S}) \to \mathrm{Fib}(E_{/T})$ has a left adjoint $f^\bullet$.

Corollary 26. *Let* $f\colon T \to S$ *be a map of* E. *The adjoint pair* $(f^\bullet, f_\bullet^{fib})$ *is a Quillen pair.*

5. The generalized model category for stacks over a site

We briefly recall from [11, Chapitre 0 Définition 1.2] the notion of site. Let E be a category. A *topology* on E is an application which associates to each $S \in Ob(E)$ a non-empty collection $J(S)$ of sieves of $E_{/S}$. This data must satisfy two axioms. The elements of $J(S)$ are called *refinements* of S. A *site* is a category endowed with a topology.

Every category E has the *discrete* topology (only $E_{/S}$ is a refinement of the object S) and the *coarse* topology (every sieve of $E_{/S}$ is a refinement of S). Any other topology on E is 'in between' the discrete one and the coarse one.

Let E be a site. Let $\mathcal{C}$ be the collection of maps $R \subset E_{/S}$ of $\mathrm{Fib}(E)$, where S ranges through $Ob(E)$ and R is a refinement of S.

Since CAT is a model category, the theory of homotopy fiber squares [14, Section 13.3.11] is available.

Definition 27. A map $F \to G$ of $\mathrm{Fib}(E)$ *has property* P if for every element $R \subset E_{/S}$ of $\mathcal{C}$, the diagram

$$
\begin{array}{ccc}
\mathbf{Cart}_E(E_{/S}, F) & \longrightarrow & \mathbf{Cart}_E(R, F) \\
\downarrow & & \downarrow \\
\mathbf{Cart}_E(E_{/S}, G) & \longrightarrow & \mathbf{Cart}_E(R, G)
\end{array}
$$

in which the horizontal arrows are the restriction functors, is a homotopy fiber square. The map $F \to G$ is a $\mathcal{C}$-*local fibration* if it is an isofibration and it has property P. An object F of $\mathrm{Fib}(E)$ is $\mathcal{C}$-*local* if the map $F \to E$ is a $\mathcal{C}$-local fibration. The map $F \to G$ is a $\mathcal{C}$-*local equivalence* if for all $\mathcal{C}$-local objects X, the map

$$
\mathbf{Cart}_E(u, X) \colon \mathbf{Cart}_E(G, X) \to \mathbf{Cart}_E(F, X)
$$

is an equivalence of categories.

It follows directly from Definition 27 and a standard property of homotopy fiber squares that a $\mathcal{C}$-local object is the same as a stack ($=(E\text{-})champ$) in the sense of [11, Chapitre II Définition 1.2.1(ii)].

Example 28. We shall recall that 'sheaves are stacks'.

Let $\widehat{E}$ be the category of presheaves on E and η be the Yoneda embedding. Let $D \colon SET \to CAT$ denote the discrete category functor; it induces a functor $D \colon \widehat{E} \to [E^{op}, CAT]$. For every objects X, Y of $\widehat{E}$ there is a natural isomorphism

$$
\mathbf{Cart}_E(\Phi D X, \Phi D Y) \cong D\mathrm{Fib}(E)(\Phi D X, \Phi D Y)
$$

The composite functor $\Phi D \colon \widehat{E} \to \mathrm{Fib}(E)$ is full and faithful, hence we obtain a natural isomorphism

$$
\mathbf{Cart}_E(\Phi D X, \Phi D Y) \cong D\widehat{E}(\Phi D X, \Phi D Y)
$$

Let now $S \in Ob(E)$ and R be a refinement of S. Let R' be the sub-presheaf of $\eta(S)$ which corresponds to R. Since $E_{/S} = \Phi D \eta(S)$ and $R = \Phi D R'$, the previous natural isomorphism shows that a presheaf X on E is a sheaf if and only if $\Phi D X$ is a stack. In particular, $\eta(S)$ is a sheaf if and only if $E_{/S}$ is a stack.

Theorem 29. *There is a proper generalized model category* $\mathrm{Champ}(E)$ *on the category* $\mathrm{Fib}(E)$ *in which the weak equivalences are the* $\mathcal{C}$-*local equivalences and the cofibrations are the maps that are injective on objects. The fibrant objects of* $\mathrm{Champ}(E)$ *are the stacks.*

The proof of Theorem 29 will be given after some preparatory results.

Proposition 30. (1) *Every* E-*equivalence is a* $\mathcal{C}$-*local equivalence.*

(2) *The class of maps having property* P *is invariant under* E-*equivalences.*

(3) *The class of maps having property* P *contains* E-*equivalences and all maps between stacks.*

(4) *The class of maps having property* P *is closed under compositions, pullbacks along isofibrations and retracts.*

Proof. (1) follows from Proposition 14. (2) says that for every commutative diagram

$$
\begin{array}{ccc}
F & \longrightarrow & H \\
{\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\
G & \longrightarrow & K
\end{array}
$$

in which the horizontal maps are E-equivalences, u has property P if and only if v has it. This is so by Proposition 14 and [14, Proposition 13.3.13]. (3) follows from a standard property of homotopy fiber squares. (4) follows from standard properties of homotopy fiber squares and the fact that equivalences are closed under retracts. $\square$

Lemma 31. *A map between stacks has the right lifting property with respect to all maps that are both cofibrations and* $\mathcal{C}$-*local equivalences if and only if it is an isofibration.*

Proof (sufficiency). Let $H \to K$ be an isofibration between stacks and $F \to G$ a map that is both a cofibration and a $\mathcal{C}$-local equivalence. A commutative diagram

$$
\begin{array}{ccc}
F & \longrightarrow & H \\
\downarrow & & \downarrow \\
G & \longrightarrow & K
\end{array}
$$

has a diagonal filler if and only if the functor

$$
\mathbf{Cart}_E(G, H) \to \mathbf{Cart}_E(G, K) \times_{\mathbf{Cart}_E(F, K)} \mathbf{Cart}_E(F, H)
$$

is surjective on objects. We show that it is a surjective equivalence. The functor is an isofibration by Proposition 21. Hence it suffices to show that it is an equivalence. The maps $\mathbf{Cart}_E(G, K) \to \mathbf{Cart}_E(F, K)$ and $\mathbf{Cart}_E(G, H) \to \mathbf{Cart}_E(F, H)$ are surjective equivalences by assumption and Proposition 21. Since surjective equivalences are stable under pullback, the required functor is an equivalence by the two out of three property of equivalences. $\square$

For the notion of bicovering (=*bicouvrant*) map in $\mathrm{Fib}(E)$ we refer the reader to [11, Chapitre II Définition 1.4.1]. As in [*loc. cit.*, Chapitre II 1.4.1.1], we informally say that a map is bicovering if it is 'locally bijective on arrows' and 'locally essentially surjective on objects'.

Example 32. For every $S \in Ob(E)$ and every refinement R of S, $R \subset E_{/S}$ is a bicovering map.

By [11, Chapitre II Proof of Théorème d'existence 2.1.3] there are a 2-functor $\mathsf{A} \colon \mathscr{F}ib(E) \to \mathscr{F}ib(E)$ and a 2-natural transformation $a \colon Id_{\mathscr{F}ib(E)} \to \mathsf{A}$ such that $\mathsf{A}F$ is a stack and aF is bicovering for every object F of $\mathscr{F}ib(E)$. By [11, Chapitre II Corollaire 2.1.4] the class of bicovering maps coincides with the class of $\mathcal{C}$-local equivalences in the sense of Definition 27.

Lemma 33. *Bicovering maps are closed under pullbacks along isofibrations.*

Proof. Let

$$
\begin{array}{ccc}
F \times_H G & \xrightarrow{\ u'\ } & G \\
\downarrow & & \downarrow{\scriptstyle v} \\
F & \xrightarrow[\ u\ ]{} & H
\end{array}
$$

be a pullback diagram in $\mathrm{Fib}(E)$ with v an isofibration (see Lemma 17(1)).

Step 1. Suppose that the above pullback diagram is a pullback diagram of split fibrations and split functors with v an arbitrary split functor and u 'locally bijective on arrows'. We prove that u' is 'locally bijective on arrows'. Let $S \in Ob(E)$ and $(x, y), (x', y')$ be two objects of $(F \times_H G)_S$. Then, in the notation of [11, Chapitre I 2.6.2.1] and the terminology of [11, Chapitre 0 Définition 3.5] we have to show that the map

$$\mathrm{Hom}_S((x, y), (x', y')) \longrightarrow \mathrm{Hom}_S(y, y')$$

of presheaves on $E_{/S}$ is bicovering, where $E_{/S}$ has the induced topology [11, Chapitre 0 3.1.4]. This map is the pullback of the map

$$\mathrm{Hom}_S(x, x') \longrightarrow \mathrm{Hom}_S(u(x), u(x'))$$

which is by assumption bicovering. But bicovering maps of presheaves are stable under pullbacks [11, Chapitre 0 3.5.1].

Step 2. Suppose that in the above pullback diagram the map u is 'locally essentially surjective on objects'. We prove that u' is 'locally essentially surjective on objects'. Let $S \in Ob(E)$ and $y \in Ob(G_S)$. Let R' be the set of maps $f \colon T \to S$ such that there are $x \in Ob(F_T)$ and $y' \in Ob(G_T)$ with $u_T(x) = v_T(y')$ and $y' \cong f^*(y)$ in G_T. We have to show that R' is a refinement of S. Let R be the set of maps $f \colon T \to S$ such that there is $x \in Ob(F_T)$ with $u_T x \cong f^* v_S(y)$ in H_T. By assumption R is a refinement of S. Since $v_T f^*(y) \cong f^* v_S(y)$ we have $R' \subset R$. Conversely, let $f \colon T \to S$ be in R and x as above. Let ξ be the isomorphism $u_T(x) \cong v_T f^*(y)$. By assumption there are $y' \in Ob(G_T)$ and an isomorphism $y' \cong f^*(y)$ in G_T which is sent by v_T to ξ. In particular $u_T(x) = v_T(y')$ and so $R \subset R'$.

Step 3. Suppose that in the above pullback diagram the map u is bicovering. We can form the cube diagram

One clearly has $\mathsf{S}(F \times_H G) \cong \mathsf{S}F \times_{\mathsf{S}H} \mathsf{S}G$. By 2.3(5) the vertical arrows of the cube diagram are trivial fibrations and the map $\mathsf{S}v$ is an isofibration. By Proposition 30(1) $\mathsf{S}u$ is bicovering, hence by Steps 1 and 2 the map $(\mathsf{S}u)'$ is bicovering, so u' is bicovering. $\square$

Corollary 34. *Let* $F \in \mathrm{Fib}(E)$ *and* u *be a bicovering map. Then* $F \times u$ *is a bicovering map.*

Proof. This follows from Example 19(1) and Lemma 33. $\square$

The next result is the first part of [11, Chapitre II Corollaire 2.1.5], with a different proof.

Corollary 35. *If* G *is a stack then so is* $\mathsf{CART}(F, G)$ *for every* $F \in \mathrm{Fib}(E)$.

Proof. This follows from 2.3(4), Example 32 and Corollary 34. $\square$

Lemma 36. *An object of* $\mathrm{Fib}(E)$ *that has the right lifting property with respect to all maps that are both cofibrations and* $\mathcal{C}$-*local equivalences is a stack.*

Proof. Let F be as in the statement of the Lemma. Using Theorem 13 we factorize the map $aF \colon F \to \mathsf{A}F$ as a cofibration $F \to G$ followed by a trivial fibration $G \to \mathsf{A}F$. By Proposition 30(2) G is a stack. By hypothesis the diagram

$$
\begin{array}{ccc}
F & = & F \\
\downarrow & & \\
G & &
\end{array}
$$

has a diagonal filler, therefore F is a retract of G. By Proposition 30(4) F is a stack. $\square$

Proof of Theorem 29. We shall apply Lemma 9 to the natural generalized model category $\mathrm{Fib}(E)$ (Theorem 13). Since we have Proposition 30(1), it only remains to prove that every map $F \to G$ of $\mathrm{Fib}(E)$ can be factorized as a map that is both a cofibration and a $\mathcal{C}$-local equivalence followed by a map that has the right lifting property with respect to all maps that are both cofibrations and $\mathcal{C}$-local equivalences. Consider the diagram

$$
\begin{array}{ccc}
F & \xrightarrow{\ aF\ } & \mathsf{A}F \\
\downarrow & & \downarrow \\
G & \xrightarrow{\ aG\ } & \mathsf{A}G
\end{array}
$$

We can factorize the map $\mathsf{A}F \to \mathsf{A}G$ as a map $\mathsf{A}F \to H$ that is an E-equivalence followed by an isofibration $H \to \mathsf{A}G$. By Proposition 30(2) H is a stack, so by Lemma 31 the map $H \to \mathsf{A}G$ has the right lifting property with respect to all maps that are both cofibrations and $\mathcal{C}$-local equivalences. Therefore the pullback map $G \times_{\mathsf{A}G} H \to G$ has the right lifting property with respect to all maps that are both cofibrations and $\mathcal{C}$-local equivalences. By Lemma 33 the map $G \times_{\mathsf{A}G} H \to H$ is bicovering, therefore the canonical map $F \to G \times_{\mathsf{A}G} H$ is bicovering. We factorize it as a cofibration $F \to K$ followed by a trivial fibration $K \to G \times_{\mathsf{A}G} H$. The desired factorization is $F \to K$ followed by the composite $K \to G \times_{\mathsf{A}G} H \to G$.

The fact that the fibrant objects of $\mathrm{Champ}(E)$ are the stacks follows from Lemmas 31 and 36. Left properness of $\mathrm{Champ}(E)$ is a consequence of the left properness of $\mathrm{Fib}(E)$ and right properness is a consequence of Lemma 33. $\square$

Proposition 37. *Every fibration of* $\mathrm{Champ}(E)$ *is a* $\mathcal{C}$-*local fibration.*

Proof. Let $F \to G$ be a fibration of $\mathrm{Champ}(E)$. The argument used in the proof of Theorem 29 shows that $F \to G$ is a retract of the composite $K \to G \times_{\mathsf{A}G} H \to G$. We conclude by Proposition 30((3) and (4)). $\square$

Proposition 38 (Compatibility with the 2-category structure). *Let* $u\colon F \to G$ *be a cofibration and* $v\colon H \to K$ *a fibration in* $\mathrm{Champ}(E)$. *Then the canonical map*

$$\mathbf{Cart}_E(G, H) \longrightarrow \mathbf{Cart}_E(G, K) \times_{\mathbf{Cart}_E(F,K)} \mathbf{Cart}_E(F, H)$$

is an isofibration that is a surjective equivalence if either u *or* v *is a* $\mathcal{C}$-*local equivalence.*

Proof. The first part is contained in Proposition 21 since every fibration of $\mathrm{Champ}(E)$ is an isofibration. If v is a $\mathcal{C}$-local equivalence then v is a trivial fibration and the Proposition is contained in Proposition 21. Suppose that u is a $\mathcal{C}$-local equivalence. By adjunction it suffices to prove that for every injective on objects functor $A \to B$, the canonical map

$$A \times G \sqcup_{A \times F} B \times F \longrightarrow B \times G$$

is a cofibration and a $\mathcal{C}$-local equivalence (the pushout in the displayed arrow exists by Lemma 17(2)). This follows, for example, from Corollary 34. $\square$

Corollary 39 (Compatibility with the 'internal hom'). *Let* $u\colon F \to G$ *be a cofibration and* $v\colon H \to K$ *a fibration in* $\mathrm{Champ}(E)$. *Then the canonical map*

$$\mathsf{CART}(G, H) \longrightarrow \mathsf{CART}(G, K) \times_{\mathsf{CART}(F,K)} \mathsf{CART}(F, H)$$

is a trivial fibration if either u *or* v *is a* $\mathcal{C}$-*local equivalence.*

Proof. If v is a $\mathcal{C}$-local equivalence then v is a trivial fibration and the Corollary is Corollary 23. If u is a $\mathcal{C}$-local equivalence the result follows from Proposition 38. $\square$

Proposition 40. *The classes of bicoverings and isofibrations make* $\mathrm{Fib}(E)$ *a category of fibrant objects* [7].

Proof. A path object was constructed in Remark 20. Since we have Lemma 17(1), we conclude by the next result. $\square$

Lemma 41. *The maps that are both bicoverings and isofibrations are closed under pullbacks.*

Proof. A proof entirely similar to the proof of Lemma 33 can be given. We shall give a proof that uses Lemma 33. Let

$$
\begin{array}{ccc}
F \times_H G & \xrightarrow{\ u'\ } & G \\
\downarrow & & \downarrow v \\
F & \xrightarrow{\ u\ } & H
\end{array}
$$

be a pullback diagram in $\mathrm{Fib}(E)$ with u both an isofibration and a bicovering map. We factorize v as $v = pj\colon G \to K \to H$, where j is an E-equivalence and p is an isofibration and then we take successive pullbacks. The map $F \times_H K \to K$ is a bicovering map by Lemma 33 and an isofibration. The map $F \times_H G \to F \times_H K$ is an E-equivalence. By Proposition 30(1) the map u' is bicovering. $\qquad\square$

We give now, as Lemma 42, the analogues, in our context, of [16, Lemma 2.2 and Remark 2.3].

Let E be a category. Let

$$F \xrightarrow{\ u\ } H \xleftarrow{\ v\ } G$$

be a digram in $\mathrm{Fib}(E)$. The discussion from Section 3.4 and Remark 20 suggest the following model for the homotopy pullback of the previous diagram. The objects of the fibre category over $S \in Ob(E)$ are triples (x, y, θ) with $x \in Ob(F_S), y \in Ob(G_S)$ and $\theta\colon u(x) \to v(y)$ an isomorphism in H_S. The arrows are pairs of arrows making the obvious diagram commute. This model is commonly known as the 2-pullback or the iso-comma object of u and v and from now on we shall designate it by $F \times_H^h G$.

Lemma 42 (Homotopy pullbacks in $\mathrm{Champ}(E)$)**.** *Suppose that E is a site.*

(1) If F, G and H are stacks, then $F \times_H^h G$ is weakly equivalent in $\mathrm{Fib}(E)$ to the homotopy pullback in $\mathrm{Champ}(E)$ of the previous diagram.

(2) $F \times_H^h G$ is weakly equivalent in $\mathrm{Champ}(E)$ to the homotopy pullback in $\mathrm{Champ}(E)$ of the previous diagram.

Proof. (1) follows from Proposition 11(1). (2) follows from Proposition 11(2) and Lemma 41. $\qquad\square$

Let E and E' be two sites and $f\colon E \to E'$ be a category fibred in groupoids. Then for every $S \in Ob(E)$, the induced map $f_{/S}\colon E_{/S} \to E'_{/f(S)}$ sends sieves to sieves. Recall from Section 2.3 the adjoint pair $(f^\bullet, f_\bullet^{fib})$.

Proposition 43 (A change of base)**.** *Let E and E' be two sites and $f\colon E \to E'$ be a category fibred in groupoids. Suppose that for every $S \in Ob(E)$, the map $f_{/S}$ sends a refinement of S to a refinement of $f(S)$. Then the adjoint pair $(f^\bullet, f_\bullet^{fib})$ is a Quillen pair between $\mathrm{Champ}(E)$ and $\mathrm{Champ}(E')$.*

Proof. Since we have Proposition 25, it suffices to show that $f_\bullet^{fib}$ preserves stacks (Sections 3.5 and 3.3). Let F be a stack in $\mathrm{Fib}(E')$, $S \in Ob(E)$ and R be a refinement of S. The map $f_{/S}$ is an E'-equivalence and its restriction to R is an E'-equivalence $f_{/S}\colon R \to f_{/S}(R)$. It follows from Proposition 14 that the maps $\mathbf{Cart}_{E'}(f_{/S}, F)$ are equivalences. We have the following commutative diagram (see 2.3(7))

$$
\begin{array}{ccccc}
\mathbf{Cart}_{E'}(E'_{/f(S)}, F) & \longrightarrow & \mathbf{Cart}_{E'}(E_{/S}, F) & \xrightarrow{\ \cong\ } & \mathbf{Cart}_E(E_{/S}, f_\bullet^{fib}F) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{Cart}_{E'}(f_{/S}(R), F) & \longrightarrow & \mathbf{Cart}_{E'}(R, F) & \xrightarrow{\ \cong\ } & \mathbf{Cart}_E(R, f_\bullet^{fib}F)
\end{array}
$$

The left vertical arrow is an equivalence by assumption, hence $f_\bullet^{fib}F$ is a stack. $\quad\square$

6. Categories fibred in groupoids

Let E be a category. In this section we give the analogues of Theorems 13 and 29 for the category $\mathrm{Fibg}(E)$ defined in Section 2.3.

Theorem 44. *The category* $\mathrm{Fibg}(E)$ *is a proper generalized model category with the E-equivalences as weak equivalences, the maps that are injective on objects as cofibrations and the isofibrations as fibrations.*

Proof. It only remains to check axiom A2 from Definition 8. This is satisfied by Remark 18. □

Suppose now that E is a site. Notice that for every $S \in Ob(E)$ and every refinement R of S, $E_{/S}$ and R are objects of $\mathrm{Fibg}(E)$. We recall from [11, Chapitre II Définition 1.2.1(ii)] that an object F of $\mathrm{Fib}(E)$ is a *prestack* if for every $S \in Ob(E)$ and every refinement $R \subset E_{/S}$ of S, the restriction functor

$$\mathbf{Cart}_E(E_{/S}, F) \to \mathbf{Cart}_E(R, F)$$

is full and faithful.

Lemma 45. ([11] and [21, Proposition 4.20]) *If an object F of $\mathrm{Fib}(E)$ is a stack, so is F^{cart}. The converse holds provided that F is a prestack.*

Proof. We recall that $max\colon CAT \to GRPD$ denotes the maximal groupoid functor. We recall that an arbitrary functor f is essentially surjective if and only if the functor $max(f)$ is so and that if f is full and faithful then so is $max(f)$. The Lemma follows then from the following commutative diagram (see 2.3(6))

$$
\begin{array}{ccc}
\mathbf{Cartg}_E(E_{/S}, F^{cart}) & \xrightarrow{\cong} & max\mathbf{Cart}_E(E_{/S}, F) \\
\downarrow & & \downarrow \\
\mathbf{Cartg}_E(R, F^{cart}) & \xrightarrow{\cong} & max\mathbf{Cart}_E(R, F)
\end{array}
$$

□

Let F be an object of $\mathrm{Fibg}(E)$, G an object of $\mathrm{Fib}(E)$ and $u\colon F \to G$ a bicovering map. We claim that u^{cart} is a bicovering map as well. For, consider the diagram

$$
\begin{array}{ccc}
F^{cart} & = & F \\
{\scriptstyle u^{cart}}\downarrow & & \downarrow{\scriptstyle u} \\
G^{cart} & \longrightarrow & G
\end{array}
$$

One can readily check that the inclusion map $G^{cart} \to G$ is an isofibration and that by Lemma 17(1) the above diagram is a pullback. We conclude by Lemma 33. If, in addition, G is a stack, then the map $G^{cart} \to G$ is a bicovering map between stacks (see Lemma 45), hence by [11, Chapitre II Proposition 1.4.5] it is an E-equivalence. It follows that G is an object of $\mathrm{Fibg}(E)$.

Theorem 46. *There is a proper generalized model category* $\mathrm{Champg}(E)$ *on the category* $\mathrm{Fibg}(E)$ *in which the weak equivalences are the bicovering maps, the cofibrations are the maps that are injective on objects and the fibrantions are the fibrations of* $\mathrm{Champ}(E)$.

Proof. Using Theorem 44 and Lemma 9 it only remains to prove the factorization of an arbitrary map of $\mathrm{Fibg}(E)$ into a map that is both a cofibration and bicovering followed by a map that has the right lifting property with respect to all maps that are both cofibrations and bicoverings. The argument is the same as the one given in the proof of Theorem 29. For it to work one needs the functor A to send objects of $\mathrm{Fibg}(E)$ to objects of $\mathrm{Fibg}(E)$. This is so by the considerations preceding the statement of the Theorem, applied to the map $F \to \mathsf{A}F$. $\qquad\square$

7. Sheaves of categories

We begin by recalling the notion of sheaf of categories.

Let E be a small site. We recall that $\widehat{E}$ is the category of presheaves on E and $\eta\colon E \to \widehat{E}$ is the Yoneda embedding. We denote by $\widetilde{E}$ the category of sheaves on E and by a the associated sheaf functor, left adjoint to the inclusion functor $i\colon \widetilde{E} \to \widehat{E}$.

We denote by $\underline{Hom}$ the internal CAT-hom of the 2-category $[E^{op}, CAT]$ and by $X^{(A)}$ the cotensor of $X \in [E^{op}, CAT]$ with a category A. Let $Ob\colon CAT \to SET$ denote the set of objects functor; it induces a functor $Ob\colon [E^{op}, CAT] \to \widehat{E}$.

Lemma 47. *Let X be an object of $[E^{op}, CAT]$. The following are equivalent.*
  (a) *For every category A, $ObX^{(A)}$ is a sheaf* [1, Exposé ii Définition 6.1].
  (b) *For every $S \in Ob(E)$ and every refinement R of S, the natural map*

$$\underline{Hom}(D\eta(S), X) \to \underline{Hom}(DR', X)$$

is an isomorphism, where R' is the sub-presheaf of $\eta(S)$ which corresponds to R.
  (c) *For every $S \in Ob(E)$ and every refinement R of S, the natural map*

$$X(S) \to \lim_{R^{op}}(X|R)$$

is an isomorphism, where $(X|R)$ is the composite $R^{op} \to (E_{/S})^{op} \to E^{op} \xrightarrow{X} CAT$.

An object X of $[E^{op}, CAT]$ is a *sheaf on E with values in CAT* (simply, *sheaf of categories*) if it satisfies one of the conditions of Lemma 47. We denote by $\mathrm{Faisc}(E; CAT)$ the full subcategory of $[E^{op}, CAT]$ whose objects are the sheaves of categories. The category $[E^{op}, CAT]$ is equivalent to the category $\mathrm{Cat}(\widehat{E})$ of internal categories and internal functors in $\widehat{E}$ and $\mathrm{Faisc}(E; CAT)$ is equivalent to the category $\mathrm{Cat}(\widetilde{E})$ of internal categories and internal functors in $\widetilde{E}$ [1, Exposé ii Proposition 6.3.1].

Consider now the adjunctions

$$\mathrm{Fib}(E) \underset{\Phi}{\overset{\mathsf{L}}{\rightleftarrows}} \mathrm{Cat}(\widehat{E}) \underset{i}{\overset{a}{\rightleftarrows}} \mathrm{Cat}(\widetilde{E})$$

(see Section 2.3 for the adjoint pair (L, Φ)). We denote the unit of the adjoint pair (a, i) by k.

Theorem 48. *There is a right proper model category $\mathrm{Stack}(\widetilde{E})_{proj}$ on the category $\mathrm{Cat}(\widetilde{E})$ in which the weak equivalences and the fibrations are the maps that Φ takes into weak equivalences and fibrations of $\mathrm{Champ}(E)$. The adjoint pair $(a\mathsf{L}, \Phi i)$ is a Quillen equivalence between $\mathrm{Champ}(E)$ and $\mathrm{Stack}(\widetilde{E})_{proj}$.*

The prove the existence of the model category $\mathrm{Stack}(\widetilde{E})_{proj}$ we shall use Lemma 49 below and the following facts:

(1) if X is a sheaf of categories then ΦX is a prestack [11, Chapitre II 2.2.1];

(2) for every $X \in [E^{op}, CAT]$, the natural map $\Phi\mathsf{k}(X)\colon \Phi X \to \Phi ia X$ is bicovering [11, Chapitre II Lemme 2.2.2(ii)];

(3) if X is a sheaf of categories then $\Phi ia \mathcal{S}\Phi X$ is a stack (which is a consequence of) [11, Chapitre II Lemme 2.2.2(iv)].

See the end of this section for another proof of (1).

We recall that the weak equivalences of $\mathrm{Stack}(\widetilde{E})_{proj}$ have a simplified description. Let f be a map of $\mathrm{Cat}(\widetilde{E})$. By [11, Chapitre II Proposition 1.4.5] the map Φf is bicovering if and only if Φf is full and faithful and Φf is 'locally essentially surjective on objects'. Given any map u of $\mathrm{Fib}(E)$, the underlying functor of u is full and faithful if and only if for every $S \in Ob(E)$, u_S is full and faithful [13, Exposé VI Proposition 6.10]. Hence f is a weak equivalence if and only if f is full and faithful and Φf is 'locally essentially surjective on objects'.

Lemma 49. *Let $\mathcal{M}$ be a generalized model category. Suppose that there is a set I of maps of $\mathcal{M}$ such that a map of $\mathcal{M}$ is a trivial fibration if and only if it has the right lifting property with respect to every element of I. Let $\mathcal{N}$ be a complete and cocomplete category and let $F\colon \mathcal{M} \rightleftarrows \mathcal{N}\colon G$ be a pair of adjoint functors. Assume that*

(1) the set $F(I) = \{F(u) \mid u \in I\}$ permits the small object argument [14, Definition 10.5.15];

(2) $\mathcal{M}$ is right proper;

(3) $\mathcal{N}$ has a fibrant replacement functor, which means that there are

(i) a functor $\widehat{\mathrm{F}}\colon \mathcal{N} \to \mathcal{N}$ such that for every object X of $\mathcal{N}$ the object $G\widehat{\mathrm{F}}X$ is fibrant and

(ii) a natural transformation from the identity functor of $\mathcal{N}$ to $\widehat{\mathrm{F}}$ such that for every object X of $\mathcal{N}$ the map $GX \to G\widehat{\mathrm{F}}X$ is a weak equivalence;

(4) every fibrant object of $\mathcal{N}$ has a path object, which means that for every object X of $\mathcal{N}$ such that GX is fibrant there is a factorization

$$X \xrightarrow{\ s\ } Path X \xrightarrow{\ p_0 \times p_1\ } X \times X$$

of the diagonal map $X \to X \times X$ such that $G(s)$ is a weak equivalence and $G(p_0 \times p_1)$ is a fibration.

Then $\mathcal{N}$ becomes a right proper model category in which the weak equivalences and the fibrations are the maps that G takes into weak equivalences and fibrations.

The adjoint pair (F, G) is a Quillen equivalence if and only if for every cofibrant object A of $\mathcal{M}$, the unit map $A \to GFA$ of the adjunction is a weak equivalence.

Proof. Let f be a map of $\mathcal{N}$. We say that f is a *trivial fibration* if $G(f)$ is a trivial fibration and we say that f is a *cofibration* if it is an $F(I)$-cofibration in the sense of [14, Definition 10.5.2(2)]. By (1) and [14, Corollary 10.5.23] every map of $\mathcal{N}$ can be factorized into a cofibration followed by a trivial fibration and every cofibration has the left lifting property with respect to every trivial fibration.

Let $f\colon X \to Y$ be a map of $\mathcal{N}$ such that GX and GY are fibrant. Then (4) implies that we can construct the mapping path factorization of f (see Section 3.0.6, for instance), that is, f can be factorized into a map $X \to Pf$ that is a

weak equivalence followed by a map $Pf \to Y$ that is a fibration. Moreover, GPf is fibrant.

We show that every map $f\colon X \to Y$ of $\mathcal{N}$ can be factorized into a map that is both a cofibration and a weak equivalence followed by a map that is a fibration. By (3) we have a commutative diagram

$$
\begin{array}{ccc}
X & \longrightarrow & \widehat{\mathrm{F}}X \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \widehat{\mathrm{F}}f} \\
Y & \longrightarrow & \widehat{\mathrm{F}}Y
\end{array}
$$

The map $\widehat{\mathrm{F}}f$ can be factorized into a map $\widehat{\mathrm{F}}X \to P\widehat{\mathrm{F}}f$ that is a weak equivalence followed by a map $P\widehat{\mathrm{F}}f \to \widehat{\mathrm{F}}Y$ that is a fibration. Let Z be the pullback of $P\widehat{\mathrm{F}}f \to \widehat{\mathrm{F}}Y$ along $Y \to \widehat{\mathrm{F}}Y$. By (2) the map $Z \to P\widehat{\mathrm{F}}f$ is a weak equivalence, therefore the canonical map $X \to Z$ is a weak equivalence. We factorize $X \to Z$ into a map $X \to X'$ that is a cofibration followed by a map $X' \to Z$ that is a trivial fibration. The desired factorization of f is $X \to X'$ followed by the composite $X' \to Z \to Y$.

We show that every commutative diagram in $\mathcal{N}$

$$
\begin{array}{ccc}
A & \longrightarrow & X \\
{\scriptstyle j}\downarrow & & \downarrow{\scriptstyle p} \\
B & \longrightarrow & Y
\end{array}
$$

where j is both a cofibration and a weak equivalence and p is a fibration has a diagonal filler. We shall construct a commutative diagram

$$
\begin{array}{ccccc}
A & \longrightarrow & X' & \longrightarrow & X \\
{\scriptstyle j}\downarrow & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\
B & \longrightarrow & Y' & \longrightarrow & Y
\end{array}
$$

with q a trivial fibration. We factorize the map $B \to Y$ into a map $B \to Y'$ that is both a cofibration and a weak equivalence followed by a map $Y' \to Y$ that is a fibration. Similarly, we factorize the canonical map $A \to Y' \times_Y X$ into a map $A \to X'$ that is both a cofibration and a weak equivalence followed by a map $X' \to Y' \times_Y X$ that is a fibration. Let q be the composite map $X' \to Y'$. Then q is a trivial fibration.

The model category $\mathcal{N}$ is right proper since $\mathcal{M}$ is right proper.

Suppose that (F, G) is a Quillen equivalence. Let A be a cofibrant object of $\mathcal{M}$. We can find a weak equivalence $f\colon FA \to X$ with X fibrant. The composite map $A \to GFA \to GX$ is the adjunct of f, hence it is a weak equivalence. Thus, $A \to GFA$ is a weak equivalence. Conversely, let A be a cofibrant object of $\mathcal{M}$ and X a fibrant object of $\mathcal{N}$. If $FA \to X$ is a weak equivalence then its adjunct is the composite $A \to GFA \to GX$, which is a weak equivalence. If $f\colon A \to GX$ is a weak equivalence, then it factorizes as $A \to GFA \xrightarrow{Gf'} GX$, where f' is the adjunct of f. Hence Gf' is a weak equivalence, which means that f' is a weak equivalence. $\qquad\square$

Proof of Theorem 48. In Lemma 49 we take $\mathcal{M} = \mathrm{Champ}(E)$, $\mathcal{N} = \mathrm{Cat}(\widetilde{E})$, $F = a\mathsf{L}$, $G = \Phi i$ and I to be the set of maps $\{f \times E_{/S}\}$ with $S \in Ob(E)$ and $f \in \mathscr{M}$, where

$\mathscr{M}$ is the set of functors such that a functor is a surjective equivalence if and only if it has the right lifting property with respect to every element of $\mathscr{M}$.

By Theorem 13 and Proposition 16(2) a map of $\mathrm{Fib}(E)$ is a trivial fibration if and only if it has the right lifting property with respect to every element of I.

We shall now check the assumptions (1)-(4) of Lemma 49. (1) and (2) are clear. We check (3). Let X be a sheaf of categories. We put $\widehat{F}X = ia\mathbb{S}\Phi X$ and the natural transformation from the identity functor of $\mathrm{Cat}(\widetilde{E})$ to $\widehat{F}$ to be the composite map

$$X \longrightarrow \mathbb{S}\Phi X \xrightarrow{\mathsf{k}(\mathbb{S}\Phi X)} ia\mathbb{S}\Phi X$$

Assumption (3) of Lemma 49 is fulfilled by the facts (2) and (3) mentioned right after the statement of Theorem 48. We check (4). Let X be a sheaf of categories such that ΦX is a stack. Let J be the groupoid with two objects and one isomorphism between them. The diagonal $X \to X \times X$ factorizes as

$$X \xrightarrow{\ s\ } X^{(J)} \xrightarrow{p_0 \times p_1} X \times X$$

Since Φ preserves cotensors and the cotensor of a stack and a category is a stack (2.3(2)), (4) follows from Remark 20.

We now prove that $(a\mathsf{L}, \Phi i)$ is a Quillen equivalence. For this we use Lemma 49. For every object F of $\mathrm{Fib}(E)$, the unit $F \to \Phi ia\mathsf{L}F$ of this adjoint pair is the composite

$$F \xrightarrow{\ \mathsf{I}_F\ } \Phi\mathsf{L}F \xrightarrow{\Phi\mathsf{k}(\mathsf{L}F)} \Phi ia\mathsf{L}F$$

which is a bicovering map. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Theorem 50. *The model category* $\mathrm{Cat}(\widehat{E})_{proj}$ *admits a proper left Bousfield localization* $\mathrm{Stack}(\widehat{E})_{proj}$ *in which the weak equivalences and the fibrations are the maps that Φ takes into weak equivalences and fibrations of* $\mathrm{Champ}(E)$. *The adjoint pair* (L, Φ) *is a Quillen equivalence between* $\mathrm{Champ}(E)$ *and* $\mathrm{Stack}(\widehat{E})_{proj}$.

Proof. The proof is similar to the proof of Theorem 48, using the adjoint pair (L, Φ) and the fibrant replacement functor

$$X \longrightarrow \mathbb{S}\Phi X \xrightarrow{\mathbb{S}(a\Phi X)} \mathbb{S}A\Phi X$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proposition 51 (Compatibility with the 2-category structure)**.** *Let $A \to B$ be an injective on objects functor and $X \to Y$ a fibration of* $\mathrm{Stack}(\widetilde{E})_{proj}$. *Then the canonical map*

$$X^{(B)} \longrightarrow X^{(A)} \times_{Y^{(A)}} Y^{(B)}$$

is a fibration that is a trivial fibration if $A \to B$ is an equivalence of categories or $X \to Y$ is a weak equivalence.

Proof. Since Φ preserves cotensors, the Proposition follows from Proposition 38. $\quad\square$

We recall [19, Theorem 4] that $\mathrm{Cat}(\widetilde{E})$ is a model category in which the weak equivalences are the maps that Φ takes into bicovering maps and the cofibrations are the internal functors that are monomorphisms on objects. See Appendix 3 for another approach to this result. We denote this model category by $\mathrm{Stack}(\widetilde{E})_{inj}$.

Proposition 52. *The identity functors on* $\mathrm{Cat}(\widetilde{E})$ *form a Quillen equivalence between* $\mathrm{Stack}(\widetilde{E})_{proj}$ *and* $\mathrm{Stack}(\widetilde{E})_{inj}$.

Proof. We show that the identity functor $\mathrm{Stack}(\widetilde{E})_{proj} \to \mathrm{Stack}(\widetilde{E})_{inj}$ preserves cofibrations. For that, it suffices to show that for every object F of $\mathrm{Fib}(E)$ and every injective on objects functor f, the map $a\mathsf{L}(f \times F)$ is a cofibration of $\mathrm{Stack}(\widetilde{E})_{inj}$. The map $\mathsf{L}(f \times F)$ is objectwise injective on objects (see the proof of Proposition 24), which translates in $\mathrm{Cat}(\widehat{E})$ as: $\mathsf{L}(f \times F)$ is an internal functor having the property that is a monomorphism on objects. But the associated sheaf functor a is known to preserve this property.

Since the classes of weak equivalences of the two model categories are the same, the result follows. $\square$

Let E' be another small site and $f^{-1}\colon E \to E'$ be the functor underlying a morphisms of sites $f\colon E' \to E$ [11, Chapitre 0 Définition 3.3]. The adjoint pair $f^*\colon \widetilde{E} \rightleftarrows \widetilde{E'}\colon f_*$ induces an adjoint pair $f^*\colon \mathrm{Cat}(\widetilde{E}) \rightleftarrows \mathrm{Cat}(\widetilde{E'})\colon f_*$.

Proposition 53 (Change of site)**.** *The adjoint pair* (f^*, f_*) *is a Quillen pair between* $\mathrm{Stack}(\widetilde{E})_{proj}$ *and* $\mathrm{Stack}(\widetilde{E'})_{proj}$.

Proof. Consider the diagram

$$
\begin{array}{ccc}
\mathrm{Fib}(E) & \xleftarrow{\;\;f_{\bullet}^{fib}\;\;} & \mathrm{Fib}(E') \\[4pt]
\Big\uparrow{\scriptstyle \Phi} & & \Big\uparrow{\scriptstyle \Phi'} \\[4pt]
[E^{op}, CAT] & \xleftarrow{\;\;f_*\;\;} & [E'^{op}, CAT] \\[4pt]
\Big\uparrow{\scriptstyle i} & & \Big\uparrow{\scriptstyle i'} \\[4pt]
\mathrm{Cat}(\widetilde{E}) & \xleftarrow{\;\;f_*\;\;} & \mathrm{Cat}(\widetilde{E'})
\end{array}
$$

where $f_{\bullet}^{fib}$ was defined in Section 2.3 and $f_*\colon [E'^{op}, CAT] \to [E^{op}, CAT]$ is the functor obtained by composing with f. It is easy to check that the functor $f_{\bullet}^{fib}$ preserves isofibrations and trivial fibrations. By [11, Chapitre II Proposition 3.1.1] it also preserves stacks. Since $f_{\bullet}^{fib}\Phi' = \Phi f_*$, it follows that f_* preserves trivial fibrations and the fibrations between fibrant objects. $\square$

Let $p\colon C \to I$ be a fibred site [2, Exposé vi 7.2.1] and $\tilde{p}\colon \tilde{C}^{/I} \to I$ be the (bi)fibred topos associated to p [2, Exposé vi 7.2.6]. Using the above considerations we obtain a bifibration $\mathrm{Cat}(\tilde{C}^{/I}) \to I$ whose fibres are isomorphic to $\mathrm{Cat}(\widetilde{C_i})$, hence by they are model categories. Moreover, by Proposition 53 the inverse and direct image functors are Quillen pairs.

An elementary example of a fibred site is the Grothendieck construction associated to the functor that sends a topological space X to the category $\mathcal{O}(X)$ whose objects are the open subsets of X and whose arrows are the inclusions of subsets.

Proposition 54. *Let* E *and* E' *be two small sites and* $f\colon E \to E'$ *be a category fibred in groupoids. Suppose that for every* $S \in Ob(E)$, *the map* $E_{/S} \to E'_{/f(S)}$ *sends a refinement of* S *to a refinement of* $f(S)$. *Then* f *induces a Quillen pair between* $\mathrm{Stack}(\widetilde{E})_{proj}$ *and* $\mathrm{Stack}(\widetilde{E'})_{proj}$.

Proof. The proof is similar to the proof of Proposition 53. Consider the solid arrow diagram

$$
\begin{array}{ccc}
\mathrm{Fib}(E) & \xleftarrow{\;f^{fib}_{\bullet}\;} & \mathrm{Fib}(E') \\
\Big\uparrow{\scriptstyle\Phi} & & \Big\uparrow{\scriptstyle\Phi'} \\
[E^{op}, CAT] & \underset{f^*}{\overset{f_!}{\rightleftarrows}} & [E'^{op}, CAT] \\
{\scriptstyle i}\Big\Updownarrow{\scriptstyle a} & & {\scriptstyle i'}\Big\Updownarrow{\scriptstyle a'} \\
\mathrm{Cat}(\widetilde{E}) & \xleftarrow{\quad f^*\quad} & \mathrm{Cat}(\widetilde{E'})
\end{array}
$$

where $f_!$ is the left adjoint to the functor f^* obtained by composing with f. We claim that the composition with f functor $f^* \colon \widehat{E'} \to \widehat{E}$ preserves sheaves. Let X be a sheaf on E'. By Example 28 it suffices to show that $\Phi D f^* X$ is a stack. But $\Phi D f^* X = f^{fib}_{\bullet} \Phi' D X$, so $f^* X$ is a sheaf by Proposition 43. Therefore, f^* induces a functor $f^* \colon \mathrm{Cat}(\widetilde{E'}) \to \mathrm{Cat}(\widetilde{E})$. Since $f^* i' = i f^*$, a formal argument implies that $a' f_! i$ is left adjoint to f^*.

The fact that $(a' f_! i, f^*)$ is a Quillen pair follows from Proposition 43. $\qquad\square$

Here is an application of Proposition 54. For every $S \in Ob(E)$, the category $E_{/S}$ has the induced topology [11, Chapitre 0 3.1.4]. A map $T \to S$ of E induces a category fibred in groupoids $E_{/T} \to E_{/S}$. The assumption of Proposition 54 is satisfied. By [11, Chapitre II Proposition 3.4.4] we obtain a stack over E whose fibres are model categories and such that the inverse and direct image functors are Quillen pairs.

Sheaves of categories are prestacks. Let E be a site. We recall that an object F of $\mathrm{Fib}(E)$ is a prestack if for every $S \in Ob(E)$ and every refinement $R \subset E_{/S}$ of S, the restriction functor

$$
\mathbf{Cart}_E(E_{/S}, F) \to \mathbf{Cart}_E(R, F)
$$

is full and faithful.

We give here an essentially-from-the-definition proof of [11, Chapitre II 2.2.1], namely that if $X \in [E^{op}, CAT]$ is a sheaf of categories then ΦX is a prestack.

Let first $X \in [E^{op}, CAT]$. Let $D \colon SET \to CAT$ be the discrete category functor; it induces a functor $D \colon \widehat{E} \to [E^{op}, CAT]$. Let R' be the sub-presheaf of $\eta(S)$ which corresponds to R. We have the following commutative diagram

$$
\begin{array}{ccc}
\mathbf{Cart}_E(E_{/S}, \Phi X) & \longrightarrow & (\Phi X)_S = \underline{Hom}(D\eta(S), X) \\
\Big\downarrow & & \Big\downarrow{\scriptstyle(I)} \\
& & \underline{Hom}(DR', X) \\
\Big\downarrow & & \Big\downarrow{\scriptstyle(II)} \\
\mathbf{Cart}_E(R, \Phi X) & \longrightarrow & \underline{Hom}(DR', \mathcal{S}\Phi X)
\end{array}
$$

The top horizontal arrow is a surjective equivalence $(2.3(5))$. Since $(\Phi, \mathcal{S})$ is a 2-adjunction, the bottom horizontal arrow is an isomorphism. We will show below

that the map (II) is full and faithful. If X is now a sheaf of categories, then the map (I) is an isomorphism by Lemma 47, therefore in this case ΦX is a prestack.

Let $P \in \widehat{E}$. We denote by $E_{/P}$ the category $\Phi D P$. Let $m \colon E_{/P} \to E$ be the canonical map. The natural functor

$$m^* \colon [E^{op}, CAT] \to [(E_{/P})^{op}, CAT]$$

has a left adjoint $m_!$ that is the left Kan extension along m^{op}. Since m^{op} is an opfibration, $m_!$ has a simple description. For example, let A be a category and let $cA \in [(E_{/P})^{op}, CAT]$ be the constant object at A; then $m_! cA$ is the tensor between A and DP in the 2-category $[E^{op}, CAT]$. It follows that for every $X \in [E^{op}, CAT]$ we have an isomorphism

$$\lim_{(E_{/P})^{op}} m^* X \cong \underline{Hom}(DP, X)$$

The map $X \to \mathcal{S}\Phi X$ is objectwise both an equivalence of categories and injective on objects, hence so is the map $m^* X \to m^* \mathcal{S}\Phi X$. Therefore the map

$$\lim_{(E_{/P})^{op}} m^* X \to \lim_{(E_{/P})^{op}} m^* \mathcal{S}\Phi X$$

is both full and faithful and injective on objects.

8. Appendix 1: Stacks vs. the homotopy sheaf condition

Throughout this section E is a site whose topology is generated by a pretopology.

8.1.   We recall that the model category CAT is a simplicial model category. The cotensor $A^{(K)}$ between a category A and a simplicial set K is constructed as follows.

Let $\mathbf{S}$ be the category of simplicial sets. Let $cat \colon \mathbf{S} \to CAT$ be the fundamental category functor, left adjoint to the nerve functor. Let $(-)_1^{-1} \colon CAT \to GRPD$ be the free groupoid functor, left adjoint to the inclusion functor. Then

$$A^{(K)} = [(catK)_1^{-1}, A]$$

One has $A^{(\Delta[n])} = [J^n, A]$, where J^n is the free groupoid on $[n]$.

Let $\mathbf{X}$ be a cosimplicial object in CAT. The total object of $\mathbf{X}$ [14, Definition 18.6.3] is calculated as

$$\mathrm{Tot}\mathbf{X} = \underline{Hom}(J, \mathbf{X})$$

where $\underline{Hom}$ is the CAT-hom of the 2-category $[\Delta, CAT]$ and J is the cosimplicial object in CAT that J^n defines. The category $\underline{Hom}(J, \mathbf{X})$ has a simple description. For $n \geq 2$, J^n is constructed from J^1 by iterated pushouts, so by adjunction an object of $\underline{Hom}(J, \mathbf{X})$ is a pair (x, f), where $x \in Ob(\mathbf{X}^0)$ and $f \colon d^1(x) \to d^0(x)$ is an isomorphism of $\mathbf{X}^1$ such that $s^0(f)$ is the identity on x and $d^1(f) = d^0(f)d^2(f)$. An arrow $(x, f) \to (y, g)$ is an arrow $u \colon x \to y$ of $\mathbf{X}^0$ such that $d^0(u)f = gd^1(u)$.

If $\mathbf{X}$ moreover a coaugmented cosimplicial object in CAT with coaugmentation $\mathbf{X}^{-1}$, there is a natural map

$$\mathbf{X}^{-1} \to \mathrm{Tot}\mathbf{X}$$

We recall [14, Theorem 18.7.4(2)] that if $\mathbf{X}$ is Reey fibrant in $[\Delta, CAT]$, then the natural map

$$\mathrm{Tot}\mathbf{X} \to \mathrm{holim}\mathbf{X}$$

is an equivalence of categories.

8.2. For each $S \in Ob(E)$ and each covering family $\mathscr{S} = (S_i \to S)_{i \in I}$ there is a simplicial object $E_{/\mathscr{S}}$ in $\mathrm{Fib}(E)$ given by

$$(E_{/\mathscr{S}})_n = \coprod_{i_0,\dots,i_n \in I^{n+1}} E_{/S_{i_0,\dots,i_n}}$$

where $S_{i_0,\dots,i_n} = S_{i_0} \times_S \dots \times_S S_{i_n}$. $E_{/\mathscr{S}}$ is augmented with augmentation $E_{/S}$.

Proposition 55. *An object F of $\mathrm{Fib}(E)$ is a stack if and only if for every $S \in Ob(E)$ and every covering family $\mathscr{S} = (S_i \to S)$, the natural map*

$$\mathbf{Cart}_E(E_{/S}, F) \to \mathrm{Tot}\mathbf{Cart}_E(E_{/\mathscr{S}}, F)$$

is an equivalence of categories.

Proof. The proof consists of unraveling the definitions. $\qquad\square$

Following [15], we say that an object F of $\mathrm{Fib}(E)$ *satisfies the homotopy sheaf condition* if for every $S \in Ob(E)$ and every covering family $\mathscr{S} = (S_i \to S)$, the natural map

$$\mathbf{Cart}_E(E_{/S}, F) \to \mathrm{holim}\mathbf{Cart}_E(E_{/\mathscr{S}}, F)$$

is an equivalence of categories.

Proposition 56. [15, Theorem 1.1] *An object of $\mathrm{Fib}(E)$ satisfies the homotopy sheaf condition if and only if it is a stack.*

Proof. This follows from Proposition 55 and Lemma 57. $\qquad\square$

Lemma 57. *For every $S \in Ob(E)$, every covering family $\mathscr{S} = (S_i \to S)_{i \in I}$ and every object F of $\mathrm{Fib}(E)$, the cosimplicial object in CAT $\mathbf{Cart}_E(E_{/\mathscr{S}}, F)$ is Reedy fibrant.*

Proof. By Proposition 21 the adjoint pair

$$F^{(-)} \colon CAT \rightleftarrows \mathrm{Fib}(E)^{op} \colon \mathbf{Cart}_E(-, F)$$

is a Quillen pair. Therefore, to prove the Lemma it suffices to show that $E_{/\mathscr{S}}$ is Reedy cofibrant, by which we mean that for every $[n] \in Ob(\Delta)$ the latching object of $E_{/\mathscr{S}}$ at $[n]$, denoted by $L_n E_{/\mathscr{S}}$, exists and the natural map $L_n E_{/\mathscr{S}} \to (E_{/\mathscr{S}})_n$ is injective on objects. A way to prove this is by using Lemma 58. $\qquad\square$

8.3. **Latching objects of simplicial objects.** In general, the following considerations may help deciding whether a simplicial object in a generalized model category is Reedy cofibrant.

Let $\mathcal{M}$ be a category and $\mathbf{X}$ a simplicial object in $\mathcal{M}$. We recall that the latching object of $\mathbf{X}$ at $[n] \in Ob(\Delta)$ is

$$L_n \mathbf{X} = \operatorname*{colim}_{\partial([n] \downarrow \overleftarrow{\Delta})^{op}} \mathbf{X}$$

provided that the colimit exists. Here $\overleftarrow{\Delta}$ is the subcategory of Δ consisting of the surjective maps and $\partial([n] \downarrow \overleftarrow{\Delta})$ is the full subcategory of $([n] \downarrow \overleftarrow{\Delta})$ containing all the objects except the identity map of $[n]$. Below we shall review the construction of $L_n \mathbf{X}$.

The category $([n] \downarrow \overleftarrow{\Delta})$ has the following description [12, VII 1]. The identity map of $[n]$ is its initial object. Any other object is of the form $s^{i_1} \dots s^{i_k} \colon [n] \to [n-k]$, where s^j denotes a codegeneracy operator, $1 \le k \le n$ and $0 \le i_1 \le \dots \le i_k \le n-1$.

For $n \geq 0$ we let $\underline{n}$ be the set $\{1, 2, ..., n\}$, with the convention that $\underline{0}$ is the empty set. We denote by $\mathcal{P}(\underline{n})$ the power set of $\underline{n}$. $\mathcal{P}(\underline{n})$ is a partially ordered set. We set $\mathcal{P}_0(\underline{n}) = \mathcal{P}(\underline{n}) \setminus \{\emptyset\}$ and $\mathcal{P}_1(\underline{n}) = \mathcal{P}(\underline{n}) \setminus \{\underline{n}\}$. There is an isomorphism

$$([n] \downarrow \overleftarrow{\Delta}) \cong \mathcal{P}(\underline{n})$$

which sends the identity map of $[n]$ to $\emptyset$ and the object $s^{i_1}...s^{i_k} \colon [n] \to [n-k]$ as above to $\{i_1 + 1, ..., i_k + 1\}$. Under this isomorphism the category $\partial([n] \downarrow \overleftarrow{\Delta})$ corresponds to $\mathcal{P}_0(\underline{n})$, therefore $\partial([n] \downarrow \overleftarrow{\Delta})^{op}$ is isomorphic to $\mathcal{P}_1(\underline{n})$. The displayed isomorphism is natural in the following sense. Let $Dec^1 \colon \Delta \to \Delta$ be $Dec^1([n]) = [n] \sqcup [0] \cong [n+1]$. Then we have a commutative diagram

$$
\begin{array}{ccc}
([n] \downarrow \overleftarrow{\Delta}) & \xrightarrow{\cong} & \mathcal{P}(\underline{n}) \\
{\scriptstyle Dec^1}\downarrow & & \downarrow \\
([n+1] \downarrow \overleftarrow{\Delta}) & \xrightarrow{\cong} & \mathcal{P}(\underline{n+1})
\end{array}
$$

in which the unlabelled vertical arrow is the inclusion. Restricting the arrow Dec^1 to ∂ and then taking the opposite category we obtain a commutative diagram in which the unlabelled vertical arrow becomes $- \cup \{n+1\} \colon \mathcal{P}_1(\underline{n}) \to \mathcal{P}_1(\underline{n+1})$.

For $n \geq 1$ the category $\mathcal{P}_1(\underline{n})$ is constructed inductively as the Grothendieck construction applied to the functor $(2 \leftarrow 1 \rightarrow 0) \to CAT$ given by $* \leftarrow \mathcal{P}_1(\underline{n-1}) = \mathcal{P}_1(\underline{n-1})$. Therefore colimits indexed by $\mathcal{P}_1(\underline{n})$ have the following description. Let $\mathbf{Y} \colon \mathcal{P}_1(\underline{n}) \to \mathcal{M}$. We denote by $\mathbf{Y}$ the precomposition of $\mathbf{Y}$ with the inclusion $\mathcal{P}_1(\underline{n-1}) \subset \mathcal{P}_1(\underline{n})$; then $\underset{\mathcal{P}_1(\underline{n})}{\operatorname{colim}}\mathbf{Y}$ is the pushout of the diagram

$$
\begin{array}{ccc}
\underset{\mathcal{P}_1(\underline{n-1})}{\operatorname{colim}} \mathbf{Y} & \longrightarrow & \underset{\mathcal{P}_1(\underline{n-1})}{\operatorname{colim}} \mathbf{Y}(- \cup \{n\}) \\
\downarrow & & \\
\mathbf{Y}_{\underline{n-1}} & &
\end{array}
$$

provided that the pushout and all the involved colimits exist.

Let $\overleftarrow{\mathbf{X}}$ be the restriction of $\mathbf{X}$ to $(\overleftarrow{\Delta})^{op}$. Notice that the definition of the latching object of $\mathbf{X}$ uses only $\overleftarrow{\mathbf{X}}$. Summing up, $L_n\mathbf{X}$ is the pushout of the diagram

$$
\begin{array}{ccc}
L_{n-1}\overleftarrow{\mathbf{X}} & \longrightarrow & L_{n-1}Dec^1(\overleftarrow{\mathbf{X}}) \\
\downarrow & & \\
\mathbf{X}_{n-1} & &
\end{array}
$$

provided that the pushout and all the involved colimits exist, where $\overleftarrow{\mathbf{X}} \to Dec^1(\overleftarrow{\mathbf{X}})$ is induced by $s^n \colon [n] \sqcup [0] \to [n]$. Thus, we have

Lemma 58. *Let $\mathcal{M}$ be a generalized model category and $\mathbf{X}$ a simplicial object in $\mathcal{M}$. Let $n \geq 1$. If $L_{n-1}\overleftarrow{\mathbf{X}}$ and $L_{n-1}Dec^1(\overleftarrow{\mathbf{X}})$ exist and the map $L_{n-1}\overleftarrow{\mathbf{X}} \to \mathbf{X}_{n-1}$ is a cofibration, then $L_n\mathbf{X}$ exists.*

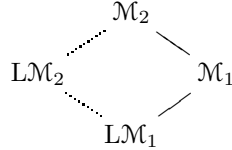## 9. Appendix 2: Left Bousfield localizations and change of cofibrations

In this section we essentially propose an approach to the existence of left Bousfield localizations of 'injective'-like model categories. The approach is based on the existence of both the un-localized 'injective'-like model category and the left Bousfield localization of the 'projective'-like model category. We give a full description of the fibrations of these localized 'injective'-like model categories; depending on one's taste, the description may or may not be satisfactory. The approach uses only simple factorization and lifting arguments.

Let $\mathcal{M}_1 = (W, C_1, F_1)$ and $\mathcal{M}_2 = (W, C_2, F_2)$ be two model categories on a category $\mathcal{M}$, where, as usual, W stands for the class of weak equivalences, C stands for the class of cofibrations, and F for the class of fibrations. We assume that $C_1 \subset C_2$. Let W' be a class of maps of $\mathcal{M}$ that contains W and has the two out of three property. We define F'_1 to be the class of maps having the right lifting property with respect to every map of $C_1 \cap W'$, and we define F'_2 to be the class of maps having the right lifting property with respect to every map of $C_2 \cap W'$. One can think of $\mathcal{M}_1$ as the 'projective' model category, of $\mathcal{M}_2$ as the 'injective' model category, and of W' as the class of 'local', or 'stable, equivalences'. Of course, other adjectives can be used. Recall from Section 3.3 the notion of left Bousfield localization of a (generalized) model category.

Theorem 59. *(1)* (Restriction) *If* $\mathrm{L}\mathcal{M}_2 = (W', C_2, F'_2)$ *is a left Bousfield localization of* $\mathcal{M}_2$*, then the class of fibrations of* $\mathrm{L}\mathcal{M}_2$ *is the class* $F_2 \cap F'_1$ *and* $\mathrm{L}\mathcal{M}_1 = (W', C_1, F'_1)$ *is a left Bousfield localization of* $\mathcal{M}_1$*.*

(2) (Extension) *If* $\mathrm{L}\mathcal{M}_1 = (W', C_1, F'_1)$ *is a left Bousfield localization of* $\mathcal{M}_1$ *that is right proper, then* $\mathrm{L}\mathcal{M}_2 = (W', C_2, F'_2)$ *is a left Bousfield localization of* $\mathcal{M}_2$*.*

For future purposes we display the conclusion of Theorem 59(2) in the diagram

The proofs of the existence of the left Bousfield localizations in parts (1) and (2) are different from one another. As it will be explained below, the existence of the left Bousfield localization in part (1) is actually well-known, but perhaps it has not been formulated in this form. Also, the right properness assumption in part (2) is dictated by the method of proof.

Proof of Theorem 59. We prove part (1). We first show that $F'_2 = F_2 \cap F'_1$. Clearly, we have $F'_2 \subset F_2 \cap F'_1$. Conversely, we must prove that every commutative diagram in $\mathcal{M}$

where j is in $C_2 \cap W'$ and p is in $F_2 \cap F'_1$, has a diagonal filler. The idea, which we shall use again, is very roughly that a commutative diagram

$$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & & \downarrow \\ \bullet & \longrightarrow & \bullet \end{array}$$

in an arbitrary category has a diagonal filler when, for example, viewed as an arrow going from left to right in the category of arrows, it factors through an isomorphism.

We first construct a commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & X' & \longrightarrow & X \\ j \downarrow & & q \downarrow & & p \downarrow \\ B & \longrightarrow & Y' & \longrightarrow & Y \end{array}$$

with q in $F_2 \cap W$. Then, since j is in C_2, the left commutative square diagram has a diagonal filler.

We factorize the map $B \to Y$ into a map $B \to Y'$ in $C_2 \cap W'$ followed by a map $Y' \to Y$ in F'_2. We factorize the canonical map $A \to Y' \times_Y X$ into a map $A \to X'$ in $C_2 \cap W'$ followed by a map $X' \to Y' \times_Y X$ in F'_2. Let q be the composite map $X' \to Y'$; then q is in F_2 being the composite of two maps in F_2. On the other hand, q is in F'_1 since $F'_2 \subset F'_1$ and since F'_1 is stable under pullbacks and compositions. By the two out of three property q is in W', therefore q belongs to $F'_1 \cap W' = F_1 \cap W$. In all, q is in $F_2 \cap W$.

We now prove the existence of LM_1. This can be seen as a consequence of a result of M. Cole [8, Theorem 2.1] (or of B.A. Blander [4, Proof of Theorem 1.5]). In our context however, since we have Lemma 9 we only need to check the factorization of an arbitrary map of M into a map in $C_1 \cap W'$ followed by a map in F'_1. This proceeds as in [8, 4]; for completeness we reproduce the argument.

Let $f \colon X \to Y$ be a map of M. We factorize it as a map $X \to Z$ in $C_2 \cap W'$ followed by a map $Z \to Y$ in F'_2. We further factorize $X \to Z$ into a map $X \to Z'$ in C_1 followed by a map $Z' \to Z$ in $F_1 \cap W$. The desired factorization of f is $X \to Z'$ followed by the composite $Z' \to Y$.

We prove part (2). By Lemma 9, it only remains to check the factorization of an arbitrary map of M into a map in $C_2 \cap W'$ followed by a map in F'_2. Mimicking the argument given in part (1) for the existence of LM_1 does not seem to give a solution. We shall instead expand on an argument due to A.K. Bousfield [6, Proof of Theorem 9.3], that's why we assumed right properness of LM_1.

Step 1. We give an example of a map in F'_2. We claim that every commutative diagram in M

$$\begin{array}{ccc} A & \longrightarrow & X \\ j \downarrow & & p \downarrow \\ B & \longrightarrow & Y \end{array}$$

where j is in $C_2 \cap W'$, p is in F_2, and X and Y are fibrant in LM_1, has a diagonal filler. For this we shall construct a commutative diagram

$$
\begin{array}{ccccc}
A & \longrightarrow & X' & \longrightarrow & X \\
{\scriptstyle j}\downarrow & & {\scriptstyle q}\downarrow & & {\scriptstyle p}\downarrow \\
B & \longrightarrow & Y' & \longrightarrow & Y
\end{array}
$$

with q in W. Factorizing then q as a map in C_2 followed by a map in $F_2 \cap W$ and using two diagonal fillers, we obtain the desired diagonal filler. We factorize the map $B \to Y$ into a map $B \to Y'$ in $C_1 \cap W'$ followed by a map $Y' \to Y$ in F'_1. We factorize the canonical map $A \to Y' \times_Y X$ into a map $A \to X'$ in $C_1 \cap W'$ followed by a map $X' \to Y' \times_Y X$ in F'_1. Let q be the composite map $X' \to Y'$. By the two out of three property q is in W'. Since Y is fibrant in LM_1, so is Y'. The map $Y' \times_Y X \to X$ is in F'_1 and X is fibrant in LM_1, therefore $Y' \times_Y X$, and hence X', are fibrant in LM_1. It follows that the map q is in W. The claim is proved.

Step 2. Let $f\colon X \to Y$ be a map of M. We can find a commutative diagram

$$
\begin{array}{ccc}
X & \longrightarrow & X' \\
{\scriptstyle f}\downarrow & & {\scriptstyle f'}\downarrow \\
Y & \longrightarrow & Y'
\end{array}
$$

in which the two horizontal arrows are in W' and both X' and Y' are fibrant in LM_1. We can find a commutative diagram

$$
\begin{array}{ccc}
X' & \longrightarrow & X'' \\
{\scriptstyle f'}\downarrow & & {\scriptstyle g}\downarrow \\
Y' & \longrightarrow & Y''
\end{array}
$$

in which the two horizontal arrows are in W, g is in F_2, and both X'' and Y'' are fibrant in M_1. It follows that both X'' and Y'' are fibrant in LM_1. The map g is a fibration in LM_1, since $F_2 \subset F_1$. Putting the two previous commutative diagrams side by side we obtain a commutative diagram

$$
\begin{array}{ccc}
X & \longrightarrow & X'' \\
{\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow \\
Y & \longrightarrow & Y''
\end{array}
$$

in which the two horizontal arrows are in W'. Since LM_1 is right proper, the map $Y \times_{Y''} X'' \to X''$ is in W', therefore the canonical map $X \to Y \times_{Y''} X''$ is in W'. By the claim, the map $Y \times_{Y''} X'' \to Y$ is in F'_2. We factorize the map $X \to Y \times_{Y''} X''$ into a map $X \to Z$ that is in C_2 followed by a map $Z \to Y \times_{Y''} X''$ that is in $F_2 \cap W$. Since $F_2 \cap W \subset F'_2$, we obtain the desired factorization of f into a map in $C_2 \cap W'$ followed by a map in F'_2. The proof of the existence of LM_2 is complete. $\qquad\square$

Some results in the subject of 'homotopical sheaf theory' can be seen as consequences of Theorem 59. Here are a couple of examples.

Let C be a small category. The category of presheaves on C with values in simplicial sets is a model category in two standard ways: it has the so-called projective and injective model structures. The class of cofibrations of the projective model

category is contained in the class of cofibrations of the injective model category. If $\mathcal{C}$ is moreover a site, a result of Dugger-Hollander-Isaksen [10, Theorem 6.2] says that the projective model category admits a left Bousfield localization $U\mathcal{C}_{\mathcal{L}}$ at the class $\mathcal{L}$ of local weak equivalences. The fibrations of $U\mathcal{C}_{\mathcal{L}}$ are the objectwise fibrations that satisfy descent for hypercovers [10, Theorem 7.4]. The model category $U\mathcal{C}_{\mathcal{L}}$ is right proper (for an interesting proof, see [9, Proposition 7.1]). Therefore, by Theorem 59(2), Jardine's model category, denoted by $sPre(\mathcal{C})_{\mathcal{L}}$ in [10], exists. Moreover, by Theorem 59(1) its fibrations are the injective fibrations that satisfy descent for hypercovers: this is exactly the content of the first part of [10, Theorem 7.4]. As suggested in [9], this approach to $sPre(\mathcal{C})_{\mathcal{L}}$ reduces the occurence of stalks and Boolean localization technique. The category of presheaves on $\mathcal{C}$ with values in simplicial sets also admits the so-called flasque model category [18, Theorem 3.7(a)]. The class of cofibrations of the projective model category is contained in the class of cofibrations of the flasque model category [18, Lemma 3.8]. Using $U\mathcal{C}_{\mathcal{L}}$ and Theorem 59 it follows that the local flasque model category [18, Definition 4.1] exists.

Other examples can be found on page 199 of [17]: the existence of both therein called the S model and the injective stable model structures, together with the description of their fibrations, can be seen as consequences of Theorem 59.

10. Appendix 3: Strong stacks of categories revisited

Let E be a small site. Recall from Theorem 48 the model category $Stack(\widetilde{E})_{proj}$.

Lemma 60. *The class of weak equivalences of $Stack(\widetilde{E})_{proj}$ is accessible.*

Proof. Let $f\colon X \to Y$ be a map of $\mathrm{Cat}(\widetilde{E})$. Consider the commutative square diagram

$$
\begin{array}{ccc}
X & \longrightarrow & \widehat{F}X \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \widehat{F}f} \\
Y & \longrightarrow & \widehat{F}Y
\end{array}
$$

with $\widehat{F}X$ defined in the proof of Theorem 48. Then f is a weak equivalence if and only if $i\widehat{F}f$ is a weak equivalence in $[E^{op}, CAT]_{proj}$. The functors i and $\widehat{F}$ preserve κ-filtered colimits for some regular cardinal κ. Since the class of weak equivalences of $[E^{op}, CAT]_{proj}$ is accessible, the result follows. $\square$

Recall from Theorem 50 the (right proper) model category $Stack(\widehat{E})_{proj}$. By Theorem 59(2) we have the model category $Stack(\widehat{E})_{inj}$, which we display in the diagram

$$
\begin{array}{ccc}
 & [E^{op}, CAT]_{inj} & \\
Stack(\widehat{E})_{inj} & & [E^{op}, CAT]_{proj} \\
 & Stack(\widehat{E})_{proj} &
\end{array}
$$

By Theorem 59(1), an object X of $[E^{op}, CAT]$ is fibrant in $Stack(\widehat{E})_{inj}$ if and only if ΦX is a stack and X is fibrant in $[E^{op}, CAT]_{inj}$.

Theorem 61. [19, Theorem 4] *There is a model category* $\mathrm{Stack}(\widetilde{E})_{inj}$ *on the category* $\mathrm{Cat}(\widetilde{E})$ *in which the weak equivalences are the maps that* Φ *takes into bicovering maps and the cofibrations are the internal functors that are monomorphisms on objects. A sheaf of categories* X *is fibrant in* $\mathrm{Stack}(\widetilde{E})_{inj}$ *(aka* X *is a strong stack) if and only if* ΦX *is a stack and* X *is fibrant in* $[E^{op}, CAT]_{inj}$.

Proof. We shall use J. Smith's recognition principle for model categories [3, Theorem 1.7]. We take in *op. cit.* the class W to be the class of weak equivalences of $Stack(\widetilde{E})_{proj}$. By Lema 60, W is accessible. Let I_0 be a generating set for the class C of cofibrations of $[E^{op}, CAT]_{inj}$, so that $\mathrm{C} = \mathrm{cof}(I_0)$. We put $I = aI_0$. The functors a and i preserve the property of internal functors of being a monomorphism on objects. Using that i is full and faithful it follows that $a\mathrm{C}$ is the class of internal functors that are monomorphisms on objects, and that moreover $a\mathrm{C} = \mathrm{cof}(I)$. By adjunction, every map in $\mathrm{inj}(I)$ is objectwise an equivalence of categories, so in particular every such map is in W. Recall that for every object X of $[E^{op}, CAT]$, the natural map $X \to iaX$ is a weak equivalence (see fact (2) stated below Theorem 48). Thus, by Lemma 62 all the assumptions of Smith's Theorem are satisfied, so $\mathrm{Cat}(\widetilde{E})$ is a model category, which we denote by $\mathrm{Stack}(\widetilde{E})_{inj}$.

Let f be a fibration in this model category. Then clearly if is a fibration in $Stack(\widehat{E})_{inj}$. Conversely, if if is a fibration in $Stack(\widehat{E})_{inj}$, then, since i is full and faithful, f is a fibration in $\mathrm{Stack}(\widetilde{E})_{inj}$. $\square$

Lemma 62. *Let* $\mathcal{M}$ *be a model category. We denote by* C *the class of cofibrations of* $\mathcal{M}$. *Let* $\mathcal{N}$ *be a category and let* $R: \mathcal{M} \rightleftarrows \mathcal{N}: K$ *be a pair of adjoint functors with* K *full and faithful. We denote by* W *the class of maps of* $\mathcal{N}$ *that* K *takes into weak equivalences. Assume that*
(1) $KR\mathrm{C} \subset \mathrm{C}$ *and*
(2) for every object X *of* $\mathcal{M}$, *the unit map* $X \to KRX$ *is a weak equivalence.*
Then the class $\mathrm{W} \cap R\mathrm{C}$ *is stable under pushouts and transfinite compositions.*

Proof. We first remark that by (2), the functor R takes a weak equivalence to an element of W. Let

$$
\begin{array}{ccc}
X & \xrightarrow{\;f\;} & Y \\
\downarrow & & \downarrow \\
Z & \xrightarrow{\;g\;} & P
\end{array}
$$

be a pushout diagram in $\mathcal{N}$ with $f \in \mathrm{W} \cap R\mathrm{C}$. Then g is obtained by applying R to the pushout diagram

$$
\begin{array}{ccc}
KX & \xrightarrow{\;Kf\;} & KY \\
\downarrow & & \downarrow \\
KZ & \xrightarrow{\quad} & P'
\end{array}
$$

By the assumptions it follows that $g \in \mathrm{W} \cap R\mathrm{C}$. The case of transfinite compositions is dealt with similarly. $\square$

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