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TWO-DIMENSIONAL SHEAF THEORY

Ross STREET
School of Mathematics and Physics, Macquarie University, North Ryde, N.S.W. 2113, Australia

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The purpose of this work is to analyse the appropriate notion of sheaf when we pass from categories to 2-categories. One reason for such an analysis is, as suggested by André Joyal, to clarify our understanding of the second cohomology construction H^2.

From the outset I must apologise to André in that this whole paper deals purely with the ordinary limits for 2-categories (as considered in [6] and called indexed limits). However, he may rest assured that all the results carry over with minor alterations when: 2-categories are replaced by bicategories; indexed limits are replaced by indexed bilimits [7]; arrows are never asked to be equal, only isomorphic; and, objects are never asked to be isomorphic, only equivalent. For example, in 1.1, delete 'injective on objects' so that 'chronic' is replaced by 'fully faithful'; and, in 1.4, the square which, when $\mathscr{X}$ is a bicategory, is only known to commute up to isomorphism, should be a bipullback. Also, equivalences and adjunctions between 2-categories should be replaced by biequivalences and biadjunctions [7].

On the other hand, I make no apology for considering categories with homs enriched in **Cat** (that is, 2-categories) rather than with homs enriched in the category **Gpd** of groupoids. André maintains that 'bicategories in which all 2-cells are invertible' is an important level of generality. Again the results carry over and in fact simplify somewhat due to the fact that the distinction between 'pullback' and 'comma object' disappears (see especially the notion of congruence (1.8)). However, I believe that, in the final analysis, the bicategorical version will be needed for the understanding of H^2.

The plan of the paper is to establish 2-dimensional versions of the notions of regular and exact categories [2], sites, sheaves, and so on. The starting point is the premise that the right notion of *cover* in **Cat** is a set of functors with the same target which are jointly surjective on objects (in the bicategorical version this becomes 'jointly surjective up to isomorphism on objects').

The main result on regular 2-categories is that the arrows which play the role of surjectives on objects (here called *acute*) are in fact regular in the sense that they are quotient maps in an appropriate sense. This leads to the appropriate exactness condition which is shown to hold in **Cat**.

Our results on 2-sheaves include the existence and exactness of the associated 2-sheaf construction and a comparison lemma.

Our main result is a 2-categorical version of the theorem of Giraud which characterizes Grothendieck toposes [1; IV§1, p. 303]. We include in the same theorem the relationship between 2-toposes and lex-total 2-categories. The final section gives the classification theorem for 2-toposes: geometric morphisms into the 2-sheaves on a 2-site classify left-exact cover-preserving 2-functors out of the 2-site.

We are concerned here with the 2-dimensional version of Grothendieck toposes. The work of [3] is closely related to ours and sheds light on the 2-dimensional versions of elementary toposes.

1. Regular 2-categories

1.1. An arrow $m:X\to Y$ in a 2-category $\mathscr{K}$ is called *chronic* when, for all objects K of $\mathscr{K}$, the functor $\mathscr{K}(K,m):\mathscr{K}(K,X)\to\mathscr{K}(K,Y)$ is injective on objects and fully faithful.

1.2. A composite of chronic arrows is chronic. A pullback of a chronic arrow is chronic.

1.3. Suppose $J:\mathscr{A}\to\mathbf{Cat}$, $S,S':\mathscr{A}\to\mathscr{K}$ are 2-functors, $\theta:S\to S'$ is a 2-natural transformation, and $\mathscr{K}$ admits the J-indexed limits $\lim(J,S)$, $\lim(J,S')$. If each component of θ is chronic then so is $\lim(J,\theta):\lim(J,S)\to\lim(J,S')$.

1.4. An arrow $e:A\to B$ in $\mathscr{K}$ is called *acute* when, for all chronic arrows $m:X\to Y$, the following diagram of categories is a pullback:

$$\begin{tikzcd}
\mathscr{K}(B,X) \arrow[r, "\mathscr{K}(B,m)"] \arrow[d, "\mathscr{K}(e,X)"'] & \mathscr{K}(B,Y) \arrow[d, "\mathscr{K}(e,Y)"] \\
\mathscr{K}(A,X) \arrow[r, "\mathscr{K}(A,m)"'] & \mathscr{K}(A,Y)
\end{tikzcd}$$

1.5. *A composite of acute arrows is acute. If ef is acute and f is either an epimorphism or acute then e is acute. Acute chronic arrows are isomorphisms.*

1.6. Each acute arrow $e:A\to B$ satisfies: *For all commutative squares*

$$\begin{tikzcd}
A \arrow[r, "e"] \arrow[d, "u"'] & B \arrow[d, "v"] \\
X \arrow[r, "m"'] & Y
\end{tikzcd}$$

in which m is chronic, there exists a unique $w:B\to X$ such that $we=u$, $mw=v$.

When K has cotensor products with the category **2**, this condition *implies* e is acute.

1.7. *If $\mathscr{K}$ is finitely complete then each acute $e:A\to B$ has the following properties:*

(a) *for all 2-cells*

$$\begin{tikzcd} B \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\theta\Downarrow\ \Downarrow\phi"] & C, \end{tikzcd}$$

if $\theta e=\phi e$ then $\theta=\phi$;

(b) *for all 2-cells*

$$\begin{tikzcd} B \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\phi"] & C, \end{tikzcd}$$

if θe is an identity 2-cell then θ is an identity 2-cell.

Proof. Let $k:K\to B$ be the universal arrow which satisfies $\theta k=\phi k$ in case (a), and $\theta k=1$ in case (b). Then k is chronic and $e=ku$ for a unique u. By 1.6 there exists a unique w such that $u=we$, $kw=1$. Since k is a monomorphism, k is an isomorphism. So $\theta=\phi$ in case (a) and $\theta=1$ in case (b). □

1.8. A *congruence E on an object A* of $\mathscr{K}$ is a functor $j:E^1\to E^2$ between categories E^1,E^2 in $\mathscr{K}$ satisfying the following conditions:

(a) $E_0^1=E_0^2=A$, $j_0=1_A$, and $j_1:E_1^1\to E_1^2$ is chronic;

(b) the span (d_0,E_1^2,d_1) from A to A is a discrete fibration;

(c) E^1 is an equivalence relation on A.

1.9. Suppose $\mathscr{K}$ is finitely complete. The *congruence $E=\mathrm{E}(f)$ associated with an arrow $f:A\to B$* is defined as follows. Form the following diagrams in which the squares with 2-cells have the comma property and the solid squares with no 2-cells have the pullback property.

$$\begin{tikzcd} E_2^2 \arrow[r, "d_2"] \arrow[d, "d_0"'] & E_1^2 \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\overset{\lambda}{\Rightarrow}"] & A \arrow[d, "f"] \\ E_1^2 \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\overset{\lambda}{\Rightarrow}"] & A \arrow[r] \arrow[d, "f"] & B \arrow[d, dashed, "1"] \\ A \arrow[r, "f"'] & B \arrow[r, dashed, "1"'] & B \end{tikzcd} \qquad \begin{tikzcd} E_1^1 \arrow[r, "d_1"] \arrow[d, "d_0"'] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

The 2-cell obtained by pasting the above left-hand diagram (with dotted arrows included) induces an arrow $d_1 : E_2^2 \to E_1^2$, and, the square on the right induces an arrow $j_1 : E_1^1 \to E_1^2$, using the comma property of E_1^2. One easily checks that this does indeed define a congruence on A.

1.10. Let $\mathrm{Cng}(\mathscr{K})$ denote the 2-category of congruences in $\mathscr{K}$: an object is a congruence $j : E^1 \to E^2$ while the arrows and 2-cells are those appropriate for diagrams of this form in $\mathscr{K}$. The universal property of limits allows us to extend the definition of 1.9 to arrows and 2-cells to obtain a 2-functor

$$\mathbf{E} : [2, \mathscr{K}] \to \mathrm{Cng}(\mathscr{K}).$$

1.11. *An arrow* $m : X \to Y$ *is chronic if and only if* $\mathbf{E}(1, m) : \mathbf{E}(1_K) \to \mathbf{E}(m)$ *is an isomorphism.*

1.12. Let $\mathscr{D}$ denote the category generated by the graph:

$$\begin{tikzcd}
2' \arrow[r, shift left=2, "d_0"] \arrow[r, "d_1"] \arrow[r, shift right=2, "d_2"'] \arrow[d, "j_2"'] & 1' \arrow[d, "j_1"] \arrow[dr, shift left, "d_0"] \arrow[dr, shift right, "d_1"'] & \\
2 \arrow[r, shift left=2, "d_0"] \arrow[r, "d_1"] \arrow[r, shift right=2, "d_2"'] & 1 \arrow[r, shift left=2, "d_0"] \arrow[r, "d_1"] & 0 \arrow[l, shift left=2, "i"] \arrow[ul, "i"]
\end{tikzcd}$$

subject to the equations which are the simplicial identities for the two sets of d's and i, and the commutativity conditions with the j's. Let $J : \mathscr{D}^{op} \to \mathbf{Cat}$ denote the functor which takes the above diagram to the following diagram

$$\begin{tikzcd}
\mathbf{1} \arrow[d] & \mathbf{1} \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \arrow[d] \arrow[dr] & \\
\mathbf{3} & \mathbf{2} \arrow[l, shift left=2, "\partial_0"'] \arrow[l, "\partial_1"'] \arrow[l, shift right=2, "\partial_2"] \arrow[r, shift right=2] & \mathbf{1} \arrow[l, shift left=2, "\partial_0"'] \arrow[l, "\partial_1"'] \arrow[ul, shift left] \arrow[ul, shift right]
\end{tikzcd}$$

Each congruence E on A in $\mathscr{K}$ can be regarded as a functor $E : \mathscr{D} \to \mathscr{K}$. A 2-natural transformation $J \to \mathscr{K}(E, X)$ amounts to an arrow $g : A \to X$ and a 2-cell $\gamma : gd_0 \Rightarrow gd_1$ such that the following equalities hold:

$$\begin{tikzcd}
E_1^1 \arrow[r, "j_1"] & E_1^2 \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\stackrel{\gamma}{\Rightarrow}"] & A \arrow[d, "g"] \\
& A \arrow[r, "g"'] & X
\end{tikzcd}
=
\begin{tikzcd}
E_1^1 \arrow[r, "d_1"] \arrow[d, "d_0"'] & A \arrow[d, "g"] \\
A \arrow[r, "g"'] & X
\end{tikzcd}$$

$$
\begin{tikzcd}
E_2^2 \arrow[r, "d_2"] \arrow[d, "d_0"'] & E_1^2 \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\overset{\gamma}{\Rightarrow}"] & A \arrow[d, "g"] \\
E_1^2 \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\overset{\gamma}{\Rightarrow}"] & A \arrow[r, "g"'] \arrow[d, "g"] & X \arrow[d, "1"] \\
A \arrow[r, "g"'] & X \arrow[r, "1"'] & X
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
E_2^2 \arrow[r, "d_1"] & E_1^2 \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\overset{\gamma}{\Rightarrow}"] & A \arrow[d, "g"] \\
& A \arrow[r, "g"'] & X
\end{tikzcd}
$$

When $\mathscr{X}$ is finitely complete, such a pair g, γ amounts to an arrow $g: A \to X$ together with an arrow of congruences $h: E \to \mathbf{E}(g)$ with $h_0 = 1_A$ (where $h_1: E_1^2 \to \mathbf{E}(g)_1^2$ corresponds to γ via $\lambda h = \gamma$).

1.13. A *quotient for a congruence E on A* is a J-indexed colimit $\operatorname{col}(J, E)$ for $E: \mathscr{D} \to \mathscr{X}$. Explicitly, a quotient for E consists of an object Q of $\mathscr{X}$, an arrow $q: A \to Q$ and a 2-cell $\tau: qd_0 \Rightarrow qd_1$ which are universal amongst those X, g, γ satisfying the conditions of 1.12. It follows from [6] that every congruence has a quotient if $\mathscr{X}$ is finitely cocomplete.

1.14. An arrow $q: A \to Q$ is called a *quotient map* when there exists a congruence E on A and a 2-cell $\tau: qd_0 \Rightarrow qd_1$ such that Q, q, τ form a quotient for E. When $\mathscr{X}$ is finitely complete, *an arrow q is a quotient map if and only if Q, q, λ provide a quotient for the congruence $E(q)$ associated with q* (1.9, 1.10).

1.15. *If a congruence is a congruence associated with some arrow then it is a congruence associated with its quotient map provided this quotient map exists.*

1.16. Notice that a quotient for a congruence E provides the value $q: A \to Q$ of a left adjoint $\mathbf{Q}$ to the 2-functor $\mathbf{E}$ (1.10) at the object E.

1.17. *Every quotient map is acute.*

Proof. In the notation of 1.13, consider a commutative square as in 1.6 with m chronic and e replaced by $q: A \to Q$. Then there exists a unique $\theta: ud_0 \Rightarrow ud_1$ such that $m\theta = vT: mud_0 \Rightarrow mud_1$ since m is chronic. Since q, τ satisfy the properties of g, γ in 1.11, so too do $vq = mu$, $v\tau = m\theta$. Since m is chronic, so too do u, θ. By the universal property of quotients, there exists a $w: Q \to X$ such that $wq = u$ and $w\tau = \theta$. By the uniqueness property of quotients, we also get $mw = v$. Since m is chronic, w is unique as required. $\square$

1.18. The converse of 1.17 is generally *not* true. Our aim is to examine a class of 2-categories where this converse will be shown to hold.

1.19. A 2-category $\mathscr{K}$ will be called *regular* when:

(a) all finite limits exist [6];

(b) each arrow f factors as $f=me$ where m is chronic and e is acute;

(c) each pullback of an acute arrow is acute.

1.20. *In a regular 2-category $e\times e' : A\times A' \to B\times B'$ is acute if e, e' are.*

Proof. For any X, the arrow $e\times X : A\times X\to B\times X$ is the pullback of e along the projection $B\times X\to B$. By 1.19(c), $e\times X$ is acute. Similarly, $Y\times e'$ is acute. By 1.5, $e\times e' = (e\times A')(A\times e')$ is acute. □

1.21. *In a regular 2-category, suppose $e:A\to B$ is acute. If $n:B\to C$ is such that $\mathbf{E}(1,n):\mathbf{E}(e)\to\mathbf{E}(ne)$ is an isomorphism then n is chronic.*

Proof. Take $b, b' : X\to B$ and form the pullback:

$$\begin{tikzcd} Y \arrow[r, "r"] \arrow[d, "\binom{a}{a'}"'] & X \arrow[d, "\binom{b}{b'}"] \\ A\times A \arrow[r, "e\times e"'] & B\times B \end{tikzcd}$$

By 1.20 and 1.19(c), r is acute. Take $\gamma : nb\Rightarrow nb'$. Since $\mathbf{E}(e)\cong\mathbf{E}(ne)$, there exists a unique $\delta : ea\Rightarrow ea'$ such that $n\delta=\gamma r : nea\Rightarrow nea'$; furthermore, if γr is an identity, so too is $\delta : br\Rightarrow b'r$.

Suppose $\beta, \beta' : b\Rightarrow b'$ are such that $n\beta = n\beta'$. Taking $\gamma = n\beta$ and applying the above argument, we get $\delta=\beta r=\beta' r$. By 1.7(a), $\beta=\beta'$. *So n is faithful.*

Suppose $\beta : b\Rightarrow b'$ is such that $n\beta = 1$. Taking $\gamma = n\beta$ and applying the above argument, we get $\delta = \beta r = 1$. By 1.7(b), $\beta = 1$. *So n reflects identities.*

From the left-hand pullback

$$\begin{tikzcd} ne/ne \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & n/n \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & 2\pitchfork C \arrow[d, "\binom{d_0}{d_1}"] \\ A\times A \arrow[r, "e\times e"'] & B\times B \arrow[r, "n\times n"'] & C\times C \end{tikzcd}$$

we deduce that the right-hand arrow in the following commutative square is acute.

$$\begin{tikzcd} e/e \arrow[r] \arrow[d] & ne/ne \arrow[d] \\ 2\pitchfork B \arrow[r] & n/n \end{tikzcd}$$

The top arrow is an isomorphism since $\mathbf{E}(e) \cong \mathbf{E}(ne)$. Since n is faithful it follows that the bottom arrow of the square is chronic. It follows then from 1.6 that this bottom arrow is an isomorphism which precisely says that n is fully faithful.

Since n is fully faithful and reflects identities, it is chronic. □

1.22. Theorem. *In a regular* 2*-category, every acute arrow is a quotient map.*

Proof. Suppose $f: A \to B$ is acute. By 1.14, we must show that f, λ provide a quotient for $\mathbf{E}(f)$.

Take $g: A \to X$ and $\gamma: gd_0 \Rightarrow gd_1$ as in 1.12 with $E = \mathbf{E}(f)$. Apply 1.19(b) to $\binom{f}{g}: A \to B \times X$ to obtain an acute $s: A \to Y$ and chronic $\binom{u}{v}: Y \to B \times X$ with $f = us$, $g = vs$. Now $\binom{f}{g}$, $\binom{\lambda}{\gamma}$ satisfy the equalities of 1.12 with $E = \mathbf{E}(f)$; and since $\binom{f}{g} = \binom{u}{v}s$ with $\binom{u}{v}$ chronic, there exists a unique $\sigma: sd_0 \Rightarrow sd_1$ such that s, σ satisfy those equalities and $u\sigma = \lambda$, $v\sigma = \lambda$. Since $f = us$, it follows that $\mathbf{E}(s) \cong \mathbf{E}(f)$. Since s is acute, u is chronic by 1.21. But $us = f$ and s are acute, so, by 1.5, u is acute. By 1.5, u is an isomorphism. Thus $g = vu^{-1}f$ and $\gamma = vu^{-1}\lambda$.

If $g = wf$, $\gamma = w\lambda$ then $wus = g = vs$, $wu\sigma = \gamma = v\sigma$ imply $wu = v$; so $w = vu^{-1}$ is unique, as required.

The 2-cell condition for the colimit is a consequence of the arrow condition since $\mathscr{K}$ admits cotensors with 2. □

2. Exact 2-categories

2.1. A 2-category $\mathscr{K}$ is *exact* when it is regular (1.19) and each congruence (1.8) is the congruence associated with some arrow (1.9).

2.2. *In an exact* 2*-category, every congruence has a quotient* (1.13).

Proof. By definition each congruence E is associated with some arrow $f: A \to B$. By 1.19(b), $f = me$ with m chronic and e acute. So E is also associated with e. By Theorem 1.22, e is a quotient map. By 1.14, e, λ provide a quotient for E. □

2.3. Theorem. Cat *is an exact* 2*-category.*

Proof. 1.19(a) is well known. An arrow in **Cat** is chronic if and only if it is injective on objects and fully faithful. An arrow in **Cat** is acute if and only if it is surjective on objects. Then 1.19(b), (c) follow easily.

Suppose E is a congruence on A in **Cat**. We regard E^2 as a double category whose objects are the objects of A, whose horizontal arrows are the arrows of A, whose vertical arrows are the objects of E_1^2, and whose squares are the arrows of E_1^2. We regard E_1^1 as a full subcategory of E_1^2 whose objects we will call *trite* vertical arrows (1.8(a)). The trite vertical arrows are invertible and there is at most one from one object to another (1.8(c)). For each horizontal arrow α and each vertical arrow ξ with target equal to the source of α, there exists a unique square of the form

$$\begin{tikzcd} x \arrow[r, "1"] \arrow[d, "\xi"'] \arrow[dr, phantom, "\pi"] & x \arrow[d, "\xi'"] \\ a \arrow[r, "\alpha"'] & a' \end{tikzcd}$$

and similarly for squares with identities at the bottom (1.8(b)).

Two objects of A are *equivalent* when there is a trite vertical arrow between them. Two vertical arrows ξ, η are *equivalent* when there are trite κ, τ with $\tau\xi = \eta\kappa$. These equivalence relations are compatible with source, target, identities and composition of vertical arrows.

Let Q be the category of equivalence classes of objects of A and equivalence classes of vertical arrows.

Let $q: A \to Q$ be the functor defined as follows. For each object a, take qa to be the equivalence class of a. For each arrow $\alpha: a \to a'$ in A, consider the above square π with ξ the identity, and take $q\alpha$ to be the equivalence class of ξ'.

Let $\pi: qd_0 \Rightarrow qd_1$ be the natural transformation whose component at ξ is the equivalence class of ξ.

The verification that q, τ form a quotient for E is left to the reader. □

2.4. Corollary. *For any 2-category $\mathscr{C}$, the 2-functor 2-category $[\mathscr{C}^{\mathrm{op}}, \mathbf{Cat}]$ is exact.* □

3. Two-dimensional sheaves

3.1. A *topology on a 2-category $\mathscr{C}$* is merely a (Grothendieck) topology on the underlying category of $\mathscr{C}$ in the usual sense [1,4]. A *U-crible* for the underlying category of $\mathscr{C}$ can be identified with a chronic arrow $R \to \mathscr{C}(-, U)$ in $[\mathscr{C}^{\mathrm{op}}, \mathbf{Cat}]$. A 2-category with a topology will be called a *2-site*.

A 2-sheaf for a topology on $\mathscr{C}$ is a 2-functor $F: \mathscr{C}^{\mathrm{op}} \to \mathbf{Cat}$ such that each covering crible $\mathscr{R} \to \mathscr{C}(-, U)$ induces an isomorphism

$$[\mathscr{C}^{\mathrm{op}}, \mathbf{Cat}](\mathscr{C}(-, U), F) \cong [\mathscr{C}^{\mathrm{op}}, \mathbf{Cat}](R, F).$$

For a 2-category $\mathscr{X}$, a *$\mathscr{X}$-valued 2-sheaf on $\mathscr{C}$* is a 2-functor $F: \mathscr{C}^{\mathrm{op}} \to \mathscr{X}$ such that, for all objects X of $\mathscr{X}$, the 2-functor $\mathscr{X}(X, F): \mathscr{C}^{\mathrm{op}} \to \mathbf{Cat}$ is a 2-sheaf.

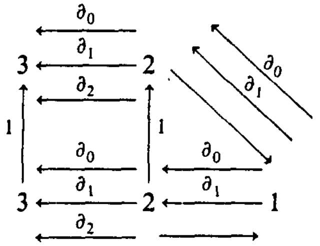
3.2. It is often convenient, when $\mathscr{C}$ has pullbacks, to work in terms of a *pretopology*; that is, a function which assigns, to each object U of $\mathscr{C}$, a set $\mathrm{Cov}(U)$ whose elements are sets of arrows into U, subject to the following axioms:

(1) for each $u: V \to U$ and each $\mathscr{U}$ in $\mathrm{Cov}(U)$, there exists $\mathscr{V}$ in $\mathrm{Cov}(V)$ consisting of a pullback along u for each element of $\mathscr{U}$;

(2) for each U, the singleton set consisting of the identity arrow of U is in $\mathrm{Cov}(U)$;

(3) given $\mathscr{U}$ in $\mathrm{Cov}(U)$ and $\mathscr{U}_u$ in $\mathrm{Cov}(V)$ for each $u: V \to U$ in $\mathscr{U}$, the set $\{uv \mid u \in \mathscr{U}, v \in \mathscr{U}_u\}$ is in $\mathrm{Cov}(U)$.

3.3. Let $\mathscr{D}$ be the category of 1.12 and let $N: \mathscr{D}^{\mathrm{op}} \to \mathbf{Set}$ be the functor corresponding to the following diagram in **Set**:



For each set $\mathscr{U}$, define the category $\mathscr{D}_{\mathscr{U}}$ to be the dual of the comma category $N/\ulcorner \mathscr{U} \urcorner$:

$$\begin{tikzcd}
\mathscr{D}_{\mathscr{U}}^{\mathrm{op}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\Rightarrow"] & 1 \arrow[d, "\ulcorner \mathscr{U} \urcorner"] \\
\mathscr{D}^{\mathrm{op}} \arrow[r, "\alpha"'] & \mathbf{Set}
\end{tikzcd}$$

Objects of $\mathscr{D}_{\mathscr{U}}$ are pairs (x, u) where x is an object of $\mathscr{D}$ and u is a sequence of elements of $\mathscr{U}$ of length Nx. Let $J_{\mathscr{U}}: \mathscr{D}^{\mathrm{op}} \to \mathbf{Cat}$ denote the composite of the left-hand side of the above square with the functor J of 1.12.

3.4. Suppose u, v, w are arrows into an object U of a finitely complete 2-category $\mathscr{C}$. Form the following diagrams in which the squares with 2-cells have the comma property and those solid squares without 2-cells are pullbacks.

$$\begin{tikzcd}
V_{uvw} \arrow[r, "d_2"] \arrow[d, "d_0"'] & V_{vw} \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\overset{\lambda}{\Rightarrow}"] & V_w \arrow[d, "w"] \\
V_{uv} \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\overset{\lambda}{\Rightarrow}"] & V_v \arrow[r, "v"'] \arrow[d, "v"] & U \arrow[d, dashed] \\
V_u \arrow[r, "u"'] & U \arrow[r, dashed] & U
\end{tikzcd}
\qquad
\begin{tikzcd}
V'_{uvw} \arrow[r, "d_2"] \arrow[d, "d_0"'] & V'_{vw} \arrow[r, "d_1"] \arrow[d, "d_0"'] & V_w \arrow[d, "w"] \\
V'_{uv} \arrow[r, "d_1"] \arrow[d, "d_0"'] & V_v \arrow[r, "v"'] \arrow[d, "v"] & U \arrow[d, dashed] \\
V_u \arrow[r, "u"'] & U \arrow[r, dashed] & U
\end{tikzcd}$$

Arrows $j_1 : V'_{uv} \to V_{uv}$, $j_2 : V'_{uvw} \to V_{uvw}$ are induced which, together with the identities on U, V_u, give maps of the right-hand diagram above into the left-hand diagram above. The above two diagrams induce arrows $d_1 : V_{uvw} \to V_{uw}$, $d_1 : V'_{uvw} \to V'_{uw}$ using the universal properties of V_{uw}, V'_{uw}, respectively.

3.5. Suppose $\mathscr{U}$ is a set of arrows into an object U of a finitely complete 2-category $\mathscr{C}$. A functor $S_{\mathscr{U}} : \mathscr{S}_{\mathscr{U}} \to \mathscr{C}$ is defined as follows. The objects $(0,u)$, $(1,(u,v))$, $(2,(u,v,w))$, $(1',(u,v))$, $(2',(u,v,w))$ are taken to $V_u, V_{uv}, V_{uvw}, V'_{uv}, V'_{uvw}$, in the terminology of 3.4. An arrow $d_n : (x,u) \to (y,v)$ is taken to d_n as in 3.4 and $j_n : (p',u) \to (p,u)$ is taken to j_n as in 3.4.

The arows u in $\mathscr{U}$ and the 2-cells $\lambda : ud_0 \Rightarrow vd_1$ defined above determine a 2-natural transformation $\kappa_{\mathscr{U}} : J_{\mathscr{U}} \to \mathscr{C}(S_{\mathscr{U}}, U)$.

Let R be the U-crible generated by $\mathscr{U}$; that is, RV consists of the arrows $V \to U$ which factor through some arrow in $\mathscr{U}$. *Then $\kappa_{\mathscr{U}}$ induces a 2-natural transformation*

$$\begin{tikzcd} \mathscr{S}_{\mathscr{U}}^{\mathrm{op}} \arrow[rr, "S_{\mathscr{U}}"] \arrow[dr, "J_{\mathscr{U}}"'] \arrow[rr, phantom, shift right=6, "\Rightarrow"] & & \mathscr{C}^{\mathrm{op}} \arrow[dl, "R"] \\ & \mathbf{Cat} & \end{tikzcd}$$

which exhibits R as a left Kan extension of $J_{\mathscr{U}}$ along $S_{\mathscr{U}}$.

3.6. Proposition. *Suppose $\mathscr{C}$ is a finitely complete 2-site. A 2-functor $F : \mathscr{C}^{\mathrm{op}} \to \mathscr{K}$ is a $\mathscr{K}$-valued 2-sheaf if and only if, for all objects U of $\mathscr{C}$ and all covers $\mathscr{U}$ of U the 2-natural transformation*

$$J_{\mathscr{U}} \xrightarrow{\kappa_{\mathscr{U}}} \mathscr{C}(S_{\mathscr{U}}, U) \xrightarrow{F} \mathscr{K}(FU, FS_{\mathscr{U}})$$

exhibits FU as a $J_{\mathscr{U}}$-indexed limit for $FS_{\mathscr{U}}$ [6].

Proof. Since $\mathscr{K}$-valued 2-sheaves and indexed limits are defined representably, one must verify only the case $\mathscr{K} = \mathbf{Cat}$. The Kan extension of 3.5 gives

$$[\mathscr{C}^{\mathrm{op}}, \mathbf{Cat}](R, F) \cong [\mathscr{S}_{\mathscr{U}}^{\mathrm{op}}, \mathbf{Cat}](J_{\mathscr{U}}, FS_{\mathscr{U}}) \cong \lim(J_{\mathscr{U}}, FS_{\mathscr{U}}).$$

The result follows from the Yoneda lemma:

$$[\mathscr{C}^{\mathrm{op}}, \mathbf{Cat}](\mathscr{C}(-, U), F) \cong FU. \qquad \square$$

3.7. For a 2-site $\mathscr{C}$, we write $\mathscr{S}h(\mathscr{C}, \mathscr{K})$ for the full sub-2-category of $[\mathscr{C}^{\mathrm{op}}, \mathscr{K}]$ consisting of the $\mathscr{K}$-valued 2-sheaves on $\mathscr{C}$. We remind the reader that a 2-functor is said to be *left exact* when it preserves the finite limits appropriate to 2-category theory (including cotensors with $\mathbf{2}$).

3.8. Theorem. *Suppose $\mathscr{E}$ is a 2-site with small hom-categories. Suppose $\mathscr{C}$ is a small 2-category and $\mathscr{J}:\mathscr{C}\to\mathscr{E}$ is a fully faithful 2-functor such that, for each object X of $\mathscr{E}$, there exists a set $\mathscr{U}$ of arrows into X which covers X and for which the source of each arrow in $\mathscr{U}$ is in the image of $\mathscr{J}$. Then*

(i) (associated 2-sheaf) *the inclusion 2-functor $\mathscr{S}h(\mathscr{E},\mathbf{Cat})\to[\mathscr{E}^{\mathrm{op}},\mathbf{Cat}]$ has a left-exact left adjoint A;*

(ii) (comparison lemma) *when $\mathscr{C}$ is enriched with the largest topology such that $F\mathscr{J}^{\mathrm{op}}:\mathscr{C}^{\mathrm{op}}\to\mathbf{Cat}$ is a 2-sheaf for all 2-sheaves F on $\mathscr{E}$, the 2-functor $\mathscr{J}$ induces an equivalence of 2-categories*

$$\mathscr{S}h(\mathscr{E},\mathbf{Cat})\simeq\mathscr{S}h(\mathscr{C},\mathbf{Cat}).$$

Proof. (i) Let $C(X)$ denote the ordered set of covering cribles of X and let $C_{\mathscr{J}}(X)$ denote the subset consisting of those cribles generated by covering sets $\mathscr{U}$ whose arrows all have source in the image of $\mathscr{J}$. Then $C_{\mathscr{J}}(X)$ is a small initial subset of $C(X)$ so the following colimit exists:

$$(LP)X=\operatorname{col}\left(C(X)^{\mathrm{op}}\to[\mathscr{E}^{\mathrm{op}},\mathbf{Cat}]\xrightarrow[{[\mathscr{E}^{\mathrm{op}},\mathbf{Cat}](-,P)}]{}\mathbf{Cat}\right).$$

This puts us in a position to attempt to mimic the construction of the associated sheaf as given in [1; p. 230]. The details are as follows.

Put $\mathscr{P}=[\mathscr{E}^{\mathrm{op}},\mathbf{Cat}]$. For any 2-cell σ in $\mathscr{E}$, we can define a 2-cell $(LP)\sigma$ by the condition that, for all commutative diagrams

$$\begin{tikzcd} S \arrow[r, shift left=1.5, "g'"] \arrow[r, shift right=1.5, "h'"'] \arrow[r, phantom, "\Downarrow\sigma'"] \arrow[d] & R \arrow[d] \\ Y \arrow[r, shift left=1.5, "g"] \arrow[r, shift right=1.5, "h"'] \arrow[r, phantom, "\Downarrow\sigma"] & X \end{tikzcd}$$

in which the vertical arrows are covering cribles (and we are identifying $\mathscr{E}$ with its image in $\mathscr{P}$ under the Yoneda embedding), the following diagram commutes:

$$\begin{tikzcd} \mathscr{P}(R,P) \arrow[r, shift left=1.5, "\mathscr{P}(g'{,}1)"] \arrow[r, shift right=1.5, "\mathscr{P}(h'{,}1)"'] \arrow[r, phantom, "\Downarrow\mathscr{P}(\sigma'{,}1)"] \arrow[d, "\mathrm{copr}_R"'] & \mathscr{P}(S,P) \arrow[d, "\mathrm{copr}_S"] \\ (LP)X \arrow[r, shift left=1.5, "(LP)g"] \arrow[r, shift right=1.5, "(LP)h"'] \arrow[r, phantom, "\Downarrow(LP)\sigma"] & (LP)Y \end{tikzcd}$$

Thus $LP\in\mathscr{P}$, and we have $l_p:P\to LP$ in $\mathscr{P}$ whose component at X is obtained from the coprojection of the maximum crible on X. Clearly LP is 2-functorial in P and we obtain a 2-natural transformation $l:1_{\mathscr{P}}\Rightarrow L$ between endo-2-functors on $\mathscr{P}$.

Since $C(X)^{\mathrm{op}}$ is filtered and filtered colimits in **Cat** commute with finitary limits, *L is left exact.*

We say that $P \in \mathscr{P}$ is 1-*separated* when the functor $\mathscr{P}(i,P): \mathscr{P}(X,P) \to \mathscr{P}(R,P)$ is injective on objects and faithful for all covering cribles $i: R \to X$. We say that P is 2-*separated* when $\mathscr{P}(i,P)$ is chronic for all such i.

For each $P \in \mathscr{P}$, LP is 1-separated. Suppose $i: R \to X$ is a covering crible and $f,g: X \to LP$ are such that $fi = gi$. Now f,g correspond to objects of $(LP)X$ and as such are represented by $u: R' \to P$, $v: R'' \to P$ where R', R'' are covering cribles on X. Since f,g are also represented by the restrictions of u,v to $R \cap R' \cap R''$, we may suppose $R' = R'' \subset R$. From the definition of LP on arrows, one sees that the composite

$$R'Y \longrightarrow RY \xrightarrow{iY} \mathscr{E}(Y,X) \xrightarrow{fY} (LP)Y$$

takes $s: Y \to X$ in R' to the equivalence class of

$$Y \xrightarrow{\bar{s}} R \xrightarrow{u} P$$

where $i\bar{s} = s$. The condition $fi = gi$ thus yields that, for each s in R', there exists a covering crible S_s on Y such that

$$\left(S_s \longrightarrow Y \xrightarrow{\bar{s}} R \xrightarrow{u} P\right) = \left(S_s \longrightarrow Y \xrightarrow{\bar{s}} R \xrightarrow{u} P\right).$$

Let T denote the crible on X consisting of the arrows st where $s \in R'$ and $t \in S_s$. Then T is covering by one of the axioms for a topology (corresponding to 3.2(3) for a pretopology) and u,v agree when restricted to T. Thus $f = g$. So $\mathscr{P}(i, LP)$ is injective on objects. Since L preserves cotensoring with **2**, the latter result applied to $\mathbf{2} \pitchfork P$ in place of P implies that $\mathscr{P}(i, LP)$ is faithful. So LP is 1-separated.

Since passing to filtered colimits in **Cat** preserves monomorphisms, *P is 1-separated if and only if $l_p: P \to LP$ is a monomorphism.*

If P is 1-separated then LP is 2-separated. Begin as in the proof that LP is 1-separated; this time, instead of $fi = gi$, we merely have a 2-cell $\theta: fi \Rightarrow gi$. So, for each s in R'', we obtain a covering crible S_s on Y and a 2-cell

$$\begin{tikzcd}
S_s \arrow[r] \arrow[d] & Y \arrow[r, "\bar{s}"] \arrow[d, phantom, "\Downarrow \phi_s"] & R \arrow[dr] & \\
Y \arrow[r, "\bar{s}"'] & R \arrow[rr, "v"'] & & P
\end{tikzcd}$$

Let T denote the covering crible on X obtained from the S_s as before. From the fact that θ is a modification it follows that, for all $s: Y \to X$ in R' and $t: Z \to Y$ in S_s, the 2-cells ϕ_{st} and $\phi_s \bar{t}$ (where $\bar{t}$ composed with $S_s \to Y$ is t) agree when restricted to some covering subcrible of S_{st}. Since P is 1-separated, $\phi_{st} = \phi_s \bar{t}$. Thus we have a 2-cell

$$\begin{tikzcd}
T \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow\phi"] & R \arrow[d, "u"] \\
R \arrow[r, "v"'] & P
\end{tikzcd}$$

well-defined by the equation $(\phi Z)(st) = (\phi_s Z)(t)$. Clearly ϕ represents a 2-cell $\psi : f \Rightarrow g$ with $\psi i = \theta$. Thus $\mathscr{P}(i, LP)$ is full. So LP is 2-separated.

Since passing to filtered colimits in **Cat** preserves chronicness, *P is 2-separated if and only if $l_p : P \to LP$ is chronic.*

If P is 2-separated then LP is a 2-sheaf. The proof that $\mathscr{P}(i, LP)$ is surjective on objects proceeds precisely as in the familiar proof that P separated implies LP a sheaf for set-valued P [1; pp. 233–4]. This is all that is needed since LP is 2-separated.

P is a 2-sheaf if and only if $l_p : P \to LP$ is an isomorphism. The proof is as usual.

With these ingredients one now sees that $A = L^3$ is the sought left exact left adjoint.

(ii) For simplicity (and in fact, this is all that will be required later in this paper), we shall suppose $\mathscr{C}$, $\mathscr{E}$ finitely complete and that $\mathscr{J}$ is left exact. Then a set of arrows with common target in $\mathscr{C}$ covers if and only if it is taken by $\mathscr{J}$ to a cover in $\mathscr{E}$. It follows that, for each object U of $\mathscr{C}$, we have an initial functor $C(U) \to C_{\mathscr{J}}(\mathscr{J}U)$ which takes each covering U-crible in $\mathscr{C}$ to the covering $\mathscr{J}U$-crible in $\mathscr{E}$ generated by it.

Consider the following diagram.

$$\begin{tikzcd}
{[\mathscr{E}^{\mathrm{op}}, \mathbf{Cat}]} \arrow[r, shift right, "\mathscr{J}^*"'] \arrow[d, shift right, "\mathscr{A}"'] & {[\mathscr{C}^{\mathrm{op}}, \mathbf{Cat}]} \arrow[l, shift right, "\exists_{\mathscr{J}}"'] \arrow[d, shift right] \\
\mathscr{S}h(\mathscr{E}, \mathbf{Cat}) \arrow[r, shift right, "\mathscr{J}^s"'] \arrow[u, shift right] & \mathscr{S}h(\mathscr{C}, \mathbf{Cat}) \arrow[l, shift right, "\mathscr{J}_s"'] \arrow[u, shift right]
\end{tikzcd}$$

Here $\mathscr{J}^*(P) = P\mathscr{J}^{\mathrm{op}}$, so $\mathscr{J}^*$ takes 2-sheaves to 2-sheaves and so induces $\mathscr{J}^s$. Left Kan extension along $\mathscr{J}^{\mathrm{op}}$ is denoted by $\exists_{\mathscr{J}}$, and $A\exists_{\mathscr{J}}$ gives a left adjoint $\mathscr{J}_s$ for $\mathscr{J}^s$.

We shall show that the unit $1 \to \mathscr{J}^s\mathscr{J}_s$ is an isomorphism. Observe that the following diagram commutes up to isomorphism (3.5).

$$\begin{tikzcd}
C(U) \arrow[r] \arrow[d] & {[\mathscr{C}^{\mathrm{op}}, \mathbf{Cat}]} \arrow[d, "\exists_{\mathscr{J}}"] \\
C(\mathscr{J}U) \arrow[r] & {[\mathscr{E}^{\mathrm{op}}, \mathbf{Cat}]}
\end{tikzcd}$$

Hence $(LP)(\mathscr{J}U) \cong L(P\mathscr{J}^{\mathrm{op}})U$ where the L on the left-hand side is that of (i) and that on the right-hand side is the corresponding L for $\mathscr{C}$. Thus $\mathscr{J}^*L \cong L\mathscr{J}^*$. Since $\mathscr{J}$ is fully faithful, the unit $1 \to \mathscr{I}^*\exists_{\mathscr{I}}$ is an isomorphism. Hence $\mathscr{I}^s\mathscr{I}_s(G) = \mathscr{I}^*L^3\exists_{\mathscr{I}}(G) \cong L^3\mathscr{I}^*\exists_{\mathscr{I}}(G) \cong L^3(G) \cong G$ where the last isomorphism comes from the fact that G is a 2-sheaf.

It remains to show that $\mathscr{I}^s$ reflects isomorphisms. Suppose $\theta: F \to F'$ is an arrow in $\mathscr{S}h(\mathscr{E}, \mathbf{Cat})$ such that $\theta\mathscr{I}U$ is an isomorphism for all U of $\mathscr{C}$. We must show that X is an isomorphism for all θX in $\mathscr{E}$. This is done in three steps. First, suppose there is a chronic $X \to \mathscr{I}U'$. Cover X by arrows with sources in the image of $\mathscr{I}$. The comma objects and pullbacks of these arrows are then in the image of $\mathscr{I}$. Using the fact that F, F' are 2-sheaves so that $FX, F'X$ are limits, we deduce that θX is an isomorphism in this case. Second, suppose there is a faithful $X \to \mathscr{I}U'$ and repeat the procedure this time observing that the comma objects and pullbacks have chronics into objects in the image of $\mathscr{I}$. Finally, in the general case, the comma objects and pullbacks have faithfuls into objects in the image of $\mathscr{I}$. □

4. Characterization of 2-toposes

4.1. A 2-category $\mathscr{X}$ is said to be a 2-*topos* when there exists a small 2-site $\mathscr{C}$ and an equivalence of 2-categories:

$$\mathscr{X} \simeq \mathscr{S}h(\mathscr{C}, \mathbf{Cat}).$$

4.2. Suppose $\mathscr{C}$ is a finitely complete 2-category and U is an object of $\mathscr{C}$. A set $\mathscr{U}$ of arrows into U is said to be *acute* when the following property holds:

If $m: V \to U$ is chronic and each arrow $W \to U$ in $\mathscr{U}$ factors through m then m is an isomorphism.

A singleton set of arrows is acute if and only if the arrow it contains is acute (1.4, 1.6).

4.3. A set $\mathscr{G}$ of objects of $\mathscr{C}$ is said to be *acutely generating* when, for each object U of $\mathscr{C}$, the set of arrows into U with sources in $\mathscr{G}$ is acute.

4.4. The *canonical topology on a* 2-*category* is the largest topology for which the representable 2-functors are all 2-sheaves. *Canonical* 2-*sheaves* are 2-sheaves for the canonical topology.

4.5. *For any finitely complete* 2-*site for which the representables are* 2-*sheaves, all covering families are acute.* To see this, take a cover $\mathscr{U}$ of U. If the representables are 2-sheaves then, by (3.6), $\kappa_{\mathscr{U}}$ exhibits U as $\mathrm{col}(J_{\mathscr{U}}, S_{\mathscr{U}})$. Any chronic through which the arrows of $\mathscr{U}$ all factor must therefore be a retraction and so an isomorphism. □

4.6. *A set of arrows with common target in* **Cat** *is acute when the arrows are jointly surjective on objects. Every set of arrows with common target in* **Cat** *factors into an* *acute set followed by a chronic arrow. A pullback of an acute set in* **Cat** *is acute. For each acute set* $\mathscr{U}$ *of arrows into* U *in* **Cat**, $\kappa_{\mathscr{U}}$ *(3.5) exhibits* U *as a* $J_{\mathscr{U}}$*-indexed colimit of* $S_{\mathscr{U}}$.

To prove these statements, take a large enough version of **Cat** so that the given sets of arrows are small and so can be replaced by a single acute arrow out of a coproduct; then apply Theorem 2.3. (See the proof of (iv) $\Rightarrow$ (iii) for Theorem 4.11.)

4.7. A 2-category $\mathscr{X}$ is called *lex-total* when it has small hom-categories and the Yoneda embedding 2-functor $Y: \mathscr{X} \to [\mathscr{X}^{\mathrm{op}}, \mathbf{Cat}]$ has a left adjoint Z which preserves finite limits.

4.8. Proposition. *Every* **Cat***-valued canonical* 2*-sheaf on a lex-total* 2*-category* $\mathscr{X}$ *is representable.*

Proof. The statements of 4.6 transfer pointwise to $[\mathscr{X}^{\mathrm{op}}, \mathbf{Cat}]$. Using Z, we find that, in $\mathscr{X}$, a pullback of an acute set $\mathscr{U}$ is acute and that $\kappa_{\mathscr{U}}$ exhibits a $J_{\mathscr{U}}$-indexed colimit of $S_{\mathscr{U}}$. It follows that acute sets form the covers for a topology on $\mathscr{X}$ for which the representables are sheaves. By 4.5 this must be the canonical topology 4.4.

From (3.6) and Yoneda's lemma, the fully faithful $\mathscr{X} \to \mathscr{S}h(\mathscr{X}, \mathbf{Cat})$ preserves $\mathrm{col}(J_{\mathscr{U}}, S_{\mathscr{U}})$ for each acute $\mathscr{U}$.

We now turn to the problem of showing that each F of $\mathscr{S}h(\mathscr{X}, \mathbf{Cat})$ is representable.

Let $\mathscr{U}$ denote the set of arrows $\sigma: YU \to F$, $U \in \mathscr{X}$, from representables into F. Clearly $\mathscr{U}$ is acute in both $\mathscr{S}h(\mathscr{X}, \mathbf{Cat})$ and $[\mathscr{X}^{\mathrm{op}}, \mathbf{Cat}]$. Thus $\kappa_{\mathscr{U}}$ exhibits F as a $J_{\mathscr{U}}$-indexed colimit of $s_{\mathscr{U}}$ in $[\mathscr{X}^{\mathrm{op}}, \mathbf{Cat}]$ and $\mathscr{S}h(\mathscr{X}, \mathbf{Cat})$ (for the latter, use 3.8(i) with larger version of **Cat**).

Apply Z to $\mathrm{col}(J_{\mathscr{U}}, S_{\mathscr{U}}) \cong F$ in $[\mathscr{X}^{\mathrm{op}}, \mathbf{Cat}]$ to obtain $\mathrm{col}(J_{\mathscr{U}}, ZS_{\mathscr{U}}) \cong ZF$ since Z preserves all colimits. Since Z preserves finite limits, the latter colimit is also $\mathrm{col}(J_{\mathscr{V}}, S_{\mathscr{V}})$ where $\mathscr{V}$ is the set of all arrows into ZF in $\mathscr{X}$. Apply $\mathscr{X} \to \mathscr{S}h(\mathscr{X}, \mathbf{Cat})$ to obtain $\mathrm{col}(J_{\mathscr{U}}, YZS_{\mathscr{U}}) \cong YZF$ in $\mathscr{S}h(\mathscr{X}, \mathbf{Cat})$. The unit of the adjunction $Z - Y$ induces a commutative diagram:

$$\begin{tikzcd}
\mathrm{col}(J_{\mathscr{U}}, S_{\mathscr{U}}) \arrow[r, phantom, "\cong"] \arrow[d] & F \arrow[d] \\
\mathrm{col}(J_{\mathscr{U}}, YZS_{\mathscr{U}}) \arrow[r, phantom, "\cong"] & YZF
\end{tikzcd}$$

The final argument breaks up into three cases.

First, suppose F has a chronic into a representable. Since Z preserves chronics, the unit $F \to YZF$ must be chronic. From the construction of $S_{\mathscr{U}}$ it follows that $S_{\mathscr{U}} \to YZS_{\mathscr{U}}$ is an isomorphism. Hence from the above square, $F \cong YZF$. So F is representable.

Second, suppose F has a faithful into a representable. Then, from the construction of $S_{\mathscr{U}}$, $S_{\mathscr{U}}$ has a chronic into a representable. By the first case, $S_{\mathscr{U}} \cong YZS_{\mathscr{U}}$. So again F is representable.

Finally, for any F, there is a faithful from $S_{\mathscr{U}}$ into a representable. By the second part, $S_{\mathscr{U}} \cong YZS_{\mathscr{U}}$. So again F is representable. □

4.9. Coproducts in a 2-category which are preserved by pullback are said to be *universal*. If any two distinct coprojections into a coproduct have an initial comma object then the coproduct is said to be *disjoint*.

4.10. A set of cardinality no greater than the cardinality of the set of objects of Set is said to be *moderate*.

4.11. Theorem. *For a 2-category $\mathscr{X}$ with small hom-categories, the following conditions are logically equivalent:*

(i) *$\mathscr{X}$ is a 2-topos;*

(ii) *$\mathscr{X}$ is lex-total and there exists a moderate set $\mathscr{M}$ of objects of $\mathscr{X}$ such that, for each X of $\mathscr{X}$, there exists an acute arrow $M \to X$ with M in M;*

(iii) *every* **Cat**-*valued canonical 2-sheaf on $\mathscr{X}$ is representable and $\mathscr{X}$ has an acutely generating small set of objects;*

(iv) *$\mathscr{X}$ is an exact 2-category which has disjoint universal small coproducts and has an acutely generating small set of objects;*

(v) *there exists a finitely complete, small canonical 2-site $\mathscr{C}$ and and equivalence $\mathscr{X} \simeq \mathscr{S}h(\mathscr{C}, \mathbf{Cat})$.*

Proof. (i)⇒(ii). This follows from the generalities of [10] using Theorem 2.8. In more detail, suppose $\mathscr{X} = \mathscr{S}h(\mathscr{C}, \mathbf{Cat})$. The Yoneda embedding of $\mathscr{X}$ is the composite

$$\mathscr{X} \xrightarrow[\mathscr{I}]{} [\mathscr{C}^{\mathrm{op}}, \mathbf{Cat}] \xrightarrow[Y]{} [[\mathscr{C}^{\mathrm{op}}, \mathbf{Cat}]^{\mathrm{op}}, \mathbf{Cat}] \xrightarrow[{[\mathscr{I}^{\mathrm{op}}, 1]}]{} [\mathscr{X}^{\mathrm{op}}, \mathbf{Cat}].$$

By Theorem 3.8, $\mathscr{I}$ has a left-exact left adjoint A. So $[\mathscr{I}^{\mathrm{op}}, 1]$ has a left-exact left adjoint $[A^{\mathrm{op}}, 1]$. The above Yoneda embedding Y has a left-exact left adjoint Z given by $Z(P)U = P(\mathscr{C}(-, U))$. So $\mathscr{X}$ is lex-total. The second clause of (ii) is fulfilled by taking $\mathscr{M}$ to consist of all objects of $\mathscr{X}$.

(ii)⇒(iii) Proposition (4.8) gives the first clause of (iii). The production of a small acutely generating set as will now be described is a 2-version of an argument of Freyd [9].

Suppose $\mathscr{X}$ has no acutely generating small set of objects. Then $\mathscr{M}$ is not small. We may assume $\mathscr{X}$ is skeletal. Well order the objects of $\mathscr{M}$ so that each $[B \in \mathscr{M} \mid B \le A]$ is small for each A in $\mathscr{M}$; this is possible since $\mathscr{M}$ is moderate. The latter set cannot acutely generate $\mathscr{X}$. So there exists a chronic $m_A : X_A \to Y_A$ in $\mathscr{X}$ which is not an isomorphism and yet $\mathscr{X}(B, m_A)$ is an isomorphism for all $B \le A$. Form the following colimit of category-valued 2-functors:

$$\begin{tikzcd} \mathscr{X}(-,X_A) \arrow[r, "\mathscr{X}(-,m_A)"] \arrow[d] & \mathscr{X}(-,Y_A) \arrow[d, "\gamma_A"] \\ 1 \arrow[r, "\omega"'] & P \end{tikzcd} \qquad A \in \mathscr{M}.$$

More explicitly, for $C \in \mathscr{X}$, the set of objects of PC is the disjoint union of 1 and, for each $A \in \mathscr{M}$, the set of arrows $C \to Y_A$ which do not factor through m_A. Using (ii), we have an acute arrow $B \to C$ with B in $\mathscr{M}$. It follows that $PC \to PB$ is chronic. Since $\mathscr{X}(B, m_A)$ is an isomorphism for all $A \geq B$, it follows that PB is small. So PC is small. So P lands in **Cat**. Applying the left-exact left adjoint of the Yoneda embedding, we obtain a colimit in $\mathscr{X}$:

$$\begin{tikzcd} X_A \arrow[r, "m_A"] \arrow[d] & Y_A \arrow[d, "\xi_A"] \\ 1 \arrow[r, "\pi"'] & Q \end{tikzcd} \qquad A \in M.$$

For each object M of $\mathscr{M}$, define $\varepsilon_M : Q \to Q$ by $\varepsilon_M \cdot \pi = \pi$, $\varepsilon_M \xi_A = \xi_A$ when $M = A$, and $\varepsilon_M \xi_A$ is $Y_A \to 1 \xrightarrow[\pi]{} Q$ when $M \neq A$.

Suppose $\varepsilon_M = \varepsilon_{M'}$ and $M \neq M'$. Then $\varepsilon_M \xi_M = \varepsilon_{M'} \xi_M$. So ξ_M factors through π. It follows from the above colimit property of Q that the following is a pushout in $\mathscr{X}$:

$$\begin{tikzcd} X_M \arrow[r, "m_M"] \arrow[d] & Y_M \arrow[d] \\ 1 \arrow[r] & 1 \end{tikzcd}$$

In a lex-total 2-category it is easy to see that a pushout of a chronic is automatically a pullback. So m_M is an isomorphism, a contradiction.

So $M \mapsto \varepsilon_M$ is a monomorphism into the set of objects of $\mathscr{X}(Q, Q)$. So $\mathscr{M}$ is small, a contradiction.

(iv) ⇒ (iii) Regard $\mathscr{X}$ as a canonical 2-site. The Yoneda embedding factors via a 2-functor $R : \mathscr{X} \to \mathscr{S}h(\mathscr{X}, \mathbf{Cat})$ which we must show is an equivalence. Clearly R preserves limits.

For any small acute set $\mathscr{U}$ of arrows into U of $\mathscr{X}$, the 2-natural transformation $\kappa_{\mathscr{U}} : J_{\mathscr{U}} \to \mathscr{X}(S_{\mathscr{U}}, U)$ of 3.5 exhibits U as a $J_{\mathscr{U}}$-indexed colimit of $S_{\mathscr{U}}$. To see this, in the notation of 3.4, consider the arrow $e : \sum_{u \in \mathscr{U}} V_u \to U$ whose composite with the u-th coprojection is u. Since $\mathscr{U}$ is acute, e is acute. Using the universality of coproducts, we see that the congruence $E = \mathbf{E}(e)$ is given by:

$$E_1^2 = \sum_{u,v \in \mathscr{U}} V_{u,v}, \qquad E_2^2 = \sum_{u,v,w \in \mathscr{U}} V_{u,v,w}, \qquad E_1^1 = \sum_{u,v \in \mathscr{U}} V'_{u,v}.$$

By Theorem 1.22 and 1.14, *e is a quotient map for the congruence E*. Reinterpretation of this gives that U is the $J_{\mathscr{U}}$-indexed colimit of $S_{\mathscr{U}}$.

It follows that every small acute set must cover in the canonical topology. From the definition of a 2-sheaf and the Yoneda lemma, it follows that R preserves the $J_{\mathscr{U}}$-indexed colimit of $S_{\mathscr{U}}$ for each small acute set $\mathscr{U}$. In particular, *R preserves quotients of congruences, small coproducts* (this uses disjointness of coproducts), *and acuteness of small sets.*

For each $F \in \mathscr{S}h(\mathscr{X}, \mathbf{Cat})$, there exists $X \in \mathscr{X}$ and an acute arrow $RX \to F$. To see this, first observe that the set of arrows $RU \to F$, $U \in \mathscr{X}$, is acute. Let $\mathscr{G}$ be a small acutely generating set of objects of $\mathscr{X}$. For each $U \in \mathscr{X}$, the set of arrows $G \to U$, $G \in \mathscr{G}$, is a small acute set. Since R preserves acuteness of small sets and a 'composite' of acute sets is acute, the set of arrows $RG \to F$, $G \in \mathscr{G}$, is small acute. Take $X = \sum_{G \in \mathscr{G}} G$. Since R preserves small coproducts, we have an acute arrow $RX \to F$.

We must show that $F \in \mathscr{S}h(\mathscr{X}, \mathbf{Cat})$ is isomorphic to some RZ. Select an acute $e: RX \to F$ and consider the associated congruence $E = \mathbf{E}(e)$. If we can show that E is isomorphic to the image under R of a congruence in $\mathscr{X}$ then a quotient of the latter congruence will have image under R isomorphic to F (since R preserves quotients of congruences), and we will be done. Consider three cases.

First, suppose there is a chronic $k: F \to RY$. Then $\mathbf{E}(e) \cong \mathbf{E}(ke)$, and $ke = Rf$ for some $f: X \to Y$. So $\mathbf{E}(e) \cong R\mathbf{E}(f)$.

Second, suppose there is a faithful $k: F \to RY$. Then $\mathbf{E}(e) \to \mathbf{E}(ke)$ is chronic in each component. By the first case, $E(e)$ is isomorphic to a congruence in the image of R.

Third, suppose F is arbitrary. Each object of E has a faithful arrow into an object in the image of R: for example.

$$E_1^2 = e/e \xrightarrow[\binom{d_0}{d_1}]{} RX \times RX \cong R(X \times X)$$

is faithful. So again the result follows using the previous case.

(iii) ⇒ (v) *First observe that the canonical topology on $\mathscr{X}$ consists of the acute sets as covers.* To see this, one only needs the equivalence $\mathscr{X} \simeq \mathscr{S}h(\mathscr{X}, \mathbf{Cat})$. The only problem is to see that the acute sets do give a topology; that is, that a pullback of an acute set is acute. The latter follows easily by employing Theorem 3.8 with some larger version of **Cat**.

Next observe that the chronic subobjects of a given object form a small set. This uses a standard argument in the presence of the appropriate small generating set [5; p. 92].

It is thus possible to find a small full sub-2-category $\mathscr{C}$ of $\mathscr{X}$ which is closed under finite limits and chronic subobjects, and, which contains an acutely generating set of objects. Enrich $\mathscr{C}$ with the largest topology such that the restriction to $\mathscr{C}$ of each canonical 2-sheaf on $\mathscr{X}$ is a 2-sheaf. It is easy to see that the covers for this topology are the acute sets in $\mathscr{C}$, and, indeed, that this is the canonical topology on $\mathscr{C}$.

The result now follows from the comparison lemma Theorem 3.8(ii).

(v) $\Rightarrow$ (i) Trivial.

(i) $\Rightarrow$ (iv) Easy consequence of 2.4, 3.8(i). $\square$

5. Classification by 2-toposes

5.1. Suppose $\mathscr{C}, \mathscr{E}$ are two finitely complete 2-sites. Write $\mathrm{Lexcov}(\mathscr{C}, \mathscr{E})$ for the full sub-2-category of $[\mathscr{C}, \mathscr{E}]$ consisting of the 2-functors $T: \mathscr{C} \to \mathscr{E}$ which take covers to covers and which preserve finite limits (3.7).

5.2. A 2-functor $M: \mathscr{X} \to \mathscr{X}'$ is called a *geometric morphism* when it has a left exact (3.7), left adjoint M^*. Write $\mathrm{Geom}(\mathscr{X}, \mathscr{X}')$ for the full sub-2-category of $[\mathscr{X}, \mathscr{X}']^{\mathrm{op}}$ consisting of the geometric morphisms.

5.3. A 2-topos, when regarded as a 2-site, is always understood (unless otherwise stated) to have the canonical topology. The covers are acute sets.

5.4. Theorem. *Suppose $\mathscr{C}$ is a finitely complete small site and $\mathscr{X}$ is a 2-topos. The assignment $M \mapsto M^*AY$ (3.8) is an equivalence of 2-categories.*

$$\mathrm{Geom}(\mathscr{X}, \mathscr{S}h(\mathscr{C}, \mathbf{Cat})) \simeq \mathrm{Lexcov}(\mathscr{C}, \mathscr{X}).$$

Proof. The proof follows the general method of [10; p. 374] once one observes that the left Kan extension of a left exact 2-functor $T: \mathscr{C} \to \mathbf{Cat}$ is left exact. The latter follows from the observation of Max Kelly (in his work on 2-enriched theories) that such a 2-functor T is a filtered colimit of $\mathbf{Cat}$-valued representables and then the observation that filtered colimits commute with finite indexed limits in $\mathbf{Cat}$. $\square$

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