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FLAT SEMILATTICES[1]

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ABSTRACT. Let $\mathbf{S}$ (respectively $\mathbf{S}_0$) denote the category of all join-semilattices (resp. join-semilattices with 0) with (0-preserving) semilattice homomorphisms. For $A \in \mathbf{S}$ let A_0 represent the object of $\mathbf{S}_0$ obtained by adjoining a new 0-element. In either category the tensor product of two objects may be constructed in such a manner that the tensor product functor is left adjoint to the hom functor. An object $A \in \mathbf{S}$ ($\mathbf{S}_0$) is called *flat* if the functor $-\otimes_{\mathbf{S}} A$ $(-\otimes_{\mathbf{S}_0} A)$ preserves monomorphisms in $\mathbf{S}$ ($\mathbf{S}_0$).

THEOREM. *For $A \in \mathbf{S}$ ($\mathbf{S}_0$) the following conditions are equivalent*: (1) *A is flat in* $\mathbf{S}$ ($\mathbf{S}_0$), (2) A_0 (A) *is distributive* (*see Grätzer, Lattice theory, p.* 117), (3) *A is a directed colimit of a system of f.g. free algebras in* $\mathbf{S}$ ($\mathbf{S}_0$). The equivalence of (1) and (2) in $\mathbf{S}$ was previously known to James A. Anderson. (1) $\Leftrightarrow$ (3) is an analogue of Lazard's well-known result for R-modules.

1. Introduction. For any algebras A and B in a variety $\mathcal{V}$ the tensor product $A \otimes B$ can be constructed, and has the defining property that any bi-homomorphism from $A \times B$ to an algebra in $\mathcal{V}$ factors uniquely through $A \otimes B$. An algebra $A \in \mathcal{V}$ is called *flat* if the functor $-\otimes A$ preserves monomorphisms of $\mathcal{V}$. This terminology is consistent with that used for R-modules, and appears also in **[1]**, **[4]** and **[12]**, while the notion of flatness appearing in **[23]** is somewhat stronger. In this paper we are concerned with the varieties $\mathbf{S}$ (of ($\bigvee$-) semilattices) and $\mathbf{S}_0$ (of semilattices with least element 0, henceforth called 0-*semilattices*). $\mathbf{S}$ and $\mathbf{S}_0$ will also be considered as categories, the morphisms being all homomorphisms and all 0-preserving homomorphisms, respectively. Tensor products in $\mathbf{S}$ and $\mathbf{S}_0$, as well as in semigroups, commutative semigroups, distributive lattices and M-sets, have been extensively studied in recent years. Some idea of the existing literature in this area is given by the references at the end of this paper.

In 1969 Lazard **[21]** proved that an R-module M is flat iff it is the directed colimit of a system of finitely generated free R-modules. If we call an algebra in a variety $\mathcal{V}$ *L-flat* if it is the directed colimit of a system of finitely generated $\mathcal{V}$-free algebras, then Lazard's theorem simply states that an R-module is L-flat iff it is flat. L-flatness has been studied for M-sets (M a monoid) in **[23]**, for commutative semigroups in **[15]**, and for arbitrary varieties in **[22]**. Examples show that the analogue of Lazard's theorem is not true in general: **[4]** shows that there are M-sets (taking M to be the free

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1-generated commutative monoid) which are flat but not L-flat.

The main results of this paper are that for the varieties $\mathbf{S}$ and $\mathbf{S}_0$ Lazard's theorem is true, and that the flat ($= L$-flat) objects in either case are simply those semilattices which, when a new 0 is adjoined, are distributive in the sense of [**10**]. Our results are proved using Shannon's description [**22**] of L-flatness in terms of the Killing Interpolation Property (KIP), and using a version of Lambek's characterization [**20**] of a flat module in terms of its character module.

We acknowledge a private communication [**1**], with James A. Anderson of Northern Arizona University, Flagstaff, Arizona, announcing a characterization of flat semilattices which is tantamount to the equivalence of conditions (1) and (2) of our Theorem 2.3. So far as we know there is little similarity between Anderson's approach and that used here.

2. Flat semilattices. It is well known that for any $A \in \mathbf{S}$ the functor $-\otimes A$: $\mathbf{S} \to \mathbf{S}$ is left adjoint to the functor $\mathrm{Hom}_{\mathbf{S}}(A, -)$: $\mathbf{S} \to \mathbf{S}$; in fact, for any $A, B, C \in \mathbf{S}$ there is a bijection

$$\mathrm{Hom}_{\mathbf{S}}(B \otimes A, C) \cong \mathrm{Hom}_{\mathbf{S}}(B, \mathrm{Hom}_{\mathbf{S}}(A, C))$$

natural in all three arguments. Using the above adjointness together with the fact that the 2-element semilattice $\mathbf{2}$ is an injective cogenerator of $\mathbf{S}$, one easily proves the following analogue of a well-known result of Lembek [**20**]:

PROPOSITION 2.1. *A semilattice A is flat iff* $\mathrm{Hom}_{\mathbf{S}}(A, \mathbf{2})$ *is injective in* $\mathbf{S}$.

Now $\mathrm{Hom}_{\mathbf{S}}(A, \mathbf{2})$ is dually isomorphic to the $\wedge$-semilattice (under intersection) $I_0(A)$ of all ideals of A together with the empty set. Using the characterization [**3**], [**18**] of injective $\wedge$-semilattices S as complete Brouwerian lattices and the fact that for any $A \in \mathbf{S}$, $I_0(A)$ is a complete join-continuous lattice (see [**10**, p. 152]), we quickly see that $I_0(A)$ is an injective $\wedge$-semilattice iff it is a distributive lattice. Therefore A is flat iff $I_0(A)$ is a distributive lattice.

A semilattice is called *distributive* (see for example [**10**]) if $a \leqslant b \vee c$ implies the existence of elements $\beta \leqslant b$, $\gamma \leqslant c$ such that $a = \beta \vee \gamma$. One easily shows that a semilattice A is distributive iff its set $I(A)$ of nonempty ideals is a distributive lattice and that for a lattice, distributivity *qua* lattice coincides with distributivity *qua* semilattice. This notion of distributivity seems to us to be a little too strong for many applications: for example, free semilattices, although they differ only by a 0 element from distributive lattices, are not distributive semilattices. One can weaken the definition of distributive semilattice in the following way to overcome such difficulties:

DEFINITION 2.2. A semilattice *satisfies condition* (D) if $a \leqslant b \vee c$, $a \nleqslant b$, $a \nleqslant c$ implies the existence of elements $\beta \leqslant b$, $\gamma \leqslant c$ such that $a = \beta \vee \gamma$.

One can easily see that a semilattice A satisfies (D) iff A_0 (obtained by adjoining a new least element to A) is distributive, and also iff $I_0(A)$ is a distributive lattice. Thus we have the following characterization of flat semilattices:

THEOREM 2.3. *For any semilattice A the following conditions are equivalent*:
(1) *A is flat*;
(2) *A satisfies condition* (D);
(3) *$I_0(A)$ is a distributive lattice*.

We next turn our attention to L-flatness. Grillet **[14]** devised a criterion for L-flatness of commutative semigroups known as the *Killing Interpolation Property* (KIP) which was later shown by Shannon **[22]** to be a criterion for L-flatness in arbitrary varieties:

DEFINITION 2.4. An algebra A in a variety $\mathcal{V}$ has (KIP) if whenever $p(\vec{a}) = q(\vec{a})$ for n-ary polynomials p, q and for $\vec{a} \in A^n$ $(n \in \mathbf{N})$ then there exist $m \in \mathbf{N}$, $\vec{c} \in A^m$, and m-ary polynomials $r_1, \ldots, r_n$ such that:

$$r_i(\vec{c}) = a_i \qquad (1 \leqslant i \leqslant n) \tag{1}$$

and

$$p(r_1, \ldots, r_n) = q(r_1, \ldots, r_n) \text{ is an identity of } \mathcal{V}. \tag{2}$$

It is clear that any $\mathcal{V}$-free algebra has (KIP) and that (KIP) is preserved by directed colimits.

THEOREM 2.5 (SHANNON **[22]**). *An algebra A in a variety $\mathcal{V}$ is L-flat* (*i.e. the directed colimit of a system of finitely generated $\mathcal{V}$-free algebras*) *iff A has* (*KIP*).

Referring to Stenström **[23]** we remark that condition (b) of Theorem 5.3 gives a form of (KIP) intrinsic to right S-systems, and that for any variety with onto epimorphisms the analogues of conditions (c)–(f) of this theorem remain equivalent to (KIP). Grillet **[15]** gives conditions intrinsic to commutative monoids which are equivalent to (KIP). We shall show in the remainder of this section that, for semilattices, (KIP) is equivalent to (D), and thus that flatness and L-flatness for semilattices coincide.

PROPOSITION 2.6. *A semilattice A satisfies* (D) *iff A satisfies* (D_n) *for all $n \geqslant 2$, where*

$$\begin{array}{l} a \leqslant b_1 \vee \cdots \vee b_n \text{ and } a \nleqslant b_{i_1} \vee \cdots \vee b_{i_m} \text{ for all} \\ \{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\},\ m < n, \text{ implies the existence of} \\ \beta_i \leqslant b_i\ (1 \leqslant i \leqslant n) \text{ with } a = \beta_1 \vee \cdots \vee \beta_n. \end{array} \tag{D_n}$$

PROOF. Clearly (D) is equivalent to (D_2). An easy induction on n establishes the result. □

PROPOSITION 2.7. *A semilattice A has* (*KIP*) *iff A satisfies* (D).

PROOF. Since free semilattices satisfy (D) and (D) is preserved by directed colimits, any semilattice with (KIP) satisfies (D).

Conversely, suppose A satisfies (D), and thus (D_n) for all $n \geqslant 2$, by

Proposition 2.6. Let $a_1 \vee \cdots \vee a_n = b_1 \vee \cdots \vee b_m$ hold in A. For each i $(1 \leqslant i \leqslant n)$ choose a minimal subset J_i of $\{1, \ldots, m\}$ for which $a_i \leqslant \bigvee_{j \in J_i} b_j$. Similarly for each j $(1 \leqslant j \leqslant m)$ choose a minimal subset $I_j \subseteq \{1, \ldots, n\}$ for which $b_j \leqslant \bigvee_{i \in I_j} a_i$. By use of appropriate conditions (D_n) we find for each i $(1 \leqslant i \leqslant n)$ elements $\beta_{ij} \leqslant b_j$ $(j \in J_i)$ and for each j $(1 \leqslant j \leqslant m)$ elements $\alpha_{ij} \leqslant a_i$ $(i \in I_j)$ such that

$$a_i = \bigvee_{j \in J_i} \beta_{ij} \quad (1 \leqslant i \leqslant n) \quad \text{and} \quad b_j = \bigvee_{i \in I_j} \alpha_{ij} \quad (1 \leqslant j \leqslant m).$$

For each i $(1 \leqslant i \leqslant n)$ define $K_i = \{j | i \in I_j, 1 \leqslant j \leqslant m\}$ and for each j $(1 \leqslant j \leqslant m)$ define $L_j = \{i | j \in J_i, 1 \leqslant i \leqslant n\}$. Introduce variables x_{ij} $(1 \leqslant i \leqslant n, j \in J_i)$ and y_{ij} $(1 \leqslant j \leqslant m, i \in I_j)$, and define polynomials r_i $(1 \leqslant i \leqslant n)$ and s_j $(1 \leqslant j \leqslant m)$ by

$$r_i = \bigvee_{j \in J_i} x_{ij} \vee \bigvee_{j \in K_i} y_{ij} \quad \text{and} \quad s_j = \bigvee_{i \in L_j} x_{ij} \vee \bigvee_{i \in I_j} y_{ij}.$$

One shows without difficulty that $r_i(\vec{\beta}, \vec{\alpha}) = a_i$ $(1 \leqslant i \leqslant n)$, $s_j(\vec{\beta}, \vec{\alpha}) = b_j$ $(1 \leqslant j \leqslant m)$, and that $\bigvee_{i=1}^n r_i = \bigvee_{j=1}^m s_j$ is a semilattice identity. Thus (D) implies (KIP) and the proof is complete. $\square$

THEOREM 2.8. *A semilattice is flat iff it is L-flat.*

PROOF. Combine 2.3, 2.5 and 2.7. $\square$

3. Flat 0-semilattices. For the variety $\mathbf{S}_0$ methods similar to those employed in §2 yield the following result:

THEOREM 3.1. *For any* 0*-semilattice A the following conditions are equivalent*:

(1) *A is flat*;

(2) *A is a distributive semilattice*;

(3) *$I(A)$ is a distributive lattice*;

(4) *A is L-flat* (*i.e. the directed colimit of a system of finitely generated free* 0*-semilattices*).

PROOF. The equivalence of (1)–(3) can be accomplished using an argument similar to that used in obtaining Theorem 2.3, plus the fact that the injective objects in the variety of $\wedge$-semilattices with 1 are exactly the complete Brouwerian lattices, as in the semilattice case (see [**17**, p. 87]). That (2) and (4) are equivalent is seen by noting that, for $\mathbf{S}_0$, (KIP) is equivalent to distributivity. In fact, for $\mathbf{S}_0$, (KIP) is equivalent to the Riesz Interpolation Property (RIP) in its semigroup form (see [**15**] for the definition of (RIP), and [**15**, Lemma 1.12], for what is essentially a proof of the equivalence of (KIP) and (RIP) for $\mathbf{S}_0$). However, it is easily seen that (RIP) is equivalent to distributivity (in any semilattice). $\square$

COROLLARY 3.2. *For any semilattice A, A is a flat semilattice iff A_0 is a flat* 0*-semilattice.*

PROOF. Observe that $I_0(A) \cong I(A_0)$ and apply 2.3, 3.1. $\square$

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