# A category-theoretic version of the identity type weak factorization system

Jacopo Emmenegger

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### Abstract

Gambino and Garner proved that the syntactic category of a dependent type theory with identity types can be endowed with a weak factorization system structure, called identity type weak factorization system. In this paper we consider an enrichment of Joyal's notion of tribe which allow us to prove a purely category-theoretic version of the identity type weak factorization system, thus generalizing Gambino and Garner's result. We investigate then how it relates with other well-known weak factorization systems, namely those arising from Quillen model structures on the category of topological spaces and on the category of small groupoids.

## 1   Introduction

In recent years much interest has been devoted to investigating the relation between paths in a topological space, and proofs of identity in Martin-Löf type theory. In particular, Nicola Gambino and Richard Garner have proved in [3] that the syntactic category (or category of contexts) of a dependent type theory with identity types carries a weak factorization system structure.

Weak factorization systems traditionally arise in abstract homotopy theory: given a Quillen model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ on a category $\mathsf{C}$, both $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are weak factorization systems. A prime example is the Quillen model structure on $\mathsf{Top}$ (the category of topological spaces and continuous functions) presented by Strøm in [13], in which the class of weak equivalences $\mathcal{W}$ consists of the homotopy equivalences and the classes of fibrations $\mathcal{F}$ and cofibrations $\mathcal{C}$ consist of the Hurewicz fibrations and cofibrations, respectively.

A Quillen model structure can also be given to the category $\mathsf{Gpd}$ of small groupoids and functors [1, 8]. In this case the arrows in $\mathcal{W}$, $\mathcal{F}$ and $\mathcal{C}$ are the categorical equivalences, the Grothendieck fibrations and the functors injective on objects, respectively.

The goal of this paper is to abstract from the particular setting of the syntactic category in order to generalize the result of Gambino and Garner and investigate how it relates with the weak factorization systems of the form $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ given by the Quillen model structures on $\mathsf{Top}$ and $\mathsf{Gpd}$ mentioned before. In particular, this paper can be regarded as a development of the results already obtained in the thesis [2] carried out under the supervision of Nicola Gambino. The main improvements consist in the application of such results to the category $\mathsf{Top}$, worked out in Section 5, and in a reformulation of the stability condition imposed on the factorization of the diagonal arrow (Definition 7) which has been weakened in order to cover also the topological case.

Our starting point is a particular category, introduced by André Joyal in [6], called tribe. This is a category with a distinguished class of arrows $\mathcal{A}$ which, roughly speaking,

can be thought of as a class of fibrations. This class of arrows is reminiscent of the (stronger) notion of class of small maps introduced by André Joyal and Ieke Moerdijk in the context of Algebraic Set Theory [7], which was subsequently used by Ieke Moerdijk and Erik Palmgren to abstract on Martin-Löf type theory [10].

We shall consider additional structure on the class $\mathcal{A}$, defining a tribe with stable path objects, and prove that such category admits a weak factorization system, thus giving rise to Joyal's notion of h-tribe.

Independently of this work, Benno van den Berg has announced similar results in a recent talk in Oxford [14], achieved by considering a structure, called identity tribe, closely related to a tribe with stable path objects. The main difference between our two notions lies in the condition of stability under pullbacks imposed on the factorization of the diagonal arrow. Nevertheless, their similarities suggest that we can regard them as leading to a natural enrichment of the tribe structure on a category.

We begin the second section recalling some basic definitions and facts about weak factorization systems. The notion of tribe with stable path objects is then defined, and its relation with the notions introduced by Joyal and van den Berg is discussed. We conclude the section proving in Theorem 8 the existence of a weak factorization system in every tribe with stable path objects.

The third section provides a first application of the main result. There we show that the category of contexts of a type theory with identity types is a tribe with stable path objects and that Theorem 8 yields exactly the identity type weak factorization system.

In the fourth and fifth sections we show that the categories Gpd and Top are two other examples of the notion of tribe, in which the class $\mathcal{A}$ consists of the Grothendieck fibrations and of the Hurewicz fibrations, respectively. In order to prove that such categories can also be equipped with stable path objects, however, we shall need to require the arrows in $\mathcal{A}$ to satisfy an additional property. For this reason, the weak factorization system provided by Theorem 8 does not coincide exactly with the weak factorization system $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ of the two Quillen model structures mentioned above, since we have $\mathcal{A} \subset \mathcal{F}$. Nevertheless, they are closely related, as the factorization of an arrow given by our theorem coincide with that one given by the structure $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$.

# 2 Weak factorization systems and stable path objects

In this section we begin recalling the definition of weak factorization systems and a couple of elementary results. We present then the definition of tribe and introduce the richer structure of tribe with stable path objects, also discussing its relation with Joyal's h-tribe and van den Berg's identity tribe. Finally, we prove that every tribe with stable path objects can be endowed with a weak factorization system.

**Definition 1.** Let $\mathsf{C}$ be a category. Given two arrows $f \colon A \to X$ and $g \colon B \to Y$, a *left lifting problem for $f$ over $g$* (or a *right lifting problem for $g$ over $f$*) is a commutative square

$$
\begin{array}{ccc}
A & \longrightarrow & B \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\
X & \longrightarrow & Y
\end{array}
$$

The arrow $f$ has the *left lifting property (l.l.p.)* with respect to $g$, if every left lifting problem for $f$ over $g$ has a *diagonal filler*, that is an arrow $j \colon X \to B$ such that the

diagram

$$A \longrightarrow B$$

commutes. Symmetrically, $g$ has the *right lifting property (r.l.p.)* with respect to $f$, if every right lifting problem for $g$ over $f$ has a diagonal filler. Of course, $f$ has the l.l.p. with respect to $g$ if and only if $g$ has the r.l.p. with respect to $f$, we write $f \boxtimes g$ to denote this situation.

If $\mathcal{A}$ is a class of arrows, we write $\mathcal{A}^\boxtimes$ for the class of all the arrows that have the r.l.p. with respect to every arrow in $\mathcal{A}$, and $^\boxtimes\mathcal{A}$ for the class of all the arrows that have the l.l.p. with respect to every arrow in $\mathcal{A}$. We shall also write $f \boxtimes \mathcal{A}$ instead of $f \in {}^\boxtimes\mathcal{A}$.

**Definition 2.** A *weak factorization system (w.f.s.)* on $\mathsf{C}$ consists of a pair of classes of arrows $(\mathcal{L}, \mathcal{R})$ such that:

(*a*) every arrow $f$ in $\mathsf{C}$ admits a factorization $f = pi$, where $i \in \mathcal{L}$ and $p \in \mathcal{R}$,

(*b*) $\mathcal{L}^\boxtimes = \mathcal{R}$ and $^\boxtimes\mathcal{R} = \mathcal{L}$.

These properties are called Factorization Axiom and Weak Orthogonality Axiom, respectively.

**Lemma 3.** *Let $\mathsf{C}$ be a category and $\mathcal{A} \subset \mathrm{Ar}(\mathsf{C})$ a class of arrows in $\mathsf{C}$. Define $\mathcal{L} = {}^\boxtimes\mathcal{A}$ and $\mathcal{R} = \mathcal{L}^\boxtimes$. Thus $\mathcal{L} = {}^\boxtimes\mathcal{R}$.*

*Proof.* Let $f \in \mathcal{L}$. Since every arrow in $\mathcal{R}$ has the r.l.p. with respect to all the arrows in $\mathcal{L}$, in particular every $g \in \mathcal{R}$ has the r.l.p. with respect to $f$. Hence, from the symmetry of lifting properties, $f$ has the l.l.p with respect to any $g \in \mathcal{R}$. Thus, $f \in {}^\boxtimes\mathcal{R}$.

If we apply the same argument to an arrow in $\mathcal{A}$, we obtain $\mathcal{A} \subset \mathcal{R}$. Thus we have the opposite inclusion $^\boxtimes\mathcal{R} \subset {}^\boxtimes\mathcal{A} = \mathcal{L}$. $\qquad\square$

**Lemma 4.** *Let $\mathsf{C}$ be a category and $\mathcal{A} \subset \mathrm{Ar}(\mathsf{C})$ a class of arrows in $\mathsf{C}$. If $\mathsf{C}$ has pullbacks and $\mathcal{A}$ is closed under base change, then $f \boxtimes \mathcal{A}$ if and only if a diagonal filler exists for every lifting problem of the form*

$$A \longrightarrow B$$

*Proof.* If $f \boxtimes \mathcal{A}$, then obviously a diagonal filler exists. To prove the opposite implication notice that, given a left lifting problem for $f$ over an arrow in $\mathcal{A}$, taking the pullback of the bottom and the right-hand arrow yields a left lifting problem with the identity arrow as the bottom arrow, as illustrated by the following diagram

$$A \xrightarrow{\langle f,h\rangle} P \longrightarrow B$$

where the right-hand square is a pullback, $\langle f, h \rangle$ is the arrow given by its universal property and the outer square is the original lifting problem. Since $g \in \mathcal{A}$ and $\mathcal{A}$ is closed under base change, the left-hand square has a diagonal filler. Composing it with the base change of $k$ along $g$ yields, in turn, a diagonal filler for the original lifting problem. $\qquad\square$

**Definition 5.** Let $\mathsf{C}$ be a category and $\mathcal{A} \subset \mathrm{Ar}(\mathsf{C})$ a class of arrows in $\mathsf{C}$. The pair $(\mathsf{C}, \mathcal{A})$ is a *tribe* if $\mathsf{C}$ has a terminal object $\mathbf{1}$ and the following hold:

($a$) for every pair of arrows in $\mathsf{C}$ with the same codomain, we have a choice of a pullback square if at least one of them is in $\mathcal{A}$,

($b$) $\mathcal{A}$ is closed under composition and base change,

($c$) all the iso and terminal arrows are in $\mathcal{A}$.

Given two arrows $p \in \mathcal{A}$ and $f$ with the same codomain, we write as usual $f^*p$ to denote the (chosen) base change arrow of $p$ along $f$ and similarly for $p^*f$. Notice that point ($b$) of Definition 5 implies $f^*p \in \mathcal{A}$. The arrow defined by the universal property of a pullback is denoted by $\langle f, g \rangle$. Furthermore, we call *products* those pullback squares involving a single arrow $p \in \mathcal{A}$ or two terminal arrows, *projections* the base change arrows and denote them by

$$
\begin{array}{ccc}
E \times_p E & \xrightarrow{\ \mathrm{pr}_1^p\ } & E \\
{\scriptstyle \mathrm{pr}_0^p} \downarrow & & \downarrow {\scriptstyle p} \\
E & \xrightarrow{\ p\ } & Y
\end{array}
\qquad\qquad
\begin{array}{ccc}
X \times Y & \xrightarrow{\ \mathrm{pr}_1\ } & Y \\
{\scriptstyle \mathrm{pr}_0} \downarrow & & \downarrow \\
X & \longrightarrow & \mathbf{1}
\end{array}
$$

respectively. We shall also denote by $f \times g := \langle f\mathrm{pr}_0, g\mathrm{pr}_1 \rangle$ the product arrow between products. Notice that points ($b$) and ($c$) of Definition 5 imply that both the projections are in $\mathcal{A}$. We shall drop superscripts from the projections whenever they are clear from the context.

**Definition 6.** We say that a tribe $(\mathsf{C}, \mathcal{A})$ has *path objects* if, for every arrow $p \colon E \to Y$ in $\mathcal{A}$, we have a choice of a factorization of the diagonal $\Delta_p := \langle \mathrm{id}_E, \mathrm{id}_E \rangle \colon E \to E \times_p E$, denoted by

$$
E \xrightarrow{\ \mathrm{r}_p\ } \mathrm{Path}(p) \xrightarrow{\ \partial_p\ } E \times_p E,
$$

such that

($a$) $\partial_p \in \mathcal{A}$,

($b$) every base change of $\mathrm{r}_p$ along an arrow in $\mathcal{A}$ is in $^\boxtimes\mathcal{A}$.

Before introducing the notion of a tribe with stable path objects, let us again fix the notation for a particular kind of pullback. Let $g \colon X \to E$ be an arrow in $\mathsf{C}$, and let $p \colon E \to Y$ be an arrow in $\mathcal{A}$. The pullback of $g$ along $\partial_p^0$ will be denoted by

$$
\begin{array}{ccc}
\mathrm{Map}_p(g) & \xrightarrow{\ \mathrm{pm}_1^{p,g}\ } & \mathrm{Path}(p) \\
{\scriptstyle \mathrm{pm}_0^{p,g}} \downarrow & & \downarrow {\scriptstyle \partial_p^0} \\
X & \xrightarrow{\ g\ } & E
\end{array}
$$

and the object $\mathrm{Map}_p(g)$ will be called *mapping path object of $f$ along $p$*. As already said for products, we shall usually drop superscripts from the base change arrows.

If $p \colon E \to \mathbf{1}$ is a terminal arrow, we shall write $\mathrm{Path}(E)$, $\mathrm{r}_E$, $\partial_E$ and $\mathrm{Map}(f)$ in place of $\mathrm{Path}(p)$, $\mathrm{r}_p$, $\partial_p$ and $\mathrm{Map}_p(f)$, respectively, and simply say "mapping path object of $f$" for the latter.

**Definition 7.** We say that a tribe $(\mathsf{C}, \mathcal{A})$ with path objects has *stable path objects* if, for every arrow $p \colon E \to Y$ in $\mathcal{A}$ and every arrow $f \colon X \to Y$, there exists an arrow

$$
i \colon \mathrm{Map}_p(p^*f) \to \mathrm{Path}(f^*p)
$$

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such that the diagram

$$
\begin{array}{ccc}
F & =\!=\!=\!=\!=\!=\!= & F \\
{\scriptstyle\langle \mathrm{id}_F,\, \mathrm{r}_p p^*f\rangle}\Big\downarrow & & \Big\downarrow{\scriptstyle\mathrm{r}_{f^*p}} \\
\mathrm{Map}_p(p^*f) & \xrightarrow{\;\;i\;\;} & \mathrm{Path}(f^*p) \\
{\scriptstyle\langle \mathrm{pm}_0,\, \langle (f^*p)\mathrm{pm}_0,\, \partial_p^1 \mathrm{pm}_1\rangle\rangle}\Big\downarrow & & \Big\downarrow{\scriptstyle\partial_{f^*p}} \\
F \times_{f^*p} F & =\!=\!=\!=\!=\!=\!= & F \times_{f^*p} F
\end{array}
\tag{1}
$$

commutes, where $F$ is the pullback object of $f$ and $p$.

Joyal defines a h-tribe as a tribe $(\mathsf{C}, \mathcal{A})$ in which every arrow can be factored through an arrow in $^{\boxtimes}\mathcal{A}$ followed by an arrow in $\mathcal{A}$, and such that the class $^{\boxtimes}\mathcal{A}$ is closed under base change [6]. Since it is possible to prove that in a tribe with stable path objects the class $^{\boxtimes}\mathcal{A}$ is closed under base change, Lemma 9 implies that every tribe with stable path objects is a h-tribe.

The identity tribe proposed by van den Berg's seems to be instead a tighter notion, as it is defined as a tribe with path objects in which the factorization of the diagonal arrow is required to be stable under pullback along any arrow [14]. It appears that this is equivalent, in our formulation, to requiring the arrow $i$ to be an iso.

According to this remark, every identity tribe is a tribe with stable path objects, and both of them are h-tribes. Moreover, it seems that not every tribe with stable path objects is an identity tribe: an example of this should be the category of topological spaces with the structure described in Section 5, where the arrow $i$ is not an iso but just a mono (cfr. Theorem 21).

**Theorem 8.** *Every tribe with stable path objects $(\mathsf{C}, \mathcal{A})$ admits a weak factorization system $(\mathcal{L}, \mathcal{R})$ defined by $\mathcal{L} := {}^{\boxtimes}\mathcal{A}$ and $\mathcal{R} := \mathcal{L}^{\boxtimes}$.*

*Proof.* The Weak Orthogonality Axiom directly follows from Lemma 3, whereas the Factorization Axiom is given by Lemma 9 and $\mathcal{A} \subset \mathcal{R}$. $\qquad\square$

**Lemma 9.** *Let $(\mathsf{C}, \mathcal{A})$ be a tribe with stable path objects. Every arrow $f\colon X \to Y$ in $\mathsf{C}$ admits a factorization through an arrow in $^{\boxtimes}\mathcal{A}$ followed by an arrow in $\mathcal{A}$.*

The proof relies on the following lemma, which we prove first.

**Lemma 10.** *Let $(\mathsf{C}, \mathcal{A})$ be a tribe with path objects and $p\colon Y \to X$ be an arrow in $\mathcal{A}$. Let then $q\colon E \to Y$, $e\colon Z \to E$ and $u\colon Z \to \mathrm{Path}(p)$ be three arrows such that $qe = \partial_p^0 u$. If $q \in \mathcal{A}$, there exists an arrow $u_! e\colon Z \to E$ such that*

*(i) $q(u_! e) = \partial_p^1 u$,*

*(ii) $(u_! e)f = ef$, for every commutative square*

$$
\begin{array}{ccc}
W & \xrightarrow{\;\;f\;\;} & Z \\
{\scriptstyle g}\Big\downarrow & & \Big\downarrow{\scriptstyle u} \\
Y & \xrightarrow[\;\mathrm{r}_p\;]{} & \mathrm{Path}(p)
\end{array}
$$

*Proof.* Let us consider the following commutative diagram

$$
\begin{array}{ccccc}
E & \xrightarrow{\;\langle \mathrm{id}_E,\, \mathrm{r}_p q\rangle\;} & \mathrm{Map}_p(q) & \xrightarrow{\;\mathrm{pm}_0\;} & E \\
{\scriptstyle q}\Big\downarrow & & {\scriptstyle\mathrm{pm}_1}\Big\downarrow & & \Big\downarrow{\scriptstyle q} \\
Y & \xrightarrow[\;\mathrm{r}_p\;]{} & \mathrm{Path}(p) & \xrightarrow[\;\partial_p^0\;]{} & Y
\end{array}
$$

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where the right-hand square defines the mapping path object of $q$ along $\partial_p^0$. Since the composites of the horizontal arrows are identities, the left-hand square is a pullback. Furthermore, the commutativity of the latter implies that the square

$$
\begin{array}{ccc}
E & =\!=\!=\!=\!= & E \\
{\scriptstyle \langle \mathrm{id}_E,\, \mathrm{r}_p q \rangle} \downarrow & & \downarrow {\scriptstyle q} \\
\mathrm{Map}_p(q) & \xrightarrow[\;\partial_p^1 \mathrm{pm}_1\;]{} & Y
\end{array} \tag{2}
$$

commutes.

Suppose now $q \in \mathcal{A}$. Point $(b)$ of Definition 5 implies $\mathrm{pm}_1 \in \mathcal{A}$ and, in turn, point $(b)$ of Definition 6 implies $\langle \mathrm{id}_E, \mathrm{r}_p q \rangle \boxtimes \mathcal{A}$. Therefore, the square (2) admits a diagonal filler $j \colon M \to E$.

Define
$$
u_! e := j\langle e, u \rangle \colon Z \to \mathrm{Map}_p(q) \to E.
$$

The property *(i)* follows easily from the definition and from the fact that $j$ is a diagonal filler for the square (2):
$$
q(u_! e) = \partial_p^1 \mathrm{pm}_1 \langle e, u \rangle = \partial_p^1 u.
$$

To prove the second one, suppose we are given a commutative square as in the statement *(ii)* and observe that

$$
qef = \partial_p^0 \mathrm{r}_p g = g.
$$

Therefore, we have

$$
(u_! e)f = j\langle ef, \mathrm{r}_p g \rangle = j\langle \mathrm{id}_E, \mathrm{r}_p q \rangle ef = ef. \qquad \square
$$

Notice that writing $u_! e$ is actually an abuse of notation. Indeed, the arrow denoted by $u_! e$ is not uniquely determined by $u$ and $e$, as there could be several different diagonal fillers for the square (2). Nevertheless, the two properties *(i)* and *(ii)* do not depend on a particular choice of $j$, therefore we prefer to use such notation in order to increase readability of the following proof.

*Proof of Lemma 9.* Let us consider the mapping path object of $f$

$$
\begin{array}{ccc}
\mathrm{Map}(f) & \longrightarrow & \mathrm{Path}(Y) \\
\downarrow & & \downarrow {\scriptstyle \partial_Y^0} \\
X & \xrightarrow{\;f\;} & Y
\end{array} \tag{3}
$$

and define $p := \mathrm{pm}_0^{Y \to \mathbf{1}, f}$ and $m := \mathrm{pm}_1^{Y \to \mathbf{1}, f}$. For the stability property of pullbacks, the following square is a pullback too

$$
\begin{array}{ccc}
\mathrm{Map}(f) & \xrightarrow{\;m\;} & \mathrm{Path}(Y) \\
{\scriptstyle \langle p,\, \partial_Y^1 m \rangle} \downarrow & & \downarrow {\scriptstyle \partial_Y} \\
X \times Y & \xrightarrow[\;f \times \mathrm{id}_Y\;]{} & Y \times Y
\end{array} \tag{4}
$$

In particular, point $(b)$ of Definition 5 implies $\langle p, \partial_Y^1 m \rangle \in \mathcal{A}$ and, in turn, $\partial_Y^1 m \in \mathcal{A}$.

Observe now that
$$
f = \partial_Y^1 m \langle \mathrm{id}_X, \mathrm{r}_Y f \rangle \colon X \to \mathrm{Map}(f) \to Y.
$$

where $\langle \mathrm{id}_X, \mathrm{r}_Y f \rangle \colon X \to \mathrm{Map}(f)$ is the arrow defined by the universal property of the pullback (3). We now prove that this is the required factorization. As we already know that $\partial_Y^1 m \in \mathcal{A}$, we only need to show that $\langle \mathrm{id}_X, \mathrm{r}_Y f \rangle$ has the left lifting property with respect to all the arrows in $\mathcal{A}$ but, thanks to Lemma 4, it suffices to consider only lifting problems involving the identity as bottom arrow

Let then $q \colon E \to \mathrm{Map}(f)$ be an arrow in $\mathcal{A}$ and let

$$
\begin{array}{ccc}
X & \xrightarrow{\ \ g\ \ } & E \\
{\scriptstyle \langle \mathrm{id}_X, \mathrm{r}_Y f \rangle} \big\downarrow & & \big\downarrow {\scriptstyle q} \\
\mathrm{Map}(f) & =\!=\!=\!= & \mathrm{Map}(f)
\end{array}
\tag{5}
$$

be a commutative square. In the following commutative diagram

$$
\begin{array}{ccccc}
E & \xrightarrow{\ \ l\ \ } & \mathrm{Map}(\partial_Y^1 mq) & \xrightarrow{\ \ \mathrm{pm}_0\ \ } & E \\
{\scriptstyle \partial_Y^1 mq} \big\downarrow & & \big\downarrow {\scriptstyle \mathrm{pm}_1} & & \big\downarrow {\scriptstyle \partial_Y^1 mq} \\
Y & \xrightarrow[\ \ \mathrm{r}_Y\ \ ]{} & \mathrm{Path}(Y) & \xrightarrow[\ \ \partial_Y^0\ \ ]{} & Y
\end{array}
\tag{6}
$$

the right-hand square defines the mapping path object of $\partial_Y^1 mq$, and $l = \langle \mathrm{id}_E, \mathrm{r}_Y \partial_Y^1 mq \rangle$. As in the proof of Lemma 10, the composites of the horizontal arrows are identities, so the left-hand square is a pullback of $\mathrm{r}_Y$ along $\mathrm{pm}_1 \in \mathcal{A}$, and $l \boxslash \mathcal{A}$ by point $(b)$ of Definition 6. Thus we have a diagonal filler $h \colon \mathrm{Map}(\partial_Y^1 mq) \to E$ for the following commutative square

$$
\begin{array}{ccc}
E & =\!=\!=\!= & E \\
{\scriptstyle l} \big\downarrow & & \big\downarrow {\scriptstyle q} \\
\mathrm{Map}(\partial_Y^1 mq) & \xrightarrow[\ \ q\mathrm{pm}_0\ \ ]{} & \mathrm{Map}(f)
\end{array}
$$

Define now
$$
k := h \langle gp, m \rangle \colon \mathrm{Map}(f) \to E
$$
where $\langle gp, m \rangle \colon \mathrm{Map}(f) \to \mathrm{Map}(\partial_Y^1 mq)$ is the universal arrow given by the right-hand pullback in (6). We have

$$
\begin{aligned}
k \langle \mathrm{id}_X, \mathrm{r}_Y f \rangle &= h \langle g, \mathrm{r}_Y f \rangle \\
&= hlg \\
&= g,
\end{aligned}
\tag{7}
$$

where the second equality follows from commutativity of the square (5) and definition of $l$, but

$$
\begin{aligned}
qk &= q\mathrm{pm}_0 \langle gp, m \rangle \\
&= qgp \\
&= \langle \mathrm{id}_X, \mathrm{r}_Y f \rangle p
\end{aligned}
\tag{8}
$$

which is not $\mathrm{id}_{\mathrm{Map}(f)}$ in general, hence k is in the wrong fiber over $q$. Nevertheless, using Lemma 10, we can "transport" it in the right one using a "path in $\mathrm{Map}(f)$ from $qk$ to $\mathrm{id}_{\mathrm{Map}(f)}$", i.e. an arrow $\psi \colon \mathrm{Map}(f) \to \mathrm{Path}(p)$ such that $\partial_p \psi = \langle qk, \mathrm{id}_{\mathrm{Map}(f)} \rangle$. To obtain such an arrow we use the stability of path objects stated in Definition 7.

First of all, take a diagonal filler $\varphi\colon \mathrm{Path}(Y) \to \mathrm{Path}(\partial_Y^0)$ of the square in

$$
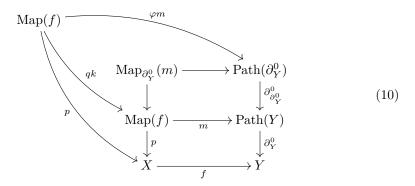\begin{array}{ccc}
Y & \xrightarrow{\;\;\mathrm{r}_{\partial_Y^0}\,\mathrm{r}_Y\;\;} & \mathrm{Path}(\partial_Y^0) \\[2pt]
{\scriptstyle \mathrm{r}_Y}\downarrow & \nearrow{\scriptstyle \varphi} & \downarrow{\scriptstyle \partial_{\partial_Y^0}} \\[2pt]
\mathrm{Path}(Y) & \xrightarrow[\langle \mathrm{r}_Y\,\partial_Y^0,\,\mathrm{id}_{\mathrm{Path}(Y)}\rangle]{} & \mathrm{Path}(Y)\times_{\partial_Y^0}\mathrm{Path}(Y)
\end{array}
\tag{9}
$$

Observe then that commutativity of the lower triangle in (9), of (3) and Eq. (8), entail the commutativity of the following diagram

$$
\tag{10}
$$

where both the upper and the lower squares define mapping path objects. Therefore there exists the universal arrow $\langle qk, \varphi m\rangle\colon \mathrm{Map}(f) \to \mathrm{Map}_{\partial_Y^0}(m)$.

Now, Definition 7 applied to $\partial_Y^0$ and $f$ (i.e. to the lower square in (10)), yields an arrow $i\colon \mathrm{Map}_{\partial_Y^0}(m) \to \mathrm{Path}(p)$. Therefore, we can define

$$
\psi := i\langle qk, \varphi m\rangle\colon \mathrm{Map}(f) \to \mathrm{Map}_{\partial_Y^0}(m) \to \mathrm{Path}(p).
$$

This is the path we are looking for. Indeed, commutativity of the lower square in (1), that one of (10) and of (9), in order, imply

$$
\begin{aligned}
\partial_p \psi &= \langle \mathrm{pm}_0^{\partial_Y^0,m}, \langle p\mathrm{pm}_0^{\partial_Y^0,m}, \partial_{\partial_Y^0}^1 \mathrm{pm}_1^{\partial_Y^0,m}\rangle\rangle\langle qk,\varphi m\rangle \\
&= \langle qk, \langle p, \partial_{\partial_Y^0}^1 \varphi m\rangle\rangle \\
&= \langle qk, \mathrm{id}_{\mathrm{Map}(f)}\rangle.
\end{aligned}
\tag{11}
$$

We are now in a position to apply Lemma 10 to the arrows $q\colon E \to \mathrm{Map}(f)$, $k\colon \mathrm{Map}(f) \to E$ and $\psi\colon \mathrm{Map}(f) \to \mathrm{Path}(p)$, thus obtaining

$$
\psi_! k\colon \mathrm{Map}(f) \to E.
$$

Let us prove that $\psi_! k$ is a diagonal filler for the square (5). The commutativity of the lower triangle follows immediately from point *(i)* of Lemma 10 and from Eq. (11). To prove the commutativity of the upper one, observe that Eq. (8), commutativity of (9), of (5) and of the upper square in (1), entail

$$
\begin{aligned}
\psi\langle \mathrm{id}_X, \mathrm{r}_Y f\rangle &= i\langle qg, \mathrm{r}_{\partial_Y^0}\,\mathrm{r}_Y f\rangle \\
&= i\langle \mathrm{id}_{\mathrm{Map}(f)}, \mathrm{r}_{\partial_Y^0}\,m\rangle qg \\
&= \mathrm{r}_p qg,
\end{aligned}
$$

that is, the commutativity of

$$
\begin{array}{ccc}
X & \xrightarrow{\langle \mathrm{id}_X, \mathrm{r}_Y f \rangle} & \mathrm{Map}(f) \\
{\scriptstyle qg}\downarrow & & \downarrow{\scriptstyle \psi} \\
\mathrm{Map}(f) & \xrightarrow[\mathrm{r}_p]{} & \mathrm{Path}(p)
\end{array}
$$

Therefore, point *(ii)* of Lemma 10 and Eq. (7) imply

$$(\psi_! k)\langle \mathrm{id}_X, \mathrm{r}_Y f \rangle = k\langle \mathrm{id}_X, \mathrm{r}_Y f \rangle = g.$$

Hence $\langle \mathrm{id}_X, \mathrm{r}_Y f \rangle \boxdot \mathcal{A}$ and the proof is complete. $\qquad\square$

## 3 The syntactic category

We consider a version of Martin-Löf type theory [9], which we denote by $\mathbb{T}_0$, equipped the usual structural rules and with the only identity type as type former, whose rules are given in Table 1. For more details about the syntax of Martin-Löf type theory see [11]. We write $[x]b$ to denote the term $b$ with the variable $x$ bound and $b[^a/_x]$ $(B[^a/_x])$ to denote the term (type) obtained by substituting the term $a$ for all the free occurrences of the variable $x$ in $b$ (in $B$).

As usual, we assume that derivations given by all the rules we present can be made also relative to chains of assumptions, called contexts. Recall that a *context* $\Phi$ of length $n \geq 0$ is a list of declarations of the form

$$\Phi = (x_1 : A_1,\, x_2 : A_1, \ldots,\, x_n : A_n)$$

such that all the variables $x_1, \ldots, x_n$ are different and the following judgments are derivable.

$$
\begin{aligned}
& A_1 \ \mathsf{Type}, \\
& (x_1 : A_1)\ A_1 \ \mathsf{Type}, \\
& \quad \vdots \\
& (x_1 : A_1, \ldots, x_{n-1} : A_{n-1})\ A_n \ \mathsf{Type}.
\end{aligned}
\tag{12}
$$

When $n = 0$ we have the empty context (). We write $\Phi\ \mathsf{Cxt}$ to mean that $\Phi$ is a context, that is, as a shorthand for the judgments in (12). If the first judgment $A_1 \ \mathsf{Type}$ (and then all the others) depends on a context $\Gamma$, we write $(\Gamma)\,\Phi\ \mathsf{Cxt}$ and say that $\Phi$ is a *dependent context*. Notice that, given a dependent context $(\Gamma)\,\Phi\ \mathsf{Cxt}$, we then obtain $(\Gamma, \Phi)\ \mathsf{Cxt}$ by concatenating the judgments they stands for.

Since we are working in a theory with the only identity type, we assume an elimination rule, called parametric, which is stronger than that one presented in [11], for it allows the presence of a parametric context $\Theta$ within the derivation. This parametric elimination can be deduced from the usual one in presence of $\Pi$ types [5].

In addition to the rules in Table 1, there are also rules expressing the congruence of definitional equality with respect to the constants Id, refl and J, which we leave implicit. Instead, we explicitly state in Table 2 the rules for substituting terms for variables in such constants. As usual, stating these rules, we assume that no occurrence of free variables in $a$ becomes bound when substituting $a$ in another term, just renaming variables if needed. To increase readability, we have omitted from the premises of the last rule of Table 2 the judgment

$$(x : A, y_1 : B, y_2 : B, u : \mathrm{Id}_B(y_1, y_2), z : \Theta)\ C \ \mathsf{Type}.$$

Table 1: Derivation rules for identity types

Formation

$$\frac{A\ \mathsf{Type} \qquad a : A \qquad b : A}{\mathrm{Id}_A(a, b)\ \mathsf{Type}}$$

Introduction

$$\frac{A\ \mathsf{Type} \qquad a : A}{\mathrm{refl}_A(a) : \mathrm{Id}_A(a, a)}$$

Elimination

$$\frac{(x : A,\ y : A,\ u : \mathrm{Id}_A(x, y),\ z : \Theta)\ C\ \mathsf{Type}}{\quad p : \mathrm{Id}_A(a, b) \qquad (x : A,\ z : \Theta[^{x,x,\mathrm{refl}(x)}/_{x,y,u}])\ d : C[^{x,x,\mathrm{refl}(x),z}/_{x,y,u,z}] \quad}{(w : \Theta(a, b, p))\ \mathrm{J}(a, b, p, d, z) : C[^{a,b,p,w}/_{x,y,u,z}]}$$

Computation

$$\frac{(x : A,\ y : A,\ u : \mathrm{Id}_A(x, y),\ z : \Theta)\ C\ \mathsf{Type}}{\quad a : A \qquad (x : A,\ z : \Theta[^{x,x,\mathrm{refl}(x)}/_{x,y,u}])\ d : C[^{x,x,\mathrm{refl}(x),z}/_{x,y,u,z}] \quad}{(w : \Theta(a, a, \mathrm{refl}(a)))\ \mathrm{J}(a, a, \mathrm{refl}(a), d, w) = d[^{a,w}/_{x,z}] : C[^{a,a,\mathrm{refl}(a),w}/_{x,y,u,z}]}$$

Using the structural rules of Martin-Löf type theory it is possible to define a category, called syntactic category, whose construction we briefly recall. For details see [12].

Given two contexts $\Phi$ and $\Psi = (y_1 : B_1, \ldots, y_m : B_m)$, a *context morphism* from $\Phi$ to $\Psi$ in an m-tuple $(b_1, \ldots, b_m)$ such that all the following judgments are derivable.

$$(\Phi)\ b_1 : B_1,$$
$$(\Phi)\ b_2 : B_2[^{b_1}/_{y_1}],$$
$$\vdots$$
$$(\Phi)\ b_m : B_m[^{b_1, \ldots, b_{m-1}}/_{y_1, \ldots, y_{m-1}}].$$

We write $(\Phi)\ (b_1, \ldots, b_m) : \Psi$ to mean that $(b_1, \ldots, b_m)$ is a context morphism from $\Phi$ to $\Psi$.

An important class of context morphisms, called *dependent projections*, are those which drop some variables of a context, that is, context morphisms of the form

$$(\Phi, \Psi)\ (x_1, \ldots, x_n) : \Phi$$

where $\Psi$ is a context depending on $\Phi$, and $(x_1, \ldots, x_n)$ are the variables in $\Phi$.

We can then define an equivalence relation on contexts and context morphisms, which extends definitional equality, identifying two contexts (contexts morphisms) if

Table 2: Substitution rules for identity types

Id-substitution

$$\frac{(x:A)\ B\ \mathsf{Type} \qquad (x:A)\ b_1:B \qquad (x:A)\ b_2:B \qquad a:A}{\mathrm{Id}_B(b_1,b_2)[^a/_x] = \mathrm{Id}_{B[^a/_x]}(b_1[^a/_x], b_2[^a/_x])\ \mathsf{Type}}$$

refl-substitution

$$\frac{(x:A)\ B\ \mathsf{Type} \qquad (x:A)\ b:B \qquad a:A}{\mathrm{refl}_B(b)[^a/_x] = \mathrm{refl}_{B[^a/_x]}(b[^a/_x]) : \mathrm{Id}_{B[^a/_x]}(b[^a/_x], b[^a/_x])}$$

J-substitution

$$\frac{\begin{array}{c}(x:A)\ B\ \mathsf{Type} \quad (x:A)\ b_1:B \quad (x:A)\ b_2:B \quad (x:A)\ p:\mathrm{Id}_B(b_1,b_2) \\ a:A \qquad (x:A,y:B,z:\Theta[^{y,y,\mathrm{refl}(y)}/_{y_0,y_1,u}])\ d:C[^{y,y,\mathrm{refl}(y),z}/_{y_0,y_1,u,z}] \end{array}}{\begin{array}{c}(w:\Theta(a,b_1[^a/_x],b_2[^a/_x],p[^a/_x]))\ \mathrm{J}(b_1,b_2,p,[y,z]d,z)[^{a,w}/_{x,z}] = \\ = \mathrm{J}(b_1[^a/_x],b_2[^a/_x],p[^a/_x],[y,z](d[^a/_x]),w):C[^{a,b_1[^a/_x],b_2[^a/_x],p[^a/_x],w}/_{x,y_0,y_1,u,z}]\end{array}}$$

they coincide up to variable renaming. Objects and arrows of the *syntactic category* $\mathsf{C}(\mathbb{T})$ of a type theory $\mathbb{T}$ are equivalence classes of contexts and context morphisms respectively. Identity arrows are defined in the obvious way, whilst the composition of two arrows is defined by means of substitution within representatives of equivalence classes: if $(\Phi)\ a:\Theta$ and $(\Theta)\ b:\Psi$ are such representatives and $z_1,\dots,z_k$ are the free variables in the context $\Theta$, the composite arrow is the equivalence class of the context morphism given by the judgments

$$(\Phi)\ b_i[^{a_1,\dots,a_k}/_{z_0,\dots,z_k}] : B_i[^{b_1,\dots,b_{i-1}}/_{y_1,\dots,y_{i-1}}][^{a_1,\dots,a_k}/_{z_0,\dots,z_k}],$$

for $i = 1,\dots,m$, where $\Psi = (y_1:B_1,\dots,y_m:B_m)$, $a = (a_1,\dots,a_k)$ and $b = (b_1,\dots,b_m)$.

In what follows, we will not use equivalence classes but just identify two contexts (context morphisms) up to variable renaming. Also, since we will only work with contexts, and not with types, we denote the former with capital letters of Latin alphabet, instead of Greek one.

Gambino and Garner have proved in [3] that, exploiting the rules for the identity type, it is possible to endow the syntactic category $\mathsf{C}(\mathbb{T}_0)$ associated to the type theory $\mathbb{T}_0$ with a weak factorization system, called identity type weak factorization system.

Let $\mathcal{D}$ denotes the set of all the dependent projections in $\mathsf{C}(\mathbb{T}_0)$, the identity type weak factorization system is the pair $(\mathcal{L}, \mathcal{R})$ where $\mathcal{L} = {}^{\boxtimes}\mathcal{D}$ and $\mathcal{R} = \mathcal{L}^{\boxtimes}$. The Weak Orthogonality Axiom follows then from Lemma 3 whereas, given two contexts $X$ and $Y$, the factorization of a context morphism $(x:X)\ f:Y$ is

$$X \xrightarrow{(x,f,\mathrm{refl}(f))} (x:X,\ y:Y,\ u:\mathrm{Id}_Y(f,y)) \xrightarrow{\quad (y) \quad} Y \qquad (13)$$

where $\mathrm{Id}_Y(f,y)$ is the so-called identity context, which can be defined from the rules for identity types as shown in [4]. The right-hand arrow is obviously a dependent projections, the hard part of the proof is in proving that the left-hand one is in $\mathcal{L}$.

Let us now show that the pair $(\mathsf{C}(\mathbb{T}_0), \mathcal{D})$ is a tribe with stable path objects, and that the weak factorization system given by Theorem 8 is the identity type weak factorization system.

**Theorem 11.** *The pair $(\mathsf{C}(\mathbb{T}_0), \mathcal{D})$ is a tribe with stable path objects. In particular, the choice of path objects is given by identity contexts.*

*Proof.* First of all, the empty context () Cxt is a terminal object in $\mathsf{C}(\mathbb{T}_0)$, terminal arrows are dependent projections dropping all the variables of a context, whilst identity arrows are dependent projections which do not drop any variable.

Given a dependent projection $(x : X, y : E) \to (x : X)$ and a context morphism $t \colon X' \to X$, the following square is a pullback

$$
\begin{array}{ccc}
(x' : X',\, y' : E[^t/_x]) & \xrightarrow{\ (t,y')\ } & (x : X,\, y : E) \\
\big\downarrow & & \big\downarrow \\
(x' : X') & \xrightarrow{\quad t \quad} & (x : X)
\end{array}
\tag{14}
$$

where the two vertical arrows are dependent projections. Indeed, if $f \colon Z \to X'$ and $(g_0, g_1) \colon Z \to (X, E)$ are context morphisms such that $g_0 = tf$, the universal arrow is the context morphism $(f, g_1) \colon Z \to (X, E[^t/_x])$.

Finally, the composite of two dependent projections is obviously a dependent projection. Therefore, $(\mathsf{C}(\mathbb{T}_0), \mathcal{D})$ is a tribe.

We now show how to associate a path object to every dependent projection. Let

$$
p \colon (x : X,\, y : E) \to (x : X),
$$

be a dependent projection. The diagonal $\Delta_p$ is the context morphism

$$
(x, y, y) \colon (x : X, y : E) \to (x : X, y_1 : E, y_2 : E).
$$

Define

$$
\mathrm{Path}(p) := (x : X,\, y_1 : E,\, y_2 : E,\, u : \mathrm{Id}_E(y_1, y_2)),
$$

$$
\mathrm{r}_p := (x, y, y, \mathrm{refl}(y)) \qquad \text{and} \qquad \partial_p := (x, y_1, y_2),
$$

where $\mathrm{Id}_E(y_1, y_2)$ is the (dependent) identity context of the context $E$, for which one can derive rules analogous to those given in Table 1, as proved in [4]. These definitions yield of course a factorization of the diagonal, and $\partial_p \in \mathcal{D}$ follows from definition.

Let us prove that every left lifting problem for a pullback of $\mathrm{r}_p$ over $\mathcal{D}$ has a solution in $\mathsf{C}(\mathbb{T}_0)$. Let $(v : \mathrm{Path}(p), z : B) \to \mathrm{Path}(p)$ be a dependent projection (where we use $v : \mathrm{Path}(p)$ as a shorthand for $(x, y_1, y_2, u) : \mathrm{Path}(p)$) and

$$
\begin{array}{ccc}
A & \xrightarrow{\quad k \quad} & (v : \mathrm{Path}(p),\, z : B) \\
\big\downarrow & & \big\downarrow \\
(x : X,\, y : E) & \xrightarrow{\quad \mathrm{r}_p \quad} & \mathrm{Path}(p)
\end{array}
$$

a pullback. The commutativity of the square implies

$$
A = (x : X,\, y : E,\, z' : B[^{x,y,y,\mathrm{refl}(y)}/_v])
$$

and

$$
k = (\mathrm{r}_p, z') = (x, y, y, \mathrm{refl}(y), z').
$$

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Let us then consider the following lifting problem

$$(x : X, \, y : E, \, z' : B[^{x,y,y,\mathrm{refl}(y)}/_v]) \xrightarrow{\ d\ } (v : \mathrm{Path}(p), \, z : B, \, w : C)$$

$$\Big\downarrow{\scriptstyle k} \qquad\qquad\qquad\qquad\qquad\qquad \Big\downarrow \qquad\qquad\qquad (15)$$

$$(v : \mathrm{Path}(p), \, z : B) =\!=\!=\!=\!=\!=\!=\!= (v : \mathrm{Path}(p), \, z : B)$$

where the right arrow is the dependent projection dropping the variable $w$. Commutativity of (15) implies

$$d = (x, y, y, \mathrm{refl}(y), z', d_C),$$

for some term $d_C$ such that the judgment

$$(x : X, \, y : E(x), \, z' : B[^{x,y,y,\mathrm{refl}(y)}/_v]) \ d_C : C[^{x,y,y,\mathrm{refl}(y),z'}/_{v,z}].$$

holds. Applying the elimination for identity contexts to it, we obtain

$$\mathrm{J}(y_1, y_2, u, [y, z']d_C, z) : C$$

for $(x, y_1, y_2, u) : \mathrm{Path}(p)$ and $z : B$, and the computation rule yields

$$\mathrm{J}(y, y, \mathrm{refl}(y), [y, z']d_C, z') = d_C : C[^{x,y,y,\mathrm{refl}(y),z'}/_{v,z}].$$

for $x : X, \, y : E(x), \, z' : B[^{x,y,y,\mathrm{refl}(y)}/_v]$. Thus the context morphism

$$(x, y_1, y_2, u, z, \mathrm{J}(y_1, y_2, u, [y, z']d_C, z))$$

is a diagonal filler for the lifting problem (15): commutativity of the lower triangle is obvious whereas commutativity of the upper one follows from the computation rule. Therefore, the tribe $(\mathsf{C}(\mathbb{T}_0), \, \mathcal{D})$ has path objects.

To see that is is also stable in the sense of Definition 7, take a dependent projection $p \colon (y : Y, \, z : E) \to (y : Y)$ and a context morphism $f \colon X \to Y$. According to the choice for the pullback square in (14), we have

$$F = (x : X, \, z' : E[^f/_y]), \quad f^*p = (x), \quad \text{and} \quad p^*f = (f, z'),$$

the mapping path object of $p^*f$ along $p$ is

$$\mathrm{Map}_p(p^*f) = (x : X, \, z_1' : E[^f/_y], \, (z_2 : E, \, u' : \mathrm{Id}_E(z_1, z_2))[^{f,z_1'}/_{y,z_1}])$$
$$= (x : X, \, z_1' : E[^f/_y], \, z_2' : E[^f/_y], \, u' : (\mathrm{Id}_E(z_1, z_2))[^{f,z_1',z_2'}/_{y,z_1,z_2}])$$

and the two associated base change arrows are

$$\mathrm{pm}_0 = (x, z_1') \qquad \text{and} \qquad \mathrm{pm}_1 = (f, z_1', z_2', u').$$

Therefore, from the construction in the syntactic category of the universal arrow of a pullback, we have

$$\langle \mathrm{id}_F, \mathrm{r}_p p^*f \rangle = \langle (x, z'), (f, z', z', (\mathrm{refl}_E(z))[^{f,z'}/_{y,z}]) \rangle$$
$$= (x, z', z', (\mathrm{refl}_E(z))[^{f,z'}/_{y,z}])$$

and

$$\langle \mathrm{pm}_0, \langle f^*p\mathrm{pm}_0, \partial_p^1 \mathrm{pm}_1 \rangle \rangle = \langle (x, z_1'), \langle (x), (f, z_2') \rangle \rangle$$
$$= \langle (x, z_1'), (x, z_2') \rangle$$
$$= (x, z_1', z_2')$$

On the other hand, the factorization of $\Delta_{f^*p}$ is given by

$$\mathrm{Path}(f^*p) = (x : X,\, z_1' : E[^f/_y],\, z_2' : E[^f/_y],\, u' : \mathrm{Id}_{E[^f/_y]}(z_1', z_2'))$$

$$\mathrm{r}_{f^*p} = (x, z', z', \mathrm{refl}_{E[^f/_y]}(z')) \qquad \text{and} \qquad \partial_{f^*p} = (x, z_1', z_2')$$

and the substitution rules for identity types ensure

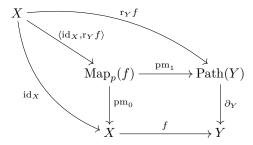$$(\mathrm{Id}_E(z_1, z_2))[^{f, z_1', z_2'}/_{y, z_1, z_2}] = \mathrm{Id}_{E[^f/_y]}(z_1', z_2')$$

and

$$(\mathrm{refl}_E(z))[^{f, z'}/_{y, z}] = \mathrm{refl}_{E[^f/_y]}(z').$$

Therefore $\mathrm{Map}_p(p^*f) = \mathrm{Path}(f^*p)$ and it suffices to take the identity context morphism as the arrow $i$ in order to make the diagram (1) commute. $\qquad\square$

The third substitution rule for identity types, which we did not use here, is needed in the case a choice of diagonal filler is assumed in the definition of tribe with path objects and such choice required to be stable under pullback in the definition of tribe with stable path objects. We did so in [2], also assuming a stronger notion of stability under pullback of the structure given by path objects. This assumption amounts, in the current formulation, to requiring the arrow $i$ of Definition 7 to be in isomorphism.

Let us conclude this section by showing that the weak factorization system given by Theorem 8 is precisely the identity type weak factorization system. The two classes of arrows $\mathcal{L}$ and $\mathcal{R}$ are obviously the same. Consider then a context morphism $f \colon X \to Y$. The factorization in Lemma 9 is obtained through the mapping path object of $f$ over the terminal arrow $Y \to ()$:



where the mapping path object is the context

$$\mathrm{Map}_p(f) = (x : X, y : Y, u : \mathrm{Id}_Y(f, y))$$

because of the choice of pullback and the first substitution rule for identity types.

The arrow $\partial_Y^1 \mathrm{pm}_1$ is the dependent projection

$$(y) \colon (x : X, y : Y, u : \mathrm{Id}_Y(f, y)) \to Y$$

and the universal arrow $\langle \mathrm{id}_X, \mathrm{r}_Y f \rangle \colon X \to \mathrm{Map}_p(f)$ is

$$\langle (x), (f, f, \mathrm{refl}_Y(f)) \rangle = (x, f, \mathrm{refl}_Y(f)).$$

We have then obtained exactly the factorization of $f$ described in (13). Thus, the weak factorization system on $(\mathsf{C}(\mathbb{T}_0), \mathcal{D})$ given by Theorem 8 is the identity type weak factorization system.

# 4   Small groupoids

In this section we provide another application of Theorem 8, showing that the category of small groupoids is a tribe which can be endowed with stable path objects. The class $\mathcal{A}$ of arrows of the tribe is given by the so-called Grothendieck fibrations, which we consider with some additional properties in order to prove Lemma 15. We conclude this section exhibiting the factorization of a functor between groupoids, which turns out to be that one given by the weak factorization system $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ of the Quillen model structure on $\mathsf{Gpd}$ mentioned in [1] and [8].

Recall that a groupoid is a category in which every arrow is invertible. We denote by $\mathsf{Gpd}$ the category of small groupoids and functors between them. We write $A \cong B$ if $A$ and $B$ are isomorphic groupoids, denote by $\mathrm{id}_X$ the identity functor of a groupoid, and by $1_x$ the identity arrow of an object $x \in X$. Since we will only consider small groupoids, from now on we omit that adjective.

**Definition 12.** Let $p\colon E \to X$ be a functor between groupoids and $\gamma\colon x \to x'$ an arrow in $X$. A *lifting of $\gamma$ along $p$* is an arrow $\varepsilon\colon e \to e'$ in $E$ such that $p(\varepsilon) = \gamma$. $p$ is a *Grothendieck fibration* if every arrow in $X$ has a lifting along $p$.

A *cloven fibration* is a Grothendieck fibration $p\colon E \to X$ with an operation which assigns to every arrow $\gamma\colon p(e) \to x$ in $X$ a lifting of $\gamma$ along $p$, denoted by $p^{-1}\gamma\colon e \to p^{-1}x$, such that it is stable under pullback and composition, i.e. $(pq)^{-1}\gamma = q^{-1}p^{-1}\gamma$ and $(f^*p)^{-1}\gamma = (\gamma, p^{-1}f(\gamma))$. A cloven fibration is *normal* if $p^{-1}1_{p(a)} = 1_a$.

Let $\mathcal{G}$ be the class of all normal cloven fibrations in $\mathsf{Gpd}$.

**Lemma 13.** *The pair $(\mathsf{Gpd}, \mathcal{G})$ is a tribe.*

*Proof.* The terminal object in $\mathsf{Gpd}$ is given by the groupoid with one object and the only identity arrow. Terminal and identity arrows are trivially normal cloven fibrations.

Since $\mathsf{Gpd}$ has arbitrary pullbacks, in particular it has pullbacks along arrows in $\mathcal{G}$. Let $f\colon A \to X$ and $g\colon B \to X$ be two arrows in $\mathsf{Gpd}$. A choice for the pullback object is given by the groupoid $A \times_X B$, whose objects are pairs $(a, b)$ such that

$$a \in A, \quad b \in B \quad \text{and} \quad f(a) = g(b),$$

and whose arrows $(a, b) \to (a', b')$ are pairs $(\alpha, \beta)$ such that

$$\alpha\colon a \to a', \quad \beta\colon b \to b' \quad \text{and} \quad f(\alpha) = g(\beta).$$

Identities and composition are defined componentwise. The two base change arrows $f^*g$ and $g^*f$ are the projections from $A \times_X B$ onto $A$ and $B$, respectively.

Grothendieck fibrations are closed under composition and base change, and the stability requirement on the choice of lifting ensures that the class of normal cloven fibrations is closed under such operations as well. $\qquad\square$

**Theorem 14.** *The pair $(\mathsf{Gpd}, \mathcal{G})$ is a tribe with stable path objects.*

*Proof.* Let $p\colon A \to X$ be a normal cloven fibration and define $\mathrm{Path}(p)$ to be the groupoid whose objects are the triplets $(a, b, \alpha)$ such that $\alpha\colon a \to b$ is an arrow in $A$,

$$p(a) = p(b) \qquad \text{and} \qquad p(\alpha) = 1_{p(a)},$$

and whose arrows from $(a, b, \alpha)$ to $(a', b', \alpha')$ are pairs $(\bar{\alpha}, \bar{\beta})$ of arrows in $A$ such that the square

$$
\begin{array}{ccc}
a & \xrightarrow{\;\alpha\;} & b \\
{\scriptstyle\bar{\alpha}}\big\downarrow & & \big\downarrow{\scriptstyle\bar{\beta}} \\
a' & \xrightarrow{\;\alpha'\;} & b'
\end{array}
\tag{16}
$$

15

commutes. Identity arrows and composition are defined componentwise.

Define then, for $a \in A$ and $\alpha \in \mathrm{Ar}(A)$,

$$\mathrm{r}_p(a) := (a, a, 1_a) \qquad \text{and} \qquad \mathrm{r}_p(\alpha) := (\alpha, \alpha),$$

and $\partial_p \colon \mathrm{Id}_p(A) \to A \times_X A$ by

$$(a, b, \alpha) \longmapsto (a, b)$$

$$(\bar{\alpha}, \bar{\beta}) \colon (a, b, \alpha) \to (a', b', \alpha') \longmapsto (\bar{\alpha}, \bar{\beta}) \colon (a, b) \to (a', b').$$

While functoriality of both $\mathrm{r}_p$ and $\partial_p$ is straightforward from the definitions, we must verify that $\partial_p$ is well-defined on arrows, i.e. that $p(\bar{\alpha}) = p(\bar{\beta})$. But this equality holds because $p(\alpha) = 1_{p(a)}$, $p(\alpha') = 1_{p(a')}$, $p$ is a functor and the square (16) commutes.

The diagonal $\Delta_p \colon A \to A \times_X A$ is the functor sending an object $a \in A$ in $(a, a) \in A \times_X A$ and an arrow $\alpha$ in $(\alpha, \alpha)$, therefore the functors $\mathrm{r}_p$ and $\partial_p$ yield obviously a factorization of it.

Let us now prove that $\partial_p$ is a Grothendieck fibration and that we can endow it with a choice of liftings which is normal. Let $(\bar{\alpha}, \bar{\beta}) \colon \partial_p(a, b, \alpha) \to (a', b')$ be an arrow in $A \times_X A$, in order to obtain a lifting along $\partial_p$ we need to define an arrow $\alpha' \colon a' \to b'$ such that the square (16) commutes. Thus we simply take $\bar{\beta}\alpha\bar{\alpha}^{-1}$ and define

$$\partial_p^{-1}(\bar{\alpha}, \bar{\beta}) := (\bar{\alpha}, \bar{\beta}) \colon (a, b, \alpha) \to (a', b', \bar{\beta}\alpha\bar{\alpha}^{-1}).$$

This choice of a lifting is of course normal, hence $\partial_p$ is a normal cloven fibration. Furthermore, point ($b$) of Definition 6 follows from lemmas 15 and 4, therefore $(\mathsf{Gpd}, \mathcal{G})$ is a tribe with path objects.

To see that our choice of path objects is also stable, let $p \colon E \to Y$ be a normal cloven fibration, $f \colon X \to Y$ a functor between groupoids and consider the following pullback square

$$
\begin{array}{ccc}
X \times_Y E & \longrightarrow & E \\
\downarrow & & \downarrow{\scriptstyle p} \\
X & \underset{f}{\longrightarrow} & Y
\end{array}
\tag{17}
$$

The objects of the groupoid $\mathrm{Path}(f^*p)$ are the triplets $((x_0, e_1), (x_1, e_1), (\gamma, \varepsilon))$ such that $x_0 = f^*p(x_0, e_0) = f^*p(x_1, e_1) = x_1$ and $\gamma = f^*p(\gamma, \varepsilon) = 1_{x_0}$, and similarly for the arrows. Hence $\mathrm{Path}(f^*p) \cong X \times_Y \mathrm{Path}(p)$.

On the other hand, by the choice of the pullback object, we have $\mathrm{Map}_p(p^*f) = (X \times_Y E) \times_E \mathrm{Path}(p)$, which is isomorphic to $X \times_Y \mathrm{Path}(p)$ as well. Therefore, it suffices to define $i \colon \mathrm{Map}_p(p^*f) \to \mathrm{Path}(f^*p)$ as the isomorphism defined by

$$((x, e), (e, e', \varepsilon)) \mapsto ((x, e), (x, e'), (1_x, \varepsilon))$$

on objects and by

$$((\zeta, \eta), (\eta, eta')) \mapsto ((\zeta, \eta), (\zeta, \eta'))$$

on arrows. The commutativity of the diagram (1) is then straightforward. $\qquad \square$

**Lemma 15.** *Let* $p \colon A \to X$ *and* $q \colon B \to \mathrm{Path}(p)$ *be normal cloven fibrations and let*

$$
\begin{array}{ccc}
C & \xrightarrow{\ \pi_1\ } & B \\
\downarrow{\scriptstyle \pi_0} & & \downarrow{\scriptstyle q} \\
A & \xrightarrow{\ \mathrm{r}_p\ } & \mathrm{Path}(p)
\end{array}
$$

*be a pullback. Every left lifting problem*

$$
\begin{array}{ccc}
C & \xrightarrow{\;f\;} & E \\
{\scriptstyle \pi_0}\big\downarrow & & \big\downarrow{\scriptstyle \bar{q}} \\
B & =\!=\!= & B
\end{array}
\qquad (18)
$$

*has a diagonal filler if $\bar{q} \in \mathcal{G}$.*

*Proof.* We may suppose, up to isomorphism, $C = A \times_{\mathrm{Path}(p)} B$, $\pi_0 = \mathrm{r}_p^* q$ and $\pi_1 = q^* \mathrm{r}_p$. We begin defining a diagonal filler $j$ on objects of $B$. Let $b \in B$ and $(a, a', \alpha) = q(b) \in \mathrm{Path}(p)$, the domain of the arrow

$$
(1_a, \alpha^{-1}) \colon (a, a', \alpha) \to (a, a, 1_a).
$$

is image of $q$, therefore we can lift it along $q$ obtaining

$$
\gamma := q^{-1}(1_a, \alpha^{-1}) \colon b \to b_0,
$$

where $b_0 := q^{-1}(a, a, 1_a)$. In particular, $(a, b_0) \in C$. Again, since $b_0 = \pi_1(a, b_0) = \bar{q}f(a, b_0)$, the domain of $\gamma^{-1}$ is image of $\bar{q}$ and we can lift it along $\bar{q}$. Thus, we define $j(b) := \bar{q}^{-1}b$.

We now define $j$ on the arrows of $B$. Let $\beta \colon b \to b'$ be an arrow in $B$ and $(\alpha_0, \alpha_1) := q(\beta) \colon (a_0, a_1, \alpha) \to (a_0', a_1', \alpha')$ its image under $q$. Let us consider the following commutative square

$$
\begin{array}{ccc}
(a_0, a_1, \alpha) & \xrightarrow{\;(\alpha_0, \alpha_1)\;} & (a_0', a_1', \alpha') \\
{\scriptstyle (1_{a_0}, \alpha^{-1})}\big\downarrow & & \big\downarrow{\scriptstyle (1_{a_0'}, \alpha'^{-1})} \\
(a_0, a_0, 1_{a_0}) & \xrightarrow{\;(\alpha_0, \alpha_0)\;} & (a_0', a_0', 1_{a_0'})
\end{array}
\qquad (19)
$$

As in the object case, we have liftings of the vertical arrows in (19)

$$
\gamma_0 := q^{-1}(1_{a_0}, \alpha^{-1}) \colon b \to b_0 \qquad \text{and} \qquad \gamma_1 := q^{-1}(1_{a_0'}, \alpha'^{-1}) \colon b' \to b_1,
$$

where $b_0 := q^{-1}\mathrm{r}_p(a_0)$ and $b_1 := q^{-1}\mathrm{r}_p(a_0')$. In particular

$$
(\alpha_0, \gamma_1 \beta \gamma_0^{-1}) \colon (a_0, b_0) \to (a_0', b_1)
$$

is an arrow in C. Again, since $b_0 = \pi_1(a_0, b_0) = \bar{q}f(a_0, b_0)$ and, similarly, $b_1 = \bar{q}f(a_0', b_1)$, we can lift $\gamma_0^{-1}$ and $\gamma_1^{-1}$ along $\bar{q}$ obtaining

$$
\eta_0 := \bar{q}^{-1}\gamma_0^{-1} \colon f(a_0, b_0) \to c_0 \qquad \text{and} \qquad \eta_1 := \bar{q}^{-1}\gamma_1^{-1} \colon f(a_0', b_1) \to c_1,
$$

where $c_0 = \bar{q}^{-1}b$ and $c_1 = \bar{q}^{-1}b'$. Then, we define

$$
j(\beta) := \eta_1 f(\alpha_0, \gamma_1 \beta \gamma_0^{-1}) \eta_0^{-1},
$$

We now verify that $j$ is a functor. Functoriality on identity arrows follows from functoriality of $f$, once we observe that, when $\beta = 1_b$, we have $\alpha_0 = 1_{a_0}$ and the uniqueness of lifting implies $\gamma_0 = \gamma_1$ and $\eta_0 = \eta_1$.

Let then $\beta \colon b \to b'$, $\beta' \colon b' \to b''$ be two arrows in $B$ and

$$
\begin{aligned}
(\alpha_0, \alpha_1) &= q(\beta) \colon (a_0, a_1, \alpha) \to (a_0', a_1', \alpha'), \\
(\alpha_0', \alpha_1') &= q(\beta') \colon (a_0', a_1', \alpha') \to (a_0'', a_1'', \alpha''),
\end{aligned}
$$

their images under $q$. According to the construction of $j$, we have $b_0, b_1, b_2 \in B$ such that $q(b_0) = (a_0, a_0, 1_{a_0})$, $q(b_1) = (a_0', a_0', 1_{a_0'})$, $q(b_2) = (a_0'', a_0'', 1_{a_0''})$ and

$$\gamma_0 := q^{-1}(1_{a_0}, \alpha^{-1}) \colon b \to b_0,$$
$$\gamma_1 := q^{-1}(1_{a_0'}, \alpha'^{-1}) \colon b' \to b_1,$$
$$\gamma_2 := q^{-1}(1_{a_0''}, \alpha''^{-1}) \colon b'' \to b_2.$$

In particular $(a_0, b_0) \in C$ and $b_0 = \bar{q}f(a_0, b_0)$, and similarly for $(a_0', b_1)$ and $(a_0'', b_2)$. Again, we have $c_0, c_1, c_2 \in E$ such that $\bar{q}(c_0) = b$, $\bar{q}(c_1) = b'$, $\bar{q}(c_2) = b''$ and

$$\eta_0 := \bar{q}^{-1}\gamma_0^{-1} \colon f(a_0, b_0) \to c_0,$$
$$\eta_1 := \bar{q}^{-1}\gamma_1^{-1} \colon f(a_0', b_1) \to c_1,$$
$$\eta_2 := \bar{q}^{-1}\gamma_2^{-1} \colon f(a_0'', b_2) \to c_2.$$

Therefore, functoriality of $f$ implies

$$j(\beta')j(\beta) = (\eta_2 f(\alpha_0', \gamma_2\beta'\gamma_1^{-1})\eta_1^{-1})(\eta_1 f(\alpha_0, \gamma_1\beta\gamma_0^{-1})\eta_0^{-1})$$
$$= \eta_2 f(\alpha_0'\alpha_0, \gamma_2\beta'\beta\gamma_0^{-1})\eta_0^{-1}$$
$$= j(\beta'\beta)$$

Let us observe that, to prove functoriality of $j$, we have extensively used the choice of a unique lifting, but not the normality of that choice.

We now verify that the functor $j$ is indeed a diagonal filler for (18). Let $\beta$ be an arrow in $B$, from functoriality of $\bar{q}$ and from commutativity of (18) we have

$$\bar{q}j(\beta) = \bar{q}(\eta_1)\bar{q}f(\alpha_0, \gamma_1\beta\gamma_0^{-1})\bar{q}(\eta_0^{-1})$$
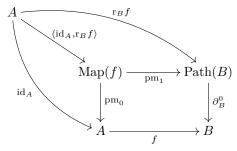$$= \gamma_1^{-1}\gamma_1\beta\gamma_0^{-1}(\gamma_0^{-1})^{-1}$$
$$= \beta$$

that is, commutativity of the lower triangle. In order to prove commutativity of the upper one, we use the fact that every identity lifts to an identity. Let $(\alpha, \beta)$ be an arrow in $C$. We have $q\pi_1(\alpha, \beta) = \mathrm{r}_p(\alpha) = (\alpha, \alpha) \colon (a, a, 1_a) \to (a', a', 1_{a'})$, hence the vertical arrows in (19) are identities, their liftings $\gamma_0$ and $\gamma_1$ are so, and $\eta_0$ and $\eta_1$ are identity arrows too. Thus

$$j\pi_1(\alpha, \beta) = f(\alpha, \beta)$$

follows from the definition of $j$. $\qquad\square$

Let us now show the factorization on $\mathsf{Gpd}$ obtained from Lemma 9. Given a functor $f \colon A \to B$, it is constructed through the mapping path object for $f$ over the terminal arrow $B \to \mathbf{1}$:



by $f = \partial_B^1 \mathrm{pm}_1 \langle \mathrm{id}_A, \mathrm{r}_B f\rangle$.

We can regard objects in $\mathrm{Map}(f)$ as triplets $(a, b, \gamma)$ such that

$$a \in A, \quad b \in B, \quad \gamma \colon f(a) \to b,$$

and arrows $(a, b, \gamma) \to (a', b', \gamma')$ in $\mathrm{Map}(f)$ as pairs $(\alpha, \beta)$ such that $\alpha \colon a \to a'$ is in $A$, $\beta \colon b \to b'$ is in $B$ and

$$
\begin{array}{ccc}
f(a) & \xrightarrow{f(\alpha)} & f(a') \\
\gamma \downarrow & & \downarrow \gamma' \\
b & \xrightarrow{\beta} & b'
\end{array}
$$

commutes. The normal cloven fibration $\partial_B^1 \mathrm{pm}_1 \colon \mathrm{Map}(f) \to B$ is the projection onto the second component, and lifts an arrow

$$
\beta \colon \partial_B^1 \mathrm{pm}_1(a, b, \gamma) \to b'
$$

in $B$ to the arrow $(1_a, \beta) \colon (a, b, \gamma) \to (a, b', \beta\gamma)$ in $\mathrm{Map}(f)$. Finally, the universal arrow $\langle \mathrm{id}_A, \mathrm{r}_B f \rangle$ is defined by

$$
a \longmapsto (a, f(a), 1_{f(a)}) \qquad \text{and} \qquad \alpha \longmapsto (\alpha, f(\alpha)).
$$

This is the factorization of functors between groupoids given by the weak factorization system $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ of the Quillen model structure on $\mathsf{Gpd}$ presented in [1] and [8].

# 5 Topological spaces

In this section we begin recalling some basic definitions about homotopies and paths in topological spaces. We then show that the category $\mathsf{Top}$ of topological spaces and continuous functions (maps) between them is a tribe with stable path objects whose class $\mathcal{A}$ is the class of Hurewicz fibrations satisfying a normality requirement similar to that one assumed in the previous section for Grothendieck fibrations, which is used to prove Lemma 20. We conclude the section by showing that the factorization of a map given by the path object structure is the same as that one given by the weak factorization system $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ of the Quillen model structure on $\mathsf{Top}$ showed by Strøm [13].

Let $f, g \colon X \to Y$ be two maps between topological spaces, a *homotopy* between $f$ and $g$ is a map $H \colon X \times I \to Y$, where $I$ is the unit real interval, such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. If $f|_A = g|_A$ for a subspace $A \subset X$, a homotopy $H$ between $f$ and $g$ is *relative to $A$* if $H(x, t) = f(x)$ for every $x$ in $A$ and $t$ in $I$.

Let $X$ be a topological space, every map $f \colon I \to X$ will be regarded as a *path in $X$* from $f(0)$ to $f(1)$ (also written $f(0) \rightsquigarrow f(1)$). The *path space $X^I$* is the set of all maps $I \to X$ equipped with the compact-open topology, that is, the topology generated by the subbase

$$
U(K, V) = \{ f \in X^I \mid f(K) \subset V \}
$$

for $K \subset I$ compact and $V \subset X$ open.

We denote by $\mathrm{u}_X \colon X \to X^I$ the map $x \mapsto 1_x$, where $1_x$ is the constant path over $x \in X$, and by $\partial_X \colon X^I \to X \times X$ the map $f \mapsto (f(0), f(1))$, and call them the *unit map* and the *boundary map* of $X$, respectively. We shall also write $\partial_X^0$ and $\partial_X^1$ for the first and second component of $\partial_X$.

Given two paths $f$ and $g$ in $X$ such that $f(0) = g(0)$ and $f(1) = g(1)$, we write $f \simeq g$ if $f$ and $g$ are *homotopic relative to endpoints*, i.e. if there exists a homotopy between $f$ and $g$ relative to the discrete subspace $\{0, 1\} \subset I$. The relation $\simeq$ is an equivalence relation on the path space $X^I$.

Observe that every homotopy $I^2 \to X$ defines a path $I \to X^I$, and vice versa. In particular, the compact-open topology ensures that this correspondence is a homeomorphism between the spaces $X^{I^2}$ and $X^{I^I}$.

In this section, we shall denote the composite map by $f \circ g$ instead of $fg$.

**Definition 16.** A map $p\colon E \to B$ is a *Hurewicz fibration* if it has the homotopy lifting property, that is, if every commutative square

$$
\begin{array}{ccc}
X \times \{0\} & \longrightarrow & E \\
{\scriptstyle \mathrm{id}_X \times 0}\big\downarrow & & \big\downarrow{\scriptstyle p} \\
X \times I & \longrightarrow & B
\end{array}
$$

has a diagonal filler $H\colon X \times I \to E$, called lifting of the homotopy $K\colon X \times I \to B$. For every topological space $X$, we will write $X \hookrightarrow X \times I$ for the inclusion $\mathrm{id}_X \times 0$.

We say that a Hurewicz fibration is *normal* if it lifts constant paths to constant paths, i.e. if, for some $x \in X$, $K(x,t) = K(x,0)$ for all $t \in I$, then $H(x,t) = H(x,0)$ for all $t \in I$. In particular, a normal Hurewicz fibration lifts a constant homotopy to a constant homotopy.

We denote by $\mathcal{H}$ the class of all normal Hurewicz fibrations.

**Definition 17.** Let $p\colon E \to B$ be a normal Hurewicz fibration. A path $\varepsilon \in E^I$ is *h-vertical* (resp. *vertical*) *over* $p$, if $p \circ \varepsilon \simeq 1_{p \circ \varepsilon(0)}$ (resp. $p \circ \varepsilon = 1_{p \circ \varepsilon(0)}$).

The subspace of $E^I$ given by all *h*-vertical paths over $p$ is called the *path space of E over p*:

$$
E_p^I := \big\{\, \varepsilon \in E^I \;\big|\; p \circ \varepsilon \simeq 1_{p \circ \varepsilon(0)} \,\big\}
$$

Of course, $\mathrm{u}_E(e) \in E_p^I$ for every $e \in E$ and every $p \in \mathcal{H}$, let then

$$
\mathrm{u}_p \colon E \to E_p^I
$$

denote the unit map $\mathrm{u}_E$ with codomain restricted to the subspace $E_p^I$. Similarly, let $\partial_p$ be the restriction of $\partial_E$ to the subspace $E_p^I$. Since $\varepsilon$ is *h*-vertical over $p$, we have $p \circ \varepsilon(0) = p \circ \varepsilon(1)$, hence

$$
\partial_p \colon E_p^I \to E \times_p E.
$$

Notice that all the identity and terminal arrows are normal Hurewicz fibrations. In particular, if $p\colon E \to \mathbf{1}$ is a terminal arrow, we have again $E_p^I = E^I$, $\mathrm{u}_p = \mathrm{u}_E$ and $\partial_p = \partial_E$.

We now prove some lemmas, which are instrumental in the proof of Theorem 21, which asserts that $(\mathsf{Top}, \mathcal{H})$ is a tribe with stable path objects.

**Lemma 18.** *Let $p\colon E \to B$ be a normal Hurewicz fibration, then $\partial_p \in \mathcal{H}$.*

*Proof.* We need to define a diagonal filler for every commutative square like

$$
\begin{array}{ccc}
X & \overset{\varepsilon}{\longrightarrow} & E_p^I \\
\big\uparrow{\scriptstyle\cap} & & \big\downarrow{\scriptstyle \partial_p} \\
X \times I & \underset{\langle \eta_0, \eta_1 \rangle}{\longrightarrow} & E \times_p E
\end{array}
\qquad (20)
$$

In order to do so, we need to sightly modify the classical construction of the lifting by means of a retraction $I^2 \to J$, where $J := I \times \{0,1\} \cup \{0\} \times I \subset I^2$, because of the normality requirement.

For $x \in X$ and $i = 0, 1$, define

$$
l_i(x) := \min \big\{\, t \in I \mid (\forall t' \ge t)\, \eta_i(x, t') = \eta_i(x, 1) \,\big\}
$$

(it is easy to see that, for a fixed $x$, the set involved in this definition is a non-empty closed subinterval of $I$), and

$$
l(x) := \max(l_0(x), l_1(x)).
$$

Then $l \colon X \to I$ is a continuous function vanishing on $x$ if and only if both $\eta_0(x)$ and $\eta_1(x)$ are constant paths.

This is the function we will use to fulfil normality. Indeed, since $l(x)$ is the minimum number such that both $\eta_0(x)$ and $\eta_1(x)$ are constant on $[l(x), 1]$ we can, roughly speaking, use $l$ to reparametrize $\eta_0(x)$ and $\eta_1(x)$ cutting away all the points of $I$ on which both those paths are constant, thus obtaining two "paths of length $l(x)$". In this way, if both $\eta_0(x)$ and $\eta_1(x)$ are constant, we then obtain two "paths of length 0", that is, just two points.

Let now $m \colon X \times I \to X \times I$ be the minimum between $t \in I$ and $l(x)$:

$$m(x, t) := (x, \min(t, l(x))),$$

and let $r \colon I^2 \to J$ be a retraction of $I^2$ onto $J$.

Finally, define $g \colon X \times J \to E$ by

$$g(x, t, s) := \begin{cases} \eta_0(x, t), & \text{if } s = 0 \\ \varepsilon(x)(s), & \text{if } t = 0 \\ \eta_1(x, t), & \text{if } s = 1 \end{cases}$$

and

$$h := g(\mathrm{id}_X \times r)(m \times \mathrm{id}_I) \colon X \times I^2 \to X \times J \to E$$

We now prove that the map $H \colon X \times I \to E^I$ given by

$$H(x, t)(s) := h(x, t, s),$$

is a diagonal filler for the square (20). We begin proving that $H$ is a diagonal filler for the square with $\partial_E \colon E^I \to E \times E$ in place of $\partial_p \colon E_p^I \to E \times_p E$, the general result will then follow from $\partial_p = \partial_E|_{E_p^I}$, once we have proved that the codomain of $H$ is indeed $E_p^I$. From the definition of $l$ we have

$$\min(0, l(x)) = 0 \qquad \text{and} \qquad \eta_i(m(x, t)) = \eta_i(x, t) \tag{21}$$

for $i = 0, 1$, where the second equality holds because $\eta_i(x, t) = \eta_i(x, l(x))$ for every $t \geq l(x)$. Commutativity of both the triangles then follows from (21), $r|_J = \mathrm{id}_J$ and the definition of $g$:

$$\begin{aligned} H(x, 0)(s) &= g(\mathrm{id}_X \times r)(x, 0, s) = g(x, 0, s) \\ &= \varepsilon(x)(s), \end{aligned} \tag{22}$$

$$\begin{aligned} H(x, t)(i) &= g(\mathrm{id}_X \times r)(m(x, t), i) = g(m(x, t), i) \\ &= \eta_i(x, t). \end{aligned} \tag{23}$$

for $i = 0, 1$ and $t, s \in I$.

In order to prove $H(x, t) \in E_p^I$ for $(x, t) \in X \times I$, observe that the definition of $g$, $p \circ \eta_0 = p \circ \eta_1$ and $p \circ \varepsilon(x) \simeq 1_{p \circ \varepsilon(x)(0)}$ imply

$$p \circ (H(x, t)) \simeq 1_{p \circ \eta_0(x, t)},$$

and the latter is precisely $1_{p \circ (H(x, t))(0)}$ because of Eq. (23). Hence, $H(x, t)$ is $h$-vertical over $p$ and $H \colon X \times I \to E_p^I$.

Finally, suppose that, for a fixed $x \in X$, $\eta_0(x) = 1_{\varepsilon(0)}$ and $\eta_1(x) = 1_{\varepsilon(1)}$. Thus $l = 0$, $m = \mathrm{id}_X \times 0$ and

$$\begin{aligned} H(x, t)(s) &= g(x, 0, s) \\ &= \varepsilon(x)(s), \end{aligned}$$

that is, $H(x, t) = H(x, 0)$ for every $t \in I$. Hence the normality condition is fulfilled and $\partial_p \in \mathcal{H}$. $\qquad \square$

The next lemma proves that every $h$-vertical path is homotopic relative to endpoints to a vertical one, via a homotopy within $E_p^I$.

**Lemma 19.** *Let $p\colon E \to B$ be a normal Hurewicz fibration. For every $\varepsilon \in E_p^I$ and every homotopy $K\colon p \circ \varepsilon \simeq 1_{p \circ \varepsilon(0)}$ in $B$, we can construct a path $h\colon I \to E_p^I$ such that*

$$h(t)\colon \varepsilon(0) \rightsquigarrow \varepsilon(1) \qquad \text{for every } t \in I, \tag{24}$$

$$h(0) = \varepsilon \qquad \text{and} \qquad p \circ (h(1)) = 1_{(p \circ h(1))(0)} \tag{25}$$

*Furthermore, if $\varepsilon$ is vertical and $K$ is the constant homotopy, the construction yields $h = 1_\varepsilon$.*

*Proof.* Consider the following commutative square.

$$\begin{array}{ccc}
I & \xrightarrow{\;\varepsilon\;} & E \\
\big\downarrow & & \big\downarrow{\scriptstyle p} \\
I^2 & \xrightarrow{\;K'\;} & B
\end{array} \tag{26}$$

where $K'(t,s) := K(s,t)$. Since $p \in \mathcal{H}$, there exists a a homotopy $H\colon I^2 \to E$ such that

$$H(t,0) = \varepsilon(t) \qquad \text{and} \qquad p \circ H = K'.$$

We then define

$$h(t)(s) := H(s,t)$$

Let us prove that this is the required path.

First of all, $h(0)(s) = H(s,0) = \varepsilon(s)$. Secondly, since $K$ is relative to endpoints, $K'(0,s) = p \circ \varepsilon(0) = K'(1,s)$ for all $s \in I$, and the normality of $p$ ensures

$$h(t)(0) = H(0,t) = H(0,0) = \varepsilon(0)$$

and

$$h(t)(1) = H(1,t) = H(1,0) = \varepsilon(1).$$

for all $t \in I$.

Finally, we have $h(t) \in E_p^I$ for every $t \in I$ because $(p \circ h(t))(s) = K(t,s)$, and the path $s \mapsto K(t,s)$ is of course homotopic to $1_{p \circ \varepsilon(0)} = 1_{(p \circ h(t))(0)}$ relative to endpoints. In particular, $p \circ h(1) = 1_{p \circ \varepsilon(0)} = 1_{(p \circ h(1))(0)}$ ☐

**Lemma 20.** *Let $p\colon E \to B$ and $q\colon A \to E_p^I$ be two normal Hurewicz fibrations, and let*

$$\begin{array}{ccc}
C & \xrightarrow{\;i\;} & A \\
\big\downarrow & & \big\downarrow{\scriptstyle q} \\
E & \xrightarrow{\;\mathrm{u}_p\;} & E_p^I
\end{array} \tag{27}$$

*be a pullback. Every lifting problem*

$$\begin{array}{ccc}
C & \xrightarrow{\;f\;} & F \\
{\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle \bar{q}} \\
A & =\!=\!= & A
\end{array} \tag{28}$$

*has a diagonal filler if $\bar{q} \in \mathcal{H}$.*

*Proof.* We may suppose, up to homeomorphisms, that

$$C = \{ (e, a) \in E \times A \mid \bar{q}(a) = 1_e \}$$

and that $i$ and $C \to E$ are the projections.

In order to be able to exploit the lifting property of $q$ we need, for every $a \in A$, a pair $(e, a') \in C$ and a path $\alpha \colon a' \rightsquigarrow a$.

Let $a \in A$, $e \coloneqq q(a)(0)$ and let $K \colon p \circ (q(a)) \simeq 1_{p(e)}$ be a homotopy in $B$. Lemma 19 provides us with a path

$$h_0 \colon q(a) \rightsquigarrow \varepsilon$$

within $E_p^I$, such that $p \circ \varepsilon = 1_{p(e)}$. We can then define the arrow $\langle 1_e, \varepsilon \rangle \colon I \to E \times_p E$ and take a lifting $h_1 \colon I \to E_p^I$ of it along $\partial_p$ such that

$$
\begin{array}{ccc}
\mathbf{1} & \xrightarrow{\ 1_e\ } & E_p^I \\
\downarrow & {\Large\nearrow}^{h_1} & \downarrow{\partial_p} \\
I & \xrightarrow[\langle 1_e, \varepsilon \rangle]{} & E \times_p E
\end{array}
\tag{29}
$$

commutes ($\mathbf{1} \coloneqq \{ * \}$ is the terminal object in $\mathsf{Top}$ and it is customary to denote a map $* \mapsto f$ by $f$ itself). The construction of $h_1$ in Lemma 18 imply that $h_1(1)$ is a path which stands in $e = \varepsilon(0)$ for some time and goes then to $\varepsilon(1)$ along $\varepsilon$. Thus $h_1(1)$ is just a reparametrization of $\varepsilon$ respecting its orientation and, in particular, is homotopic to $\varepsilon$ relative to endpoints. Therefore we may suppose

$$h_1 \colon 1_e \rightsquigarrow \varepsilon.$$

Let $h \colon I \to E_p^I$ be the following concatenation of $h_0$ and $h_1$

$$
h(t) \coloneqq \begin{cases} h_0(2t), & t \in [0, \tfrac{1}{2}] \\ h_1(2(1-t)), & t \in [\tfrac{1}{2}, 1] \end{cases}
$$

which is a path in $E_p^I$ from $q(a)$ to $1_e$, and let $\alpha(t) \coloneqq \alpha'(1-t)$, where $\alpha'$ is defined by the lifting property of $q$:

$$
\begin{array}{ccc}
\mathbf{1} & \xrightarrow{\ a\ } & A \\
\downarrow & {\Large\nearrow}^{\alpha'} & \downarrow{q} \\
I & \xrightarrow[h]{} & E_p^I
\end{array}
\tag{30}
$$

Hence $(e, \alpha(0)) \in C$, the square

$$
\begin{array}{ccc}
\mathbf{1} & \xrightarrow{f(e, \alpha(0))} & F \\
\downarrow & & \downarrow{\bar{q}} \\
I & \xrightarrow[\alpha]{} & A
\end{array}
\tag{31}
$$

commutes, and it then admits a diagonal filler $\varphi \colon I \to F$.

Define $j(a) \coloneqq \varphi(1)$ and let us prove that the map $j \colon A \to F$ is a diagonal filler for the square 28. Commutativity of the lower triangle is immediate:

$$\bar{q} \circ j(a) = \bar{q} \circ \varphi(1) = \alpha(1) = a,$$

whereas commutativity of the upper one follows from normality of the Hurewicz fibrations involved. Indeed, if $a = i(e, a)$ for some $(e, a) \in C$, then $q(a) = 1_e$. Therefore we may take, in Lemma 19, $K \colon p \circ (q(a)) \simeq 1_{p(e)}$ to be the constant homotopy, thus obtaining $\varepsilon = 1_e$ and $h_0 = 1_{1_e}$. Again, from the normality of $\partial_p$, we have $h_1 = 1_{1_e}$ and $h = 1_{1_e}$. For the same reason, $\alpha = 1_a$, $\varphi = 1_{f(e,a)}$ and finally

$$j(a) = \varphi(1) = f(e, a). \qquad \square$$

**Theorem 21.** *The pair* $(\mathsf{Top}, \mathcal{H})$ *is a tribe with stable path objects.*

*Proof.* That it is a tribe is easily verified, similarly to the case of small groupoids. The factorization of the diagonal arrow is straightforward once we define

$$\mathrm{Path}(p) \coloneqq E_p^I, \qquad \mathrm{r}_p \coloneqq \mathrm{u}_p \qquad \text{and} \qquad \partial_p \coloneqq \partial_p,$$

whereas lemmas 18 and 20 establish points $(a)$ and $(b)$ of Definition 6, respectively. Therefore we only need to verify the stability requirement of Definition 7. Given $p \colon E \to Y$ in $\mathcal{H}$ and $f \colon X \to Y$ we have, up to homeomorphism, that

$$\mathrm{Map}_p(p^*f) = \left\{ (x, \varepsilon) \in X \times E_p^I \mid p \circ \varepsilon(0) = f(x) \right\}$$
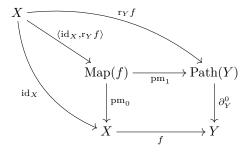
and

$$\mathrm{Path}(f^*p) = \left\{ (\gamma, \varepsilon) \in X^I \times E^I \mid p \circ \varepsilon = f \circ \gamma \quad \text{and} \quad \gamma \simeq 1_{\gamma(0)} \right\}$$

and that the first one is a deformation retract of the second one. In particular, the map $i \colon \mathrm{Map}_p(p^*f) \to \mathrm{Path}(f^*p)$ is just the inclusion

$$(x, \varepsilon) \mapsto (1_x, \varepsilon).$$

Commutativity of diagram (1) is then immediate. $\qquad \square$

We conclude by showing the factorization of maps obtained from Lemma 9. A map $f \colon X \to Y$ is factored using the universal property of the mapping path object for $f$ over the terminal arrow $Y \to \mathbf{1}$:

and $f = \partial_Y^1 \mathrm{pm}_1 \langle \mathrm{id}_X, \mathrm{r}_Y f \rangle$. But the object $\mathrm{Map}(f)$ is precisely the mapping path space of $f$

$$\left\{ (x, \varepsilon) \in X \times E^I \mid f(x) = \varepsilon(0) \right\},$$

whilst the universal arrow $\langle \mathrm{id}_X, \mathrm{r}_Y f \rangle$ and the normal Hurewicz fibration $\partial_Y^1 \mathrm{pm}_1 \colon \mathrm{Map}(f) \to Y$ are

$$x \longmapsto (x, 1_{f(x)}) \qquad \text{and} \qquad (x, \varepsilon) \longmapsto \varepsilon(1)$$

respectively. Therefore, we obtain the factorization of maps given by the weak factorization system $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ of Strøm's Quillen model structure on $\mathsf{Top}$.

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