A model structure on the cartesian cubical sets

### 1 The cartesian cube category

We consider the cartesian cube category  $\mathbb{C}$ , defined as the free finite product category on an interval  $\delta_0, \delta_1 : 1 \rightrightarrows I$ . As a classifying category for an algebraic theory  $\mathbb{T} = \{0,1\}$ ,  $\mathbb{C}$  has a covariant presentation by Lawvere duality, namely as the dual of the full subcategory of finitely-generated, free  $\mathbb{T}$ -algebras  $\mathsf{Alg}(\mathbb{T})_{\mathrm{fg}}$ . In this case, the algebras are simply bipointed sets  $(A, a_0, a_1)$ , and the free ones are the strictly bipointed sets  $a_0 \neq a_1$ . Thus  $\mathsf{Alg}(\mathbb{T})_{\mathrm{fg}}$  consists of the finite, strictly bipointed sets and all bipointed maps between them.

**Definition 1.** The objects of the cartesian cube category  $\mathbb C$  are themselves called cubes, and will be written

$$[n] = \{x_1, ..., x_n\},\$$

where the  $x_i$  may be regarded as coordinate axes. The arrows,

$$f:[n]\longrightarrow [m]$$
,

are then taken to be m-tuples of elements drawn from the set

$$[n]^+ = \{0, x_1, ..., x_n, 1\},\$$

regarded as formal terms representing composites of faces, degeneracies, permutations, and diagonals (alternately, the arrows are arbitrary bipointed maps  $[m]^+ \longrightarrow [n]^+$ ).

#### 2 Cubical sets

The category cSet of *cubical sets* is the category of presheaves on the cartesian cube category  $\mathbb{C}$ . It is generated by the representable presheaves y([n]), which will be written  $I^n = y([n])$  and called the *standard n-cubes*.

# 3 Partial map classification and the +-algebra weak factorization system

Cofibrations, partial map classification, the functor  $X^+$ , the awfs of +-algebras.

### 4 Partial path lifting (biased version)

We first recall the specification of the trivial-cofibration/fibration WFS from [?], and show that the resulting fibrations are equivalent to those specified in the "logical style" given in [?, ?].

The generating trivial cobrations are all maps of the form

$$m \otimes \delta_{\epsilon} : U \longrightarrow \mathbf{I}^{n+1}$$
, (1)

where:

- 1.  $n \ge 0$ ,
- 2.  $\delta_{\epsilon}: 1 \longrightarrow I$  is one of the two endpoint inclusions, where  $\epsilon = 0, 1$ ,
- 3.  $m \otimes \delta_{\epsilon}$  is the push-out product, resp. "Leibniz tensor", of any cofibration  $m: M \longrightarrow I^n$  and a  $\delta_{\epsilon}: 1 \longrightarrow I$ ,
- 4. U is  $I^n +_M (M \times I)$ , the domain of  $m \otimes \delta_{\epsilon}$ .

Let  $\mathcal{C} \otimes \delta_{\epsilon}$  be the set of all such maps; the *fibrations* are defined to be the elements of the right class of these,

$$\mathcal{F} = (\mathcal{C} \otimes \delta_{\epsilon})^{\pitchfork}$$
 .

A fibration structure on a map  $f: Y \longrightarrow X$  is a choice of diagonal fillers,

$$\begin{array}{ccc}
I^{n} +_{M} (M \times I) & \longrightarrow X \\
\downarrow^{m \otimes \delta_{\epsilon}} & \downarrow^{f} \\
I^{n} \times I & \longrightarrow Y.
\end{array} \tag{2}$$

that is uniform with respect to arbitrary pullbacks of the cofibration m, as in the case of the +algebra factorization system.

Fixing the argument  $\delta_{\epsilon}$ , the Leibniz tensor functor

$$(-)\otimes \delta_{\epsilon}:\widehat{\mathbb{C}}^2\longrightarrow\widehat{\mathbb{C}}^2$$

has a right adjoint, the "Leibniz exponential", which for a map  $f: X \longrightarrow Y$  we will write as,

$$(\delta_{\epsilon} \Rightarrow f) : X^{\mathrm{I}} \longrightarrow (Y^{\mathrm{I}} \times_{Y} X)$$
.

Using this adjunction on arrow categories, one can easily show the following:

**Proposition 2.** An object X is fibrant if and only if both of the pathspace projections  $X^{\delta_{\epsilon}}: X^{\mathrm{I}} \longrightarrow X$  are +algebras.

An analogous statement also holds for maps  $f: X \longrightarrow Y$  in place of objects X.

\*\*\* ToDo: Spell out the condition for a map  $f: X \longrightarrow Y$ , namely that the maps  $(\delta_{\epsilon} \Rightarrow f): X^I \longrightarrow Y^I \times_Y X$  are both +algebras (for  $\epsilon = 0, 1$ ), and then show that it is equivalent to the following "local" partial path lifting condition. \*\*\*

#### 4.1 Local partial path lifting

To make the connection to the logical style of presentation used in [?, ?], suppose we want to describe a (uniform) filling structure on an arbitrary  $f: X \longrightarrow Y$  with respect to all generating trivial cofibrations  $m \otimes \delta_{\epsilon}: I^n +_M (M \times I) \longrightarrow I^{n+1}$ ,

$$\begin{array}{ccc}
I^{n} +_{M} (M \times I) & \longrightarrow X \\
 & \downarrow f \\
I^{n} \times I & \longrightarrow Y.
\end{array} \tag{3}$$

By pulling back along c, it suffices to consider the case  $Y = \mathbf{I}^n \times \mathbf{I}$  and c the identity map. Moreover, since we shall internalize the quantification over all cofibrations  $m: M \to \mathbf{I}^n$  using the classifier  $\Phi$ , it suffices to consider just the following case internally,

$$\begin{array}{ccc}
1 +_{[\phi]} ([\phi] \times I) & \xrightarrow{[a_0, s]} X \\
\downarrow^{\phi \otimes \delta_{\epsilon}} & \downarrow & \downarrow \\
1 \times I & \xrightarrow{\simeq} & I
\end{array}$$
(4)

where the cofibration  $[\phi] \rightarrow 1$  is classified by  $\phi : 1 \rightarrow \Phi$ .

Using a universe Set in the internal language of  $\widehat{\mathbb{C}}$ , we can regard the family  $X \longrightarrow I$  internally as a map  $P: I \to Set$  (switching notation from X to P to agree with [?]). Thus we arrive at the following local specification, expressed logically in the internal language of  $\widehat{\mathbb{C}}$ , of the object of "(0-directed) lifting structures"  $L^0(P)$  on a family  $P: I \to Set$ :

$$L^{0}(P) = \prod_{\phi:\Phi} \prod_{s:\prod_{i:I}(P_{i})^{\phi}} \prod_{a_{0}:P_{0}} a_{0}|_{\phi} = s_{0} \longrightarrow \sum_{a:\prod_{i:I}P_{i}} (a_{0} = a_{0}) \times (a|_{\phi} = s).$$
(5)

Here the variables  $s: \prod_{i:I} (Pi)^{\phi}$  and  $a_0: P0$ , and the condition  $a_0|_{\phi} = s0$ , give the domain  $1 + [\phi] ([\phi] \times I)$  of the arrow  $[a_0, s]$  in (??), and  $a: \prod_{i:I} Pi$  is the diagonal filler, with  $(a_0 = a_0) \times (a|_{\phi} = s)$  expressing the commutativity of the top triangle.

There is an analogous condition  $L^1(P)$  in which 1 replaces 0 everywhere, describing ("directed") filling from the other end of the interval. Note that [?, ?] derive the "filling" conclusion

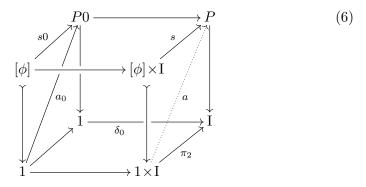
$$\sum_{a:\prod_{i:I} Pi} (a0 = a_0) \times (a|_{\phi} = s)$$

from (connections on I and) a weaker "composition operation"

$$\sum_{a_1:P1} a_1|_{\phi} = s_1 \,,$$

but we will not take this approach.

The specification of the type  $L^0(P)$  of (??) can also be represented diagrammatically as follows:



Here the left-hand vertical square is determined as a pullback of the right-hand one along the endpoint  $\delta_0: 1 \longrightarrow I$ .

Now write

$$\widetilde{P} = \prod_{i:I} Pi$$

for the type of sections of the projection  $P = \sum_{i:I} Pi \longrightarrow I$ , and write

$$\pi_0: \widetilde{P} \longrightarrow P0$$

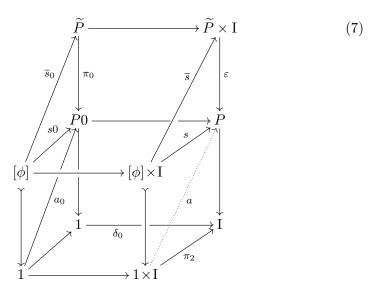
for the  $0^{th}$ -projection (i.e. the evaluation of  $P: I \longrightarrow \mathsf{Set}$  at 0: I).

Then the (0-directed) lifting structures on P correspond to +-algebra structures on the projection  $\pi_0: \widetilde{P} \longrightarrow P0$ , as follows.

**Proposition 3.** For any  $P : Set^{I}$ , there is an isomorphism

$$L^0(P) \cong {}^+ Alg(\pi_0 : \widetilde{P} \longrightarrow P0)$$
.

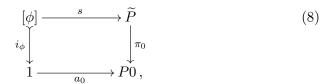
*Proof.* Consider the following diagram,



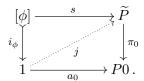
which is  $(\ref{eq:initial})$ , extended by the counit (evaluation)  $\varepsilon: \widetilde{P} \times I \longrightarrow P$  over I on the right, and with 1 still representing the domain of a variable to reason internally. The pullback of  $\varepsilon$  over I along  $\delta_0$  is then the map  $\pi_0: \widetilde{P} \longrightarrow P0$  that we are interested in.

Given an  $L^0(P)$ -structure, reasoning internally we construct a <sup>+</sup>Alg-structure on  $\pi_0: P \longrightarrow P0$  as follows: for any cofibration  $i_\phi: [\phi] \rightarrowtail 1$ 

and any commutative square,



we require a diagonal filler,



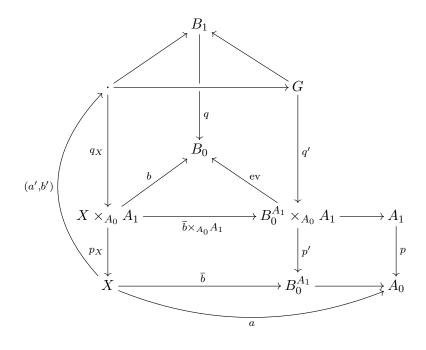
Transposing the top left span in  $(\ref{eq:condition})$  formed by  $i_{\phi}$  and s along the adjunction  $I^* \dashv \prod_I$  gives the right-hand square in  $(\ref{eq:condition})$ , and the commutative square in  $(\ref{eq:condition})$  formed by  $a_0$  and  $\pi_0$  gives the rest of the data in  $(\ref{eq:condition})$ . Thus the assumed  $L^0(P)$ -structure gives an  $a: 1 \times I \longrightarrow P$  as indicated in  $(\ref{eq:condition})$ . But then a lifts uniquely across  $\varepsilon$  to a map  $\overline{a}: 1 \times I \longrightarrow \widetilde{P} \times I$  over I, by the universal property of  $\varepsilon: \widetilde{P} \times I \longrightarrow P$ . We can therefore set

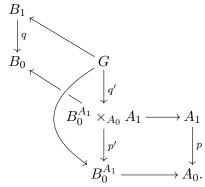
$$j = \delta_0^*(\overline{a}) : 1 \longrightarrow \widetilde{P}$$
.

Suppose conversely that we have a <sup>+</sup>Alg-structure on  $\pi_0: \widetilde{P} \longrightarrow P0$ , and we want to build a (0-directed) lifting structure on P. Take any  $\phi, s, a_0$  as indicated in (??), and we require an  $a: 1 \times I \longrightarrow P$  over I. From s we get  $\overline{s}$  by the universal property of  $\varepsilon$ , and we therefore have  $\overline{s}_0$  by pullback. From  $\overline{s}_0$  and  $a_0$  and the <sup>+</sup>Alg structure on  $\pi_0$  we obtain a map  $j: 1 \longrightarrow \widetilde{P}$  over P0 which is a diagonal filler of the indicated square formed by  $i_{\phi}, \overline{s}_0, a_0$  and  $\pi_0$ . Finally, we obtain the required map  $a: 1 \times I \longrightarrow P$  over I as the  $(I^* \dashv \prod_I)$ -transpose of j,

$$a = \varepsilon \circ (j \times I)$$
.

We leave to the reader the verification that these assignments are mutually inverse.  $\Box$ 





## 5 Unbiased partial path lifting

# 6 A left-induced model structure on the Cartesian cubical sets

We make use of the Sattler model structure [?] on the *Dedekind cubical* sets  $\widehat{\mathbb{D}} = \mathsf{Set}^{\mathbb{D}^{\mathrm{op}}}$ , where  $\mathbb{D}$  is the category of *Dedekind cubes*, defined as

the Lawvere theory of distributive lattices. The unique product-preserving functor

$$i: \mathbb{C} \longrightarrow \mathbb{D}$$

classifying the Dedekind interval  $I_{\mathbb{D}} \in \mathbb{D}$  induces an adjunction,

$$i_! \dashv i^* \dashv i_* : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{D}},$$

where  $i^*(Q) = Q \circ i$ , for  $Q \in \mathbb{D}$ .

**Lemma 4.** Observe that  $i_!$  is left exact since the Dedekind interval  $I_{\mathbb{D}}$  is strict,  $0 \neq 1 : 1 \rightrightarrows I_{\mathbb{D}}$ . Thus we have geometric morphisms:

$$(i_! \dashv i^*): \widehat{\mathbb{D}} \longrightarrow \widehat{\mathbb{C}},$$

classifying the bipointed object  $i_!(I_{\mathbb{C}}) = I_{\mathbb{D}}$ ,

$$(i^* \dashv i_*): \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{D}},$$

classifying the dLat  $i^*(I_{\mathbb{D}}) := \mathbb{I}$ , where  $\eta : I_{\mathbb{C}} \longrightarrow \mathbb{I}$  can be described pointwise as the distributive lattice completion of the corresponding bipointed set.

Also, since i is faithful so is  $i_!$ , and since i is surjective on objects  $i^*$  is also faithful.

It follows that:

- $\widehat{\mathbb{C}}$  is  $(i_! \circ i^*)$ -coalgebras on  $\widehat{\mathbb{D}}$ ,
- $\widehat{\mathbb{D}}$  is  $(i^* \circ i_*)$ -coalgebras on  $\widehat{\mathbb{C}}$ ,
- $\widehat{\mathbb{D}}$  is  $(i^* \circ i_!)$ -algebras on  $\widehat{\mathbb{C}}$ .

We will use the following transfer theorem for QMSs from  $\cite{MSs}$  from  $\cite{M$ 

**Theorem** ([?, ?]). Suppose  $\widehat{\mathbb{D}}$  has a (cofibrantly generated) model structure  $(\mathcal{C}_{\mathbb{D}}, \mathcal{W}_{\mathbb{D}}, \mathcal{F}_{\mathbb{D}})$ . Given an adjunction

$$i_! \dashv i^* : \widehat{\mathbb{D}} \longrightarrow \widehat{\mathbb{C}},$$

there is a left-induced model structure on  $\widehat{\mathbb{C}}$  if the following acyclicity condition holds:

$$(i_!^{-1}\mathcal{C}_{\mathbb{D}})^{\pitchfork} \subset i_!^{-1}\mathcal{W}_{\mathbb{D}}.$$

For the left-induced model structure  $(\mathcal{C}_{\mathbb{C}}, \mathcal{W}_{\mathbb{C}}, \mathcal{F}_{\mathbb{C}})$  on  $\widehat{\mathbb{C}}$  we then have:

$$\mathcal{C}_{\mathbb{C}} = i_{!}^{-1} \mathcal{C}_{\mathbb{D}},$$
 $\mathcal{W}_{\mathbb{C}} = i_{!}^{-1} \mathcal{W}_{\mathbb{D}}.$ 

The Sattler model structure on  $\widehat{\mathbb{D}}$  is given as follows (for a constructive treatment a smaller class of "pointwise decidable cofibrations" is used, but we consider the classical case first):

$$\begin{array}{ll} \mathcal{C} &=& \text{monomorphisms} \,, \\ \mathcal{W} &=& \left\{ f \mid f = p \circ i, \ p \in \mathcal{F} \cap \mathcal{W}, \ i \in \mathcal{C} \cap \mathcal{W} \right\}, \\ \mathcal{F} &=& \left( \mathcal{C} \otimes \delta \right)^{\pitchfork}. \end{array}$$

where  $\delta: 1 \longrightarrow I$  is either endpoint inclusion.

For the left-induced model structure on  $\widehat{\mathbb{C}}$  we therefore have the following specification:

$$\mathcal{C} = \text{monomorphisms},$$

$$\mathcal{W} = \{ f \mid i_! f = p \circ i, \ p \in \mathcal{F}_{\mathbb{D}} \cap \mathcal{W}_{\mathbb{D}}, \ i \in \mathcal{C}_{\mathbb{D}} \cap \mathcal{W}_{\mathbb{D}} \},$$

$$\mathcal{F} = (\mathcal{C} \cap \mathcal{W})^{\pitchfork}.$$

The determination of C follows from the fact that  $i_!: \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{D}}$  is conservative. To check the acyclicity condition,

$$(i_!^{-1}\mathcal{C}_{\mathbb{D}})^{\pitchfork} \subset i_!^{-1}\mathcal{W}_{\mathbb{D}},$$

we know that  $i_!^{-1}\mathcal{C}_{\mathbb{D}}$  consists of the monos in  $\mathbb{C}$ , so take  $f: Y \longrightarrow X$  in  $(i_!^{-1}\mathcal{C}_{\mathbb{D}})^{\pitchfork}$ , apply  $i_!$ , and factor the result as  $i_!f = p \circ m: i_!Y \longrightarrow Z \longrightarrow i_!X$  with  $p \in \mathcal{F}_{\mathbb{D}} \cap \mathcal{W}_{\mathbb{D}}$  and  $m: i_!Y \longrightarrow Z$  monic. We then need to show that m is in  $\mathcal{W}_{\mathbb{D}}$ .

We can apply Theorem 2.2.1 of [?], with  $K = \widehat{\mathbb{C}}$ ,  $M = \widehat{\mathbb{D}}$ ,  $V = i_!$ ,  $k = i^*$ , and:

- 1. QX = X and  $\epsilon = 1_X : X \longrightarrow X$ , so that  $i_! 1_X = 1_{i_!X}$  and therefore in  $\mathcal{W}_{\mathbb{D}}$ , while all objects are cofibrant,
- 2. Qf = f for any  $f: X \longrightarrow Y$  in  $\widehat{\mathbb{C}}$ , so that the naturality condition is similarly trivial,
- 3. factor the codiagonal  $X + X \longrightarrow X$  as  $\pi_2 \circ j : X + X \longrightarrow I \times X \longrightarrow X$  with  $j = (\partial I \times X) : X + X \longrightarrow I \times X$ .

It remains only to show that  $i_!p: i_!(I \times X) \longrightarrow i_!X$  is in  $\mathcal{W}_{\mathbb{D}}$  and  $i_!j: i_!(X + X) \longrightarrow i_!(I \times X)$  is in  $\mathcal{C}_{\mathbb{D}}$ . The latter is clear, since j is monic. To show the former, observe that for any  $D \in \widehat{\mathbb{D}}$ , the projection  $\pi_2: I_{\mathbb{D}} \times D \longrightarrow D$  is in  $\mathcal{W}_{\mathbb{D}}$  by 3-for-2, since the "cylinder end" inclusion  $D \longrightarrow I_{\mathbb{D}} \times D$ , as a pullback of an endpoint inclusion, is a cofibration, and a strong deformation retract (using the connection on I), and hence is in  $\mathcal{W}_{\mathbb{D}}$  by [?].

Thus we have shown:

**Theorem 5.** There is a Quillen model structure  $(C, W, \mathcal{F})$  on the category  $\widehat{\mathbb{C}}$  of cartesian cubical sets, in which

$$\mathcal{C} = monomorphisms,$$

$$\mathcal{W} = \{ f \mid i_! f = p \circ i, \ p \in \mathcal{F}_{\mathbb{D}} \cap \mathcal{W}_{\mathbb{D}}, \ i \in \mathcal{C}_{\mathbb{D}} \cap \mathcal{W}_{\mathbb{D}} \},$$

$$\mathcal{F} = (\mathcal{C} \cap \mathcal{W})^{\pitchfork}.$$

where  $i_!: \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{D}}$  is the left adjoint of precomposition along the canonical map  $i: \mathbb{C} \longrightarrow \mathbb{D}$  from Cartesian cubes to Dedekind cubes, and  $(\mathcal{C}_{\mathbb{D}}, \mathcal{W}_{\mathbb{D}}, \mathcal{F}_{\mathbb{D}})$  is the Sattler model structure on  $\widehat{\mathbb{D}}$ .

#### References:

- Gambino-Sattler
- Sattler
- Hess, Kedziorek, Riehl, Shipley
- Garner, Kedziorek, Riehl