

# Cartesian cubical model categories

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## Abstract

Add an abstract.

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## Introduction

novelties: univalence, universes, classifying types, uniformity, constructive,  
...

# 1 Cartesian cubical sets

There are now many treatments of cubical sets in the literature, including [?, ?, ?, ?, ?, ?, ?, ?]. Our construction is intended to work in *all* of these, insofar as the axioms in Definition ?? below are satisfied. For the sake of concreteness, however, we shall consider what may be called the *cartesian* cube category  $\square$ , defined as the free finite product category on an interval  $\delta_0, \delta_1 : 1 \rightrightarrows I$ .

**Definition 1.** The objects of the cartesian cube category  $\square$ , called *n*-cubes, are finite sets of the form

$$[n] = \{0, x_1, \dots, x_n, 1\},$$

where  $x_1, \dots, x_n$ , are formal *generators*. The arrows,

$$f : [n] \rightarrow [m],$$

may be taken to be *m*-tuples of elements drawn from the set  $\{0, x_1, \dots, x_n, 1\}$ , regarded as formal terms representing composites of faces, degeneracies, permutations, and diagonals. Equivalently, the arrows  $[n] \rightarrow [m]$  are arbitrary bipointed maps  $[m] \rightarrow [n]$ .

See [?] for further details.

**Definition 2.** The category **cSet** of *cubical sets* is the category of presheaves on the cartesian cube category  $\square$ ,

$$\mathbf{cSet} = \mathbf{Set}^{\square^{\text{op}}}.$$

It is of course generated by the representable presheaves  $y[n]$ , to be written

$$I^n = y[n]$$

and called the *geometric n-cubes*.

Note that the representables  $I^n$  are closed under finite products,  $I^n \times I^m = I^{n+m}$ . We of course write  $I$  for  $I^1$  and  $1$  for  $I^0$ . We will need the following basic fact about the cubes  $I^n$  in **cSet**.

**Proposition 3.** *For each  $n$ , the  $n$ -cube  $I^n$  is tiny, in the sense that the exponential (or “internal Hom”) functor  $(-)^{I^n} : \mathbf{cSet} \rightarrow \mathbf{cSet}$  has a right adjoint.*

(See [?] for more on such “amazing right adjoints”.)

*Proof.* It clearly suffices to prove the claim for  $n = 1$ . For any cubical set  $X$ , the exponential  $X^I$  is a “shift by one dimension”,

$$X^I(n) \cong \text{Hom}(I^n, X^I) \cong \text{Hom}(I^{n+1}, X) \cong X(n+1).$$

Thus  $X^I$  is given by precomposition with the “successor” functor  $\square \rightarrow \square$  with  $[n] \mapsto [n+1]$ . Precomposition always has a right adjoint, which in this case we write as

$$X^I \dashv X_I$$

and call the  $I$ -root. We can calculate the values of the  $I$ -root to be,

$$\begin{aligned} X_I(n) &\cong \text{Hom}(I^n, X_I) \\ &\cong \text{Hom}((I^n)^I, X) \\ &\cong \text{Hom}(I^I, X) \\ &\cong \text{Hom}((I+1)^n, X) \\ &\cong \text{Hom}\left(\sum_{k=0}^n \binom{n}{k} I^k, X\right) \\ &\cong \prod_{k=0}^n X(k)^{\binom{n}{k}}, \end{aligned}$$

using the fact that  $I^I \cong (I+1)$ , much as in [?].  $\square$

The exponential object  $X^I$  will be called the *pathobject* of  $X$ , and plays a special role. As we have just seen, it classifies “paths” in  $X$ , i.e. maps  $p : I \rightarrow X$ , corresponding to 1-cubes  $p \in X_1$  between the endpoints (0-cubes)  $p_0, p_1 \in X_0$ . More generally, higher cubes  $c \in X_{n+1}$  correspond to maps  $c : I^{n+1} \cong I^n \times I \rightarrow X$ , which can be regarded as paths between the  $n$ -cubes  $c_0, c_1 : I^n \rightarrow X$ , respectively  $c_0, c_1 \in X_n$ .

We conclude this section with two facts that will be useful later on concerning the base change functors,

$$f_! \dashv f^* \dashv f_* : \mathbf{cSet}/_X \longrightarrow \mathbf{cSet}/_Y,$$

associated to a map  $f : X \rightarrow Y$  in  $\mathbf{cSet}$ .

**Lemma 4.** *The pulled-back interval  $I^*I = I \times I \rightarrow I$  in  $\mathbf{cSet}/_I$  is also tiny.*

*Proof.* Since the interval  $I = y[1]$  is representable, the slice category  $\mathbf{cSet}/_I$  is also a category of presheaves, namely over the sliced cube category  $\square/[1]$ ,

$$\mathbf{cSet}/_I = \mathbf{Set}^{\square^{\text{op}}}/_{y[1]} \cong \mathbf{Set}^{(\square/[1])^{\text{op}}} = \widehat{\square/[1]}.$$

However, since  $\square$  does not have all finite limits, the sliced index category does not have all finite products, and so we cannot simply repeat the proof from Proposition 3. But as in that proof, we do have a “successor” functor

$$s_{[1]} : \square/[1] \rightarrow \square/[1],$$

resulting from the “predecessor” natural transformation  $s \Rightarrow 1_{\square}$  given by the projection  $I \times X \rightarrow X$ . Evaluating  $s$  at each object  $f : [n] \rightarrow [1]$  in  $\square/[1]$ , we obtain a commutative diagram:

$$\begin{array}{ccccc} s[n] & \xrightarrow{\cong} & [1] \times [n] & \xrightarrow{p_n} & [n] \\ sf \downarrow & & & & \downarrow f \\ s[1] & \xrightarrow{\cong} & [1] \times [1] & \xrightarrow{p_1} & [1] \end{array} \quad (1)$$

We can then set  $s_{[1]}(f) = p_1 \circ sf = f \circ p_n$ . As in the foregoing proof, we can then calculate the values of the adjoints on presheaves, associated to  $s_{[1]}$ ,

$$s_{[1]!} \dashv s_{[1]}^* : \widehat{\square/[1]} \longrightarrow \widehat{\square/[1]}$$

to be, successively,

$$\begin{aligned} s_{[1]!}(X) &= I^*I \times X, \\ s_{[1]}^*(X) &= X^{I^*I}. \end{aligned}$$

The first equation follows from the observation that the diagram (1) is a pullback, and so the object  $s_{[1]}(f) : s[n] \rightarrow [1]$  of  $\widehat{\square/[1]}$  given by the evident composite is just  $I^*I \times f$ , and the diagram itself represents the counit map  $(I^*I \times f) \rightarrow f$  over  $I$ . The second line then follows by adjointness, as does the fact that we have a further right adjoint, namely, the  $I^*I$ -root:

$$s_{[1]*}(X) =: X_{I^*I}.$$

□

**Lemma 5.** *The pushforward functor along any map  $f : X \rightarrow Y$  preserves pathspaces; for any object  $A \rightarrow X$  over  $X$ , the pathobject of the pushforward  $f_*A$  is (canonically isomorphic to) the pushforward of the pathobject,*

$$(f_*A)^I \cong f_*(A^I)$$

over  $Y$ .

*Proof.* This true for any constant family  $X^*C = X \times C \rightarrow X$  with  $C$  in place of  $I$ , as the reader can easily verify, using the Beck-Chevalley condition.  $\square$

## 2 The cofibration weak factorization system

**Definition 6** (Cofibration). The *cofibrations*, written

$$c : A \rightarrowtail B,$$

are any class  $\mathcal{C}$  of monomorphisms in  $\mathbf{cSet}$  satisfying the following axioms:

- (C0) The map  $0 \rightarrow C$  is always a cofibration.
- (C1) All isomorphisms are cofibrations.
- (C2) The composite of two cofibrations is a cofibration.
- (C3) Any pullback of a cofibration is a cofibration.

We also require the cofibrations to be classified by a subobject  $\Phi \hookrightarrow \Omega$  of the standard subobject classifier  $\top : 1 \rightarrow \Omega$  of  $\mathbf{cSet}$ :

- (C4) There is a terminal object  $t : 1 \rightarrowtail \Phi$  in the category of cofibrations and cartesian squares.

Two further axioms for cofibrations will be added in Section 3.1, one in Section 3.2, and a final one in Section 6.4 (see Appendix 8 for a summary). Note that we also permit the case  $\Phi = \Omega$ , so that all monos are cofibrations, in which case no axioms are required.

**The cofibrant partial map classifier.** Write

$$X^+ := \sum_{\varphi: \Phi} X^{[\varphi]} = \Phi_! t_*(X), \quad (2)$$

for the polynomial endofunctor  $\mathbf{cSet} \rightarrow \mathbf{cSet}$  determined by the cofibration classifier  $t : 1 \rightarrow \Phi$  (see [?]). The reader familiar with type theory will recognize the similarity to the “partiality” or “lifting” monad.

Observe that since  $t$  is monic there is a pullback square,

$$\begin{array}{ccc} X & \xrightarrow{\eta} & X^+ \\ \downarrow \lrcorner & & \downarrow t_* X \\ 1 & \xrightarrow{t} & \Phi. \end{array}$$

Let  $\eta : X \rightarrow X^+$  be the indicated top horizontal map; we call this map the *cofibrant partial map classifier* of  $X$ .

**Proposition 7.** *The map  $\eta : X \rightarrow X^+$  classifies partial maps into  $X$  with cofibrant domain, in the following sense.*

1. *The map  $\eta : X \rightarrow X^+$  is a cofibration.*
2. *For any object  $Z$  and any partial map  $(s, g) : Z \leftarrow S \rightarrow X$ , with  $s : S \rightarrow Z$  a cofibration, there is a unique  $f : Z \rightarrow X^+$  making a pullback square as follows.*

$$\begin{array}{ccc} S & \xrightarrow{g} & X \\ \downarrow s \lrcorner & & \downarrow \eta \\ Z & \xrightarrow{f} & X^+ \end{array}$$

*Proof.* The map  $\eta : X \rightarrow X^+$  is a cofibration since it is a pullback of  $t : 1 \rightarrow \Phi$ . Observe that  $(\eta, 1_X) : X^+ \leftarrow X \rightarrow X$  is therefore a partial map into  $X$  with cofibrant domain. The second statement is the universal property of  $X^+$  as a polynomial (see [?], prop. 7).  $\square$

**Proposition 8.** *The pointed endofunctor determined by  $\eta_X : X \rightarrow X^+$  has a natural multiplication  $\mu_X : X^{++} \rightarrow X^+$  making it a monad.*

*Proof.* Since the cofibrations are closed under composition, the monad structure on  $X^+$  follows as in [?], proposition XY. Explicitly,  $\mu_X$  is determined by proposition 7 as the unique map making the following a pullback diagram.

$$\begin{array}{ccc}
 X & \xrightarrow{=} & X \\
 \eta_X \downarrow & & \downarrow \eta \\
 X^+ & & \\
 \eta_{X^+} \downarrow & & \\
 X^{++} & \xrightarrow{\mu} & X^+
 \end{array}$$

□

**Relative partial map classifier.** For any object  $X \in \mathbf{cSet}$  the usual pullback functor

$$X^* : \mathbf{cSet} \rightarrow \mathbf{cSet}/_X,$$

taking any  $A$  to the second projection  $A \times X \rightarrow X$ , not only preserves the subobject classifier  $\Omega$ , but also the cofibration classifier  $\Phi \hookrightarrow \Omega$ , where a map in  $\mathbf{cSet}/_X$  is defined to be a cofibration if it is one in  $\mathbf{cSet}$ . Thus in  $\mathbf{cSet}/_X$  the *(relative) cofibration classifier* is the map

$$t \times X : 1 \times X \rightarrow \Phi \times X \quad \text{over } X$$

which we may also write  $t_X : 1_X \rightarrow \Phi_X$ . Like  $t : 1 \rightarrow \Phi$ , this map determines a polynomial endofunctor

$$+_X : \mathbf{cSet}/_X \rightarrow \mathbf{cSet}/_X,$$

which commutes (up to natural isomorphism) with  $+$  :  $\mathbf{cSet} \rightarrow \mathbf{cSet}$  and  $X^* : \mathbf{cSet} \rightarrow \mathbf{cSet}/_X$  in the evident way:

$$\begin{array}{ccc}
 \mathbf{cSet}/_X & \xrightarrow{+_X} & \mathbf{cSet}/_X \\
 X^* \uparrow & & \uparrow X^* \\
 \mathbf{cSet} & \xrightarrow{+} & \mathbf{cSet}
 \end{array} \tag{3}$$

The endofunctor  $+_X$  is also pointed  $\eta_Y : Y \rightarrow Y^+$  and has a natural monad multiplication  $\mu_Y : Y^{++} \rightarrow Y^+$ , for any  $Y \rightarrow X$ , for the same reason that  $+$  has this structure. Summarizing, we may say that *the polynomial monad  $+$  :  $\mathbf{cSet} \rightarrow \mathbf{cSet}$  is indexed (or fibered) over  $\mathbf{cSet}$ .*

**Definition 9.** A  $+$ -algebra in  $\mathbf{cSet}$  is a cubical set  $A$  together with a retraction  $\alpha : A^+ \rightarrow A$  of  $\eta_A : A \rightarrow A^+$ , i.e. an algebra for the pointed endofunctor  $(+ : \mathbf{cSet} \rightarrow \mathbf{cSet}, \eta : 1 \rightarrow +)$ . Algebras for the monad  $(+, \eta, \mu)$  will be referred to specifically as  $(+, \eta, \mu)$ -algebras, or  $+$ -monad algebras.

A *relative  $+$ -algebra* in  $\mathbf{cSet}$  is a map  $A \rightarrow X$  together with an algebra structure over the codomain  $X$  for the pointed endofunctor  $+_X : \mathbf{cSet}/_X \rightarrow \mathbf{cSet}/_X$ .

### The cofibration weak factorization system.

**Proposition 10.** *There is an (algebraic) weak factorization system on  $\mathbf{cSet}$  with the cofibrations as the left class and as the right class, the maps underlying the relative  $+$ -algebras. Thus a right map is one  $f : A \rightarrow X$  for which there is a retract  $\alpha : A' \rightarrow A$  over  $X$  of the canonical map  $\eta_f : A \rightarrow A'$ ,*

$$\begin{array}{ccccc}
 & & \overset{=}{\curvearrowright} & & \\
 A & \xrightarrow{\eta_f} & A' & \xrightarrow{\alpha} & A \\
 & \searrow f & \downarrow f^+ & \swarrow f & \\
 & & X & & 
 \end{array}$$

*Proof.* The factorization of a map  $f : Y \rightarrow X$  is given by applying the relative  $+$ -functor over the codomain,

$$\begin{array}{ccc}
 Y & \xrightarrow{\eta_f} & Y' \\
 & \searrow f & \downarrow f^+_X \\
 & & X.
 \end{array}$$

We know by proposition 7 that the unit  $\eta_f$  is always a cofibration, and since  $f^+_X$  is the free algebra for the  $+_X$ -monad, it is in particular a  $+_X$ -algebra.

For the lifting condition, consider a cofibration  $c : B \rightarrowtail C$ , a right map  $A \rightarrow X$ , with  $+_X$ -algebra structure map  $\alpha : A^+_X \rightarrow A$  over  $X$ , and a



commutative square as indicated in the following.

$$\begin{array}{ccc}
 B & \xrightarrow{g} & A \\
 \downarrow c & & \downarrow \eta \\
 C & \xrightarrow{f} & X
 \end{array}
 \quad
 \begin{array}{c}
 \nearrow \alpha \\
 A \searrow \eta \\
 A^{+X}
 \end{array}$$

Thus over  $X$ , we have the situation

$$\begin{array}{ccc}
 B & \xrightarrow{g} & A \\
 \downarrow c & \nearrow d & \downarrow \eta \\
 C & & A^+
 \end{array}
 \quad
 \begin{array}{c}
 \nearrow \alpha \\
 A \searrow \eta \\
 A^+
 \end{array}$$

and we seek a diagonal filler  $d$  as indicated. Since  $(c, g) : B \hookrightarrow C \rightarrow A$  is a cofibrant partial map into  $A$ , there is a map  $\varphi : C \rightarrow A^+$  (over  $X$ ) making a (pullback) square,

$$\begin{array}{ccc}
 B & \xrightarrow{g} & A \\
 \downarrow c & & \downarrow \eta \\
 C & \xrightarrow{\varphi} & A^+
 \end{array}
 \quad
 \begin{array}{c}
 \nearrow \alpha \\
 A \searrow \eta \\
 A^+
 \end{array}$$

We thus have  $d := \alpha \circ \varphi : C \rightarrow A$  as the required diagonal filler.

The closure of the cofibrations under retracts follows from their classification by a universal object  $t : 1 \rightarrow \Phi$ , and the closure of the right maps under retracts follows from their being the algebras for a pointed endofunctor underlying a monad (cf. [?]). Algebraicity of this weak factorization system is immediate, since  $+$  is a monad.  $\square$

Summarizing, we have an algebraic weak factorization system  $(\mathcal{C}, \mathcal{C}^\natural)$  on the category  $\mathbf{cSet}$  of cubical sets, where:

$$\begin{aligned}
 \mathcal{C} &= \text{the cofibrations} \\
 \mathcal{C}^\natural &= \text{the maps underlying relative } +- \text{algebras}
 \end{aligned}$$

We shall call this the *cofibration weak factorization system*. The right maps will be denoted

$$\mathbf{TFib} = \mathcal{C}^\natural$$

and called *trivial fibrations*.

The cofibration algebraic weak factorization system is a refinement of the one defined in [?] and mentioned in [?].

**Uniform filling structure.** It is convenient to relate relative +-algebra structure with the more familiar diagonal filling condition of cofibrantly generated weak factorization systems, and specifically the special form occurring in [CCHM16] under the name *uniform filling structure* (this notion is closely related to that of an *algebraic weak factorization system*, cf. [GKR18, Rie14]).

Consider a generating subset of cofibrations consisting of those with representable codomain  $c : C \rightarrow I^n$ , and call these the *basic cofibrations*.

$$\mathbf{BCof} = \{c : C \rightarrow I^n \mid c \in \mathcal{C}, n \geq 0\}. \quad (4)$$

**Proposition 11.** *For any object  $X$  in  $\mathbf{cSet}$  the following are equivalent:*

1.  $X$  admits a +-algebra structure: a retraction  $\alpha : X^+ \rightarrow X$  of the unit  $\eta : X \rightarrow X^+$ .
2.  $X \rightarrow 1$  is a trivial fibration: it has the right lifting property with respect to all cofibrations,

$$\mathcal{C} \pitchfork X.$$

3.  $X$  admits a uniform filling structure: for each basic cofibration  $c : C \rightarrow I^n$  and map  $x : C \rightarrow X$  there is given an extension  $j(c, x)$ ,

$$\begin{array}{ccc} C & \xrightarrow{x} & X, \\ c \downarrow & \nearrow j(c, x) & \\ I^n & & \end{array} \quad (5)$$

and the choice is uniform in  $I^n$  in the following sense.

Given any cubical map  $u : I^m \rightarrow I^n$ , the pullback  $u^*c : u^*C \rightarrow I^m$ , which is again a basic cofibration, fits into a commutative diagram of the form

$$\begin{array}{ccccc} u^*C & \xrightarrow{c^*u} & C & \xrightarrow{x} & X. \\ u^*c \downarrow & \lrcorner & c \downarrow & \nearrow j(c, x) & \\ I^m & \xrightarrow{u} & I^n & & \end{array} \quad (6)$$

For the pair  $(u^*c, x \circ c^*u)$  in (6), the chosen extension  $j(u^*c, x \circ c^*u) : I^m \rightarrow X$ , is required to be equal to  $j(c, x) \circ u$ ,

$$j(u^*c, x \circ c^*u) = j(c, x) \circ u. \quad (7)$$

*Proof.* Let  $(X, \alpha)$  be a  $+-$ -algebra and suppose given the span  $(c, x)$  as below, with  $c$  a cofibration.

$$\begin{array}{ccc} C & \xrightarrow{x} & X \\ c \downarrow & & \\ Z & & \end{array}$$

Let  $\chi(c, x) : Z \rightarrow X^+$  be the classifying map of the evident partial map  $(c, x) : Z \rightarrow X$ , so that we have a pullback square as follows.

$$\begin{array}{ccc} C & \xrightarrow{x} & X \\ c \downarrow \lrcorner & & \downarrow \eta \\ Z & \xrightarrow{\chi(c, x)} & X^+ \end{array} \quad (8)$$

Then set

$$j = \alpha \circ \chi(c, x) : Z \rightarrow X \quad (9)$$

to get a filler,

$$\begin{array}{ccc} C & \xrightarrow{x} & X \\ c \downarrow & \nearrow j & \downarrow \eta \\ Z & \xrightarrow{\chi(c, x)} & X^+ \end{array} \quad \begin{array}{c} \alpha \\ \curvearrowright \end{array} \quad (10)$$

since

$$j \circ c = \alpha \circ \chi(c, x) \circ c = \alpha \circ \eta \circ x = x.$$

Thus (1) implies (2). To see that it also implies (3), observe that in the case where  $Z = I^n$  and we specify, in (9), that

$$j(c, x) = \alpha \circ \chi(c, x) : I^n \rightarrow X, \quad (11)$$

then the assignment is natural in  $I^n$ . Indeed, given any  $u : I^m \rightarrow I^n$ , we have

$$j(c', xu') = \alpha \circ \chi(c', xu') = \alpha \circ \chi(c, x) \circ u = j(c, x)u, \quad (12)$$

by the uniqueness of the classifying maps.

It is clear that (2) implies (1), since if  $\mathcal{C} \dashv X$  then we can take as an algebra structure  $\alpha : X^+ \rightarrow X$  any filler for the span

$$\begin{array}{ccc} X & \xrightarrow{=} & X \\ \eta \downarrow & \nearrow \alpha & \\ X^+ & & \end{array}$$

To see that (3) implies (1), suppose that  $X$  has a uniform filling structure  $j$  and we want to define an algebra structure  $\alpha : X^+ \rightarrow X$ . By Yoneda, for every  $y : I^n \rightarrow X^+$  we need a map  $\alpha(y) : I^n \rightarrow X$ , naturally in  $I^n$ , in the sense that for any  $u : I^m \rightarrow I^n$ , we have

$$\alpha(yu) = \alpha(y)u. \quad (13)$$

Moreover, to ensure that  $\alpha\eta = 1_X$ , for any  $x : I^n \rightarrow X$  we must have  $\alpha(\eta \circ x) = x$ . So take  $y : I^n \rightarrow X^+$  and let

$$\alpha(y) = j(y^*\eta, y'),$$

as indicated on the right below.

$$\begin{array}{ccccc} u^*C & \xrightarrow{u'} & C & \xrightarrow{y'} & X \\ \downarrow u^*y^*\eta & \lrcorner & \downarrow y^*\eta & \nearrow j(y^*\eta, y') & \downarrow \eta \\ I^m & \xrightarrow{u} & I^n & \xrightarrow{y} & X^+ \end{array} \quad (14)$$

Then for any  $u : I^m \rightarrow I^n$ , we indeed have

$$\alpha(yu) = j((yu)^*\eta, y'u') = j(y^*\eta, y') \circ u = \alpha(y)u,$$

by the uniformity of  $j$ . Finally, if  $y = \eta \circ x$  for some  $x : I^n \rightarrow X$  then

$$\alpha(\eta x) = j((\eta x)^*\eta, (\eta x)') = j(1_X, x) = x,$$

because the defining diagram for  $\alpha(\eta x)$ , i.e. the one on the right in (14), then factors as

$$\begin{array}{ccccc} I^n & \xrightarrow{x} & X & \xrightarrow{=} & X \\ \downarrow \lrcorner & & \downarrow \lrcorner & & \downarrow \eta \\ I^n & \xrightarrow{x} & X & \xrightarrow{\eta} & X^+ \end{array} \quad (15)$$

and the only possible extension  $j(1_X, x)$  for the span  $(1_{I^n}, x)$  is  $x$  itself.  $\square$

*Remark 12.* Observe that the uniformity condition (3) can be extended to the *class of all* cofibrations, in the form:

4.  $X$  admits a (large) *uniform filling structure*: for each cofibration  $c : C \rightarrowtail Z$  and map  $x : C \rightarrow X$  there is given an extension  $j(c, x)$ ,

$$\begin{array}{ccc} C & \xrightarrow{x} & X, \\ \downarrow c & \nearrow j(c, x) & \\ Z & & \end{array} \quad (16)$$

and the choice is *uniform in  $Z$*  in the following sense: Given any map  $u : Y \rightarrow Z$ , the pullback  $u^*c : u^*C \rightarrowtail Y$ , which is again a cofibration, fits into a commutative diagram of the form

$$\begin{array}{ccccc} u^*C & \xrightarrow{c^*u} & C & \xrightarrow{x} & X. \\ \downarrow u^*c & \lrcorner & \downarrow c & \nearrow j(c, x) & \\ Y & \xrightarrow{u} & Z & & \end{array} \quad (17)$$

For the pair  $(u^*c, x \circ c^*u)$  in (17), the chosen extension  $j(u^*c, x \circ c^*u) : I^m \rightarrow X$ , is required to be equal to  $j(c, x) \circ u$ ,

$$j(u^*c, x \circ c^*u) = j(c, x) \circ u. \quad (18)$$

Indeed, the proof that (1) implies (2) and (3) works just as well to infer (4), which in turn implies (2) and (3) as special cases.

The relative version of the foregoing is entirely analogous, since the  $+$ -functor is fibered over  $\mathbf{cSet}$  in the sense of diagram (3). We can therefore omit the entirely analogous proof of the following.

**Proposition 13.** *For any map  $f : Y \rightarrow X$  in  $\mathbf{cSet}$  the following are equivalent:*

1.  $f : Y \rightarrow X$  admits a relative  $+$ -algebra structure over  $X$ , i.e. there is a retraction  $\alpha : Y' \rightarrow Y$  over  $X$  of the unit  $\eta : Y \rightarrow Y'$ , where  $f^+ : Y' \rightarrow X$  is the result of the relative  $+$ -functor applied to  $f$ , as in definition 9.
2.  $f : Y \rightarrow X$  is a trivial fibration,

$$\mathcal{C} \Vdash f.$$

3.  $f : Y \rightarrow X$  admits a (small) uniform filling structure: for each basic cofibration  $c : C \rightarrow I^n$  and maps  $x : C \rightarrow X$  and  $y : I^n \rightarrow Y$  making the square below commute, there is given a diagonal filler  $j(c, x, y)$ ,

$$\begin{array}{ccc} C & \xrightarrow{x} & X \\ c \downarrow & \nearrow j(c, x, y) & \downarrow f \\ I^n & \xrightarrow{y} & Y, \end{array} \quad (19)$$

and the choice is uniform in  $I^n$  in the following sense: given any cubical map  $u : I^m \rightarrow I^n$ , the pullback  $u^*c : u^*C \rightarrow I^m$  is again a basic cofibration and fits into a commutative diagram of the form

$$\begin{array}{ccccc} u^*C & \xrightarrow{c^*u} & C & \xrightarrow{x} & X \\ u^*c \downarrow \lrcorner & & c \downarrow & \nearrow j(c, x, y) & \downarrow f \\ I^m & \xrightarrow{u} & I^n & \xrightarrow{y} & Y. \end{array} \quad (20)$$

For the evident triple  $(u^*c, x \circ c^*u, y \circ u)$  in (20) the chosen diagonal filler

$$j(u^*c, x \circ c^*u, y \circ u) : I^m \rightarrow X$$

is equal to  $j(c, x, y) \circ u$ ,

$$j(u^*c, x \circ c^*u, y \circ u) = j(c, x, y) \circ u. \quad (21)$$

We next collect some basic facts about trivial fibrations: they have sections, they are closed under composition and retracts, and they are closed under pullback and pushforward along all maps.

**Corollary 14.** 1. Every trivial fibration  $A \rightarrow X$  has a section  $s : X \rightarrow A$ , which is a cofibration. *[add this!]*

2. If  $a : A \rightarrow X$  is a trivial fibration and  $b : B \rightarrow A$  is a trivial fibration, then  $a \circ b : B \rightarrow X$  is a trivial fibration.

3. If  $a : A \rightarrow X$  is a trivial fibration and  $a' : A' \rightarrow X'$  is a retract of  $a$  in the arrow category, then  $a'$  is a trivial fibration.

4. For any map  $f : X \rightarrow Y$  and any trivial fibration  $B \rightarrow Y$ , the pullback  $f^*B \rightarrow X$  is a trivial fibration.

5. For any map  $f : X \rightarrow Y$  and any trivial fibration  $A \rightarrow X$ , the push-forward  $f_*A \rightarrow Y$  is a trivial fibration.

*Proof.* (1) holds because all objects are cofibrant by (C0). (5) is a consequence of (C3), stability of cofibrations under pullback, by a standard argument using the adjunction  $f^* \dashv f_*$ . The rest hold for the right maps in any weak factorization system.  $\square$

*Remark 15.* The structured notion of trivial fibration, *vis.* relative  $+$ -algebra, can also be shown algebraically (i.e. not using Proposition 13) to be preserved by pullback, pushforward, composition, and to be closed under retracts. We do just the case of pushforward as an example. Thus consider the following situation with  $A \rightarrow X$  a  $+$ -algebra with structure  $\alpha$ , as indicated.

$$\begin{array}{ccc}
 A & \xleftarrow{\alpha} & A^+ \\
 \downarrow \eta_A & \searrow & \downarrow \eta_{f_*A} \\
 X & \xrightarrow{f} & Y
 \end{array}
 \quad , \quad
 \begin{array}{ccc}
 f_*A & \xleftarrow{\beta} & (f_*A)^+ \\
 \downarrow \eta_{f_*A} & \searrow & \downarrow \eta_{f_*A} \\
 Y & \xrightarrow{f} & Y
 \end{array}
 \quad (22)$$

A  $+$ -algebra structure for  $f_*A \rightarrow Y$  would be a retract  $\beta : (f_*A)^+ \rightarrow f_*A$  of  $\eta_{f_*A} : f_*A \rightarrow (f_*A)^+$  over  $Y$ , which corresponds under  $f^* \dashv f_*$  to a map  $\tilde{\beta} : f^*((f_*A)^+) \rightarrow A$  over  $X$  with

$$\tilde{\beta} \circ f^* \eta_{f_*A} = \epsilon_A$$

as indicated below.

$$\begin{array}{ccc}
 f^* f_* A & \xrightarrow{f^* \eta_{f_*A}} & f^*((f_*A)^+) \\
 \searrow \epsilon_A & & \downarrow \tilde{\beta} \\
 & & A \xrightarrow{\eta_A} A^+
 \end{array}
 \quad (23)$$

But since pullback  $f^*$  commutes with  $+$ , there is a canonical iso  $c : f^*((f_*A)^+) \cong$

$(f^* f_* A)^+$  with  $c \circ f^* \eta_{f_* A} = \eta_{f^* f_* A}$ . So we can set  $\tilde{\beta} := \alpha \circ (\epsilon_A)^+ \circ c$ .

$$\begin{array}{ccc}
 & \xrightarrow{\eta_{f^* f_* A}} & \\
 f^* f_* A & \xrightarrow{f^* \eta_{f_* A}} f^*((f_* A)^+) \xrightarrow{\sim_c} (f^* f_* A)^+ & \\
 & \searrow \epsilon_A & \downarrow (\epsilon_A)^+ \\
 & A & \xrightarrow{\eta_A} A^+ \\
 & \nwarrow \tilde{\beta} & \nearrow \alpha
 \end{array} \quad (24)$$

### 3 The fibration weak factorization system

We now specify a second weak factorization system, with a restricted class of “trivial” cofibrations on the left, and an expanded class of right maps, the *fibrations*. As explained in the introduction, we first recall from [GS17] what we shall call the “biased” notion of fibration, before giving the “unbiased” one used in our more general setting. The biased version makes use of *connections*,

$$\vee, \wedge : I \times I \longrightarrow I,$$

on the cubes, which are also used in [?] to give a model structure. We do not assume connections here. In [AGH21] it is shown that the fibrations of [GS17, ?] agree with those specified in the “logical style” of [CCHM16, OP17].

#### 3.1 Partial box filling (biased version)

The *generating biased trivial cofibrations* are all maps of the form

$$c \otimes \delta_\epsilon : D \twoheadrightarrow Z \times I, \quad (25)$$

where:

1.  $c : C \twoheadrightarrow Z$  is an arbitrary cofibration,
2.  $\delta_\epsilon : 1 \rightarrow I$  is one of the two *endpoint inclusions*, for  $\epsilon = 0, 1$ .



3.  $c \otimes \delta_\epsilon$  is the *pushout-product* indicated in the following diagram.

$$\begin{array}{ccc}
 C \times 1 & \xrightarrow{C \times \delta_\epsilon} & C \times I \\
 \downarrow C \times 1 & & \downarrow \\
 Z \times 1 & \longrightarrow & Z +_C (C \times I) \\
 & \searrow Z \times \delta_\epsilon & \nearrow c \otimes \delta_\epsilon \\
 & & Z \times I
 \end{array}
 \quad (26)$$

4.  $D = Z +_C (C \times I)$  is the indicated domain of the map  $c \otimes \delta_\epsilon$ .

In order to ensure that such maps are indeed cofibrations, we assume two further axioms in addition to (C1)–(C4) from Definition 6:

(C5) The endpoint inclusions  $\delta_\epsilon : 1 \rightarrow I$  are cofibrations, for  $\epsilon = 0, 1$ .

(C6) The cofibrations are closed under pushout-products  $- \otimes -$  in the arrow category.

Note that if we assume that  $\delta_0$  and  $\delta_1$  are disjoint (as they are in most categories of cubical sets), then by (C5) (and (C3), stability under pullback) we have that  $0 \rightarrow 1$  is a cofibration, so that, again by (C3),  $0 \rightarrow A$  is a cofibration, for all objects  $A$ , so that (C0) is no longer required. In place of (C6), we could require the cofibrations to be closed under the join operation  $A \vee B$  in the lattice of subobjects of any object (as is done in [CCHM16, OP17]).

**Fibrations (biased version).** Now let

$$\mathcal{C} \otimes \delta_\epsilon = \{c \otimes \delta_\epsilon : D \rightarrow Z \times I \mid c \in \mathcal{C}, \epsilon = 0, 1\}$$

be the class of all generating biased trivial cofibrations. The *biased fibrations* are defined to be the right class of these maps,

$$(\mathcal{C} \otimes \delta_\epsilon)^\text{h} = \mathcal{F}.$$

Thus a map  $f : Y \rightarrow X$  is a biased fibration just if for every commutative square of the form

$$\begin{array}{ccc}
 Z +_C (C \times I) & \longrightarrow & Y \\
 \downarrow c \otimes \delta_\epsilon & \nearrow j & \downarrow f \\
 Z \times I & \longrightarrow & X
 \end{array}
 \quad (27)$$

with a generating biased trivial cofibration on the left, there is a diagonal filler  $j$  as indicated.

To relate this notion of fibration to the cofibration weak factorization system, fix any map  $u : A \rightarrow B$ , and recall (e.g. from [?, ?]) that the pushout-product with  $u$  is a functor on the arrow category

$$(-) \otimes u : \mathbf{cSet}^2 \rightarrow \mathbf{cSet}^2.$$

This functor has a right adjoint, the *pullback-hom*, which for a map  $f : X \rightarrow Y$  we shall write as

$$(u \Rightarrow f) : Y^B \longrightarrow (X^B \times_{X^A} Y^A).$$

The pullback-hom is determined as indicated in the following diagram.

$$\begin{array}{ccc} Y^B & \xrightarrow{Y^u} & Y^A \\ \text{\scriptsize $u \Rightarrow f$} \searrow \text{dotted} & & \downarrow \text{\scriptsize $f^A$} \\ & X^B \times_{X^A} Y^A & \longrightarrow Y^A \\ \text{\scriptsize $f^B$} \searrow \text{curved} & \downarrow & \downarrow \text{\scriptsize $f^A$} \\ & X^B & \xrightarrow{X^u} X^A \end{array} \quad (28)$$

The  $\otimes \dashv \Rightarrow$  adjunction on the arrow category has the following useful relation to weak factorization systems (cf. [GS17, Rie14, ?]), where, as usual, for any maps  $a : A \rightarrow B$  and  $f : X \rightarrow Y$  we write

$$a \pitchfork f$$

to mean that for every solid square of the form

$$\begin{array}{ccc} A & \longrightarrow & X \\ a \downarrow & \nearrow \text{\scriptsize $j$} \text{ dotted} & \downarrow f \\ B & \longrightarrow & Y \end{array} \quad (29)$$

there exists a diagonal filler  $j$  as indicated.

**Lemma 16.** *For any maps  $a : A_0 \rightarrow A_1, b : B_0 \rightarrow B_1, c : C_0 \rightarrow C_1$  in  $\mathbf{cSet}$ ,*

$$(a \otimes b) \pitchfork c \quad \text{iff} \quad a \pitchfork (b \Rightarrow c).$$

The following is now a direct corollary.

**Proposition 17.** *An object  $X$  is fibrant if and only if both of the endpoint projections  $X^I \rightarrow X$  from the pathspace are trivial fibrations. More generally, a map  $f : Y \rightarrow X$  is a fibration iff both of the maps*

$$(\delta_\epsilon \Rightarrow f) : Y^I \rightarrow X^I \times_X Y$$

*are trivial fibrations (for  $\epsilon = 0, 1$ ).*

**Fibration structure (biased version).** The  $\otimes \dashv \Rightarrow$  adjunction determines the fibrations in terms of the trivial fibrations, which in turn can be determined by *uniform* lifting against a *small category* consisting of basic cofibrations and pullback squares between them, by proposition 13. The fibrations are similarly determined by *uniform* lifting against the *small category* of basic, biased trivial cofibrations, consisting of all those  $c \otimes \delta_\epsilon$  in  $\mathcal{C} \otimes \delta_\epsilon$  where  $c : C \rightarrow I^n$  is a *basic* cofibration, *i.e.* one with representable codomain. Thus the set of *basic biased trivial cofibrations* is

$$\text{BCof} \otimes \delta_\epsilon = \{c \otimes \delta_\epsilon : B \rightarrow I^{n+1} \mid c : C \rightarrow I^n, \epsilon = 0, 1, n \geq 0\}, \quad (30)$$

where the pushout-product  $c \otimes \delta_\epsilon$  now takes the simpler form

$$\begin{array}{ccc} C & \longrightarrow & C \times I \\ \downarrow & & \downarrow \\ I^n & \longrightarrow & I^n +_C (C \times I) \\ & \searrow & \downarrow c \otimes \delta_\epsilon \\ & & I^n \times I \end{array} \quad (31)$$

for a basic cofibration  $c : C \rightarrow I^n$ , an endpoint  $\delta_\epsilon : 1 \rightarrow I$ , and with domain  $B = (I^n +_C (C \times I))$ . These subobjects  $B \rightarrow I^{n+1}$  can be seen geometrically as generalized open box inclusions.

For any map  $f : Y \rightarrow X$  a *uniform, biased fibration structure* on  $f$  is a choice of diagonal fillers  $j_\epsilon(c, x, y)$ ,

$$\begin{array}{ccc} I^n +_C (C \times I) & \xrightarrow{x} & X \\ c \otimes \delta_\epsilon \downarrow & \searrow j_\epsilon(c, x, y) & \downarrow f \\ I^n \times I & \xrightarrow{y} & Y, \end{array} \quad (32)$$

for each basic biased trivial cofibration  $c \otimes \delta_\epsilon : B = (\mathbb{I}^n +_C (C \times \mathbb{I})) \rightarrow \mathbb{I}^{n+1}$  and maps  $x : B \rightarrow X$  and  $y : \mathbb{I}^{n+1} \rightarrow Y$ , which is *uniform in*  $\mathbb{I}^n$  in the following sense: Given any cubical map  $u : \mathbb{I}^m \rightarrow \mathbb{I}^n$ , the pullback  $u^*c : u^*C \rightarrow \mathbb{I}^m$  of  $c : C \rightarrow \mathbb{I}^n$  along  $u$  determines another basic biased trivial cofibration

$$u^*c \otimes \delta_\epsilon : B' = (\mathbb{I}^m +_{u^*C} (u^*C \times \mathbb{I})) \rightarrow \mathbb{I}^{m+1},$$

which fits into a commutative diagram of the form

$$\begin{array}{ccccc} \mathbb{I}^m +_{u^*C} (u^*C \times \mathbb{I}) & \xrightarrow{(u \times \mathbb{I})'} & \mathbb{I}^n +_C (C \times \mathbb{I}) & \xrightarrow{x} & X \\ u^*c \otimes \delta_\epsilon \downarrow & & c \otimes \delta_\epsilon \downarrow & \nearrow j_\epsilon(c, x, y) & \downarrow f \\ \mathbb{I}^m \times \mathbb{I} & \xrightarrow{u \times \mathbb{I}} & \mathbb{I}^n \times \mathbb{I} & \xrightarrow{y} & Y, \end{array} \quad (33)$$

by applying the functor  $(-) \otimes \delta_\epsilon$  to the pullback square relating  $u^*c$  to  $c$ . For the outer rectangle in (33) there is then a chosen diagonal filler

$$j_\epsilon(u^*c, x \circ (u \times \mathbb{I})', y \circ (u \times \mathbb{I})) : \mathbb{I}^m \times \mathbb{I} \rightarrow X,$$

and for this map we require that

$$j_\epsilon(u^*c, x \circ (u \times \mathbb{I})', y \circ (u \times \mathbb{I})) = j_\epsilon(c, x, y) \circ (u \times \mathbb{I}). \quad (34)$$

This can be seen to be a reformulation of the logical specification given in [CCHM16] (see [AGH21]).

**Definition 18.** A *uniform, biased fibration structure* on a map  $f : Y \rightarrow X$  is a choice of fillers  $j_\epsilon(c, x, y)$  as in (32) satisfying (34) for all maps  $u : \mathbb{I}^m \rightarrow \mathbb{I}^n$ .

Finally, we have the analogue of proposition 11 for fibrant objects. The analogous statement of proposition 13 for fibrations is omitted, as is the entirely analogous proof.

**Corollary 19.** *For any object  $X$  in  $\mathbf{cSet}$  the following are equivalent:*

1.  $X$  is biased fibrant, in the sense that every map  $D \rightarrow X$  from the domain of a generating biased trivial cofibration  $D \rightarrow Z \times \mathbb{I}$  extends to a total map  $Z \times \mathbb{I} \rightarrow X$ ,

$$\mathcal{C} \otimes \delta_\epsilon \dashv X.$$

2. The canonical maps  $(\delta_\epsilon \Rightarrow X) : X^I \rightarrow X$  are trivial fibrations.
3.  $X \rightarrow 1$  admits a uniform biased fibration structure. Explicitly, for each basic biased trivial cofibration  $c \otimes \delta_\epsilon : B \rightarrow I^{n+1}$  and map  $x : B \rightarrow X$ , there is given an extension  $j_\epsilon(c, x)$ ,

$$\begin{array}{ccc} B & \xrightarrow{x} & X, \\ c \otimes \delta_\epsilon \downarrow & \nearrow j_\epsilon(c, x) & \\ I^{n+1} & & \end{array} \quad (35)$$

and, moreover, the choice is uniform in  $I^n$  in the following sense: Given any cubical map  $u : I^m \rightarrow I^n$ , the pullback  $u^*c \otimes \delta_\epsilon : B' \rightarrow I^m \times I$  fits into a commutative diagram of the form

$$\begin{array}{ccccc} B' & \xrightarrow{(u \times I)'} & B & \xrightarrow{x} & X. \\ u^*c \otimes \delta_\epsilon \downarrow \lrcorner & & c \otimes \delta_\epsilon \downarrow & \nearrow j(c, x) & \\ I^m \times I & \xrightarrow{u \times I} & I^n \times I & & \end{array} \quad (36)$$

For the pair  $(u^*c \otimes \delta_\epsilon, x \circ (u \times I)')$  in (36) the chosen extension

$$j(u^*c \otimes \delta_\epsilon, x \circ (u \times I)') : I^m \times I \rightarrow X$$

is equal to  $j(c, x) \circ (u \times I)$ ,

$$j(u^*c \otimes \delta_\epsilon, x \circ (u \times I)') = j(c, x)(u \times I). \quad (37)$$

### 3.2 Partial box filling (unbiased version)

Rather than building a weak factorization system based on the foregoing notion of biased fibration (as is done in [GS17]), we shall first eliminate the “bias” with respect to the endpoints  $\delta_\epsilon : 1 \rightarrow I$ , for  $\epsilon = 0, 1$ . This will have the effect of adding more trivial cofibrations, and thus more weak equivalences, to our model structure. Consider first the simple path-lifting condition for a map  $f : Y \rightarrow X$ , which is a special case of (27) with  $c = ! : 0 \rightarrow 1$ , so that  $! \otimes \delta_\epsilon = \delta_\epsilon$ .

$$\begin{array}{ccc} 1 & \longrightarrow & Y \\ \delta_\epsilon \downarrow & \nearrow j_\epsilon & \downarrow f \\ I & \longrightarrow & X \end{array}$$

In topological spaces, for instance, rather than requiring lifts  $j_\epsilon$  for each of the endpoints  $\epsilon = 0, 1$  of the real interval  $I = [0, 1]$ , one could equivalently require there to be a lift  $j_i$  for each point  $i : 1 \rightarrow I$ . Such “unbiased path-lifting” can be formulated in  $\mathbf{cSet}$  by introducing a “generic point”  $\delta : 1 \rightarrow I$  by passing to  $\mathbf{cSet}/_I$  via the pullback functor  $I^* : \mathbf{cSet} \rightarrow \mathbf{cSet}/_I$ , and then requiring path-lifting for  $I^*f$  with respect to  $\delta : I \rightarrow I \times I$ , regarded as a map  $\delta : 1 \rightarrow I^*I$  in  $\mathbf{cSet}/_I$ . We shall therefore define  $f$  to be an unbiased fibration just if  $I^*f$  is a  $\delta$ -biased fibration for the generic point  $\delta$ . The following specification implements that idea, while also adding cofibrant partiality, as in the biased case.

We first replace axiom (C5) with the following stronger assumption.

(C7) The diagonal map  $\delta : I \rightarrow I \times I$  of the interval  $I$  is a cofibration.

The unbiased notion of a fibration for  $\mathbf{cSet}$  is now as follows.

**Definition 20** (unbiased fibration). Let  $\delta : I \rightarrow I \times I$  be the diagonal map.

1. An object  $X$  is *unbiased fibrant* if the map

$$(\delta \Rightarrow X) = \langle \text{eval}, p_2 \rangle : X^I \times I \rightarrow X \times I$$

is a trivial fibration.

2. A map  $f : Y \rightarrow X$  is an *unbiased fibration* if the map

$$(\delta \Rightarrow f) = \langle f^I \times I, \langle \text{eval}, p_2 \rangle \rangle : Y^I \times I \rightarrow (X^I \times I) \times_{(X \times I)} (Y \times I)$$

is a trivial fibration.

Let us (temporarily) write  $\mathbb{I} = I^*I$  for the pulled-back interval in the slice category  $\mathbf{cSet}/_I$ , so that the generic point is written  $\delta : 1 \rightarrow \mathbb{I}$ . Condition (1) above (which of course is a special case of (2)) then says that evaluation at the generic point  $\delta : 1 \rightarrow \mathbb{I}$ , the map  $(I^*X)^\delta : (I^*X)^\mathbb{I} \rightarrow I^*X$ , constructed in the slice category  $\mathbf{cSet}/_I$ , is a trivial fibration. Condition (2) says that the pullback-hom of the generic point  $\delta : 1 \rightarrow \mathbb{I}$  with  $I^*f$ , constructed in the slice category  $\mathbf{cSet}/_I$ , is a trivial fibration. Thus a map  $f : Y \rightarrow X$  is an *unbiased fibration* just if its base change  $I^*f$  is a  $\delta$ -biased fibration in the slice category  $\mathbf{cSet}/_I$ . The latter condition can also be reformulated as follows.

**Proposition 21.** *A map  $f : Y \rightarrow X$  is an unbiased fibration if and only if the canonical map  $u$  to the pullback, in the following diagram in  $\mathbf{cSet}$ , is a trivial fibration.*

$$\begin{array}{ccccc}
 Y^I \times I & \xrightarrow{\quad \text{eval} \quad} & Y & & \\
 \downarrow f^I \times I & \searrow u & \downarrow Y_{\text{eval}} & \xrightarrow{\quad} & \downarrow f \\
 & & Y_{\text{eval}} & \xrightarrow{\quad} & Y \\
 & & \downarrow \lrcorner & & \downarrow \\
 & & X^I \times I & \xrightarrow{\quad \text{eval} \quad} & X
 \end{array} \tag{38}$$

*Proof.* We interpolate another pullback into the rectangle in (40) to obtain

$$\begin{array}{ccccc}
 Y_{\text{eval}} & \xrightarrow{\quad} & Y \times I & \xrightarrow{\quad} & Y \\
 \downarrow \lrcorner & & \downarrow \lrcorner & & \downarrow f \\
 X^I \times I & \xrightarrow{\quad} & X \times I & \xrightarrow{\quad} & X
 \end{array} \tag{39}$$

with the evident maps. The left hand square is therefore a pullback, so we indeed have that

$$Y_{\text{eval}} \cong (X^I \times I) \times_{(X \times I)} (Y \times I) \cong (X^I \times I) \times_X Y$$

and  $u = (\delta \Rightarrow f)$ . □

As a special case, we have:

**Corollary 22.** *An object  $X$  is unbiased fibrant if and only if the canonical map  $u$  to the pullback, in the following diagram in  $\mathbf{cSet}$ , is a trivial fibration.*

$$\begin{array}{ccccc}
 X^I \times I & \xrightarrow{\quad \text{eval} \quad} & X & & \\
 \downarrow p_2 & \searrow u & \downarrow I \times X & \xrightarrow{\quad} & \downarrow \\
 & & I & \xrightarrow{\quad} & 1
 \end{array} \tag{40}$$

Now we can run the proof of Proposition 17 backwards in order to determine a class of generating trivial cofibrations for the unbiased case. Consider

pairs of maps  $c : C \rightarrowtail Z$  and  $i : Z \rightarrow I$ , where the former is a cofibration and the latter is regarded as an “I-indexing”, so that

$$\begin{array}{ccc} C & \xrightarrow{c} & Z \\ & \searrow & \downarrow i \\ & & I \end{array}$$

is regarded as an “I-indexed family of cofibrations  $c_i : C_i \rightarrowtail Z_i$ ”. We shall use the notation

$$\langle i \rangle := \langle 1_Z, i \rangle : Z \longrightarrow Z \times I, \quad (41)$$

for the graph of the indexing map  $i : Z \rightarrow I$ . Then write

$$c \otimes_i \delta := [\langle i \rangle, c \times I] : Z +_C (C \times I) \longrightarrow Z \times I,$$

which is easily seen to be well-defined on the indicated pushout below.

$$\begin{array}{ccccc} C & \xrightarrow{\langle ic \rangle} & C \times I & & \\ \downarrow c & & \downarrow & \searrow c \times I & \\ Z & \longrightarrow & Z +_C (C \times I) & \xrightarrow{c \otimes_i \delta} & Z \times I \\ & \searrow \langle i \rangle & & \nearrow & \end{array} \quad (42)$$

*Remark 23.* The specification (42) differs from the similar (26) by using the graph  $\langle i \rangle : Z \rightarrowtail Z \times I$  for the inclusion of  $Z$  into the cylinder over  $Z$ , rather than one of the two “ends”,

$$\langle 1_Z, \delta_\epsilon ! \rangle : Z \cong Z \times 1 \xrightarrow{Z \times \delta_\epsilon} Z \times I \quad (43)$$

arising from the endpoint inclusions  $\delta_\epsilon : 1 \rightarrow I$ , for  $\epsilon = 0, 1$ . Note that as an arrow over  $I$ , the graph  $\langle i \rangle : Z \rightarrowtail Z \times I$  also takes the form (43), namely

$$\langle i \rangle = \langle 1_Z, \delta ! \rangle : Z \rightarrowtail Z \times I.$$

If we also regard  $c : C \rightarrowtail Z$  as an arrow over  $I$  via  $i : Z \rightarrow I$ , and use the generic point  $\delta : 1 \rightarrow I$  over  $I$  in place of  $\delta_\epsilon : 1 \rightarrow I$ , then (42) agrees with (26), up to those changes. Thus the indicated map  $c \otimes_i \delta$  in (42) is the pushout-product constructed over  $I$  of the generic point  $\delta$  with the map  $c$  regarded as an  $I$ -indexed family of cofibrations via the indexing  $i : Z \rightarrow I$ .



Observe that for any map  $f : X \rightarrow Y$ , the graph  $\langle f \rangle = \langle 1_X, f \rangle : X \rightarrow X \times Y$  is a cofibration, as a pullback of the diagonal of  $Y$  along  $f \times Y$ . The subobject

$$c \otimes_i \delta \rightharpoonup Z \times I$$

constructed in (42) is then the join, in the lattice  $\mathbf{Sub}(Z \times I)$ , of the (cofibrant) subobjects  $\langle i \rangle \rightharpoonup Z \times I$  and  $C \times I \rightharpoonup Z \times I$ , where the latter is the “cylinder over  $C \rightharpoonup Z$ ”.

**Definition 24.** The maps of the form  $c \otimes_i \delta : Z +_C (C \times I) \rightharpoonup Z \times I$  now form the class of *generating unbiased trivial cofibrations*,

$$\mathcal{C} \otimes \delta = \{c \otimes_i \delta : D \rightharpoonup Z \times I \mid c : C \rightharpoonup Z, i : Z \rightarrow I\}. \quad (44)$$

The unbiased fibrations are exactly the right class of these maps,

$$(\mathcal{C} \otimes \delta)^\natural = \mathcal{F}.$$

**Proposition 25.** A map  $f : Y \rightarrow X$  is an unbiased fibration iff for every pair of maps  $c : C \rightharpoonup Z$  and  $i : Z \rightarrow I$ , where the former is a cofibration, every commutative square of the following form has a diagonal filler, as indicated in the following.

$$\begin{array}{ccc} Z +_C (C \times I) & \xrightarrow{\quad} & Y \\ c \otimes_i \delta \downarrow & \swarrow j & \downarrow f \\ Z \times I & \xrightarrow{\quad} & X. \end{array} \quad (45)$$

*Proof.* Suppose that for all  $c : C \rightharpoonup Z$  and  $i : Z \rightarrow I$ , we have  $(c \otimes_i \delta) \natural f$  in  $\mathbf{cSet}$ . Pulling  $f$  back over  $I$ , this is equivalent to the condition  $c \otimes \delta \natural I^*f$  in  $\mathbf{cSet}/_I$ , for all cofibrations  $c : C \rightharpoonup Z$  over  $I$ , which is equivalent to  $c \natural (\delta \Rightarrow I^*f)$  in  $\mathbf{cSet}/_I$  for all cofibrations  $c : C \rightharpoonup Z$ . But this in turn means that  $\delta \Rightarrow I^*f$  is a trivial fibration, which by definition means that  $f$  is an unbiased fibration.  $\square$

*Remark 26.* Note that the endpoints  $\delta_\epsilon : 1 \rightarrow I$ , in particular, are of the form  $c \otimes_i \delta$  by taking  $Z = 1$  and  $i = \delta_\epsilon$  and  $c = ! : 0 \rightarrow 1$ , so that the case of biased filling is subsumed. Moreover, for any  $i : Z \rightarrow I$  the graph  $\langle i \rangle \rightharpoonup Z \times I$  is itself of the form  $0 \otimes_i \delta$  for the cofibration  $0 \rightharpoonup Z$ , so the graph of any “ $I$ -indexing” map  $i : Z \rightarrow I$  is also a trivial cofibration.

The following sanity check will be needed later.

**Proposition 27.** *Let  $f : F \rightarrow X$  be an unbiased fibration in  $\mathbf{cSet}$ . Then for the endpoints  $\delta_0, \delta_1 : 1 \rightarrow \mathbf{I}$ , the associated pullback-homs,*

$$\delta_\epsilon \Rightarrow f : F^{\mathbf{I}} \rightarrow X^{\mathbf{I}} \times_X F \quad (\epsilon = 0, 1) \quad (46)$$

*are also trivial fibrations. Thus unbiased fibrations are also  $\delta_\epsilon$ -biased fibrations, for  $\epsilon = 0, 1$ .*

*Proof.* This follows from Remark 26 and the  $\otimes \dashv \Rightarrow$  adjunction, but we give a different proof. Consider the case  $X = 1$ , the general one  $f : F \rightarrow X$  being analogous. Thus let  $F$  be an unbiased fibrant object in  $\mathbf{cSet}$ . So by definition  $(\mathbf{I}^*F)^\delta : (\mathbf{I}^*F)^\mathbb{I} \rightarrow \mathbf{I}^*F$  in  $\mathbf{cSet}/_{\mathbf{I}}$  is a trivial fibration. Pulling back  $\delta : 1 \rightarrow \mathbb{I}$  in  $\mathbf{cSet}/_{\mathbf{I}}$  along the base change  $\delta_\epsilon : 1 \rightarrow \mathbf{I}$  takes it to  $\delta_\epsilon : 1 \rightarrow \mathbf{I}$  in  $\mathbf{cSet}$ , by the universal property of the generic point  $\delta : 1 \rightarrow \mathbb{I}$ ; that is  $\delta_\epsilon^*(\delta) = \delta_\epsilon : 1 \rightarrow \mathbf{I}$ . So  $(\mathbf{I}^*F)^\delta : (\mathbf{I}^*F)^\mathbb{I} \rightarrow \mathbf{I}^*F$  is taken by  $\delta_\epsilon^*$  to

$$\delta_\epsilon^*((\mathbf{I}^*F)^\delta) = (\delta_\epsilon^*\mathbf{I}^*F)^{\delta_\epsilon^*\delta} = F^{\delta_\epsilon} : F^{\mathbf{I}} \rightarrow F,$$

as shown in the following.

$$\begin{array}{ccccc} F^{\mathbf{I}} & \xrightarrow{\quad} & (\mathbf{I}^*F)^\mathbb{I} & & \\ \downarrow F^{\delta_\epsilon} \lrcorner & & \downarrow (\mathbf{I}^*F)^\delta & & \\ F & \xrightarrow{\quad} & \mathbf{I}^*F & \xrightarrow{\quad} & F \\ \downarrow \lrcorner & & \downarrow \lrcorner & & \downarrow \\ 1 & \xrightarrow{\quad \delta_\epsilon \quad} & \mathbf{I} & \xrightarrow{\quad} & 1 \end{array} \quad (47)$$

And pullback preserves trivial fibrations.  $\square$

**Unbiased fibration structure.** As in the biased case, the fibrations can be determined by *uniform* right-lifting against a *small category* of unbiased trivial cofibrations, now consisting of all those  $c \otimes_i \delta$  in  $\mathcal{C} \otimes \delta$  for which  $c : C \rightarrow \mathbf{I}^n$  is basic, *i.e.* has representable codomain. Call these maps the *basic unbiased trivial cofibrations*, and let

$$\mathbf{BCof} \otimes \delta = \{c \otimes_i \delta : B \rightarrow \mathbf{I}^{n+1} \mid c : C \rightarrow \mathbf{I}^n, i : \mathbf{I}^n \rightarrow \mathbf{I}, n \geq 0\}, \quad (48)$$

where the pushout-product  $c \otimes_i \delta$  now has the form

$$\begin{array}{ccc}
 C & \xrightarrow{\langle ic \rangle} & C \times I \\
 \downarrow c & & \downarrow \\
 I^n & \xrightarrow{\quad} & I^n +_C (C \times I) \\
 & \searrow \langle i \rangle & \downarrow c \otimes_i \delta \\
 & & I^n \times I
 \end{array}
 \quad \begin{array}{l} \text{curved arrow } C \times I \rightarrow I^n \times I \\ \text{dotted arrow } I^n +_C (C \times I) \rightarrow I^n \times I \end{array}
 \quad (49)$$

for a basic cofibration  $c : C \hookrightarrow I^n$  and an indexing map  $i : I^n \rightarrow I$ , and with domain  $B = (I^n +_C (C \times I))$ . These subobjects  $B \hookrightarrow I^{n+1}$  can again be seen geometrically as “generalized open box inclusions”, but now the floor and lid of the open box are generalized to the graph of an arbitrary map  $i : I^n \rightarrow I$ .

For any map  $f : Y \rightarrow X$  a uniform, unbiased fibration structure on  $f$  is then a choice of diagonal fillers  $j(c, i, x, y)$ ,

$$\begin{array}{ccc}
 B & \xrightarrow{x} & X \\
 \downarrow c \otimes_i \delta & \nearrow j(c, i, x, y) & \downarrow f \\
 I^n \times I & \xrightarrow{y} & Y,
 \end{array}
 \quad (50)$$

for each basic trivial cofibration  $c \otimes_i \delta : B \hookrightarrow I^{n+1}$ , which is *uniform in  $I^n$*  in the following sense: Given any cubical map  $u : I^m \rightarrow I^n$ , the pullback  $u^*c : u^*C \hookrightarrow I^m$  and the reindexing  $iu : I^m \rightarrow I^n \rightarrow I$  determine another basic trivial cofibration  $u^*c \otimes_{iu} \delta : B' = (I^m +_{u^*C} (u^*C \times I)) \hookrightarrow I^{m+1}$ , which fits into a commutative diagram of the form

$$\begin{array}{ccccc}
 B' & \xrightarrow{(u \times I)'} & B & \xrightarrow{x} & X \\
 \downarrow u^*c \otimes_{iu} \delta & \lrcorner & \downarrow c \otimes_i \delta & \nearrow j(c, i, x, y) & \downarrow f \\
 I^m \times I & \xrightarrow{u \times I} & I^n \times I & \xrightarrow{y} & Y.
 \end{array}
 \quad (51)$$

For the outer rectangle in (51) there is a chosen diagonal filler

$$j(u^*c, iu, x(u \times I)', y(u \times I)) : I^m \times I \rightarrow X,$$

and for this map we require that

$$j(u^*c, iu, x(u \times I)', y(u \times I)) = j(c, i, x, y) \circ (u \times I). \quad (52)$$

**Definition 28.** A *uniform, unbiased fibration structure* on a map

$$f : Y \rightarrow X$$

is a choice of fillers  $j(c, i, x, y)$  as in (50) satisfying (52) for all  $u : I^m \rightarrow I^n$ .

In these terms, we have the following analogue of corollary 19.

**Proposition 29.** *For any object  $X$  in  $\mathbf{cSet}$  the following are equivalent:*

1.  *$X$  is an unbiased fibrant object in the sense of Definition 20: the canonical map  $\delta \Rightarrow X : X^I \times I \rightarrow X \times I$  is a trivial fibration.*
2.  *$X$  has the right lifting property with respect to all generating unbiased trivial cofibrations,*

$$(\mathcal{C} \otimes \delta) \pitchfork X.$$

3.  *$X$  has a uniform, unbiased fibration structure in the sense of Definition 28.*

*Proof.* The equivalence between (1) and (2) is proposition 25. So assume (1). Then in  $\mathbf{cSet}/_I$ , the evaluation at  $\delta : 1 \rightarrow I$ ,

$$(I^*X)^\delta : (I^*X)^I \longrightarrow X$$

is a trivial fibration. By Proposition 13 it therefore has a uniform filling structure with respect to all basic cofibrations  $c : C \rightarrowtail I^n$  over  $I$ . Transposing by the  $\otimes \dashv \Rightarrow$  adjunction and unwinding then gives exactly a uniform fibration structure on  $X$ .  $\square$

A statement analogous to the foregoing also holds for maps  $f : Y \rightarrow X$  in place of objects  $X$ . Indeed, as before, we have the following sharper formulation.

**Corollary 30.** *Fibration structures on a map  $f : Y \rightarrow X$  correspond uniquely to relative  $+$ -algebra structures on the map  $(\delta \Rightarrow f)$  (cf. definition 20),*

$$(\delta \Rightarrow f) : Y^I \times I \longrightarrow (X^I \times I) \times_{(X \times I)} (Y \times I).$$

### 3.3 Factorization

**Definition 31.** Summarizing the foregoing definitions, we have the following classes of maps:

- The *generating unbiased trivial cofibrations* were determined in (44) as

$$\mathcal{C} \otimes \delta = \{c \otimes_i \delta : D \rightarrowtail Z \times I \mid c : C \rightarrowtail Z, i : Z \rightarrow I\}, \quad (53)$$

where  $D = (Z +_C (C \times I))$  and the pushout-product  $c \otimes_i \delta$  has the form

$$\begin{array}{ccc} C & \xrightarrow{\langle ic \rangle} & C \times I \\ c \downarrow & & \downarrow \\ Z & \longrightarrow & Z +_C (C \times I) \end{array} \quad \begin{array}{c} \searrow c \times I \\ \nearrow c \otimes_i \delta \\ \searrow \langle ic \rangle \end{array} \quad \begin{array}{c} \\ \\ Z \times I \end{array} \quad (54)$$

for any cofibration  $c : C \rightarrowtail Z$  and indexing map  $i : Z \rightarrow I$ .

- The class  $\mathcal{F}$  of *unbiased fibrations*, which can be characterized as the right-lifting class of the generating unbiased trivial cofibrations,

$$(\mathcal{C} \otimes \delta)^\pitchfork = \mathcal{F}.$$

- The class of *unbiased trivial cofibrations* is then defined to be left-lifting class of the fibrations,

$$\text{TCof} = {}^\pitchfork \mathcal{F}.$$

It follows that the classes  $\text{TCof}$  and  $\mathcal{F}$  are closed under retracts and are mutually weakly orthogonal,  $\text{TCof} \pitchfork \mathcal{F}$ . Thus in order to have a weak factorization system  $(\text{TCof}, \mathcal{F})$  it just remains to show the following.

**Lemma 32.** *Every map  $f : X \rightarrow Y$  in  $\mathbf{cSet}$  can be factored as  $f = p \circ i$ ,*

$$\begin{array}{ccc} X & \xrightarrow{i} & X' \\ & \searrow f & \downarrow p \\ & & Y \end{array} \quad (55)$$

with  $i : X \rightarrowtail X'$  an unbiased trivial cofibration and  $p : X' \twoheadrightarrow Y$  an unbiased fibration.

*Proof.* We can use a standard argument (the “algebraic small objects argument”, cf. [GKR18]), which can be further simplified using the fact that the codomains of the basic trivial cofibrations  $c \otimes_i \delta : B \rightarrow I^{n+1}$  are not just representable, but *tiny* in the sense of Proposition 3, and the domains are not merely “small”, but *finitely presented*. The reader is referred to [?] for details in a similar case.  $\square$

*Remark 33.* The proof in *ibid.* actually produces a stronger result than we need, namely an *algebraic* weak factorization system. This follows from the small generating category  $\mathcal{C} \otimes \delta$  of basic unbiased trivial cofibrations (and pullback squares of the form on the left in (51)). The relationship between this stronger condition and the *classifying types* used in Section 6 is studied in [?], which also gives an even more “constructive” proof of the factorization Lemma 32, not requiring quotients, exactness, or impredicativity [?]. With this modification, the present approach can also be used in a *quasitopos*, as occurs in *e.g.* realizability and sheaves.

**Proposition 34.** *There is a weak factorization system on the category  $\mathbf{cSet}$  in which the right maps are the unbiased fibrations and the left maps are the unbiased trivial cofibrations, both as specified in definition 31. This will be called the (unbiased) fibration weak factorization system.*

Hereafter, unless otherwise stated, all fibrations in  $\mathbf{cSet}$  are assumed to be unbiased.

## 4 The weak equivalences

Our approach to proving that the classes  $\mathcal{C}$  of cofibrations from Section 2 and  $\mathcal{F}$  of fibrations from Section 3 determine a model structure will be to first build a *premodel structure* in the sense of [?], and then turn to the question of the 3-for-2 property for the weak equivalences.

**Definition 35** (Weak equivalence). A map  $f : X \rightarrow Y$  in  $\mathbf{cSet}$  is a *weak equivalence* if it can be factored as  $f = g \circ h$ ,

$$\begin{array}{ccc} X & \xrightarrow{h} & W \\ & \searrow f & \downarrow g \\ & & Y \end{array}$$

with  $h : X \rightarrow W$  a trivial cofibration and  $g : W \rightarrow Y$  a trivial fibration. Let

$$\begin{aligned}\mathcal{W} &= \text{TFib} \circ \text{TCof} \\ &= \{f : X \rightarrow Y \mid f = g \circ h \text{ for } g \in \text{TFib} \text{ and } h \in \text{TCof}\}\end{aligned}$$

be the class of weak equivalences.

Observe first that every trivial fibration  $f \in \text{TFib} = \mathcal{C}^{\text{h}}$  is indeed a fibration, because the generating trivial cofibrations  $c \otimes_i \delta$  are cofibrations. Moreover, every trivial fibration  $f : X \rightarrow Y$  is also a weak equivalence  $f = f \circ 1_X$ , since the identity map  $1_X$  is, well, trivially a trivial cofibration  $\text{TCof} = {}^{\text{h}}\mathcal{F}$ . Thus we have

$$\text{TFib} \subseteq (\mathcal{F} \cap \mathcal{W}).$$

Similarly, because  $\text{TFib} \subseteq \mathcal{F}$ , we have  $\text{TCof} \subseteq \mathcal{C}$ . Moreover, since identity maps are also trivial fibrations we have  $\text{TCof} \subseteq \text{TFib} \circ \text{TCof} = \mathcal{W}$ . Thus we also have

$$\text{TCof} \subseteq (\mathcal{C} \cap \mathcal{W}).$$

**Lemma 36.**  $(\mathcal{C} \cap \mathcal{W}) \subseteq \text{TCof}$ .

*Proof.* Let  $c : A \rightarrow B$  be a cofibration with a factorization

$$c = tf \circ tc : A \rightarrow W \rightarrow B$$

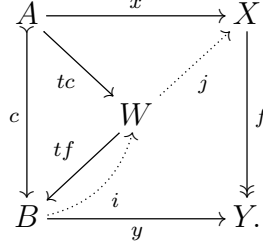
where  $tc \in \text{TCof}$  and  $tf \in \text{TFib}$ . Let  $f : X \rightarrow Y$  be a fibration and consider a commutative diagram,

$$\begin{array}{ccc} A & \xrightarrow{x} & X \\ c \downarrow & & \downarrow f \\ B & \xrightarrow{y} & Y. \end{array}$$

Inserting the factorization of  $c$ , from  $tc \dashv f$  we obtain  $j : W \rightarrow X$  as indicated, with  $j \circ tc = x$  and  $f \circ j = y \circ tf$ .

$$\begin{array}{ccccc} A & & \xrightarrow{x} & & X \\ & \searrow tc & & \nearrow j & \\ & W & & & \\ & \swarrow tf & & & \\ B & & \xrightarrow{y} & & Y. \end{array}$$

Moreover, since  $c \pitchfork tf$  there is an  $i : B \rightarrow W$  as indicated, with  $i \circ c = tc$  and  $tf \circ i = 1_B$ .



Let  $k = j \circ i$ . Then  $k \circ c = j \circ i \circ c = j \circ tc = x$ , and  $f \circ k = f \circ j \circ i = y \circ tf \circ i = y$ .  $\square$

The proof of the following is exactly dual.

**Lemma 37.**  $(\mathcal{F} \cap \mathcal{W}) \subseteq \mathbf{TFib}$ .

**Proposition 38.** *The three classes of maps  $\mathcal{C}, \mathcal{W}, \mathcal{F}$  in  $\mathbf{cSet}$  constitute a premodel structure in the sense of [?]. In particular, we have*

$$\begin{aligned}\mathcal{F} \cap \mathcal{W} &= \mathbf{TFib}, \\ \mathcal{C} \cap \mathcal{W} &= \mathbf{TCof},\end{aligned}$$

and therefore two interlocking weak factorization systems:

$$(\mathcal{C}, \mathcal{W} \cap \mathcal{F}) \quad , \quad (\mathcal{C} \cap \mathcal{W}, \mathcal{F}).$$

It now “only” remains to show that the weak equivalences  $\mathcal{W}$  satisfy the 3-for-2 axiom in order to finish the proof that  $(\mathcal{C}, \mathcal{W}, \mathcal{F})$  is a model structure. Perhaps surprisingly, this requires a considerable amount of work, which will occupy the rest of these lectures! We shall follow roughly the approach of [?]. We expect the weak equivalences between fibrant objects to be exactly the *homotopy equivalences* (Definition ??), which can be seen directly to satisfy 3-for-2. So we reduce to this case, using the fact that  $K^X$  is fibrant whenever  $K$  is. It suffices, namely, to show that weak equivalences  $w : X \rightarrow Y$  induce homotopy equivalences  $K^w : K^Y \simeq K^X$  for fibrant  $K$ , and that any map with this property is a weak equivalence. Such maps are termed *weak homotopy equivalences* (Definition 43), and our task is therefore reduced to showing that a map is a weak equivalence if and only if it is a weak homotopy equivalence.



### Weak homotopy equivalence.

**Definition 39.** A homotopy  $\vartheta : f \sim g$  between maps  $f, g : X \rightrightarrows Y$  is a map,

$$\vartheta : I \times X \longrightarrow Y,$$

such that  $\vartheta \circ \iota_0 = f$  and  $\vartheta \circ \iota_1 = g$ ,

$$\begin{array}{ccccc} X & \xrightarrow{\iota_0} & I \times X & \xleftarrow{\iota_1} & X, \\ & \searrow f & \downarrow \vartheta & \swarrow g & \\ & & Y & & \end{array} \quad (56)$$

where  $\iota_0, \iota_1$  are the canonical inclusions into the ends of the cylinder,

$$\iota_\epsilon : X \cong 1 \times X \xrightarrow{\delta_\epsilon \times X} I \times X, \quad \epsilon = 0, 1.$$

**Proposition 40.** *The relation of homotopy  $f \sim g$  between maps  $f, g : X \rightrightarrows Y$  is preserved by pre- and post-composition. If  $Y$  is fibrant, then  $f \sim g$  is an equivalence relation.*

*Proof.* Inspecting (56), preservation of  $f \sim g$  under post-composing with any  $h : Y \rightarrow Z$  is obvious: we have  $h \circ \vartheta : h \circ f \sim h \circ g$ . Now observe that a homotopy  $f \overset{\vartheta}{\sim} g : X \times I \rightarrow Y$  determines a (unique) path  $\tilde{\vartheta} : I \rightarrow Y^X$  in the function space, with endpoints  $\vartheta_0 = \vartheta \circ \delta_0 = \tilde{f} : 1 \rightarrow Y^X$  and  $\vartheta_1 = \vartheta \circ \delta_1 = \tilde{g}$ . Precomposing maps  $f, g : X \rightrightarrows Y$  with any  $e : W \rightarrow X$  is induced by post-composing  $\tilde{f}, \tilde{g} : 1 \rightarrow Y^X$  with the map  $Y^e : Y^X \rightarrow Y^W$ , which then also takes the path  $\tilde{\vartheta} : I \rightarrow Y^X$  to a path  $\tilde{\varphi} = Y^e \circ \tilde{\vartheta} : I \rightarrow Y^W$  corresponding to a (unique) homotopy  $\varphi : f \circ e \sim g \circ e$ .

Now note that  $Y^X$  is fibrant if  $Y$  is fibrant, since the generating trivial cofibrations  $c \times_i \delta$  are preserved by the functor  $X \times (-)$ . So we can use “box-filling” in  $Y^X$  to verify the claimed equivalence relation.

- Reflexivity  $f \sim f$  is witnessed by the homotopy  $\rho : I \rightarrow 1 \xrightarrow{f} Y^X$ .
- For symmetry  $f \sim g \Rightarrow g \sim f$  take  $\vartheta : I \rightarrow Y^X$  with  $\vartheta_0 = f$  and  $\vartheta_1 = g$  and we want to build  $\vartheta' : I \rightarrow Y^X$  with  $\vartheta'_0 = g$  and  $\vartheta'_1 = f$ . Take an open 2-box in  $Y^X$  of the following form.

$$\begin{array}{ccc} & g & f \\ \vartheta \uparrow & & \uparrow \rho \\ f & \xrightarrow{\rho} & f \end{array}$$

This box is a map  $b : I +_1 I +_1 I \rightarrow Y^X$  with the indicated components, and it has a filler  $c : I \times I \rightarrow Y^X$ , i.e. an extension along the canonical map  $I +_1 I +_1 I \rightarrow I \times I$ , which is a trivial cofibration of the form  $\partial I \otimes \delta_0$ . Let  $t : I \rightarrow I \times I$  be the top face of the 2-cube (the bipointed map  $\{0, x_1, x_2, 1\} \rightarrow \{0, x, 1\}$  that is constantly 1). We can set  $\vartheta' = c \circ t : I \rightarrow Y^X$  to get a homotopy  $\vartheta' : I \rightarrow Y^X$  with  $\vartheta'_0 = g$  and  $\vartheta'_1 = f$  as required.

- For transitivity,  $f \stackrel{\vartheta}{\sim} g, g \stackrel{\varphi}{\sim} h \Rightarrow f \sim h$ , an analogous construction will fill the open box:

$$\begin{array}{ccc} f & & h \\ \uparrow \rho & & \uparrow \varphi \\ f & \xrightarrow{\vartheta} & g \end{array}$$

□

**Definition 41** (Connected components). The functor

$$\pi_0 : \mathbf{cSet} \rightarrow \mathbf{Set}$$

is defined on a cubical set  $X$  as the coequalizer

$$X_1 \rightrightarrows X_0 \rightarrow \pi_0 X,$$

where the two parallel arrows are the maps  $X_{\delta_0}, X_{\delta_1} : X_1 \rightrightarrows X_0$  for the endpoints  $\delta_0, \delta_1 : 1 \rightrightarrows I$ . If  $K$  is fibrant, then by the foregoing Proposition 40, for any  $X$  we have

$$\pi_0(K^X) = \text{Hom}(X, K)/\sim .$$

That is,  $\pi_0(K^X)$  is the set  $[X, K]$  of homotopy equivalence classes of maps  $X \rightarrow K$ .

*Remark 42.* One can show that in fact  $\pi_0 X = \varinjlim X_n$  where the colimit is taken over *all* objects  $[n]$  in the index category  $\square^{\text{op}} = \mathbb{B}$ , rather than just the “last” two  $[1] \rightrightarrows [0]$ . Since the category  $\mathbb{B}$  of finite strictly bipointed sets is sifted, the functor  $\pi_0 : \mathbf{cSet} \rightarrow \mathbf{Set}$  preserves finite products.

As usual, a *homotopy equivalence* is a map  $f : X \rightarrow Y$  together with a map  $g : Y \rightarrow X$  and homotopies  $\vartheta : 1_X \sim g \circ f$  and  $\varphi : 1_Y \sim f \circ g$ . We call  $g$  a *quasi-inverse* of  $f$ .

**Definition 43** (Weak homotopy equivalence). A map  $f : X \rightarrow Y$  is called a *weak homotopy equivalence* if for every fibrant object  $K$ , the canonical map  $K^f : K^Y \rightarrow K^X$  is bijective on connected components,

$$\pi_0 K^f : \pi_0 K^Y \cong \pi_0 K^X .$$

**Lemma 44.** *Every homotopy equivalence is a weak homotopy equivalence.*

*Proof.* Let  $f : X \rightarrow Y$  be a homotopy equivalence. Then  $K^f : K^Y \rightarrow K^X$  is also a homotopy equivalence for any  $K$ , since homotopy respects (post-) composition by all maps. If  $K$  is fibrant, then so is  $K^X$  and  $\pi_0$  is well defined on homotopy classes of maps, by Proposition 40. It clearly takes homotopy equivalences to isomorphisms of sets, since it identifies homotopic maps.  $\square$

**Lemma 45.** *The weak homotopy equivalences  $f : X \rightarrow Y$  satisfy the 3-for-2 condition.*

*Proof.* This follows by applying the **Set**-valued functors  $\pi_0 K^{(-)}$ , for all fibrant objects  $K$ , and the corresponding fact about bijections of sets.  $\square$

In virtue of Lemma 45 it suffices to show that a map is a weak equivalence if and only if it is a weak homotopy equivalence. This will be shown by splitting into four cases: a cofibration/fibration is a weak equivalence if/only if it is a weak homotopy equivalence. The following will be useful thereby.

**Lemma 46.** *A map  $f : X \rightarrow Y$  is a weak homotopy equivalence iff it satisfies the following two conditions.*

1. *For every fibrant object  $K$  and every map  $x : X \rightarrow K$  there is a map  $y : Y \rightarrow K$  such that  $y \circ f \sim x$ ,*

$$\begin{array}{ccc} X & \xrightarrow{x} & K \\ f \downarrow & \nearrow \sim & \uparrow y \\ Y & & \end{array}$$

*We say that  $x$  “extends along  $f$  up to homotopy”.*

2. *For every fibrant object  $K$  and maps  $y, y' : Y \rightarrow K$  such that  $yf \sim y'f$ , there is a homotopy  $y \sim y'$ ,*

$$\begin{array}{ccc} X & \longrightarrow & K^1 \\ f \downarrow & \nearrow & \downarrow \\ Y & \xrightarrow{\langle y, y' \rangle} & K \times K \end{array}$$

*Proof.* Condition (1) says exactly that the internal precomposition map  $K^f : K^Y \rightarrow K^X$  is surjective under connected components  $\pi_0$ , while (2) says just that it is injective under  $\pi_0$ .  $\square$

**Lemma 47.** *Any trivial cofibration  $c : A \rightarrowtail B$  is a weak homotopy equivalence.*

*Proof.* Applying Lemma 46, condition (1) is immediate, and (2) results from the fact that  $K^1 \twoheadrightarrow K \times K$  is a fibration when  $K$  is fibrant, since  $\partial : 1 + 1 \rightarrowtail I$  is a cofibration.

[show that  $\partial : 1 + 1 \rightarrowtail I$  is a cofibration!]

$\square$

**Lemma 48.** *Any trivial fibration  $p : Y \twoheadrightarrow X$  is a weak homotopy equivalence.*

*Proof.* By Corollary 14, the trivial fibration  $p : Y \twoheadrightarrow X$  has a section  $s : X \rightarrowtail Y$ .

[show that sections are cofibrations!]

Moreover, there is a homotopy  $\vartheta : sf \sim 1_Y$ , resulting from the lifting problem

$$\begin{array}{ccc} Y + Y & \xrightarrow{[sf, 1]} & Y \\ \downarrow [\iota_0, \iota_1] & & \downarrow f \\ I \times Y & \xrightarrow{f\pi_2} & X. \end{array}$$

Thus  $f$  is a homotopy equivalence, and therefore a weak homotopy equivalence by Lemma 44.  $\square$

**Corollary 49.** *Any weak equivalence  $e : X \simeq Y$  is a weak homotopy equivalence, since  $e$  factors into a trivial cofibration followed by a trivial fibration, each of which is a weak homotopy equivalence, and these are closed under composition by Lemma 45.*

**Lemma 50.** *If  $K$  is fibrant, then any fibration  $f : Y \twoheadrightarrow K$  that is a homotopy equivalence is a weak equivalence.*

*Proof.* This is a standard argument, which we just sketch. It suffices to show that any diagram of the form

$$\begin{array}{ccc} C & \xrightarrow{y} & Y \\ c \downarrow & & \downarrow f \\ K & \xrightarrow{=} & K, \end{array} \tag{57}$$

with  $c : C \rightarrowtail X$  a cofibration, has a diagonal filler, for then  $f$  is a trivial fibration. Since  $f$  is a homotopy equivalence, it has a quasi-inverse  $s : X \rightarrow Y$  with  $\vartheta : fs \sim 1_K$ , which we claim can be corrected to a section  $s' : K \rightarrow Y$ . Indeed, consider

$$\begin{array}{ccc} K & \xrightarrow{s} & Y \\ \downarrow \iota_0 & \nearrow \vartheta' & \downarrow f \\ K \times I & \xrightarrow{\vartheta} & K \\ \uparrow \iota_1 & \nearrow = & \\ K & & \end{array}$$

where  $\vartheta'$  results from  $\iota_0 \pitchfork f$ . Let  $s' = \vartheta' \iota_1$ , so that  $\vartheta' : s \sim s'$  and  $fs' = 1_K$ .

Thus we can assume that  $s = s' : K \rightarrow Y$  is a section, which fills the diagram (57) up to a homotopy in the upper triangle.

$$\begin{array}{ccc} C & \xrightarrow{y} & Y \\ \downarrow c & \nearrow \sim & \downarrow f \\ K & \xrightarrow{s} & K \end{array}$$

Now we can correct  $s : K \rightarrow Y$  to a homotopic  $t : K \rightarrow Y$  over  $f$  by using the homotopy  $\varphi : sc \sim y$  to get a map  $\varphi : C \rightarrow Y^I$  over  $f$ . Since  $f$  is a fibration, the projections  $p_0, p_1 : Y^I \rightarrow Y$  over  $f$  are trivial fibrations, and so there is a lift  $\varphi' : K \rightarrow Y^I$  for which  $t := p_1 \varphi'$  has  $tc = y$  and  $ft = 1_K$ , and so is a filler for (57).  $\square$

**Lemma 51.** *If  $K$  is fibrant, then any fibration  $f : Y \twoheadrightarrow K$  that is a weak homotopy equivalence is a weak equivalence.*

*Proof.* Since  $K$  is fibrant, so is  $Y$ , and since  $f$  is a weak homotopy equivalence, by lemma 46(1) there is then a map  $s : K \rightarrow Y$  and a homotopy  $\theta : sf \sim 1_Y$ . Postcomposing with  $f$  gives a homotopy  $f\theta : fsf \sim f$ , forming the outer commutative square in

$$\begin{array}{ccc} Y & \xrightarrow{f\theta} & K^I \\ \downarrow f & \nearrow \varphi & \downarrow \\ K & \xrightarrow{\langle fs, 1_K \rangle} & K \times K \end{array}$$

By lemma 46(2) there is a diagonal filler  $\varphi : fs \sim 1_K$ , and so  $f$  is a homotopy equivalence. Now apply lemma 50.  $\square$

**Lemma 52.** *If  $K$  is fibrant, then any cofibration  $c : A \rightarrowtail K$  that is a weak homotopy equivalence is a weak equivalence.*

*Proof.* Let  $c : A \rightarrowtail K$  be a cofibration weak homotopy equivalence and factor it into a trivial cofibration  $i : A \rightarrowtail Z$  followed by a fibration  $p : Z \twoheadrightarrow K$ . By lemma 46, any trivial cofibration is clearly a weak homotopy equivalence. So both  $c$  and  $i$  are weak homotopy equivalence, and therefore so is  $p$  by 3-for-2 for weak homotopy equivalences. Since  $K$  is fibrant,  $p$  is a trivial fibration by lemma 51, and thus  $c$  is a weak equivalence.  $\square$

**Lemma 53** ([?], x.n.m). *A cofibration  $c : A \rightarrowtail B$  weak homotopy equivalence lifts against all fibrations  $f : Y \twoheadrightarrow K$  with fibrant codomain.*

*Proof.* Let  $c : A \rightarrowtail B$  be a cofibration weak homotopy equivalence and  $f : Y \twoheadrightarrow K$  a fibration with fibrant codomain  $K$ , and consider a lifting problem

$$\begin{array}{ccc} A & \xrightarrow{a} & Y \\ c \downarrow & & \downarrow f \\ B & \xrightarrow{b} & K. \end{array}$$

Let  $\eta : B \rightarrowtail B'$  be a fibrant replacement of  $B$ , since  $K$  is fibrant,  $b$  extends along  $\eta$  to give  $b' : B' \rightarrow K$  as shown below.

$$\begin{array}{ccc} A & \xrightarrow{a} & Y \\ c \downarrow & & \downarrow f \\ B & \xrightarrow{b} & K \\ \eta \downarrow & \nearrow b' & \\ B' & & \end{array}$$

Since  $\eta$  is a trivial cofibration, it is a weak homotopy equivalence. So the composite  $\eta c$  is also a weak homotopy equivalence. But since  $B'$  is fibrant,  $\eta c$  is then a trivial cofibration by lemma 52. Thus there is a lift  $j : B' \rightarrow Y$ , and therefore also one  $k = j\eta : B \rightarrow Y$ .  $\square$

To complete the proof that a cofibration weak homotopy equivalence is a weak equivalence, we shall use the following *fibration extension property*, the proof of which is deferred to section 8.

**Definition 54** (Fibration extension property). For any fibration  $f : Y \twoheadrightarrow X$  and trivial cofibration  $\eta : X \rightarrow X'$ , there is a fibration  $f' : Y' \twoheadrightarrow X'$  that pulls back to  $f$  along  $\eta$ , as shown below.

$$\begin{array}{ccc} Y & \longrightarrow & Y' \\ f \downarrow \lrcorner & & \downarrow f' \\ X & \xrightarrow{\eta} & X' \end{array} \quad (58)$$

**Lemma 55.** *Assuming the fibration extension property, a cofibration that lifts against every fibration  $f : Y \twoheadrightarrow K$  with fibrant codomain is a weak equivalence.*

*Proof.* Let  $c : A \rightarrowtail B$  be a cofibration and consider a lifting problem against an arbitrary fibration  $f : Y \twoheadrightarrow X$ ,

$$\begin{array}{ccc} A & \xrightarrow{a} & Y \\ c \downarrow & & \downarrow f \\ B & \xrightarrow{b} & X. \end{array} \quad (59)$$

Let  $\eta : X \rightarrow X'$  be a fibrant replacement, so  $\eta$  is a trivial cofibration and  $X'$  is fibrant. By the fibration extension property of definition 54, there is a fibration  $f' : Y' \twoheadrightarrow X'$  such that  $f$  is a pullback of  $f'$  along  $\eta$ . So we can extend diagram (59) to obtain the following, in which the righthand square is a pullback.

$$\begin{array}{ccccc} A & \xrightarrow{a} & Y & \xrightarrow{y} & Y' \\ c \downarrow & & \downarrow f & \lrcorner & \downarrow f' \\ B & \xrightarrow{b} & X & \xrightarrow{\eta} & X'. \end{array} \quad (60)$$

By assumption, there is a lift  $j' : B \rightarrow Y'$  with  $f'j' = \eta b$  and  $j'c = yb$ . Therefore, since  $f$  is a pullback, there is a map  $j : B \rightarrow Y$  with  $fj = b$  and  $yj = j'$ .

$$\begin{array}{ccccc} A & \xrightarrow{a} & Y & \xrightarrow{y} & Y' \\ c \downarrow & j \nearrow & \downarrow f & \lrcorner & \downarrow f' \\ B & \xrightarrow{b} & X & \xrightarrow{\eta} & X'. \end{array} \quad (61)$$

Thus  $yjc = j'c = ya$ . But as a trivial cofibration,  $\eta$  is monic, and as a pullback of  $\eta$ ,  $y$  is also monic. So  $jc = a$ .  $\square$

Combining the previous two lemmas 53 and 55 we now have the following.

**Corollary 56.** *Assuming the fibration extension property, a cofibration  $c : A \rightarrowtail B$  weak homotopy equivalence is a weak equivalence.*

The following is not required, but we state it anyway for the record:

**Lemma 57.** *Assuming the fibration extension property, a fibration  $f : Y \twoheadrightarrow X$  weak homotopy equivalence is a weak equivalence.*

*Proof.* Factor  $f : Y \twoheadrightarrow X$  into a cofibration  $i : Y \rightarrowtail Z$  followed by a trivial fibration  $p : Z \twoheadrightarrow X$ . Then  $f$  is a trivial fibration if  $i \pitchfork f$ , for then  $f$  is a retract of  $p$ . Since  $p$  is a trivial fibration, it is a weak homotopy equivalence by lemma 48. Since  $f$  is also a weak homotopy equivalence, so is  $i$  by 3-for-2. Thus  $i$  is a trivial cofibration by corollary 56. Since  $f$  is a fibration,  $i \pitchfork f$  as required.  $\square$

**Proposition 58.** *Assuming the fibration extension property, a map  $f : X \rightarrow Y$  is a weak homotopy equivalence if and only if it is a weak equivalence. The weak equivalences  $\mathcal{W}$  then satisfy the 3-for-2 condition.*

*Proof.* Let  $f : X \rightarrow Y$  be a weak equivalence and factor it into a trivial cofibration  $i : X \rightarrowtail Z$  followed by a trivial fibration  $p : Z \twoheadrightarrow Y$ . Then both  $i$  and  $p$  are weak homotopy equivalence, whence so is  $f$ . Conversely, let  $f$  be a weak homotopy equivalence and factor it into a cofibration  $i : X \rightarrowtail Z$  followed by a trivial fibration  $p : Z \twoheadrightarrow Y$ . Since  $p$  is then a weak homotopy equivalence, as is  $f$ , it follows that  $i$  is as well. Thus  $i$  is also a weak equivalence, by lemma 56, hence a trivial cofibration. So  $f$  is a weak equivalence.  $\square$

Our results can now be summarized as follows.

**Theorem 59.** *Assume the fibration weak factorization system of Definition 31 satisfies the fibration extension property of Definition 54 (as will be shown in Corollary 96). Then the weak equivalences  $\mathcal{W}$  have the 3-for-2 property, and so by Proposition 38, the classes  $(\mathcal{C}, \mathcal{W}, \mathcal{F})$  form a Quillen model structure on the category  $\mathbf{cSet}$  of cubical sets. Moreover, the weak equivalences can be characterized as those maps  $f : X \rightarrow Y$  for which  $K^f : K^Y \rightarrow K^X$  is bijective on connected components whenever  $K$  is fibrant.*

The proof of the fibration extension property will occupy the second half of these lectures, concluding in Section 8. It uses the equivalence extension property (Section 7), a universal fibration (Section 6), and the Frobenius condition (Section 5), to which we now turn.



## 5 The Frobenius condition

In this section, we show that the (unbiased) fibration weak factorization system from section ?? satisfies what has been called the *Frobenius condition*: the left maps are stable under pullback along the right maps (see [?]). This will imply the *right properness* of our model structure: the weak equivalences are preserved by pullback along fibrations. In our setting, it then follows that the entire model structure is stable under such a base change. The Frobenius condition will be used in the proof of the equivalence extension property in Section 7.

An “algebraic” proof of Frobenius in the related setting of cubical sets *with connections* was given in [GS17] using conventional, functorial methods. By contrast, the type theoretic approach of [CCHM16] provides a proof that is much more direct, and can also be modified to work without connections (as in [?]). That approach proves the dual fact that the *pushforward* operation, which is right adjoint to pullback and always exists in a topos, preserves fibrations when applied along a fibration. This corresponds to the type-theoretic  $\Pi$ -formation rule, and the proof given in *op.cit.* is entirely in type theory. It also employs a reduction of box filling (in all dimensions) to an apparently weaker condition of *Kan composition* (in all dimensions), which merely “puts a lid on” the open box, rather than filling it. This aspect of the type theoretic proof can also be described algebraically, but is not used in the proof given here, and will therefore not be discussed further (see [?] for an algebraic description of Kan composition with connections, and [?, ?, ?, ?] for the same without connections).

This section applies the same strategy as that used in defining the unbiased fibrations, namely we first establish something in the *biased but generic* setting, and then transfer it to the unbiased setting by pulling back along the base change  $\mathbf{cSet} \rightarrow \mathbf{cSet}/_I$ . In order to emphasize this approach, we give the second step first, in the form of a conditional statement.

**Proposition 60.** *Suppose the  $\delta$ -biased fibrations in  $\mathbf{cSet}/_I$  satisfy the Frobenius condition. Then the unbiased fibrations in  $\mathbf{cSet}$  also satisfy the Frobenius condition.*

*Proof.* This follows almost immediately from the fact that the pullback functor  $I^* : \mathbf{cSet} \rightarrow \mathbf{cSet}/_I$  preserves the locally cartesian closed structure, takes unbiased fibrations to  $\delta$ -biased ones, and reflects  $\delta$ -biased fibrations to unbiased ones. In detail, let unbiased fibrations  $B \twoheadrightarrow A$  and  $A \twoheadrightarrow X$  in  $\mathbf{cSet}$  be

given, and we wish to find  $C \rightarrow X$  and  $e : A \times_X C \rightarrow B$  over  $A$ , universal in the way recalled in the diagram below.

$$\begin{array}{ccc}
 A \times_X C & \xrightarrow{\quad} & C \\
 \downarrow e & & \downarrow \\
 B & & \\
 \downarrow & & \downarrow \\
 A & \xrightarrow{\quad} & X
 \end{array} \tag{62}$$

Take the pushforward  $C := A_* B \rightarrow X$ , and its associated map  $e : A \times_X C \rightarrow B$ , in the locally cartesian closed category  $\mathbf{cSet}$ . Since fibrations are stable under (all) pullbacks, it then suffices to show that  $C \rightarrow X$  is a fibration.

By definition,  $C \rightarrow X$  is an unbiased fibration in  $\mathbf{cSet}$  just in case the base change  $I^*C \rightarrow I^*X$  is a  $\delta$ -biased fibration in the slice category  $\mathbf{cSet}/_I$ . Since the pullback functor  $I^* : \mathbf{cSet} \rightarrow \mathbf{cSet}/_I$  preserves all lcc structure, over  $I^*X$  we have an iso,

$$I^*C = I^*(A_* B) \cong (I^*A)_* I^*B,$$

where the pushforward  $(I^*A)_* I^*B$  is taken in the topos  $\mathbf{cSet}/_I$ . But  $I^*B \rightarrow I^*A$  and  $I^*A \rightarrow I^*X$  are  $\delta$ -biased fibrations in  $\mathbf{cSet}/_I$  because  $B \rightarrow A$  and  $A \rightarrow X$  were assumed to be unbiased fibrations in  $\mathbf{cSet}$ . Since we are assuming the Frobenius condition for  $\delta$ -biased fibrations in  $\mathbf{cSet}/_I$ , the pushforward  $I^*C \cong (I^*A)_* I^*B \rightarrow I^*X$  is also a  $\delta$ -biased fibration, as required.  $\square$

**Frobenius for biased fibrations.** The results proved in this section will be applied to the slice category  $\mathbf{cSet}/_I$  and the generic point  $\delta : 1 \rightarrow I = I^*I$ , but nothing depends on this particular case, and so we shall write simply  $\delta : 1 \rightarrow I$  for a chosen pointed object in an arbitrary topos  $\mathcal{E}$ . (Indeed, in this section  $\mathcal{E}$  may even be taken to be just a locally cartesian closed category with a representable class of cofibrations.)

Recall from Definition 20 that a map  $f : A \rightarrow X$  is a  $\delta$ -biased fibration just if the map  $\delta \Rightarrow f$  admits a relative  $+$ -algebra structure, and is therefore

a trivial fibration. The definition of the pullback-hom  $\delta \Rightarrow f$  is recalled below.

$$\begin{array}{ccc}
 A^I & \xrightarrow{A^\delta} & A \\
 \delta \Rightarrow f \swarrow & & \downarrow f \\
 X^I \times_X A & \longrightarrow & A \\
 \downarrow & & \downarrow \\
 X^I & \xrightarrow{X^\delta} & X
 \end{array}
 \quad (63)$$

Let us write this condition schematically as follows:

$$\begin{array}{ccccc}
 A^I & \dashrightarrow & A_\epsilon & \longrightarrow & A \\
 & & \downarrow \lrcorner & & \downarrow f \\
 & & X^I & \xrightarrow{\epsilon} & X
 \end{array}
 \quad (64)$$

where  $\epsilon = X^\delta$  and  $A_\epsilon = X^I \times_X A$ , and the struck-through arrow indicates that it admits a  $+$ -algebra structure.

**Lemma 61.** *Let  $A \rightarrow X$  be a  $\delta$ -biased fibration and  $t : Y \rightarrow X$  any map, then the pullback  $t^*A \rightarrow Y$  is also a  $\delta$ -biased fibration.*

*Proof.* This is clear from the fact that the  $\delta$ -biased fibrations can be made into the right class of a weak factorization system (by reasoning analogous to that for Proposition 34), but it will be useful to see how the structure indicated in (64) is itself stable under pullback. Indeed, consider the following commutative diagram, in which the front face of the cube is the pullback in question, and the right and left sides are the respective versions of the

construction in (64).

$$\begin{array}{ccccc}
 (t^*A)^I & \xrightarrow{\quad} & A^I & & \\
 \downarrow & & \downarrow & & \\
 (t^*A)_\epsilon & \xrightarrow{\quad\cdots\quad} & A_\epsilon & & \\
 \downarrow & \searrow & \downarrow & \searrow & \\
 & t^*A & & A & \\
 \downarrow & \downarrow & \downarrow & \downarrow & \\
 Y^I & \xrightarrow{\quad} & X^I & & \\
 \searrow \epsilon & & \searrow \epsilon & & \\
 Y & \xrightarrow{\quad t \quad} & X & & 
 \end{array} \tag{65}$$

The rear square of solid arrows is the image of the front face under the pathobject functor and is therefore also a pullback. The base commutes by the naturality of the maps  $\epsilon$ , as does a corresponding top square involving further such  $\epsilon$ 's not shown. Note that these naturality squares need not be pullbacks, but the vertical squares on the sides are, by construction. It follows that there is a dotted arrow as shown, making the resulting lower rear square commute. That lower square is then also a pullback, since the other vertical faces of the resulting cube are pullbacks, and thus finally, the upper rear square is also a pullback.

Now if  $A \rightarrow X$  is a  $\delta$ -biased fibration, then  $A^I \rightarrow A_\epsilon$  is a trivial fibration, and then so is its pullback  $(t^*A)^I \rightarrow (t^*A)_\epsilon$  since relative  $+$ -algebras are stable under pullback. Therefore the pullback  $t^*A \rightarrow Y$  is also a  $\delta$ -biased fibration.  $\square$

*Remark 62.* In this way we can show algebraically that the pullback of a  $\delta$ -biased fibration is again one by pulling back the structure that makes it so. In Section 6.3, the pullback stability of the fibration structure will be used in the construction of a universal fibration via a closely related argument.

**Lemma 63.** *Let  $\alpha : A \rightarrow X$  and  $\beta : B \rightarrow A$  be  $\delta$ -biased fibrations, then the composite  $\alpha \circ \beta : B \rightarrow X$  is also a  $\delta$ -biased fibration.*

*Proof.* Again for maps in the right class of a weak factorization system this is immediate. But let us see how the fibration structures also compose. We

have the following diagram for the fibration structures on  $B \rightarrow A$  and  $A \rightarrow X$  (with obvious notation).

$$\begin{array}{ccccc}
 B^I & \xrightarrow{+} & B_{\epsilon_A} & \xrightarrow{\quad} & B \\
 & & \downarrow & & \downarrow \\
 & & A^I & \xrightarrow{+} & A_{\epsilon_X} & \xrightarrow{\quad} & A \\
 & & & & \downarrow & & \downarrow \\
 & & & & X^I & \xrightarrow{\epsilon_X} & X,
 \end{array} \tag{66}$$

Pulling back  $B \rightarrow A$  in two steps we therefore obtain the intermediate map  $B_{\epsilon_X} \rightarrow A_{\epsilon_X}$  indicated in the following diagram.

$$\begin{array}{ccccccc}
 B^I & \xrightarrow{+} & B_{\epsilon_A} & \xrightarrow{\quad} & B_{\epsilon_X} & \xrightarrow{\quad} & B \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & A^I & \xrightarrow{+} & A_{\epsilon_X} & \xrightarrow{\quad} & A \\
 & & & & \downarrow & & \downarrow \\
 & & & & X^I & \xrightarrow{\epsilon_X} & X
 \end{array} \tag{67}$$

Now use the fact that a trivial fibration structure (*i.e.* a  $+$ -algebra structure) has a canonical pullback along any map, and that two such structures have a canonical composition (cf. Remark 15), to obtain a trivial fibration structure for the indicated composite map  $B^I \rightarrow B_{\epsilon_X}$ , which is then a fibration structure for the composite  $B \rightarrow A \rightarrow X$ .  $\square$

**Proposition 64** (Biased Frobenius). *If  $\alpha : A \rightarrow X$  and  $\beta : B \rightarrow A$  are  $\delta$ -biased fibrations, then the pushforward  $\alpha_*\beta : \Pi_A B \rightarrow X$  is also a  $\delta$ -biased fibration.*

*Proof.* Given  $\delta$ -biased fibrations  $\alpha : A \rightarrow X$  and  $\beta : B \rightarrow A$ , let  $a : A^I \rightarrow A_\epsilon$  and  $b : B^I \rightarrow a^*B_\epsilon$  be the associated trivial fibrations, so that we have the

situation of diagram (67), with all three squares pullbacks.

$$\begin{array}{ccccccc}
 B^I & \xrightarrow{b} & a^* B_\epsilon & \longrightarrow & B_\epsilon & \longrightarrow & B \\
 & \searrow \beta^I & \downarrow & & \downarrow & & \downarrow \beta \\
 & & A^I & \xrightarrow{a} & A_\epsilon & \longrightarrow & A \\
 & & & \searrow \alpha^I & \downarrow & & \downarrow \alpha \\
 & & & & X^I & \xrightarrow{\epsilon} & X
 \end{array} \tag{68}$$

Taking the pushforward of the righthand vertical column give a map,

$$\gamma := \alpha_* \beta : \Pi_A B \rightarrow X,$$

and placing it underneath, along with the corresponding construction from (64), we then have the following commutative diagram.

$$\begin{array}{ccccccc}
 B^I & \xrightarrow{b} & a^* B_\epsilon & \longrightarrow & B_\epsilon & \longrightarrow & B \\
 & \searrow \beta^I & \downarrow & & \downarrow & & \downarrow \beta \\
 & & A^I & \xrightarrow{a} & A_\epsilon & \longrightarrow & A \\
 & & & \searrow \alpha^I & \downarrow & & \downarrow \alpha \\
 & & & & X^I & \xrightarrow{\epsilon} & X \\
 & & \nearrow \gamma^I & & \uparrow & & \uparrow \gamma \\
 (\Pi_A B)^I & \xrightarrow{c} & (\Pi_A B)_\epsilon & \longrightarrow & \Pi_A B & & 
 \end{array} \tag{69}$$

We wish to show that the indicated map  $c : (\Pi_A B)^I \rightarrow (\Pi_A B)_\epsilon$  admits a  $+$ -algebra structure. This we will do by showing that it is a retract of a known  $+$ -algebra. Namely, we can apply the pushforward along the map  $\alpha^I : A^I \rightarrow X^I$  to the  $+$ -algebra  $b : B^I \rightarrow a^* B_\epsilon$  regarded as an arrow over  $A^I$ . We obtain an arrow over  $X^I$  of the form

$$\Pi_{A^I} b : \Pi_{A^I} B^I \longrightarrow \Pi_{A^I} a^* B_\epsilon \tag{70}$$

which is indeed a  $+$ -algebra, since these are preserved under pushing forward, by Remark 15.

Next, observe that by the Beck-Chevalley condition for the central pullback, for the codomain of  $c$  we have an isomorphism

$$(\Pi_A B)_\epsilon \cong \Pi_{A_\epsilon} B_\epsilon \quad \text{over } X^I.$$

And since  $\Pi_{A^I} \cong \Pi_{A_\epsilon} \circ a_*$ , for the codomain of our +-algebra  $\Pi_{A^I} b$  from (70) we also have

$$\Pi_{A^I} a^* B_\epsilon \cong \Pi_{A_\epsilon} a_* a^* B_\epsilon.$$

Thus the image of the unit  $\eta : B_\epsilon \rightarrow a_* a^* B_\epsilon$  under  $\Pi_{A_\epsilon}$  provides a map  $\sigma := \Pi_{A_\epsilon} \eta$  over  $X^I$  of the form:

$$\begin{array}{ccc} & & X^I \\ & \nearrow & \uparrow \\ (\Pi_A B)^I & \xrightarrow{c} & \Pi_{A_\epsilon} B_\epsilon \\ & & \downarrow \sigma \\ \Pi_{A^I} B^I & \xrightarrow{\Pi_{A^I} b} & \Pi_{A_\epsilon} a_* a^* B_\epsilon \end{array} \quad (71)$$

Our goal is now to determine further arrows  $\varphi, \psi, \tau$  as indicated below, exhibiting  $c$  as a retract of  $\Pi_{A^I} b$  in the arrow category over  $X^I$ .

$$\begin{array}{ccc} & & X^I \\ & \nearrow & \uparrow \\ (\Pi_A B)^I & \xrightarrow{c} & \Pi_{A_\epsilon} B_\epsilon \\ \downarrow \varphi & & \downarrow \sigma \\ \Pi_{A^I} B^I & \xrightarrow{\Pi_{A^I} b} & \Pi_{A_\epsilon} a_* a^* B_\epsilon \\ \downarrow \psi & & \downarrow \tau \\ (\Pi_A B)^I & \xrightarrow{c} & \Pi_{A_\epsilon} B_\epsilon \end{array} \quad (72)$$

- For  $\varphi$ , we require a map

$$\varphi : (\Pi_A B)^I \rightarrow \Pi_{A^I} B^I \quad \text{over } X^I.$$

(73)

The map  $e$  is the counit at  $\beta : B \rightarrow A$  of the pullback-pushforward adjunction along  $\alpha : A \rightarrow X$ . The right-hand side of the diagram, including  $e$  and the associated pullback square, reappears (mirrored) on the left under the functor  $(-)^I$ , which preserves the pullback. Thus we can take  $\varphi$  to be the transpose of  $e^I$  under the pullback-pushforward adjunction along  $\alpha^I : A^I \rightarrow X^I$ ,

An easy diagram chase involving the pullback-pushforward adjunction along  $A_\epsilon \rightarrow X^I$  shows that the upper square in (72) then commutes.

$$o^*a^*B_\epsilon \cong B_\epsilon \quad \text{over } A_\epsilon$$
$$a^*B_\epsilon \rightarrow o_*B_\epsilon \quad \text{over } A^I$$
$$t : a_* a^* B_\epsilon \rightarrow a_* o_* B_\epsilon \cong B_\epsilon \quad \text{over } A_\epsilon.$$
$$\tau := \Pi_{A_\epsilon} t : \Pi_{A_\epsilon} a_* a^* B_\epsilon \rightarrow \Pi_{A_\epsilon} B_\epsilon.$$



- For  $\psi$ , we require a map

$$\psi : \Pi_{A^I} B^I \rightarrow (\Pi_A B)^I \quad \text{over } X^I.$$

Consider the following diagram resulting from combining (69) and (72), in which all solid arrows are those already introduced. The dotted arrow labelled  $p$  is the evident composite.

$$\begin{array}{ccccc}
 & & X^I & \xrightarrow{\epsilon} & X \\
 & \nearrow & \uparrow & & \uparrow \\
 (\Pi_A B)^I & \longrightarrow & \Pi_{A_\epsilon} B_\epsilon & \longrightarrow & \Pi_A B \\
 \downarrow & & \downarrow & & \downarrow \\
 \Pi_{A^I} B^I & \longrightarrow & \Pi_{A_\epsilon} a_* a^* B_\epsilon & & = \\
 & \searrow & \downarrow & \nearrow p & \downarrow \\
 (\Pi_A B)^I & \longrightarrow & \Pi_{A_\epsilon} B_\epsilon & \longrightarrow & \Pi_A B
 \end{array} \tag{74}$$

The lower horizontal composite is the evaluation of the pathobject  $(\Pi_A B)^I$  at the point  $\delta : 1 \rightarrow I$ ,

$$\epsilon_{\Pi_A B} = (\Pi_A B)^\delta : (\Pi_A B)^I \longrightarrow (\Pi_A B)^1 \cong \Pi_A B.$$

This is constructed from the (cartesian closed) evaluation,

$$\text{eval} : I \times (\Pi_A B)^I \longrightarrow \Pi_A B$$

which is the counit of  $I \times (-) \dashv (-)^I$ , as the composite shown below.

$$\begin{array}{ccc}
 (\Pi_A B)^I & \xrightarrow{\epsilon_{\Pi_A B}} & \Pi_A B \\
 \cong \downarrow & & \uparrow \text{eval} \\
 1 \times (\Pi_A B)^I & \xrightarrow{\delta \times (\Pi_A B)^I} & I \times (\Pi_A B)^I
 \end{array} \tag{75}$$

Let us analyse this evaluation at  $\delta$  further, in terms of the *locally* cartesian closed structure associated to the base changes along the section  $\delta : 1 \rightarrow I$  and retraction  $I \rightarrow 1$  in  $\mathcal{E}$ . Since  $\text{id} \cong \delta^* I^* : \mathcal{E} \rightarrow \mathcal{E}/_I \rightarrow \mathcal{E}$ , the map  $\epsilon_{\Pi_A B}$  can

be rewritten as follows.

$$\begin{array}{ccc}
(\Pi_A B)^I & \xrightarrow{\epsilon_{\Pi_A B}} & \Pi_A B \\
\cong \downarrow & & \downarrow \cong \\
\delta^* I^* ((\Pi_A B)^I) & \xrightarrow{\delta^* I^* \epsilon_{\Pi_A B}} & \delta^* I^* \Pi_A B \\
\cong \downarrow & & \downarrow = \\
\delta^* I^* I_* I^* \Pi_A B & \xrightarrow{\delta^* \epsilon} & \delta^* I^* \Pi_A B
\end{array} \tag{76}$$

where the map  $\delta^* \epsilon$  across the bottom is the counit of the adjunction  $I^* \dashv I_*$ , taken at  $I^* \Pi_A B$ , and then pulled back along  $\delta : 1 \rightarrow I$ . Before taking the pullback, we therefore have the following iso over  $I$  between that counit  $\epsilon_{I^*}$  and the image under  $I^*$  of the previously considered evaluation  $\epsilon : (\Pi_A B)^I \rightarrow \Pi_A B$  from (75).

$$\begin{array}{ccc}
I^* ((\Pi_A B)^I) & \xrightarrow{I^* \epsilon} & I^* \Pi_A B \\
\cong \downarrow & & \downarrow = \\
I^* I_* I^* \Pi_A B & \xrightarrow{\epsilon_{I^*}} & I^* \Pi_A B
\end{array} \tag{77}$$

Now let us apply  $I^*$  to (74) to get the map  $I^* p$  in the diagram below, which therefore factors (up to (77)) through the counit  $\epsilon_{I^*}$  as  $\epsilon_{I^*} \circ I^*(\widetilde{I^* p})$ , where  $\widetilde{I^* p}$  is the adjoint transpose of  $I^* p$ , as shown.

$$\begin{array}{ccccc}
I^* \Pi_{A^I} B^I & \longrightarrow & I^* \Pi_{A_\epsilon} a_* a^* B_\epsilon & & \\
I^*(\widetilde{I^* p}) \downarrow & & \downarrow & \searrow I^* p & \\
I^* I_* I^* \Pi_A B & \longrightarrow & I^* \Pi_{A_\epsilon} B_\epsilon & \longrightarrow & I^* \Pi_A B \\
& & \searrow \epsilon_{I^*} & & 
\end{array} \tag{78}$$

We can therefore set

$$\psi := \widetilde{I^* p},$$

and we obtain  $\epsilon \circ \psi = p$ , from which it follows that the square in (78) commutes by the definition of  $\Pi_{A_\epsilon} B_\epsilon$  as a pullback. The same square without  $I^*$  then also commutes by applying the retraction  $\delta^*$ .

We have now defined all the maps indicated below, the squares involving  $\varphi$  and  $\psi$  commute, and the composite of  $\sigma$  and  $\tau$  is the identity.

$$\begin{array}{ccccc}
 & & X^I & \longrightarrow & X \\
 & \nearrow & \uparrow & & \uparrow \\
 (\Pi_A B)^I & \longrightarrow & \Pi_{A_\epsilon} B_\epsilon & \longrightarrow & \Pi_A B \\
 \downarrow \varphi & & \downarrow \sigma & & \downarrow = \\
 \Pi_{A^I} B^I & \longrightarrow & \Pi_{A_\epsilon} p_* p^* B_\epsilon & & \\
 \downarrow \psi & & \downarrow \tau & \nearrow p & \\
 (\Pi_A B)^I & \longrightarrow & \Pi_{A_\epsilon} B_\epsilon & \longrightarrow & \Pi_A B \\
 & \searrow \epsilon & & & 
 \end{array} \tag{79}$$

To see that  $\psi \circ \varphi = 1$ , an easy chase through the diagram (79) shows that

$$\epsilon \circ \psi \circ \varphi = p \circ \varphi = \epsilon.$$

Thus by applying  $I^*$  and using (77) we have  $\varepsilon_{I^*} \circ I^*(\psi \circ \varphi) = \varepsilon_{I^*}$ , and so  $\psi \circ \varphi = \widetilde{\varepsilon_{I^*}} = 1$ . □

From Proposition 60 we then have:

**Corollary 65** (Unbiased Frobenius). *The unbiased fibration weak factorization system on  $\mathbf{cSet}$  satisfies the Frobenius condition.*

*Remark 66.* We note in passing that the proof just given for the  $\delta$ -biased case of Frobenius, Proposition 64, made no use of the fact that  $\delta : 1 \rightarrow I$  is generic, nor even that we were working in the slice category over  $I$ . Indeed the same algebraic argument works for  $p$ -biased fibrations for any point  $p : 1 \rightarrow I$  of any object  $I$  in any (quasi-)topos  $\mathcal{E}$ .

## 6 A universal fibration

We next construct a *universal small fibration*  $\mathcal{U} \rightarrow \mathcal{U}$ . It will be shown in Section 8 that the base object  $\mathcal{U}$  is fibrant, using the equivalence extension property to be proved in Section 7.<sup>1</sup>

<sup>1</sup>With only minor changes the position of this section and Section 7 could be exchanged.

Our construction of  $\dot{\mathcal{U}} \rightarrow \mathcal{U}$  makes use, first of all, of a new description of the well-known Hofmann-Streicher universe in a category  $\widehat{\mathbb{C}} = [\mathbb{C}^{\text{op}}, \mathbf{Set}]$  of presheaves on a small category  $\mathbb{C}$ , which was used in [HS97] to interpret dependent type theory. See [?] for further details.

## 6.1 Classifying families

**Definition 67** ([HS97]). Let  $\mathbb{C}$  be a small category. A (type-theoretic) *universe*  $(U, \text{El})$  consists of  $U \in \widehat{\mathbb{C}}$  and  $\text{El} \in \widehat{\int_{\mathbb{C}} U}$  with:

$$U(c) = \text{Cat}(\mathbb{C}/_c^{\text{op}}, \mathbf{Set}) \quad (80)$$

$$\text{El}(c, A) = A(\text{id}_c) \quad (81)$$

with the evident associated action on morphisms.

A few comments are required:

- In contrast to [HS97], in (80) we take the underlying set of objects of the functor category  $\widehat{\mathbb{C}/_c} = [\mathbb{C}/_c^{\text{op}}, \mathbf{Set}]$ .
- As in [HS97], (81) adopts the “categories with families” point of view in describing an arrow  $E \rightarrow U$  in  $\widehat{\mathbb{C}}$  equivalently as a presheaf on the category of elements  $\int_{\mathbb{C}} U$ , using

$$\widehat{\mathbb{C}/_U} \simeq \widehat{\int_{\mathbb{C}} U} \quad (82)$$

where

$$E(c) = \coprod_{A \in U(c)} \text{El}(c, A).$$

The argument  $(c, A) \in \int_{\mathbb{C}} U$  in (81) thus consists of an object  $c \in \mathbb{C}$  and an element  $A \in U(c)$ .

- To account for size issues, the authors of [HS97] assume a Grothendieck universe  $u$  in  $\mathbf{Set}$ , the elements of which are called *small*. The category  $\mathbb{C}$  is assumed to be small, as are the values of the presheaves, unless otherwise stated.

The presheaf  $U$ , which is not small, is then regarded as the Grothendieck universe  $u$  “lifted” from  $\mathbf{Set}$  to  $[\mathbb{C}^{\text{op}}, \mathbf{Set}]$ . We first analyse this specification of  $(U, \text{El})$  from a different perspective, in order to establish its basic property as a classifier for small families in  $\widehat{\mathbb{C}}$ .

**A realization-nerve adjunction.** For a presheaf  $X$  on  $\mathbb{C}$ , recall that the category of elements is the comma category,

$$\int_{\mathbb{C}} X = \mathbf{y}_{\mathbb{C}}/X,$$

where  $\mathbf{y}_{\mathbb{C}} : \mathbb{C} \rightarrow \mathbf{Set}^{\mathbb{C}^{\text{op}}}$  is the Yoneda embedding, which we sometimes suppress and write simply  $\mathbb{C}/_X$  for  $\mathbf{y}_{\mathbb{C}}/X$ .

**Proposition 68** ([Gro83], §28). *The category of elements functor*

$$\int_{\mathbb{C}} : \widehat{\mathbb{C}} \longrightarrow \mathbf{Cat}$$

*has a right adjoint,*

$$\nu_{\mathbb{C}} : \mathbf{Cat} \longrightarrow \widehat{\mathbb{C}}.$$

*For a small category  $\mathbb{A}$ , we shall call the presheaf  $\nu_{\mathbb{C}}(\mathbb{A})$  the  $(\mathbb{C}\text{-})$ nerve of  $\mathbb{A}$ .*

*Proof.* The adjunction  $\int_{\mathbb{C}} \dashv \nu_{\mathbb{C}}$  is an instance of the usual “realization/nerve” adjunction, here with respect to the covariant slice category functor  $\mathbb{C}/- : \mathbb{C} \rightarrow \mathbf{Cat}$ , as indicated below.

$$\begin{array}{ccc} \widehat{\mathbb{C}} & \begin{array}{c} \xleftarrow{\nu_{\mathbb{C}}} \\ \xrightarrow{\int_{\mathbb{C}}} \end{array} & \mathbf{Cat} \\ \uparrow \mathbf{y} & \nearrow \mathbb{C}/- & \\ \mathbb{C} & & \end{array} \quad (83)$$

In detail, for  $\mathbb{A} \in \mathbf{Cat}$  and  $c \in \mathbb{C}$ , let  $\nu_{\mathbb{C}}(\mathbb{A})(c)$  be the Hom-set of functors,

$$\nu_{\mathbb{C}}(\mathbb{A})(c) = \mathbf{Cat}(\mathbb{C}/_c, \mathbb{A}),$$

with contravariant action on  $h : d \rightarrow c$  given by pre-composing a functor  $P : \mathbb{C}/_c \rightarrow \mathbb{A}$  with the post-composition functor

$$\mathbb{C}/_h : \mathbb{C}/_d \longrightarrow \mathbb{C}/_c.$$

For the adjunction, observe that the slice category  $\mathbb{C}/_c$  is the category of elements of the representable functor  $\mathbf{y}_c$ ,

$$\int_{\mathbb{C}} \mathbf{y}_c \cong \mathbb{C}/_c.$$

Thus for representables  $yc$ , we have the required natural isomorphism

$$\widehat{\mathbb{C}}(yc, \nu_{\mathbb{C}}(\mathbb{A})) \cong \nu_{\mathbb{C}}(\mathbb{A})(c) = \mathbf{Cat}(\mathbb{C}/_c, \mathbb{A}) \cong \mathbf{Cat}(\int_{\mathbb{C}} yc, \mathbb{A}).$$

For arbitrary presheaves  $X$ , one uses the presentation of  $X$  as a colimit of representables over the index category  $\int_{\mathbb{C}} X$ , and the easy to prove fact that  $\int_{\mathbb{C}}$  itself preserves colimits. Indeed, for any category  $\mathbb{D}$ , we have an isomorphism in  $\mathbf{Cat}$ ,

$$\varinjlim_{d \in \mathbb{D}} \mathbb{D}/_d \cong \mathbb{D}.$$

□

When  $\mathbb{C}$  is fixed, we may omit the subscript in the notation  $y_{\mathbb{C}}$  and  $\int_{\mathbb{C}}$  and  $\nu_{\mathbb{C}}$ . The unit and counit maps of the adjunction  $\int \dashv \nu$ ,

$$\begin{aligned} \eta : X &\longrightarrow \nu \int X, \\ \epsilon : \int \nu \mathbb{A} &\longrightarrow \mathbb{A}, \end{aligned}$$

are then as follows. At  $c \in \mathbb{C}$ , for  $x : yc \rightarrow X$ , the functor  $(\eta_X)_c(x) : \mathbb{C}/_c \rightarrow \mathbb{C}/_X$  is just composition with  $x$ ,

$$(\eta_X)_c(x) = \mathbb{C}/_x : \mathbb{C}/_c \longrightarrow \mathbb{C}/_X. \quad (84)$$

For  $\mathbb{A} \in \mathbf{Cat}$ , the functor  $\epsilon : \int \nu \mathbb{A} \rightarrow \mathbb{A}$  takes a pair  $(c \in \mathbb{C}, f : \mathbb{C}/_c \rightarrow \mathbb{A})$  to the object  $f(1_c) \in \mathbb{A}$ ,

$$\epsilon(c, f) = f(1_c).$$

**Lemma 69.** *For any  $f : Y \rightarrow X$ , the naturality square below is a pullback.*

$$\begin{array}{ccc} Y & \xrightarrow{\eta_Y} & \nu \int Y \\ f \downarrow & & \downarrow \nu f f \\ X & \xrightarrow[\eta_X]{} & \nu \int X. \end{array} \quad (85)$$

*Proof.* It suffices to prove this for the case  $f : X \rightarrow 1$ . Thus consider the square

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & \nu \int X \\ \downarrow & & \downarrow \\ 1 & \xrightarrow[\eta_1]{} & \nu \int 1. \end{array} \quad (86)$$

Evaluating at  $c \in \mathbb{C}$  and applying (84) gives the following square in  $\mathbf{Set}$ .

$$\begin{array}{ccc} Xc & \xrightarrow{\mathbb{C}/-} & \mathbf{Cat}(\mathbb{C}/_c, \mathbb{C}/_X) \\ \downarrow & & \downarrow \\ 1c & \xrightarrow[\mathbb{C}/-]{} & \mathbf{Cat}(\mathbb{C}/_c, \mathbb{C}/_1) \end{array} \quad (87)$$

The image of  $* \in 1c$  along the bottom is the forgetful functor  $U_c : \mathbb{C}/_c \rightarrow \mathbb{C}$ , and its fiber under the map on the right is the set of functors  $F : \mathbb{C}/_c \rightarrow \mathbb{C}/_X$  such that  $U_X \circ F = U_c$ , where  $U_X : \mathbb{C}/_X \rightarrow \mathbb{C}$  is also a forgetful functor. But any such  $F$  is uniquely of the form  $\mathbb{C}/_x$  for  $x = F(1_c) : yc \rightarrow X$ .  $\square$

**A universal family.** For the terminal presheaf  $1 \in \widehat{\mathbb{C}}$  we have an iso  $\int 1 \cong \mathbb{C}$ , so for every  $X \in \widehat{\mathbb{C}}$  there is a canonical projection  $\int X \rightarrow \mathbb{C}$ , which is a discrete fibration. It follows that for any map  $Y \rightarrow X$  of presheaves, the associated map  $\int Y \rightarrow \int X$  is also a discrete fibration. Ignoring size issues temporarily, recall that discrete fibrations in  $\mathbf{Cat}$  are classified by the forgetful functor  $\mathbf{Set}^{\text{op}} \rightarrow \mathbf{Set}^{\text{op}}$  from (the opposites of) the category of pointed sets to that of sets (cf. [Web07]). For every presheaf  $X \in \widehat{\mathbb{C}}$ , we therefore have a pullback diagram in  $\mathbf{Cat}$ ,

$$\begin{array}{ccc} \int X & \longrightarrow & \mathbf{Set}^{\text{op}} \\ \downarrow & \lrcorner & \downarrow \\ \mathbb{C} & \xrightarrow[X]{} & \mathbf{Set}^{\text{op}}. \end{array} \quad (88)$$

Using  $\mathbb{C} \cong \int 1$  and transposing by the adjunction  $\int \dashv \nu$  then gives a commutative square in  $\widehat{\mathbb{C}}$  of the form:

$$\begin{array}{ccc} X & \longrightarrow & \nu \mathbf{Set}^{\text{op}} \\ \downarrow & & \downarrow \\ 1 & \xrightarrow[\tilde{X}]{} & \nu \mathbf{Set}^{\text{op}}. \end{array} \quad (89)$$

**Lemma 70.** *The square (89) is a pullback in  $\widehat{\mathbb{C}}$ . More generally, for any*

map  $Y \rightarrow X$  in  $\widehat{\mathbb{C}}$ , there is a canonical pullback square

$$\begin{array}{ccc} Y & \longrightarrow & \nu \dot{\mathbf{Set}}^{\text{op}} \\ \downarrow \lrcorner & & \downarrow \\ X & \longrightarrow & \nu \mathbf{Set}^{\text{op}}. \end{array} \quad (90)$$

*Proof.* Apply the right adjoint  $\nu$  to the pullback square (88) and paste the naturality square (85) from Lemma 69 on the left, to obtain the transposed square (90) as a pasting of two pullbacks.  $\square$

Let us write  $\dot{\mathcal{V}} \rightarrow \mathcal{V}$  for the vertical map on the right in (90), setting

$$\begin{aligned} \dot{\mathcal{V}} &:= \nu \dot{\mathbf{Set}}^{\text{op}} \\ \mathcal{V} &:= \nu \mathbf{Set}^{\text{op}}. \end{aligned} \quad (91)$$

We summarize our results so far as follows.

**Proposition 71.** *The nerve  $\dot{\mathcal{V}} \rightarrow \mathcal{V}$  of the classifier for discrete fibrations  $\dot{\mathbf{Set}}^{\text{op}} \rightarrow \mathbf{Set}^{\text{op}}$ , as defined in (91), classifies natural transformations  $Y \rightarrow X$  in  $\widehat{\mathbb{C}}$ , in the sense that there is always a pullback square,*

$$\begin{array}{ccc} Y & \longrightarrow & \dot{\mathcal{V}} \\ \downarrow \lrcorner & & \downarrow \\ X & \xrightarrow{\tilde{Y}} & \mathcal{V}. \end{array} \quad (92)$$

The classifying map  $\tilde{Y} : X \rightarrow \mathcal{V}$  is determined by the adjunction  $\int \dashv \nu$  as the transpose of the classifying map of the discrete fibration  $\int Y \rightarrow \int X$ .

Given a natural transformation  $Y \rightarrow X$ , the classifying map  $\tilde{Y} : X \rightarrow \mathcal{V}$  is of course not in general unique. Nonetheless, we can use the construction of  $\dot{\mathcal{V}} \rightarrow \mathcal{V}$  as the nerve of the discrete fibration classifier  $\dot{\mathbf{Set}}^{\text{op}} \rightarrow \mathbf{Set}^{\text{op}}$ , for which classifying functors  $\mathbb{C} \rightarrow \mathbf{Set}^{\text{op}}$  are unique up to natural isomorphism, to infer the following proposition, which will be required below (cf. [Shu15, GSS22]).

**Proposition 72** (Realignment for families). *Given a monomorphism  $c : C \rightarrow X$  and a family  $Y \rightarrow X$ , let  $y_c : C \rightarrow \mathcal{V}$  classify the pullback  $c^*Y \rightarrow C$ .*



Then there is a classifying map  $y : X \rightarrow \mathcal{V}$  for  $Y \rightarrow X$  with  $y \circ c = y_c$ .

$$\begin{array}{ccc}
 c^*Y & \xrightarrow{\quad} & \dot{\mathcal{V}} \\
 \downarrow & \searrow & \vdots \\
 & Y & \\
 \downarrow & & \downarrow \\
 C & \xrightarrow{y_c} & \mathcal{V} \\
 \downarrow c & & \downarrow y \\
 & X &
 \end{array}
 \quad (93)$$

*Proof.* Transposing the realignment problem (93) for presheaves across the adjunction  $\int \dashv \nu$  results in the following realignment problem for discrete fibrations.

$$\begin{array}{ccc}
 \int c^*Y & \xrightarrow{\quad} & \mathbf{Set}^{\text{op}} \\
 \downarrow & \searrow & \vdots \\
 & \int Y & \\
 \downarrow & & \downarrow \\
 \int C & \xrightarrow{\tilde{y}_c} & \mathbf{Set}^{\text{op}} \\
 \downarrow \int c & & \downarrow \tilde{y} \\
 & \int X &
 \end{array}
 \quad (94)$$

The category of elements functor  $\int$  is easily seen to preserve pullbacks, hence monos; thus let us consider the general case of a functor  $C : \mathbb{C} \rightarrow \mathbb{D}$  which is monic in  $\mathbf{Cat}$ , a pullback of discrete fibrations as on the left below, and a

presheaf  $E : \mathbb{C} \rightarrow \mathbf{Set}^{\text{op}}$  with  $\int E \cong \mathbb{E}$  over  $\mathbb{C}$ .

$$\begin{array}{ccc}
 \mathbb{E} & \xrightarrow{\quad} & \mathbf{Set}^{\text{op}} \\
 \downarrow & \searrow & \uparrow \text{dotted} \\
 & \mathbb{F} & \\
 \downarrow & \downarrow & \downarrow \\
 \mathbb{C} & \xrightarrow{E} & \mathbf{Set}^{\text{op}} \\
 \downarrow C & & \uparrow \text{dotted} \\
 & \mathbb{D} & \xrightarrow{F}
 \end{array}
 \tag{95}$$

We seek  $F : \mathbb{D} \rightarrow \mathbf{Set}^{\text{op}}$  with  $\int F \cong \mathbb{F}$  over  $\mathbb{D}$  and  $F \circ C = E$ . Let  $F_0 : \mathbb{D} \rightarrow \mathbf{Set}^{\text{op}}$  with  $\int F_0 \cong \mathbb{F}$  over  $\mathbb{D}$ , which exists since  $\mathbb{F} \rightarrow \mathbb{D}$  is a discrete fibration. Since  $F_0 \circ C$  and  $E$  both classify  $\mathbb{E}$ , there is a natural iso  $e : F_0 \circ C \cong E$ . Consider the following diagram

$$\begin{array}{ccccc}
 \mathbb{C} & \xrightarrow{e} & (\mathbf{Set}^{\cong})^{\text{op}} & \xrightarrow{p_2} & \mathbf{Set}^{\text{op}} \\
 \downarrow C & & \uparrow \text{dotted} f & \downarrow p_1 & \\
 \mathbb{D} & \xrightarrow{F_0} & \mathbf{Set}^{\text{op}} & & 
 \end{array}
 \tag{96}$$

where  $\mathbf{Set}^{\cong}$  is the category of isos in  $\mathbf{Set}$ , with  $p_1, p_2$  the (opposites of the) domain and codomain projections. There is a well-known weak factorization system on  $\mathbf{Cat}$  (part of the “canonical model structure”) with injective-on-objects functors on the left and isofibrations on the right. Thus there is a diagonal filler  $f$  as indicated. The functor  $F := p_2 \circ f : \mathbb{D} \rightarrow \mathbf{Set}^{\text{op}}$  is then the one we seek.  $\square$

**Small maps.** Of course, as defined in (91), the classifier  $\dot{\mathcal{V}} \rightarrow \mathcal{V}$  cannot be a map in  $\widehat{\mathbb{C}}$ , for reasons of size; we now address this. Let  $\alpha$  be a cardinal number, and call the sets strictly smaller than it  $\alpha$ -small. Let  $\mathbf{Set}_\alpha \hookrightarrow \mathbf{Set}$  be the full subcategory of  $\alpha$ -small sets. Call a presheaf  $X : \mathbb{C}^{\text{op}} \rightarrow \mathbf{Set}$   $\alpha$ -small if all of its values are  $\alpha$ -small sets, and thus if, and only if, it factors through  $\mathbf{Set}_\alpha \hookrightarrow \mathbf{Set}$ . Call a map  $f : Y \rightarrow X$  of presheaves  $\alpha$ -small if all of the

fibers  $f_c^{-1}\{x\} \subseteq Yc$  are  $\alpha$ -small sets (for all  $c \in \mathbb{C}$  and  $x \in Xc$ ). The latter condition is of course equivalent to saying that, in the pullback square over the element  $x : yc \rightarrow X$ ,

$$\begin{array}{ccc} Y_x & \longrightarrow & Y \\ \downarrow & \lrcorner & \downarrow f \\ yc & \xrightarrow{x} & X, \end{array} \quad (97)$$

the presheaf  $Y_x$  is  $\alpha$ -small.

Now let us restrict the specification (91) of  $\dot{\mathcal{V}} \rightarrow \mathcal{V}$  to the  $\alpha$ -small sets:

$$\begin{aligned} \dot{\mathcal{V}}_\alpha &:= \nu \dot{\text{Set}}_\alpha^{\text{op}} \\ \mathcal{V}_\alpha &:= \nu \text{Set}_\alpha^{\text{op}}. \end{aligned} \quad (98)$$

Then the evident forgetful map  $\dot{\mathcal{V}}_\alpha \rightarrow \mathcal{V}_\alpha$  is a map in the category  $\widehat{\mathbb{C}}$  of presheaves, and it is in fact  $\alpha$ -small. Moreover, it has the following basic property, which is just a restriction of the basic property of  $\dot{\mathcal{V}} \rightarrow \mathcal{V}$  stated in Proposition 71.

**Proposition 73.** *The map  $\dot{\mathcal{V}}_\alpha \rightarrow \mathcal{V}_\alpha$  classifies  $\alpha$ -small maps  $f : Y \rightarrow X$  in  $\widehat{\mathbb{C}}$ , in the sense that there is always a pullback square,*

$$\begin{array}{ccc} Y & \longrightarrow & \dot{\mathcal{V}}_\alpha \\ \downarrow & \lrcorner & \downarrow \\ X & \xrightarrow{\tilde{Y}} & \mathcal{V}_\alpha. \end{array} \quad (99)$$

The classifying map  $\tilde{Y} : X \rightarrow \mathcal{V}_\alpha$  is determined by the adjunction  $\int \dashv \nu$  as (the factorization of) the transpose of the classifying map of the discrete fibration  $\int X \rightarrow \int Y$ .

*Proof.* If  $Y \rightarrow X$  is  $\alpha$ -small, its classifying map  $\tilde{Y} : X \rightarrow \mathcal{V}$  factors through  $\mathcal{V}_\alpha \hookrightarrow \mathcal{V}$ , as indicated below,

$$\begin{array}{ccccc} & & \curvearrowright & & \\ Y & \longrightarrow & \nu \dot{\text{Set}}_\alpha^{\text{op}} & \hookrightarrow & \nu \text{Set}^{\text{op}} \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & \nu \text{Set}_\alpha^{\text{op}} & \hookrightarrow & \nu \text{Set}^{\text{op}}, \\ & & \curvearrowleft & & \\ & & \tilde{Y} & & \end{array} \quad (100)$$

in virtue of the following adjoint transposition,

$$\begin{array}{ccccc}
 \int Y & \xrightarrow{\quad} & \mathbf{Set}_\alpha^{\text{op}} & \hookrightarrow & \mathbf{Set}^{\text{op}} \\
 \downarrow & & \downarrow & & \downarrow \\
 \int X & \xrightarrow{\quad} & \mathbf{Set}_\alpha^{\text{op}} & \hookrightarrow & \mathbf{Set}^{\text{op}}.
 \end{array}
 \quad (101)$$

Note that the square on the right is evidently a pullback, and so the one on the left is, too, because the outer rectangle is the classifying pullback of the discrete fibration  $\int Y \rightarrow \int X$ , as stated. Thus the left square in (100) is also a pullback.  $\square$

**Examples of universal families  $\dot{\mathcal{V}}_\alpha \rightarrow \mathcal{V}_\alpha$ .**

1. Let  $\alpha = \kappa$  a strongly inaccessible cardinal, so that  $\mathbf{ob}(\mathbf{Set}_\kappa)$  is a Grothendieck universe. Then the Hofmann-Streicher universe of Definition 67 is re-covered as the  $\kappa$ -small map classifier

$$E \cong \dot{\mathcal{V}}_\kappa \rightarrow \mathcal{V}_\kappa \cong U$$

in the sense of Proposition 73. Indeed, for  $c \in \mathbb{C}$ , we have

$$\mathcal{V}_\kappa c = \nu(\mathbf{Set}_\kappa^{\text{op}})(c) = \mathbf{Cat}(\mathbb{C}/_c, \mathbf{Set}_\kappa^{\text{op}}) = \mathbf{ob}(\widehat{\mathbb{C}/_c}) = U c. \quad (102)$$

For  $\dot{\mathcal{V}}_\kappa$  we then have,

$$\begin{aligned}
 \dot{\mathcal{V}}_\kappa c &= \nu(\dot{\mathbf{Set}}_\kappa^{\text{op}})(c) = \mathbf{Cat}(\mathbb{C}/_c, \dot{\mathbf{Set}}_\kappa^{\text{op}}) \\
 &\cong \coprod_{A \in \mathcal{V}_\kappa c} \mathbf{Cat}_{\mathbb{C}/_c}(\mathbb{C}/_c, A^* \mathbf{Set}_\kappa^{\text{op}})
 \end{aligned}
 \quad (103)$$

where the  $A$ -summand in (103) is defined by taking sections of the pullback indicated below.

$$\begin{array}{ccc}
 A^* \mathbf{Set}_\kappa^{\text{op}} & \longrightarrow & \dot{\mathbf{Set}}_\kappa^{\text{op}} \\
 \downarrow \lrcorner & \nearrow & \downarrow \\
 \mathbb{C}/_c & \xrightarrow{A} & \mathbf{Set}_\kappa^{\text{op}}
 \end{array}
 \quad (104)$$

But  $A^*\mathbf{Set}_\kappa^{\text{op}} \cong \int_{\mathbb{C}/c} A$  over  $\mathbb{C}/c$ , and sections of this discrete fibration in  $\mathbf{Cat}$  correspond uniquely to natural maps  $1 \rightarrow A$  in  $\widehat{\mathbb{C}/c}$ . Since  $1$  is representable in  $\widehat{\mathbb{C}/c}$  we can continue (103) by

$$\begin{aligned} \dot{\mathcal{V}}_\kappa c &\cong \coprod_{A \in \mathcal{V}_\kappa c} \mathbf{Cat}_{\mathbb{C}/c}(\mathbb{C}/c, A^*\mathbf{Set}_\kappa^{\text{op}}) \\ &\cong \coprod_{A \in \mathcal{V}_\kappa c} \widehat{\mathbb{C}/c}(1, A) \\ &\cong \coprod_{A \in \mathcal{V}_\kappa c} A(1_c) \\ &= \coprod_{A \in \mathcal{V}_\kappa c} \mathbf{El}(\langle c, A \rangle) \\ &= Ec. \end{aligned}$$

2. By functoriality of the nerve  $\nu : \mathbf{Cat} \rightarrow \widehat{\mathbb{C}}$ , a sequence of Grothendieck universes

$$\mathbf{Set}_\alpha \subseteq \mathbf{Set}_\beta \subseteq \dots$$

in  $\mathbf{Set}$  gives rise to a (cumulative) sequence of type-theoretic universes

$$\mathcal{V}_\alpha \rightarrowtail \mathcal{V}_\beta \rightarrowtail \dots$$

in  $\widehat{\mathbb{C}}$ . More precisely, there is a sequence of cartesian squares,

$$\begin{array}{ccccc} \dot{\mathcal{V}}_\alpha & \rightarrowtail & \dot{\mathcal{V}}_\beta & \rightarrowtail & \dots \\ \downarrow \lrcorner & & \downarrow \lrcorner & & \\ \mathcal{V}_\alpha & \rightarrowtail & \mathcal{V}_\beta & \rightarrowtail & \dots, \end{array} \tag{105}$$

in the image of  $\nu : \mathbf{Cat} \rightarrow \widehat{\mathbb{C}}$ , classifying small maps in  $\widehat{\mathbb{C}}$  of increasing size, in the sense of Proposition 73.

3. Let  $\alpha = 2$  so that  $1 \rightarrow 2$  is the subobject classifier of  $\mathbf{Set}$ , and

$$\mathbb{1} = \mathbf{Set}_2^{\text{op}} \longrightarrow \mathbf{Set}_2^{\text{op}} = \mathbb{2}$$

is then a classifier in  $\mathbf{Cat}$  for *sieves*, i.e. full subcategories  $\mathbb{S} \hookrightarrow \mathbb{A}$  closed under the domains of arrows  $a \rightarrow s$  for  $s \in \mathbb{S}$ . The nerve  $\dot{\mathcal{V}}_2 \rightarrow \mathcal{V}_2$  is then the usual subobject classifier  $1 \rightarrow \Omega$  of  $\widehat{\mathbb{C}}$ ,

$$1 = \nu \mathbb{1} = \dot{\mathcal{V}}_2 \longrightarrow \mathcal{V}_2 = \nu \mathbb{2} = \Omega.$$

4. For any  $X \in \widehat{\mathbb{C}}$ , we have an equivalence

$$\widehat{\mathbb{C}}/X \simeq \widehat{\int_{\mathbb{C}} X} \simeq \mathbf{dFib}/_{\int_{\mathbb{C}} X}$$

where, generally,  $\mathbf{dFib}/_{\mathbb{D}}$  is the category of discrete fibrations over a category  $\mathbb{D}$ . This equivalence commutes with composition along discrete fibrations, in the sense that the forgetful functor

$$X_! : \widehat{\mathbb{C}}/X \rightarrow \widehat{\mathbb{C}}$$

given by composition along  $X \rightarrow 1$  agrees (up to canonical isomorphism) with the base change  $(p_X)_! \dashv (p_X)^*$  of presheaves along the projection  $p_X : \int_{\mathbb{C}} X \rightarrow \mathbb{C}$ , and with composition along the discrete fibration  $p_X$ , as indicated in:

$$\begin{array}{ccccc} \widehat{\mathbb{C}}/X & \xrightarrow{\sim} & \widehat{\int_{\mathbb{C}} X} & \xrightarrow{\sim} & \mathbf{dFib}/_{\int_{\mathbb{C}} X} \\ X_! \downarrow & & (p_X)_! \downarrow & & \downarrow p_X \circ (-) \\ \widehat{\mathbb{C}} & \xrightarrow{\sim} & \widehat{\mathbb{C}} & \xrightarrow{\sim} & \mathbf{dFib}/_{\mathbb{C}}. \end{array} \quad (106)$$

It follows that the pullback functor  $X^* : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}/X$  commutes with the corresponding right adjoints (one of which is the nerve), and therefore preserves the respective universes,

$$X^* \mathcal{V}_{\mathbb{C}} \cong (p_X)^* \nu_{\mathbb{C}}(\mathbf{Set}^{\text{op}}) \cong \nu_{\int_{\mathbb{C}} X}(\mathbf{Set}^{\text{op}}) \cong \mathcal{V}_{\int_{\mathbb{C}} X}.$$

**Corollary 74.** *Let  $\dot{\mathcal{V}}_{\alpha} \rightarrow \mathcal{V}_{\alpha}$  classify  $\alpha$ -small maps in  $\widehat{\mathbb{C}}$ , as in Proposition 73. Then for any  $X \in \mathbb{C}$ , the pullback  $X^* \dot{\mathcal{V}}_{\alpha} \rightarrow X^* \mathcal{V}_{\alpha}$  classifies  $\alpha$ -small maps in  $\widehat{\mathbb{C}}/X$*

## 6.2 Classifying trivial fibrations

Returning now to the presheaf category  $\mathbf{cSet} = \mathbf{Set}^{\square^{\text{op}}}$  of cubical sets, recall from section 2 that (uniform) trivial fibration structures on a map  $A \rightarrow X$  correspond bijectively to relative  $+$ -algebra structures over  $X$  (definition 9). A relative  $+$ -algebra structure on  $A \rightarrow X$  is an algebra structure for the pointed endofunctor  $+_X : \mathbf{cSet}/X \rightarrow \mathbf{cSet}/X$ , where recall from (2),

$$A^+ = \sum_{\varphi : \Phi} A^{[\varphi]} \quad \text{over } X.$$

A  $+$ -algebra structure is then a retract  $\alpha : A^+ \rightarrow A$  over  $X$  of the canonical map  $\eta_A : A \rightarrow A^+$ ,

$$\begin{array}{ccccc}
 & & \xrightarrow{\quad \quad} & & \\
 & \curvearrowright & & \curvearrowleft & \\
 A & \xrightarrow{\eta_A} & A^+ & \xrightarrow{\alpha} & A \\
 & \searrow & \downarrow & \swarrow & \\
 & & X & & 
 \end{array}
 \tag{107}$$

In more detail, let us write  $A \rightarrow X$  as a family  $(A_x)_{x \in X}$ , so that  $A = \sum_{x:X} A_x \rightarrow X$ . Since the  $+$ -functor acts fiberwise, the object  $A^+$  in (107) is then the indexing projection

$$\sum_{x:X} A_x^+ \rightarrow X.$$

Working in the slice  $\mathbf{cSet}/X$ , the (relative) exponentials (internal Hom's)  $[A^+, A]$  and  $[A, A]$  and the “precomposition by  $\eta_A$ ” map  $[\eta_A, A]$ , fit into the following pullback diagram

$$\begin{array}{ccc}
 +\mathbf{Alg}(A) & \longrightarrow & [A^+, A] \\
 \downarrow \lrcorner & & \downarrow [\eta_A, A] \\
 1 & \xrightarrow{\quad id_A \quad} & [A, A].
 \end{array}
 \tag{108}$$

The constructed object  $+\mathbf{Alg}(A) \rightarrow X$  over  $X$  is then the *object of  $+$ -algebra structures on  $A \rightarrow X$* , in the sense that sections  $X \rightarrow +\mathbf{Alg}(A)$  correspond uniquely to  $+$ -algebra structures on  $A \rightarrow X$ . Moreover,  $+\mathbf{Alg}(A) \rightarrow X$  is stable under pullback, in the sense that for any  $f : Y \rightarrow X$ , we have two pullback squares,

$$\begin{array}{ccc}
 f^*A & \longrightarrow & A \\
 \downarrow & & \downarrow \\
 Y & \xrightarrow{\quad f \quad} & X \\
 \uparrow & & \uparrow \\
 +\mathbf{Alg}(f^*A) & \longrightarrow & +\mathbf{Alg}(A)
 \end{array}
 \tag{109}$$

because the  $+$ -functor, exponentials and pullbacks occurring in the construction of  $+\mathbf{Alg}(A) \rightarrow X$  are themselves all stable.

It then follows from Proposition 73 that, if  $A \rightarrow X$  is small, then  $+Alg(A) \rightarrow X$  is itself a pullback of the analogous object  $+Alg(\dot{\mathcal{V}}) \rightarrow \mathcal{V}$  constructed from the universal small family  $\dot{\mathcal{V}} \rightarrow \mathcal{V}$  of Proposition 73, so there are two pullback squares:

$$\begin{array}{ccc}
 A & \longrightarrow & \dot{\mathcal{V}} \\
 \downarrow & & \downarrow \\
 X & \xrightarrow{\chi_A} & \mathcal{V} \\
 \uparrow & & \uparrow \\
 +Alg(A) & \longrightarrow & +Alg(\dot{\mathcal{V}})
 \end{array} \tag{110}$$

**Proposition 75.** *There is a universal small trivial fibration*

$$TFib \rightarrow TFib.$$

*Every small trivial fibration  $A \rightarrow X$  is a pullback of  $TFib \rightarrow TFib$  along a canonically determined classifying map  $X \rightarrow TFib$ .*

$$\begin{array}{ccc}
 A & \longrightarrow & TFib \\
 \downarrow \lrcorner & & \downarrow \\
 X & \longrightarrow & TFib
 \end{array} \tag{111}$$

*Proof.* We can take

$$TFib := +Alg(\dot{\mathcal{V}}),$$

which comes with its projection  $+Alg(\dot{\mathcal{V}}) \rightarrow \mathcal{V}$  as in diagram (110). Now define  $p_t : TFib \rightarrow TFib$  by pulling back the universal small family,

$$\begin{array}{ccc}
 TFib & \longrightarrow & \dot{\mathcal{V}} \\
 p_t \downarrow \lrcorner & & \downarrow p \\
 TFib & \longrightarrow & \mathcal{V}
 \end{array}$$

Consider the following diagram, in which all the squares (including the distorted ones) are pullbacks, with the outer one coming from proposition 73



and the lower one from (110).

$$\begin{array}{ccc}
 A & \xrightarrow{q_A} & \dot{\mathcal{V}} \\
 \downarrow & \nearrow & \downarrow p \\
 & \text{TFib} & \\
 \downarrow p_t & & \\
 \text{TFib}(A) & \xrightarrow{\quad} & \text{TFib} \\
 \downarrow \alpha & \nearrow \alpha' & \downarrow \\
 X & \xrightarrow{\chi_A} & \mathcal{V}
 \end{array} \tag{112}$$

A trivial fibration structure  $\alpha$  on  $A \rightarrow X$  is a section the object of  $+$ -algebra structures on  $A$ , occurring in the diagram as

$$\text{TFib}(A) := +\text{Alg}(A),$$

the pullback of  $\text{TFib} = +\text{Alg}(\dot{\mathcal{V}})$  along the classifying map  $\chi_A : X \rightarrow \mathcal{V}$  for the small family  $A \rightarrow X$ . Such sections correspond uniquely to factorizations  $\alpha'$  of  $\chi_A$  as indicated, which in turn induce pullback squares of the required kind (111).

Note that the map  $p_t : \text{TFib} \rightarrow \text{TFib}$  has a canonical trivial fibration structure. Indeed, consider the following diagram, in which both squares are pullbacks.

$$\begin{array}{ccc}
 \text{TFib} & \longrightarrow & \dot{\mathcal{V}} \\
 p_t \downarrow & & \downarrow \\
 \text{TFib} & \longrightarrow & \mathcal{V} \\
 \uparrow & & \uparrow \\
 \text{TFib}(\text{TFib}) & \longrightarrow & \text{TFib}(\dot{\mathcal{V}})
 \end{array} \tag{113}$$

$\text{TFib}(\dot{\mathcal{V}})$  is the object of trivial fibration structures on  $\dot{\mathcal{V}} \rightarrow \mathcal{V}$ , and its pullback  $\text{TFib}(\text{TFib})$  is therefore the object of trivial fibration structures on  $p_t : \text{TFib} \rightarrow \text{TFib}$ . Thus we seek a section of  $\text{TFib}(\text{TFib}) \rightarrow \text{TFib}$ . But recall that  $\text{TFib} = \text{TFib}(\dot{\mathcal{V}})$  by definition, so the lower pullback square is the pullback of  $\text{TFib}(\dot{\mathcal{V}}) \rightarrow \mathcal{V}$  against itself, which does indeed have a distinguished section, namely the diagonal

$$\Delta : \text{TFib}(\dot{\mathcal{V}}) \rightarrow \text{TFib}(\dot{\mathcal{V}}) \times_{\mathcal{V}} \text{TFib}(\dot{\mathcal{V}}).$$

□

We record the following notation and corresponding fact from the foregoing proof for future reference:

**Lemma 76.** *The classifying type  $\mathbf{TFib}(A) := +\mathbf{Alg}(A) \rightarrow X$  for trivial fibration structures on a map  $A \rightarrow X$  is stable under pullback, in the sense that for any  $f : Y \rightarrow X$ , we have two pullback squares,*

$$\begin{array}{ccc}
 f^*A & \longrightarrow & A \\
 \downarrow & & \downarrow \\
 Y & \xrightarrow{f} & X \\
 \uparrow & & \uparrow \\
 \mathbf{TFib}(f^*A) & \longrightarrow & \mathbf{TFib}(A)
 \end{array} \tag{114}$$

Since the universal small trivial fibration  $\mathbf{TFib} \rightarrow \mathbf{TFib}$  in  $\mathbf{cSet}$  from Proposition 75 was constructed as  $\mathbf{TFib} = \mathbf{TFib}(\dot{\mathcal{V}})$  for the universal small family  $\dot{\mathcal{V}} \rightarrow \mathcal{V}$ , which in turn is stable under pullback by Corollary 74, we also have:

**Corollary 77.** *The base change of the universal small trivial fibration*

$$\mathbf{TFib} \rightarrow \mathbf{TFib}$$

*in  $\mathbf{cSet}$  along  $I^* : \mathbf{cSet} \rightarrow \mathbf{cSet}/_I$  is a universal small trivial fibration in  $\mathbf{cSet}/_I$ .*

### 6.3 Classifying fibrations

In order to classify fibrations  $A \rightrightarrows X$ , we shall proceed as for trivial fibrations by constructing, for any map  $A \rightarrow X$ , an object  $\mathbf{Fib}(A) \rightarrow X$  of fibration structures which, moreover, is stable under pullback. We then apply the construction to the universal small family  $\dot{\mathcal{V}} \rightarrow \mathcal{V}$  of Proposition 73 to obtain a universal small fibration. Here we will of course need to distinguish between biased and unbiased fibrations. In Proposition ??, we first construct a stable classifying type  $\mathbf{Fib}(A) \rightarrow X$  for  $\delta$ -biased fibration structures on any map  $A \rightarrow X$  in  $\mathbf{cSet}/_I$  where  $\delta$  is the generic point. In Proposition ?? we then transfer the construction along the base change  $I^* : \mathbf{cSet} \rightarrow \mathbf{cSet}/_I$  to obtain a classifier  $\mathbf{Fib}(A) \rightarrow X$  for unbiased fibration structures on any  $A \rightarrow X$  in  $\mathbf{cSet}$ .

The construction of  $\text{Fib}(A) \rightarrow X$  for biased fibration structures with respect to a point  $\delta : 1 \rightarrow I$  is already a bit more involved than was that of  $\text{TFib}(A) \rightarrow X$ . In particular, it requires the codomain  $I$  of  $\delta$  to be *tiny*, which is indeed the case for the generic point  $\delta : 1 \rightarrow I^*I$  in  $\mathbf{cSet}/_I$  by Lemma 4.

**The classifying type of biased fibration structures.** A classifying type  $\text{Fib}(A) \rightarrow X$  of (uniform,  $\delta$ -biased) fibration structures on a map  $p : A \rightarrow X$ , as defined in Section 3.1, can be constructed as follows.

1. First form the pullback-hom  $\delta \Rightarrow p : A^I \rightarrow X^I \times_X A$  with the point  $\delta : 1 \rightarrow I$ , as indicated in the following diagram.

$$\begin{array}{ccc}
 A^I & \xrightarrow{A^\delta} & A \\
 \delta \Rightarrow p \swarrow & & \downarrow p \\
 X^I \times_X A & \longrightarrow & A \\
 \downarrow & & \downarrow p \\
 X^I & \xrightarrow{X^\delta} & X \\
 (p_A)^I \searrow & & \\
 & & 
 \end{array} \tag{115}$$

2. A fibration structure on  $p : A \rightarrow X$  is then a relative  $+$ -algebra structure on  $\delta \Rightarrow p$  in the slice category over its codomain  $X^I \times_X A$ . To construct a classifier for such structures, let us first relabel the objects and arrows in diagram (115) as follows:

$$\begin{aligned}
 \epsilon &:= X^\delta : X^I \rightarrow X \\
 A_\epsilon &:= X^I \times_X A \\
 \epsilon_A &:= \delta \Rightarrow p
 \end{aligned}$$

so that the working part of (115) becomes:

$$\begin{array}{ccc}
 A^I & & \\
 \epsilon_A \searrow & & \\
 A_\epsilon & \longrightarrow & A \\
 p_\epsilon \downarrow & \lrcorner & \downarrow p \\
 X^I & \xrightarrow{\epsilon} & X
 \end{array} \tag{116}$$

3. Now a relative  $+$ -algebra structure on  $\epsilon_A$  (Definition 9) is a retract  $\alpha$  over  $A_\epsilon$  of the unit  $\eta$ , as indicated below, where  $D$  is simply the domain of the map  $(\epsilon_A)^+$  resulting from applying the relative  $+$ -functor in the slice category over  $A_\epsilon$  to the object  $\epsilon_A$ .

$$\begin{array}{ccc}
 A^I & \xrightarrow[\eta]{\overset{\alpha}{k}} & D \\
 \searrow \epsilon_A & & \downarrow (\epsilon_A)^+ \\
 & & A_\epsilon \longrightarrow A \\
 & & \downarrow p \\
 & & X^I \xrightarrow{\epsilon} X
 \end{array}
 \quad (117)$$

4. As in the construction (108), there is an object  $\mathbf{TFib}(\epsilon_A) = +\mathbf{Alg}(\epsilon_A)$  over  $A_\epsilon$  of relative  $+$ -algebra structures on  $\epsilon_A$ , the sections of which correspond uniquely to relative  $+$ -algebra structures on  $\epsilon_A$  (and thus to fibration structures on  $A$ ).

$$\begin{array}{ccccc}
 & & A^I & \xrightarrow[\eta]{\overset{\alpha}{k}} & D \\
 & & \downarrow \epsilon_A & \nearrow (\epsilon_A)^+ & \\
 \mathbf{TFib}(\epsilon_A) & \longrightarrow & A_\epsilon & \longrightarrow & A \\
 & & \downarrow p_\epsilon & & \downarrow p \\
 & & X^I & \xrightarrow{\epsilon} & X
 \end{array}
 \quad (118)$$

5. Sections of  $\mathbf{TFib}(\epsilon_A) \longrightarrow A_\epsilon$  then correspond to sections of its push-forward along  $p_\epsilon$ , which we shall call  $F_A$ :

$$F_A := (p_\epsilon)_* \mathbf{TFib}(\epsilon_A).$$

$$\begin{array}{ccccc}
 & & A^I & \xrightarrow[\eta]{\overset{\alpha}{k}} & D \\
 & & \downarrow \epsilon_A & \nearrow (\epsilon_A)^+ & \\
 \mathbf{TFib}(\epsilon_A) & \longrightarrow & A_\epsilon & \longrightarrow & A \\
 & & \downarrow p_\epsilon & & \downarrow p \\
 F_A & \longrightarrow & X^I & \xrightarrow{\epsilon} & X
 \end{array}
 \quad (119)$$

6. One might now try taking another pushforward of  $F_A \rightarrow X^I$  along  $\epsilon : X^I \rightarrow X$  to get the object  $\mathbf{Fib}(A) \rightarrow X$  that we seek, but unfortunately, this would not be stable under pullback along arbitrary maps  $Y \rightarrow X$ , because the evaluation  $\epsilon = X^\delta : X^I \rightarrow X$  is not stable in that way. Instead we use the *root* functor, i.e. the right adjoint of the pathspace,  $(-)^I \dashv (-)_I$  (Proposition 3).

Let  $f : F_A \rightarrow X^I$  be the map  $(p_\epsilon)_* \mathbf{TFib}(\epsilon_A)$  indicated in (119), and let  $\eta : X \rightarrow (X^I)_I$  be the unit of the root adjunction at  $X$ . Then define  $\mathbf{Fib}(A) \rightarrow X$  by

$$\mathbf{Fib}(A) := \eta^* f_I$$

as indicated in the following pullback diagram.

$$\begin{array}{ccc} \mathbf{Fib}(A) & \longrightarrow & (F_A)_I \\ \downarrow & \lrcorner & \downarrow f_I \\ X & \xrightarrow{\eta} & (X^I)_I \end{array} \quad (120)$$

By adjointness, sections of  $\mathbf{Fib}(A) \rightarrow X$  then correspond bijectively to sections of  $f : F_A \rightarrow X^I$ .

**Lemma 78.** *For any map  $A \rightarrow X$  in  $\mathbf{cSet}/_I$ , the map  $\mathbf{Fib}(A) \rightarrow X$  in (120) is a classifying type for  $\delta$ -biased fibration structures: sections of  $\mathbf{Fib}(A) \rightarrow X$  correspond bijectively to  $\delta$ -biased fibration structures on  $A \rightarrow X$ , and the construction is stable under pullback in the sense that for any  $f : Y \rightarrow X$ , we have two pullback squares,*

$$\begin{array}{ccc} f^* A & \longrightarrow & A \\ \downarrow & & \downarrow \\ Y & \xrightarrow{f} & X \\ \uparrow & & \uparrow \\ \mathbf{Fib}(f^* A) & \longrightarrow & \mathbf{Fib}(A) \end{array} \quad (121)$$

*Proof.* It is clear from the construction that fibration structures on  $A \rightarrow X$  correspond bijectively to sections of  $\mathbf{Fib}(A) \rightarrow X$ . We show that  $\mathbf{Fib}(A) \rightarrow X$  is also stable under pullback. To that end, the relevant steps of the

construction are recalled schematically below.

$$\begin{array}{ccccc}
 & & A^I & & \\
 & & \downarrow \epsilon_A & & \\
 \mathrm{TFib}(\epsilon_A) & \longrightarrow & A_\epsilon & \longrightarrow & A \\
 & & \downarrow p_\epsilon & \lrcorner & \downarrow p \\
 F_A & \longrightarrow & X^I & \xrightarrow{\epsilon} & X \longleftarrow \mathrm{Fib}(A)
 \end{array} \tag{122}$$

Now consider the following diagram, in which the right hand side consists of the data from (122), and the front, central square is a pullback.

$$\begin{array}{ccccccc}
 & & B^I & \xrightarrow{\quad} & A^I & & \\
 & & \downarrow \epsilon_B & & \downarrow \epsilon_A & & \\
 \mathrm{TFib}(\epsilon_B) & \longrightarrow & B_\epsilon & \xrightarrow{\quad \cdots \quad} & A_\epsilon & \longleftarrow & \mathrm{TFib}(\epsilon_A) \\
 & & \downarrow & \searrow & \swarrow & & \\
 F_B & \longrightarrow & Y^I & \longrightarrow & B & \longrightarrow & A \\
 & & \downarrow & \searrow & \downarrow & & \downarrow \\
 & & \mathrm{Fib}(B) & \longrightarrow & Y & \xrightarrow{f} & X \longleftarrow \mathrm{Fib}(A)
 \end{array}$$

As in the proof of Lemma 61, on the left side we repeat the construction with  $B \rightarrow Y$  in place of  $A \rightarrow X$ . The left face of the indicated (distorted) cube is then also a pullback, whence the back (dotted) face is a pullback, since the two-story square in back is the image of the front pullback square under the right adjoint  $(-)^I$ . Finally, the top rectangle in the back is therefore also a pullback.

It follows that  $\mathrm{TFib}(\epsilon_B)$  is a pullback of  $\mathrm{TFib}(\epsilon_A)$  along the upper dotted arrow, as in Lemma 76, and so the pushforward  $F_B$  is a pullback of the corresponding  $F_A$ , along the lower dotted arrow (which is  $f^I$ ), by the Beck-Chevalley condition for the dotted pullback square. Let us record this for later reference:

$$F_B \cong (f^I)^* F_A. \tag{124}$$

It remains to show that  $\mathrm{Fib}(B)$  is a pullback of  $\mathrm{Fib}(A)$  along  $f : Y \rightarrow X$ , and now it is good that we did not take these to be pushforwards of  $F_B$

and  $F_A$ , because the floor of the cube need not be a pullback, and so the Beck-Chavalley condition would not apply. Instead, consider the following diagram.

$$\begin{array}{ccccc}
 & \text{Fib}(B) & \longrightarrow & \text{Fib}(A) & \\
 & \swarrow & & \searrow & \\
 (F_B)_I & \xleftarrow{\quad} & & \xrightarrow{\quad} & (F_A)_I \\
 \downarrow & & \downarrow & & \downarrow \\
 & Y & \xrightarrow{f} & X & \\
 \swarrow \eta & & & & \searrow \eta \\
 (Y^I)_I & \xrightarrow{\quad} & (f^I)_I & \xrightarrow{\quad} & (X^I)_I
 \end{array} \tag{125}$$

The sides of the cube are pullbacks by the construction of  $\text{Fib}(A)$  and  $\text{Fib}(B)$ . The front face is the root of the pullback (124) and is thus also a pullback, since the root is a right adjoint. The base commutes by naturality of the unit of the adjunction, and so the back face is also a pullback, as required.  $\square$

Now let us apply the foregoing construction of  $\text{Fib}(A)$  to the universal family  $\dot{\mathcal{V}} \rightarrow \mathcal{V}$  to get  $\text{Fib}(\dot{\mathcal{V}}) \rightarrow \mathcal{V}$ , and define the universal small ( $\delta$ -biased) fibration in  $\mathbf{cSet}/_I$  by setting  $\dot{\text{Fib}} := \text{Fib}(\dot{\mathcal{V}})$  and  $\dot{\text{Fib}} \rightarrow \text{Fib}$  by pulling back the universal family,

$$\begin{array}{ccc}
 \dot{\text{Fib}} & \longrightarrow & \dot{\mathcal{V}} \\
 \downarrow \lrcorner & & \downarrow p \\
 \text{Fib} & \longrightarrow & \mathcal{V}.
 \end{array} \tag{126}$$

The proof of the following then proceeds just as that given for  $\text{TFib} \rightarrow \text{TFib}$  in Proposition 75.

**Proposition 79.** *The map  $\dot{\text{Fib}} \rightarrow \text{Fib}$  constructed in (126) is a universal small  $\delta$ -biased fibration in  $\mathbf{cSet}/_I$ : every small  $\delta$ -biased fibration  $A \twoheadrightarrow X$  in  $\mathbf{cSet}/_I$  is a pullback of  $\dot{\text{Fib}} \rightarrow \text{Fib}$  along a canonically determined classifying map  $X \rightarrow \text{Fib}$ .*

$$\begin{array}{ccc}
 A & \longrightarrow & \dot{\text{Fib}} \\
 \downarrow \lrcorner & & \downarrow \\
 X & \longrightarrow & \text{Fib}
 \end{array} \tag{127}$$

*Remark 80.* Proposition 79 made no use of the fact that we were working in the slice category  $\mathbf{cSet}/_I$  with  $\delta : 1 \rightarrow I$  the generic point. It holds equally for  $\delta$ -biased fibrations with respect to any point  $\delta : 1 \rightarrow I$  of a tiny object  $I$ . Thus *e.g.* it could be used (with obvious adjustment) to construct a classifier for the  $\{\delta_0, \delta_1\}$ -biased fibrations of Section 3.1 in (Cartesian, Dedekind, or other varieties of) cubical sets  $\mathbf{cSet}$ .

**The classifying type of unbiased fibration structures.** In order to classify *unbiased* fibration structures on maps  $A \rightarrow X$  in  $\mathbf{cSet}$ , we first apply the pullback  $I^* : \mathbf{cSet} \rightarrow \mathbf{cSet}/_I$  and take the classifier  $\mathbf{Fib}(I^*A) \rightarrow I^*X$  for  $\delta$ -biased fibration structures, then apply the pushforward  $I_* : \mathbf{cSet}/_I \rightarrow \mathbf{cSet}$  and pull the result  $I_*\mathbf{Fib}(I^*A) \rightarrow I_*I^*X$  back along the unit  $X \rightarrow I_*I^*X$ .

To show that this indeed classifies unbiased fibration structures on  $A \rightarrow X$ , let us first rename the classifying type from Lemma 78, which was constructed over  $I$ , to  $\mathbf{Fib}_i(I^*A) \rightarrow I^*X$ , and then apply  $I_*$  to get the map,

$$\Pi_{i:I}\mathbf{Fib}_i(I^*A) := I_*(\mathbf{Fib}_i(I^*A)) \longrightarrow X^I$$

in  $\mathbf{cSet}$ . Then, as just said, we define the desired map  $\mathbf{Fib}(A) \rightarrow X$  as the pullback along the unit  $\rho : X \rightarrow X^I$  of  $I^* \dashv I_*$  as indicated below.

$$\begin{array}{ccc} \mathbf{Fib}(A) & \longrightarrow & \Pi_{i:I}\mathbf{Fib}_i(I^*A) \\ \downarrow \lrcorner & & \downarrow \\ X & \xrightarrow{\rho} & X^I \end{array} \quad (128)$$

It now immediately from the adjunction  $I^* \dashv I_*$  that sections of  $\mathbf{Fib}(A) \rightarrow X$  correspond bijectively to sections of  $\mathbf{Fib}_i(I^*A) \rightarrow I^*X$  over  $I$ , and thus to *unbiased* fibration structures on  $A \rightarrow X$ .

**Lemma 81.** *For any map  $A \rightarrow X$  in  $\mathbf{cSet}$ , the map  $\mathbf{Fib}(A) \rightarrow X$  in (128) is a classifying type for unbiased fibration structures: sections of  $\mathbf{Fib}(A) \rightarrow X$  correspond bijectively to unbiased fibration structures on  $A \rightarrow X$ , and the construction is stable under pullback in the expected sense (as in Lemma 78).*

*Proof.* It remains only to check the stability, but since both of the adjoints in  $I^* \dashv I_* : \mathbf{cSet}/_I \rightarrow \mathbf{cSet}$  preserve pullbacks, this follows easily from the fact that the classifying types  $\mathbf{Fib}_i$  are stable under pullback by Lemma 78.  $\square$



Finally, we can again take  $\mathbf{Fib} := \mathbf{Fib}(\dot{\mathcal{V}})$  to now obtain a universal small *unbiased* fibration  $\mathbf{Fib} \rightarrow \mathbf{Fib}$  in  $\mathbf{cSet}$ , as in (126), and the proof can conclude just as in that for Proposition 75.

**Proposition 82.** *The map  $\mathbf{Fib} \rightarrow \mathbf{Fib}$  just constructed is a universal small unbiased fibration in  $\mathbf{cSet}$ : every small unbiased fibration  $A \twoheadrightarrow X$  is a pull-back of  $\mathbf{Fib} \twoheadrightarrow \mathbf{Fib}$  along a canonically determined classifying map  $X \rightarrow \mathbf{Fib}$ .*

$$\begin{array}{ccc} A & \longrightarrow & \mathbf{Fib} \\ \downarrow \lrcorner & & \downarrow \\ X & \longrightarrow & \mathbf{Fib} \end{array} \quad (129)$$

*Remark 83.* Recall from Proposition 74 that the universe in the slice category  $\mathbf{cSet}/_I$  is the pullback of the universe  $\mathcal{V}$  from  $\mathbf{cSet}$  along the base change  $I^* : \mathbf{cSet} \rightarrow \mathbf{cSet}/_I$ . Thus in the construction just given of the classifier  $\mathbf{Fib} \rightarrow \mathbf{Fib}$  for unbiased fibrations in  $\mathbf{cSet}$  we are first building the classifying type

$$\mathbf{Fib}_i(I^*\dot{\mathcal{V}}) \rightarrow I^*\mathcal{V}$$

for  $\delta$ -biased fibration structures on the universal family in  $\mathbf{cSet}/_I$ , and then taking a pushforward  $I_* : \mathbf{cSet}/_I \rightarrow \mathbf{cSet}$  to obtain the (base of the) classifier for unbiased fibrations as the pullback along the unit:

$$\begin{array}{ccc} \mathbf{Fib}(\dot{\mathcal{V}}) & \longrightarrow & \Pi_{i:I} \mathbf{Fib}_i(I^*\dot{\mathcal{V}}) \\ \downarrow \lrcorner & & \downarrow \\ \mathcal{V} & \xrightarrow{\rho} & \mathcal{V}^I \end{array} \quad (130)$$

We remark for later reference that this classifying type  $\mathbf{Fib} = \mathbf{Fib}(\dot{\mathcal{V}}) \rightarrow \mathcal{V}$  for unbiased fibration structures can therefore be constructed as the pushforward of the classifier  $\mathbf{Fib}_i(I^*\dot{\mathcal{V}}) \rightarrow I^*\mathcal{V}$  for  $\delta$ -biased fibration structures along the projection  $q : I^*\mathcal{V} = I \times \mathcal{V} \rightarrow \mathcal{V}$  indicated below.

$$\begin{array}{ccccc} \mathbf{Fib}_i(I^*\dot{\mathcal{V}}) & & \mathbf{Fib}(\dot{\mathcal{V}}) & \longrightarrow & \Pi_{i:I} \mathbf{Fib}_i(\dot{\mathcal{V}}) \\ \downarrow & & \downarrow \lrcorner & & \downarrow \\ I^*\mathcal{V} & \xrightarrow{q} & \mathcal{V} & \xrightarrow{\rho} & \mathcal{V}^I \\ \downarrow \lrcorner & & \downarrow & & \\ I & \longrightarrow & 1 & & \end{array} \quad (131)$$

We record this fact as:

*Corollary 84.*  $\mathbf{Fib} = \Sigma_{\mathcal{V}} q_* \mathbf{Fib}_i(I^* \dot{\mathcal{V}}).$

The reader may also find it illuminating to reconsider the construction of the universal small unbiased fibration in more type theoretic terms. It was defined to be  $\mathbf{Fib} \rightarrow \mathbf{Fib} = \mathbf{Fib}(\dot{\mathcal{V}})$ , for the universal family  $\dot{\mathcal{V}} \rightarrow \mathcal{V}$ , with  $\mathbf{Fib}$  the pullback of  $\dot{\mathcal{V}} \rightarrow \mathcal{V}$  along the canonical projection  $\mathbf{Fib}(\dot{\mathcal{V}}) \rightarrow \mathcal{V}$ . Since, type theoretically, we have  $\dot{\mathcal{V}} = \Sigma_{A:\mathcal{V}} A$ , by the stability of the classifying type  $\mathbf{Fib}(-)$  we can write  $\mathbf{Fib} = \Sigma_{A:\mathcal{V}} \mathbf{Fib}(A)$  so that:

$$\mathbf{Fib} = \Sigma_{A:\mathcal{V}} \mathbf{Fib}(A) \times A \longrightarrow \Sigma_{A:\mathcal{V}} \mathbf{Fib}(A) = \mathbf{Fib}.$$

## 6.4 Realignment for fibration structure

The realignment for families of Proposition 72 will need to be extended to (structured) fibrations. Our approach makes use of the notion of a *weak proposition*. Informally, a map  $P \rightarrow X$  may be said to be a weak proposition if it is “conditionally contractible”, in the sense that it is contractible if it has a section (recall that a proposition may be defined as a fibration that is “contractible if inhabited”). More formally, we have the following.

**Definition 85.** A map  $P \rightarrow X$  is said to be a *weak proposition* if the projection  $P \times_X P \rightarrow P$  is a trivial fibration.

$$\begin{array}{ccc} P^2 & \longrightarrow & P \\ \sim \downarrow \lrcorner & & \downarrow \\ P & \longrightarrow & X. \end{array} \quad (132)$$

Note that if either projection is a trivial fibration, then both are.

As an object over the base, a weak proposition is thus one that “thinks it is contractible”. The key fact needed for realignment is the following.

**Lemma 86.** *For any  $A \rightarrow X$ , the classifying type  $\mathbf{TFib}(A) \rightarrow X$  is a weak proposition. Moreover, the same is true for  $\mathbf{Fib}(A) \rightarrow X$  (both the biased and unbiased versions) if the cofibrations are closed under exponentiation by the interval  $I$ .*

*Proof.* Let  $A \rightarrow X$  and consider the following diagram, in which we have written  $A' = \mathbf{TFib}(A) \times_X A$  and  $\mathbf{TFib}(A)^2 = \mathbf{TFib}(A) \times_X \mathbf{TFib}(A)$ .

$$\begin{array}{ccccc}
 A' & \xrightarrow{\quad} & A & & \\
 \downarrow & \lrcorner & \downarrow & & \\
 & & \mathbf{TFib}(A)^2 & \xrightarrow{\quad} & \mathbf{TFib}(A) \\
 & \swarrow & \lrcorner & \searrow & \\
 \mathbf{TFib}(A) & \xrightarrow{\quad} & X & & 
 \end{array} \tag{133}$$

Since  $\mathbf{TFib}$  is stable under pullback (by Lemma 76), we have  $\mathbf{TFib}(A)^2 \cong \mathbf{TFib}(A')$ , and since  $\mathbf{TFib}(A)^2$  has a canonical section,  $A' \rightarrow \mathbf{TFib}(A)$  is therefore a trivial fibration. Inspecting the definition of  $\mathbf{TFib}(A) = +\mathbf{Alg}(A)$  in (108), we see that if a map  $A \rightarrow X$  is a trivial fibration, then so is  $\mathbf{TFib}(A) \rightarrow X$  (since  $\eta : A \rightarrow A^+$  is always a cofibration). Thus  $\mathbf{TFib}(A)^2 \cong \mathbf{TFib}(A') \rightarrow \mathbf{TFib}(A)$  is also a trivial fibration.

For  $\mathbf{Fib}(A) \rightarrow X$ , with reference to the construction (122) we use the foregoing to infer that  $\mathbf{TFib}(\epsilon_A) \rightarrow A_\epsilon$  is a weak proposition, and so therefore is its pushforward  $F_A = (p_\epsilon)_* \mathbf{TFib}(\epsilon_A) \rightarrow X^I$  along the projection  $p_\epsilon : A_\epsilon = X^I \times_X A \rightarrow X^I$ , since pushforward clearly preserves weak propositions. Applying the root  $(-)_I$  preserves trivial fibrations, by the assumption that its left adjoint  $(-)^I$  preserves cofibrations, and so, as a right adjoint, it also preserves weak propositions. Therefore  $(F_A)_I \rightarrow (X^I)_I$  is a weak proposition, but then so is its pullback along the unit  $X \rightarrow (X^I)_I$ , which is  $\mathbf{Fib}_i(A) \rightarrow X$ , the classifier for  $\delta$ -biased fibration structures. The same reasoning shows that  $\mathbf{Fib}(A) = \rho^* \Pi_{i:I} \mathbf{Fib}_i(I^* A)$  (as in (128)) is also a weak proposition.  $\square$

In light of Lemma 86 we shall assume as a final axiom on cofibrations:

(C8) The pathobject functor preserves cofibrations: thus  $c : A \rightarrowtail B$  implies  $c^I : A^I \rightarrowtail B^I$ .

Now, by Propositions 79 and 82 we have universal small  $\delta$ -biased and unbiased fibrations, the former in  $\mathbf{cSet}/_I$ , the latter in  $\mathbf{cSet}$ . The following remarks apply to both, which we refer to neutrally as  $\dot{\mathcal{U}} \rightarrow \mathcal{U}$ . The base object  $\mathcal{U}$  is (the domain of) the classifying type  $\mathbf{Fib}(\dot{\mathcal{V}}) \rightarrow \mathcal{V}$ , where  $\dot{\mathcal{V}} \rightarrow \mathcal{V}$

is the universal small family. Type theoretically, this object can be written as

$$\mathcal{U} = \Sigma_{E:\mathcal{V}} \mathbf{Fib}(E),$$

which comes with the canonical projection

$$\mathcal{U} = \Sigma_{E:\mathcal{V}} \mathbf{Fib}(E) \longrightarrow \mathcal{V}.$$

In these terms, a fibration  $E \rightarrow X$  is a pair  $\langle E, e \rangle$ , consisting of the underlying family  $E \rightarrow X$ , equipped with a fibration structure  $e : \mathbf{Fib}(E)$ . Lemma 86 then allows us to establish the following, which was first isolated in [?] (as condition (2'), also see [Shu15]). It holds for both biased and unbiased fibrations, and will be used in the sequel to “correct” the fibration structure on certain maps.

**Lemma 87** (Realignment for fibrations). *Given a fibration  $F \rightarrow X$  and a cofibration  $c : C \rightarrow X$ , let  $f_c : C \rightarrow \mathcal{U}$  classify the pullback  $c^*F \rightarrow C$ . Then there is a classifying map  $f : X \rightarrow \mathcal{U}$  for  $F$  with  $f \circ c = f_c$ .*

$$\begin{array}{ccc}
 c^*F & \xrightarrow{\quad} & \mathcal{U} \\
 \downarrow & \searrow & \downarrow \\
 & F & \\
 \downarrow & & \downarrow \\
 C & \xrightarrow{f_c} & \mathcal{U} \\
 \downarrow c & & \downarrow f \\
 & X & 
 \end{array}
 \quad (134)$$

*Proof.* First, let  $|f_c| : C \rightarrow \mathcal{V}$  be the composite of  $f_c : C \rightarrow \mathcal{U}$  with the canonical projection  $\mathcal{U} \rightarrow \mathcal{V}$ , thus classifying the underlying family  $c^*F \rightarrow C$ . Next, let  $f_0 : X \rightarrow \mathcal{V}$  classify the underlying family  $F \rightarrow X$ . We may assume

that  $f_0 \circ c = |f_c|$  by realignment for families, Proposition 72.

$$\begin{array}{ccccc}
 c^*F & \xrightarrow{\quad} & \mathcal{U} & \xrightarrow{\quad} & \dot{\mathcal{V}} \\
 \downarrow & \searrow & \downarrow & \nearrow & \downarrow \\
 & & F & & \\
 C & \xrightarrow{f_c} & \mathcal{U} & \xrightarrow{\quad} & \mathcal{V} \\
 \downarrow c & & \downarrow & \nearrow f_0 & \\
 & & X & & 
 \end{array}
 \quad (135)$$

Since  $F \twoheadrightarrow X$  is a fibration, there is a lift  $f_1 : X \rightarrow \mathcal{U}$  of  $f_0$  classifying the fibration structure. We thus have the following commutative diagram in the base of (135).

$$\begin{array}{ccccc}
 & & |f_c| & & \\
 C & \xrightarrow{f_c} & \mathcal{U} & \xrightarrow{\quad} & \mathcal{V} \\
 \downarrow c & & \downarrow & & \parallel \\
 X & \xrightarrow{f_1} & \mathcal{U} & \xrightarrow{\quad} & \mathcal{V} \\
 & & f_0 & & 
 \end{array}
 \quad (136)$$

Now pull  $\mathcal{U} \rightarrow \mathcal{V}$  back against itself and rearrange the previous data to give (the solid part of) the following, which also commutes.

$$\begin{array}{ccccc}
 & & f_c & & \\
 C & \xrightarrow{\quad} & \mathcal{U} \times_{\mathcal{V}} \mathcal{U} & \xrightarrow{\pi_2} & \mathcal{U} \\
 \downarrow c & \nearrow \langle f_1 c, f_c \rangle & \downarrow \pi_1 & \lrcorner & \downarrow \\
 X & \xrightarrow{f_1} & \mathcal{U} & \xrightarrow{\quad} & \mathcal{V} \\
 & & f_0 & & 
 \end{array}
 \quad (137)$$

Since  $\mathcal{U} = \text{Fib}(\dot{\mathcal{V}}) \rightarrow \mathcal{V}$  is a weak proposition by Lemma 86 and (C8), the projection  $\pi_1 : \mathcal{U} \times_{\mathcal{V}} \mathcal{U} \twoheadrightarrow \mathcal{U}$  is a trivial fibration, so there is a diagonal filler  $f_2 : X \rightarrow \mathcal{U} \times_{\mathcal{V}} \mathcal{U}$  as indicated. Taking  $f := \pi_2 \circ f_2 : X \rightarrow \mathcal{U} \times_{\mathcal{V}} \mathcal{U} \rightarrow \mathcal{U}$  gives another classifying map for the fibration structure on  $F \rightarrow X$ , for which  $f \circ c = f_c$  as required.  $\square$



1. If  $f : X \rightarrow Y$  is a map over a base  $Z$  in  $\mathbf{cSet}$ , we instead use the relative pathobject  $Y^I \rightarrow Z$ , where the exponential is taken in the slice over  $Z$ , and the interval in the exponent is the result of pulling the interval  $\delta_0, \delta_1 : 1 \rightarrow I$  from  $\mathbf{cSet}$  back along  $Z \rightarrow 1$ . The factorization  $t \circ s : X \rightarrow P_f \rightarrow Y$  is then stable under pullback along any map  $g : Z' \rightarrow Z$ , in the sense that  $g^*(Y^I) \cong g^*(Y)^I$  and so  $g^*P_f = P_{g^*f}$ , where  $g^*f : g^*X \rightarrow g^*Y$ , and similarly for the factors  $g^*s$  and  $g^*t$ .
2. The retraction  $p_0 \circ r = 1_Y$  pulls back along  $f$  to a retraction  $p_f \circ s = 1_X$ .
3. If  $Y$  is a fibrant object, then  $p_0, p_1 : Y^I \rightarrow Y$  are both trivial fibrations by Proposition 27. If  $f : X \rightarrow Y$  is over a base  $Z$  and  $Y$  is fibrant over  $Z$ , then the maps  $p_0, p_1 : Y^I \rightarrow Y$  over  $Z$  are also trivial fibrations, again by Proposition 27[*add a proof*]. In that case, the pullback  $p_f : P_f \rightarrow X$  (which is also a map over  $Z$ ) is also a trivial fibration.
4. If  $X$  and  $Y$  are both fibrant (over any base  $Z \in \mathbf{cSet}$ ), then  $t = p_1 \circ p_0^*f : P_f \rightarrow Y$  is a fibration. This can be seen by factoring the maps  $p_0, p_1 : Y^I \rightrightarrows Y$  through the product projections as  $\pi_0 \circ p, \pi_1 \circ p : Y^I \rightarrow Y \times Y \rightrightarrows Y$ , with  $p = (p_0, p_1)$ , and then interpolating the pullback along the map  $(f, 1_Y) : X \times Y \rightarrow Y \times Y$  into (138) as indicated below.

$$\begin{array}{ccccc}
 X & \xrightarrow{f} & Y & & \\
 \downarrow s & \lrcorner & \downarrow r & & \\
 P_f & \xrightarrow{p_0^*f} & Y^I & & \\
 \downarrow f^*p & \lrcorner & \downarrow p & \searrow p_1 & \\
 Y & \xleftarrow{\pi'_1} X \times Y & \xrightarrow{(f, 1_Y)} Y \times Y & \xrightarrow{\pi_1} Y & \\
 & \downarrow \pi'_0 & & \downarrow \pi_0 & \\
 & X & \xrightarrow{f} & Y & 
 \end{array} \tag{139}$$

The second factor  $t = p_1 \circ p_0^*f : P_f \rightarrow Y$  now appears also as  $\pi_1 \circ (f, 1_Y) \circ f^*p$ , which is the pullback  $f^*p : P_f \rightarrow X \times Y$  followed by the second projection  $\pi'_1 : X \times Y \rightarrow Y$  (which is not a pullback). But if  $Y$  is fibrant, then  $p : Y^I \rightarrow Y \times Y$  is a fibration, and then so is  $f^*p$ . And if  $X$  is fibrant, then the projection  $\pi'_1 : X \times Y \rightarrow Y$  is a fibration. Thus in this case,  $t = \pi'_1 \circ f^*p : P_f \rightarrow Y$  is a fibration, as claimed.

5. Summarizing (1)-(4), for any map  $f : X \rightarrow Y$  (over any base  $Z \in \mathbf{cSet}$ ), we have a stable factorization  $f = t \circ s : X \rightarrow P_f \rightarrow Y$ , in which  $s$  has a retraction  $p_f$ , which is a trivial fibration when  $Y$  is fibrant, and  $t$  is a fibration when both  $X$  and  $Y$  are fibrant.

$$\begin{array}{ccc}
 & p_f & \\
 & \curvearrowright & \\
 X & \xrightarrow{s} & P_f \\
 & \searrow f & \downarrow t \\
 & & Y
 \end{array} \tag{140}$$

Note that the retraction  $p_f : P_f \rightarrow X$  is not over  $Y$ .

The following simple fact concerning just the cofibration weak factorization system will also be needed.

**Lemma 88.** *Let  $p : E \rightarrow B$  be a trivial fibration and  $c : C \rightarrow B$  a cofibration. Then the unit  $\eta : E \rightarrow c_*c^*E$  of the base change  $c^* \dashv c_*$  along  $c$  is a trivial fibration.*

*Proof.* The unit map  $\eta : E \rightarrow c_*c^*E$  is the pullback-hom  $c \Rightarrow p$ , as is easily checked. By lemma 16, for any map  $a : A \rightarrow Z$  we have the equivalence of diagonal filling conditions,

$$a \dashv c \Rightarrow p \quad \text{iff} \quad a \otimes c \dashv p.$$

But since  $c : C \rightarrow B$  is a cofibration,  $a \otimes c$  is also a cofibration if  $a : A \rightarrow Z$  is one, by axiom (C6), which says that cofibrations are closed under pushout-products. So  $a \otimes c \dashv p$  indeed holds, since  $p$  is a trivial fibration.  $\square$

**Proposition 89 (EEP).** *Weak equivalences extended along cofibrations in the following sense: given a cofibration  $c : C' \rightarrow C$  and fibrations  $A' \rightarrow C'$  and  $B \rightarrow C$ , and a weak equivalence  $w' : A' \simeq c^*B$  over  $C'$ ,*

$$\begin{array}{ccccc}
 A' & \xrightarrow{\quad} & A & & \\
 \downarrow & \searrow w' & \downarrow & \searrow w & \\
 & \sim & & \sim & \\
 & c^*B & \xrightarrow{\quad} & B & \\
 \downarrow & \swarrow & \downarrow & \swarrow & \\
 C' & \xrightarrow{\quad c \quad} & C & & 
 \end{array} \tag{141}$$

*there is a fibration  $A \rightarrow C$  and a weak equivalence  $w : A \simeq B$  over  $C$  that pulls back along  $c : C' \rightarrow C$  to  $w'$ , so  $c^*w = w'$ .*



*Proof.* Call the given fibration  $q : B \rightarrow C$  and let  $b := q^*c : c^*B \rightarrow B$  be the indicated pullback, which is thus also a cofibration. Let  $w := b_*w' : A \rightarrow B$  be the pushforward of  $w'$  along  $b$ . Composing with  $q$  gives the map  $p := q \circ w : A \rightarrow C$ . Since  $b$  is monic, we indeed have  $b^*w = w'$ , thus filling in all the dotted arrows in (141). Note moreover that  $c^*w = b^*w = w'$ , as required. It remains to show that  $p : A \rightarrow C$  is a fibration and  $w : A \rightarrow B$  is a weak equivalence.

$$\begin{array}{ccccc}
 A' & \xrightarrow{\quad} & A & \xrightarrow{s} & P_w \\
 \downarrow & \searrow w' & \downarrow & \searrow w & \downarrow t \\
 & & c^*B & \xrightarrow{b} & B \\
 \downarrow & \swarrow & \downarrow p & \swarrow q & \\
 C' & \xrightarrow{c} & C & & 
 \end{array}
 \quad (142)$$

Let us rename  $p' := c^*p : A' \rightarrow C'$  and  $B' := c^*B$  and  $q' := c^*q$ . Now let  $w = t \circ s$  be the pathspace factorization (138) of  $w$ , as a map over  $C$ . Since  $q : B \rightarrow C$  is a fibration, by the foregoing remarks on pathspace factorizations, we know that  $s : A \rightarrow P_w$  has a retraction  $p_w : P_w \rightarrow A$  which is a trivial fibration. The retraction  $p_w$  is a map over  $C$ .

The pathspace factorization  $w = t \circ s : A \rightarrow P_w \rightarrow B$  is stable under pullback along  $c$ , providing a pathspace factorization  $w' = t' \circ s' : A' \rightarrow P_{w'} \rightarrow B'$  over  $C'$ . Since both  $p'$  and  $q'$  are fibrations, the retraction  $p_{w'} : P_{w'} \rightarrow A'$  is a trivial fibration, and now  $t' : P_{w'} \rightarrow B'$  is a fibration.

$$\begin{array}{ccc}
 A' & \xrightarrow{s'} & P_{w'} \\
 \downarrow p' & \searrow w' & \downarrow t' \\
 & & B' \\
 & \swarrow q' & \\
 C' & & 
 \end{array}
 \quad (143)$$

Thus the composite  $q' \circ t' : P_{w'} \rightarrow B' \rightarrow C'$  is a fibration and therefore, by the retraction over  $C'$  with the trivial fibration  $p_{w'}$ , we have that  $s' : A' \rightarrow P_{w'}$  is a weak equivalence, by 3-for-2 for weak equivalences between fibrations. For the same reason,  $t'$  is then a weak equivalence, and therefore a trivial fibration.

Since  $t' = c^*t = b^*t$  is a trivial fibration, its pushforward  $b_*b^*t$  along  $b$  is also one by Corollary 14. Moreover,  $b_*b^*t : b_*b^*P_w \rightarrow B$  admits a unit  $\eta : P_w \rightarrow b_*b^*P_w$  (over  $B$ ).

$$\begin{array}{ccccc}
 A' & \xrightarrow{\quad} & A & \xrightarrow{s} & P_w & \xrightarrow{\eta} & b_*b^*P_w \\
 \downarrow w' & \searrow \sim & \downarrow w & & \downarrow t & \swarrow b_*b^*t & \\
 & & B' & \xrightarrow{b} & B & & \\
 \downarrow & \swarrow & \downarrow p & & \downarrow q & & \\
 C' & \xrightarrow{c} & C & & & & 
 \end{array}
 \quad (144)$$

We now *claim* that  $\eta : P_w \rightarrow b_*b^*P_w$  is a trivial fibration. Given that, the composite  $t = b_*b^*t \circ \eta$  is also a trivial fibration, whence  $q \circ t : P_w \rightarrow C$  is a fibration, and so its retract  $p : A \rightarrow C$  is a fibration. Moreover, since  $s$  is a section of the trivial fibration  $p_w : P_w \rightarrow A$  between fibrations, as before it is also a weak equivalence. Thus  $w = t \circ s$  is a weak equivalence, and we are finished.

To prove the remaining claim that  $\eta : P_w \rightarrow b_*b^*P_w$  is a trivial fibration, we shall use lemma 88. It does not apply directly, however, since  $t : P_w \rightarrow B$  is not yet known to be a trivial fibration. Instead, we show that  $\eta$  is a pullback of the corresponding unit at the trivial fibration  $p_1 : B^I \rightarrow B$ .

Consider the following cube (viewed with  $b : B' \rightarrow B$  at the front).

$$\begin{array}{ccccc}
 P_{w'} & \xrightarrow{\bar{a}} & P_w & & \\
 \downarrow p_{w'} & \searrow (p'_0)^* w' & \downarrow p_w & \searrow p_0^* w & \\
 & & B^I & \xrightarrow{\bar{b}} & B^I \\
 & & \downarrow p'_0 & & \downarrow p_0 \\
 A' & \xrightarrow{a} & A & & \\
 \downarrow w' & & \downarrow w & & \\
 B' & \xrightarrow{b} & B & & 
 \end{array} \tag{145}$$

The right hand face is a pullback by definition, and the remainder results from pulling the right face back along  $b$ , by the stability of the pathspace factorization (140). Thus all faces are pullbacks. The base is also a pushforward,  $b_* w' = w$ , again by definition. Thus the top face is also a pushforward,  $\bar{b}_*((p'_0)^* w') = p_0^* w$ . Indeed, since the front face is a pullback, the Beck-Chevalley condition applies, and so we have  $\bar{b}_*((p'_0)^*(w')) = p_0^* b_*(w') = p_0^* w$ .

Now consider the following, in which the top square remains the same as in (145), but  $p_0$  has been relaxed by  $p_1 : B^I \rightarrow B$ , so the composite at right is by definition  $t = p_1 \circ p_0^* w$ .

$$\begin{array}{ccccc}
 P_{w'} & \xrightarrow{\bar{a}} & P_w & & \\
 \downarrow & \searrow (p'_0)^* w' & \downarrow & \searrow p_0^* w & \\
 & & B^I & \xrightarrow{\bar{b}} & B^I \\
 & & \downarrow p'_1 & & \downarrow p_1 \\
 & & B' & \xrightarrow{b} & B \\
 & \swarrow t' & & \swarrow t & 
 \end{array} \tag{146}$$

The horizontal direction is still pullback along  $b$ ; let us rename  $p_0^* w =: u$  so that  $(p'_0)^* w' = b^* u$  and  $t' = b^* t$  and  $p'_1 = b^* p_1$  to make this clear. We then

add the pushforward along  $b$  on the right, in order to obtain the two units  $\eta$ .

$$\begin{array}{ccccccc}
 b^*P_w & \xrightarrow{\bar{a}} & P_w & \xrightarrow{\eta_t} & b_*b^*P_w & & \\
 \searrow b^*u & & \searrow u & & \searrow b_*b^*u & & \\
 & b^*B^I & \xrightarrow{\bar{b}} & B^I & \xrightarrow{\eta_{p_1}} & b_*b^*B^I & \\
 \searrow b^*t & \downarrow b^*p_1 & & \downarrow p_1 & \searrow b_*b^*t & \downarrow b_*b^*p_1 & \\
 & B' & \xrightarrow{b} & B & \xrightarrow{=} & B & 
 \end{array} \tag{147}$$

By the usual calculation of pushforwards in slice categories,  $\bar{b}_* \cong \eta_{p_1}^* \circ b_*$ , and so for  $b^*u$  we have  $\bar{b}_*b^*u = \eta_{p_1}^* b_*b^*u$ . But as we just determined in (145) the top left square is already a pushforward, and therefore  $u = \eta_{p_1}^* b_*b^*u$ , so the top right naturality square is a pullback.

To finish the proof as planned,  $p_1 : B^I \rightarrow B$  is a trivial fibration because  $q : B \rightarrow C$  is a fibration, and  $b : B' \rightarrow B$  is a cofibration because it is a pullback of  $c : C' \rightarrow C$ . Thus by lemma 88, we have that  $\eta_{p_1} : B^I \rightarrow b_*b^*B^I$  is a trivial fibration, and so its pullback  $\eta_t : P_w \rightarrow b_*b^*P_w$  is a trivial fibration, as claimed.  $\square$

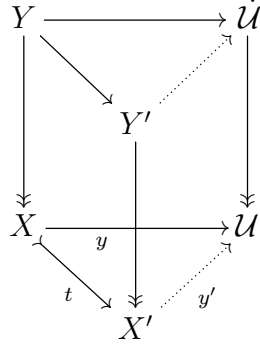
*Remark 90.* Note that  $p : A \rightarrow C$  is small if  $q : B \rightarrow C$  is small.

## 8 The fibration extension property

Given a universal fibration  $\dot{\mathcal{U}} \twoheadrightarrow \mathcal{U}$ , such as  $\mathbf{Fib} \twoheadrightarrow \mathbf{Fib}$  of Proposition 82, the fibration extension property (Definition 54) is closely related to the statement that the base object  $\mathcal{U}$  is fibrant. For Kan simplicial sets, Voevodsky proved the latter directly using minimal fibrations [?]. In a more general setting, Shulman [?] gives a proof using univalence (in the form of the equivalence extension property as stated in section 7), but that proof also uses the 3-for-2 property for weak equivalences, which we do not yet have. For cubical sets, Coquand [CCHM16] uses the equivalence extension property to prove that  $\mathcal{U}$  is fibrant, without assuming 3-for-2 for weak equivalences, via a neat argument reducing box-filling to an operation of *Kan-composition*. We shall prove that  $\mathcal{U}$  is fibrant using the equivalence extension property, also without

assuming 3-for-2 for weak equivalences, via a different argument than that in [CCHM16], one not using Kan composition.

Returning to the relation between the fibration extension property and the fibrancy of the base object of the universal fibration  $\dot{\mathcal{U}} \twoheadrightarrow \mathcal{U}$ , it is easy to see that the latter implies the former. Indeed, let  $t : X \rightarrowtail X'$  be a trivial cofibration and  $Y \twoheadrightarrow X$  a fibration. To extend  $Y$  along  $t$ , take a classifying map  $y : X \rightarrow \mathcal{U}$ , so that  $Y \cong y^* \dot{\mathcal{U}}$  over  $X$ . If  $\mathcal{U}$  is fibrant then we can extend  $y$  along  $t : X \rightarrowtail X'$  to get  $y' : X' \rightarrow \mathcal{U}$  with  $y = y' \circ t$ . The pullback  $Y' = (y')^* \dot{\mathcal{U}} \twoheadrightarrow X'$  is then a (small) fibration such that  $t^* Y' \cong t^* (y')^* \dot{\mathcal{U}} \cong y^* \dot{\mathcal{U}} \cong Y$  over  $X$ .



Thus, for the record, we have:

**Proposition 91.** *If the base object  $\mathcal{U}$  of the universal fibration  $\dot{\mathcal{U}} \twoheadrightarrow \mathcal{U}$  is fibrant, then the fibration weak factorization system has the fibration extension property.*

Conversely, given the Realignment Lemma 87, the fibration extension property also implies the fibrancy of  $\mathcal{U}$ :

**Corollary 92.** *The fibration extension property implies that the base  $\mathcal{U}$  of the universal fibration  $\dot{\mathcal{U}} \twoheadrightarrow \mathcal{U}$  is fibrant: given any  $y : X \rightarrow \mathcal{U}$  and trivial cofibration  $t : X \rightarrowtail X'$ , there is a map  $y' : X' \rightarrow \mathcal{U}$  with  $y' \circ t = y$ .*

*Proof.* Take the pullback of  $\dot{\mathcal{U}} \twoheadrightarrow \mathcal{U}$  along  $y : X \rightarrow \mathcal{U}$  to get a (small) a fibration  $Y \twoheadrightarrow X$ , which extends along the (trivial) cofibration  $t : X \rightarrowtail X'$  by the fibration extension property, to a (small) fibration  $Y' \twoheadrightarrow X'$  with  $Y \cong t^* Y'$  over  $X$ . By realignment there is a classifying map  $y' : X' \rightarrow \mathcal{U}$  for  $Y'$  with  $y' \circ t = y$ .  $\square$

Now let us show the following.

**Proposition 93.** *The base  $\mathcal{U}$  of the universal fibration  $\mathcal{U}^I \twoheadrightarrow \mathcal{U}$  in  $\mathbf{cSet}$ , as constructed in section 6.3, is a fibrant object.*

*Proof.* By Corollary 22,  $\mathcal{U}$  is an unbiased fibrant object if the canonical map  $u = \langle p_2, \text{eval} \rangle$  in the following diagram in  $\mathbf{cSet}$ , is a trivial fibration.

$$\begin{array}{ccc}
 \mathcal{U}^I \times I & \xrightarrow{\quad \text{eval} \quad} & \mathcal{U} \\
 \downarrow p_2 & \searrow u & \downarrow \\
 I \times \mathcal{U} & \xrightarrow{\quad \text{eval} \quad} & \mathcal{U} \\
 \downarrow & \lrcorner & \downarrow \\
 I & \xrightarrow{\quad \text{eval} \quad} & 1
 \end{array} \tag{148}$$

Thus consider a filling problem of the following form, with an arbitrary cofibration  $c : C \hookrightarrow Z$ .

$$\begin{array}{ccc}
 C & \xrightarrow{\quad \quad} & \mathcal{U}^I \times I \\
 \downarrow c & \nearrow & \downarrow \langle p_2, \text{eval} \rangle \\
 Z & \xrightarrow{\quad \quad} & I \times \mathcal{U}
 \end{array} \tag{149}$$

The horizontal maps may be written in the form  $\langle z, b \rangle : Z \rightarrow I \times \mathcal{U}$  and  $\langle \tilde{a}, zc \rangle : C \rightarrow \mathcal{U}^I \times I$ , regarding  $z : Z \rightarrow I$  as an  $I$ -indexing.

Transposing  $\tilde{a}$  to  $a : C \times I \rightarrow \mathcal{U}$  we obtain the new problem

$$\begin{array}{ccccc}
 C & \xrightarrow{\langle zc \rangle} & C \times I & \xrightarrow{\quad a \quad} & \mathcal{U} \\
 \downarrow c & & \downarrow c \times I & & \downarrow \\
 Z & \xrightarrow{\langle z \rangle} & Z \times I & \xrightarrow{\quad d \quad} & \mathcal{U} \\
 & \searrow b & & & \uparrow
 \end{array} \tag{150}$$

in which we recall from (41) the graph notation  $\langle z \rangle = \langle 1_Z, z \rangle : Z \rightarrow Z \times I$ . Given a map  $d$  as shown in (150), we can obtain the indicated diagonal filler in (149) as  $\langle \tilde{d}, z \rangle : Z \rightarrow \mathcal{U}^I \times I$ .

As a sanity check, note that  $b \circ c = a \circ \langle zc \rangle$  turns the problem (150) into that of extending the copair  $[b, a]$  along the unique map

$$Z +_C (C \times I) \longrightarrow Z \times I,$$

which is exactly the (trivial cofibration) pushout-product  $c \otimes_z \delta$  from (42), recalled below for the reader's convenience.

$$\begin{array}{ccc}
 C & \xrightarrow{\langle zc \rangle} & C \times I \\
 \downarrow c & & \downarrow \\
 Z & \xrightarrow{\quad} & Z +_C (C \times I) \\
 & \searrow \langle z \rangle & \downarrow c \otimes_z \delta \\
 & & Z \times I
 \end{array}
 \quad (151)$$

$\text{Curved arrow from } Z \text{ to } Z \times I \text{ is labeled } \langle z \rangle.$ 
 $\text{Curved arrow from } C \times I \text{ to } Z \times I \text{ is labeled } c \times I.$

Returning to (150), take pullbacks of  $\dot{\mathcal{U}} \rightarrow \mathcal{U}$  along  $a$  and  $b$  to get fibrations  $p_a : A \rightarrow C \times I$  and  $p_b : B \rightarrow Z$  respectively, and let

$$p_c := \langle zc \rangle^* p_a : A_c \rightarrow C$$

be the corresponding “fiber of  $A$  over the graph of  $zc$ ”. We then have  $c^* B \cong A_c$  over  $C$  by the commutativity of the outer square of (149).

$$\begin{array}{ccccc}
 A_c & \xrightarrow{\quad} & A & & \\
 \downarrow p_c & \lrcorner & \downarrow p_a & & \\
 C & \xrightarrow{\langle zc \rangle} & C \times I & & \\
 \downarrow c & & \downarrow c \times I & & \\
 B & \xrightarrow{\quad} & Z & \xrightarrow{\langle z \rangle} & Z \times I \\
 \downarrow p_b & & & & \\
 & & & & 
 \end{array}$$

The diagonal filler sought in (149) corresponds, again by transposition and pullback of  $\dot{\mathcal{U}} \rightarrow \mathcal{U}$ , to a fibration  $p_d : D \rightarrow Z \times I$  with  $\langle z \rangle^* D \cong B$  over

$Z$  and  $(c \times I)^*D \cong A$  over  $C \times I$ , as indicated below.

$$\begin{array}{ccccc}
 & A_c & \xrightarrow{\quad} & A & \\
 & \downarrow p_c & & \downarrow p_a & \\
 & C & \xrightarrow{\langle zc \rangle} & C \times I & \\
 & \downarrow c & & \downarrow c \times I & \\
 B & \xrightarrow{\quad} & D & & \\
 \downarrow p_b & & \downarrow p_d & & \\
 Z & \xrightarrow{\langle z \rangle} & Z \times I & & 
 \end{array}
 \tag{152}$$

We shall construct  $p_d : D \rightarrow Z \times I$  using the equivalence extension property (Proposition 89) as follows. First apply the functor  $(-) \times I$  to the left vertical (pullback) face of the cube in (152) to get the following, with a new pullback square on the right with the indicated fibrations.

$$\begin{array}{ccccc}
 & A_c & \xrightarrow{\quad} & A & & A_c \times I \\
 & \downarrow p_c & & \downarrow p_a & & \downarrow p_c \times I \\
 & C & \xrightarrow{\langle zc \rangle} & C \times I & & \\
 & \downarrow c & & \downarrow c \times I & & \\
 B & \xrightarrow{\quad} & D & & & B \times I \\
 \downarrow p_b & & \downarrow p_d & & & \downarrow p_b \times I \\
 Z & \xrightarrow{\langle z \rangle} & Z \times I & & & 
 \end{array}
 \tag{153}$$

We now *claim* that there is a weak equivalence  $e : A \simeq A_c \times I$  over  $C \times I$ . From this it follows by the equivalence extension property (Proposition 89) that there are:

- (i) a fibration  $p_d : D \rightarrow Z \times I$  with  $(c \times I)^*D \cong A$  over  $C \times I$ , and
- (ii) a weak equivalence  $f : D \simeq B \times I$  over  $Z \times I$  with  $(c \times I)^*f \cong e$  over  $C \times I$ .

It then remains only to show that  $B \cong \langle z \rangle^*D$  over  $Z$  to complete the proof.



To obtain the claimed weak equivalence  $e$ , consider the following square,

$$\begin{array}{ccc} A_c & \xrightarrow{\langle zcp_c \rangle} & A_c \times I \\ \downarrow & & \downarrow p_c \times I \\ A & \xrightarrow[p_a]{} & C \times I, \end{array} \quad (154)$$

in which the top horizontal map is the graph of the composite,

$$A_c \xrightarrow{p_c} C \xrightarrow{c} Z \xrightarrow{z} I,$$

and the others are the evident ones from (153). The square is easily seen to commute, and the top map is a trivial cofibration (by Remark 26), because it is the graph of a map into  $I$ . The left map is also a trivial cofibration by Frobenius (Proposition 64), because by its definition in (152) it is the pullback of another such graph  $\langle zc \rangle$  along the fibration  $p_a$ . A simple lemma (Lemma 94 below) provides the claimed weak equivalence  $e : A \simeq A_c \times I$  over  $C \times I$ .

To see that  $B \cong \langle z \rangle^* D$  over  $Z$ , recall from the proof of the equivalence extension property that the map  $f : D \cong B \times I$  is the pushforward of  $e : A \simeq A_c \times I$  along the cofibration  $b_c \times I : A_c \times I \rightarrow B \times I$ , where we have christened the evident map in (153)  $b_c : A_c \rightarrow B$ . Thus by construction  $f = (b_c \times I)_* e$ . We can then apply the Beck-Chevalley condition for the pushforward using the pullback square on the left below.

$$\begin{array}{ccccc} A_c & \xrightarrow{\langle zcp_c \rangle} & A_c \times I & \xleftarrow{e} & A \\ \downarrow b_c & \lrcorner & \downarrow b_c \times I & & \\ B & \xrightarrow{\langle zp_b \rangle} & B \times I & \xleftarrow{f} & D \end{array} \quad (155)$$

The pullback of  $e$  along the top of the square is the identity on  $A_c$ , as can be seen by pulling back  $e$  as a map over  $C \times I$  along  $\langle zc \rangle : C \rightarrow C \times I$ . Thus the same is true up to isomorphism for the pullback of  $f$  along the bottom.

An application of the Realignment Lemma 87 along the trivial cofibration  $c \otimes_z \delta$  completes the proof.  $\square$

**Lemma 94.** *Suppose the following square commutes and the indicated cofibrations are trivial.*

$$\begin{array}{ccc} A & \longrightarrow & C \\ \downarrow & & \downarrow \\ B & \twoheadrightarrow & D \end{array} \quad (156)$$

Then there is a weak equivalence  $e : B \simeq C$  over  $D$  (and under  $A$ ).

*Proof.* Use the fact that any two diagonal fillers are homotopic to get a homotopy equivalence  $e : B \simeq C$  filling the square.  $\square$

*Remark 95.* The foregoing proof also works, *mutatis mutandis*, for the case of *biased* fibrations, as in the setting of [CCHM16].

Applying proposition 91 now yields the following.

**Corollary 96.** *The fibration weak factorization system has the fibration extension property (definition 54).*

By Theorem 59, finally, we have the following.

**Theorem 97.** *There is a Quillen model structure  $(\mathcal{C}, \mathcal{W}, \mathcal{F})$  on the category  $\mathbf{cSet}$  of cubical sets for which:*

1. *the cofibrations  $\mathcal{C}$  are any class of maps satisfying (C0)-(C8) (equivalently, the simplified axioms in the Appendix),*
2. *the fibrations  $\mathcal{F}$  are the maps  $f : Y \rightarrow X$  for which the canonical map*

$$(f^1 \times \mathbf{I}, \text{eval}_Y) : Y^1 \times \mathbf{I} \longrightarrow (X^1 \times \mathbf{I}) \times_X Y$$

*lifts on the right against  $\mathcal{C}$ .*

3. *the weak equivalences  $\mathcal{W}$  are the maps  $w : X \rightarrow Y$  for which, for every fibrant object  $K$ , the internal precomposition  $K^w : K^Y \rightarrow K^X$  is bijective on connected components.*

## Appendix A: Axioms for cartesian cofibrations

A system of maps satisfying the axioms (C0)-(C8) in the main text, for the cofibrations in a cartesian cubical model category, will be called *cartesian cofibrations*. The axioms can be restated equivalently as follows.

- (A0) All cofibrations are monomorphisms.
- (A1) All isomorphisms are cofibrations.
- (A2) The composite of two cofibrations is a cofibration.

- (A3) Any pullback of a cofibration is a cofibration.
- (A4) The category of cofibrations and cartesian squares has a terminal object.
- (A5) The join of two cofibrant subobjects is a cofibration.
- (A6) The diagonal map  $\delta : I \rightarrow I \times I$  is a cofibration.
- (A7) If  $c : A \rightarrow B$  is a cofibration, then so is  $c^I : A^I \rightarrow B^I$ .

**Proposition 98.** *In any topos, the locally decidable subobjects satisfy the axioms for cartesian cofibrations.*

*Proof.* [fill in ...]

□

## Appendix B: Cubical sets as a classifying topos

The objects of the Cartesian cube category  $\square$  may be taken concretely to be finite sets of the form

$$[n] = \{\perp, x_1, \dots, x_n, \top\},$$

and the arrows  $f : [n] \rightarrow [m]$  to be all bipointed maps  $[m] \rightarrow [n]$  (note the direction). These maps are evidently just  $m$ -tuples of elements from the set  $[n]$ , which are easily shown to be composites of faces, degeneracies, permutations, and diagonals.

The category of (*Cartesian*) *cubical sets* is the presheaf topos

$$\mathbf{cSet} = \mathbf{Set}^{\square^{\text{op}}}.$$

It is of course generated by the representable presheaves  $I^n := y[n]$ , called the *n-cubes*. The 0-cube is  $I^0 = y[0] = 1$ ; the 1-cube is  $I = y[1]$ ; and  $I^n \times I^m \cong I^{n+m}$  in virtue of preservation of products by the Yoneda embedding  $y : \square^{\text{op}} \hookrightarrow \mathbf{cSet}$ . For a cubical set  $X : \square^{\text{op}} \rightarrow \mathbf{Set}$  we write  $X_n = X[n]$  and call this the *set of n-cubes in X*, for which we have the usual Yoneda correspondence,

$$\{c \in X_n\} \cong \{c : I^n \rightarrow X\}.$$

In particular,  $I_m^n = \mathbf{cSet}([m], [n])$  is the set of  $m$ -cubes in the  $n$ -cube.<sup>2</sup>

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<sup>2</sup>Note that the cardinality of  $I_m^n$  is therefore just  $(m+2)^n$ , in comparison to the *Dedekind* cubes, for which *e.g.*  $\mathbf{cSet}([1], [n])$  the  $n^{\text{th}}$  *Dedekind number*, the number of elements in the free distributive lattice on  $n$  generators, which in general is a number so large that it is not known, even for values of  $n > 7$ .

**Proposition 99.** *The category  $\mathbf{cSet}$  of Cartesian cubical sets is the classifying topos for strictly bipointed objects: objects  $A$  with points  $a_0, a_1 : 1 \rightarrow A$  the equalizer of which is  $0 \rightarrow 1$ .*

*Proof.* Consider the covariant presentation  $\mathbf{cSet} = \mathbf{Set}^{\mathbb{B}}$  where  $\mathbb{B}$  is the category of finite, strictly bipointed sets and bipointed maps. We can extend  $\mathbb{B} \hookrightarrow \mathbb{B}_=$  by freely adjoining coequalizers, making  $\mathbb{B}_=$  the free finite *colimit* category on a co-bipointed object. A concrete presentation of  $\mathbb{B}_=$  is the finite bipointed sets, including those with  $\perp = \top$ . Let us write  $(n)$  for the bipointed set  $\{x_1, \dots, x_n, \perp = \top\}$ , with  $n$  (non-constant) elements and a further element  $\perp = \top$ . There is an evident coequalizer  $[1] \rightrightarrows [n] \rightarrow (n)$ , which just identifies the distinguished points, and every coequalizer has either the form  $[m] \rightrightarrows [n] \rightarrow [k]$  or  $[m] \rightrightarrows [n] \rightarrow (k)$ , for a suitable choice of  $k$ . Note that there are no maps of the form  $(m) \rightarrow [n]$ , and that every map  $[m] \rightarrow (n)$  factors uniquely as  $[m] \rightarrow (m) \rightarrow (n)$  with  $[m] \rightarrow (m)$  the canonical coequalizer of  $\perp$  and  $\top$ . The category  $\mathbb{B}_=$  can therefore be decomposed into two “levels”, the upper one of which is essentially  $\mathbb{B}$ , the lower one consisting of just the objects  $(n)$  and thus essentially the finite *pointed* sets, and for each  $n$ , there is the canonical coequalizer  $[n] \rightarrow (n)$  going from the upper level to the lower one.

$$\begin{array}{ccccc} \dots & \longrightarrow & [m] & \longrightarrow & [n] & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \\ \dots & \longrightarrow & (m) & \longrightarrow & (n) & \longrightarrow & \dots \end{array}$$

Write  $u : \mathbb{B} \rightarrow \mathbb{B}_=$  for the upper inclusion, which is the classifying functor of generic co-bipointed object in  $\mathbb{B}_=$  (which is strict).

Now consider the induced geometric morphism:

$$\begin{array}{ccc} \mathbf{Set}^{\mathbb{B}} & \xrightleftharpoons[u_!]{u_*} & \mathbf{Set}^{\mathbb{B}_=} \\ & u^* \longleftarrow & \end{array} \quad u_! \dashv u^* \dashv u_*$$

Since  $u^*$  is the restriction along  $u$ , the right adjoint  $u_*$  must be “prolongation by 1”,

$$\begin{aligned} u_*(P)[n] &= P[n], \\ u_*(P)(n) &= \{*\}, \end{aligned}$$

with the obvious maps,

$$\begin{array}{ccccccc} \dots & \longrightarrow & P[m] & \longrightarrow & P[n] & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \\ \dots & \longrightarrow & \{*\} & \longrightarrow & \{*\} & \longrightarrow & \dots \end{array}$$

as is easily seen by considering maps in  $\mathbf{Set}^{\mathbb{B}^=}$  of the form

$$\begin{array}{ccc} Q[n] & \longrightarrow & P[n] \\ \downarrow & & \downarrow \\ Q(n) & \longrightarrow & \{*\}. \end{array}$$

Since  $u_* : \mathbf{Set}^{\mathbb{B}} \rightarrow \mathbf{Set}^{\mathbb{B}^=}$  is evidently full and faithful, it is the inclusion part of a sheaf subtopos  $\mathbf{sh}(\mathbb{B}_{=}^{\text{op}}, j) \hookrightarrow \mathbf{Set}^{\mathbb{B}^=}$  for a suitable Grothendieck topology  $j$  on  $\mathbb{B}_{=}^{\text{op}}$ . We claim that  $j$  is the closed complement topology of the subobject  $[\perp = \top] \rightarrowtail 1$  represented by the coequalizer  $[0] \rightarrow (0)$ . Indeed, in  $\mathbf{Set}^{\mathbb{B}^=}$  we have the representable functors:

$$\begin{aligned} I &= y[1], \\ 1 &= y[0], \\ [\perp = \top] &= y(0) \end{aligned}$$

fitting into an equalizer  $[\perp = \top] \rightarrow 1 \rightrightarrows I$ , which is the image under Yoneda of the canonical coequalizer  $[1] \rightrightarrows [0] \rightarrow (0)$  in  $\mathbb{B}_{=}$ . The closed complement topology for  $[\perp = \top] \rightarrowtail 1$  is generated by the single cover  $0 \rightarrow [\perp = \top]$ , which can be described logically as forcing the sequent  $(\perp = \top \vdash \perp)$  to hold. Recall from [?], Proposition 3.53, the following simple characterization of the sheaves for a closed topology generated by an object  $U \rightarrowtail 1$ : an object  $X$  is a sheaf iff  $X \times U \cong U$ . In the present case, it therefore suffices to show that for any  $P : \mathbb{B}_{=} \rightarrow \mathbf{Set}$  we have:

$$P \times [\perp = \top] \cong [\perp = \top] \quad \text{iff} \quad P(n) = 1 \text{ for all } n.$$

For any object  $b \in \mathbb{B}_{=}$ , consider the map

$$\text{Hom}(yb, P \times [0 = 1]) \cong \text{Hom}(yb, P) \times \text{Hom}(yb, [0 = 1]) \rightarrow \text{Hom}(yb, [\perp = \top]).$$

If  $b = [k]$ , then  $\text{Hom}(yb, [\perp = \top]) \cong \text{Hom}_{\mathbb{B}=}((0), [k]) \cong 0$ , and so we always have an iso

$$\begin{aligned} \text{Hom}(yb, P \times [\perp = \top]) &\cong \text{Hom}(yb, P) \times \text{Hom}(yb, [\perp = \top]) \\ &\cong \text{Hom}(yb, P) \times 0 \cong 0. \end{aligned}$$

If  $b = (k)$ , then  $\text{Hom}(y(k), [\perp = \top]) \cong \text{Hom}_{\mathbb{B}=}((0), (k)) \cong 1$ , and we have an iso

$$\begin{aligned} \text{Hom}(y(k), P \times [\perp = \top]) &\cong \text{Hom}(y(k), P) \times \text{Hom}(y(k), [\perp = \top]) \\ &\cong \text{Hom}(y(k), P) \times 1 \cong \text{Hom}(y(k), P) \cong P(k). \end{aligned}$$

Thus we will have an iso  $P \times [\perp = \top] \cong [\perp = \top]$  iff  $P(k) \cong 1$ .

Thus the presheaf topos  $\mathbf{Set}^{\mathbb{B}}$  is the closed complement of the open subtopos

$$\mathbf{Set}^{\mathbb{B}=} /_{[\perp = \top]} \hookrightarrow \mathbf{Set}^{\mathbb{B}=}$$

given by forcing the proposition  $\perp \neq \top$ . Since  $\mathbf{Set}^{\mathbb{B}=}$  is clearly the classifying topos for *arbitrary* bipointed objects  $\perp, \top : 1 \rightarrow B$ , the subtopos  $\mathbf{Set}^{\mathbb{B}}$  indeed classifies *strictly* bipointed objects, as claimed.  $\square$

**Corollary 100.** *The geometric realization functor to topological spaces*

$$R : \mathbf{cSet} \rightarrow \mathbf{Top}$$

*preserves cartesian products,  $R(X \times Y) \cong R(X) \times R(Y)$ .*<sup>3</sup>

*Proof.* This can of course be shown directly, but it follows immediately by composing the inverse image of the classifying geometric morphism  $\mathbf{sSets} \rightarrow \mathbf{cSet}$  of the 1-simplex  $\Delta^1$  with the standard geometric realization  $\mathbf{sSets} \rightarrow \mathbf{Top}$ , each of which preserves finite products.  $\square$

**Definition 101.** Let  $\square \rightarrow \mathbf{Cat}$  be the unique product-preserving functor taking the interval  $[1]$  to the one arrow category  $\mathbf{2} = (0 \leq 1)$ . This functor then takes  $[n]$  to  $\mathbf{2}^n$ , the  $n$ -fold product in  $\mathbf{Cat}$ , and maps  $[m] \rightarrow [n]$  to the corresponding monotone functions of the posets  $\mathbf{2}^n$ . The *cubical nerve* functor

$$N : \mathbf{Cat} \rightarrow \mathbf{cSet}$$

---

<sup>3</sup>This fact and the next one are to be contrasted with the case of monoidal cubical sets, *e.g.* as studied by [?, ?]

is then defined by:

$$N(\mathbb{C})_n = \mathbf{Cat}(\mathbb{2}^n, \mathbb{C}).$$

Thus  $N(\mathbb{C})_0$  is the set of objects of  $\mathbb{C}$ ;  $N(\mathbb{C})_1$  is the set of arrows;  $N(\mathbb{C})_2$  consists of all commutative squares;  $N(\mathbb{C})_3$  all commutative cubes, etc.

**Proposition 102.** *The nerve functor  $N : \mathbf{Cat} \rightarrow \mathbf{cSet}$  is full and faithful.*

*Proof.* Given categories  $\mathbb{C}$  and  $\mathbb{D}$  and functors  $F, G : \mathbb{C} \rightarrow \mathbb{D}$ , suppose  $F(f) \neq G(f)$  for some  $f : A \rightarrow B$  in  $\mathbb{C}$ . Take  $f^\sharp : \mathbb{2} \rightarrow \mathbb{C}$  with image  $f$ . Then  $N(F)_1(f^\sharp) = F(f) \neq G(f) = N(G)_1(f^\sharp)$ , and so  $N(F) \neq N(G) : N(\mathbb{C}) \rightarrow N(\mathbb{D})$ . So  $N$  is faithful.

For fullness, let  $\varphi : N(\mathbb{C}) \rightarrow N(\mathbb{D})$  be a natural transformation, and define a proposed functor  $F : \mathbb{C} \rightarrow \mathbb{D}$  by

$$\begin{aligned} F_0 &= \varphi_0 : \mathbb{C}_0 = N(\mathbb{C})_0 \rightarrow N(\mathbb{D})_0 = \mathbb{D}_0 \\ F_1 &= \varphi_1 : \mathbb{C}_1 = N(\mathbb{C})_1 \rightarrow N(\mathbb{D})_1 = \mathbb{D}_1. \end{aligned}$$

We just need to show that  $F$  preserves identity arrows and composition. Consider the following diagram.

$$\begin{array}{ccc} \mathbf{Cat}(\mathbb{2}^1, \mathbb{C}) = N(\mathbb{C})_1 & \xrightarrow{F_1} & N(\mathbb{D})_1 = \mathbf{Cat}(\mathbb{2}^1, \mathbb{D}) \\ \uparrow !^* & & \uparrow !^* \\ \mathbf{Cat}(\mathbb{2}^0, \mathbb{C}) = N(\mathbb{C})_0 & \xrightarrow{F_0} & N(\mathbb{D})_0 = \mathbf{Cat}(\mathbb{2}^0, \mathbb{D}). \end{array}$$

Here  $!^* : \mathbf{Cat}(\mathbb{2}^0, \mathbb{C}) \rightarrow \mathbf{Cat}(\mathbb{2}, \mathbb{C})$  is precomposition with  $! : \mathbb{2} = \mathbb{2}^1 \rightarrow \mathbb{2}^0 = \mathbb{1}$ , so the diagram commutes. But since  $! : \mathbb{2} \rightarrow \mathbb{1}$  is a functor,

$$\mathbb{C}_0 = \mathbf{Cat}(\mathbb{1}, \mathbb{C}) \xrightarrow{!^*} \mathbf{Cat}(\mathbb{2}, \mathbb{C}) = \mathbb{C}_1$$

takes objects in  $\mathbb{C}$  to their identity arrows. Thus  $F$  preserves identity arrows. Similarly, for composition, consider

$$\begin{array}{ccc} \mathbf{Cat}(\mathbb{2}^2, \mathbb{C}) = N(\mathbb{C})_2 & \xrightarrow{\varphi_2} & N(\mathbb{D})_2 = \mathbf{Cat}(\mathbb{2}^2, \mathbb{D}) \\ \downarrow d^* & & \downarrow d^* \\ \mathbf{Cat}(\mathbb{2}, \mathbb{C}) = N(\mathbb{C})_1 & \xrightarrow{F_1} & N(\mathbb{D})_1 = \mathbf{Cat}(\mathbb{2}, \mathbb{D}). \end{array}$$

where  $\varphi_2 : N(\mathbb{C})_2 \rightarrow N(\mathbb{D})_2$  is the action of  $\varphi$  on commutative squares of arrows, and  $d^* : \mathbf{Cat}(\mathbb{2}^2, \mathbb{C}) \rightarrow \mathbf{Cat}(\mathbb{2}, \mathbb{C})$  is precomposition with the diagonal

map  $d : \mathbb{2} \rightarrow \mathbb{2}^2 = \mathbb{2} \times \mathbb{2}$ , so the diagram commutes. For any composable pair of arrows  $A \xrightarrow{f} B \xrightarrow{g} C$  in  $\mathbb{C}$  there is a commutative square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ f \downarrow & & \downarrow g \\ B & \xrightarrow{g} & C, \end{array}$$

and the effect of  $d^* : \mathbf{Cat}(\mathbb{2}^2, \mathbb{C}) \rightarrow \mathbf{Cat}(\mathbb{2}, \mathbb{C})$  on this square is exactly  $g \circ f : A \rightarrow C$ , and similarly for  $d^* : \mathbf{Cat}(\mathbb{2}^2, \mathbb{D}) \rightarrow \mathbf{Cat}(\mathbb{2}, \mathbb{D})$ . Thus the commutativity of the above diagram implies that  $F$  preserves composition. Since clearly  $N(F) = \varphi$ , we indeed have that  $N$  is also full.  $\square$

**Proposition 103.** *For any cubical set  $X$ , the exponential  $X^{\mathbf{I}}$  is calculated as the “shift by one dimension”,*

$$X^{\mathbf{I}}(n) \cong X(n+1).$$

*Proof.*

$$\begin{aligned} X^{\mathbf{I}}(n) &\cong \mathrm{Hom}(y[n], X^{\mathbf{I}}) \cong \mathrm{Hom}(\mathbf{I}^n, X^{\mathbf{I}}) \cong \mathrm{Hom}(\mathbf{I}^n \times \mathbf{I}, X) \\ &\cong \mathrm{Hom}(\mathbf{I}^{n+1}, X) \cong \mathrm{Hom}(y[n+1], X) \cong X(n+1). \end{aligned}$$

$\square$

**Corollary 104.** *The functor  $X \mapsto X^{\mathbf{I}}$  has a right adjoint.*

*Proof.* The functor  $X \mapsto X^{\mathbf{I}}$  is given by precomposition with the “successor” functor  $S : \square \rightarrow \square$  with  $S[n] = [n+1]$ . Thus  $X^{\mathbf{I}}([n]) = X(S[n]) = (S^*(X))([n])$ . Precomposition always has a right adjoint  $S^* \dashv S_*$ , which can be calculated as:

$$S_*(X)(n) \cong \mathrm{Hom}(y[n], S_*X) \cong \mathrm{Hom}(S^*(y[n]), X) \cong \mathrm{Hom}(\square(S(-), [n]), X).$$

$\square$

We need the following fact in order to calculate the right adjoint further.

**Lemma 105.** *In  $\mathbf{cSet}$ , we have  $\mathbf{I}^{\mathbf{I}} \cong \mathbf{I} + 1$ .*



*Proof.* For any  $[n] \in \square$  we have:

$$(I^I)(n) \cong I(n+1) \cong \text{Hom}(I^{(n+1)}, I) \cong \square([n+1], [1]) \cong \mathbb{B}([1], [n+1]) \cong n+3.$$

On the other hand,

$$(I+1)(n) \cong I(n) + 1(n) \cong \text{Hom}(I^n, I) + 1 \cong \mathbb{B}([1], [n]) + 1 \cong (n+2) + 1.$$

The isomorphism is natural in  $n$ .  $\square$

We mention that a similar fact holds for the generic object in the object classifier topos, and in the Schanuel topos, and is used in the theory of “abstract higher-order syntax” [?, ?].

**Definition 106.** Let us write

$$X_I = S_*(X)$$

for the right adjoint of the path object functor  $X^I = S^*X$ .

**Corollary 107.** *We have the following calculation for the right adjoint  $X_I$ :*

$$\begin{aligned} X_I(n) &\cong \text{Hom}(I^n, X_I) \\ &\cong \text{Hom}((I^n)^I, X) \\ &\cong \text{Hom}((I^I)^n, X) \\ &\cong \text{Hom}((I+1)^n, X) \\ &\cong \text{Hom}(I^n + C_{n-1}^n I^{n-1} + \dots + C_1^n I + 1, X) \\ &\cong X(n) \times X(n-1)^{C_{n-1}^n} \times \dots \times X(1)^{C_1^n} \times X(0), \end{aligned}$$

where  $C_k^n = \binom{n}{k}$  is the usual binomial coefficient.

**Corollary 108.** *There is a natural transformation  $X_I \rightarrow X$ , given by the first projection from  $X_I(n) \cong X(n) \times X(n-1)^{C_{n-1}^n} \times \dots \times X(1)^{C_1^n} \times X(0)$ .*

Finally, we observe that the path object functor  $X^I$  itself, as a left adjoint, preserves all colimits. This does not hold in general in type theory, but will be a special property of the cubical model. (Cf. Lawvere [?] on the notion of “tiny” objects and the “amazing right adjoint”.)

**Example.** (P. Aczel) The cubical set  $P$  of polynomials (over the integers, say), is defined by:

$$P_n = \{p(x_1, \dots, x_n) \mid \text{polynomials in at most } x_1, \dots, x_n\}$$

with the evident maps  $P_m \rightarrow P_n$  for each function  $[m] \rightarrow [n]$ .

This is a ring object in the category of cubical sets, and the interval  $I = y[1]$  embeds into  $P$ . The same is true for any algebraic theory  $\mathbb{T}$  with two constants, such as boolean algebras: there is a cubical  $\mathbb{T}$ -algebra  $A$  and a monic  $I \hookrightarrow A$ .

Let  $\square[I] = \square$  be the cube category, classifying intervals, and  $\square[\mathbb{T}]$  the classifying category for  $\mathbb{T}$ -algebras. There is an interval  $J$  in  $\square_{\mathbb{T}}$  consisting of the generic  $\mathbb{T}$ -algebra and its two constants. This  $J$  has a classifying functor  $J : \square_I \rightarrow \square_{\mathbb{T}}$ , inducing functors on presheaves

$$J_! \dashv J^* \dashv J_* : \mathbf{Set}^{\square_I^{\text{op}}} \rightarrow \mathbf{Set}^{\square_{\mathbb{T}}^{\text{op}}}$$

as usual, where  $J_! \circ y_{\square_I} = y_{\square_{\mathbb{T}}} \circ J$ , with  $y$  the respective Yoneda embeddings.

We can calculate:

$$\begin{aligned} J^* J_!(I)([n]) &= J^* J_!(Y[1])([n]) \\ &= J^* Y(J[1])([n]) = Y(J[1])(J[n]) \\ &= \square_{\mathbb{T}}(J[n], J[1]) = \mathbb{T}\text{-Alg}(J[1], J[n]) \\ &= \mathbb{T}\text{-Alg}(F(1), F(n)) = |F(n)|, \end{aligned} \tag{157}$$

where  $F(n)$  is the free  $\mathbb{T}$ -algebra on  $n$  generators. So in the case of polynomials we indeed have

$$P = J^* J_!(I).$$

The unit of the adjunction  $I \rightarrow J^* J_!(I)$  is faithful, since  $J$  itself is faithful and therefore the left adjoint  $J_!$  is faithful.  $P$  is a ring in  $\mathbf{Set}^{\square_I^{\text{op}}}$  since  $J_!(I)$  is a ring in  $\mathbf{Set}^{\square_{\mathbb{T}}^{\text{op}}}$  and  $J^*$  is left exact.

A closely related example is the cubical set of “boolean polynomials”,

$$B_n = \{\varphi(p_1, \dots, p_n) \mid \text{propositional formulas in at most } p_1, \dots, p_n\}$$

which is the free boolean algebra  $2^{2^n}$ .

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