Illinois Institute of Technology
Department of Computer Science

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CS 330 Discrete Structures
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1 Combinations

We have examined the question of how many ways there are to arrange (permute) n different items. Suppose instead we want to arrange only k of the n items, $k < n$. For example, if we have ten empty, single-bed hotel rooms numbered $1, 2, \ldots, 10$ and four guests arrive on a stormy night, in how many ways can we assign each guest to a room?

The first guest can be given any of the ten rooms; the second guest can be given any of the remaining nine rooms; the third guest can be given any of the remaining eight rooms; finally, the fourth guest can be given any of the remaining seven rooms. The rule of product tells us that there are

$$10 \times 9 \times 8 \times 7 = 5040$$

different ways to make the room assignments. In general, if there are n rooms and $k < n$ guests, the room assignments can be made in

$$n \times (n-1) \times (n-2) \times \cdots \times (n-k+1)$$

ways. This number can be rewritten conveniently using the factorial notation:

$$n \times (n-1) \times (n-2) \times \cdots \times (n-k+1) = \frac{n!}{(n-k)!}; \tag{1}$$

this is called the *number of permutations of n things taken k at a time*, and is sometimes denoted $P(n, k)$.

The rewritten form $n!/(n-k)!$ is also valid for $k = n$ since it yields

$$\frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$$

(by our definition that $0! = 1$). This agrees with our discovery that there are $n!$ permutations of n different items. Furthermore, when $k = 0$, $n!/(n-k)! = 1$, in agreement with our convention that there is a unique (empty) permutation of zero items. Rewriting the product in the form $n!/(n-k)!$ also suggests an alternative proof of the result based on the variation on the rule of product: Let E be the event of forming a permutation of all n items, viewed as a compound event $E = E_1 \cap E_2$ in which E_2 is the event of entering the first k of the n elements and E_1 is the event of arranging the remaining $n-k$ elements. The variation on the rule of product then tells us that E_2, that is forming a permutation of k of n items, can happen in $e_2 = e/e_1$. But, we know that $e = n!$ and $e_1 = (n-k)!$ since these are simply the numbers of permutations of n and $n-k$ items, respectively. It follows that $e_2 = n!/(n-k)!$.

Returning to the problem of how to assign guests to empty hotel rooms, let is consider the point of view of the hotel cleaning staff. As far as the cleaning staff is concerned, guests are indistinguishable from one another—the staff only cares about which rooms have been occupied and need cleaning. Thus, if there are n hotel rooms and k guests, we might want to ask: how many different arrangements of the k rooms might the cleaning staff be asked to clean? Again we use the variation on the rule of product. Let E be the event of assigning k guests to k of n hotel rooms; we have seen from (1) that this can be done in $e = n!/(n-k)!$

ways. View E as a compound event $E = E_1 \cap E_2$ in which E_2 is the event of choosing which k of the n rooms will be occupied and E_1 is the event of arranging the k guests in the rooms to be occupied. The event E_1 can happen in $e_1 = k!$ ways since there are $k!$ different orders in which the guests can be assigned to occupied rooms. Thus event E_2, choosing k of n rooms, can occur in

$$e_2 = \frac{e}{e_1} = \frac{n!}{k!(n-k)!}$$

ways. In our example of 10 hotel rooms and 4 guests, the rooms to be occupied can be chosen in

$$\frac{10!}{4!(10-4)!} = \frac{10!}{6!}/4! = \frac{5040}{24} = 210$$

ways.

The value $\frac{n!}{k!(n-k)!}$ is so important and occurs so often in the solution of combinatorial problems (and hence in the analysis of algorithms for sorting, searching, and merging) that the shorthand notation

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

is almost always used. The symbol "$\binom{n}{k}$" is read "n choose k," since its value counts the number of ways to choose k items from a set of n items; it is also sometimes written $C(n,k)$ meaning the number of combinations of n elements, k taken at a time. In the remainder of this section we examine a few of the remarkable properties of the values $\binom{n}{k}$ and some applications. For reasons that will become clear in a few pages, the values $\binom{n}{k}$ are called *binomial coefficients.*

2 Myrioramas

A *myriorama* is a set of illustrated cards that can be arranged to form different pictures; they date from early nineteenth century France. The term "myriorama" was coined by John Clark of London who, in 1824, designed a set of cards called an "Italian landscape" of 24 cards, 18 of which are shown in Figure 1. The cards are cleverly designed so that they can be aligned next to each other in any order to form a coherent image. Given a myriorama of n cards, how many images can be formed? If k cards are used we can choose the cards in $\binom{n}{k}$ ways and then arrange them in $k!$ orders giving a total of $\binom{n}{k}k!$ pictures by the rule of product. But we can use any number of cards, so the rule of sum tells us that a total of

$$\sum_{k=1}^{n} \binom{n}{k} k!$$

pictures can be formed. We have

$$\sum_{k=1}^{n} \binom{n}{k} k! = \sum_{k=1}^{n} \frac{n!}{(n-k)!} = n! \sum_{k=0}^{n-1} \frac{1}{k!} \approx en!$$

using Taylor series. The 24 cards of "Italian landscape" can thus form 1686553615927922354187744 scenes of one or more cards.[1]

[1]One septillion, six hundred eighty six sextillion, five hundred fifty three quintillion, six hundred fifteen quadrillion, nine hundred twenty seven trillion, nine hundred twenty two billion, three hundred fifty four million, one hundred eighty seven thousand, seven hundred forty four.

Figure 1: Eighteen of the 24 cards of John Clark's "Italian landscape."

Exercise Some myrioramas were printed on both sides of the cards, so each card could be used with either side facing up. In such a myriorama of n cards, how many pictures can be formed?

3 Pascal's Triangle

Binomial coefficients hide some very beautiful and powerful properties behind their simple form. We will now study some of these properties in the form of various identities.

The algebraic formula

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

is symmetric in k and $n-k$, suggesting the identity

$$\binom{n}{k} = \binom{n}{n-k}. \tag{2}$$

This identity has a simple *combinatorial interpretation*: To choose a k-element subset of a set with n elements we can either choose the k elements of the subset, or we can choose the $n-k$ elements not in the subset. In this case, an algebraic proof of (2) can be easily realized by substituting $n-k$ for k in the definition of $\binom{n}{k}$ and working through the arithmetic to see that we get the same expression; however, combinatorial interpretation of formulas is a basic technique that has wide application and is generally easier to realize than its algebraic counterpart; we will examine a number of instances of it in this section.

Viewing $\binom{n}{k}$ as the number of ways to choose k of n different objects, we can use the rule of sum to calculate a different powerful identity

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}. \tag{3}$$

To see that this identity is correct, let us call the n different objects be $O_1, O_2, \ldots, O_n$. We split the choice of k of these objects into two separate cases:

1. The object O_1 is one of the chosen k objects.
2. The object O_1 is *not* one of the chosen k objects.

In the first case, there remain $k-1$ objects to be chosen from among the $n-1$ objects $O_2, O_3, \ldots, O_n$; we know that this event can occur in $\binom{n-1}{k-1}$ different ways. In the second case, all k of the objects must be chosen from the remaining $n-1$ objects $O_2, O_3, \ldots, O_n$; this event can occur in $\binom{n-1}{k}$ different ways. Applying the rule of sum then proves the identity.

Equation (3) is, perhaps, the single most important identity satisfied by the binomial coefficients. We could also have proven it algebraically from definitions:

$$\begin{aligned}
\binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)!}{(k-1)![(n-1)-(k-1)]!} + \frac{(n-1)!}{k![(n-1)-k]!} \\
&= \frac{(n-1)!}{(k-1)!(n-1-k)!}\left[\frac{1}{n-k} + \frac{1}{k}\right] \\
&= \frac{(n-1)!}{(k-1)!(n-1-k)!}\left[\frac{n}{(n-k)k}\right] \\
&= \frac{n!}{k!(n-k)!}
\end{aligned}$$

$$= \binom{n}{k}.$$

Equation (3) also allows us to calculate the value of $\binom{n}{k}$ without having to compute large factorials. Specifically, we calculate all the values of $\binom{n}{k}$ (for $k = 0, 1, 2, \ldots, n$) iteratively from the values of $\binom{n-1}{k}$ (for $k = 0, 1, 2, \ldots, n-1$). Such calculation can be organized into a visually pleasing structure known as Pascal's triangle:

							(k=0)					
($n=0$)						1		(k=1)				
($n=1$)					1		1		(k=2)			
($n=2$)				1		2		1		(k=3)		
($n=3$)			1		3		3		1		(k=4)	
($n=4$)		1		4		6		4		1		(k=5)
($n=5$)	1		5		10		10		5		1	

Pascal's triangle is organized into rows that correspond to different values of n and right-to-left diagonals corresponding to different values of k. Thus, we can read off $\binom{4}{2} = 6$ which is in the $(n = 4)$ row and the $(k = 2)$ diagonal. Notice that the entry $\binom{0}{0} = 1$ has been added to the top of the triangle for completeness. We see that in solving the Abracadabra problem we were actually computing binomial coefficients.

Many fascinating identities on the binomial coefficients can be found by examining Pascal's triangle. For example, we see that if we sum across the nth row of the triangle, we get 2^n. Specifically, we can see this for the first few rows: $1 = 2^0$, $1 + 1 = 2^1$, $1 + 2 + 1 = 2^2$, $1 + 3 + 3 + 1 = 2^3$, and so on. This observation can be proved in several ways. An *algebraic proof* notices that each element in the $n-1$st row is used twice in computing the elements of the nth row (we can think of imaginary rows of zeroes along the sides of the triangle, making this statement true for the ones also). In other words, the total across the nth row will be twice the total across the $n-1$st row. Since the zeroth row has sum 2^0, the result follows.

A combinatorial argument is also possible. We want to prove that

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n. \tag{4}$$

We may consider the kth term on the left hand side to be the number of ways a subset of size k can be chosen from a set of size n. We may apply the rule of sum to note that the terms on the left hand side add to give all possible ways of picking a subset from a set of size n. We have shown earlier, by the rule of product, that there are 2^n different ways of picking a subset from a set of size n, and so the identity is proved.

Exercise Prove (4) by induction.

Another identity can be observed by summing along any diagonal from *upper left to lower to right*:

$$\sum_{i=0}^{k} \binom{n+i}{i} = \binom{n}{0} + \binom{n+1}{1} + \cdots + \binom{n+k}{k} = \binom{n+k+1}{k}; \tag{5}$$

for example, 1+3+6 = 10, 1+6+21+56+126 = 210, and so on. Again, two proofs are possible. An algebraic proof of equation (5) follows by mathematical induction from equation (3):

Base Case $k = 0$. Equation (5) is true from the definition that $\binom{n}{0} = \binom{n+1}{0} = 1$. **Inductive Hypothesis** Assume that equation (5) is true for k. **To show:** Equation (5) is true for $k + 1$:

$$\begin{aligned}
\sum_{i=0}^{k+1} \binom{n+i}{i} &= \sum_{i=0}^{k} \binom{n+i}{i} + \binom{n+k+1}{k+1}; \\
&= \binom{n+k+1}{k} + \binom{n+k+1}{k+1} \qquad \text{(by induction)} \\
&= \binom{n+k+2}{k+1} \qquad \text{[by equation (3)].}
\end{aligned}$$

Thus, the induction is complete.

The combinatorial proof of (5) observes that the right hand side of equation (5) is the number of ways to choose k of $n + k + 1$ distinct objects. Let those objects be O_1, O_2, $\ldots$, O_{n+k+1}. If we insist that object O_1 *not* be chosen, then all k elements must be chosen from the $n + k$ elements O_2, O_3, $\ldots$, O_{n+k+1}; this can be done in $\binom{n+k}{k}$ ways. On the other hand, if we insist that O_1 be chosen and O_2 *not* be chosen then the remaining $k - 1$ elements (aside from O_1) must be chosen from $O_3, O_4, \ldots, O_{n+k+1}$; this can be done in $\binom{n+k-1}{k-1}$ ways. If we insist that both O_1 and O_2 be chosen, but not O_3, this can be done in $\binom{n+k-2}{k-2}$ ways. We finally arrive at the situation of insisting that O_1, O_2, $\ldots$, O_k all must be chosen; this can be done in $\binom{n}{0}$ ways. (Notice that we stop here, since the first element not chosen can not appear after O_{k+1}, in which case at least $k + 1$ elements are chosen). Equation (5) now follows from the rule of sum. Note that to make this combinatorial proof rigorous, we would either have to formulate our left-hand side terms more carefully, or use induction.

To get Equation (5) we summed along diagonals from left to right. Using the left-right symmetry of Pascal's triangle that comes (algebraically) from equation (2) (i.e. we can flip the triangle around its middle) we see that if instead we sum along diagonals from right to left we get a similar identity, namely

$$\sum_{i=k}^{n} \binom{i}{k} = \binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}. \tag{6}$$

This identity can be proved algebraically by iterating equation (3). It can also be proven combinatorially; but most easily it can be proven by using equation (2) to transform each term of equation (5). It is convenient to define $\binom{i}{k} = 0$ for $k > i$ (i.e. there is *no* way to choose k distinct elements from a smaller set of i elements) and to extend the limits of summation in equation (6) to run from $i = 0$ to $i = n$. We obtain

$$\sum_{i=0}^{n} \binom{i}{k} = \binom{0}{k} + \binom{1}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1} \tag{7}$$

Exercise Prove (7) by induction. Prove it by a combinatorial argument.

The special case $k = 1$ of equation (7) is

$$
\begin{aligned}
\binom{0}{1} + \binom{1}{1} + \binom{2}{1} + \cdots + \binom{n}{1} &= 0 + 1 + 2 + \cdots + n \\
\sum_{k=1}^{n} k &= \binom{n+1}{2} = \frac{n(n+1)}{2}.
\end{aligned}
$$

In a similar fashion, we can write

$$
\begin{aligned}
i^2 &= 2\binom{i}{2} + \binom{i}{1} \\
\sum_{i=0}^{n} i^2 &= \sum_{i=0}^{n} \left[2\binom{i}{2} + \binom{i}{1} \right] \\
&= 2\sum_{i=0}^{n} \binom{i}{2} + \sum_{i=0}^{n} \binom{i}{1}
\end{aligned}
$$

and use equation (7) to get:

$$
\begin{aligned}
\sum_{i=0}^{n} i^2 &= 2\binom{n+1}{3} + \binom{n+1}{2} \\
&= \frac{n(2n+1)(n+1)}{6}
\end{aligned}
$$

Exercise Find the sum $\sum_{i=0}^{n} i^3$ using this technique.

4 An observation on Pascal's triangle

$$
\begin{array}{lcccccccccccc}
 & & & & & & (k{=}0) & & & & & & \\
(n=0) & & & & & & 1 & & (k{=}1) & & & & \\
(n=1) & & & & & 1 & & 1 & & (k{=}2) & & & \\
(n=2) & & & & 1 & & 2 & & 1 & & (k{=}3) & & \\
(n=3) & & & 1 & & 3 & & 3 & & 1 & & (k{=}4) & \\
(n=4) & & 1 & & 4 & & 6 & & 4 & & 1 & & (k{=}5) \\
(n=5) & 1 & & 5 & & 10 & & 10 & & 5 & & 1 &
\end{array}
$$

By adding the numbers along diagonals as indicated by the above figure, we find the relation between binomial coefficients and Fibonacci numbers, $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$, $n > 1$:

$$\binom{0}{0} = 1 = F_1$$

$$\binom{1}{1} = 1 = F_2$$

$$\binom{2}{0} + \binom{1}{1} = 2 = F_3$$

$$\binom{3}{0} + \binom{2}{1} = 3 = F_4$$

$$\binom{4}{0} + \binom{3}{1} + \binom{2}{2} = 5 = F_5$$

$$\binom{5}{0} + \binom{4}{1} + \binom{3}{2} = 8 = F_6$$

$$\cdots$$

Generally, we have

$$\sum_{k \geq 0} \binom{n-k}{k} = F_{n+1}$$

Exercise Prove this by induction.

Exercise Prove this by a combinatorial argument.