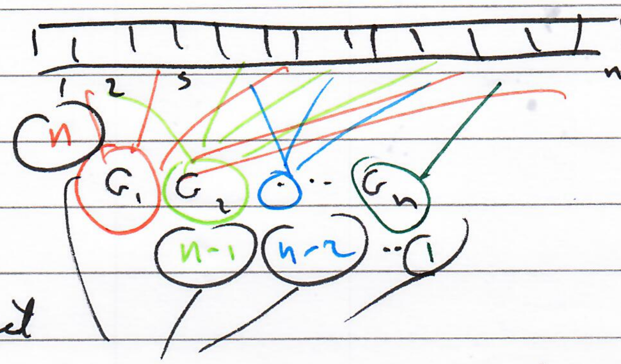
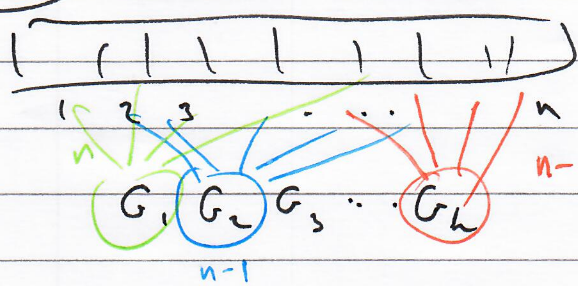
Permutations / $n!$



Rule of Product

$$n(n-1)(n-2)\cdots 1 = n!$$



$n-k+1 = n-(k-1)$

$$n(n-1)(n-2)\cdots(n-k+1) = \boxed{\frac{n!}{(n-k)!}} = P_{n,k}$$

= # of permutations of n items taken k at a time

$$E = E_1 \wedge E_2$$

$$|E| = |E_1|\,|E_2| \Rightarrow |E_2| = \frac{|E|}{|E_1|}$$

How many ways can guests be assigned to rooms where guests are indistinguishable?

$$\frac{n!}{(n-k)!} \quad E = E_2 \wedge \underbrace{\text{shuffling } k \text{ guests}}_{k!}$$

E: assign rooms to k dist guests

E_2: assign INDISTING. guests

$$\boxed{\frac{n!}{(n-k)!\,k!}}$$

BINOMIAL COEF

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!} = \text{\# of ways to select } k \text{ out of } n \text{ items}$$

"n choose k"

choose k cards —— $\binom{n}{k}$
select an order —— $k!$
$\approx en!$

rule of product $\Longrightarrow$ $\binom{n}{k}k!$ lands ways of k cards

$$\sum_{k=0}^{n} \binom{n}{k} k!$$

$$= \sum_{k=0}^{n} \frac{n!}{\cancel{k!}(n-k)!}\cancel{k!}$$

$$= \sum_{k=0}^{n} \frac{n!}{(n-k)!}$$

$n, n-1, n-2, \dots 0$

$$= n! \boxed{\sum_{k=0}^{n} \frac{1}{k!}}$$

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$$

$\approx e$

Binomial coef

$$a_i x^i$$

$$(1+x)^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

what's a_i ?

$$\underbrace{(1+x)(1+x)(1+x)\cdots(1+x)}_{n \text{ Times}}$$

$$\left.\begin{array}{l} i \ x\text{'s} \\ (n-i) \ 1\text{'s} \end{array}\right\} x^i$$

$$a_i = \binom{n}{i}$$

Binomial Thm

$$(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i \qquad x=1$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$\parallel$$

$$\binom{n}{n-k} = \frac{n!}{(n-k)!\,(n-(n-k))!}$$

$$= \frac{n!}{(n-k)!\,k!}$$

$$\boxed{\binom{n}{k}} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

$$\frac{n\,\cancel{n!}}{\cancel{k!}(n-k)!} \stackrel{?}{=} \underbrace{\frac{\cancel{(n-1)!}\,k}{\cancel{(k-1)!}\,(n-1-(k-1))!}} + \frac{\cancel{(n-1)!}}{\cancel{k!}\,(n-1-k)!}$$

choose k of n items

<u>item 1</u>

$$\binom{n-1}{k-1} + \binom{n-1}{k}$$

$$\boxed{\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}}$$

$$\binom{n+k+1}{k} = \binom{n}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \cdots + \binom{n+k}{k}$$

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$$= \binom{n}{k}.$$

Equation (3) also allows us to calculate the value of $\binom{n}{k}$ without having to compute large factorials. Specifically, we calculate all the values of $\binom{n}{k}$ (for $k = 0, 1, 2, \ldots, n$) iteratively from the values of $\binom{n-1}{k}$ (for $k = 0, 1, 2, \ldots, n-1$). Such calculation can be organized into a visually pleasing structure known as Pascal's triangle:

PASCAL'S TRIANGLE

							$(k{=}0)$					
$(n = 0)$ 1						1		$(k{=}1)$				
$(n = 1)$ 2					1		1		$(k{=}2)$			
$(n = 2)$ 4				1		2		1		$(k{=}3)$		
$(n = 3)$ 8			1		3		3		1		$(k{=}4)$	
$(n = 4)$ 16		1		$\boxed{4}$		6		4		1		$(k{=}5)$
$(n = 5)$ 32	1		5		10		10		5		1	

$\sum_{k=0}^{n} \binom{n}{k}$ row n

ALL SUBSETS $= 2^n$

Pascal's triangle is organized into rows that correspond to different values of n and right-to-left diagonals corresponding to different values of k. Thus, we can read off $\binom{4}{2} = 6$ which is in the $(n = 4)$ row and the $(k = 2)$ diagonal. Notice that the entry $\binom{0}{0} = 1$ has been added to the top of the triangle for completeness. We see that in solving the Abracadabra problem we were actually computing binomial coefficients.

Many fascinating identities on the binomial coefficients can be found by examining Pascal's triangle. For example, we see that if we sum across the nth row of the triangle, we get 2^n. Specifically, we can see this for the first few rows: $1 = 2^0$, $1 + 1 = 2^1$, $1 + 2 + 1 = 2^2$, $1 + 3 + 3 + 1 = 2^3$, and so on. This observation can be proved in several ways. An *algebraic proof* notices that each element in the $n-1$st row is used twice in computing the elements of the nth row (we can think of imaginary rows of zeroes along the sides of the triangle, making this statement true for the ones also). In other words, the total across the nth row will be twice the total across the $n-1$st row. Since the zeroth row has sum 2^0, the result follows.

A combinatorial argument is also possible. We want to prove that

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n. \tag{4}$$

We may consider the kth term on the left hand side to be the number of ways a subset of size k can be chosen from a set of size n. We may apply the rule of sum to note that the terms on the left hand side add to give all possible ways of picking a subset from a set of size n. We have shown earlier, by the rule of product, that there are 2^n different ways of picking a subset from a set of size n, and so the identity is proved.

Exercise Prove (4) by induction.

The special case $k = 1$ of equation (7) is

$$\binom{0}{1} + \binom{1}{1} + \binom{2}{1} + \cdots + \binom{n}{1} = 0 + 1 + 2 + \cdots + n$$

$$\sum_{k=1}^{n} k = \binom{n+1}{2} = \frac{n(n+1)}{2}.$$

In a similar fashion, we can write

$$i^2 = 2\binom{i}{2} + \binom{i}{1}$$

$$\sum_{i=0}^{n} i^2 = \sum_{i=0}^{n}\left[2\binom{i}{2} + \binom{i}{1}\right]$$

$$= 2\sum_{i=0}^{n}\binom{i}{2} + \sum_{i=0}^{n}\binom{i}{1}$$

and use equation (7) to get:

$$\sum_{i=0}^{n} i^2 = 2\binom{n+1}{3} + \binom{n+1}{2}$$

$$= \frac{n(2n+1)(n+1)}{6}$$

Exercise Find the sum $\sum_{i=0}^{n} i^3$ using this technique.

4 An observation on Pascal's triangle

									(k=0)				
$(n=0)$					1		1			$(k{=}1)$			
$(n=1)$					1	1		1			$(k{=}2)$		
$(n=2)$				2	1		2		1			$(k{=}3)$	
$(n=3)$			3	1		3		3		1			$(k{=}4)$
$(n=4)$		5	1		4		6		4		1		$(k{=}5)$
$(n=5)$		1		5		10		10		5		1	
	8												