

Lecture 5: January 30, 2019

CS 330 Discrete Structures
Spring Semester, 2019

1 How big is $n!$?

In the preceding material we have seen that $n!$ occurs in many contexts when counting arrangements of elements. Furthermore, in subsequent sections we will find that it figures centrally in most of the counting problems that we pursue. It is reasonable, then, to ask about the behavior of $n!$ as a function n . Specifically, how quickly does $n!$ grow as n becomes large? The answer to this question will give us asymptotic information about how the number of configurations grows in various cases; we will use such information in the next section, for example, to establish a benchmark for the performance of sorting algorithms.

Brief computation reveals that $n!$ grows fast as n increases:

n	0	1	2	3	4	5	6	7	8	9	10
$n!$	1	1	2	6	24	120	720	5,040	40,320	362,880	3,628,800

For example, if a code-cracking program requires the examination of each of the permutations of n items, the program will (most likely) be practical for $n \leq 7$, but will start to get expensive for $n = 8, 9, 10$ and become impossible to use for larger values of n , again, because $n!$ grows fast.

Exactly how does the growth rate of $n!$ compare with other functions of n ? In the remainder of this section we will prove *Stirling's approximation* that will help us get a handle on this growth rate:

$$n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n, \quad (1)$$

where

$$e = \lim_{k \rightarrow \infty} \left(1 + \frac{1}{k}\right)^k \approx 2.7182824590452354$$

is the base of the natural logarithms, and

$$\pi \approx 3.1415926535897932385$$

is the ratio of the circumference of a circle to its diameter, as usual.

Actually, we will establish only that $n!$ grows proportionately to $\sqrt{n}(n/e)^n$, without proving that the constant of proportionality is $\sqrt{2\pi}$, because proving things with $\sqrt{2\pi}$ in them usually involves some complex analysis which is beyond the scope of this course. As an interesting side note, one may also ask how big are the Harmonic numbers H_n that we studied in previous lectures.

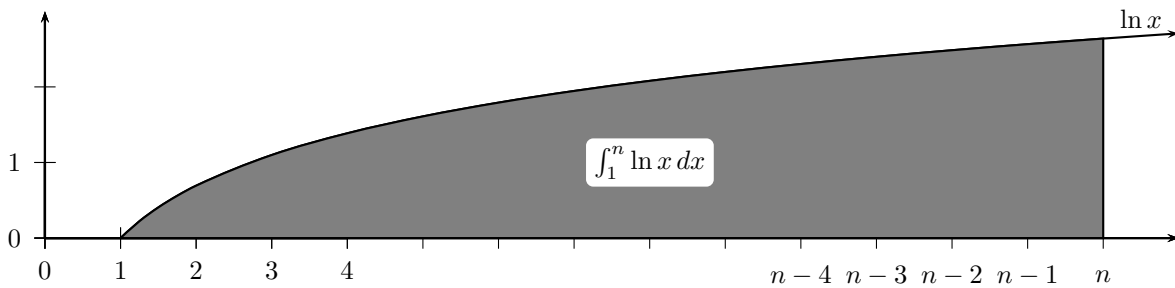
The technique we use to compute the approximate value of $n!$ is to transform the product to a sum by taking logarithms and then to estimate the size of the sum by comparing it to an integral. Taking logarithms gives

$$\begin{aligned} \ln n! &= \ln(1 \times 2 \times 3 \times \cdots \times n) \\ &= \ln 1 + \ln 2 + \cdots + \ln n \\ &= \sum_{k=1}^n \ln k. \end{aligned}$$

Approximating the sum by an integral gives

$$\ln n! \approx \int_1^n \ln x \, dx$$

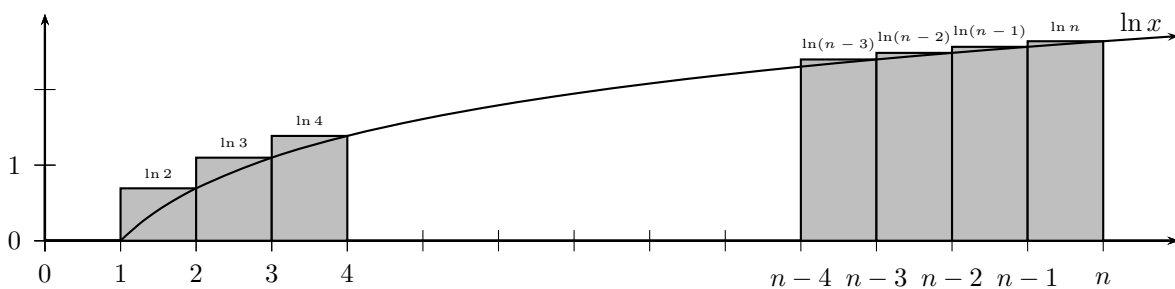
Consider $\int_1^n \ln x \, dx$; it can be taken to mean *the area under the curve $f(x) = \ln x$ from $x = 1$ to n* , or graphically:



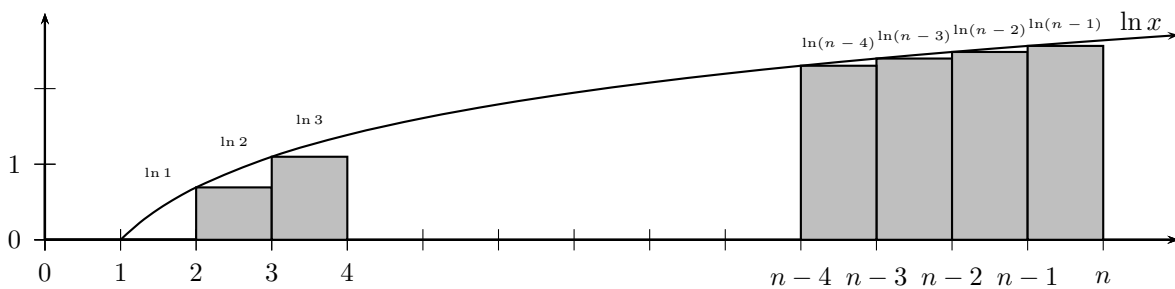
But if we split the area up into many rectangles, we notice that:

$$\int_1^n \ln x \, dx \leq \sum_{i=2}^n \ln i \quad (2)$$

Graphically, we see that the area under $f(x) = \ln x$ is *always* less than or equal to the area of the rectangles:



If we were to use rectangles below the curve, however, we see:



Which leads to the following inequality:

$$\int_1^n \ln x \, dx \geq \sum_{i=1}^{n-1} \ln i$$

Combining the two results we have:

$$\ln n + \int_1^n \ln x \, dx \geq \sum_{i=1}^n \ln i \geq \ln 1 + \int_1^n \ln x \, dx$$

Which can be reduced to:

$$\ln n + [x \ln x - x]_{x=1}^n \geq \ln n! \geq [x \ln x - x]_{x=1}^n,$$

or

$$n \ln n - n + \ln n - 1 \geq \ln n! \geq n \ln n - n + 1,$$

so that

$$\ln n! = n \ln n - n + O(\log n).$$

an error margin of $O(\log n)$ in our approximation. However, this is much worse than Stirling's approximation which narrowed the margin to $O(1)$:

$$\begin{aligned} n! &\longrightarrow \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \\ \ln n! &\longrightarrow \frac{\ln 2\pi}{2} + \frac{\ln n}{2} + n \ln \left(\frac{n}{e}\right) \\ &= \frac{\ln 2\pi}{2} + \frac{\ln n}{2} + n(\ln n - 1) \\ &= \frac{\ln n}{2} + n \ln n - n + O(1) \end{aligned}$$

Thus, the approximation we showed with our rectangles was rather crude. We can, however, salvage our idea with a minor twist; if we use trapezoids to approximate the area rather than rectangles, we can get a much better approximation:

$$\ln n! = \left(n + \frac{1}{2}\right) \ln n - n + O(1) \tag{3}$$

Exercise Use (??) to determine the growth rate of $\log_2 \frac{(2n)!}{n!n!}$.

2 How big is H_n

Let's try to approximate harmonic numbers using rectangles. Similar to the case of $n!$, we have

$$\int_1^n \frac{1}{x} \, dx \leq H_n - \frac{1}{n}$$

and

$$\int_1^n \frac{1}{x} \, dx \geq H_n - 1$$

Therefore we get

$$\int_1^n \frac{1}{x} dx + \frac{1}{n} \leq H_n \leq \int_1^n \frac{1}{x} dx + 1$$

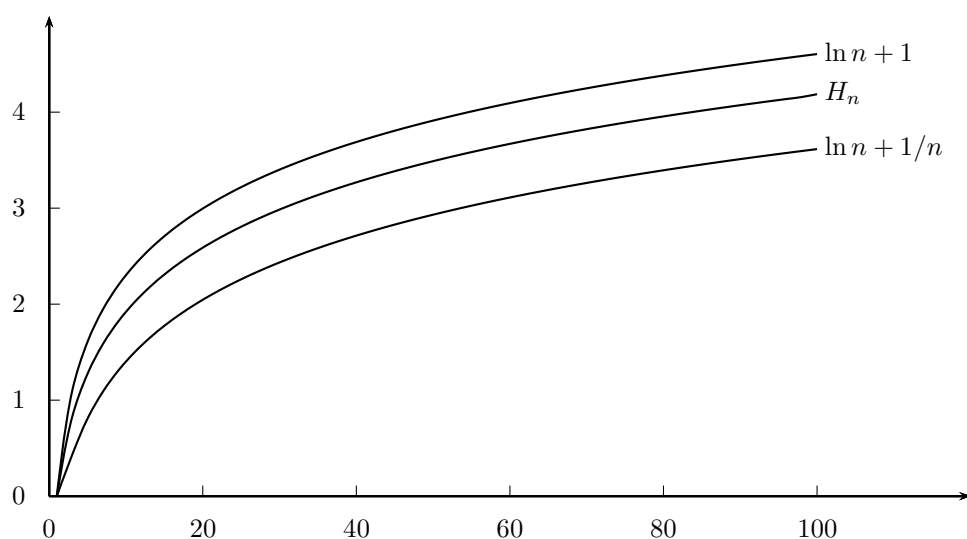
We know that

$$\int_1^n \frac{1}{x} dx = \ln n$$

So

$$\ln n + \frac{1}{n} \leq H_n \leq \ln n + 1$$

This is a good approximation as we can see:



If we use trapezoids instead of rectangles, it would be even better.

Exercise Use the ideas above to determine the growth rate of $\sum_{i=1}^n i^k$.

What we have done is a very simple version of “Euler Summation”. For an in-depth treatment, see section 1.2.11.2 of Donald Knuth’s *Art of Computer Programming*.