Illinois Institute of Technology
Department of Computer Science

Lecture 1: January 14, 2019

CS 330 Discrete Structures
Fall Semester, 2019

1 Why Study Mathematics and Theoretical Computer Science?

This question is best answered by an example. Consider yourself an employee of a factory. Your boss comes up to you with the following dilemma: “Because of the tough economic times, we need to operate at peak efficiency. We need you to determine what our peak efficiency is at the lowest cost possible.”

After further discussion with your employer, you determine the following facts:

- The function relating efficiency to the number of employees is **unimodal**. That is, efficiency will increase to a maximum, then decrease as the number of employees increases.
- The only way to determine efficiency is very expensive—perhaps building a pilot plant.
- Your boss will be unhappy with any solution that is more expensive than absolutely necessary.

There is an obvious, stupid solution to this problem: you could build a plant for one employee then find the efficiency of that plant, then build a plant with two employees and find its efficiency, then build a plant for three employees, and so on, until the efficiency goes down. You will then have the peak.

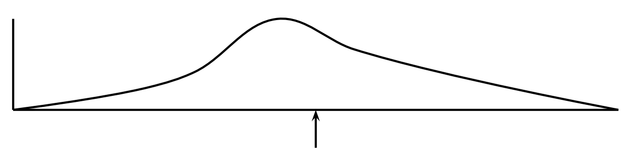
2 Finding a Better Solution

Unfortunately, this is not a very efficient solution. It runs in **linear** time—that is, time proportional to the number of employees. (Consider that the peak may be at a few thousand employees. Your boss won’t be happy building thousands of pilot plants if he can do better.) But, as you are familiar with the **binary search algorithm** and that it is a faster way to search for items. You may even remember that this process is *logarithmic* is nature.

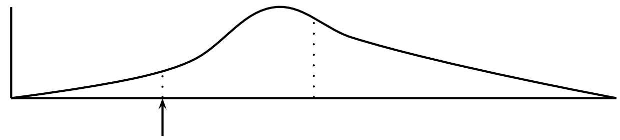
In binary search, we probe the center point and can immediately discard either the first half or the second half of the search space. This problem does not exactly match those required for a binary search, but we can modify it somewhat.

For example, we consider the range of employee numbers in our experiments to be between 1 and a zillion: the total number of employees will not exceed the world population. Then we begin probing certain points to collect information.

- Probe the center point ($\frac{1}{2}$)

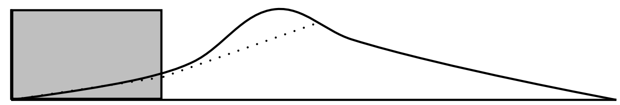


- Probe the center of one resulting segment ($\frac{1}{4}$)

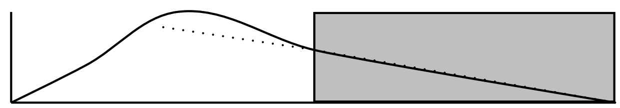


- Determine if the efficiency is rising or falling within the segments and discard inappropiate segment

  If the line between $\frac{1}{4}$ and $\frac{1}{2}$ is ascending, as shown in the figure below, the lower $\frac{1}{4}$ is discarded because the curve is rising from 0 to $\frac{1}{4}$, and from $\frac{1}{4}$ to $\frac{1}{2}$. The peak can not exist in the lower $\frac{1}{4}$ segment.



  If the line between $\frac{1}{4}$ and $\frac{1}{2}$ is descending, as shown in the figure below, the upper $\frac{1}{2}$ is discarded (why?).



- Start again (recursively) on the new, shorter segment

  In case 1 (curve is ascending), the shorter segment is the upper $\frac{3}{4}$ and the probe we have at $\frac{1}{2}$ is not at the center of the new segment. Therefore we still need to make two probes, at the center and the $\frac{1}{4}$ point of the new segment respectively, to apply our approach.

  In case 2 (curve is descending), the shorter segment is the lower $\frac{1}{2}$ and the probe we have at $\frac{1}{4}$ is at the center of the new segment. We can reuse this probe and only make one more probe at the $\frac{1}{4}$ point of the new segment.

To analyze the cost of this approach, let us consider the worst case. That is, for every recursion, we need to make two probes and can only remove $\frac{1}{4}$ of the current segment. Notice that the size of the resulting segment is $\frac{3}{4}$ the size of the original. With this piece of information, we can create a **recurrence relation** to describe this process.

Consider the cost of determining the peak efficiency of n employees. If you define the function cost(n) as the dollars required, after each step you can make 2 probe and get a segment $\frac{1}{4}$ shorter. This can be written as:

$$\text{cost}(N) = \text{cost}\left(\frac{3}{4}N\right) + 2$$

There is also an initial step, in which you probed in the center. This is the base case:

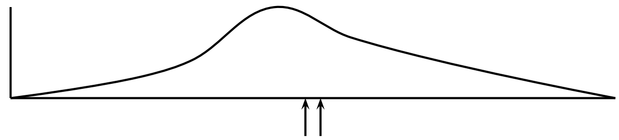
$$\text{cost}(1) = 1$$

From these facts, and through methods learned later in the course, you will show that:

$$\text{cost}(N) \approx 2\log_{\frac{4}{3}} N \ \approx 4.8188 \log_2 N$$

This approach actually works better since there is a possibility that $\frac{1}{2}$ will be removed with one probe.

We can use a better approach (suggested during the lecture by a former student!) which makes two probes near the center of the segment, hereby removing almost $\frac{1}{2}$ of the segment everytime.



The recurrence relation for this approach is:

$$\text{cost}(N) \approx \text{cost}\left(\frac{1}{2}N\right) + 2$$

From this relation we get:

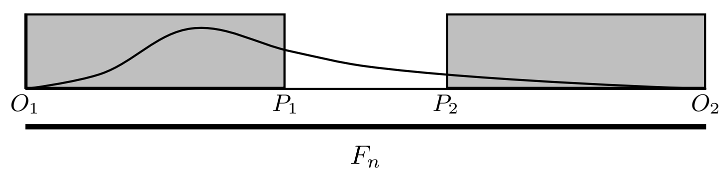
$$\text{cost}(N) \approx 2\log_2 N$$

We can see this is better than the previous method. But is it the best one? The answer is no, as we will see soon.

3 Finding the Best Solution

The question that you should now ask is, "Where is the best place to make my probes?" The best solution will try to make as few probes as possible. If during every step, we can use a probe made previously, we only need to make one more probe. This requires that we pick the probe points very carefully so that if a probe point falls into the shortened segment, it is still at a "good" position.

Suppose the interval to be searched is O_1O_2. First we make two probes, at points P_1 and P_2:



For the search to continue symmetrically with the same cost whether it goes left or right, we need

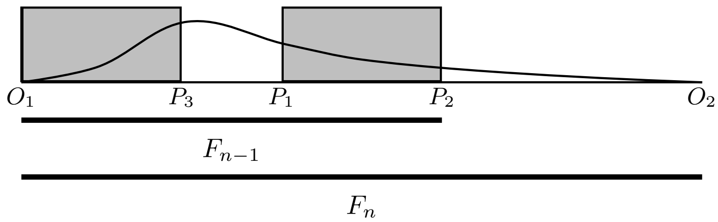
$$\overline{O_1P_2} = \overline{P_1O_2}.$$

But $\overline{P_2O_2} = \overline{O_1O_2} - \overline{O_1P_2}$ and $\overline{O_1P_1} = \overline{O_1O_2} - \overline{P_1O_2}$ which implies

$$\overline{O_1P_1} = \overline{P_2O_2} \tag{1}$$

We use the notation: F_n to represent the problem size currently being worked on. F_{n-1} is the next size to be worked on and F_{n-2} is the below after that; we have that $F_n \geq F_{n-1} \geq F_{n-2}$.

Suppose that we keep the lower part of the segment. We hope that we need only another one probe at P_3 and can reuse P_1, as shown by the figure below:



We have:

$$F_n = O_1O_2$$

and

$$F_{n-1} = O_1P_2$$

so the search can continue recursively. By the same reasoning, and using equation (1), we must have

$$F_{n-2} = O_1P_1 = P_2O_2$$

for the next iteration to work correctly. Clearly,

$$F_n = O_1O_2 = O_1P_2 + P_2O_2$$

These last three equations give us:

$$F_n = F_{n-1} + F_{n-2}$$

This formula is used to generate the Fibonacci sequence of numbers $1, 1, 2, 3, 5, 8, 13, 21, 34 \cdots$, and as we will see in a few weeks, $F_n \approx \phi F_{n-1}$, where

$$\phi = \frac{1+\sqrt{5}}{2} \approx 1.6180$$

ϕ is the **golden ratio** (which, incidentally, bears an uncanny resemblance to the zip code of Urbana, Illinois).

This approach has the property

$$\text{cost}(F_n) = \text{cost}(F_{n-1}) + 1$$

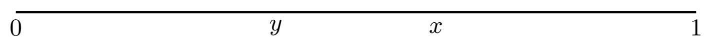
Letting $N = F_n$, we have the recursive relation

$$\text{cost}(N) = \text{cost}\left(\frac{N}{\phi}\right) + 1,$$

$$\text{cost}(N) = \log_\phi N \approx 1.4405 \log_2 N$$

This cost is **optimal**, that is, no probing strategy can do better.

Another way to analyze things is to scale the diagram to have unit length (that is, make $O_1O_2 = 1$), let $x = O_1P_2$, and $y = O_1P_1$. If we already have a probe at x (P_2), we must choose y (P_1) for the next probe so that (depending on the result) we can continue the search recursively in either O_1P_2 or in P_1O_2. The diagram is then



For the recursion to work in the interval $(0, x)$ with the (new) probe y we must have the same ratio for the location of the new point vis-à-vis the original probe x in the interval $(0, 1)$. Thus,

$$\frac{y}{x} = \frac{x}{1},$$

implying $y = x^2$. Similarly, on the other hand, to search recursively in the interval $(y, 1)$ we must have

$$\frac{1-x}{1-y} = \frac{x}{1},$$

or,

$$\frac{1-x}{1-x^2} = x,$$

which implies $x^2 + x - 1 = 0$, and hence

$$x = \frac{-1 \pm \sqrt{5}}{2}.$$

Because $x > 0$, the negative root is extraneous, so that

$$x = \frac{-1 + \sqrt{5}}{2} \approx 0.61801.$$

In searching an interval of length N, the recursion gives us:

$$\text{cost}(N) = \text{cost}\left(\frac{N}{\phi}\right) + 1,$$

as before.