$\langle a_i \rangle$

a_i is in the i^{th} position — starting at $i=0$

$= \langle a_0, a_1, a_2, \dots \rangle$

$\langle a_{i+2} \rangle = \langle a_2, a_3, a_4, \dots \rangle$ LEFT SHIFT BY 2 POSITIONS

$E\langle a_i \rangle = \langle a_{i+1} \rangle$

$E(E\langle a_i \rangle) = E^{(2)}\langle a_i \rangle = E^2 \langle a_i \rangle$

Mult by a constant $c\langle a_i \rangle = \langle c a_i \rangle$

$\langle 2^i \rangle = \langle 1, 2, 4, 8, 16, \dots \rangle$

$E\langle 2^i \rangle = \langle 2^{i+1} \rangle$

$\underbrace{E\langle 2^i \rangle - 2\langle 2^i \rangle}_{(E-2)\langle 2^i \rangle} = \langle 0 \rangle$

$(E-3)[(E-2)\langle a_i \rangle] = (E^2 - 5E + 6)\langle a_i \rangle$

$(E-a)[(E-b)]$

$= E^2 - (a+b)E + ab$

$(E-a)$ <u>ANNIHILATES</u> $\langle c\,a^i \rangle$

$(E-a)\langle a^i \rangle = E\langle a^i \rangle - a\langle a^i \rangle$

$= \langle a^{i+1} \rangle - \langle a^{i+1} \rangle$

$= \langle 0 \rangle$

$(E-b)$ $b \neq a$

DOES <u>NOT</u> ANNIHILATE $\langle a^i \rangle$

$$\underbrace{(E-a)\left[(E-b)}\langle\ \rangle\right]$$

ANNIHILATE
POWERS OF a and b

$F_0 = 0 \quad F_1 = 1$

$F_i = F_{i-1} + F_{i-2} \Longrightarrow F_{i+2} = F_{i+1} + F_i$

$E^2\langle F_i\rangle = E\langle F_i\rangle + 1\langle F_i\rangle$

$\boxed{(E^2 - E - 1)}\langle F_i\rangle = \langle 0\rangle$

$F_{i+2} - F_{i+1} - F_i \Longrightarrow = 0$

$x^2 - x - 1 = 0$

$\boxed{\left(E - \frac{1+\sqrt{5}}{2}\right)\left(E - \frac{1-\sqrt{5}}{2}\right)}$

$i \to \infty$

$i \to \infty$

$\boxed{F_i = a\left(\frac{1+\sqrt{5}}{2}\right)^i + b\left(\left|\frac{1-\sqrt{5}}{2}\right|\right)^i}$

$0 \quad < 1$

$i = 0 \qquad 0 = a + b$

$i = 1 \qquad 1 = a\left(\frac{1+\sqrt{5}}{2}\right) + b\left(\frac{1-\sqrt{5}}{2}\right)$

$F_i = \Theta\left(\left(\frac{1+\sqrt{5}}{2}\right)^i\right)$

$(1.61801\ldots)^i$

$E-1$ ANNIHILATES $\langle a_i \rangle$ A CONSTANT SEQUENCE

$\langle a_{i+1} \rangle - \langle a_i \rangle = \langle 0 \rangle$

$\langle a_{i+1} - a_i \rangle = \langle 0 \rangle \Rightarrow a_{i+1} = a_i \quad \forall i \geq 0$

$(E-1)(E-1)$ ANNIHILATES $\langle ci + d \rangle$

$E^2 - 2E + 1$

$(E-1)\langle a_i \rangle = \langle c \rangle$

$\Rightarrow a_{i+1} - a_i = c$

$$
\begin{aligned}
a_{i+1} &= a_i + c \\
&= a_{i-1} + 2c \\
&= a_{i-2} + 3c \\
&\vdots \\
&= (i+1)c
\end{aligned}
$$

$\Rightarrow a_i = \boxed{c}\, i$ $\quad 1^i \quad 1^i$

$$\boxed{(E-1)^2 \text{ ANNIHATES } \langle ci + d \rangle}$$

WHAT ABOUT $(E-2)^2$ $\cancel{\langle ci \rangle}$

$\langle ci2^i + d2^i \rangle$

$\langle (ci+d)2^i \rangle$

$(E-2)(E-2)\langle (ci+d)2^i \rangle = (E-2)\left[(E-2)\langle ci2^i \rangle + (E-2)\langle d2^i \rangle\right.$

$c(\cancel{i}+1)2^{i+1} - 2c\cancel{i}2^i$ $\quad 0$

$c2^i$

$(E-c)^k$ ANNIHILATES (polynomial of degree $k-1$) c^i

Proof by induction on k

$$\boxed{\begin{aligned} r_0 &= 1 \\ r_1 &= 5 \\ r_2 &= 17 \end{aligned}}$$

$$\underbrace{r_{i+3} = 7r_{i+2} - 16r_{i+1} + 12r_i}\ + \cancel{3}\ 2^i$$

$$(E-2)\cancel{(E-1)^2}\underbrace{(E^3 - 7E^2 + 16E - 12)}\langle r_i\rangle = \cancel{\langle 0\rangle}\ \cancel{\langle c\rangle}\ \langle 2^i\rangle$$

$$x^3 - 7x^2 + 16x - 12 = (E-2)^3(E-3) \Rightarrow r_i =$$

$$r_i = (\cancel{ai+b}\ ai^2+bi+c)\,2^i + d\,3^i \Longrightarrow r_i = \Theta(3^i)$$

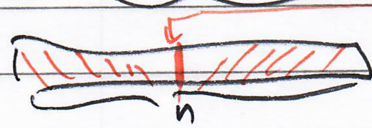
$$+ (di+e)\,1^i$$

$$a_n = (ri+s)^n + (-ri+s)^n$$

$$r^n(\sin\theta + i\cos\theta)^n$$

$$r^n(\sin n\theta + i\cos n\theta)$$

$$\cos n\theta + i\sin n\theta$$



$S(n) = 1 + S(n/2)$

HOMOGENEOUS PART

NON-HOMOGENEOUS PART

$$\underbrace{a_n = a_{n-1} + a_{n-2}} + 2^n$$

$$(E-2)(E^2 - E - 1)\langle a_i \rangle = \cancel{\langle 2^n \rangle}\ \langle 0 \rangle$$

$$\cancel{H_0 = 0}$$

NO $\cancel{H_n = H_{n-1} + 1/n}$ (HARMONIC NUMBERS)

$(E - c)$

$$H_n = H_{n-1} + e^n$$

$(E - e)$

$$\cancel{H_0 = 0}$$

$$\cancel{H_n = H_{n-1} + \ln n}$$

NO