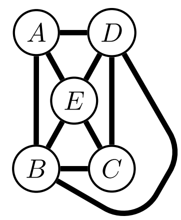
Illinois Institute of Technology
Department of Computer Science

Lecture 21: April 10, 2019

CS 330 Discrete Structures
Spring Semester, 2019

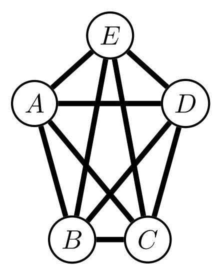
1 Planar graphs

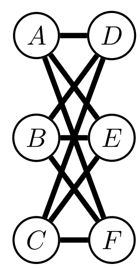
A **planar graph** is defined as a graph that can be drawn in the plane so that no edges cross. For example the graph on the right is planar, while there is no way to add the edge from (A, C) and still have it planar. Any graph of less than five nodes must be planar.



1.1 Kuratowski's theorem

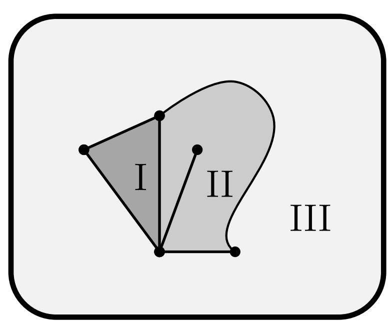
The fully connected graph with five vertices is called K_5 and is isomorphic to the graph on the near right. There is no way that this can be drawn as in the plane with no crossing edges — that is, it is a nonplanar graph. Another nonplanar graph, called $K_{3,3}$, the complete bipartite graph, is on the far right. As it turns out, **Kuratowski's theorem** states that a graph is nonplanar iff it contains a **homeomorphic image** of $K_{3,3}$ or K_5. The proof of this theorem is beyond the scope of the course.





1.2 Euler's formula

Notice that when we draw a planar graph, it divides the space (plane) into faces. These faces are regions delimited by edges. On the right are three faces. When edges cross, faces are not well-defined. Also notice that a cycle in a graph determines a face. If there are no cycles, there are no bounded faces (we consider the region outside any cycles to be an unbounded face). This example has five vertices, six edges, and three faces.



It turns out that there is a relationship between the number of vertices $|V|$, the number of edges $|E|$, and the number of faces $|F|$ in any planar graph. This result is:

Euler's formula: for any simple, connected planar graph $\quad |V| - |E| + |F| = 2$

This is proved by induction on the number of edges, $|E|$:

Base case: $|E| = 1$. A graph with no cycles has only one face. A graph with only one edge must have two vertices (since this is a connected simple graph), so $|V| - |E| + |F| = 2 - 1 + 1 = 2$.

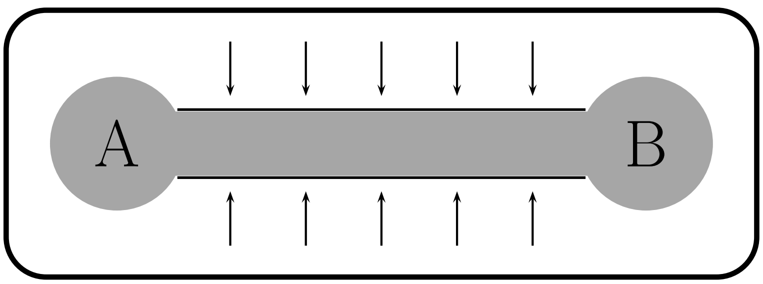
Inductive step: If $|E| > 1$ there are two cases: either the graph has a cycle or it doesn't.

In a graph without a cycle, there is at least one vertex of degree one. You can see this if you consider DFS; it will not stop until it reaches a vertex without any exits. That vertex cannot have back edges, as there are no cycles. If we remove this vertex, $|V|$ decreases by 1, $|E|$ decreases by 1 and $|F|$ remains unchanged, so $(|V| - 1) - (|E| - 1) + |F| = |V| - |E| + |F| = 2$ and Euler's formula still holds.

In a graph with a cycle, we remove one edge from the cycle. $|E|$ decreases by 1, $|F|$ decreases by 1 and $|V|$ remains unchanged so $|V| - (|E| - 1) + (|F| - 1) = |V| - |E| + |F| = 2$ and Euler's formula still holds.

Corollary: In a planar graph, $|E| \leq 3|V| - 6$.

Proof: Consider the "rims" of a blown-up edge:



Each edge has exactly two rims. At least three edges are required to make a face, so each face has at least three edge rims. We can then derive the inequality $2|E| \geq 3|F|$, which can be substituted into Euler's formula to give us the corollary:

$$|V| - |E| + |F| = 2,$$

so

$$-|V| + |E| + 2 = |F| \leq \frac{2}{3}|E|,$$

and the corollary follows.

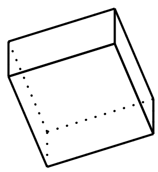
Consider K_5. There are five vertices and ten edges. Is $10 \leq 3(5) - 6$? No.

Consider $K_{3,3}$, There are six vertices and nine edges. Is $9 \leq 3(6) - 6$? Yes! Does this mean that $K_{3,3}$ is in fact planar? Not necessarily, since Euler's formula gives a condition that must hold for all planar graphs but may also hold for some nonplanar graphs. This discrepancy comes from the assumption that the minimum number of edge rims around a face is three. In a bipartite graph, each face has at least two vertices from each partition of the graph, so the inequality is $2|E| \geq 4|F|$, yielding the result $2|V| - 4 \geq |E|$ for a planar bipartite graphs. $K_{3,3}$ fails this test, as we expect.

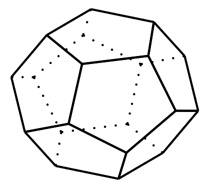
1.2.1 Platonic solids

Platonic solids are regular solids: every vertex has the same number of edges, and every face has the same number of edges. This was know to the ancient Greeks. There are only five Platonic solids:

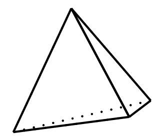
Cube



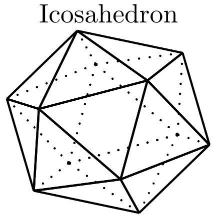
Dodecahedron



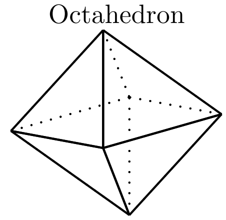
Tetrahedron



Icosahedron



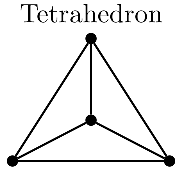
Octahedron



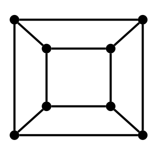
Let p be the number of edges that surround a face and q be the degree of each vertex.

Why are there only five Platonic solids? Consider another property of the planar solid: it can be represented as a planar graph. If you could put a hole in one face of the solid and stretch it until it was flat, you would end up with a planar graph where each vertex has a degree q. The number of regions would equal the number of faces. "Stretched out," the Platonic solids look like:

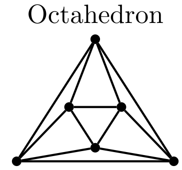
Tetrahedron



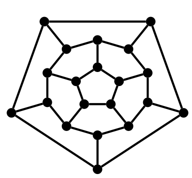
Cube



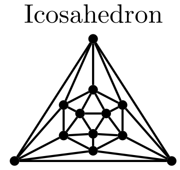
Octahedron



Dodecahedron



Icosahedron



Now we are going to count "tips of edges." There are two edge tips per edge, and there are q edge tips per vertex, so we know that

$$2|E| = q|V|$$
$$|V| = \frac{2}{q}|E|$$

Now consider the rims of the edges. There are (still) two rims per edge, and there are p rims per face.

$$2|E| = p|F|$$

$$|F| = \frac{2}{p}|E|$$

Since this is a planar graph, Euler's formula holds, so we have

$$\begin{aligned} |V| - |E| + |F| &= 2 \\ \frac{2}{q}|E| - |E| + \frac{2}{p}|E| &= 2 \\ |E|\left(\frac{2p + 2q - pq}{pq}\right) &= 2 \end{aligned}$$

We notice that $|E|$ is positive, 2 is positive and pq is positive. This implies that $2p + 2q - pq$ is positive. So:

$$\begin{aligned} 2p + 2q - pq &> 0 \\ -2p - 2q + pq &< 0 \\ 4 - 2p - 2q + pq &< 4 \\ (p-2)(q-2) &< 4 \end{aligned}$$

But we have some more constraints; p must be greater than 2, or we would not have a face, and q must also be greater than 2, or we could not create a 3-dimensional solid.

This leaves us with only 5 possible integer results for the inequality: $(1)(1) < 4$, $(1)(2) < 4$, $(2)(1) < 4$, $(1)(3) < 4$, and $(3)(1) < 4$, giving the five solutions: $p = 3$, $q = 3$ (tetrahedron) $p = 3$, $q = 4$ (octahedron) $p = 4$, $q = 3$ (cube) $p = 5$, $q = 3$ (dodecahedron) and $p = 3$, $q = 5$ (icosahedron).

1.2.2 Graph coloring

Consider a map of the Continental United States. The mapmaker would like to color the states on the map in such a way that no two adjacent states have the same color. With that in mind, he would like to minimize the number of colors he uses. What is the fewest number of colors he can use?

This can be easily transformed into a graph-theoretic problem: states are vertices and state borders are edges. Is there a maximum number of colors that will suffice to color all graphs?

Clearly three colors will not suffice: K_4 is a planar graph but it cannot be colored in fewer than four colors.

It is not difficult to prove the five-color theorem:

Theorem: Given a planar graph, five colors are sufficient to color the graph such that no two connected vertices share the same color.

We will use induction on the number of vertices.

Base case: When $|V| \leq 5$, we only have five vertices, so five colors are certainly sufficient.

Inductive step: When $|V| \geq 6$, there must be a vertex of degree ≤ 5. This follows from the corollary to Euler's formula: $|E| \leq 3|V| - 6$. If all vertices had degree ≥ 6 then by counting the number of edge tips, we have the inequality $2|E| \geq 6|V|$, or $|E| \geq 3|V|$, which contradicts the corollary to Euler's formula.

Let's examine the vertex with degree ≤ 5. There are two cases, that the degree is less than five or the degree is five. If the degree ≤ 4, remove that vertex from the graph, color the rest of the graph recursively (using the inductive hypothesis), and you will be able to color the graph with whatever color is not used by the four connected vertices. When the degree is 5, we know that there have to be two of those 5 vertices connected to the vertex that are not connected (otherwise there would be a K_5 subgraph and it would not be planar). We will cut out the two nonconnected vertices and the vertex with degree 5, replace them with one "mega-vertex," and color the graph recursively. We will then expand the vertex that we shrunk, giving the nonconnected vertices the color of the shrunken vertex, and then color the vertex with degree 5 with the remaining color not used by the five vertices connected to it, two of which have the same color.