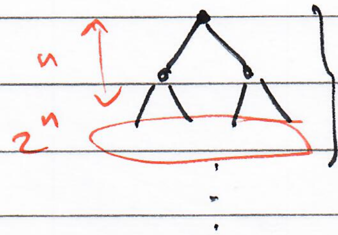
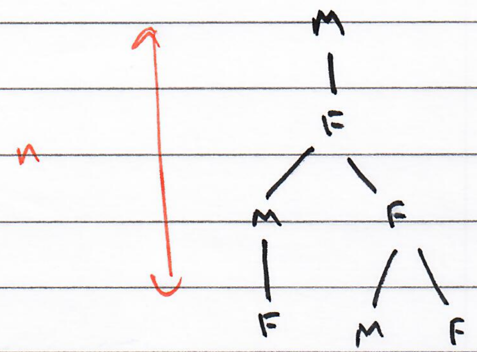
Bees

MALE — MOTHER

FEMALE — MOTHER
— FATHER



HUMAN ANCESTRY



BEE ANCESTRY

$(F_0 = 0)$

$F_0 = F_1 = 1$

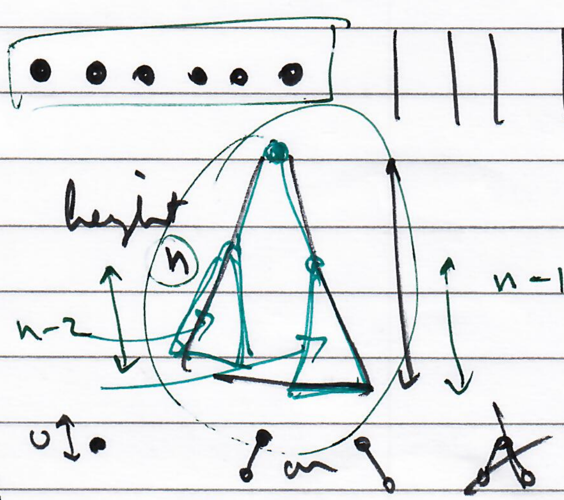
$F_n = F_{n-1} + F_{n-2}$

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...

Height-Balanced Trees (AVL)

all nodes

$h_l \approx h_r$ $\left(\underline{|h_l - h_r|} \le 1 \right)$



$T_0 = 1$

Principle of Dynamic Prog

Subsolutions of an optimal solution are themselves optimal

T_n = # of nodes in minimal HB AVL of height n

$= T_{n-1} + T_{n-2} + 1$

$T_0 = 1$

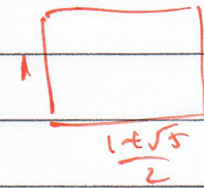
$T_1 = 2$

$T_n = T_{n-1} + T_{n-2} + 1$

$\boxed{T_n \text{ as a function of } n}$

$T_n \approx \left(\frac{1+\sqrt{5}}{2}\right)^n$ $\quad \frac{1+\sqrt{5}}{2} = 1.61801\ldots$

(Golden ratio)



$\langle T_n \rangle = \langle T_0, T_1, T_2, T_3, \cdots \rangle$

$= \langle 1, 2, 4, 7, 12, 20, 33, \cdots \rangle$

$\langle T_{n+1} \rangle = \langle 2, 4, 7, 12, \cdots \rangle = E\langle T_n \rangle$

$\langle T_{n+2} \rangle = \langle 4, 7, 12, \cdots \rangle$

OPERATOR $\boxed{E \text{ cuts off first elt}}$

$\langle T_n \rangle \pm \langle S_n \rangle = \langle T_n \pm S_n \rangle = \langle T_0 \pm S_0, T_1 \pm S_1, T_2 \pm S_2, \cdots \rangle$

$c\langle T_n \rangle = \langle cT_n \rangle = \langle cT_0, cT_1, cT_2, \cdots \rangle$

$\langle a_i \rangle = \langle a_0, a_1, a_2, \cdots \rangle$

$E\left(E\langle a_i \rangle\right) = \langle a_2, a_3, a_4 \cdots \rangle$

$EE\langle a_i \rangle$

$E^2\langle a_i \rangle$

What's $E^k\langle a_i \rangle = \langle a_{k}, a_{k+1}, \cdots \rangle$

$$\underbrace{\underbrace{E c}\langle a_i \rangle = \underbrace{c E}\langle a_i \rangle}_{\text{COMPOSITION}}$$

$$\langle 2^i \rangle = \langle 1, 2, 4, 8, 16, 32, \cdots \rangle$$

$$2\langle 2^i \rangle = \langle 2^{i+1} \rangle = \langle 2, 4, 8, 16, \cdots \rangle = E\langle 2^i \rangle$$

$$2\langle 2^i \rangle = E\langle 2^i \rangle$$

$$E\langle 2^i \rangle - 2\langle 2^i \rangle = \langle 0 \rangle$$

$$\boxed{(E-2)}\langle 2^i \rangle = \langle 0 \rangle$$

Shift left
& subtract 2 times
original elt

$$(E-3)\langle a_i \rangle$$

$$(E-3)\langle 3^i \rangle = \langle 0 \rangle$$

$$\begin{aligned}
(E-3)\langle 2^i \rangle &= E\langle 2^i \rangle - 3\langle 2^i \rangle \\
&= \langle 2^{i+1} \rangle + \langle -3 \cdot 2^i \rangle \\
&= \langle 2, 4, 8, 16, \cdots \rangle \\
&\quad + \langle -3, -6, -12, -24, \cdots \rangle \\
&= \langle -1, -2, -4, -8, \cdots \rangle \\
&= \boxed{-\langle 2^i \rangle}
\end{aligned}$$

$$(E-a)\langle b^i\rangle = \langle b^{i+1}\rangle - a\langle b^i\rangle$$

$$= \langle (b-a)\, b^i\rangle$$

$$= (b-a)\langle b^i\rangle \qquad \begin{array}{ll} a=b & \langle 0\rangle \\ a\neq b & k\langle b^i\rangle \end{array}$$

$(E-a)$ ANNIHILATES powers of a

(cannot annihilate powers of anything else)

$\langle x_i\rangle$ — annihilated by $E-2$

$$\Longrightarrow x_i = k2^i$$

$$\boxed{(E-a)}\left(\boxed{(E-b)}\langle x_i\rangle\right)$$

$$(E-a)\langle x_{i+1} - b x_i\rangle = \langle x_{i+2} - b x_{i+1} - a x_{i+1} + ab x_i\rangle$$

$$= \langle x_{i+2} - (a+b)x_{i+1} + ab x_i\rangle$$

$$= (EE - (a+b)E + ab)\langle x_i\rangle$$

$$= \boxed{(E^2 - (a+b)E + ab)}\langle x_i\rangle$$

$$\Longrightarrow (E-a)(E-b) = (E^2 - (a+b)E + ab)$$

$$= \boxed{(E-b)(E-a)} \quad \langle a^i\rangle$$

$(E-a)(E-b)$ ANNIHILATES $k_1 a^i + k_2 b^i$

$T_{n+2} = T_{n+1} + T_n + 1$

$$\boxed{T_n = T_{n-1} + T_{n-2} + 1}$$

$$\langle T_{n+2} - T_{n+1} - T_n - 1 \rangle = \langle 0 \rangle$$

1

$$E^2 \langle T_n \rangle = \langle T_{n+2} \rangle$$
$$-E \langle T_n \rangle = \langle -T_{n+1} \rangle$$
$$-1 \langle T_n \rangle = \langle -T_n \rangle$$

$$(E^2 - E - 1)\langle T_n \rangle = \langle 1 \rangle$$

ANNIHILATES $\boxed{(E-1)\underbrace{(E^2-E-1)}}\langle T_n \rangle = \langle 0 \rangle$

$$(E-a)(E-b)$$

$x^2 - x - 1 = 0$

$x = \frac{1 \pm \sqrt{5}}{2}$

$$\boxed{(E-1)\left(E - \frac{1+\sqrt{5}}{2}\right)\left(E - \frac{1-\sqrt{5}}{2}\right)}$$

$$\Rightarrow T_n = \boxed{a}\, 1^n + \boxed{b}\left(\frac{1+\sqrt{5}}{2}\right)^n + \boxed{c}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

1.61801 -0.61801

$n \to \infty$ 1^n $\left(\frac{1+\sqrt{5}}{2}\right)^n$

What does $(E-1)^2$ annihilate?