Illinois Institute of Technology
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Lecture 2: January 16, 2019

CS 330 Discrete Structures
Spring Semester, 2019

1 Proof by contradiction

We will now look at an important but easily misunderstood proof technique. The essence of this technique is that to prove $A \Rightarrow B$, we will instead show $\bar{B} \Rightarrow \bar{A}$.

1.1 Why does a proof by contradiction work?

It can be shown that these two forms are equivalent by examining the truth table of both functions. Since $A \Rightarrow B \iff \bar{B} \Rightarrow \bar{A}$ is a **tautology**, proving one implication proves the other implication, and disproving one implication disproves the other.

A	B	$A \Rightarrow B$	$\bar{B}$	$\bar{A}$	$\bar{B} \Rightarrow \bar{A}$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

1.2 A sample proof by contradiction

Theorem: $\sqrt{2}$ is irrational. That is, $\sqrt{2}$ cannot be written as a/b, where a and b are integers with no common factors.

We first need to convert this to the $A \Rightarrow B$ form as above. One simple conversion is T $\Rightarrow \sqrt{2}$ is irrational. (Convince yourself by examining the truth table above that this is indeed a valid conversion.)

Proof by contradiction: We will show that $\sqrt{2}$ is rational implies F — that is, that if we assume that $\sqrt{2}$ is rational, we can derive a contradiction.

By the definition of rationality, $\sqrt{2} = \frac{a}{b}$, for two relatively prime integers a and b. Thus $\sqrt{2}b = a$. It follows that $2b^2 = a^2$ and, by the definition of an even number, that a^2 is even.

We now take a small diversion to help us arrive at our goal.

Lemma: a^2 is even $\Rightarrow a$ is even

Proof by contradiction: We show a is odd $\Rightarrow a^2$ is odd. Since a is odd, it has the form $2n + 1$, for some integer n. Thus $a^2 = (2n + 1)^2 = 4n^2 + 4n + 1 = 2(2n^2 + 2n) + 1$. Thus a^2 is odd.

Now that we have concluded that a^2 is even $\Rightarrow a$ is even, we can resume our original proof. Since a is even, it can be written as $2c$, where c is an integer. Thus $2b^2 = (2c)^2$, so $2b^2 = 4c^2$, and $b^2 = 2c^2$. Thus b^2 is even, and by the lemma we know that b is even.

So both a and b are even. They share the common factor 2. But we originally assumed that a and b had no common factors! Thus we have arrived at a contradiction and proved the original theorem, that $\sqrt{2}$ is irrational.

1.3 Bertrand Russell's Proof

Bertrand Russell, the famous philosopher/mathematician, was challenged that because a false proposition implies any proposition, could he prove that if $2 + 2 = 5$, then he is the pope. "Yes," he responded:

- Suppose $2 + 2 = 5$.
- Subtract 2 from each side, giving $2 = 3$.
- Transpose to $3 = 2$.
- Subtract 1 from each side, giving $2 = 1$.

"Now, the pope and I are two, but two equals one. Therefore, I am the pope."

2 Proof by induction

2.1 Growth of harmonic numbers

We define the sequence of **harmonic numbers** as:

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} = \sum_{i=1}^{n} \frac{1}{i}$$

The recursion for harmonic numbers is

$$H_0 = 0$$

$$H_n = H_{n-1} + \frac{1}{n}, n \geq 1$$

They are called "harmonic numbers" because each term beyond the first is the *harmonic mean* of the two neighbors, where the harmonic mean of $x_1, x_2, \ldots, x_k$ is defined as

$$\frac{k}{1/x_1 + 1/x_2 + \cdots + 1/x_k},$$

and we have

$$\frac{2}{\frac{1}{1/(n-1)} + \frac{1}{1/(n+1)}} = \frac{2}{n - 1 + n + 1} = \frac{1}{n}.$$

The harmonic mean gives the correct "average" in many situations involving ratios: if you connect k resistors in *parallel*, $1/k$ times the harmonic mean gives the effective resistance. The name "harmonic" comes the ancient Greek use in music because it gave "harmonious" ratios.

Before we proceed any further, let's tie this into a recurring concept in computer science.

2.1.1 Rate of growth

Computer scientists find it useful to describe how fast functions grow as their input grows. Consider the function $f(n) = n$. This function grows at a **linear** rate: roughly speaking, if n is doubled, $f(n)$ will also be doubled. Other examples of functions exhibiting linear growth are $f(n) = 4n$ and $f(n) = \frac{n}{7}$.

Now consider $f(n) = n^2$. If n is doubled, $f(n)$ is quadrupled. This is a **quadratic** rate of growth. Likewise, $f(n) = n^3$ has a **cubic** rate of growth: if n is doubled, $f(n)$ increases by a factor of 8. We can similarly speak of rates of growth as quartic, quintic, and so on.

With the function $f(n) = 2^n$, the situation differs. If n is doubled, $f(n)$ is squared. We call this an **exponential** rate of growth. Comparing an exponential function with any of the polynomial functions discussed above will make it clear that the exponential grows much faster than any polymonial.

If you have seen the quicksort algorithm, you may recall that it makes $2nH_n$ comparisons in the average case. Where does that fall in the hierarchy presented above? We will attempt to obtain some information about the rate of growth of the harmonic numbers, and to do that we will use the technique of *mathematical induction*.

2.1.2 A result about the growth of harmonic numbers

Theorem: $H_{2^n} \geq 1 + \frac{n}{2}$, where $n = 0, 1, \ldots$

Proof by induction: First we consider the *base case*. (In a domino setting, this case is analogous to the tapping of the first domino.) We need to show that $H_{2^0} \geq 1 + \frac{0}{2}$, or, simplified, that $H_1 \geq 1 + 0$. By the definition of H_n, H_1 is 1. As $1 = 1$, the inequality holds.

Now we consider the *inductive step*. (This is analogous to the contact of each domino with the next domino.) For some arbitrary n, we first assume that $H_{2^n} \geq 1 + \frac{n}{2}$. We must then show that, based on this assumption, $H_{2^{n+1}} \geq 1 + \frac{n+1}{2}$.

Expanding H_{2^n} as per the definition, we arrive at:

$$1 + \frac{1}{2} + \cdots + \frac{1}{2^n}$$

Similarly expanding $H_{2^{n+1}}$ produces:

$$1 + \frac{1}{2} + \cdots + \frac{1}{2^n} + \frac{1}{2^n + 1} + \cdots + \frac{1}{2^{n+1}}$$

Notice that the first 2^n terms of these expansions are identical. That is, the sum of the first 2^n terms of $H_{2^{n+1}}$ is H_{2^n}. From our inductive hypothesis, we know that $H_{2^n} \geq 1 + \frac{n}{2}$. Let's look at the remaining terms. It is not difficult to show that $\frac{1}{2^n+1} \geq \frac{1}{2^{n+1}}$. (Consider that n is positive and manipulate the inequality algebraically.) Likewise, $\frac{1}{2^n+2} \geq \frac{1}{2^{n+1}}$, and so on. Of course, $\frac{1}{2^{n+1}} \geq \frac{1}{2^{n+1}}$. So each of the terms of $H_{2^{n+1}}$ past the nth is at least $\frac{1}{2^{n+1}}$.

How many of these terms are there? The entire harmonic number has 2^{n+1} terms, and we're not looking at the first 2^n of them right now. That leaves $2^{n+1} - 2^n = 2^n$ terms. So the sum of the last 2^n terms is at least $\frac{2^n}{2^{n+1}}$. This is simply $\frac{1}{2}$.

Adding the first 2^n terms to the remaining terms, we now know that $H_{2^{n+1}} \geq 1 + \frac{n}{2} + \frac{1}{2} = 1 + \frac{n+1}{2}$. This is precisely what we needed to show in the inductive step, so our proof is complete.

Similarly, we can prove

Theorem: $H_{2^n} \leq 1 + n$, where $n = 0, 1, \ldots$

Proof by induction: First we consider the *base case*. We need to show that $H_{2^0} \leq 1 + 0$. By the definition of H_n, H_1 is 1. As $1 = 1$, the inequality holds.

Now we consider the *inductive step*. For some arbitrary n, we first assume that $H_{2^n} \leq 1 + n$. We must then show that, based on this assumption, $H_{2^{n+1}} \leq 1 + (n + 1)$.

Expanding $H_{2^{n+1}}$ produces:

$$1 + \frac{1}{2} + \cdots + \frac{1}{2^n} + \frac{1}{2^n+1} + \cdots + \frac{1}{2^{n+1}}$$

Notice that the sum of the first 2^n terms of $H_{2^{n+1}}$ is H_{2^n}. From our inductive hypothesis, we know that $H_{2^n} \leq 1 + n$. Let's look at the remaining terms.

Similar to last proof, $\frac{1}{2^{n+1}} \leq \frac{1}{2^n}$. Likewise, $\frac{1}{2^{n+1}-1} \leq \frac{1}{2^n}$, and so on. Of course, $\frac{1}{2^n+1} \leq \frac{1}{2^n}$. So each of the terms of $H_{2^{n+1}}$ past the nth is at most $\frac{1}{2^n}$ and there are totally 2^n these terms. Therefore the sum of the last 2^n terms cannot exceed $\frac{2^n}{2^n} = 1$.

Adding the first 2^n terms to the remaining terms, we now know that $H_{2^{n+1}} \leq (1+n) + 1 = 1 + (n+1)$. This is precisely what we needed to show in the inductive step, so our proof is complete.

2.1.3 Extending this result to an arbitrary n

We have shown an inequality that gives us information about H_{2^n}, but we were looking for information about H_n, where n may or may not be a power of 2.

Let $k = 2^n$. Equivalently, $\ln k = \ln 2^n = n \ln 2$. So $n = \frac{\ln k}{\ln 2}$, and we can conclude that $H_k \geq 1 + \frac{\ln k}{2 \ln 2}$ and $H_k \leq 1 + \frac{\ln k}{\ln 2}$. These two bounds tell us that $H_k = \Theta(\ln k)$. We will sharpen this result considerably in a later lecture.

2.1.4 Similar sums

Instead of sums of reciprocals of integers, what about sums of reciprocals of the odd numbers? Call this O_n, with the corresponding sum of reciprocals of even numbers being E_n. Then, $H_n = O_n + E_n$ and $E_n = H_n/2$, implying that as n gets large both O_n and E_n diverge, tending to $(\ln n)/2$.

What about sums of reciprocals of *powers of integers*? Define

$$H_n^{(2)} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots + \frac{1}{n^2} = \sum_{i=1}^{n} \frac{1}{i^2}$$

Does $H_n^{(2)}$ also grow unboundedly as n gets large? No, for we can write

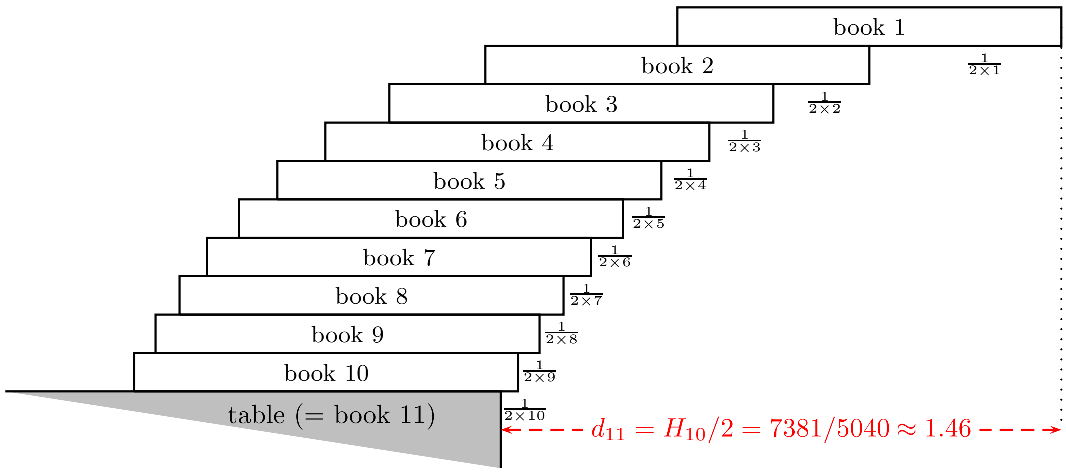
$$\begin{aligned} H_n^{(2)} &= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \\ &= 1 + \left(\frac{1}{2^2} + \frac{1}{3^2}\right) + \left(\frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2}\right) + \cdots \end{aligned}$$

where each parenthesized term contains 2^k terms beginning with $1/2^{2k}$. Thus,

$$\begin{aligned} H_n^{(2)} &< 1 + \frac{2}{2^2} + \frac{4}{4^2} + \cdots \\ &= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \\ &= 2. \end{aligned}$$

2.1.5 Stacking books

How far can a stack of n books extend over the edge of a table without the stack falling over? For n books we can achieve $H_n/2$ book lengths:



Assume that each book has unit length and unit weight. Label the top book of the stack 1, the second book on the stack 2, ..., and the bottom book on the stack n; the table's edge is considered the $(n+1)$st book on the stack.

We want to position the books so that the center of gravity of book 1 is over the right edge of book 2; the center of gravity of books 1 and 2 is over the right edge of book 3; the center of gravity of books 1, 2, and 3 is over the right edge of book 4; and so on.

Let d_i be the distance from the right edge of book i to the right edge of book 1 (the dotted line). This makes $d_1 = 0$, d_2 the amount by which the top book overhangs book 2, the second-from-the-top book, etc. Then d_{n+1} is the amount by which the top book overhangs the table edge. We must make d_{i+1} the center of gravity of the top i books—that is, each book is placed so its center is just above the center of gravity of the stack of books below it.

The center of gravity of k objects having weights $w_1, \ldots, w_k$ with respective centers of gravity at positions $p_1, \ldots, p_k$ is

$$\frac{w_1p_1 + w_2p_2 + \cdots + w_kp_k}{w_1 + w_2 + \cdots + w_k}$$

so that if we measure from the right edge of the top-most book, the center of the ith book is at position $p_i = d_i + 1/2$. Because each book weighs 1 unit, the center of gravity of the n books is

$$\frac{1(d_1 + 1/2) + \cdots + 1(d_n + 1/2)}{n}$$

which must be at the table's edge which is d_{n+1}. Thus

$$d_{n+1} = \frac{1(d_1 + 1/2) + \cdots + 1(d_n + 1/2)}{n}$$

which we can rewrite as

$$nd_{n+1} = d_1 + \cdots + d_n + n/2.$$

But this holds for all $n \geq 0$, so it also holds for $n-1$, as long as $n \geq 1$, giving

$$(n-1)d_n = d_1 + \cdots + d_{n-1} + (n-1)/2.$$

Subtracting these two equations gives

$$nd_{n+1} - (n-1)d_n = d_n + 1/2$$

or

$$d_{n+1} = d_n + 0.5/n.$$

Because $d_1 = 0$, $d_n = H_{n-1}/2$.

In other words, a 30-volume encyclopedia can be stacked so that it overhangs the table edge by $H_{30}/2 = \frac{9227046511387}{4658179125600} \approx 2$ volume lengths!

How many volumes would the encyclopedia need to get a volume overhanging the table edge by 4 volume lengths?

For further results on book stacking see Paterson and Zwick's paper "Overhang," *Amer. Math. Monthly*, January, 2009, pp. 19–44 and Treeby's paper "Further Thoughts on a Paradoxical Tower," *Amer. Math. Monthly*, January, 2018, pp. 44–60.