Binomial coef

$$\boxed{\binom{n}{k}}\boxed{\binom{k}{r}} = \boxed{\binom{n}{r}}\boxed{\binom{n-r}{k-r}}$$

$$\frac{\cancel{n!}}{k!(n-k)!}\;\frac{k!}{\cancel{r!}(k-r)!} = \frac{\cancel{n!}}{\cancel{r!}(n-r)!}\;\frac{(n-r)!}{(k-r)!\,(n-r-(k-r))!}$$

$$1 = 1$$

# of ways to select a committee of size k from n people

# of ways to select a sub committee of size k from

# of ways to choose SUB committee

# of ways to select the committee members NOT on sub committee

Vandermonde's Identity

$$\boxed{\binom{n+m}{k}} = \boxed{\sum_{i=0}^{k}\binom{n}{i}\binom{m}{k-i}}$$

$$= \binom{n}{0}\binom{m}{k} + \binom{n}{1}\binom{m}{k-1} + \cdots \binom{n}{k}\binom{m}{0}$$

n men

m women

committee of size k

rule of SUM

i men $i = 0, 1, \ldots, k$

$\binom{n}{i}$

⇒ $k-i$ women

$\binom{m}{k-i}$

⇒ rule of PROD

$\binom{n}{i}\binom{m}{k-i}$

$\sum_{i=0}^{k}\binom{n}{i}\binom{m}{k-i}$

$\boxed{k=n}$: $\binom{n+m}{n} = \sum_{i=0}^{n}\binom{n}{i}\binom{m}{i}$

committee of size $\underline{\underline{n}}$

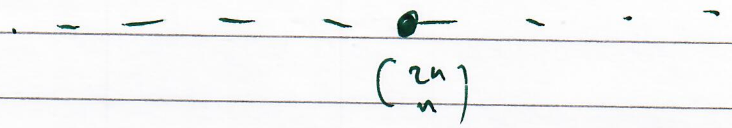
choose i women

⇒ $n-i$ men

⇒ i men EXCLUDED

$\boxed{n=m}$ → $$\boxed{\binom{2n}{n} = \sum_{i=0}^{n}\binom{n}{i}^2}$$ — COMB ?

row 2n



The Binomial Theorem

$$(1+x)^n$$

$$(1+x)^0 = \boxed{1}$$

$$(1+x)^1 = 1 + x$$

$$(1+x)^2 = 1 + 2x + x^2$$

$$(1+x)^3 = 1 + 3x + 3x^2 + x^3$$

$$\vdots$$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

. . . .

PASCAL'S TRIANGLE

$(1+x)^n \quad n \longrightarrow$

$(1+x)(1+x)^{n-1}$

$$(1+x)^{n-1} = \underbrace{\binom{n-1}{0} + \binom{n-1}{1}x + \binom{n-1}{2}x^2 + \cdots + \binom{n-1}{n-1}x^{n-1}}$$

$$x(1+x) = \underbrace{\binom{n-1}{0}x + \binom{n-1}{1}x^2 + \cdots \binom{n-1}{n-1}x^n}$$

$$(1+x)^n = (1+x)\overbrace{(1+x)^{n-1}}$$

$$= \overbrace{(1+x)^{n-1}} + x(1+x)^{n-1}$$

coef of x^k

$$\binom{n}{k}$$

$$\binom{n-1}{0} + \left[\binom{n-1}{0} + \binom{n-1}{1}\right]x + \left[\binom{n-1}{1} + \binom{n-1}{2}\right]x^2$$

$$+ \cdots$$

$$= \binom{n-1}{k-1} + \binom{n-1}{k}$$

Combinatorial View

$$\boxed{(1+x)^n} = \underbrace{1 + [\;]x + [\;]x^2 + \cdots [\;]x^k + \cdots [\;]x^n}$$

what's coef of x^k?

$$(1+x)^n = \underbrace{(1+x)(1+x)(1+x)\cdots(1+x)}_{n}$$

$$[\;]x^k$$

$\binom{n}{k}$ ways to pick x

$$2^n = \boxed{\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}} \qquad x=1$$

$$\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\cdots = \binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\cdots$$

$$\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\binom{n}{3}+\binom{n}{4}+\cdots\pm\binom{n}{n}=0$$

$$\boxed{x=-1}$$

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what is $\binom{n}{k}$, $n<k$?

$=0$

$$\sum_{i=0}^{n}\binom{i}{k} = \binom{0}{k} + \binom{1}{k} + \binom{2}{k} + \cdots + \binom{n}{k} =$$

$$\sum_{i=0}^{n}\binom{i}{k} = \sum_{i=0}^{n}\left(\text{coef of } x^k \text{ in } (1+x)^i\right)$$

$$= \text{coef of } \boxed{x^k} \text{ in } \boxed{\sum_{i=0}^{n}(1+x)^i}$$

$(1+x)^0, (1+x)^1, (1+x)^2, (1+x)^3, \cdots$

$a^0 \quad a^1 \quad a^2 \quad a^3 \ldots$

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1}-1}{r-1}$$

$$\begin{array}{r} r^n \\ r-1 \enclose{longdiv}{r^{n+1}-1} \\ \underline{r^{n+1}-r^n} \\ r^n-1 \\ \vdots \end{array}$$

$r = 1+x$

$$= \frac{(1+x)^{n+1}-1}{(1+x)-1} = \frac{\sum_i \text{coef } x^i}{x}$$

coef x^{k+1}

what is coef of ~~B~~ x^{k+1} in $(1+x)^{n+1}-1$

$= \text{coef of } x^{k+1} \text{ in } (1+x)^{n+1}$

$$\boxed{= \binom{n+1}{k+1}}$$

$$\sum_{i=0}^{k}\binom{n+i}{i} \boxed{=} \binom{n+k-1}{k}$$

$$\sum_{i=0}^{k}\binom{n+i}{i} = \sum_{i=0}^{k} \text{coef of } \underset{\underset{x^{n-i}}{\uparrow}}{\underline{x^i}} \text{ in } \underset{\underset{x^{n-i}}{\uparrow}}{\underline{(1+x)^{n+i}}}$$

$$= \sum_{i=0}^{k} \text{coef of } x^n \text{ in } (1+x)^{n+i}x^{n-i}$$

$$= \text{coef of } x^n \text{ in } \sum_{i=0}^{k}(1+x)^{n+i}x^{n-i}$$

$$= \text{coef of } x^n \text{ in } x^n(1+x)^n \underbrace{\sum_{i=0}^{k}\left(\frac{1+x}{x}\right)^i}$$

$$\boxed{\text{coef of } \underset{\uparrow}{x^n} \text{ in } x^{n+1-(k-1)}(1+x)^n\left[\frac{(1+x)^{k+1}}{x\;-1}\right]}$$

$$x^{n-k}(1+x)^n(1+x)^{k+1}$$

$$\text{coef of } x^n \text{ in } x^{n-k}(1+x)^{n+k+1}$$

$$= \text{coef of } x^k \text{ in } (1+x)^{n+k-1}$$

$$\boxed{=}\binom{n+k-1}{k}$$

How about : $\sum_{k=0}^{n} k \binom{n}{k}$?

$$_\wedge\binom{n}{k}x^{\boxed{k}}$$