Binomial Thm

$(1+x)^n$ — coef of x^k is $\binom{n}{k}$

$$\sum_{i=0}^{k} \binom{n+i}{i} = \binom{n+k-1}{k}$$

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$$

$\cdots \binom{n}{k} x^k \cdots$

$\frac{d}{dx}$

$\cdots \boxed{k\binom{n}{k}} x^{k-1}$

Exercise $\sum \binom{n}{k}/k$

integration

$x=1 \longrightarrow$ sum of all coef

$$\frac{d}{dx}(1+x)^n = n(1+x)^{n-1}\frac{d}{dx}(1+x)$$

$x=1$ $n2^{n-1}$

$$(1+x)^t = 1 + tx + \frac{t(t-1)}{2!}x^2 + \frac{t(t-1)(t-2)}{3!}x^3 + \cdots + \boxed{\frac{t(t-1)(t-2)\cdots(t-k+1)}{k!}}x^k + \cdots$$

Taylor series expansion

$\binom{t}{k}$ $= 1$ if $k=0$

$$\boxed{(1+x)^t = \sum_{k=0}^{\infty} \binom{t}{k} x^k}$$

$\boxed{t = -n}$

$$(1+x)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} x^k$$

$$\binom{-n}{k} = \frac{\overset{(n)}{(-n)}\,\overset{(n+1)}{(-n-1)}\,\overset{(n+2)}{(-n-2)}\cdots\overset{(n+k-1)}{(-n-k+1)}}{k!} \qquad \frac{(n+k-1)!}{(n-1)!}$$

$$\binom{-n}{k} = \binom{n+k-1}{k}(-1)^k$$

$$(1-x)^{-n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \cancel{(-1)^k x^k} \quad (+x)^k$$

$$\cancel{(1-x)}^{-n} = \sum_{k=0}^{\infty} \underline{\underline{\binom{n+k-1}{k}}}\, x^k$$

what is $(1-x)^{-1}$

$= \frac{1}{1-x}$

$= 1+x+x^2+x^3+\cdots$

$$(1+x+x^2+\cdots)^n =$$

$$= \overbrace{(1+x+x^2+x^3+\cdots)(1+x+x^2+\cdots)\cdots(1+x+x^2\cdots)}^{n}$$

ORANGE BALLS   GREEN BALLS   . . .   BLUE BALLS

x^{i_1} x^{i_2} x^{i_n} $\square\, x^k$

$i_1+i_2+\cdots+i_n = k$ ← # of ways

> Thm: The number of ways to choose n objects from k types <u>with replacement</u> is $\binom{n+k-1}{k}$

"Stars and bars" counting

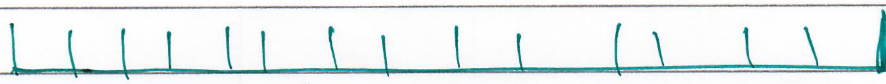
Stars & Bars

$\boxed{k \text{ stars}}$

$|\ * * * \ |\ * * \ |\ * * \quad \cdot \ | \ \cdot \cdot \cdot \quad * \ |$

invent $n-1$ "bars" — n regions

$i_1 + i_2 + \cdots + i_n$ stars $= k$



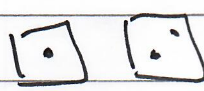
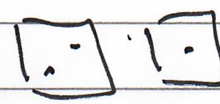
wall — $(n-1)$
star — k

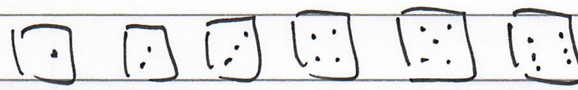
$(n-1+k)$ boxes

n ~~regions~~ segments

$n-1$

choice w/ replacement

2 dice  — same 



m ~~#~~ w/ replacement

$\boxed{\text{choosing } m \text{ items from } \underline{6} \text{ w/ replacement}}$

$$\binom{n+k-1}{k} \qquad \binom{6+m-1}{m} = \binom{m+5}{m} = \binom{m+5}{5}$$

1 die $\binom{1+5}{5} = \binom{6}{5} = \binom{6}{1} = 6$

pair of dice $\binom{2+5}{5} = \binom{7}{5} = \frac{7!}{5!\,2!} = 21$

Taylor's Thm

$$\boxed{(1+x)^t = \sum_{k=0}^{\infty} \binom{t}{k} x^k \qquad \binom{t}{k} = \frac{t(t-1)\cdots(t-k+1)}{k!}}$$

$t = 1/2$

$$\sqrt{1+x} = \sum_{k=0}^{\infty} \binom{1/2}{k} x^k \quad \begin{cases} 1 & k=0 \\ \dfrac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)\cdots(\frac{1}{2}-k+1)}{k!} & k \neq 0 \end{cases}$$

$$\binom{\frac{1}{2}}{k} = \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})\cdots(-\frac{2k-3}{2})}{k!}$$

$$= (-1)^{k-1}\left(\frac{1}{2}\right)^k \frac{1\times 3\times 5\times\cdots\times(2k-3)}{k!}$$

$2 \quad 4 \quad 6 \quad \cdots \quad 2k-2$

$$= (-1)^{k-1}\left(\frac{1}{2}\right)^k \frac{(2k-2)!}{k!\,\underbrace{2\cdot 4\cdot 6\cdots(2k-2)}_{2^{k-1}[\underbrace{1\cdot 2\cdot 3\cdots(k-1)}_{(k-1)!}]}}$$

$$= (-1)^{k-1}\frac{1}{k\,2^{2k-1}} \boxed{\frac{(2k-2)!}{\cancel{k!}\,(k-1)!}} \quad \binom{2k-2}{k-1}$$

$(k-1)!$

$(1+x)^{1/2}$

$$\boxed{\binom{1/2}{k} = \frac{(-1)^{k-1}}{2^{2k-1}}\frac{1}{k}\binom{2k-2}{k-1}}$$

$$\boxed{\sqrt{1-4x}} \qquad \cancel{\left(-\frac{1}{4}\right)^{k-1}} \boxed{\frac{1}{k}\binom{2k-2}{k-1}} \qquad \boxed{\text{CATALAN NUMBERS}}$$