

Lectures 7–8: February 6–11, 2019

CS 330 Discrete Structures
Fall Semester, 2017

Another combinatorial identity

Let us examine the identity

$$\binom{n}{k} \binom{k}{r} = \binom{n}{r} \binom{n-r}{k-r} \quad (1)$$

An algebraic proof is simple, but the combinatorial proof is more interesting. As before, we demonstrate that the combinatorial problem solved by the expression on the left-hand side of the equal sign and the combinatorial problem solved by the expression on the right-hand side of the equal sign are actually the same problem approached in two different fashions.

The left-hand side of (1) counts the number of possible outcomes of a two stage selection process of a set R of r elements from n . First, a subset $K \subseteq R$ of k elements is chosen and then r of these k are selected to form R . The rule of product says that this choice of R can happen in $\binom{n}{k} \binom{k}{r}$ ways. We can count the number of ways the same event can happen by first directly choosing r of the n elements as R (this can be done in $\binom{n}{r}$ ways) and then choosing from the other $n-r$ elements the remaining $k-r$ elements which when added to R form K (this can be done in $\binom{n-r}{k-r}$ ways). By the rule of product the compound event can occur in $\binom{n}{r} \binom{n-r}{k-r}$ ways. Since the compound event is the same in both of these applications of the rule of product, our proof is complete.

Vandermonde's identity

In the identities established so far, the algebraic proof has been very easy, almost eliminating the need for a combinatorial proof. We now present two identities for which the combinatorial proof is relatively simple and direct, but algebraic verification is not. Vandermonde's identity states that

$$\begin{aligned} \binom{n+m}{k} &= \sum_{i=0}^k \binom{n}{i} \binom{m}{k-i} \\ &= \binom{n}{0} \binom{m}{k} + \binom{n}{1} \binom{m}{k-1} + \cdots + \binom{n}{k} \binom{m}{0} \end{aligned} \quad (2)$$

The left-hand side of (2) is the number of ways to select a subcommittee of k people from a committee of n men and m women. On the other hand, the right-hand side counts the number of outcomes for the same problem: If the subcommittee is to have i men and $k-i$ women, the rule of product says that it can be chosen in $\binom{n}{i} \binom{m}{k-i}$ ways. By the rule of sum we must add this value for $i = 0, 1, \dots, k$ to count the number of ways the subcommittee can be chosen. This proves Vandermonde's identity.

A similar identity states

$$\binom{n+m}{n} = \sum_{i=0}^n \binom{n}{i} \binom{m}{i}$$

$$= \binom{n}{0}\binom{m}{0} + \binom{n}{1}\binom{m}{1} + \cdots + \binom{n}{n}\binom{m}{n} \quad (3)$$

Note that, by our convention that $\binom{i}{k} = 0$ for $k > i$, if $n > m$, the last $n - m$ terms of the sum will be zero. Again, an algebraic proof is not as simple as a combinatorial one here. The left-hand side of (3) is the number of ways to select a subcommittee of n people from a committee of n men and m women. The selection of such a committee can also be done, however, by first choosing i , $0 \leq i \leq n$, to be the number of women on the subcommittee, choosing the i women in one of the $\binom{m}{i}$ possible ways, and finally choosing the i men *not* on the subcommittee in $\binom{n}{i}$ ways—that is, the $n - i$ men *on* the committee in one of the $\binom{n}{n-i} = \binom{n}{i}$ possible ways. The rules of sum and product give the right-hand side of (3).

When $n = m$, (3) becomes the interesting identity

$$\sum_{i=0}^n \binom{n}{i}^2 = \binom{2n}{n}.$$

The Binomial Theorem

*To this day I comprehend the binomial theorem, a very rare accomplishment in an author.
For many years, indeed, I was probably the only American newspaper editor who knew
what it was.*

—H. L. Menken: *Predjudices: First, Second, and Third Series*, Library of America, 2010, page 452.

We turn now to one of the most important applications of the binomial coefficients, indeed, the justification of that name for the values $\binom{n}{k}$. We begin by applying the combinatorial reasoning developed so far to a purely algebraic problem, the evaluation of $(1+x)^n$. Writing down the first few values we find

$$\begin{aligned} (1+x)^0 &= 1 \\ (1+x)^1 &= 1+x \\ (1+x)^2 &= 1+2x+x^2 \\ (1+x)^3 &= 1+3x+3x^2+x^3 \\ (1+x)^4 &= 1+4x+6x^2+4x^3+x^4 \\ &\vdots \end{aligned}$$

The coefficients of the powers of x on the right-hand sides of this equation are a reproduction of Pascal's triangle. Why? Using the algebraic rules of polynomial multiplication, we can reason as follows. Let

$$(1+x)^{n-1} = \binom{n-1}{0} + \binom{n-1}{1}x + \binom{n-1}{2}x^2 + \cdots + \binom{n-1}{n-1}x^{n-1}.$$

Multiplying this by $(1+x)$ involves adding $(1+x)^{n-1}$ and $x(1+x)^{n-1}$:

$$\begin{aligned} (1+x)^{n-1} &= \binom{n-1}{0} + \binom{n-1}{1}x + \binom{n-1}{2}x^2 + \cdots + \binom{n-1}{n-1}x^{n-1} \\ x(1+x)^{n-1} &= \binom{n-1}{0}x + \binom{n-1}{1}x^2 + \cdots + \binom{n-1}{n-1}x^{n-1} + \binom{n-1}{n-1}x^n \end{aligned}$$

which gives

$$\begin{aligned}
 (1+x)^n &= \binom{n-1}{0} + \left[\binom{n-1}{1} + \binom{n-1}{0} \right] x + \left[\binom{n-1}{2} + \binom{n-1}{1} \right] x^2 + \cdots \\
 &\quad + \left[\binom{n-1}{n-1} + \binom{n-1}{n-2} \right] x^{n-1} + \binom{n-1}{n-1} x^n \\
 &= \binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 + \cdots + \binom{n}{n} x^n
 \end{aligned}$$

Thus the computation of $(1+x)^n$ is called the *binomial theorem*:

$$\begin{aligned}
 (1+x)^n &= \sum_{k=0}^n \binom{n}{k} x^k \\
 &= \binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 + \cdots + \binom{n}{n} x^n,
 \end{aligned} \tag{4}$$

which we just proved by induction. Now, as before, we concentrate on a combinatorial argument. We ask what the coefficient of x^k is in the simplified product

$$(1+x)^n = \underbrace{(1+x)(1+x) \cdots (1+x)}_{n \text{ times}}$$

If we expand the product into the unsimplified sum of all the monomials (products of x s and 1s), we can then ask how many of those monomials will be x^k ; that number will be the coefficient of x^k in $(1+x)^n$. We obtain the monomial x^k for each term in the unsimplified product formed by having k of the factors $(1+x)$ contribute an x and the other $n-k$ contribute a 1. Since there are n factors, this can happen in $\binom{n}{k}$ ways, verifying (4).

Equation (4) has many interesting consequences. For example, setting $x = 1$ gives

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n. \tag{5}$$

yet another way. Setting $x = -1$ gives

$$0 = (1-1)^n = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots, \tag{6}$$

so that

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots. \tag{7}$$

(Exercise: Give a combinatorial argument for this identity, showing that the number of even-size subsets of a set equals the number of odd-size subsets of the set.) The equation (from the previous lecture)

$$\sum_{i=0}^n \binom{i}{k} = \binom{0}{k} + \binom{1}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1} \tag{8}$$

can also be proved from (4) by a slightly more elaborate (and interesting) argument. We want to compute the value of

$$\sum_{i=0}^n \binom{i}{k}.$$

Now, $\binom{i}{k}$ is the coefficient x^k in $(1+x)^i$, so that the sum to be evaluated must be the coefficient of x^k in

$$\sum_{i=0}^n (1+x)^i = \frac{(1+x)^{n+1} - 1}{(1+x) - 1} = \frac{(1+x)^{n+1} - 1}{x}$$

by the formula for the sum of a geometric progression.¹ This coefficient is the coefficient of x^{k+1} in $(1+x)^{n+1} - 1$, which is $\binom{n+1}{k+1}$ by the binomial theorem, establishing (8).

Applications of the binomial theorem

Equation (4) is useful when we are faced with the evaluation of a sum of binomial coefficients, because it allows us to transform such a sum into a sum of terms in a geometric progression.

Consider

$$\sum_{i=0}^k \binom{n+i}{i} \tag{9}$$

that we talked about in the previous lecture. We know that

$$\begin{aligned} \binom{n+i}{i} &= \text{coefficient of } x^i \text{ in } (1+x)^{n+i} \\ &= \text{coefficient of } x^n \text{ in } (1+x)^{n+i} x^{n-i}. \end{aligned}$$

Therefore,

$$\sum_{i=0}^k \binom{n+i}{i} = \text{coefficient of } x^n \text{ in } \sum_{i=0}^k (1+x)^{n+i} x^{n-i}$$

and

$$\begin{aligned} \sum_{i=0}^k (1+x)^{n+i} x^{n-i} &= (1+x)^n x^n \sum_{i=0}^k (1+x)^i x^{-i} \\ &= (1+x)^n x^n \sum_{i=0}^k \left(\frac{1+x}{x} \right)^i \\ &= (1+x)^n x^n \frac{\left(\frac{1+x}{x} \right)^{k+1} - 1}{\frac{1+x}{x} - 1} \end{aligned}$$

Simplifying this polynomial, we have

$$\begin{aligned} &= (1+x)^n x^n \frac{\frac{(1+x)^{k+1}}{x^{k+1}} - 1}{1/x} \\ &= (1+x)^n x^{n+1} \left[\frac{(1+x)^{k+1} - x^{k+1}}{x^{k+1}} \right] \\ &= (1+x)^n x^{n-k} [(1+x)^{k+1} - x^{k+1}] \\ &= (1+x)^{n+k+1} x^{n-k} - (1+x)^n x^{n+1}. \end{aligned}$$

¹By long division of polynomials, $\frac{r^{n+1}-1}{r-1} = 1 + r + r^2 + \dots + r^n$.

Our sum is the coefficient of x^n in this sum. But $(1+x)^n x^{n+1}$ has no term x^n , so our sum is

$$\begin{aligned} \sum_{i=0}^k \binom{n+i}{i} &= \text{coefficient of } x^n \text{ in } (1+x)^{n+k+1} x^{n-k} \\ &= \text{coefficient of } x^k \text{ in } (1+x)^{n+k+1} \\ &= \binom{n+k+1}{k}, \end{aligned}$$

as we already knew. Other such summations require the more complicated manipulations, including the use of differentiation or integration.

Another example: What is $\sum_{k=0}^n k \binom{n}{k}$? We reason as follows. $\binom{n}{k}$ is the coefficient of x^k in $(1+x)^n$, so that $k \binom{n}{k}$ is the coefficient of x^k in $d(1+x)^n/dx = n(1+x)^{n-1}$. Thus $\sum_{k=0}^n k \binom{n}{k} = n2^{n-1}$ because setting $x=1$ sums the coefficients. We can see the result as the number of ways to select a committee of size k from a group of n , and specify a chairperson. How?

Exercise: What is $\sum_{k=0}^n \binom{n}{k}/k$?

More applications of the binomial theorem

As presented so far, the binomial theorem applies only to nonnegative integer powers of $(1+x)$. A much more general version can be derived by elementary calculus:

$$\begin{aligned} (1+x)^t &= 1 + tx + \frac{t(t-1)}{2!}x^2 + \frac{t(t-1)(t-2)}{3!}x^3 \\ &\quad + \cdots + \frac{t(t-1)(t-2)\cdots(t-k+1)}{k!}x^k + \cdots \end{aligned} \tag{10}$$

This formula, which is the Taylor series expansion of $(1+x)^t$ around $x=0$, is exactly equation (5) of last lecture, where t is a nonnegative integer. When t is negative or noninteger, the right-hand side of (10) is an infinite series that can be shown to converge for $|x| < 1$. Equation (10) suggests the generalization of the symbol $\binom{t}{k}$ to nonpositive or noninteger values of t :

$$\binom{t}{k} = \begin{cases} 1 & \text{if } k=0, \\ \frac{t(t-1)(t-2)\cdots(t-k+1)}{k!} & \text{if } k>0. \end{cases}$$

This allows us to write (10) more tersely as

$$(1+x)^t = \sum_{k=0}^{\infty} \binom{t}{k} x^k. \tag{11}$$

Notice that (11) includes (4) as a special case, since $\binom{n}{k} = 0$ for $k > n$ and integer n .

Of special interest is the case

$$(1-x)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} (-x)^k$$

where n is an integer.

By the definition of $\binom{t}{k}$, we have

$$\begin{aligned}\binom{-n}{k} &= \frac{(-n)(-n-1)(-n-2)\cdots(-n-k+1)}{k!} \\ &= \frac{n(n+1)(n+2)\cdots(n+k-1)}{k!}(-1)^k \\ &= \binom{n+k-1}{k}(-1)^k\end{aligned}$$

This gives

$$\begin{aligned}(1-x)^{-n} &= \sum_{k=0}^{\infty} \binom{n+k-1}{k} (-1)^k (-x)^k \\ &= \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k\end{aligned}\tag{12}$$

Equation (12) has a useful combinatorial interpretation. Rewriting $(1-x)^{-n} = (1+x+x^2+x^3+\cdots)^n$ by using the formula for the sum of a geometric progression, we obtain

$$\underbrace{(1+x+x^2+\cdots)(1+x+x^2+\cdots)\cdots(1+x+x^2+\cdots)}_{n \text{ times}} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k$$

The coefficient of x^k on the left-hand side (which must, of course be the same as that on the right-hand side) is the number of ways to choose k objects from a set of n objects *with unlimited repetition*; that is, a single object can be chosen 0, 1, 2, ..., or k times. Why? Examine the way a term is formed in the unsimplified product on the left-hand side: it is a power of x from the first sum times a power of x from the second sum times a power of x from the third sum and so on. If x^k is to be formed in this way, the sum of the exponents in the powers of x must equal k . Such a product of powers $x^{i_1}x^{i_2}\cdots x^{i_n}$ equaling x^k thus corresponds to a selection of k objects as follows: i_1 of the first object, i_2 of the second object, ..., i_n of the n th object, for a total of $i_1+i_2+\cdots+i_n=k$ objects. It follows that the number of ways that x^k can appear in the unsimplified product is the number of choices of k objects from n objects with unlimited repetition. Equation (12) tells us that this is $\binom{n+k-1}{k}$.

Why is there an “ $n-1$ ” in the above result? Let us think about it in another way. We want to know the number of ways to choose k objects from a set of n objects with unlimited repetition. Let $k=8$ and $n=5$. Suppose we have stars in 5 different colors, namely red, green, blue, yellow, and cyan, each with unlimited number. How many ways are there to choose 8 stars from them?

We can do this with “stars and bars counting” (Rosen, page 447); see also

http://en.wikipedia.org/wiki/Stars_and_bars_%28combinatorics%29

We can pick the stars like this: first, we put $k=8$ stars in a long box.



Then, we insert $n-1=4$ boundary bars into the long box, which will separate the long box into 5 smaller boxes. If there are m stars in the i th box, we will pick m stars with the i th color (Notice that m can be

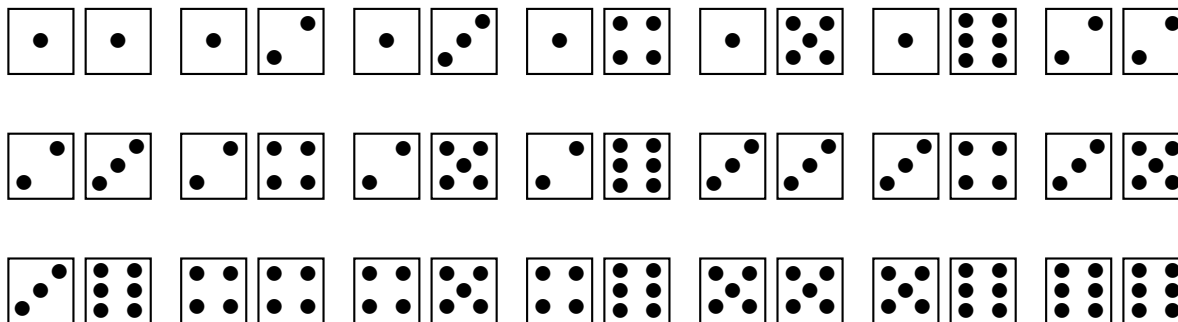


Figure 1: All possible pairs of dice.

0). In the case shown below, we will pick 2 red stars, 1 green star, 3 blue stars, no yellow stars, and 2 cyan stars.



Imagine that the long box has $n - 1 + k$ slots. In each of the slots we can put either a star or a bar. We just need to choose $n - 1$ of them to put the bars, or k of them to put the stars. Therefore, the number of ways to pick the stars should be

$$\binom{n + k - 1}{n - 1} = \binom{n + k - 1}{k}$$

Exercise: How many ways are there to choose k objects from a set of n objects with unlimited repetition if we insist that each of the n items be chosen at least once?

Let's look at another example: how many outcomes are possible if m standard dice are rolled? Each die has 6 faces that could be up, so we are choosing m faces (one for each die) from 6 possibilities, with repetition. Our discussion above showed that there are

$$\binom{6 + m - 1}{m} = \binom{m + 5}{m} = \binom{m + 5}{5}$$

ways to make the choice. For one die, there are just the

$$\binom{1 + 5}{5} = \binom{6}{5} = 6$$

obvious outcomes. For two dice there are the

$$\binom{2 + 5}{5} = \binom{7}{5} = \frac{7!}{5!2!} = 21$$

outcomes (shown in Figure 1).

As a final example of the binomial theorem, we compute the coefficients in the binomial expansion of square roots:

$$(1 + x)^{\frac{1}{2}} = \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} x^k,$$

where

$$\binom{\frac{1}{2}}{k} = \begin{cases} 1 & k = 0, \\ \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)\cdots(\frac{1}{2}-k+1)}{k!} & k = 1, 2, 3, \dots \end{cases}$$

For $k = 1, 2, 3, \dots$

$$\begin{aligned} \binom{\frac{1}{2}}{k} &= \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})\cdots(-\frac{2k-3}{2})}{k!} \\ &= \left(\frac{1}{2}\right)^k \frac{(-1)(-3)(-5)\cdots(-2k+3)}{k!} \\ &= (-1)^{k-1} \frac{1 \times 3 \times 5 \cdots (2k-3)}{2^k k!} \\ &= (-1)^{k-1} \frac{1 \times 2 \times 3 \times 4 \times 5 \cdots (2k-3)(2k-2)}{[2^k k!][2 \times 4 \times 6 \cdots (2k-2)]} \\ &= (-1)^{k-1} \frac{(2k-2)!}{[2^k k!][2^{k-1}(k-1)!]} \\ &= \frac{(-1)^{k-1}}{2^{2k-1}} \frac{(2k-2)!}{k!(k-1)!} \\ &= \frac{(-1)^{k-1}}{2^{2k-1}} \frac{1}{k} \binom{2k-2}{k-1}. \end{aligned}$$

Thus,

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{2^3}\frac{1}{2}\binom{2}{1}x^2 + \frac{1}{2^5}\frac{1}{3}\binom{4}{2}x^3 - \cdots. \quad (13)$$

Even this equation has an interesting combinatorial significance.

Exercise: What is the coefficient of x^n in $C(x) = 1/\sqrt{1-4x}$? How about in $2C'(x/4)$?