Illinois Institute of Technology
Department of Computer Science

Lecture 4: January 28, 2019

CS 330 Discrete Structures
Spring Semester, 2019

This lecture presents some elementary techniques for counting the number of configurations satisfying specified properties. Such techniques are fundamental to the analysis of algorithms, especially algorithms for sorting, searching, merging, and other problems of a combinatorial nature.

For example, consider the algorithm listed below that finds the maximum of n numbers $x[1]$, $x[2]$, $\ldots$, $x[n]$. To analyze this algorithm fully we want to know exactly how many times each of the instructions is executed, but for the moment, let us ask only how many times the statement "$m \longleftarrow i$" is executed. The answer, of course, depends not on the particular values of the $x[i]$, but rather on their relative order: If $x[1]$ is the maximum of the n elements the statement will *never* be executed. If $x[1] < x[2] < \cdots < x[n]$ then the statement will be executed $n-1$ times. For how many of the relative arrangements of the inputs $x[1], x[2], \ldots, x[n]$ will the statement be executed exactly k times? The answer to this question is at the heart of the analysis of the algorithm.

```
$m \longleftarrow 1$
FOR $i := 2$ TO $n$ DO
      IF $x[i] > x[m]$ THEN /* $x[i]$ is the largest seen so far */
         $m \longleftarrow i$
```

The Rules of Sum and Product

In order to have as wide an applicability as possible, we will talk in terms of how many ways an "event" can occur. What is an event? That depends on the context. It could be "picking an orange, speckled sock from your sock drawer"; it could be "seating a group of five cannibals, three vegetarians, and a chicken at a table in such a way that nothing gets eaten" ; it could be the "choosing an arrangement of $x[1]$, $x[2]$, $\ldots$, $x[n]$ that causes the statement "$m \longleftarrow i$" in the algorithm above to be executed k times.

In counting the number of ways an event can happen, we try to break the event down into simpler events whose combination results in the event under study. For instance, if we are trying to determine the number of ways that the roll of a pair of dice can result in a seven, we could consider the roll of each die to be a simpler event than the roll of the pair. There are two basic rules for counting the number of ways that simpler events can be combined to result in more complex events:

Rule of Sum If an event E_1 can occur in e_1 different ways and a *separate* event E_2 can occur in e_2 different ways, the compound event $E_1 \cup E_2$ can occur in $e_1 + e_2$ different ways.

Rule of Product If an event E_1 can occur in e_1 different ways and a *separate* event E_2 can occur in e_2 different ways, the compound event $E_1 \cap E_2$ can occur in $e_1 e_2$ different ways.

The word "separate" in these two rules is very important, and we will define it precisely below. These two rules are deceptively simple and, as is often true with general principles, they are easy to understand but sometimes tricky to apply. To illustrate the application of these rules, we will consider several examples designed to demonstrate them and their limitations.

Three Simple Examples

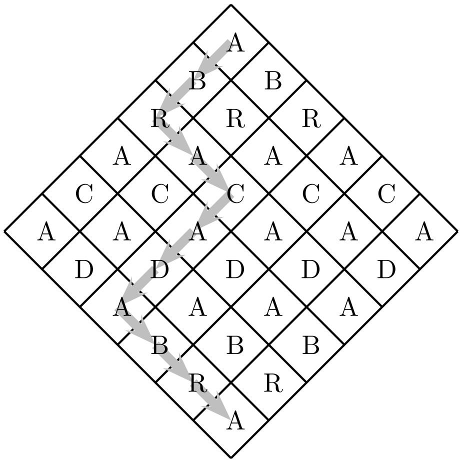
Suppose I have 5 short-sleeve shirts, 3 long-sleeve shirts, 4 pairs of pants, 7 ties, and 1 pair of shoes. Assuming that there is no issue of mis-matched colors or patterns, how many ways can I choose an ensemble each morning? For a shirt I can choose either a long- or a short-sleeve shirts ($5 + 3 = 8$ possible shirts), a pair of pants (4 possible pairs of pants), to wear a tie or not (7 possible ties or no tie at all, 8 choices in all), and only 1 pair of shoes. The total number of ensembles is thus $(5 + 3) \times 4 \times (7 + 1) \times 1 = 256$.

In a deck of cards we have 4 suits and 13 values (2 through ace), giving a total of $4 \times 13 = 52$ possible cards.

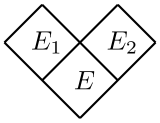
A zip code is made of 5 decimal digits, so the number of possible zip codes is $10 \times 10 \times 10 \times 10 \times 10 = 10^5 = 100000$. The number of zip codes with only odd digits is $5 \times 5 \times 5 \times 5 \times 5 = 10^5 = 3215$. The number of zipcodes with no repeated digit is $10 \times 9 \times 8 \times 7 \times 6 = 30240$.

A Bigger Example: Rule of Sum

The amulet picture below, consisting of 36 letters arranged on a 6×6 grid of diamond-shaped cells was said to have magical powers because any path of neighboring diamonds from the top "A" to the bottom "A" spells out the word "abracadabra"; furthermore, no other type of path will spell out this word. The obvious question arises: How many ways are there to spell out the sequence of letters A-B-R-A-C-A-D-A-B-R-A? The solution involves counting the number of ten-step paths in the diamond from top to bottom, with each step being "down to the left" or "down to the right", as illustrated by the arrows.



For the purpose of this example, an "event" is "arriving at a particular cell of the diamond from the top cell only by steps that are down-and-right or down-and-left." As the following diagram depicts, to reach an event E (labelled by the cell "E") we must either go through event E_1 or event E_2:

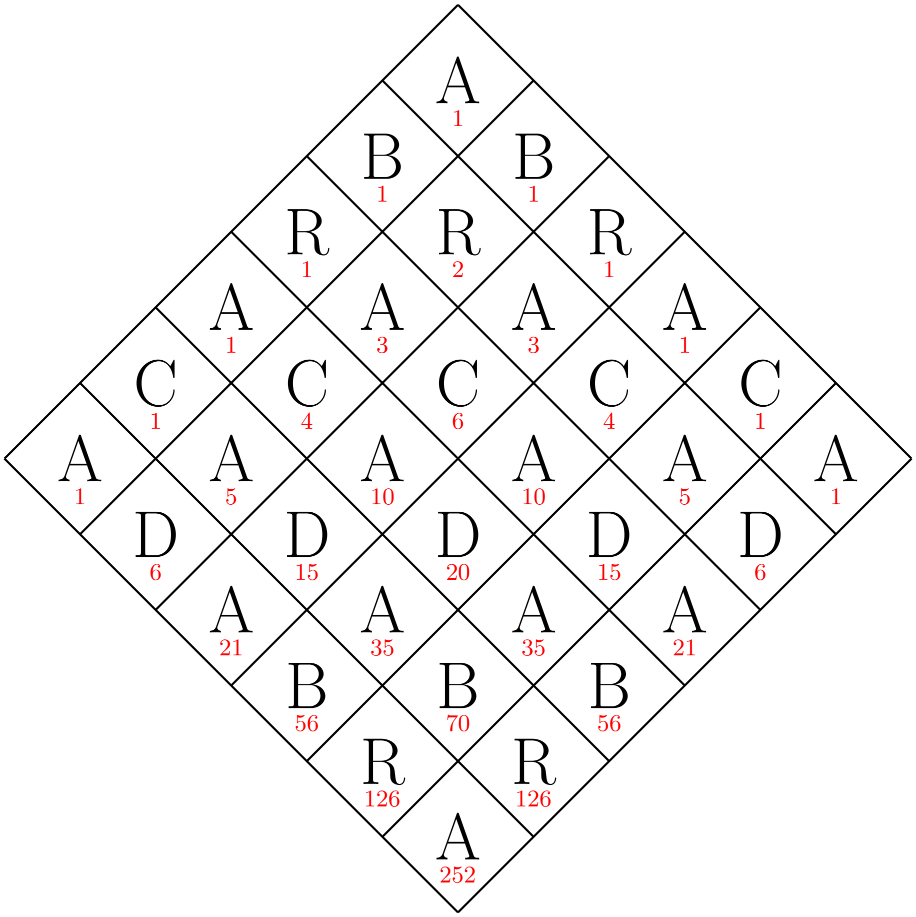


Suppose we can arrive at the cell labeled E_1 (this is the event E_1) in e_1 ways and at the cell E_2 (this is the event E_2) in e_2 ways. Then we can apply the rule of sums to see that event E can happen in exactly $e_1 + e_2$

ways. Note that there is only one way to arrive at a cell along the top left or top right boundaries. We can now compute the number of ways to spell "abracadabra" as follows:

- Fill in the 6×6 grid with ones along the top left and right boundaries
- apply the rule of sum by adding the two numbers in the cells above an empty cell
- write the resulting sum in the empty cell.

Our hard work pays off with the gleeful conclusion that "abracadabra" can be spelled out in 252 different ways. You might recognize the numbers in our diagram as the entries in Pascal's triangle.



For our second example, we choose an instance where the rule of sum is *not* applicable. Suppose that of the roughly 200 students in CS 330, 150 are taking Math 247, and 100 are taking Physics 113. How many of the CS 330 students have taken *either* Math 247 *or* Physics 113? Applying the rule of sum suggests 250 out of the 200 students in CS 330 have taken one course or the other! This nonsense results from applying the rule of sum to events that are *not* separate from one another—there are enterprising souls who are enrolled in both Math 247 and Physics 113, as well as some students who are currently taking neither. Thus, the correct answer depends on how many of the students took both courses, not just either course. This tells us how to make the notion of "separate" events precise:

Events E_1 and E_2 are *separate* if $E_1 \cap E_2 = \emptyset$.

In other words, the rule of sum can be seen as a specific case (i.e. $E_1 \cap E_2 = \emptyset$) of the familiar principle of inclusion and exclusion:

$$|E_1 \cup E_2| = |E_1| + |E_2| - |E_1 \cap E_2|$$

Examples: Rule of Product

We now turn to an example that uses the rule of product. How many permutations (arrangements) of n distinct terms $x_1, x_2, \ldots, x_n$ are there in total? We reason as follows. The first element of the permutation can be any one of the n terms (n possibilities). The second element can be any one of the n elements *except* the term that was chosen as the first, giving $n - 1$ possibilities. The third element can be any one of the n elements *except* the terms that were chosen as the first two elements ($n - 2$ possibilities), and so on. Each element of the permutation, in order, corresponds to a separate event (think about why this is true), so that the total number of permutations is thus, by the rule of product, $n \times (n-1) \times (n-2) \times \cdots \times 1$, usually written $n!$. The symbol "$n!$" is read "n factorial."

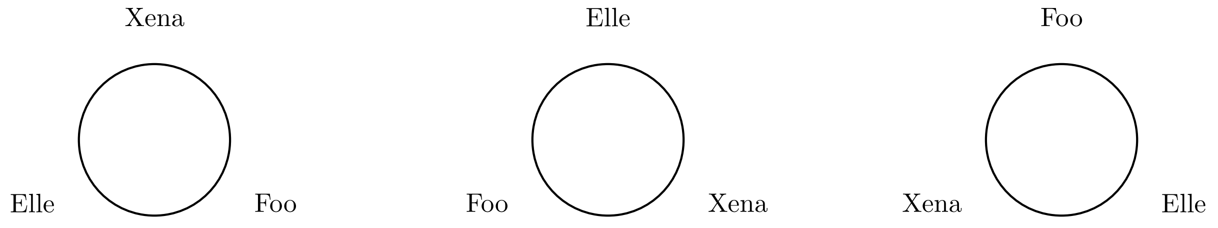
For example, there are $2! = 2 \times 1 = 2$ ways to arrange two distinct objects x_1 and x_2: either x_1, x_2 or x_2, x_1. There are $3! = 3 \times 2 \times 1 = 6$ ways to arrange the three letters A, E, T into a "word": the first letter can be either A,E,T (3 choices). Let us say that we picked A; then the second letter can be either E or T (2 choices). Let us say that we picked E; the the third letter can be only T (1 choice). Altogether, we can form the following six words: AET, ATE, EAT, ETA, TAE, TEA. There are $4! = 4 \times 3 \times 2 \times 1 = 24$ orders in which the courses of a four course meal can be served.

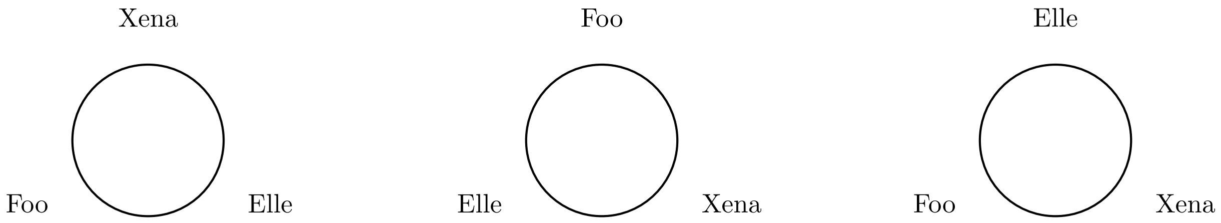
Another example where we may apply the rule of products is with subsets of a set. Specifically, how many ways are there to choose a subset of a set containing n elements? We can regard the choice of a subset as a sequence of n decisions (events) whether or not to include each of the n elements in the subset. Each of these events can happen in two ways—an element is included or is rejected. Thus the rule of product tells us that number of different compound events, each event being the choice of a subset, can happen in $2 \times 2 \times \cdots \times 2 = 2^n$ ways.

For example, if we look at the set { H, E, L, P }. The first event is whether or not we include H (two possibilities). The second event is whether or not we include E (also two possibilities), and so forth.

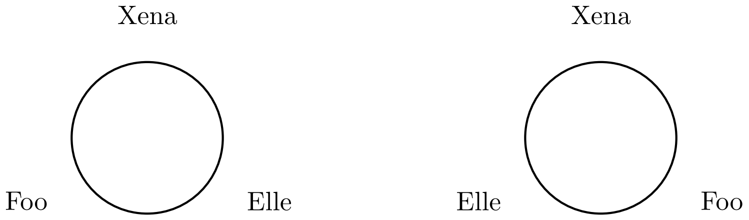
Variations of the Rule of Product

With a slight variation of the rule of product, we can solve many other interesting problems, such as the following vexing question: How many *different* ways are there to seat n people around a circular table for dinner? In this case, the table has no orientation, so rotating it does not generate a new seating. For three people, we can look at the $3! = 6$ arrangements of their names, say Xena, Elle, and Foo, around a circular table:





We see that the arrangements in the first row differ from each other only by a rotation of the table—Foo is always at Xena's left and Elle is always at Xena's right. Similarly, the arrangements in the second row differ from each other only by a rotation of the table. There are thus only two essentially different arrangements around the table:



In the general problem of n people (instead of 3), we answer the question by using an important variation on the rule of product.

Rule of Product (Variation) If an event E can occur in e different ways, and E is a compound event $E_1 \cap E_2$ in which E_1 and E_2 are separate events and E_1 can occur in e_1 different ways, then the event E_2 can occur in $e_2 = e/e_1$ different ways.

Let E be the event of seating n people in a row, which can happen in $e = n!$ ways, as we have seen. View E as a compound event $E_1 \cap E_2$ in which E_2 is the event of seating the n people in *different* ways (with respect to rotation) and E_1 is the event of starting at one of these people and reading off their neighbors clockwise around the circle. Since there are n places to start reading off neighbors, E_1 can happen in $e_1 = n$ ways. The variation on the rule of product then tells us that E_2, the event of seating the people in different circular arrangements (or "circular orders" as they are called), can happen in $e_2 = e/e_1 = n!/n = (n-1)!$ ways.

In essence, we have formed a correspondence between permutations and circular orders. Under this correspondence, each of the $n!$ permutations corresponds to a circular order, and n different permutations each give the same circular order.

How many circular arrangements are there of $n = 1$ elements? The answer is clearly one, and our formula above gives $(1-1)! = 0!$. It is thus natural for us to define $0! = 1$, a slight extension to our formula

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 1.$$

This extension is consistent with the notion of a unique "empty" permutation of zero elements.

With this extension, we can describe the value of $n!$ recursively:

$$n! = \begin{cases} 1 & \text{if } n = 0, \\ n \times (n-1)! & \text{otherwise.} \end{cases}$$

This recursive definition has a nice interpretation in terms of permutations. Suppose we want to list all of the permutations of n elements. We can do so recursively by starting with a list of all permutations of $n-1$ of the elements and then inserting the nth element into each of the n possible locations in each of the $(n-1)!$ permutations of $n-1$ elements.

We can ask a slightly different question about circular arrangements by considering n different colored beads on a loop of string (a "rosary"): In how many ways can the beads be made into a rosary? Now not only does the circular order need to be considered, but also the effect of flipping the loop, that is, taking its mirror image (which does not intrinsically change the loop). Again we use the variation on the rule of product. Let E be the event of forming a circular permutation, which we know can happen in $e = (n-1)!$ ways from the above discussion. View E as a compound event $E_1 \cap E_2$ in which E_2 is the event of forming a rosary permutation and E_1 is the event of choosing one of the two mirror images. Thus E_1 can happen in $e_1 = 2$ ways and so E_2 can happen in $e_2 = e/e_1 = (n-1)!/2$. The number of rosary permutations is thus $(n-1)!/2$.

Does this formula mean that for $n = 1$ and $n = 2$ there is half a rosary permutation? No, of course not! In our argument above we stated that E_1, the event of choosing one of the mirror images, can happen in $e_1 = 2$ ways. This is indeed true, but *only* when there are three or more beads—the two mirror images of the loop are identical if the loop contains only one or two beads. The complete answer to the question of the number of rosary permutations is thus

$$\begin{cases} (n-1)!/2 & \text{for } n \geq 3, \\ 1 & \text{otherwise.} \end{cases}$$

Food for thought:

The Problem of Derangements How many ways can you arrange n volumes of an encyclopedia in such a way that no volume is in its correct place (in the normal ordering of the volumes)?

The *Menage* problem You are have to seat n separated husband/wife couples (that's $2n$ people in total; bigamy is illegal) at your daughter's wedding. How many ways are there of seating these people around one table in such a way that no ex-couple is sitting in adjacent chairs?