Savings Models

1. Intentional Saving: $\beta R > 1$
2. Consumption Smoothing: Concavity of u when $\beta R = 1$
3. Precautionary Saving: $u''' > 0$
4. Life-Cycle Motive (Retirement): Consumption smoothing given expectation of retirement
5. Bequest Motive: Want to leave money to your children

Wealth calculation: $a_t + \sum_{\tau=t} T R^{-(\tau-t)} y_\tau = \sum_{\tau=t}^{T} R^{-(\tau-t)} c_\tau$

Asset Pricing

Real estate

Bonds

Safe asset returns a roughly constant $R^b \approx 1\%$

Stocks

Risky asset returns $R^e \approx 6\%$ with much higher variance

Equity Premium

Compensation for risk $R^e - R^b \approx 5\%$

2-period Endowment Economy with Uncertainty

$s \in \mathcal{S} = \left\{s^L, s^H\right\}$ with π a distribution over these outcome states and $y_2(s^L) < y_2(s^H)$

Bonds

$\max\limits_{c_1, c_2(s)} \frac{{c_1}^{1-\gamma}}{1-\gamma} + \beta \sum\limits_{s \in \mathcal{S}} \pi(s) \frac{c_2(s)^{1-\gamma}}{1-\gamma}$ such that $\forall s, c_1 + \frac{c_2(s)}{R^b} = y_1 + \frac{y_2(s)}{R^b}$

$$\mathcal{L} = \frac{{c_1}^{1-\gamma}}{1-\gamma} + \sum_{s \in \mathcal{S}} \pi(s) \left(\beta \frac{c_2(s)^{1-\gamma}}{1-\gamma} + \lambda_s \left(y_1 + \frac{y_2(s)}{R^b} - c_1 - \frac{c_2(s)}{R^b} \right) \right)$$

$$\frac{\partial \mathcal{L}}{\partial c_1} = 0 \Rightarrow c_1^{-\gamma} = \mathrm{Ex}\left[\lambda_s\right]$$

$$\frac{\partial \mathcal{L}}{\partial c_2(s)} = 0 \Rightarrow \pi(s) \beta c_2(s)^{-\gamma} = \pi(s) \frac{\lambda_s}{R^b}$$

$$R^b \sum_{s \in \mathcal{S}} \pi(s) \beta c_2(s)^{-\gamma} = \mathrm{Ex}\left[\lambda_s\right] = c_1^{-\gamma}$$

$$R^b \sum_{s \in \mathcal{S}} \pi(s) \left(\frac{\beta c_2(s)^{-\gamma}}{c_1^{-\gamma}} \right) = 1$$

Note that the summand is just the marginal rate of substitution, so $R^b \, \mathrm{Ex}\left[\mathrm{MRS}(s)\right] = 1$. So the decision about bond investment can be made only on the expectation of the future and not on the actual outcome.

Stocks

$\max\limits_{c_1, c_2(s)} \frac{{c_1}^{1-\gamma}}{1-\gamma} + \beta \sum\limits_{s \in \mathcal{S}} \pi(s) \frac{c_2(s)^{1-\gamma}}{1-\gamma}$ such that $\forall s, c_1 + \frac{c_2(s)}{R^e(s)} = y_1 + \frac{y_2(s)}{R^e(s)}$

$$\mathcal{L} = \frac{{c_1}^{1-\gamma}}{1-\gamma} + \sum_{s \in \mathcal{S}} \pi(s) \left(\beta \frac{c_2(s)^{1-\gamma}}{1-\gamma} + \lambda_s \left(y_1 + \frac{y_2(s)}{R^e(s)} - c_1 - \frac{c_2(s)}{R^e(s)} \right) \right)$$

$$\frac{\partial \mathcal{L}}{\partial c_1} = 0 \Rightarrow c_1^{-\gamma} = \mathrm{Ex}\left[\lambda_s\right]$$

$$\frac{\partial \mathcal{L}}{\partial c_2(s)} = 0 \Rightarrow \pi(s) \beta c_2(s)^{-\gamma} = \pi(s) \frac{\lambda_s}{R^e(s)}$$

$$\sum_{s \in \mathcal{S}} \pi(s) R^e(s) \beta c_2(s)^{-\gamma} = \mathrm{Ex}\left[\lambda_s\right] = c_1^{-\gamma}$$

$$\sum_{s \in \mathcal{S}} \pi(s) R^e(s) \left(\frac{\beta c_2(s)^{-\gamma}}{c_1^{-\gamma}} \right) = 1$$

Note that unlike before we cannot take the return rate out of the sum, so we get $\mathrm{Ex}\left[R^e(s) \, \mathrm{MRS}(s)\right] = 1$.

Equity Premium

Then by using covariance we have $\mathrm{Ex}\left[R^e(s)\right] \mathrm{Ex}\left[\mathrm{MRS}(s)\right] = 1 - \mathrm{cov}\left(R^e(s), \mathrm{MRS}(s)\right)$.

Substituting by the above we get $\frac{\mathrm{Ex}\left[R^e(s)\right] - R^b}{R^b} = -\mathrm{cov}\left(R^e(s), \mathrm{MRS}(s)\right)$ is the excess return on equity investment and contains the compensation for taking risk.