Dynamic Production Economy

1. Time: $t \in \{1, 2\}$
2. Commodity set:
   - Final good (consumption and investment)
   - Labor services
   - Capital services:
     - supply a_t units of capital (wealth)
     - firm demand for capital k_t
     - $(1-\delta)a_t$ undepreceiated capital returned where δ is the depreciation rate
     - Net rental rate r_t then rented capital $(r_t + \delta)a_t$; note $r_t + \delta$ is the gross rental rate
     - Hoseholds then get back $(1 + r_t)a_t$
   - Households have an endowment $\bar{a}_1$ of wealth
3. All markets competitive
4. Preferences:
   - Households live for 2 periods, so $U(c_1, c_2, l_1, l_2) = u(c_1, l_1) + \beta u(c_2, l_2)$ where $\beta \in [0, 1]$ is the discount factor
5. Technology
   - $y_t = z_t f(k_t, n_t)$ as before
   - Constraint: f has constant retunrs to scale
   - $k_{t+1} = (1-\delta)k_t + i_t$ where i_t is gross investment
   - Net investment is gross investment less depreciation $k_{t+1} - k_t$
6. Government
   - Taxes labor income at net τ_t and transfers lump-sum to households the tax revenuse b_t

Household Problem

- Labor income is $(1-\tau_1)w_1(1-l_1)$
- Capital income is $r_1 a_1 + d_1$ interest and dividends

$\max\limits_{a_2,\{c_t,l_t\}_{t=1}^t} u(c_1,l_2) + \beta u(c_2,l_2)$ such that:

- the first budget constraint $c_1 + a_2 = (1-\tau_1)w_1(1-l_1) + (1+r_1)\bar{a}_1 + d_1 + b_1$
- the second budget constraint $c_2 = (1-\tau_2)w_2(1-l_2) + (1+r_2)a_2 + d_2 + b_2$
- usual bound on c_t and l_t ignored due to Inada assumptions

Savings: $s_1 = y_1^D - c_1 = (1-\tau_1)w_1(1-l_1) + (1+r_1)\bar{a}_1 + d_1 + b_1 - c_1$ are exactly $a_2 - a_1$

- Lifetime budget constraint: substitute into a_2 in the first budget constraint the second budget constraint solved for a_2

$$c_1 + \frac{c_2}{1+r_2} = (1-\tau_1)w_1(1-l_1) + \frac{(1-\tau_2)w_2(1-l_2)}{1+r_2} + (1+r_1)\bar{a}_1 + d_1 + \frac{d_2}{1+r_2} + b_1 + \frac{b_2}{1+r_2}$$

The relevant Lagrangian is:

$$\begin{aligned}\mathcal{L}(c_1,c_2,l_1,l_2,\lambda) &= u(c_1,l_2) + \beta u(c_2,l_2) \\ &+ \lambda\left((1-\tau_1)w_1(1-l_1) + \frac{(1-\tau_2)w_2(1-l_2)}{1+r_2} + (1+r_1)\bar{a}_1 + d_1 + \frac{d_2}{1+r_2} + b_1 + \frac{b_2}{1+r_2} - c_1 - \frac{c_2}{1+r_2}\right)\end{aligned}$$

From consumption FOCs

$$\mathcal{L}_{c_1} = 0 \Leftrightarrow u_c(c_1^*, l_1^*) = \lambda$$

$$\mathcal{L}_{c_2} = 0 \Leftrightarrow \beta u_c(c_2^*, l_2^*) = \frac{\lambda}{1+r_2}$$

then we find the **Euler equation**:

$$\frac{u_c(c_1^*, l_1^*)}{u_c(c_2^*, l_2^*)} = \beta(1+r_2)$$

which is a general result which holds in multi-period economies: the intertemporal consumption allocation is determined by the discount factor and the interest rate.

From leisure FOCs

$$\mathcal{L}_{l_1} = 0 \Leftrightarrow u_l(l_1^*, l_1^*) = \lambda(1-\tau_1)w_1$$

$$\mathcal{L}_{l_2} = 0 \Leftrightarrow \beta u_l(l_2^*, l_2^*) = \frac{\lambda(1-\tau-2)w_2}{1+r_2}$$

then we find the intratemporal MRS:

$$\frac{u_l(c_1^*, l_1^*)}{u_c(c_2^*, l_2^*)} = (1-\tau_t)w_t$$

Note the distortionary taxes ensure a non-Pareto-optimal competitive equilibrium.

And as above the intertemporal time allocation is determined by the dicount factor, interest rate, wage, and taxes.

$$\frac{u_l(c_1^*, l_1^*)}{u_l(c_2^*, l_2^*)} = \frac{(1-\tau_1)w_1}{(1-\tau_2)w_2}\beta(1+r_2)$$

Firm Problem

$$d_t = \Pi_t = \max_{\{n_t,k_t\}_{t=1}^{2}} z_t f(k-t, n_t) - w(tn_t - (r_t+\delta)k_t$$

Marginal product of labor $z_t f_n(n_t, k-t) = w_t$

Marginal product of capital $z_t f_k(k-t, n_t) = r_t + \delta$

With constant returns to scale this only constrains the ratio.

Theorem: If $f(x,y)$ is CRS, then $f(x,y) = f_x(x) + f_y(y)$

Proof: CRS means $f(\theta x, \theta y) = \theta f(x,y) \forall \theta > 0$

Differentiate by θ, then set $\theta = 1$:

$$\frac{\partial f(\theta x, \theta y)}{\partial \theta} |_{\theta=1} = f_x(x,y)x + f_y(x,y)y$$

$$\frac{\partial \theta f(x,y)}{\partial \theta} |_{\theta=1} = f(x,y)$$

$$\Pi_t = z_t f(k_t, n_t) - z_t f_k k_t - z_t f_n N - t = z_t(f(k_t, n-t) - f_n(n_t) - f_k k_t) = 0$$

Then with CRS, $\Pi_t = 0$ always!

Government

$\tau_t w_t (1-l_t) = b_t$ at each t