Uncertainty

Normal two period problem with next income period stochastic $y_2(s) : y_2(\bar{s}_N) > \ldots > y_2(\bar{s}_1)$

Household $\max_{c_1,a_2,c_2} u(c_1) + \beta \sum s \in \mathcal{S} \pi(s) u(c_2(s))$ such that $c_1 + a_2 = y_1$ and w.p. $\pi(s)$, $c_2(s) = y_2(s) + Ra_2$.

Combine the $N+1$ budget constraints into N budget constraints: $c_1 + \frac{c_2(s)}{R} \leq y_1 + \frac{y_2(s)}{R}$

$$\mathcal{L} = u(c_1) + \sum_{s \in \mathcal{S}} \pi(s) \left(\beta u(c_2(s)) + \lambda_s \left(y_1 + \frac{y_2(s)}{R} - c_1 - \frac{c_2(s)}{R} \right) \right)$$

$$u'(c_1) = \sum_{s \in \mathcal{S}} \pi(s) \lambda_s = \mathrm{Ex}\left[\lambda_s\right]$$

$$\pi(s) \beta u'(c_2(s)) = \frac{\pi(s) \lambda_s}{R}$$

Add up the FOCs:

$$\beta \sum_{s \in \mathcal{S}} \pi(s) u'(c_2(s)) = \frac{1}{R} \sum_{s \in \mathcal{S}} \pi(s) \lambda_s \Rightarrow \beta \, \mathrm{Ex}\left[u'(c_2(s))\right] = \frac{1}{R} \mathrm{Ex}\left[\lambda_s\right]$$

Euler equation:

$$u'(c_1) = \beta R \, \mathrm{Ex}\left[u'(c_2(s))\right]$$

Risk Aversion

How can we measure the amount of risk aversion of a specific utility function? Take a lottery with payoff z with $\mathrm{Ex}[z] = 0$ with variance σ^2. What at premium π to avoid the lottery would you be indifferent about playing the lottery.

- Absolute risk aversion: how many dollars π (or absolute goods) are you willing to pay?

  $u(c - \pi) = \mathrm{Ex}\left[u(c + z)\right] \Rightarrow \pi = \frac{-vu''(c)}{2u'(c)}$

- Relative risk aversion: what fraction π of your consumption are you willing to pay?

  $u(c(1 - \pi)) = \mathrm{Ex}\left[u(c(1 + z))\right] \Rightarrow \pi = \frac{-cvu''(c)}{2u'(c)}$

With a concave function then the player is risk averse.

Intuition:

- Portfolios contain risky assets and safe assets.
- Increasing absolute risk aversion implies that as individuals get richer, they invest fewer dollars in stocks.
- Increasing relative risk aversion implies that as individuals get richer, they invest a smaller proportion of their wealth in stocks.
- Increasing absolute risk aversion is inconsistent with real world behavior.

Precautionary Savings

- If there is uncertainty about future income, agents save in order to be able to consume in the future even in the case of an undesirable income shock.
- Any u with prudence, i.e. $u''' > 0$, features this savings motive.
- To see this, note that when $u''' > 0$ then a_1 are increasing in the variance of y_2 by lecture.

Conclusion:

- What are uncertainty, random variable, probability distribution
- Write down and solve a model that includes uncertainty and solve for the Euler Equation
- What are risk aversion, precautionary saving as properties of utility functions and why are we interested, i.e. to narrow down what is a sensible functional form to describe utility functions

Permanent Income Hypothesis: Deterministic and Stochastic

a. We want to show $c_t = \text{Ex}_t\left[c_t + 1\right]$ since we know $\beta R = 1$ in order to show c_t is a Martingale.

   Introduce history notation $s^t - (s_i)_{i=0}^t$ so then $c_t(s^t) = c_t(s_0, s_1, \ldots s_t)$. The household problem is:

$$\max_{c_t(s_t), a_{t+1}(s_t)} \sum_{t=0}^{\infty}\sum_{s^t} \pi(s^t)\beta^t\left(b_1 c_t(s^t) - \frac{1}{2}b_2 c_t(s_t)^2\right)\Big| \forall t, \forall s^t, c_t(s^t) + a_{t+1}(s^t) \leq R a_t(s^{t-1}) + y_t(s^t)$$

$$\mathcal{L} = \sum_{t=0}^{\infty}\sum_{s^t} \pi(s^t)\beta^t\left(b_1 c_t(s^t) - \frac{1}{2}b_2 c_t(s_t)^2\right) - \sum_{t=0}^{\infty}\sum_{s^t} \pi(s^t)\lambda_t(s^t)\big(c_t(s^t) + a_{t+1}(s^t) - R a_t(s^{t-1}) - y_t(s^t)\big)$$

$$\frac{\partial \mathcal{L}}{\partial c_t(s^t)} = 0 \Rightarrow \pi(s^t)\beta^t\big(b_1 - b_2 c_t(s^t)\big) = \lambda_t(s^t)\pi(s^t)$$

$$\frac{\partial \mathcal{L}}{\partial a_{t+1}(s^t)} = 0 \Rightarrow \sum_{s^{t+1}} \lambda_{t+1}(s^{t+1})\pi(s^{t+1})R = \lambda_t(s^t)\pi(s^t)$$

   There is a term $\sum_{s^{t+1}} R a_{t+1}(s^t)\pi(s^{t+1})\lambda_{t+1}(s^{t+1})$ hence the $t+1$ term above. Then:

$$b_1 - b_2 c_t(s^t) = \beta R \sum_{s^{t+1}} \frac{\pi(s^{t+1})}{\pi(s^t)}\big(b_1 - b_2 c_{t+1}(s^{t+1})\big)$$

   This simplifies since $\pi(s^{t+1}) = \pi(s^t \wedge s_{t+1})$:

$$b_1 - b_2 c_t(s^t) = \beta R \sum_{s^{t+1}} \pi\big(s_{t+1}\big| s^t\big)\big(b_1 - b_2 c_{t+1}(s^{t+1})\big) = \beta R\, \text{Ex}_t\left[b_1 - b_2 c_{t+1}(s^{t+1})\right]$$

   This is exactly the Martingale statement above since $\beta R = 1$.

b.