$U(c,l) = \log(c) + \log(l)$, $f(n) = zn$

1. $c^*, l^* \Rightarrow \exists \alpha_i : \hat{c} = c^*, \hat{l} = c^*$.
2. Given $\alpha_i, (\hat{c}, \hat{l}), \Rightarrow \exists t_i : \hat{c} = c^*, \hat{l} = l^*$.

Household Problem

$\max_{\{c_i, l_i\}_i} \log(c_i) + \log(l_i)$ such that $c_i \leq (1 - l_i)\varepsilon_i w + \frac{1}{2}d + t_i$.

Lagrangian is $\mathcal{L}(c_i, l_i, \lambda_i) = \log(c_i) + \log(l_i) + \lambda_i\big((1 - l_i)\varepsilon_i w + \frac{1}{2}d + t_i - c_i\big)$.

First order conditions are $\frac{1}{c_i^*} = \lambda_i$ and $\frac{1}{l_i^*} = \lambda_i e_i w$; Euler equation is $\frac{c_i^*}{l_i^*} = \varepsilon_i w$.

Plug in to the budget constraint $\varepsilon_i l_i^* w = (1 - l_i^*)\varepsilon_i w + \frac{1}{2}d + t_i$. Then $l_i^* = \frac{\varepsilon_i w + \frac{1}{2}d + t_i}{2\varepsilon_i w}$ and $c_i^* = \frac{\varepsilon_i w + \frac{1}{2}d + t_i}{2}$.

Once we know the real wage and the dividends, we will know the equilibrium behavior of the worker.

Firm Problem

$\max_{n \in [0, \varepsilon_1 + \varepsilon_2]} (z - w)n$ will be $n^* = \varepsilon_1$ when $z > w$, anywhere when $z = w$, or 0 if $z < w$.

Thus either profits are zero everywhere or we are at a corner solution.

Competetive Equilibrium

Endogenous variables: allocations $n^*, \{c_i^*, l_i^*\}_i, d^*$ and prices w.

1. Given w and d^*, we have $\{c_i^*, l_i^*\}_i$, solve the household problem.
2. Given w, we have n^* solves the firm problem.
3. Labor market clearing: $n^* = \sum_i (1 - l_i^*)\varepsilon_i$
4. Skip goods market clearing condition by Walras' Law
5. Profits are transfered to dividends: $d^* = (z - w)n^*$

Consider Inada conditions: we know people will neither work always nor work never. Thus we know $z = w$ directly, which confirms $d^* = 0$. Then we have $l_i^* = \frac{1}{2}$ and $c_i^* = \frac{\varepsilon_i z}{2}$. By (3), we have $n^* = \frac{1}{2}\sum_i \varepsilon_i$.

Social Planner Problem

$\max_{\{c_i, l_i\}_i} \sum_i \alpha_i(\log(c_i) + \log(l_i))$ such that $\sum_i c_i \leq z \sum_i (1 - l_i)\varepsilon_i$ and $\sum_i \alpha_i = 1$.

First order conditions are $\frac{\alpha_i}{\hat{c}_i} = \lambda$ and $\frac{\alpha_i}{\hat{l}_i} = \lambda \varepsilon_i z$. Then $\frac{\hat{c}_i}{\hat{l}_i} = \varepsilon_i z$. This is exacly the same as the Euler equation from the HP, so we can definitely find $\{\alpha_i\}_i$ such that the CE is PO. Also $\frac{\hat{l}_j}{\hat{l}_i} = \frac{\alpha_j \varepsilon_i}{\alpha_i \varepsilon_j}$ and $\frac{\hat{c}_j}{\hat{c}_i} = \frac{\alpha_j}{\alpha_i}$.

Then we have four equations and four variables and elimiate all variables but c_1 and get conditions such as $\hat{l}_i = \frac{\alpha_i}{2\varepsilon_i}\sum_j \varepsilon_j$. Then by taking the CE allocation for l_i^* we solve for α_1 given $l_i^* = \hat{l}_i$ we have $\alpha_i = \frac{\varepsilon_i}{\sum_j \varepsilon_j}$.

First Welfare Theorem

To show the CE is PO, show the allocations (c-l FOCs) for a given household are the same in CE and PO.

Second Welfare Theorem

Then, take the Pareto outcome as given: $\hat{l}_i = \frac{\alpha_i}{2\varepsilon_i} \sum_j \varepsilon_j$ and $l_i^* = \frac{\varepsilon_i z + t_i}{2\varepsilon_i z}$. We know $\sum_i t_i = 0$.

Taking α_i as given and setting $\hat{l}_i = l_i^*$, $\frac{\alpha_i}{2\varepsilon_i} \sum_j \varepsilon_j = l_i^* = \frac{\varepsilon_i z + t_i}{2\varepsilon_i z}$ When $N = 2$ as throughout, $\alpha_1 z \varepsilon_2 - \alpha_2 z \varepsilon_1 = t$.