

Dynamic Production Economy

1. Time: $t \in \{1, 2\}$
2. Commodity set:
 - Final good (consumption and investment)
 - Labor services
 - Capital services:
 - supply a_t units of capital (wealth)
 - firm demand for capital k_t
 - $(1 - \delta)a_t$ undepreciated capital returned where δ is the depreciation rate
 - Net rental rate r_t then rented capital $(r_t + \delta)a_t$; note $r_t + \delta$ is the gross rental rate
 - Households then get back $(1 + r_t)a_t$
 - Households have an endowment \bar{a}_1 of wealth
3. All markets competitive
4. Preferences:
 - Households live for 2 periods, so $U(c_1, c_2, l_1, l_2) = u(c_1, l_1) + \beta u(c_2, l_2)$ where $\beta \in [0, 1]$ is the discount factor
5. Technology
 - $y_t = z_t f(k_t, n_t)$ as before
 - Constraint: f has constant returns to scale
 - $k_{t+1} = (1 - \delta)k_t + i_t$ where i_t is gross investment
 - Net investment is gross investment less depreciation $k_{t+1} - k_t$
6. Government
 - Taxes labor income at net τ_t and transfers lump-sum to households the tax revenue b_t

Household Problem

- Labor income is $(1 - \tau_1)w_1(1 - l_1)$
- Capital income is $r_1a_1 + d_1$ interest and dividends

$\max_{a_2, \{c_t, l_t\}_{t=1}^T} u(c_1, l_2) + \beta u(c_2, l_2)$ such that:

- the first budget constraint $c_1 + a_2 = (1 - \tau_1)w_1(1 - l_1) + (1 + r_1)\bar{a}_1 + d_1 + b_1$
- the second budget constraint $c_2 = (1 - \tau_2)w_2(1 - l_2) + (1 + r_2)a_2 + d_2 + b_2$
- usual bound on c_t and l_t ignored due to Inada assumptions

Savings: $s_1 = y_1^D - c_1 = (1 - \tau_1)w_1(1 - l_1) + (1 + r_1)\bar{a}_1 + d_1 + b_1 - c_1$ are exactly $a_2 - a_1$

- Lifetime budget constraint: substitute into a_2 in the first budget constraint the second budget constraint solved for a_2

$$c_1 + \frac{c_2}{1 + r_2} = (1 - \tau_1)w_1(1 - l_1) + \frac{(1 - \tau_2)w_2(1 - l_2)}{1 + r_2} + (1 + r_1)\bar{a}_1 + d_1 + \frac{d_2}{1 + r_2} + b_1 + \frac{b_2}{1 + r_2}$$

The relevant Lagrangian is:

$$\begin{aligned} \mathcal{L}(c_1, c_2, l_1, l_2, \lambda) &= u(c_1, l_2) + \beta u(c_2, l_2) \\ &+ \lambda \left((1 - \tau_1)w_1(1 - l_1) + \frac{(1 - \tau_2)w_2(1 - l_2)}{1 + r_2} + (1 + r_1)\bar{a}_1 + d_1 + \frac{d_2}{1 + r_2} + b_1 + \frac{b_2}{1 + r_2} - c_1 - \frac{c_2}{1 + r_2} \right) \end{aligned}$$

From consumption FOCs

$$\begin{aligned} \mathcal{L}_{c_1} &= 0 \Leftrightarrow u_c(c_1^*, l_1^*) = \lambda \\ \mathcal{L}_{c_2} &= 0 \Leftrightarrow \beta u_c(c_2^*, l_2^*) = \frac{\lambda}{1 + r_2} \end{aligned}$$

then we find the **Euler equation**:

$$\frac{u_c(c_1^*, l_1^*)}{u_c(c_2^*, l_2^*)} = \beta(1 + r_2)$$

which is a general result which holds in multi-period economies: the intertemporal consumption allocation is determined by the discount factor and the interest rate.

From leisure FOCs

$$\begin{aligned} \mathcal{L}_{l_1} &= 0 \Leftrightarrow u_l(c_1^*, l_1^*) = \lambda(1 - \tau_1)w_1 \\ \mathcal{L}_{l_2} &= 0 \Leftrightarrow \beta u_l(c_2^*, l_2^*) = \frac{\lambda(1 - \tau_2)w_2}{1 + r_2} \end{aligned}$$

then we find the intratemporal MRS:

$$\frac{u_l(c_1^*, l_1^*)}{u_c(c_2^*, l_2^*)} = (1 - \tau_t)w_t$$

Note the distortionary taxes ensure a non-Pareto-optimal competitive equilibrium.

And as above the intertemporal time allocation is determined by the discount factor, interest rate, wage, and taxes.

$$\frac{u_l(c_1^*, l_1^*)}{u_l(c_2^*, l_2^*)} = \frac{(1 - \tau_1)w_1}{(1 - \tau_2)w_2} \beta(1 + r_2)$$

Firm Problem

$$d_t = \Pi_t = \max_{\{n_t, k_t\}_{t=1}^2} z_t f(k_t, n_t) - w_t n_t - (r_t + \delta) k_t$$

Marginal product of labor $z_t f_n(k_t, n_t) = w_t$

Marginal product of capital $z_t f_k(k_t, n_t) = r_t + \delta$

With constant returns to scale this only constrains the ratio.

Theorem: If $f(x, y)$ is **CRS**, then $f(x, y) = f_x(x) + f_y(y)$

Proof: CRS means $f(\theta x, \theta y) = \theta f(x, y) \forall \theta > 0$

Differentiate by θ , then set $\theta = 1$:

$$\frac{\partial f(\theta x, \theta y)}{\partial \theta} \big|_{\theta=1} = f_x(x, y)x + f_y(x, y)y$$

$$\frac{\partial \theta f(x, y)}{\partial \theta} \big|_{\theta=1} = f(x, y)$$

$$\Pi_t = z_t f(k_t, n_t) - z_t f_k k_t - z_t f_n N - t = z_t (f(k_t, n - t) - f_n(n_t) - f_k k_t) = 0$$

Then with CRS, $\Pi_t = 0$ always!

Government

$\tau_t w_t (1 - l_t) = b_t$ at each t