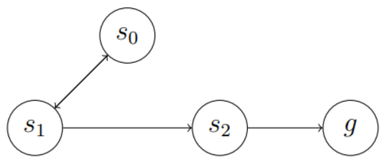
National University of Singapore
School of Computing
CS3243 Introduction to AI

Tutorial 2: Informed Search

SOLUTIONS

1. (a) Provide a counterexample to show that the **tree-search** implementation of the **Greedy Best-First Search** algorithm is **incomplete**.

Solution: Consider a search space with initial state s_0, goal state g, and where h(s_0) = 3, h(s_1) = 4, h(s_2) = 5, and h(g) = 0.



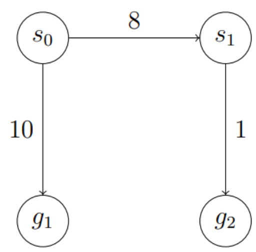
Each time s_0 is explored, we add s_1 to the front of the frontier, and each time s_1 is explored, we add s_0 to the front of the frontier. Notice that s_2 is never at the front of the frontier. This causes the greedy best-first search algorithm to continuously loop over s_0 and s_1.

(b) Briefly explain why the **graph-search** implementation of the **Greedy Best-First Search** algorithm is **complete**.

Solution: Assuming a finite search space, a graph-search implementation of the greedy best-first search algorithm will eventually visit all states within the search space. As such, the algorithm would either find a goal state and return the path to it, or else, indicate that there is no solution if a goal is not found.

(c) Provide a counterexample to show that neither the **tree-search** nor the **graph-search** implementations of the **Greedy Best-First Search** algorithm are **optimal**.

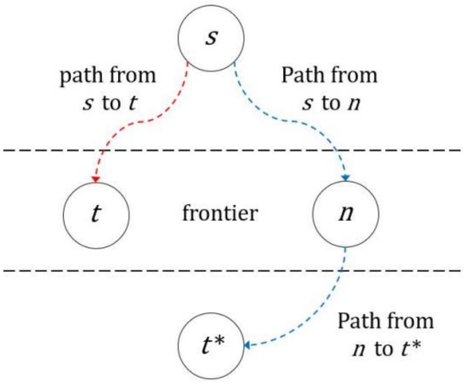
Solution: Consider the following search space with initial state s_0, goal states g_1 and g_2, and where h(s_0) = 9, h(g_1) = 0, h(s_1) = 1, and h(g_2) = 0.



With either implementation, when s_0 is explored, g_1 would be added to the front of the frontier and then explored next, resulting in the algorithm returning the non-optimal $s_0 \rightarrow g_1$ path.

2. (a) Prove that the **tree-search** implementation of the **A* Search** algorithm is optimal when an **admissible heuristic** is utilised.

Solution: Assume the following search tree below.



Let s be the initial state, n be an intermediate state along the optimal path, t be a suboptimal goal state (i.e., a goal state reached via a suboptimal path), and t^* be the goal along the optimal path.

An optimal solution implies that n must be expanded before t.

Proof by contradiction:

- Let us assume that a suboptimal solution is found – i.e., that t is expanded before n, which implies that (A): $f(t) \leq f(n)$.
- In other words, given the above frontier, only when $f(t) < f(n)$ would we expand t before n.
- However, since t is not on the optimal path but t^* is, we have:
  - $f(t) > f(t^*)$
  - $f(t) > g(t^*)$ // since $h(t^*) = 0$
  - $f(t) > g(n) + p(n, t^*)$ // where $p(n, t^*)$ is the actual cost from n to t^*
  - $f(t) > g(n) + h(n)$ // asserting admissibility
  - $f(t) > f(n)$ // this contradicts (A)
- Note: we do not consider $f(t) = f(n)$ since that would mean $f(t)$ is equally optimal.

(b) Prove that the **graph-search** implementation of the **A* Search** algorithm is optimal when a **consistent heuristic** is utilised. Assume graph search **Version 3**.

Solution: Similar to the UCS proof of optimality under graph search, we must show that when a node n is popped from the frontier, we have found the optimal path to it. (Otherwise, we might pop a non-optimal goal node off the frontier before we pop the optimal one.)

Let $f(s_k) = g(s_k) + h(s_k)$ be the minimum f value for s_k we have observed when s_k is popped.

Let the optimal path from the start node, s_0, to any node, s_g, be $\boldsymbol{P} = s_0, s_1, \ldots, s_{g-1}, s_g$. We must show that when we pop s_g, $f(s_g) = g(s_g) + h(s_g) = g^*(s_g) + h(s_g)$, where $g^*(s_g)$ denotes the optimal path cost from s_0 to s_g via $\boldsymbol{P}$.

Base case: $f(s_0) = g(s_0) + h(s_0) = g^*(s_0) + h(s_0) = h(s_0)$ as s_0 is the start node.

Induction case: Assume that for all $s_0, s_1, \ldots, s_k$, when we pop s_i, we have $f(s_i) = g(s_i) + h(s_i) = g^*(s_i) + h(s_i)$, or rather, $g(s_i) = g^*(s_i)$.

Since $g^*(s_{k+1})$ is the minimum path cost from s_0 to s_{k+1}, we know that:

- $g(s_{k+1}) + h(s_{k+1}) \geq g^*(s_{k+1}) + h(s_{k+1})$
- $g(s_{k+1}) \geq g^*(s_{k+1})$ // denote this as expression (A)

To make sure that each s_{k+1} is only popped after we pop s_k, the condition $f(s_k) \leq f(s_{k+1})$, or rather $h(s_k) \leq c(s_k, s_{k+1}) + h(s_{k+1})$, where $c(s_k, s_{k+1})$ is the action cost from s_k to s_{k+1}, is required, which leads us to assert that h is consistent.

Consequently, just after s_k is popped, we have:

- $g(s_{k+1}) + h(s_{k+1}) = \boldsymbol{min}\{g(s_{k+1}) + h(s_{k+1}), g(s_k) + c(s_k, s_{k+1}) + h(s_{k+1})\}$
- $g(s_{k+1}) = \boldsymbol{min}\{g(s_{k+1}), g(s_k) + c(s_k, s_{k+1})\}$
- $g(s_{k+1}) \leq g(s_k) + c(s_k, s_{k+1})$
- $g(s_{k+1}) = g^*(s_k) + c(s_k, s_{k+1})$ // from the inductive hypothesis
- $g(s_{k+1}) = g^*(s_{k+1})$ // denote this as expression (B)

From (A) and (B), we obtain $g(s_{k+1}) = g^*(s_{k+1})$. Hence, by induction, whenever we pop a node from the frontier, the optimal path to that node would have been found. Also, given that graph search version 3 is utilised (i.e., only nodes popped from the frontier are added to reached), the optimal path would not be excluded.

3. (a) Given a **heuristic** h, such that $h(t) = 0$, where t is any goal state, prove that if h is **consistent**, then it must be **admissible**.

Solution: The proof is by induction of $k(n)$, which denoted the number of actions required to reach the goal from a node n to the goal node t.

Base case ($k = 1$, i.e., the node n is one step from t): Since the heuristic function h is consistent, $h(n) \leq c(n, t) + h(t)$. And since $h(t) = 0$, $h(n) \leq c(n, t) = h^*(n)$. Therefore, h is admissible.

Induction case: Suppose that our assumption holds for every node that is $k - 1$ actions away from t, and let us observe a node n that is k actions away from t; that is, the least-actions optimal path from n to t has $k > 1$ steps.

We write the optimal path from n to t as: $n \rightarrow n_1 \rightarrow n_2 \rightarrow \ldots \rightarrow n_{k-1} \rightarrow t$.

Since h is consistent, we have $h(n) \leq c(n, n_1) + h(n_1)$.

Now, note that since n_1 is on a least-cost path to t from n, we must have that the path $n_1 \rightarrow n_2 \rightarrow \ldots \rightarrow n_{k-1} \rightarrow t$ is a minimal-cost path from n_1 to t as well. By our induction hypothesis, we have $h(n_1) \leq h^*(n_1)$.

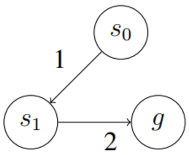
Consequently, combining the two inequalities above, we have, $h(n) \leq c(n, n_1) + h^*(n_1)$.

Note that $h^*(n_1)$ is the cost of the optimal path from n_1 to t; by our previous observation (that $n_1 \rightarrow n_2 \rightarrow \ldots \rightarrow n_{k-1} \rightarrow t$ is an optimal cost path from n_1 to t), we have that the cost of the optimal path from n to t – i.e., $h^*(n)$ – is exactly $c(n, n_1) + h^*(n_1)$, which concludes the proof.

(b) Give an example of an **admissible heuristic** that is **not consistent**.

Solution: An example of an admissible heuristic function that is not consistent is as follows.

Consider a heuristic function h, such that $h(s_0) = 3$, $h(s_1) = 1$, and $h(t) = 0$ for the following graph.



h is admissible since:

- $h(s_0) \leq h^*(s_0) = 1 + 2 = 3$
- $h(s_1) \leq h^*(s_1) = 2$

However, h is not consistent since $3 = h(s_0) > c(s_0, s_1) + h(s_1) = 1 + 1 = 2$.

4. We have seen various search strategies in class and analysed their worst-case running time. Prove that **any deterministic search algorithm** will, in the worst case, **search the entire state space**. More formally, prove the following theorem.

Theorem 1. *Let $\mathcal{A}$ be some complete, deterministic search algorithm. Then for any search problem defined by a finite connected graph $\boldsymbol{G} = \langle \boldsymbol{V}, \boldsymbol{E} \rangle$ (where $\boldsymbol{V}$ is the set of possible states and $\boldsymbol{E}$ are the transition edges between them), there exists a choice of start node s_0 and goal node g so that $\mathcal{A}$ searches through the entire graph $\boldsymbol{G}$.*

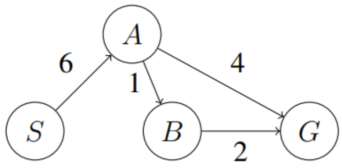
Solution: Let us begin by running $\mathcal{A}$ on the graph $\boldsymbol{G}$, without setting any goal node at all: that is, there are no goal nodes at all in $\boldsymbol{G}$. In this case, the algorithm $\mathcal{A}$ will return "False" when it explores the entire set $\boldsymbol{V}$. Let $\boldsymbol{H}_t(\mathcal{A}, s_0) \subseteq \boldsymbol{V}$ be the set of nodes that $\mathcal{A}$ explores if it starts at s_0, and does not encounter a goal node at steps 1, …, t (at $t = 1$, we have $\boldsymbol{H}_1(\mathcal{A}, s_0) = \{s_0\}$). We also let v_t be the node that $\mathcal{A}$ selects at time t given that it has observed the set $\boldsymbol{H}_{t-1}(\mathcal{A}, s_0)$ so far. We note that it is entirely possible that $\mathcal{A}$ selects $v_t \in \boldsymbol{H}_{t-1}(\mathcal{A}, s_0)$; however, we make a simple observation: the sequence $(\boldsymbol{H}_t(\mathcal{A}, s_0))_{t=1}^{\infty}$ is weakly increasing in size, and there exists some timestep t^* such that for all $t > t^*$, $\boldsymbol{H}_t(\mathcal{A}, s_0) = \boldsymbol{V}$; in other words, since $\mathcal{A}$ is a complete search algorithm, it will continue exploring the nodes in $\boldsymbol{G}$ until all nodes have been exhausted. Let us assume that t^* is the first timestep for which $\boldsymbol{H}_t(\mathcal{A}, s_0) = \boldsymbol{V}$. In other words, at time t^*-1, $|\boldsymbol{H}_{t^*-1}(\mathcal{A}, s_0)| = |\boldsymbol{V}| - 1$.

We now set the goal node to be v_{t^*}. From our previous argument, we know that when $\mathcal{A}$ starts at s_0 it will explore a set of size $|\boldsymbol{V}| - 1$ before reaching v_{t^*}, realising that it is a goal node and terminating. In other words, for any node s_0, if we select a goal node according to the above procedure, the algorithm $\mathcal{A}$ will exhaustively search through the entire graph before reaching a goal node.

Here is another, inductive proof. Let us set the goal node to some arbitrary node g_1. If $\mathcal{A}$ searches through the entire graph $\boldsymbol{G}$ when g_1 is the goal, we are done; otherwise, let $\boldsymbol{U}_1$ be the set of unsearched nodes when g_1 is the goal node. We take an arbitrary node g_2 in $\boldsymbol{U}_1$ to be the goal; since $\mathcal{A}$ is deterministic and complete it will run the same search order that it did when g_1 was the goal, and then search through the nodes in $\boldsymbol{U}_1$ until it reaches g_2. If it searched through all the nodes in $\boldsymbol{U}_1$ as well, we are done, otherwise repeat.

In general, suppose that we have set g_t to be the goal node and that $\mathcal{A}$ did not search through the entire graph until it reached g_t; let $\boldsymbol{U}_t$ be the set of unsearched nodes when g_t is the goal node. We
set g_{t+1} to be some arbitrary node in $\boldsymbol{U}_t$ and rerun $\mathcal{A}$; since $\mathcal{A}$ is deterministic, we know that when g_{t+1} is the goal we have $\boldsymbol{U}_{t+1} \subset \boldsymbol{U}_t$. Since $\boldsymbol{U}_1 \supset \boldsymbol{U}_2 \supset \ldots \supset \boldsymbol{U}_t$ and the number of nodes in $\boldsymbol{G}$ is finite, there exists some iteration t^* such that $\boldsymbol{U}_{t^*} = \emptyset$; thus g_{t^*} is a goal node for which $\mathcal{A}$ searches through the entire graph.

5. (a) In the search problem below, we have listed 5 heuristics. Indicate whether each **heuristic** is **admissible** and/or **consistent** in the table below.



Solution:

	S	A	B	G	**Admissible**	**Consistent**
h_1	0	0	0	0	True	True
h_2	8	1	1	0	True	False
h_3	9	3	2	0	True	True
h_4	6	3	1	0	True	False
h_5	8	4	2	0	False	False

(b) Write out the order of the nodes that are explored by the **A* Search** algorithm. Assume the use of a **graph search version 3** implementation that utilises heuristic h_4. You must express your answer in the form *A–B–C* (i.e., no spaces, all uppercase letters, delimited by the dash (–) character), which, for example, corresponds to the order *A*, then *B*, and then *C*.

Solution: *S–A–B–G*.

(c) Which heuristic would you use? Explain why.

Solution: The heuristic h_3 corresponds to the exact cost from each node to the goal node (i.e., $h_3 = h^*$), and therefore it is the optimal heuristic. This makes h_3 the best choice.

(d) Prove or disprove the following statement:

The heuristic $h(n) = \boldsymbol{max}\{h_3(n), h_5(n)\}$ is admissible.

Solution: This is false, since $4 = h(A) > h^*(A) = 3$.