1.6: Cobweb Diagrams and Equilibria

1 Terminology

- Suppose we have a DDS
$$m_{t+1} = f(m_t).$$

- Ex: Identify the updating functions for these DDS:

  1. $m_{t+1} = 2m_t, \quad f(x) = 2x$
  2. $m_{t+1} = -m_t, \quad f(x) = -x$
  3. $m_{t+1} = (m_t)^2, \quad f(x) = x^2$
  4. $m_{t+1} = \cos(m_t), \quad f(x) = \cos(x)$
  5. $$m_{t+1} = \frac{0.6m_t}{1 + 17m_t}, \quad f(x) = \frac{0.6x}{1 + 17x}$$

  We call the function f the updating function, because it takes as input a value m_t and "updates" it to the next value, m_{t+1}.

- Why do we care about the updating function? It's a function! What do we know how to do with functions? ☺

- Sketch bubble picture.

- We can graphically represent the sequence $m_0, m_1, m_2, m_3, \ldots$ by a *cobweb diagram*.

- spend lots of time analyzing the cobweb applet.

- Do these examples:

  1. $f(x) = 0.5x + 1$ (concentration model)
  2. $f(x) = 1.5x - 2$
  3. $f(x) = x^2 - 1$
  4. $f(x) = 4x(1 - x)$

- Cobwebbing synopsis:

  1. Draw the updating function (blue in the program)
  2. Draw the diagonal (red in the program)
  3. start at the initial condition
  4. Go vertical to the updating function (up to the update)
  5. Go horizontally to the diagonal

6. rinse and repeat.

   – up/down is *evaluating the updating function* (steps forward in time)
   – left/right is *only mechanical*; it's more of a mechanical necessity, not the actual math.

- Examples: Plot the solution m_t vs t from the cobweb diagram.
- Ex: Do the cobweb diagram for $m_{t+1} = -2m_t^3$ and use it to sketch a solution graph.

2 Equilibria

- Def: equilibrium or *equilibria* are the values of a DDS that are constant: if $m_{t+1} = f(m_t)$ is some DDS, an equilibrium, which we'll denote by m^*, has $m^* = m_t = m_{t+1} = m_{t+2} = \cdots$. In other words, if the sequence $m_0, m_1, m_2, \ldots$ stabilizes, then that stabilizing value is m^*, the equilibrium.
- Finding Equilibria: You see the equilibria on the graph as the *places where $f(x)$ crosses the diagonal*.
- Algebraically: Find m^* by solving the equation $f(m^*) = m^*$. (That's the equation corresponding to the blue curve crossing the diagonal.
- Examples:
  – (Concentration model) Verify that $C^* = 2$ mg/L is the equilibrium of the drug concentration model $C_{t+1} = 0.5C_t + 1$.
  – Find equilibria to $m_{t+1} = 6m_t - 7$. $m^* = 7/5 = 1.4$.
  – (Bacteria model) Find equilibria to $b_{t+1} = 2b_t$. $b^* = 0$.
- Important: characterize equilibria based on if whether solutions *go towards* or *go away* from the equilibria. If nearby solutions go towards m^*, we call it *stable*. If nearby solutions all go away from m^*, we call it *unstable*. If it's half-and-half, I will call it semi-stable.
- Examples:
  – $m_{t+1} = 2m_t + 1$. Find equilibrium, decide stable or not. (It's unstable)
  – $m_{t+1} = 0.2m_t + 5$. Find equilibrium, decide stable or not. (It's stable)
  – $m_{t+1} = m_t^2 + m_t$. Find equilibrium, decide stable or not. (It's semistable)