3.3: Optimization

1 Optimization Problems

- Very common use of calculus: find when a function is maximum or minimum.
- Terminology: "extremum," "extrema," or "optima" refer to extreme or optimum values, meaning maximum or minimum.
- Definition. A global maximum is a y-value on the graph of $f(t)$ that is the largest on the entire graph. Similarly, a global minimum is a y-value that is the smallest on the graph.
- smallest means most negative, not necessarily closest to 0.
- Example: What is the global maximum of the function $f(x) = -x^2 + 1$ on the whole number line? Global minimum? A: global max: $y = 1$ at $t = 0$. No global minimum.
- Notice: the maximum occurred at a *critical point*.
- Ex: Same $f(x)$, but what if we change the interval to $[-2, -1]$? A: global max is $y = 0$, global min is at $y = -3$.
- Notice: the global optima occurred at endpoints of our interval.
- Caution: a global maximum or minimum is sensitive to the choice of endpoints for x or t.

1.1 Algorithm for determining global optima

When $f(x)$ is defined on a closed interval $[a, b]$ (meaning it doesn't go off forever in one direction), the following steps find the global maxima and minima.

1. Find all critical points of $f(t)$. Remember, these are t-values.
2. Throw out any critical points not in between the endpoints.
3. Plug in the t-values found in step 1, as well as the t-values of the endpoints. You now have a list of y-values.
4. select the largest y-value from this list, and the smallest. These are the global maxima and minima.

2 Examples

- Ex: Optimize $f(x) = x^3 - 3x$ on the interval $0 \leq x \leq 2$.
    1. critical points: $f'(x) = 3x^2 - 3 = 0$ gives $x = -1$ and $x = +1$.
    2. we throw out $x = -1$ since it is not between 0 and 2.

3. we now plug in $x = 0, 1$, and 2 into $f(x)$ (NOT the derivative).
   The y-values are: $0, -2$, and 2. So, global maximum is $y = 2$ at $t = 2$, global minimum is at $y = -2$ with $t = 1$.

- Ex: Find optima of $g(x) = \frac{x}{1+x}$ on the interval $0 \le x \le 1$.
  1. critical points: this is where $g'(x) = 0$ or $g'(x)$ is undefined.
     $$g'(x) = \frac{1(1+x) - x(1)}{(1+x)^2} = \frac{1}{(1+x)^2}$$
     This is never equal to 0. But it is undefined at $x = -1$, so -1 is a critical point.
  2. Throw out $x = -1$ since -1 is not between 0 and 1.
  3. plug in critical points and endpoints: $g(0) = 0$, $g(1) = 1/2$.
  4. global maximum is at $y = 1/2$, $x = 1$, global minimum is at $y = 0$, $x = 0$.

- Extreme Value Theorem: Any continuous function $f(x)$ on a closed and bounded interval will have a global maximum and minimum.

- Closed means that the interval contains its endpoints. For example, $0 \le x \le 2$ is closed, but $0 \le x < 2$ is not closed, nor is $0 < x < 2$.

- bounded means the interval doesn't go on forever. For example, $0 \le x \le 2$ is bounded, but $0 \le x < \infty$ is not bounded.

- Ex: Find optima of $H(x) = xe^{-x}$ on the interval $0 \le x \le 2$. Also, explain why the Extereme Value Theorem applies.
  1. $H(x)$ is a product of two continuous functions, x and e^{-x}, and so it is continuous. The
  2. critical points.
     $$H'(x) = e^{-x} - xe^{-x} = e^{-x}(1 - x) = 0$$
     gives only one answer: $x = 1$.
  3. Throw out any points? no, $x = 1$ is inside the interval, so we're good.
  4. check y-values at endpoints and critical points:
     $$H(0) = 0e^0 = 0, \quad H(2) = 2e^{-2} \approx 0.271, \quad H(1) = 1e^{-1} \approx 0.369$$
  5. Global maximum is $y \approx 0.369$ at $x = 1$, global minimum is $y = 0$ at $x = 0$.

3 Application

- Consider a bee that travels from flower to flower. Question: how long should the bee spend at each flower to maximize its nectar intake?

- We will practice *modeling.*

- Problem: if bee stays longer at one flower, the nectar depletes, and the bee gets less and less as time goes on. So, staying too long is bad for the bee. (fewer flowers)

- On the other hand, it takes a while to fly between flowers, so staying too short is also bad for the bee. (less nectar)

- The bee should maximize the total amount of nectar collected. To do this, it should really maximize the *rate per visit* for collecting nectar.
- Modeling assumption 1: the time between flowers is the same constant. We will call this τ. (While not true at all, we have to make some simplifying assumptions to make the problem solvable.)
- Let t be the time spent at a flower in seconds.
- Let $F(t)$ be the amount of food collected after t seconds.
- We want to maximize the function

$$R(t) = \frac{\text{food from one flower}}{\text{flower time plus travel time}}$$

- Insert:

$$R(t) = \frac{F(t)}{t+\tau}.$$

- Specific example. Let's suppose that the bee takes two seconds to travel flower to flower. Let's also assume that each flower's nectar supply goes like

$$F(t) = \frac{t}{t+1}.$$

  [Aside: $F(t)$ approaches 1 as t gets large, meaning the bee can only get a limited amount of nectar from the flower.]

  Also assume time between flowers is 2 seconds. So, $\tau = 2$ sec.

- $$R(t) = \frac{F(t)}{t+2} = \frac{1}{t+2}F(t) = \frac{1}{t+2}\frac{t}{t+1} = \frac{t}{(t+1)(t+2)}$$

- Optimize $R(t)$:

  1. critical points:

  $$R'(t) = \frac{1(t+1)(t+2) - t(1(t+2)+(t+1)1)}{\big((t+1)(t+2)\big)^2}$$

  This simplifies to

  $$R'(t) = \frac{-t^2+2}{(\text{stuff})^2}$$

  Denominators never make a fraction 0, so set the numerator equal to 0.

  $$-t^2+2=0$$

  get: $t = \pm\sqrt{2}$ sec, but really $t = \sqrt{2} \approx 1.4$ seconds.

- Notice: the model doesn't give us bounds for t. Use graph to plot $R(t)$ and check it is a maximum.