2.1 Derivatives

1 Introduction to the Course

- Goal: discuss how quantities change in time. We want to predict the future!
- two approaches in this class: derivatives, discrete dynamical systems
- focus on derivatives first and DDS's later. But here's an overview.
- Example: two ways to discuss a bacteria colony that doubles in size each minute, starting from one bacterium.
- method one: study a differential equation:

$$\frac{dP}{dt} = \ln(2)P$$

  where P is a function of time. The solution (later) is $P(t) = 2^t$, which tells us how big the population is.
- method two: "one step of time"
- specify a rule for "updating" the population's value.
- after each minute, the population is updated to twice its current value.

$$P_{n+1} = 2P_n$$

  this is read from right to left: given a value of P_n, you get a new value for P_{n+1}. Subscript is for keeping track of how many steps of time have gone by. So, P_0 is the beginning (0 steps have gone), while P_5 would be the population after 5 steps.

$$P_0 = 1, \quad P_1 = 2P_0 = 2, \quad P_2 = 2P_1 = 2 * 2 = 4, \quad ...$$

  the rule $P_{n+1} = 2P_n$ manually forces the population to double after each time step.
- we'll start by developing some of the tools in method 1 with derivatives, and come back to DDS's later on.

2 Derivatives

- Core concept for this class AND 247: the derivative.
- To start: functions, $f(t)$: take an input in (a number, t) and returns outputs (a number, $f(t)$).

- Recall from Math 111: Average Rate of Change (ARC):

$$ARC_{[a,b]} = \frac{f(b) - f(a)}{b - a} = \frac{\Delta f}{\Delta t}$$

- Ex: $f(t) = \sin(t)$ over the interval $[0.2, 2.6]$:

$$ARC = \frac{\sin(2.6) - \sin(0.2)}{2.6 - 0.2} = \frac{0.317}{2.4} \approx 0.132$$

  This number crudely says that for each change of t by 1, $\sin(t)$ changes by 0.132.

- pictorially: the ARC is the slope of the line connecting point $t = a$ to $t = b$
- *draw picture*
- Def: we call this line a *secant line*.
- How can we make the secant line a better approximation of the curve?
- You say: make the interval smaller!
- Desmos demo
- Book-keeping change: instead of reporting $[a, b]$, I'll give it to you as basepoint t_0 and a Δt.
- you notice that when we make Δt small, the secant line solidifies.
- Def: This is called the *tangent line*.
- Def: The slope of the tangent line is called the *derivative* of $f(t)$ at t_0.
- ex: $f(t) = 0.5t^2$. (a) find secant line based at $t_0 = 1$ with $\Delta t = 1, 0.1,$ and 0.01.
- (b) Guess what the ARC is approaching.
- (c) with this guess, find the equation of the tangent line at $t_0 = 1$.
- ANS: Here's $\Delta t = 1$:

$$ARC = \frac{f(2) - f(1)}{2 - 1} = 0.5(2^2) - 0.5(1)^2 = 2 - 0.5 = 1.5.$$

  For $\Delta t = 0.1$, I use a calculator to find $ARC = 1.05$, and $\Delta t = 0.01$ gives $ARC = 1.005$.

  (b) I guess the ARC is approaching a solid 1. (c) Tangent line: slope is 1. $f(1) = 0.5$, so using these two pieces of info, find the line:

$$y = t - 0.5$$

3 Definition

The definition of the derivative is key:

$$f'(t_0) = \lim_{\Delta t \to 0} \frac{f(t_0 + \Delta t) - f(t_0)}{\Delta t}$$

This is the definition we want to understand. As such, we'll now focus on limits and what the heck they are.