5.2/5.3: Stability and Equilibria for Autonomous DE

Contents

1 Equilibria

- Def: For an autonomous diffy-Q: We call a y-value, y^*, an *Equilibrium* if $\frac{dy}{dt} = 0$.
- Ex: Temperature model:

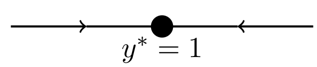
$$\frac{dT}{dt} = k(T_0 - T).$$

  Set $\frac{dT}{dt} = 0$. Get $k(T_0 - T) = 0$, so either $k = 0$ or $T^* = T_0$.
- Note: you *should* consider all possible scenarios.
- If $k = 0$, then the temperature of the object *never* changes. (unrealistic)

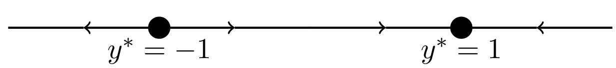
- If $T^* = T_0$, the temperature of the object is always at the ambient temp.
- Ex:
$$\frac{dy}{dt} = 3(1-y)(y^2+1).$$
Possible equilibria: set $(1-y)(y^2+1) = 0$. Get $y^* = 1$ is the only equilibrium.

2 Graphical Representation

- Introducing: phase-line diagrams.
- Key idea: encode behavior of solutions and equilibrium.
- Ex: Consider diffy-Q $\frac{dy}{dt} = (1-y)$. This is autonomous.
- Equilibrium: $y^* = 1$.
- Draw:



- Draw arrows according to whether y is increasing or decreasing (which you can tell from the diffy-Q).
- Ex: $\frac{dy}{dt} = (1-y^2)$ Equilibria: $y^* = \pm 1$.



3 Stability

- We use phase-line diagrams to assess stability of solutions.

- Ex: Using the example with $\frac{dy}{dt} = (1-y)$, we can assess that $y^* = 1$ is a stable equilibrium.
- Ex: using the example with $\frac{dy}{dt} = 1 - y^2$ we can assess that the equlibrium $y^* = -1$ is unstable while $y^* = 1$ is stable.
- Stable: all arrows go in. Unstable: all arrows go out.
- $\frac{dy}{dt} = y^2$. Equilibrium: $y^* = 0$. Phase-line diagram:

  $y^* = 0$

  This is neither stable nor unstable.
- As for discrete dynamical systems, there is a stability theorem:
- Thm: if $\frac{dy}{dt} = f(y)$ has equilibrium y^*, then:

$$\begin{array}{r|l} y^* \text{ is stable} & f'(y^*) < 0 \\ y^* \text{ is unstable} & f'(y^*) > 0 \end{array}$$

- If $f'(y^*) = 0$, the test is inconclusive; it may be stable, unstable, or neither.
- Ex: $\frac{dy}{dt} = 8 - y^3$. Equilibria: $y^* = 2$. So,

$$f(y) = 8 - y^3, \qquad f'(y) = -3y^2.$$

  We see that $f'(2) = -12 < 0$, so this equilibrium is stable.