Worksheet 6

Math 251, Summer 2017

Name: Key

Tip: Go to Desmos.com to plot implicit functions!

1. Find the derivative $y' = y'(x)$ of y with respect to x.

   (a) $x^2 + y^2 = 4$

   $$2x + 2y \cdot y' = 0$$

   $$2y\,y' = -2x$$

   $$y' = \frac{-2x}{2y}$$

   $$\boxed{y' = \frac{-x}{y}}$$

   Same!

   (b) $x^2 + y^2 = 9$

   $$2x + 2y \cdot y' = 0$$

   $$\boxed{y' = \frac{-x}{y}}$$

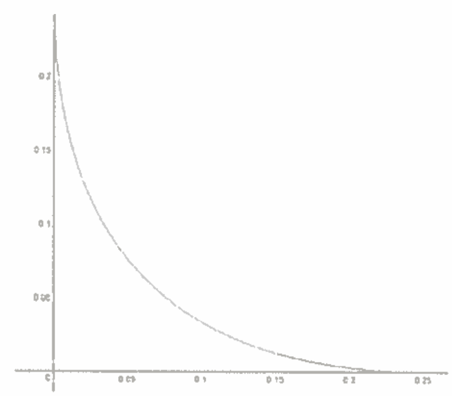
   (c) $2\sqrt{x} + 2\sqrt{y} = 1$

   $$2 \cdot \tfrac{1}{2} x^{-1/2} + 2 \cdot \tfrac{1}{2} y^{-1/2} \cdot y' = 0$$

   $$x^{-1/2} + y^{-1/2} \cdot y' = 0$$

   $$y^{-1/2}\, y' = -x^{-1/2}$$

   $$\boxed{y' = \frac{-x^{-1/2}}{y^{-1/2}}}$$



   (d) $1 + 2x = \sin(y^2)$

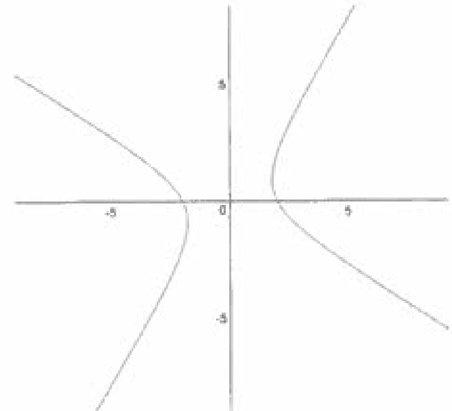
   $$(1+2x)' = (\sin(y^2))'$$

   $$2 = \cos(y^2) \cdot 2y \cdot y'$$

   $$y' = \frac{2}{\cos(y^2) \cdot 2y}$$

   $$\boxed{y' = \frac{1}{y\cos(y^2)}}$$

(e) $x^2 + xy - y^2 = 4$

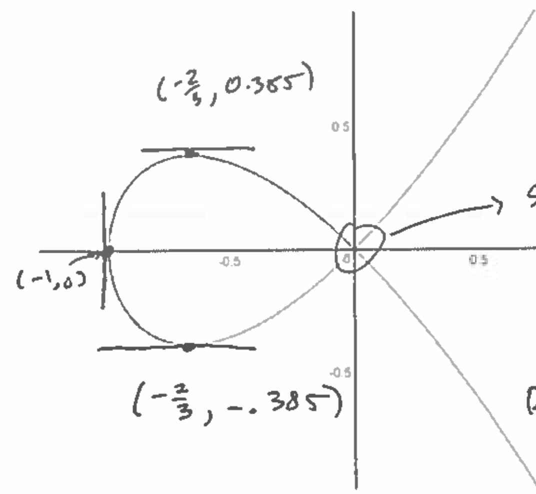


$$2x + (x)'y + x(y)' - 2yy' = 0$$
$$2x + y + xy' - 2yy' = 0$$
$$xy' - 2yy' = -2x - y$$
$$y'(x - 2y) = -2x - y$$
$$\boxed{y' = \frac{-2x-y}{x-2y}}$$

(f) For the following problem, you will be guided through the process of finding all points (x, y) where the tangent line is vertical or horizontal. We will work with the equation $y^2 = x^3 + x^2$, plotted below. (You should use this to check your answers!)

i. Find y' by implicitly differentiating.

$$2yy' = 3x^2 + 2x$$
$$\boxed{y' = \frac{3x^2+2x}{2y}}$$



Our math said (0,0), but it's actually not either hor. or ver.
Reason: both denom. and numer. are 0: $\frac{0}{0}$!

ii. Horiontal Tangents: set the numerator of your fraction equal to 0, and solve.

$$3x^2 + 2x = 0$$
$$x(3x+2) = 0 \Rightarrow \boxed{x = 0} \text{ or } \boxed{x = -\frac{2}{3}}$$

iii. Use the numbers you found to get the remaining coordinate.

$\boxed{x=0}$ $\quad y^2 = (0)^3 + (0)^2 = 0$

$y = 0$

$\boxed{x = -3/2} \Rightarrow y^2 = \left(-\frac{3}{2}\right)^3 + \left(-\frac{3}{2}\right)^2 = 0.148$

$\boxed{y = \pm 0.385}$

$$\boxed{\begin{array}{l}(0,0)\\(-3/2,\ 0.385)\\(-3/2,\ -0.385)\end{array}}$$

iv. Vertical Tangents: set the denominator equal to 0.

$$2y = 0 \longrightarrow y = 0$$

v. Use the numbers you found to get the other coordinate.

$$0^2 = x^3 + x^2$$

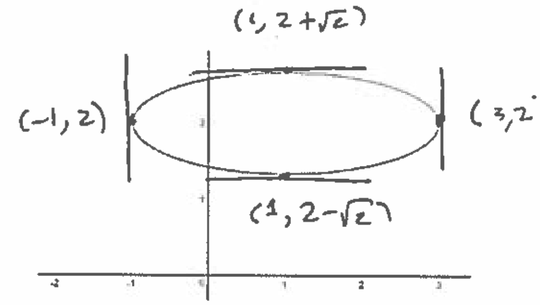
$$0 = x^2(x+1)$$

$\boxed{x=0}$ or $\boxed{x=-1}$

$\downarrow$ $(0,0)$ $\quad$ $\downarrow$ $(-1,0)$

(g) Apply the same strategy to find the locations (meaning (x, y)) where there are vertical or horizontal tangents for the ellpise:

$$(x-1)^2 + 2(y-2)^2 = 4.$$



$$2(x-1) + 2(y-2)y' = 0$$

$$y' = -\frac{(x-1)}{(y-2)}$$

Horiz:

$x = 1 \Rightarrow (1-1)^2 + 2(y-2)^2 = 4$

$$2(y-2)^2 = 4$$

$$(y-2)^2 = 2$$

$$y - 2 = \pm\sqrt{2}$$

$$y = 2 \pm \sqrt{2}$$

$(1, 2+\sqrt{2})$

$(1, 2-\sqrt{2})$

Vert: $y - 2 = 0 \rightarrow y = 2$

$$(x-1)^2 + 2(\cancel{2-2})^2 = 4$$

$$(x-1)^2 = 4$$

$$x - 1 = \pm 2$$

$$x = 1 \pm 2$$

$(3,2)$, $(-1,2)$

(h) (Challenge Problem) In advanced chemistry, you may come across the so-called *van der Waals equation*, which is a generalization of the ideal gas law $PV = nRT$. The van der Waals equation is the following:

$$\left(P + \frac{n^2a}{V^2}\right)(V - nb) = nRT$$

where P is the pressure of a gas, T its temperature, and V its volume. R is the universal gas constant, and a and b are constants depending on the chemical. If T remains constant, find $\boxed{\frac{dV}{dP}}$ using implicit differentiation.

$$\left(P + \frac{n^2a}{V^2}\right)' \cdot (V - nb) + \left(P + \frac{n^2a}{V^2}\right)(V - nb)' = 0$$

$$\left(\frac{dP}{dP} + n^2a \cdot \frac{d}{dP}\left(\frac{1}{V^2}\right)\right)(V - nb) + \left(P + \frac{n^2a}{V^2}\right) \cdot \frac{dV}{dP} = 0$$

$$\left(1 + n^2a \cdot (-2)V^{-3} \cdot \frac{dV}{dP}\right)(V - nb) + \left(P + \frac{n^2a}{V^2}\right)\frac{dV}{dP} = 0$$

$$(V - nb) - \underbrace{\frac{2n^2a}{V^3}\frac{dV}{dP} \cdot (V - nb)} + \underbrace{\left(P + \frac{n^2a}{V^2}\right)}\frac{dV}{dP} = 0$$

no dV/dP, move to other side

$$\frac{dV}{dP} \cdot \left[-\frac{2n^2a}{V^3} \cdot (V - nb) + \left(P + \frac{n^2a}{V^2}\right)\right] = -(V - nb)$$

$$\boxed{\frac{dV}{dP} = -\left[-\frac{2n^2a}{V^3}(V - nb) + \left(P + \frac{n^2a}{V^2}\right)\right]^{-1}(V - nb)}$$