Guide for Section 6 - Logarithms and Power Functions

Section Goals:

- Use the definition of a logarithm to solve an exponential equation
- Approximate solutions to an exponential equation using technology
- Apply a logarithmic model to a real-world phenomenon
- Apply properties of logarithms to solve a logarithmic equation
- Write formulas for power functions

Examples (Ex) are guided by your instructor.
Focus problems are intended to be attempted at first on your own, and then in collaboration with $1-3$ other students. I will answer questions from groups, but not individuals, in order to encourage you to use one another as resources.

Ex 1 (a) Find all real solutions to $2^t = 16$. (b) Find the two integer values that the solution to $2^t = 22$ falls between.

For (a), note that $2^4 = 16$, so $t = 4$ is a solution. The graph of 2^t is always increasing, so nowhere else can it be equal to 16; thus it is the only solution.

For (b), $t = 4$ gives $2^4 = 16$, and $t = 5$ gives $2^5 = 32$. So, the actual solution to $2^t = 22$ is between 4 and 5.

Def The logarithm with base b is the value L so that $b^L = Q$. The logarithmic equivalent to this equation is written $L = \log_b(Q)$ and read "log, base b, of Q".

The logarithm with base e ($\log_e(Q)$) is commonly abbreviated $\ln(Q)$. This is typically called the natural logarithm (due to the use of the "natural number" e in its base).

Ex 2 Compute exactly, if possible.

a) $\ln(e^3)$ $= 3$

b) $\log(\sqrt[7]{1000})$ $= \frac{3}{7}$.

Ex 3 Let $g(x) = 5 \cdot 10^{x-3} + 2$. Find all real solutions to $g(x) = 502$.

$$\begin{aligned} 5(10^{x-3}) + 2 &= 502 \\ 5(10^{x-3}) &= 500 \\ 10^{x-3} &= 100 \\ x - 3 &= \log(100) = 2 \\ x &= 5 \end{aligned}$$

$\boxed{\text{Thm}}$ (Change of Base Formula) A logarithm can be rewritten as a ratio of two other logarithms:

$$\log_b(Q) = \frac{\ln(Q)}{\ln(b)}$$

Ex 4 100 rabbits begin reproducing at a rate such that every four months the population triples. After how long will there be $1,000,000$ rabbits?

Here it is important to choose time units. Let's say t is measured in months. We know the population $P(t) = 100b^t$, and that $P(4) = 300$. So,

$$\begin{aligned} 300 &= 100b^4 \\ 3 &= b^4 \\ b &= 3^{1/4} \end{aligned}$$

Thus, $P(t) = 100 \cdot 3^{t/4}$. When does this reach 1,000,000?

$$\begin{aligned} 1,000,000 &= 100 \cdot 3^{t/4} \\ 10,000 &= 3^{t/4} \\ \log_3(10,000) &= \frac{t}{4} \\ 4 \cdot \log_3(10,000) &= t. \end{aligned}$$

To calculate this with a calculator, use change of base formula:

$$t = 4 \cdot \frac{\ln(10,000)}{\ln(3)} \approx 33.53 \text{ months.}$$

Focus 1 The half-life of a substance which exhibits exponential decay is the length of time necessary for the substance to decay to half its current amount. Nuclear reactors can create a (relatively) stable radioactive isotope of Thallium, Thallium-204, which has a half-life of 3.78 years. Consider a sample of 3 mg of ^{204}TI.

a) How much remains after 11.34 years? (Hint: How many "half-lives" is 11.34 years?)
Note that $11.34/3.78 = 3$, so 11.34 years is 3 half-lives. For each half-life, we decrease by 1/2, so we end up with $3 \cdot (1/2) \cdot (1/2) \cdot (1/2) = 0.375$ mg.

b) Write a function giving the amount of Thallium left after t years have passed.
Our initial amount is 3 mg, so we can start with the function $T(t) = 3 \cdot b^t$. We know that $T(3.78) = 1.5$ (half-life!), so

$$\begin{aligned} 1.5 &= 3 \cdot b^{3.78} \\ \frac{1}{2} &= b^{3.78} \\ b &= \left(\frac{1}{2}\right)^{1/3.78}. \end{aligned}$$

So,

$$T(t) = 3 \cdot \left(\frac{1}{2}\right)^{t/3.78}.$$

Notice that the base is 1/2 because of half-life, and the actual time for the half-life appears underneath the exponent. This is a feature of all half-life problems. You may even see in other textbooks a "formula" such as

$$T(t) = a\left(\frac{1}{2}\right)^{t/t_h}$$

where t_h is the half-life of the substance.

c) What is the continuous growth rate of the function in part (b)?
This is the k when we write $T(t) = ae^{kt}$. We must match $b = e^k$, so

$$\begin{aligned} \left(\frac{1}{2}\right)^{1/3.78} &= e^k \\ \ln\left(\left(\frac{1}{2}\right)^{1/3.78}\right) &= k \\ k &= -0.1833 \text{ per year.} \end{aligned}$$

d) Approximately how much ^{204}TI remains after 20 years have passed?

$$T(20) = 3 \cdot \left(\frac{1}{2}\right)^{20/3.78} = 0.077 \text{ mg.}$$

e) Contact with 0.1 or more milligrams of Thallium is considered dangerous. How long would exposure to this particular batch need to be avoided in order to be safe?

Let's solve $T(t) = 0.1$ for t. (We already know it should be less than 20 years!). We have

$$\begin{aligned} 3\left(\frac{1}{2}\right)^{t/3.78} &= 0.1 \\ \left(\frac{1}{2}\right)^{t/3.78} &= 0.033 \\ \left(\frac{t}{3.78}\right)\ln(0.5) &= \ln(0.033) \\ t &= 3.78\frac{\ln(0.033)}{\ln(0.5)} \approx 18.6 \text{ years.} \end{aligned}$$

Def A logarithmic function is a function of the form

$$f(t) = a + \log_b(t),$$

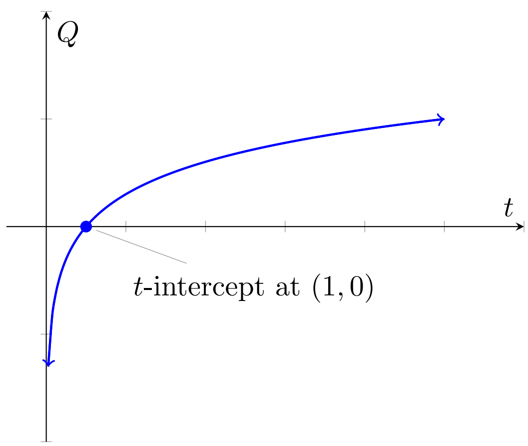
where $b > 0$. It is defined only for $t > 0$.

Thm (Basic Logarithmic Function Graphs)

Logarithmic Growth

$$Q = f(t) = a\log_b(t)$$

$$b > 1 \text{ and } a > 0$$

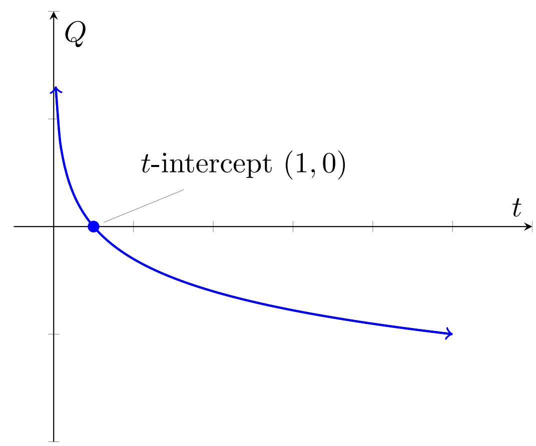


Graph rises very slowly to the right, falls dramatically as the graph approaches $t = 0$

Logarithmic Decay

$$Q = f(t) = a\log_b(t)$$

$$0 < b < 1 \text{ and } a > 0$$



Graph falls very slowly to the right, rises dramatically as the graph approaches $t = 0$

Thm (Properties of Logarithms) For positive real numbers t, u, and $b \neq 1$, and any real number n, we have

$$\log_b(t) + \log_b(u) = \log_b(t \cdot u) \quad \text{The Sum Rule}$$

$$\log_b(t) - \log_b(u) = \log_b\left(\frac{t}{u}\right) \quad \text{The Difference Rule}$$

$$n \cdot \log_b(t) = \log_b(t^n) \quad \text{The Constant Multiple Rule}$$

Ex 5 Use properties of logarithms to write each expression as the logarithm of a single quantity.

a) $\log_2(x+1) + \log_2(x)$

b) $2\ln(t-3) - \ln(t+1)$

Ex 6 The Richter scale (named after CIT scientist Charles Richter) measures the magnitude of seismic activity (read: earthquakes). Given a baseline level of activity S_0, the Richter magnitude of a seismic wave with amplitude S is

$$M = \log_{10}\left(\frac{S}{S_0}\right).$$

In February of 2004, there was an earthquake of magnitude 7.3 in Indonesia and another of magnitude of 4.7 in Burundi. How many times greater was the amplitude of the waves in the Indonesia earthquake than the one in Burundi?

Let $M_I = 7.3$ be Indonesia's magnitude, $M_B = 4.7$ be Burundi's, let S_I be Indonesia's amplitude, and S_B be Burundi's amplitude. To tell how much stronger Indonesia's earthquake was, we want to find the ratio S_I/S_B. Well,

$$\begin{aligned} M_I = 7.3 &= \log_{10}\left(\frac{S_I}{S_0}\right) \\ 10^{7.3} &= 10^{\log_{10}(S_I/S_0)} = S_I/S_0 \\ S_I &= 10^{7.3}S_0. \end{aligned}$$

Similarly,

$$\begin{aligned} M_B = 4.7 &= \log_{10}\left(\frac{S_B}{S_0}\right) \\ 10^{4.7} &= S_B/S_0 \\ S_B &= 10^{4.7}S_0. \end{aligned}$$

So,

$$\frac{S_I}{S_B} = \frac{10^{7.3}S_0}{10^{4.7}S_0} = 10^{2.6} \approx 398.1$$

which says the Indonesian earthquake was about 400 times more powerful!

Def A power function is a function of the form

$$Q = f(t) = at^b$$

for constants a (positive) and b.

It's easy to confuse a *power* function with an *exponential* function. This is an unfortunate conflict of tradition, but it is important to distinguish between the two.

Ex 7 Determine whether or not each function is a power function, and if so the values of constants a and b given by the definition.

a) $f(t) = 4\sqrt{t+2}$ b) $g(x) = 2x^{-4} \cdot 3x^{1.2}$ c) $h(t) = 2(3)^t$

Ex 8 Find a power function going through the points $(2, 1)$ and $(4, 18)$.

Start with $f(t) = at^b$, solve for a and b by plugging in the points:

$$f(2) = 1 = a(2)^b$$
$$f(4) = 18 = a(4)^b$$

Now divide the equations:

$$\frac{18}{1} = \frac{a4^b}{a2^b} = \frac{4^b}{2^b} = 2^b.$$

So, we solve $18 = 2^b$. This gives

$$b = \log_2(18) = 4.17.$$

Then

$$1 = a \cdot 2^b = a \cdot 2^{\log_2(18)} = a \cdot 18,$$

so that $a = \frac{1}{18}$. Thus,

$$f(t) = \frac{1}{18}t^{4.17}.$$