5.2 and 5.3: Equilibria and stability

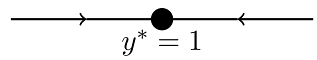
Equilibria

- Def: For an autonomous diffy-Q: We call a y-value, y^*, an *Equilibrium* if $\frac{dy}{dt} = 0$.
- Ex: Temperature model:
$$\frac{dT}{dt} = k(A - T).$$
Set $\frac{dT}{dt} = 0$. Get $k(A - T) = 0$, so either $k = 0$ or $T = A$.
- Note: you *should* consider all possible scenarios.
- If $k = 0$, then the temperature of the object *never* changes. (unrealistic)
- If $T^* = A$, the temperature of the object is always at the ambient temp.
- Ex:
$$\frac{dy}{dt} = 3(1 - y)(y^2 + 1).$$
Possible equilibria: set $(1 - y)(y^2 + 1) = 0$. Get $y^* = 1$ is the only equilibrium.

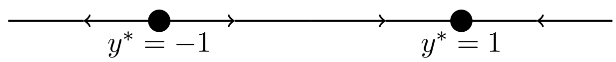
Graphical Representation

- Introducing: phase-line diagrams.
- Key idea: encode behavior of solutions and equilibrium.
- Ex: Consider diffy-Q $\frac{dy}{dt} = (1 - y)$. This is autonomous.
- Equilibrium: $y^* = 1$.

- Draw:

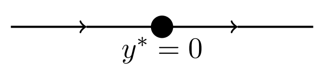


- Draw arrows according to whether y is increasing or decreasing (which you can tell from the diffy-Q).
- Ex: $\frac{dy}{dt} = (1 - y^2)$ Equilibria: $y^* = \pm 1$.



Stability

- We use phase-line diagrams to assess stability of solutions.
- Ex: Using the example with $\frac{dy}{dt} = (1 - y)$, we can assess that $y^* = 1$ is a stable equilibrium.
- Ex: using the example with $\frac{dy}{dt} = 1 - y^2$ we can assess that the equlibrium $y^* = -1$ is unstable while $y^* = 1$ is stable.
- Stable: all arrows go in. Unstable: all arrows go out.
- $\frac{dy}{dt} = y^2$. Equilibrium: $y^* = 0$. Phase-line diagram:



  This is neither stable nor unstable.

- As for discrete dynamical systems, there is a stability theorem:
- Thm: if $\frac{dy}{dt} = f(y)$ has equilibrium y^*, then:

y^* is stable	$f'(y^*) < 0$
y^* is unstable	$f'(y^*) > 0$

If $f'(y^*) = 0$, the test is inconclusive; it may be stable, unstable, or neither.

- Ex: $\frac{dy}{dt} = 8 - y^3$. Equilibria: $y^* = 2$. So,

  $$f(y) = 8 - y^3, \qquad f'(y) = -3y^2.$$

  We see that $f'(2) = -12 < 0$, so this equilibrium is stable.

Realistic Disease Model

Let I be the *fraction* of people infected. (This means I only takes decimal values btwn 0 and 1.)

- Individuals recover, but may become susceptible later (like a cold).
- More infected people = more spread. So, something like αI is a per capita infection rate.
- $1 - I$ represents uninfected, but susceptible people.
- thus, $\alpha I(1-I)$ is the infection rate (multiplying per capita by number of uninfected makes it a total infection rate).
- People recover: rate of μI.
- Equation:

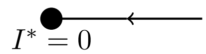
  $$\frac{dI}{dt} = \alpha I(1 - I) - \mu I.$$

- Equilibria: $I^* = 0$ and $I^* = 1 - \frac{\mu}{\alpha}$
- Now, there are two cases.
- Case 1: People recover faster than they get sick. That is, $\alpha < \mu$. Then $\frac{\mu}{\alpha} > 1$, so $I^* = 1 - \frac{\mu}{\alpha} < 0$.In this case, this equilibrium is non-realistic.
- Use stability theorem to see if it's stable or not.

  $$f(I) = \alpha I - \alpha I^2 - \mu I$$
  $$f'(I) = \alpha - \mu - 2\alpha I.$$

Plug in $I^* = 0$. You get $f'(0) = \alpha - \mu < 0$, so $I^* = 0$ is stable (expected).

- Phase-line diagram for this case:



- Case 2. People can't recover fast enough: $\alpha > \mu$. Then $I = 1 - \frac{\mu}{\alpha} > 0$, and we have a new equilibrium.

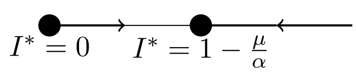
- We already got the derivative. So, calculate $f'(I^*)$:

$$f'(I^*) = \alpha - \mu - 2\alpha(1 - \frac{\mu}{\alpha}) = \alpha - \mu - 2\alpha + 2\mu = -\alpha + \mu < 0$$

so in fact the nonzero equilibrium is stable. Also, in this case,

$$f'(0) = \alpha - \mu > 0,$$

telling us that $I^* = 0$ is unstable.



the model predicts that in this case the population will come to equilibrium.

- Possible interpretation: we know in real life that diseases like the common cold don't fade out, so we must be in case 2. What we learn is that *people get sick more quickly than they recover* from these kinds of colds.