Exam 1 Practice Problems (solutions)

True/False

1. The antiderivative of $\ln(x)$ is $\frac{1}{x}$. False
2. If $f(2) < 0$, then the antiderivative of $F(2) < 0$. False
3. The integral $\int \sqrt{x^3+1}\,dx$ is computable with the strategies from our course. False
4. The integral $\int x^2\sqrt{x^3+1}\,dx$ is computable with the strategies from our course. True

Multiple Choice

1. Consider a function $f(x)$ on the interval $[2,5]$ with 3 steps. Which of the following expressions is the correct expression for the right Riemann sum?

   (a) $f(2)+f(3)+f(4)$

   (b) $f(3)+f(4)+f(5)$

   (c) $f(2)0.5+f(3)0.5+f(4)0.5$

   (d) $f(3)0.4+f(4)0.4+f(5)0.4$

   (b) is correct: $\Delta x = \frac{5-2}{3} = 1$, and the right Riemann sum contains the right endpoint. There was a typo on the original: it said "step size of 3" when it should have said "with 3 steps."

2. Suppose $R(t)$ represents the rate that air is converted into CO_2 in the lungs t seconds after inhaling. Which expression below represents the total amount of CO_2 converted in a single breath, assuming a breath lasts 1 second?

   (a) $\int R(t)\,dt$

   (b) $\int_0^{60} R(t)\,dt$

   (c) $\int_0^{1} R(t)\,dt$

   (d) $\int_0^{1} tR'(t)\,dt$

   (c) is correct.

Free Response Problems

1. Suppose $\int f(x)\cos(x)\,dx = \frac{1}{4}\sin(x)+9$. What is $f(x)$? Take the derivative and see that $f(x)\cos(x) = \frac{1}{4}\cos(x)$, so $f(x) = \frac{1}{4}$ (a constant function).
2. Evaluate the following integrals.

   (a) $\int \ln(x)\,dx$ $= x\ln(x) - x + C$ do parts.

   (b) $\int 4x\cos(x)\,dx$ $= 4x\sin(x) + 4\cos(x)$ do parts.

(c) $\int \frac{3}{6} e^{-2t}\, dt$ $\;-\frac{1}{4}e^{-2t} + C$ no fancy tricks needed

(d) $\int \pi\, dx$ $\;= \pi x + C$ remember, π is constant!

(e) $\int \frac{t^2}{t^3+1}\, dt$ $\;= \frac{1}{3}\ln(t^3+1) + C$ substitute $u = 1 + x^2$

(f) $\int dt$ $\;= t + C$

(g) $\int (x-5)^{4/3} x\, dx$ $\;= \frac{3}{10}(x-5)^{10/3} + \frac{15}{7}(x-5)^{7/3} + C$ do switcho-change-o with $u = x - 5$.

(h) $\int \frac{1}{6-2x}\, dx$ $\;= -\frac{1}{2}\ln|6-2x| + C$

3. Suppose that in a chemical reaction there is an amount $C(t)$ of water t milliseconds after the beginning of the reaction. The process is modeled with the differential equation

$$\frac{dC}{dt} = -te^{-t^2}$$

Suppose there is initially 200 mL of water.

   (a) Describe in words what happens over time with the amount of water. Looking at a graph of $\frac{dC}{dt}$, the amount of water decreases sharply for a few milliseconds and then stabilizes at a lower amount. This might mean that some of the water is used up in the chemical reaction!

   (b) When is the water being used up the fastest? At about $t = 0.707$ seconds. (set the derivative of $-te^{-t^2}$ equal to 0 and solve for t)

   (c) Find the solution of this differential equation. $C(t) = 199.5 - \frac{1}{2}e^{-t^2}$

   (d) About how much water is used up in this chemical reaction? There is initially 200 mL, and after a while the exponential piece of the solution tends to 0, leaving us with 199.5 mL of water. Therefore, about 0.5 mL of water was used in the reaction!

4. A mammal's basal metabolic rate, B, is the rate of change in energy consumed. Assume that this rate is given by $B(M) = 980M^{2/3}$ kCal per week at M kilograms. Also, mass M changes over a short period of time according to $M(t) = 100 + 0.5t$ kilograms, t weeks into a period of muscle growth. Write down, but do not evaluate, an integral that represents the total change in energy consumed over the first five weeks. What are the units of this quantity? [Hint: first write B as a function of t.]

We have that $B(t) = 980(100 + 0.5t)^{2/3}$. Since energy's rate of change is given by $B(t)$, we have

$$\text{change in Energy} = \int_0^5 980(100 + 0.5t)^{2/3}\, dt$$

This number, whatever it might come out to be, has units kCal.

5. The growth rate (in kg per year) of an animal follows the differential model $\frac{dm}{dt} = k \cdot (10 - m)$, for some constant k.

   (a) Is the differential equation pure-time, autonomous, or neither? This is autonomous.

   (b) Verify that the general solution is $m(t) = 10 - Ce^{-0.2t}$ as long as $k =$ _________. Calculate the derivative and show that it fits the differential equation. You should find that $k = 0.2$. Here are those details:

$$\begin{aligned} \frac{dm}{dt} &\overset{?}{=} k(10-m) \\ \frac{d}{dt}(10 - Ce^{-0.2t}) &\overset{?}{=} k(10 - (10 - Ce^{-0.2t}) \\ 0.2Ce^{-0.2t} &\overset{?}{=} kCe^{-0.2t} \\ 0.2 &= k \end{aligned}$$

Note that the C's cancel along with the $e^{-0.2t}$'s on both sides. So, if $k = 0.2$, this is a solution.

(c) Find the solution to the equation given the initial condition that the animal was born weighing 6 kg. Setting $m(0) = 30$, you get the equation

$$10 - C = 6,$$

so $C = 4$. Thus the solution is

$$m(t) = 10 - 6e^{-0.2t}.$$