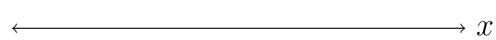
Making intervals to answer increasing/decreasing and concavity problems

We did an example in class that involved using f' and f'' to describe intervals where f was increasing/decreasing and concave up/down. The common features of this was the following process (which, oddly, no book I have ever seen clearly writes down):

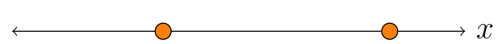
Making intervals for a function

Start with *any* function, $g(x)$. (Later, g will be f' or f''). Here are the steps:

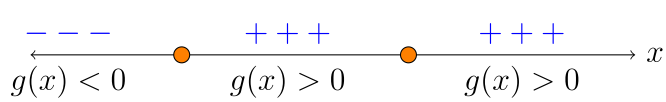
1. Make a number line:



2. Solve the equation $g(x) = 0$. You should get a few numbers, which you then plot on the number line you drew. I'll draw them in orange (pretend we found 2 answers from that equation, but in principle you could have more).



3. Plug in values of x *in between* these dots into g. Doing so tells you if g is positive or negative between these dots.



This process works when you apply it to either f' or f''. Important: when making intervals, you focus only on *one* function. Meaning, you don't go back and forth between f, f', or f'' at all. You just focus attention on one at a time.

Also, realize that this process of making intervals involves *no calculus whatsoever.* The calculus happens before this, usually.

Example from class

We found when $f(x) = x^3 - 3x + 1$ is increasing and decreasing, and when it is concave up and down. Here is how I used the above process:

Increasing/Decreasing

Apply the process to $f'(x) = 3x^2 - 3$, because $f'(x)$ tells us when f decreases or increases.

1. Make a number line with a label of the function.

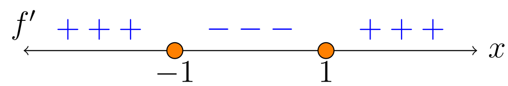
2. Solve $f'(x) = 0$:

$$\begin{aligned} 3x^2 - 3 &= 0 \\ x^2 &= 1 \\ x &= \pm 1 \end{aligned}$$

3. Plug in x values into $f'(x)$ between the orange dots: say -2, 0, and 2:

$$\begin{aligned} f'(-2) &= 3(-2)^2 - 3 = 9 \\ f'(0) &= 3(0)^2 - 3 = -3 \\ f'(2) &= 3(2)^2 - 3 = 9 \end{aligned}$$

so we get a picture like this:



From the picture, we conclude: f has critical points at -1 and 1, f increases on the intervals $(-\infty, -1)$ and $(1, \infty)$, and decreases on $(-1, 1)$.

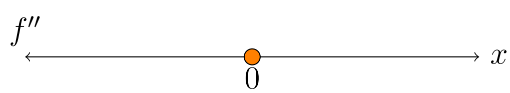
concave up/down

Apply the same process to the function $f''(x) = 6x$.

1. Make a number line with a label.

2. Solve $f''(x) = 0$:

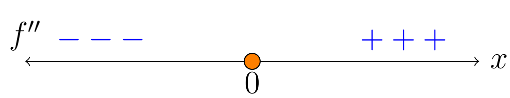
$$\begin{aligned} 6x &= 0 \\ x &= 0. \end{aligned}$$



3. Plug in x values to $f''(x) = 6x$ from both regions, say $x = -1$ and $x = 1$:

$$\begin{aligned} f''(-1) &= -6 \\ f''(1) &= 6 \end{aligned}$$

which gives us this picture:



so f is concave up on the interval $(0, \infty)$ and concave down from $(-\infty, 0)$.

Why this works

You can feel free to skip this if you're pressed for time. The reason this works is because by separating the number line into regions by where $g(x) = 0$, the function can only stay positive or negative. It must stay positive (or stay negative) because to change from positive to negative, it must cross the x-axis, which would force us to add another orange dot. So, if we find all the orange dots first (where $g(x) = 0$) then in between the orange dots $g(x)$ won't cross the axis.