4.5: Definite Integrals and FTC

Contents

1 Riemann Sums, Integrals

1.1 Definite Integrals

- We connect a new idea with an old.
- New idea: the Definite Integral.
- Suppose $\frac{dy}{dt} = f(t)$ has the solution $y = F(t)$, meaning that $F(t)$ is the antiderivative of $f(t)$.

$$\boxed{\text{Total Change in } F \text{ over interval } [a, b] \stackrel{\text{def}}{=} \int_a^b f(t)\, dt}$$

- Understand this notation as saying, $\int_a^b f(t)\, dt$ as the "total change in the antiderivative."
- Think: $f(t)$ is a growth rate of something. Then the integral $\int_a^b f(t)\, dt$ measures the total change!
- Now, if we have the actual function $F(t)$, then the total change could have been computed without calculus.

- Total change $= F(b) - F(a)$, final minus initial.
- This leads us to the following fact, which is a definition of the *definite integral*.

$$\boxed{\int_a^b f(t)\,dt = F(b) - F(a)}$$

People often call this the *Fundamental Theorem of Calculus*; I will call it FTC (part 1).

- Ex: $\int_0^2 3t^2\,dt = t^3\Big|_0^2 = 2^3 - 0^3 = 8$.
- Interpretation: The antiderivative of $3t^2$ has a net change of 8 over the interval $0 \le t \le 2$.
- Ex: $\int_{-1}^{10} 3t^2\,dt = t^3\Big|_{-1}^{10} = 10^3 - (-1)^3 = 1001$.
- Ex: $\int_{-\pi}^{0} \cos(x)\,dx = \sin(x)\Big|_{-\pi}^{0} = \sin(0) - \sin(-\pi) = 0 - 0 = 0$.
- Interpretation: $\sin(t)$ has a net change of 0 over the interval 0. Why?

1.2 Riemann Sums

- This is a general calculation: try to catch the argument.
- Return to the differential equation $\frac{dy}{dt} = f(t)$.
- We can use Euler's method to estimate the total change. Let's use $\Delta t = 0.2$.
- After one time step:

$$y(0.2) = y(0) + f(0) \cdot 0.2$$

- After two time steps:

$$y(0.4) = y(0.2) + f(0.2) \cdot 0.2 = y(0) + f(0)0.2 + f(0.2) \cdot 0.2$$

- General features of this calculation show that to get from $y(0)$ to $y(T)$, you need to add a bunch of things like $f(t) \cdot \Delta t$ to $y(0)$.

- Remember, y is the antiderivative of $f(t)$. The net change in y is then

$$\text{net change } = y(T) - y(0) \approx f(0)0.2 + f(0.2)0.2 + f(0.4)0.2 + \ldots$$

  The right hand side is a bunch of outputs of $f(t)$ multiplied by time steps Δt.

- Definition: A sum of terms of the form $f(t)\Delta t$ is called a *Riemann Sum*. The notation is tricky to get precise; I will use this notation to indicate this. A Riemann sum is of the form $\sum f(t)\,\Delta t$. The greek letter Sigma, Σ, is used to say "sum these values."

- Now: we have another name for the net change in the antiderivative:

$$\int_a^b f(t)\,dt \approx f(0)\Delta t + f(0.2)\Delta t + \ldots$$

- Main upshot: *the definite integral is approximated by Riemann sums.* This is why we use the notation $\int f(t)\,dt$, because the integral sign is a special sum of values of $f(t)$ multiplied by small changes Δt. The beautiful connection is that Euler's method justifies this connection!

- Well, we have some fixed Δt. How do we make the approximation better? Let $\Delta t \to 0$!

- Theorem (Fundamental theorem of calculus, part 0):

$$\int_a^b f(t)\,dt = \lim_{\Delta t \to 0} \sum f(t)\,\Delta t$$

- Bubble diagram: Definite integral connects to net change connects to Riemann sums. The first connection is the definition. The second is Euler's method.

2 Calculating Riemann Sums

- Calculating Riemann sums is a straightforward procedure. Start with a step size, Δt, compute $f(t)\Delta t$, then keep adding.
- Ex: Compute the Riemann sum of $f(x) = 3x^2$ with $\Delta x = 0.5$ with two steps starting at $x = 1$. Which definite integral does this approximate?
- A:

$$f(1)\Delta x + f(1.5)\Delta x = f(1)(0.5) + f(1.5)0.5 = 3(0.5) + 6.75(0.5) = 4.875.$$

- Draw picture to understand the time steps: the first one starts at $x = 1$, and the last interval ends at $x = 2$. So, we are doing a Riemann sum to approximate $\int_1^2 3x^2\,dx$.
- Ex: Compute a Riemann sum of $g(t) = \frac{1}{t}$ starting at $t = 2$ with $\Delta t = 1$ consisting of three steps. What integral does this approximate?
- A: The Riemann sum is

$$g(2) \cdot \Delta t + g(3) \cdot \Delta t + g(4) \cdot \Delta t = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \approx 1.083.$$

- The picture shows that the first interval starts at $t = 2$ and the last one ends at $t = 5$, so this Riemann sum approximates $\int_2^5 \frac{1}{t}\,dt$.