2.6: Shortcuts to Derivatives, part 1

1 Idea of Shortcuts

- The limit definition is the *true* definition of the derivative.
- However, it is clunky. We would like to know if there are shortcuts.
- There are.
- Keep in mind, however: these shortcuts are not a replacement of the definition. Also, the definition contains all of the real-world meaning. Shortcuts contain no meaning and are just a computational tool.
- Important properties of derivatives:
  1. Multiplication by constants:
  $$\big(Cf(t)\big)' = Cf'(t)$$
  2. Slopes Add:
  $$\big(f(t) + g(t)\big)' = f'(t) + g'(t)$$

2 Power shortcut

- We've seen through two calculations so far:
$$\frac{d}{dt}(mt + b) = m, \quad \frac{d}{dt}\left(t^2\right) 2t.$$
- Ex: $\frac{d}{dt}t^3 = 3t^2$ from definition.
- Trend: pull power down, then decrease the power by 1.
- This is the *power shortcut*:
$$\frac{d}{dt}t^n = nt^{n-1}.$$
- ex: compute $f'(t)$ for $f(t) = \sqrt{t}$. Do both ways. The shortcut works for $t = 1/2$ even!
- ex: $f(t) = \frac{t-1}{t^2}$.
- WARNING. Derivatives do not always work the way you guess. CANNOT take derivative of top and bottom.
- rewrite as a bunch of powers:
$$f(t) = \frac{t}{t^2} - \frac{1}{t^2} = \frac{1}{t} - \frac{1}{t^2} = t^{(-1)} - t^{(-2)}$$
$$f'(t) = (-1)t^{-2} - (-2)t^{-3}$$

- you try taking these derivatives: $\frac{1}{4}t^4$, $t^{1/3}$, x^{100}, $\frac{1}{3\sqrt{x}}$
- WARNING: again, shortcuts require very specific formats in order to be applicable.
- Ex: t^2 vs. 2^t. similar looking, but have very different derivatives.

3 Product Shortcut

- If you have a product of two functions $f(t) \cdot g(t)$, how to find the rate of change?
- Visualize the product as area of a square.
- Gives you this shortcut:

$$\big(f(t) \cdot g(t)\big)' = f'(t)g(t) + f(t)g'(t).$$

- How is t^2e^t a product of two functions? Is it a product in more than one way?
- Find the slope of the tangent line of $f(t) = (t^2+1)(14t+9)$ at $t = 3$. If you wanna foil, knock yourself out. I don't, so I'd use the product shortcut.
- A: $f'(t) = (2t)(14t+9) + (t^2+1)(14)$, $f'(4) = (6)(51) + (10)(14) = 306 + 140 = 446$.
- In homework: mass, density, volume. $\rho(t) =$ mass density of a plant/material.

$$M(t) = \rho(t)V(t)$$

  This is a naturally occurring "product" of functions in real life. Product shortcut will definitely apply here to tell you about $M'(t)$, the instantaneous growth/decay rate of mass of something.

4 Quotient shortcut

- How to take the derivative of $\frac{f(t)}{g(t)}$?
- Derivation in the book goes like this: call $h(t) = \frac{f(t)}{g(t)}$. Then $f(t) = h(t)g(t)$, use the product rule, and solve for $h'(t)$ in terms of the other stuff. You'll find:

$$\left(\frac{f(t)}{g(t)}\right) = \frac{f'(t)g(t) - f(t)g'(t)}{(g(t))^2}.$$

  The way to remember the minus sign: it goes with the denominator's rate of change. An increasing denominator should mean a decreasing fraction.
- Ex: $\frac{t-1}{t^2}$
- Ex: $\frac{14t+1}{14t-1}$
- A function we'll be taking the derivative a lot of later on looks like

$$f(t) = \frac{at}{at + b(1-t)}$$

  where a, b are constants. Find $f'(0)$ and $f'(1)$.

$$f'(0) = \frac{a}{b}, \quad f'(1) = \frac{b}{a}.$$

- Mass and density again. Rearrange to get

  $$\rho(t) = \frac{M(t)}{V(t)}$$

  can find $\rho'(t)$ in terms of $M'(t)$ and $V'(t)$. Examples in homework.