Day 14: L'Hopital's Rule

1 Revisit: Limits as $x \to \infty$

- We revisit evaluating limits.
- Ex: $\lim_{x\to\infty} \frac{x^2+2x}{1+2x^2}$.
- If you just naively plug in ∞, it gets you the expression $\frac{\infty}{\infty}$, which is not 1.
- ∞/∞ is called an *indeterminate form.*
- Ex: $\lim_{x\to 1} \frac{x^2-x}{x-1}$.
- Get: $\frac{0}{0}$, which is also an indeterminate form.
- Usually limits can be done by algebra techniques, but not always.
- Ex: $\lim_{x\to 0} \frac{\tan(x)-x}{x^3}$?

2 L'Hopital's Rule

- The rule works as follows:
- Suppose you wanna do $\lim_{x\to a} \frac{f(x)}{g(x)}$, and you get an indeterminate form, such as $\frac{0}{0}$ or $\frac{\infty}{\infty}$. THEN:

  $$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

  In other words, you are allowed to differentiate the top and the bottom *separately* and see if the problem gets easier to solve.

3 Examples

- $\lim_{x\to 0} \frac{\sin(x) - x}{x}$. Plugging in $x = 0$, you'll get $0/0$, so the rule applies.

$$= \lim_{x\to 0} \frac{\cos(x) - 1}{1} = 0/1 = 0.$$

- $\lim_{x\to\infty} \frac{x^3 + 2}{4x - x^2}$: Check that you get ∞/∞.

$$= \lim_{x\to\infty} \frac{3x^2}{4 - 2x}$$

Still get ∞/∞. Use the rule again.

$$= \lim_{x\to\infty} \frac{6x}{-2} = \frac{\infty}{-2} = -\infty.$$

You would have gotten this using the highest-order coefficients method too.

- Sometimes, L'Hopital might not be the best thing:

$$\lim_{x\to\infty} \frac{\sqrt[3]{x^9 - x^2}}{3x^3 + 5} = \lim_{x\to\infty} \frac{\frac{1}{3}(x^9 - x^2)^{-2/3}(9x^8 - 2x)}{9x^2}$$

Yuck. Instead, resort to factoring:

$$= \lim_{x\to\infty} \frac{\sqrt[3]{x^9}\sqrt[3]{1 - \frac{1}{x^7}}}{x^3(3 + \frac{5}{x^3})} = \frac{\sqrt[3]{1 - 0}}{3 + 0} = \frac{1}{3}.$$

- Be careful: it is immensely important to check that you have an indeterminate form.

$$\lim_{x\to 2} \frac{2x}{x + 3} = \frac{4}{5}.$$

It is not an indeterminate form here, so L'Hopital gives the wrong answer:

$$\lim_{x\to 2} \frac{2}{1} = 2 \quad \text{(Wrong!)}$$

4 Why does L'Hoptial work?

- We assume that $f(x)/g(x)$ has indeterminate form of $0/0$, and for simplicity, let's assume we're doing $\lim_{x\to 0}$. In other words, $f(0) = 0$ and $g(0) = 0$ (so that when you go to plug in the limit, it's $0/0$).
- Draw picture.
- Replace $f(x)$ and $g(x)$ by their tangent lines:
- $f(x) = f'(0) \cdot x$ (y-intercept is 0), $g(x) = g'(0) \cdot x$.
- $$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(0)x}{g'(0)x} = \frac{f'(0)}{g'(0)},$$
  which is exactly what we get by differentiating the top and bottom, and plugging in the limiting x value! If $f'(0)/g'(0)$ is not a well-defined number (e.g. another indeterminate form), then we're still safe to write
  $$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)}.$$
  which would still give us $f'(0)/g'(0)$ if that exists.

5 Other indeterminate forms

- Be aware of other indeterminate forms: 0^0, $0 \cdot \infty$, 1^∞, $\infty - \infty$
- Ex: $\lim_{x\to 0^+} x \ln(x)$. Looks like $0 \ln(0) = 0 \cdot \infty$. Bad!
- Algebra converts it: since $\frac{1}{1/x} = x$,
  $$\lim_{x\to 0^+} x \ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{1/x}.$$

Now, if you put in $x = 0$, it becomes $\frac{\infty}{\infty}$, so L'hopital applies:

$$= \lim_{x\to 0^+} \frac{x^{-1}}{-x^{-2}} = \lim_{x\to 0^+} \frac{x^1}{-1} = 0.$$

So, $\lim_{x\to 0^+} x\ln(x) = 0$.