Section Goals:

- Use inverse proportionality to express a function with a negative exponent
- Model a rational function as a ratio of polynomials in mathematical and non-mathematical contexts
- Use the principle of ratios of small and large numbers to infer long-term behavior of basic rational functions
- Use long-term behavior of polynomials to infer long-term behavior of non-basic rational functions

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Ex 1 Write the rule for a function which doubles the reciprocal of its input.

Def The **reciprocal function** of t is defined to be

$$Q = f(t) = \frac{1}{t}.$$

The **reciprocal square function** of t is defined to be

$$Q = f(t) = \frac{1}{t^2}.$$

Def A **rational function** is a function which can be written in the form

$$f(t) = \frac{p(t)}{q(t)}$$

where $p(t)$ and $q(t)$ are each polynomial functions (and $q(t)$ isn't always 0).

The mathematical domain of a rational function is all values of t such that $q(t) \neq 0$. As such, a rational function is only undefined at a finite list of inputs.

Ex 2 In each case, (i) identify the domain of the function $f(t)$, (ii) identify whether or not the function is rational, and then if the function *is* rational, then (iii) write possible polynomials $p(t)$ and $q(t)$ so that $f(t) = \frac{p(t)}{q(t)}$.

a) $f(t) = \dfrac{3+t^4}{t^5} - \dfrac{1}{t-1}$

b) $f(t) = \dfrac{1}{t} - \dfrac{5}{t^2}$

$\boxed{\text{Thm}}$ (Big-Little Principle)

- For any constant k and $p > 0$, we write

$$\text{As } t \to \infty, \text{ then } \frac{k}{t^p} \to 0$$

In other words, if you make the bottom of a fraction bigger and bigger (as either a large positive or large negative number), the whole thing gets closer and closer to zero.

- For any constant k and $p > 0$, we write

$$\text{As } t \to 0, \text{ then } \frac{k}{t^p} \to \pm\infty$$

In other words, if you make the bottom of a fraction a tiny number, the whole thing gets larger and larger (either in the positive or negative direction).

Ex 3 In each part, fill in the blank.

a) As $t \to \infty$, $\dfrac{10}{t^2} \to$ _____.

b) As $t \to -\infty$, $\dfrac{-1.2}{t^{0.1}} \to$ _____.

c) As t approaches 0 with $t > 0$, $\dfrac{6}{t^3} \to$ _____.

d) As t approaches 0 with $t < 0$, $\dfrac{6}{t^3} \to$ _____.

Ex 4 A data mining company uses 36 supercomputers equally in order to search 9000 terabytes of data.

a) Through how much data is each computer responsible for searching?

b) Through how much data is each computer responsible for searching if there are n supercomputers?

c) What happens to the amount of data searched by each computer as the number of supercomputers increases?

Ex 5 The relative growth rate of a microorganism can be modeled by the so-called Monod function, which can be given in the form

$$R(S) = \frac{1.35S}{0.004 + S},$$

where S is the concentration of solution (in grams per liter) available for growth of the microorganism.

a) Does the Big-Little Principle inform us about the behavior of $R(S)$ in the long term? Explain.

b) Use the formula for $R(S)$ to compute $R(0.1)$, $R(1)$, $R(10)$, and $R(100)$. Use these computations and what you know about the significance of a to guess the value these computations approach as the input grows larger.

Thm (Long-Term Behavior of a General Rational Function) Let a rational function be $f(t) = \dfrac{p(t)}{q(t)}$, where p and q are polynomial functions with leading terms $P(t)$ and $Q(t)$, respectively. Then the long-term behavior of $f(t)$ is the long-term behavior of the simplified function $\dfrac{P(t)}{Q(t)}$.

Ex 6 Identify the long-term behaviors of $f(t) = \dfrac{2t+1}{3t^3+1}$ and $g(t) = 5t + \dfrac{4t}{t+2}$.

Ex 7 Revisit Example 5. Compare degrees of the numerator and denominator in order to determine the long-term behavior of $R(S)$. Interpret this result in the context of the model.