4.5/4.6: Fundamental Theorem of Calculus and applications

1 FTC

- Recall: the Riemann integral of a function $f(x)$ on the interval $[a, b]$ computes the *total change* of $F(x)$, the antiderivative.
- Another view: Given the pure-time diffy-Q
$$\frac{dF}{dt} = f(t),$$
you are interested in the total change in F over an interval $[a, b]$. This can be computed either by taking a limit of Riemann sums, or by just computing the difference between $F(b) - F(a)$ (that is a change, after all!)
- We state this as the *Fundamental Theorem of Calculus**:
$$\int_a^b f(t)\,dt = F(b) - F(a).$$
*: the function $f(t)$ must be continuous on $[a, b]$.
- Note: the left-hand side is a very complex object: it's defined through a limit of Riemann sums! The astonishing part of this theorem is that you can instead compute this limit as a simple difference of numbers.
- Ex: Suppose that a raptor has speed
$$v(t) = t\sqrt{1+t^2} \text{ meters per minute}$$
and it runs for 5 minutes. (a) set up integral that computes its total movement.

  (b) What are the units? meters

  (c) what is the answer? 43.858 meters

- Examples: $\int_0^1 x^2\,dx$, $\int_1^4 \frac{1}{x}\,dx$.
- Non-Example: $\int_{-1}^1 \frac{1}{x}\,dx$. Non-continuous function, and this integral makes no sense!
- Important property:
$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx.$$
Why? Simple reasoning: if you change from $a \to b$, then $b \to c$, you add these changes and get the change $a \to c$.

2 Applications

- Riemann sums can be interpreted as area between a function and the x-axis.
- Example: calculate the area between the function $f(x) = x^2$ and the x-axis over the interval $[0, 4]$.
- This works for *any* function! If you can integrate it, you can find the area.
- Ex: Area of a circle: Justify geometrically that
$$\text{area} = \int_0^R 2\pi r\,dr = \pi R^2.$$
- Ex: Find the average value of a function.
$$\text{Average val(f)} = \frac{1}{b-a}\int_a^b f(x)\,dx$$
Suppose a neuron's voltage level varies continuously over three seconds according to the formula
$$V(t) = 0.5t(3-t) \text{ mV}.$$

What is the average voltage in the neuron over this time period?

A:

$$\text{Av. Voltage} = \frac{1}{3}\int_0^3 0.5t(3-t)\,dt = 0.75 \text{ mV}.$$

- A big application of integrals is finding a total amount of something given a *density* of it.

  Ex: A piece of E. coli DNA has about 4.7×10^6 nucleotides, and is about 1.6×10^6 nm long. Nucleotides are genetic code, and come in 4 types: A,C,G, and T. (A pairs with T, C pairs with G).

  Suppose the number of A's per nm increases linearly from at one end to 0.3 at the other.

  (a) Find a formula for the density of A's as a function of length along the DNA.

  (b) find the total number of A's.

  A: for (a): 150 means 150 A's per 1000 nuc. We want number of A's per nm. To do this, we find

$$\frac{150 \text{ A's}}{1000 \text{ nuc.}}\frac{4.7 \times 10^6 \text{ nuc}}{1.6 \times 10^6 \text{ nm}} \approx 0.441 \text{ A's per nm.}$$

  At the other end, we have

$$\frac{300}{1000} \times 2.94 = 0.882 \text{ A's per nm.}$$

  so the slope is

$$m = \frac{0.882 - 0.441}{1.6 \times 10^6} \approx 2.76 \times 10^{-7}$$

  Let $f(x)$ be the density of A's per nm. We know its slope. $f(0) = 0.441$, so

$$f(x) = (2.76 \times 10^{-7})x + 0.441.$$

(b): Total number of A's:

$$\begin{aligned}
\int_0^{1.6\times 10^6} f(x)\,dx &= \int_0^{1.6\times 10^6} mx + 0.441\,dx \\
&= (\frac{1}{2}mx^2 + 0.441x)\Big|_0^{1.6\times 10^6} \\
&= 1,058,880 \text{ A's.}
\end{aligned}$$

- Main point: if you integrate a density, you get a total number of stuff.

$$\int_a^b \left(\frac{\text{Amount of Bazingas}}{\text{per unit Blorb}}\right) d(\text{Blorb}) = \text{total amount of Bazingas in the range } a \to b.$$