## Section Goals:

- Identify a phenomenon having constant second differences as quadratic relationship
- Compare characteristics of a quadratic and a linear function
- Find the vertex of a parabola and interpret its value as an extremum
- Explore the definition of a higher-order polynomial function
- Classify the long-term behavior of a polynomial function

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**Examples (Ex)** are guided by your instructor.
**Focus** problems are intended to be attempted at first on your own, and then in collaboration with $1-3$ other students. I will answer questions from groups, but not individuals, in order to encourage you to use one another as resources.

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Def A **quadratic function** is a function which can be written in the form

$$Q(t) = at^2 + bt + c,$$

with constants $a$, $b$, and $c$ (where $a$ is not zero).

Def Consider a function $f(t)$ with inputs $a < b < c$. The **second difference** between these three values is

$$[f(c) - f(b)] - [f(b) - f(a)] \, .$$

Thm (Quadratic Models) Any phenomenon which exhibits nonzero, constant second differences *between inputs a fixed distance apart* can be fit to a quadratic model.

**Ex 1** Consider the table of values describing the height (in feet) of an object $t$ seconds after being launched into the air.

a) Compute the average rate of change between each adjacent point, then fill out the table to the right.

b) Compute the *second differences*, that is, the difference between adjacent rates of change. Then fill out the relevant column in the table. What do you notice about these values?

| $t$ | $h(t)$ | ARC | 2nd Diff |
|---|---|---|---|
| 0 | 5 | | |
| 1 | 69 | | |
| 2 | 101 | | |
| 3 | 101 | | |
| 4 | 69 | | |
| 5 | 5 | | |

c) What conclusion can you draw about the model for height of the object as a function of time?

Thm (The Quadratic Formula) An equation of the form $0 = at^2 + bt + c$ has solutions

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The solutions are non-real if the quantity $b^2 - 4ac$ under the radical is negative.

**Ex 2** Find all real solutions to the equation $3x - 2x^2 = -7$.

$\boxed{\text{Thm}}$ (Extremum of a Quadratic Function) A quadratic function $f(t) = at^2 + bt + c$ has either a maximum or minimum located at

$$t_{\text{vert}} = -\frac{b}{2a}.$$

The quadratic function $f(t)$ has a $\begin{bmatrix} \text{maximum} \\ \text{minimum} \end{bmatrix}$ if $\begin{bmatrix} a < 0 \\ a > 0 \end{bmatrix}$.

(*Extremum* is a fancy word for "either a maximum or a minimum")

$\boxed{\text{Def}}$ The graph of a quadratic function $f(t)$ is called a **parabola**. It is symmetric about the vertical line through the vertex: $t = -\dfrac{b}{2a}$.

**Ex 3** Find the vertex and axis intercepts of $g(t) = 2t^2 - 6t + 1$, then sketch a graph of the associated parabola.

$\boxed{\text{Thm}}$ (Vertex Form of a Quadratic) A quadratic function written in the form

$$Q = f(t) = a(t - t_{\text{vert}})^2 + Q_{\text{vert}}$$

has its maximum or minimum value at the point $(t_{\text{vert}}, Q_{\text{vert}})$.

**Ex 4** An object is thrown upwards with an initial velocity $v_0$ and initial height $h_0$. It's height $h$ (in feet) as a function of time $t$ (in seconds) will be

$$h(t) = -16t^2 + v_0 t + h_0.$$

(The 16 has to do with gravity; in fact, if we did everything in SI units with meters instead of feet, we would use $9.8m/s^2$, which you may recognize from physics as the acceleration due to gravity.)

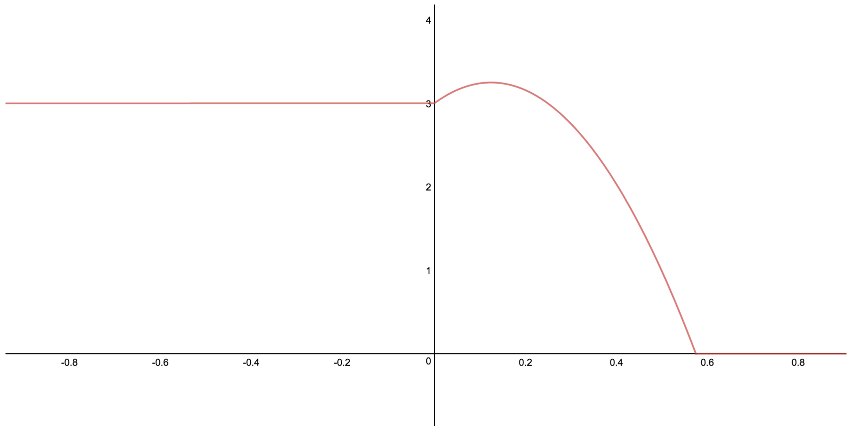
Suppose an object is thrown upwards at four feet per second, with an initial height of three feet.

a) When does the object reach its maximum height?

b) How high does the object go?

c) When does the rocket hit the ground?

d) Write the equation of the object's height for all times.


4
3
2
1
0
-0.8
-0.6
-0.4
-0.2
0.2
0.4
0.6
0.8

**Ex 5** A farmer wants to build a rectangular fence. One side costs \$30 per foot, while the other side costs \$50 per foot to build. He has \$1400 to spend. What dimensions should his fence be to maximize the area enclosed?

Def A **monomial** is an expression of the form $at^b$, where $a$ is any constant (called its **coefficient**) and $b$ is a non-negative whole number (called its **degree**).

A **polynomial** is a sum or difference of any number of monomials (including just one).

A **polynomial function** is a function whose formula can be written as a polynomial.

The **leading term** of a polynomial is the term containing the highest power on $t$.

The **leading coefficient** of a polynomial is the coefficient of the leading term.

**Ex 6** In each case, identify whether or not the function is polynomial. If so, write its degree, leading term, and leading coefficient.

a) $f(x) = \pi x^6 - 4x^4 - 3x^9$ b) $g(t) = t^4 + 8t^3 - 3t + t^{1/3}$ c) $h(n) = (n-7)(n^2+1)(2n+1)^2$

**Ex 7** Consider the function

$$P(t) = 6.75t^3 - 1639t^2 + 109732t - 471072,$$

which gives the approximate population of Detroit , Michigan $t$ years after 1900.

a) Use technology to sketch a graph of $Q = P(t)$ on the interval $[0, 110]$.

b) The censused population of Detroit in 2010 was 713,777 individuals. How well does that compare to the model's prediction of the population in that year?

c) In what year(s) does the model predict that nobody lives in Detroit? (You may use technology to find this value)

Def The **long-term behavior** or **long-run behavior** of a function $Q = f(t)$ is the value that $Q$ approaches when $t$ approaches infinity. That is, $t$ gets larger and larger with no ceiling.

This trend in values of $Q$ could be: a real number, $\infty$, $-\infty$, or the value could not exist.

Thm (Long-Term Behavior of Polynomials) The leading term of a non-constant polynomial uniquely determines its long-term behavior.

If the leading coefficient is $\begin{bmatrix} \text{positive} \\ \text{negative} \end{bmatrix}$, then the function tends toward $\begin{bmatrix} \infty \\ -\infty \end{bmatrix}$ as $t$ increases without bound.

"Tends toward infinity" or "increases without bound" are equivalent ways of says that the value keeps getting bigger and bigger with no ceiling. Similarly, "tends toward negative infinity" or "decreases without bound" mean that the value keeps getting more and more negative with no bottom.

**Ex 8** What is the long-term behavior of $P(t)$ from Example 7? Interpret this in the context of the model.