L'Hopital's Rule

Sunday, March 11, 2018 5:20 PM

Last Topic: aside on limits.

$$\lim_{x\to 2} \frac{x^2-4}{x-2} \rightsquigarrow \frac{0}{0}$$ if plug in $x=2$.

Neat trick: differentiate top & bottom!

$$\lim_{x\to 2} \frac{2x}{1} = 4!$$

Does it work?

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}} = 4!$$

Why does this trick work?

Want: $\displaystyle\lim_{x\to 2} \frac{f(x)}{g(x)} \approx \frac{0}{0}$

Assume: get $f(2) = g(2) = 0$.

Near $x=2$,

$$f(x) \approx \cancelto{0}{f(2)} + f'(2)\cdot(x-2)$$

$$g(x) \approx \cancelto{0}{g(2)} + g'(2)(x-2).$$

$$\Rightarrow \lim_{x\to 2} \frac{f(x)}{g(x)} = \lim_{x\to 2} \frac{f'(2)\cdot\cancel{(x-2)}}{g'(2)\cdot\cancel{(x-2)}}$$

$$= \lim_{x\to 2} \frac{f'(x)}{g'(x)} \quad !$$

* This always works as long as you get $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

L'Hospital's

$$\boxed{\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)} \quad \text{if } \frac{0}{0} \text{ or } \frac{\infty}{\infty}}$$

L'Hopital's Rule:

$$\lim_{x\to a} \frac{f(x)}{g(x)} \overset{L}{=} \lim_{x\to a} \frac{f'(x)}{g'(x)} \quad \text{if } \frac{0}{0} \text{ or } \frac{\infty}{\infty}$$

__Ex.__ Compute

$$\lim_{x\to\infty} \frac{x^2-3x}{2x^4-6x}. \qquad \frac{\infty}{\infty}.$$

$$\overset{L}{=} \lim_{x\to\infty} \frac{2x-3}{8x^3-6} \qquad \frac{\infty}{\infty}$$

$$\overset{L}{=} \lim_{x\to\infty} \frac{2\cdot 1}{24x^2} = \frac{2}{\text{"}\infty\text{"}} = 0.$$

__Ex.__ Compute $\lim_{x\to 0} \frac{\tan(x)}{x}$. $\frac{0}{0}$ ✓

$$\overset{L}{=} \lim_{x\to 0} \frac{\sec^2(x)}{1}$$

$$= \sec^2(0) = \boxed{1.}$$

__Ex.__ $\lim_{x\to 0^+} x\cdot \ln(x).$ $\qquad 0\cdot\infty = \frac{\infty}{\infty}$ secretly,

b/c "$0 = \frac{1}{\infty}$"!

$$= \lim \frac{\ln(x)}{} \rightsquigarrow \frac{-\infty}{}$$

1: ↑

$$\hookrightarrow \; = \lim_{x\to 0^+} \frac{\ln(x)}{1/x} \quad \rightsquigarrow \frac{-\infty}{\infty} \longleftarrow \frac{1}{x}:$$

$$\overset{L}{=} \lim_{x\to 0^+} \frac{1/x}{-1/x^2}$$

$$= \lim_{x\to 0^+} \left(-\frac{x^2}{x}\right)$$

$$= \lim_{x\to 0^+} (-x)$$

$$= 0!$$

Warning: You cannot always apply!

Ex. $\displaystyle \lim_{x\to 7} \frac{2x-7}{3-x} = \frac{14-7}{-4} = \frac{7}{-4}$.

$$\overset{L}{=} \lim_{x\to 7} \frac{2}{-1} = -2$$

oh no!

bad!

because: original wasn't $\frac{0}{0}$ or $\frac{\infty}{\infty}$!

Warning: l'Hopital isn't always helpful.

Ex. $\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}}$ $\sim \frac{\infty}{\infty}$

$$\overset{L}{=} \lim_{x\to\infty} \frac{1}{\frac{1}{\cancel{2}}\cdot(x^2+1)^{-1/2}\cdot\cancel{2}x}$$

$$\overset{alg}{=} \lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x}$$! traded places!

$$\overset{L}{=} \lim_{x\to\infty} \frac{\frac{1}{\cancel{2}}(x^2+1)^{-1/2}\cdot\cancel{2}x}{1}$$

$$\overset{alg}{=} \lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}}$$!!!

<u>Trick:</u> don't use l'Hopital factor:

$$\sqrt{x^2+1} = \sqrt{x^2\left(1+\frac{1}{x^2}\right)}$$
$$= x\cdot\sqrt{1+\frac{1}{x^2}}$$

$$\Rightarrow \lim_{x\to\infty} \frac{\cancel{x}}{\cancel{x}\sqrt{1+\frac{1}{x^2}}} = \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x^2}}}$$

$\to 0$

$$= \frac{1}{\sqrt{1+0}}$$

$$= 1.$$