5.5: Separation of Variables

Contents

1 Separable Equations

- We are now moving on from pure-time equations. Realistically, many more problems are autonomous.
- At the same time we can develop a technique to solve a generalized class of first order equations.
- Def: A *separable* first-order DE is one of the form

$$y' = f(y)g(t)$$

meaning that on the right-hand side, the state-variable and the independent variable can be "separated" into two things being multiplied.

- Which of these are separable?

1. $y' = y$
2. $y' = \cos(y)t^2$
3. $y' = y + t$
4. $y' = \frac{t}{y}$

5. $y'' = 2y$.

The first two are separable, the third is not, the fourth is, and the last one is not even a first order equation! So separability makes no sense there.

- Autonomous: $y' = f(y)$ is *always* separable.

2 Solving separable equations

- Ex: $y' = y$. What are the solutions? Guessing gives $y = Ce^t$ for a constant C, but let's see it another way.

- Method: separation of variables.

$$\begin{aligned} \frac{dy}{dt} &= y \\ \frac{1}{y}\,dy &= dt \\ \int \frac{1}{y}\,dy &= \int 1\,dt \\ \ln|y| &= t + C \\ |y| &= e^{t+C} \\ y &= \pm e^C e^t. \end{aligned}$$

Since C was arbitrary anyway, $\pm e^C$ can be any nonzero number. We just rename C as $\pm e^C$, so the general solution is

$$y = Ce^t.$$

- Don't be afraid to rename the arbitrary constants. Remember, you'll find the correct *function* at the end of the day by solving for C with an initial condition.

- Ex: Solve the DE with initial condition below.

$$y' = y \cdot t, \qquad y(0) = 1.$$

Part (b): find the solution with $y(0) = -2$. Sol: separate variables.

$$\begin{aligned}
\frac{1}{y} dy &= t\,dt \\
\int \frac{1}{y} dy &= \int t\,dt \\
\ln|y| &= \frac{1}{2}t^2 + C \\
|y| &= e^C e^{t^2/2} \\
y &= \pm e^C e^{t^2/2}
\end{aligned}$$

now, use that $y(0) = 1$, meaning $t = 0$ and $y = 1$:

$$1 = \pm e^C e^0 = \pm e^C.$$

So, we need the $+$ out of the $\pm$, and then $C = 0$. The function is

$$y(t) = e^{t^2/2}.$$

For part (b), same steps give the same general solution. This time, though, we have

$$-2 = \pm e^C,$$

so first we need to choose the negative sign for it to make sense. Then solve $2 = e^C$ by taking ln: $C = \ln(2)$. The solution now is

$$y(t) = -e^{\ln(2)} e^{t^2/2} = -2e^{t^2/2}.$$

- Application: falling objects with air resistance! Setup: air resistance applies a force in the opposite direction of the velocity.

  $$F_{\text{drag}} = -kv.$$

  What k exactly is depends on the shape of the falling object.

- using Newton's second law: $F_{\text{net}} = ma = m\frac{dv}{dt}$, we have the differential equation

  $$m\frac{dv}{dt} = -kv - 9.8\text{m/sec}^2$$

Let's make life a bit easier and assume $m = 1\,\text{kg}$. Dividing the units, we have

$$\frac{dv}{dt} = -kv - 9.8.$$

Separate and integrate!

$$\begin{aligned}
\frac{1}{-kv-9.8}\,dv &= dt \\
\int \frac{1}{-kv-9.8}\,dv &= \int dt \\
-\frac{1}{k}\ln|-kv-9.8| &= t + C \\
\ln|kv+9.8| &= -kt + C \\
|kv+9.8| &= e^C e^{-kt} \\
kv + 9.8 &= (\pm e^C)e^{-kt} \\
v &= \frac{-9.8}{k} + \left(\frac{\pm e^C}{k}\right) e^{-kt}
\end{aligned}$$

We just found the velocity as a function of time. Note, k was a parameter depending on shape, and C is a constant which turns out to be (related to) the initial velocity. As $t \to \infty$, $\lim_{t\to\infty} e^{-kt} = 0$. So,

$$\lim_{t\to\infty} v(t) = -\frac{9.8}{k}.$$

This is the idea of *terminal velocity.* We just predicted terminal velocity from mathematics, which is observed experimentally! Moreover, we also can learn from this answer that terminal velocity depends on the shape of an object, because k depends on the shape of the object!

3 Closing Remarks

- You can *only* use separation of variables on *separable* DE's. If you try this on $y' = y + t$, for example, you will get the wrong answer because you will have forced an algebra mistake upon yourself.

- Why does it work: secretly a variable substitution happens!

$$\begin{aligned}
y'(t) &= f(y)g(t) \\
\frac{1}{f(y)}y'(t) &= g(t) \\
\int \frac{1}{f(y)}y'(t)\,dt &= \int g(t)\,dt \\
\int \frac{1}{f(u)}du &= \int g(t)\,dt.
\end{aligned}$$

  This is the equation you would have ended up with had you just "multiplied by dt."

- This technique quickly becomes hard to use if the RHS is complicated.

- Ex: Logistic model, DE version.

$$\frac{dy}{dt} = ry(1-y)$$

  Compare with $x_{t+1} = rx_t(1-x_t)$.

- This is more realistic; negative rates instead of negative population values.

- You can solve this by separation of variables. Extra credit: find the solution. Hint: You'll need this integral:

$$\int \frac{1}{y-y^2}\,dy = \ln\left|\frac{y}{(1-y)}\right|$$

  This integral can be done with *partial fraction decomposition*, which is a technique we are skipping in this course.