5.5: Systems of Diffy-Q

- We will make models that can incorporate *two* state variables (or more than two).
- Ex: Predator & Prey models. Consider two populations of animals, $b(t)$ and $p(t)$ (say rabbits and foxes).
  - rabbits grow at a rate prop. to their pop.
  - with no rabbits around, foxes die off proportional to their population.
  - per capita growth rate of rabbits: positive amount λ from pop growth, but $-\epsilon f$ to indicate that each fox eats a certain portion ϵ of rabbits.
  - growth rate = (per capita growth rate)$\times$(population).

$$\frac{db}{dt} = (\lambda - \epsilon p)b.$$

  - per capita growth rate of foxes: negative amount $-\delta$ for the fact that they die with no rabbits; and a positive amount $\eta \cdot b$ since the foxes' growth rate is bigger if they eat more rabbits.
  - growth rate = (per capita growth rate)$\times$ (population):

$$\frac{dp}{dt} = (-\delta + \eta b)p.$$

  We write these as a *system of differential equations*:

$$\begin{aligned} \frac{db}{dt} &= (\lambda - \epsilon p)b \\ \frac{dp}{dt} &= (-\delta + \eta b)p. \end{aligned}$$

- Ex: Given the following system, which parameter indicates the prey and which is the predator?

$$\frac{da}{dt} = -3a + 2ab$$
$$\frac{db}{dt} = -ab + 4b$$

  A: population a is the predator.

- Note: most of the time we can never write down a solution (a pair of functions now!) to a *coupled* system of equations. (Later)
- We'll use computers to generate solutions to these equations.
- <u>Note</u>: terms with one appearance of a State variable, like $-3a$ or $4b$, correspond to stuff happening within a single species. Terms involving two state variables, like $-ab$, are called "interaction" terms and, rightfully so, correspond to interactions between the species.
- The disease model can be thought of as a system of DE:

$$\frac{dI}{dt} = \alpha IS - \mu I$$
$$\frac{dS}{dt} = -\alpha IS + \mu I$$

- Newton's law of cooling can be made more general: Let ambient temperature be its own state variable, $A(t)$. We have the system

$$\frac{dH}{dt} = \alpha(H - T)$$
$$\frac{dA}{dt} = \alpha(T - H)$$

- Ditto with diffusion.

Setting up systems from word descriptions

Helpful things to keep in mind:

- growth rate = (per capita growth rate)× (population).
- pay attention to minus signs – changing + to − drastically alters what happens.
- If two *different* mechanisms have an effect on growth rate, then separate their terms by plus or minus signs (don't multiply).

Have them work on a few.

Euler's Method

Example: Use Euler's method with $\Delta t = 1$ on the disease model

$$\frac{dI}{dt} = IS - 0.5I$$
$$\frac{dS}{dt} = -IS + 0.5I$$

with initial conditions $I(0) = 0.6$ and $S(0) = 0.4$. Perform two steps of Euler's method.

t	I'	S'	$\hat{S}$	$\hat{I}$
0	-0.06	0.06	0.54	0.46
1	0.0184	-0.0184	0.5216	0.4784

Same process, just more bookkeeping.

Note: You NEED to update both state variables in a *single* step of Euler's method. Otherwise you won't be able to compute the slopes in the next step.