2.7: The Second Derivative

1 Acceleration

- Seen so far: If $f(t)$ is a position function (height, distance, location) then $v(t) = f'(t)$ is the instantaneous velocity function.
- Acceleration is another key quantity in describing motion of objects.
- Def: Acceleration is the *rate of change* of velocity.
- In other words, if $v(t)$ is the velocity of an object, then $a(t) = v'(t)$ is the acceleration of that object.
- Ex. A water balloon is thrown up in the air from ground level. Its height in meters is described by
$$h(t) = -4.9t^2 + 12t.$$
  1. Check that the balloon starts at ground level.
  2. Find the velocity function and the acceleration function.

2 The Second Derivative

- Acceleration gives us a reason to study the *second derivative*.
- $f(t) \to f'(t) \to f''(t)$.
- Example: find second derivative of $f(t) = t + \frac{1}{2}t^2 + \frac{1}{6}t^3$. A: $f''(t) = t + 1$.
- Example: find the second derivative of $f(t) = e^{-t}\cos(t)$.
$$\begin{aligned} f'(t) &= -e^{-t}\cos(t) - e^{-t}\sin(t) \\ f''(t) &= e^{-t}\cos(t) + e^{-t}\sin(t) + e^{-t}\sin(t) - e^{-t}\cos(t) \\ &= 2e^{-t}\sin(t). \end{aligned}$$
  [side note: this function comes from modeling a mass on a spring that includes friction.]

3 Shapes of Graphs

- $f''(t)$, the second derivative, measures the change in the *slopes* of f.
- Contrast: f' measures change in the *y-values* of f.
- Graphical example: small f'' versus large f''.

- Larger f'' means curvier graph. Smaller f'' means less curvy.
- Positive f'' means the graph of f curves upward (bowl shaped)
- negative f'' means the graph of f curves downward (upside down bowl shape)
- $f''(x) = 0$ means the graph is (possibly) switching between these bowl shapes.
- If $f''(x) = 0$ and $f''(x)$ changes sign (pos to neg, or neg to pos) we call this an *inflection point*.
- $f'(x) = 0$ we've seen is a place where the graph of f is flat.
- If $f'(x) = 0$ or $f'(x)$ is undefined, we call this a *critical point*.
- Ex: For $f(x) = x^3 - 3x + 1$, (a) find all critical points. (b) find all inflection points. (c) Use this to say where f is concave up, concave down, increasing, and decreasing.
- (a): $x = \pm 1$. (b): $x = 0$ *is* an inflection point; curvature changes from neg to pos.

  (c) Use number lines.
- Find inflection points for $f(x) = 2(x-1)^4 + 3$, and describe the intervals where f is concave up/concave down.

  A: $f''(x) = 24(x-1)^2$. Setting equal to 0, you see $x = 1$. But concavity does not change sign. So there are no inflection points for this function.