2.9: The chain shortcut

1 Review: Function composition

- Recall how composition of functions works.
- $(f \circ g)(t) = f(g(t))$ "first g, then f", or f after g.
- how is this different than multiplying two functions?
- ex: $f(t) = \sin(t)$, $g(t) = e^t$.
- $f(g(t)) = \sin(e^t))$.
- $f(t)g(t) = \sin(t)e^t$; these are very different!
- Many functions are compositions of simpler functions.
- Examples: Identify the inside function and outside function for composition.

- $F(t) = e^{2t}$
- $G(t) = \cos(4t^2 + 1)$
- $H(t)e^{\ln(t)}$
- $Y(t) = (14t + 9t^4)^{17}$
- $R(t) = \dfrac{1}{9t^2 + 5}$

- Ex: consider $S(x) = -xe^x$. Why is S not a composition of functions? i.e. why is one not the inside of the other? Answer: you can check it! $f(x) = -x$, $g(x) = e^x$, composition:

$$f(g(x)) = f(e^x) = -e^x,$$

$$g(f(x)) = g(-x) = e^{-x},$$

neither of which are the same as $S(x)$!

2 Chain Shortcut

- To differentiate a composition, we use the following shortcut:

$$\boxed{\frac{d}{dt} f\Big(g(t)\Big) = f'\Big(g(t)\Big)g'(t)}$$

Why does this work? One answer: it follows from the definition (see the book). Another answer: derivative talks about change. Outputs of $f(g(t))$ are outputs of f, so naturally there needs to be something in the formula involving f'. Where do you evaluate it? Well, the inputs are coming from *outputs of* g, so it needs to be $f'(g(t))$. Finally, it should also keep track of how g changes, since it heavily relies on outputs of g. So we better throw in

$g'(t)$ somehow. Why multiply? One reason: if g doesn't change, then neither should $f(g(t))$. In other words, if $g'(t) = 0$, then so should $(f \circ g)'(t) = 0$, which can be accomplished if we *multiply* by $g'(t)$, not add. Another reason: units work out this way.

- Ex: $F(x) = e^{-x}$. Differentiate this function. A: $F'(x) = -e^{-x}$.
- Differentiate the examples we did before.
- $F(t) = e^{2t}$. $\quad F'(t) = 2e^{2t}$
- $G(t) = \cos(4t^2 + 1)$. $\quad G'(t) = -(8t)\sin(4t^2 + 1)$
- $Y(t) = (14t + 9t^4)^{17}$. $\quad Y'(t) = 17(14t + 9t^4)^{16}(14 + 36t^3)$
- $R(t) = \dfrac{1}{9t^2 + 5}$. $\quad R'(t) = -\frac{1}{(9t^2+5)^2} \cdot (18t)$
- Ex: Use the chain rule to find the derivative of $\ln(t)$.
- A: By definition, $e^{\ln(t)} = t$. These are *inverse functions.*

$$\begin{aligned} \frac{d}{dt}(e^{\ln(t)}) &= \frac{d}{dt}t \\ e^{\ln(t)} \cdot \frac{d}{dt}\ln(t) &= 1 \\ \frac{d}{dt}\ln(t) &= \frac{1}{t} \end{aligned}$$

So, we learned that

$$\boxed{\frac{d}{dt}\ln(t) = \frac{1}{t}}$$

- Ex: Find the derivative of $G(t) = b^t$, where b is any positive constant.
- A: note that $b = e^{\ln(b)}$, so $b^t = (e^{\ln(b)})^t = e^{\ln(b)\cdot t}$ by exponent properties.
- So, use chain shortcut. Inside: $\ln(b)t$ (linear! slope $= \ln(b)$). Outside: $e^{()}$.

$$\begin{aligned} \frac{d}{dt}b^t &= \frac{d}{dt}e^{\ln(b)\cdot t} \\ &= \ln(b)e^{\ln(b)\cdot t} \\ &= \ln(b)b^t. \end{aligned}$$

We learned a new derivative:

$$\boxed{\frac{d}{dt}b^t = \ln(b) \cdot b^t}.$$

Ex: $\frac{d}{dt}2^t = \ln(2) \cdot 2^t$.

3 Differentiation Overview

- At this point, you have all the techniques of differentiation. Summary:
- You know derivatives of the "building blocks:"
  - powers of x: x, x^2, x^{100}, $\frac{1}{x}$, $\sqrt{x}$,...
  - exponentials: $e^x, 2^x, 3^x, ...$
  - logs $\ln(t)$
  - trigonometric functions: $\sin(t)$, $\cos(t)$
- You also now have all of the "combo shortcuts:"
  - power: $(x^n)' = nx^{n-1}$
  - product: $(fg)' = f'g + fg'$
  - quotient: $\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$
  - chain: $(f(g(t)))' = f'(g(t))g'(t)$.

  You can now take the derivative of *any function* I give you.