Day 12: Optimization Problems

1 Distance Minimization Example

- Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1, 4)$.
- Distance d, find with pythagorean theorem:

$$d^2 = (x-1)^2 + (y-4)^2$$

We want to minimize $d = \sqrt{(x-1)^2 + (y-4)^2}$.

- We only care about finding the (x, y) on the curve, not the value of d. So, make life easier by minimizing d^2!
- $d^2 = (\frac{1}{2}y^2 - 1)^2 + (y-4)^2$. Take d/dy, set equal to 0:

$$\begin{aligned} 2\left(\frac{1}{2}y^2 - 1\right)y + 2(y-4) &= 0 \\ y^3 - 2y + 2y - 8 &= 0 \\ y^3 - 8 &= 0 \end{aligned}$$

So, we get $y = 2$. Then, $x = \frac{1}{2}(2)^2 = 2$ also. So $(2, 2)$ is the point on the parabola $y^2 = 2x$ closest to $(1, 4)$.

2 Semicircle

Find the area of the largest rectangle that can be inscribed inside a semicircle of radius r.

- Area: A. $A = 2xy$.

- Eliminate y as a variable: $x^2 + y^2 = r^2$, so $y = \sqrt{r^2 - x^2}$.
- $A = 2x\sqrt{r^2 - x^2}$. Take derivative, set equal to 0.

$$2\sqrt{r^2 - x^2} + 2x\frac{1}{2\sqrt{r^2 - x^2}}(-2x) = 0.$$

  Make it simpler by multiplying through by that denominator:

$$2(r^2 - x^2) - 2x^2 = 0$$

  This is just $-2x^2 + r^2 = 0$, or $x = r/\sqrt{2}$.
- We should check this is a maximum, but it probably will be.
- $$y = \sqrt{r^2 - x^2} = \sqrt{r^2 - \frac{1}{2}r^2} = \frac{1}{\sqrt{2}}r.$$

  So, the maximum area is $A = 2xy = r^2$.

3 Problem-Solving Meta

Problem Solving Strategy: show them in the book.

Common issues:

- Problem: “I don’t know how to start the problem.”
- Fix: you probably didn’t understand what the problem is saying, so you need to read it again (and again and again) to make sure you fully understand all components of the problem. Also try drawing pictures.
- Mistake: treating variables as constants. Ex: $A = 2xy$, so $A' = 2y$ (mistake: not realizing y depends on x).
- Fix: before doing any calculus, take time to understand your variables.

- Problem: There are too many variables in the thing I'm trying to take the derivative of!
- Fix: look in the problem for ways to eliminate variables. Ex: in the semicircle problem, we used $x^2 + y^2 = r^2$ to eliminate y as a variable.