3.1: Stability and the Derivative

1 Idea

- Imagine a cobwebbing diagram with an equilibrium m^*.
- If you cobweb *very close* to m^*, the updating function looks like a line. (Zoom in on graph)
- Use the slope at m^* to decide if it is stable or not!

2 Stability Theorem

- Consider the DDS $m_{t+1} = f(m_t)$.
- Look at $f(x)$.
- Compute $|f'(m^*)|$.
- If it's bigger than 1, m^* is definitely unstable.
- If it's smaller than 1, m^* is definitely stable.
- if it equals 1, anything can happen.

3 Examples

- $m_{t+1} = \cos(m_t)$. $f(x) = \cos(x)$, $f'(x) = -\sin(x)$. The equilibrium is at $m^* \approx 0.739$, and

  $$|f'(0.739)| = |-\sin(0.739)| = 0.674 < 1$$

  so m^* is stable.

- $m_{t+1} = \frac{0.6m_t}{0.6m_t+2(1-m_t)}$, $f(x) = \frac{0.6x}{0.6x+2(1-x)}$.
  Equilibria:

  $$\frac{0.6x}{0.6x + 2(1-x)} = x$$

  after algebra becomes

  $$0 = 1.4x(1-x)$$

  so the equilibria are $m^* = 0$ and $m^* = 1$. Stability theorem next:

  $$f'(x) = \frac{0.6(0.6x + 2(1-x)) - 0.6x(0.6-2)}{(0.6x + 2(1-x))^2}$$

$$f'(0) = \frac{0.6 * 2}{4} = \frac{0.6}{2} = 0.3.$$
$$f'(1) = \frac{0.6^2 - 0.6^2 + 2 * 0.6}{(0.6)^2} = \frac{2}{0.6} = 3.33$$

so, $m^* = 0$ is stable, while $m^* = 1$ is unstable.

- Ex: $m_{t+1} = 2m_t e^{-m_t}$.

  Equilibria: $2xe^{-x} = x$ gives $e^{-x} = \frac{1}{2}$, so

  $$m^* = -\ln(\frac{1}{2}) = \ln(2) = 0.693.$$

  Stability:

  $$f'(x) = 2e^{-x} - 2xe^{-x}$$

  So:

  $$\begin{aligned} f'(\ln(2)) &= 2e^{-\ln(2)} - 2\ln(2)e^{-\ln(2)} \\ &= 2\frac{1}{2} - 2\ln(2) \cdot \frac{1}{2} = 1 - \ln(2) \approx 0.307. \end{aligned}$$

  Since this is smaller than 1, we see that $m^* = \ln(2)$ is stable.