Day 11: Optimization

1 Local Max's and Mins

- Local Max and mins are exactly that: locally, its a peak or valley.
- These are separate from the interval-optimization problem we've been discussing.
- Draw picture.
- A *critical point* is an x-value where either $f'(x) = 0$ or $f'(x)$ does not exist (e.g. $\frac{1}{0}$).
- Ex: find all critical points of the function $f(x) = \frac{x^2-4x}{x+1}$.
- $$f'(x) = \frac{(2x-4)(x+1) - (x^2-4x)(1)}{(x+1)^2} = \frac{x^2+2x-4}{(x+1)^2} = 0$$

  $f'(x) = 0$:

  $$x^2 + 2x - 4 = 0 \implies x = -1 - \sqrt{5} or x = -1 + \sqrt{5}.$$

  $f'(x)$ undefined: $x + 1 = 0$, or $x = -1$.
- We have two important tests to say when a crit pt. is a local max or min.
- First Derivative Test: Look at slopes to the left and right of a point.
- Ex: $f(x) = x^2 - 2x + 1$. Find critical points, then decide if they are local max's or mins.

  $$f'(x) = 2x - 2 = 0 \implies x = 1.$$

  $f'(0) = -2$, $f'(2) = 2$, so $x = 1$ must be a local min.

2 Second Derivative

- The second derivative, f'', measures how the *slopes* of f' change.
- $f''(x) > 0$ means f is concave up. $f''(x) < 0$ means concave down.
- We call x's where $f''(x) = 0$ *inflection points*.
- Clearly, at local minima, $f''(x) > 0$, and at local maxima, $f''(x) < 0$.
- This is the Second-Derivative test.
- Example: $f(x) = x^3 - 3x$. Find local minima and maxima.

$$f'(x) = 3x^2 - 3 = 0 \implies x = \pm 1.$$

Check using second derivative test:

$$f''(x) = 6x.$$

$f''(1) = 6$, so 1 is local min. $f''(-1) = -6$, so -1 is local max.

3 Global Maxima/Minima

- Global maxima and minima are what we discussed in weeks 1 & 2.

4 Unbounded or open intervals

- Recall: Extreme Value Theorem: If $f(x)$ is continuous on a closed bounded interval $[a, b]$, then it attains its maximum and minimum.
- If we remove some of the conditions on the interval, we have to take limits.

- Ex: Find global maxima and minima for $f(x) = xe^{-x}$ on the interval $[0, \infty)$.
- $f'(x) = e^{-x} - xe^{-x} = 0$ for crit pts: get $x = 1$.
- Endpoints: 0 and "∞".
- $f(0) = 0$, while $\lim_{x\to\infty} xe^{-x} = 0$ (use graph for the last one; we'll prove it formally next week).
- So, global maximum is $x = 1$, $y = e^{-1}$, global min is 0.
- Ex: Optimize $f(x) = x^3 - x^2$ on $(-\infty, 1)$.
- Crit Pts: $f'(x) = 3x^2 - 2x = 0$, get $x = 0$ and $\frac{2}{3}$.
- Endpoints: $f(1) = 1 - 1 = 0$
- $\lim_{x\to-\infty} x^3 - x^2 = -\infty$
- $f(0) = 0$, and $f(2/3) = -0.148$.
- So, global max's at $x = 0$ and $x = 1$, with $y = 0$. NO GLOBAL MIN!
- Feature of these problems: when the interval is unbounded or open, you might not get global minima or maxima.