One application of autonomous DE

Contents

1 Logistic Model

- The logistic model for a population with carrying capacity is

  $$\frac{dy}{dt} = ky\left(1 - \frac{y}{N}\right)$$

  where k is a positive number. Ex: Find equilibria and apply the stability theorem to assess the stability.

- Equilibria: $y^* = 0$ and $y^* = N$.

- The updating function is $f(y) = ky\left(1 - \frac{y}{N}\right)$.

- $f'(y) = k - \frac{2ky}{N}$.

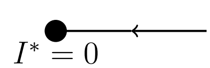
- $f'(0) = k$ is positive, so $y^* = 0$ is unstable.

- $f'(N) = k - \frac{2kN}{N} = -k$ is negative, so $y^* = 0$ is stable.

2 Realistic Disease Model

Let I be the *fraction* of people infected. (This means I only takes decimal values btwn 0 and 1.)

- Individuals recover, but may become susceptible later (like a cold).
- More infected people = more spread. So, something like αI is a per capita infection rate.
- $1 - I$ represents uninfected, but susceptible people.
- thus, $\alpha I(1 - I)$ is the infection rate (multiplying per capita by number of uninfected makes it a total infection rate).
- People recover: rate of μI.
- Equation:
$$\frac{dI}{dt} = \alpha I(1 - I) - \mu I.$$
- Equilibria: $I^* = 0$ and $I^* = 1 - \frac{\mu}{\alpha}$
- Now, there are two cases.
- Case 1: People recover faster than they get sick. That is, $\alpha < \mu$. Then $\frac{\mu}{\alpha} > 1$, so $I^* = 1 - \frac{\mu}{\alpha} < 0$.In this case, this equilibrium is non-realistic.
- Use stability theorem to see if it's stable or not.
$$\begin{aligned} f(I) &= \alpha I - \alpha I^2 - \mu I \\ f'(I) &= \alpha - \mu - 2\alpha I. \end{aligned}$$
Plug in $I^* = 0$. You get $f'(0) = \alpha - \mu < 0$, so $I^* = 0$ is stable (expected).
- Phase-line diagram for this case:



- Case 2. People can't recover fast enough: $\alpha > \mu$. Then $I = 1 - \frac{\mu}{\alpha} > 0$, and we have a new equilibrium.

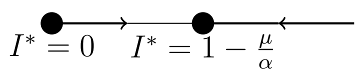
- We already got the derivative. So, calculate $f'(I^*)$:

$$f'(I^*) = \alpha - \mu - 2\alpha(1 - \frac{\mu}{\alpha}) = \alpha - \mu - 2\alpha + 2\mu = -\alpha + \mu < 0$$

  so in fact the nonzero equilibrium is stable. Also, in this case,

$$f'(0) = \alpha - \mu > 0,$$

  telling us that $I^* = 0$ is unstable.



  the model predicts that in this case the population will come to equilibrium.

- Possible interpretation: we know in real life that diseases like the common cold don't fade out, so we must be in case 2. What we learn is that *people get sick more quickly than they recover* from these kinds of colds.