Exam 2

Math 111, Summer '16

Name: ______________________________ Student Number: ______________________

Don't leave anything blank. If you don't know the entire answer, showing a formula or writing something illustrating that you understand any concept involved in the problem will allow me to give partial credit. I have to give you a 0 if you write nothing down.

Check your answers. Take the time before you turn in your test to make sure you have read the directions correctly and in their entirety, that all your work shown is correct, and that you have clearly stated your answer (by boxing or circling it where appropriate).

Pace yourself. If you're stuck on a problem, move on and come back to it later. Don't risk forcing yourself to give partial answers if you run out of time near the end of the test. Do the easy ones first. There are 50 points on this exam. That means you should budget about 1 minute for each point a problem is worth in order to complete the exam in time. After reading these instructions, circle the two words at the end of this sentence for extra credit.

Exam 2

Short Answer. Directions: Answer the following problems briefly.

1. [2 pts each] Let $f(t) = \dfrac{t^2}{10t^2 - 9t + 1}$ and $g(t) = \dfrac{3t^4}{t^7} - \dfrac{1}{t}$.

   (a) What is the long-run behavior of $f(t)$ as $t \to \infty$?
   $f(t) \sim \frac{t^2}{10t^2} = \frac{1}{10}$, so $f(t) \to \frac{1}{10}$ as $t \to \infty$.

   (b) What is the long-run behavior of $g(t)$ as $t \to \infty$?
   $g(t) \sim \frac{3}{t^3} - \frac{1}{t}$, so $g(t) \to 0$ as $t \to \infty$ by the Big-Little principle.

2. [5 pts] Solve the equation $\log_3(t^2) + \log_3(t) = 1$.

$$\begin{aligned} \log_3(t^2 \cdot t) &= 1 \\ \log_3(t^3) &= 1 \\ t^3 &= 3 \\ t &= 3^{1/3} \approx 1.442. \end{aligned}$$

3. Suppose $B(t) = 3 \cdot e^{0.12t}$ describes the number of bacteria, where t is in minutes.

   (a) [3 pts] What is the growth factor for this model?
   $b = e^{0.12} \approx 1.127.$

   (b) [3 pts] What is the growth rate of the bacteria?
   $b = 1 + r$, so $r = 0.127$, or 12.7 a minute.

Free-Response Problems (Write your answers clearly and concisely, including all work. If asked to explain something, use complete sentences. Any numerical answers may be written in approximate form as long an exact solving method is used.)

4. [6 pts] The net value of an account with initial value V_0, annual interest rate r compounded n times per year for t years is given by $V = V_0\left(1+\frac{r}{n}\right)^{nt}$. What is the annual rate of a credit line which rises from an initial balance of \$4,800 to \$6,000 in the course of five years, assuming that interest is compounded quarterly?

   $V_0 = 4.8$, $V = 6$, $n = 4$, $t = 5$. $r = ?$

$$\begin{aligned}
6 &= 4.8\left(1+\frac{r}{4}\right)^{4\cdot 5} \\
\frac{6}{4.8} &= \left(1+\frac{r}{4}\right)^{20} \\
(1.25)^{1/20} &= 1+\frac{r}{4} \\
1.01122 &= 1+\frac{r}{4} \\
\frac{r}{4} &= 0.01122 \\
r &= 0.04488,
\end{aligned}$$

   or 4.488% per year.

5. [6 pts] Find the equation of an exponential function passing through the points $(3, 90)$ and $(6, 20)$. (This problem is separate from the previous one.)

   Start with $f(t) = ab^t$. We have two equations:

$$\begin{aligned}
90 &= ab^3 \\
20 &= ab^6.
\end{aligned}$$

   Dividing gives

$$\begin{aligned}
\frac{20}{90} &= \frac{ab^6}{ab^3} = b^3 \\
b &= \left(\frac{2}{9}\right)^{1/3} \approx 0.606.
\end{aligned}$$

   Solve for a:

$$\begin{aligned}
90 &= ab^3 \\
90 &= a\cdot\frac{2}{9} \\
a &= 405
\end{aligned}$$

   So, $f(t) = 405(2/9)^{t/3} = 405(0.606)^t$.

6. The half-life of a mysterious new radioactive chemical X is 5.73 years. Suppose we started with twelve grams of chemical X.

   (a) [5 pts] Write a function that expresses the amount of chemical X in terms of the time t, *in years*.

   We have $f(t) = 12\left(\frac{1}{2}\right)^{t/5.73}$. See the review solutions for complete details.

   (b) [2 pts] Exactly how much chemical X is left after 10 years?

   $f(10) = 12\left(\frac{1}{2}\right)^{10/5.73} \approx 3.580$ grams.

   (c) [2 pts] What is the continuous decay rate of chemical X?

   $e^k = b = \left(\frac{1}{2}\right)^{1/5.73} = 0.886$, so

   $$k = \ln(b) = \ln(0.886) = -0.121,$$

   or -12.1% a year.

   (d) [5 pts] How long until there is only one gram of chemical X left from our original amount?

$$\begin{aligned}
1 &= 12\left(\frac{1}{2}\right)^{t/5.73} \\
\frac{1}{12} &= \left(\frac{1}{2}\right)^{t/5.73} \\
\ln\left(\frac{1}{12}\right) &= \ln\left(\left(\frac{1}{2}\right)^{t/5.73}\right) \\
\ln\left(\frac{1}{12}\right) &= \frac{t}{5.73}\ln\left(\frac{1}{2}\right) \\
t &= 5.73\frac{\ln(1/12)}{\ln(1/2)} \approx 20.541 \text{ years.}
\end{aligned}$$

7. Some adults were given an exam. Over the next three years, the same adults were retested several times. The average score was given by the model

$$f(t) = 91 - 5\log_{10}(t+1),$$

where t is the time in months.

(a) [2 pts] What was the average score on the original exam?

$f(0) = 91 - 5\log_{10}(1) = 91$ (remember, $\log_{\text{anything}}(1) = 0$)

(b) [2 pts] What was the average score a year and a half later?

$f(18) = 91 - 5\log(18+1) = 84.6.$

(c) [5 pts] Exactly how much time must pass for the average score to reach 85?

$$\begin{aligned} 91 - 5\log_{10}(t+1) &= 85 \\ -5\log_{10}(t+1) &= -6 \\ \log_{10}(t+1) &= \frac{6}{5} \\ t+1 &= 10^{6/5} \\ t &= 10^{6/5} - 1 \approx 14.8 \text{ months.} \end{aligned}$$

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