4.1: Euler's Method

Contents

1 Euler's Method via example

- The idea of Euler's method is rather like the idea of a discrete dynamical system: use what you know at time t to "bootstrap" yourself to time $t+1$.
- Only now, we don't need to restrict ourselves to a time increment of 1.
- Ex: Consider the equation $y' = \frac{1}{2}y$. We know the general solution is $y = Ce^{\frac{1}{2}x}$, but let's pretend we didn't.
- Say we have initial condition $y(0) = 1$ and that we want to predict by increments of $\Delta t = 0.5$. (For this example, to make things clearer, I will round everything to 2 decimals. In practice, this is unneccessary.)
- From the DE: $y'(0) = \frac{1}{2}y(0) = \frac{1}{2} \cdot 1 = \frac{1}{2}$.
- You know slope = rise/run, or that the rise = run $\times$ slope. so,

$$\text{rise} = (0.5)\left(\frac{1}{2}\right) = 0.25.$$

- predict: $y(0.5) \approx 1.25$ (add the rise to the known value of 1).

- feedback: $y'(0.5) = \frac{1}{2}(1.25) = 0.63$, so

$$\text{rise} = (0.5)(0.63) = 0.32.$$

- predict: $y(1) = 1.25 + 0.32 = 1.57$.
- can continue this cycle of predicting a rise, getting a new y-value, and feeding back into the DE to get a new slope and rise.
- Draw picture; we are obtaining y-values of the solution, predicted from the DE, the initial condition, and the time increment.
- compare to the actual solution of $e^{t/2}$ using computer.

2 Euler's Method in General

- Here's the strategy of Euler's method.
- Start with: a DE, an initial condition, and a time step Δt.
- Follow the recipe:
  1. get a slope from the DE by plugging in the current t and y-values.
  2. find the rise.
  3. use the rise and the known y-value to get the new y-value at time $t + \Delta t$.
  4. feedback: return to step 1.
- This is known as *Euler's Method*. (Note: Euler is pronounced like "Oiler".)
- Facts about Euler's method:
  - The y-values we obtain from this are known as the *numerical* solution, because we don't have a formula for them; rather we just have the y-values.
  - It can be applied to any first-order DE (and to higher order, with modifications that we might discuss later.) As such, it is *very* widely used in practice.

- The predicted y-values are only *approximate*; they may not be exactly the right y-values, but they will be close.
- Using smaller Δt makes a more accurate solution, but takes a lot more work to compute further out! So it's a tradeoff: do you want a quick answer, or an accurate answer? This is the idea of *computational complexity.*

2.1 Examples

- Ex: With the DE $\frac{dy}{dt} = -t + y$, initial value $y(0) = 0$, and step size $\Delta t = 0.5$, predict the value of $y(2)$.

t	y
0	0
0.5	0
1	-0.25
1.5	-0.875
2	-2.0625

- You try: With the DE $\frac{dy}{dt} = t^2 + 1$, initial value $y(0) = -1$, and step size $\Delta t = 0.1$, predict the value of $y(0.3)$.

t	y
0	-1
0.1	-0.9
0.2	-0.799
0.3	-0.586

- You try: with DE $y' = t^2 y$, initial condition $y(0) = 2$, and step size $\Delta t = 1$, predict $y(3)$.

t	y
0	2
1	2
2	4
3	20