1.9/3.2: Breathing Model and Logistic Model

1 Lung Model

1.1 Walking Through the Model

- Quantities: c_t = concentration of a kind of gas in the lungs. Could be O_2, could be nitrogen, or laughing gas!
- Let V be the total volume of the lungs (average is about 6 Liters)
- W - amount of volume exhaled on normal breath
- γ - outside concentration of the chemical.
- Let's get familiar with these values before modeling.
- Ex: if a person exhales 0.5 L of air, and the concentration of O_2 is 2mmol/L, then how much O_2 did they exhale?
- Ans: 0.5L$*$2mmol/L = 1 mmol of O_2. That's a number of O_2 molecules.
- The quantity we're modeling, c_t, is like a conversion factor. It converts volume (Liters) into a number of moluecules (mmol).

1.2 One Breath Computation

- Make table:

	Volume (L)	Concentration (mmol/L)	Chemical (mmol)
Before out:	6	0.03	0.18
exhale:	0.5	0.03	0.015
after exhale:	5.5	0.03	5.5(0.03) = 0.165
inhale:	0.5	2	2(0.5) = 1
after inhale:	6	$\frac{1.165}{6} \approx \underline{\underline{0.194}}$ (c_{t+1})	$1 + 0.165 = 1.165$ ($W + (V-W) \cdot C_t$)

$$\left\{ \begin{array}{l} C_t = 0.03 \\ V = 6 \\ W = 0.5 \\ \gamma = 2 \end{array} \right.$$

$$C_{t+1} = \frac{\gamma W + (V-W)C_t}{V}$$

$$C_{t+1} = \gamma\left(\frac{W}{V}\right) + \left(1 - \frac{W}{V}\right) C_t$$

$$\boxed{C_{t+1} = \gamma q + (1-q) C_t}$$

- Let $q = \frac{W}{V}$. It turns out that neither W or V are that important; what is more important is how big they are relative to each other, meaning their ratio q.

1.3 Analyze the Model

- What are the equilibria?
- Get: $c^* = \gamma$.
- Is it stable or unstable?
- Note: $f(x) = (1-q)x + q\gamma$. (where $q = W/V$)
- $f'(x) = 1 - q$. Since q is a fraction of air breathed out, q is smaller than 1, so $1 - q$ is also smaller than 1.
- Stability theorem tells us this equilibrium is always stable.
- More analysis to come in your homework.

1.4 Conclusions to Draw

- No matter the initial starting amount, we know that the concentration will come into equilibrium with the outside equilibrium, as we expected. What is cool is that the math predicts what we expected in reality; nothing told the math to do that, rather it just did that.

2 Logistic Model

2.1 Population models

- Focus on modeling a population, P, where the variable P is the number of individuals.
- The most general discrete dynamical system is

$$P_{t+1} = RP_t,$$

  where t is some time step (hours, weeks, days, years, generations...) and R is called the *growth factor*.
- Simplistic population models assume R is a pure constant, like 2. This leads to exponential growth or decay, as we've seen. [If you don't believe me, let $R = 2$, pick any seed, and plot the sequence of numbers you get. It will be exponential.]
- To make the model more realistic, we *challenge* this assumption on R.
- We now assume R is not constant. Instead, we now assume R is going to be just a function of population.
- Let N be a carrying capacity. By definition, this is the maximum allowed population size. Anything more and the population runs out of resources and collapses.
- The logistic model is when we assume R depends linearly on P, and that $R = 0$ when $P = N$:

$$\boxed{R = r\left(1 - \frac{P}{N}\right)}.$$

- Thus, we have this DDS:

$$P_{t+1} = rP_t\left(1 + \frac{P_t}{N}\right)$$

- We now define a new variable $x = P/N$. This new variable reports the percentage of carrying capacity. So, if $x = 0.57$, then that means the population has reached 57% of its carrying capacity. If $x = 1.2$, then the population is 20% *over* its carrying capacity.

- The DDS for x comes from that of P_t:

$$\boxed{x_{t+1} = rx_t(1 - x_t).}$$

- I expect you to be able to find equilibria, analyze stability, and interpret what specific values of the variable x mean in context.