## General Information

The final exam will take place on **Friday, August 12** from 11:00am-1:00pm in our usual room (Deady 306). You may have a graphing calculator. You may bring notes on the front and back of a **half** sheet of paper. This review will not be collected for credit. The exam will cover all material from section 1 to 8. Please keep in mind that this review is not comprehensive. You should also spend some time reviewing notes, reading through the sections, going over old quizzes, exams, written assignments, worksheets, and webwork.

The exam will have a few true/false problems, a few multiple choice problems, and some long answer problems.

## Topics

Here are some key words to help you study.

1. Section 1: Introduction to Functions
   a) Determine if some relationship is a function
   b) Mathematical and practical domain and image
   c) Piecewise-Defined Functions
2. Section 2: Linear Functions and Average Rate of Change
   a) Average rate of change
   b) Percent change
   c) Graphing a linear function
3. Section 3: Quadratic Functions and Higher-Order Polynomials
   a) Optimization
   b) Second differences
   c) Forms of a quadratic; finding a formula for a quadratic
   d) Polynomial functions
   e) Long term behavior of polynomials

4. Section 4: Rational Functions
    a) Determine if a function is rational
    b) Graphs of basic rational functions
    c) Long-term behavior of rational functions
5. Section 5: Exponential Functions
    a) Forms of an exponential function
    b) Graphs of basic exponential functions
    c) Long term behavior of exponential functions
    d) Domain and image of an exponential function
    e) Compound interest
    f) Continuously compounding interest
6. Section 6: Power and Logarithmic Functions
    a) Converting between logarithmic and exponential forms
    b) Graphs of basic logarithmic functions
    c) Properties of logarithms (sum, difference, and constant multiple rules)
    d) Change of base formula
7. Section 7: Composition and Arithmetic of Functions
    a) Arithmetic operations on functions
    b) Finding domains
    c) Composition of functions
    d) Domain of a composite function
8. Section 8: Function Inverses
    a) One-to-one functions
    b) Computing inverses
    c) Domain and image of an inverse function
    d) Domain restrictions

## Practice Problems

1. Give a precise definition of the following.
   a) function
   b) linear function
   c) piecewise defined function
   d) quadratic function
   e) polynomial function
   f) rational function
   g) exponential function
   h) logarithmic function

2. Determine the type of function that could model the following scenarios:
   a) The price of a cab, given how many miles I've traveled.
   b) The amount of money in a bank account gaining compound interest.
   c) A function that can optimize the area of a square, given some constraints on the sides.
   d) The height of an object after being thrown, given its initial height and velocity.
   e) The amount of money an employee makes an an hour job, given the number of hours they've worked.

3. Determine if the following statements are true or false. Provide a short explanation for your answer.
   a) The domain of $g$ is always a subset of the domain of $f \circ g$.
   False; the domain of $f \circ g$ is actually smaller, because composing with $f$ can force more restrictions.

   b) Let $f(x) = \dfrac{(x+1)^2}{x+1}$ and $g(x) = x + 1$. Then $f$ and $g$ are equal functions.
   False; their domains don't agree!

   c) If a function is not one-to-one, then it is not invertible.
   True!

   d) Every quadratic function is a polynomial function.
   True; a quadratic function is just a degree 2 polynomial function.

   e) Every function must pass the horizontal line test.
   False; a quadratic function does not pass the horizontal line test.

   f) The following is a rational function
   $$g(x) = x^{-2} + \frac{\sqrt{e}}{x+1}.$$
   True! We can rewrite $x^{-2} = 1/x^2$, and recall that the sum of any two rational functions is rational.

4. Compute the average rate of change **and** the percent change of the following functions on the specified intervals.

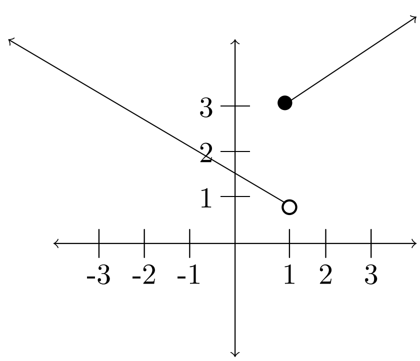
a) $f(x) = x^3 + 4$ on $[-1, 1]$.
We have,

$$\mathrm{ARC}_{[-1,1]} = \frac{f(1) - f(-1)}{1 - (-1)} = \frac{5-3}{2} = 1.$$

and

$$\mathrm{PC}_{[-1,1]} = \frac{f(1) - f(-1)}{f(-1)} \times 100\% = \frac{2}{3} \times 100\% \approx 66.67\%$$

b) $g(x)$, defined by the following graph, on $[0, 1]$.




We have,

$$\mathrm{ARC}_{[0,1]} = \frac{g(1) - g(0)}{1 - 0} = \frac{3 - \frac{3}{2}}{1} = \frac{3}{2}.$$

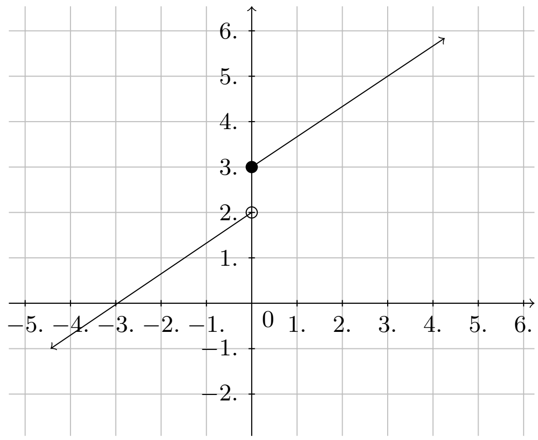
and

$$\mathrm{PC}_{[0,1]} = \frac{f(1) - f(0)}{f(0)} \times 100\% = \frac{3/2}{3/2} \times 100\% = 100\%$$

5. Find the domain of the function $f(x) = \dfrac{\sqrt{1-x}}{\sqrt{x+7}}$.

We need to make sure we have both $1 - x \geq 0$ and $x + 7 \geq 0$. This means that $-7 \leq x \leq 1$. But we also need to make sure we don't divide by 0, so $x$ cannot be 7. Thus, the domain is $(-7, 1]$.

6. Find a formula for the function below.




The first thing we should note is that our function needs to be piecewise. So, we'll have something like

$$f(x) = \begin{cases} \text{something} & \text{if } x < 0 \\ \text{some other thing} & \text{if } x \geq 0. \end{cases}.$$

Now we just need to fill in equations for each piece. Since each piece is linear, we just need slopes and intercepts. For the piece on the left, we have a slope of 2/3 and an intercept at 2. The piece on the right has a slope of 2/3 as well, and an intercept at 3. So,

$$f(x) = \begin{cases} \frac{2}{3}x + 2 & \text{if } x < 0 \\ \frac{2}{3}x + 3 & \text{if } x \geq 0. \end{cases}.$$

7. Is the function defined by the table below one-to-one? Explain.

| $x$ | -1 | 0 | 1 | 2 |
|---|---|---|---|---|
| $f(x)$ | 4 | 3 | 4 | 7 |

No; the output of 4 has two distinct inputs, which shows that $f$ is not one-to-one.

8. Determine if the following are polynomials, rational functions, or neither. Circle your choice from the options on the right.

   a) $f(x) = x^3 + 3$ polynomial function rational function neither

   Polynomial

   b) $g(t) = \dfrac{x^{1/2} - 3x^2 + 2}{2x + 1}$ polynomial function rational function neither

   neither

   c) $h(x) = \dfrac{\pi x^2 + 2e^2}{2x + x^{-1}} - \dfrac{1}{3}$ polynomial function rational function neither

   rational!

9. Find the long term behavior of each of the following functions.

   a) $f(t) = (9 - t^2 + t)(2 - t)$
   Recall that the long term behavior of a polynomial function is completely determined by its leading term. So, as $t \to \infty$, we have

$$f(t) = (9 - \mathbf{t^2} + t)(2 - \mathbf{t}) \to (-t^2)(-t) = t^3 \to \infty.$$

   b) $g(t) = 2(0.45)^{-2t}$
   Recall that the long term behavior of an exponential function is completely determined by its constant growth rate. So, let's write $g$ is standard form:

$$g(t) = 2\left(\left(\frac{1}{.45}\right)^2\right)^t = 2\left(\left(\frac{100}{45}\right)^2\right)^t.$$

   So our constant growth rate is $(100/45)^2 > 1$, hence $g(t) \to \infty$ as $t \to \infty$

   c) $h(t) = \dfrac{t^2 - 4t^5}{t^2 + 6t + 7t^8 - 2}$
   Recall that the long term behavior of a rational function is completely determined by the ratio of the leading terms of its numerator and denominator polynomials. So, as $t \to \infty$, we have

$$h(t) = \frac{t^2 - 4t^5}{t^2 + 6t + 7t^8 - 2} \to \frac{-4t^5}{7t^8} = \frac{-4}{7t^3} \to 0.$$

10. Write a sentence that describes the inverse of the function defined by: add one to the input and then double the result.
   The inverse function will half the input and subtract one from the result.

11. Find the inverse of $g$, defined by

$$g(t) = \log_3(5t+1) - 4.$$

Let $Q = g(t)$. Then, $t = g^{-1}(Q)$. So,

$$Q = \log_3(5t+1) - 4 \Rightarrow Q + 4 = \log_3(5t+1)$$

$$\Rightarrow 3^{Q+4} = 5t + 1 \Rightarrow t = \frac{3^{Q+4} - 1}{5}.$$

Hence, $g^{-1}(Q) = \dfrac{3^{Q+4} - 1}{5}$.

12. Let $f(x) = \dfrac{1}{x^2}$ and $g(x) = \sqrt{x+1}$. Find the domain of $(f \circ g)(x)$. Show all work!

Compute $f \circ g$ and along the way, note any necessary restrictions:

$$\begin{aligned}(f \circ g)(x) &= f(\sqrt{(}x+1)) \\ &= \frac{1}{(\sqrt{x+1})^2)} \\ &= \frac{1}{x+1}.\end{aligned}$$

We first need to be able to take a square root, so we need $x \geq -1$. Then, when we plug into $f$, we need to be able to divide by this, so $x \neq -1$. Thus, the domain is $(-1, \infty)$.

13. Consider the functions defined by the table below.

| x | 3 | 5 | -2 | -1 |
|---|---|---|---|---|
| g(x) | 5 | 0 | -25 | 12 |
| h(x) | 12 | 2 | 25 | 0 |

a) Evaluate each function at the specified value.

i. $(g+h)(-1) = 12$

ii. $(h \circ g \circ k)(1)$, where $k(x) = x^2 + x + 1$. $= 2$

iii. $g^{-1}(5) = 3$, since $g(3) = 5$.

iv. $\left(\frac{g}{h}\right)(-1) = \frac{12}{0}$ which is undefined!

v. $h^{-1}(2) = 5$, since $h(5) = 2$.

b) Find the domain of $g \circ h$ and $h \circ g$.
$\text{dom}(g \circ h) = \emptyset$ and $\text{dom}(h \circ g) = \{3\}$

14. The half-life of ibuprofen (used in pain-killers) is around two hours. A typical dose consists of 400 mg. If you take one dose, how long will it take for there to be 20% of the original amount of ibuprofen left? We have $f(t) = 400 \cdot (0.5)^{t/2}$. We want to solve $f(t) = 0.2 * 400 = 80$. This gives the equation

$$400 \cdot (0.5)^{t/2} = 80,$$

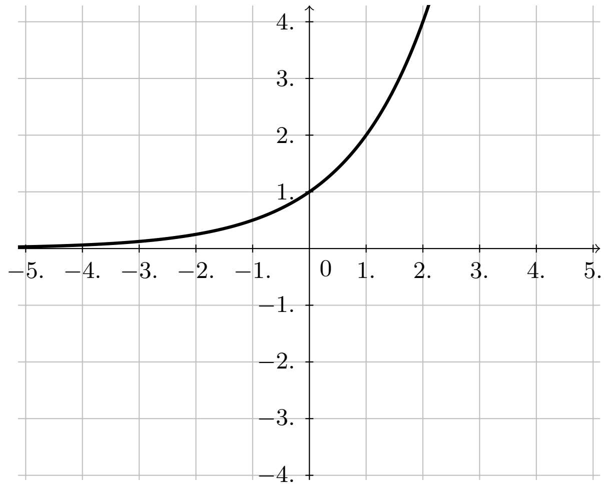
which has the solution $t = -\log_2(0.04)$.

15. Say I invest \$100 in a bank that accrues 12% interest, compounded continuously.

    a) Find a formula for the amount of money in your bank account at time $t$ in years. $f(t) = 100e^{0.12t}$.

    b) Find the inverse of the function you found in part (a). $f^{-1}(y) = \frac{1}{0.12}\ln(y/100)$.

16. Sketch the graph of the inverse to the function below.



17. Recall the formula for projectile motion,

$$h(t) = \frac{-9.8}{2}t^2 + v_0 t + h_0,$$

where $h$ is the height of an object, in meters, after $t$ seconds. If a diver reaches their maximum height 2 seconds after leaving a diving board, what must have been their initial velocity? Show all work, and be sure to include units.

Let's recall that the maximum height of the diver occurs at the vertex of $h$, which has an input of $t_{\max} = \frac{v_0}{9.8}$. Well, we know that $t_{\max} = 2$, so

$$2 = v_0/9.8 \Rightarrow v_0 = 19.6 \text{ m/s}.$$

18. The area of a circular wave expands across a still pond such that its radius increases by 2cm each second. Write a formula for the circumference of the circle as a function of time $t$ since the wave begins.

When the wave begins, there is no radius, and it is increasing at a constant rate of 2cm each second. So, $r(t) = 2t$. Now, let's recall that the circumference of a circle is given by $C(r) = 2\pi r$. To get the circumference as a function of $t$, we want the composite function $C \circ r$. So,

$$(C \circ r)(t) = 4\pi t.$$

19. [6 pts.] Say I have \$5,000 that I'd like to put into savings. When I go to my bank, they tell me that I have two options for savings accounts that will earn me money through compound interest. My first option is to open an account where interest is compounded monthly at a rate of 5%, and my second option is to open an account where interest is compounded continuously at a rate of 4%.

a) Which option will gain the most interest after 10 years?

Option 1: We have, $V_1(t) = 5000(1 + \frac{0.05}{12})^{12t}$. So,

$$V_1(10) = 5000\left(1 + \frac{0.05}{12}\right)^{120} \approx 8235.05.$$

Option 2: We have, $V_2(t) = 5000e^{0.04t}$. So,

$$V_2(10) = 5000e^{0.4} \approx 7459.12.$$

Hence, Option 1 will make me about \$775.93 more than Option 2 after 10 years.

b) What will be the difference in time between my two options to reach a \$10,000 savings goal?

Option 1: $10000 = V_1(t) = 5000(1 + \frac{0.05}{12})^{12t} \Rightarrow 2 = (1.0042)^{12t}$. So,

$$\log_{1.0042}(2) = 12t \Rightarrow t = \frac{1}{12}\frac{\ln(2)}{\ln(1.0042)} \approx 13.78.$$

Hence, it will take about 13.78 years to reach my savings goal with Option 1.

Option 2: $10000 = V_2(t) = 5000e^{0.04t} \Rightarrow 2 = e^{0.04t}$. So,

$$\ln(2) = 0.04t \Rightarrow t = \frac{1}{0.04}\ln(2) \approx 17.33.$$

So, it will take about 17.33 years to reach my savings goal with Option 2.

That means that I would save around 3.55 years with Option 1.