Day 6: Implicit Differentiation

We now embark on differentiation tricks.

1 Implicit Differentiation

- Up until now we've been in a scenario where $y = f(x)$, that is, y is explicitly a function of x.
- We can be a little more flexible: now, we only assume that y is *implicitly* a function of x, even if we can't necessarily solve for y in terms of x.
- Ex: Consider the points (x, y) such that $x^2 + y^2 = 25$. This is the graph of a circle of radius 5.
- In general, if you write an equation with x's and y's, it will define a figure in the plane. Ex: folium of Descartes: $x^3 + y^3 = 6xy$ makes a cool loop-de-loop.
- We can find the derivative (or slopes of tangent lines) implicitly:
- Ex: $x^2 + y^2 = 25$. Find slope at the point $(4, 3)$. Take the derivative with respect to x on both sides:

  $$2x + 2y \cdot y' = 0.$$

  Solve it for y'. Get:

  $$y' = -\frac{x}{y}.$$

  We are at the point $(4, 3)$, meaning $x = 4$, $y = 3$, so

  $$y' = -\frac{4}{3}.$$

- Looks like black magic, but it's just the chain rule used in a sneaky way.

- Alternative: solve for y in terms of x.

$$\begin{aligned} y &= +\sqrt{25-x^2} \\ y' &= \frac{1}{2}\left(25-x^2\right)^{-1/2}(-2x) \\ y' &= -(4)(25-16)^{-1/2} \\ y' &= -\frac{4}{3} \end{aligned}$$

- Ex: Find the tangent line to the folium of Descartes $x^3+y^3=6xy$ at the point $(3,3)$.

$$3x^2+3y^2\cdot y' = 6(x)'y+6(x)(y')$$

Solve for y'. Move all y''s to one side, move rest to other side.

$$3y^2(y')-6x(y') = -3x^2+6y$$

$$y' = \frac{-3x^2+6y}{3y^2-6x}$$

Now, at the point $(3,3)$, $x=3$ and $y=3$:

$$y' = \frac{-3(3)^3+6(3)}{3(3^2)-6(3)} = -1$$

Find the tangent line: you have slope -1, and a point $(3,3)$. Result: $y=-x+6$.

- Ex: Find the slope of the tangent line to the hyperbola $x^2-y^2=1$ at the point $(2,-\sqrt{3})$.

$$\begin{aligned} 2x-2y(y') &= 0 \\ y' &= \frac{x}{y} \\ y' &= \frac{2}{-\sqrt{3}} = -\frac{2}{\sqrt{3}}. \end{aligned}$$

What is the slope of the tangent line at the point $(1, 0)$.

$$y' = \frac{x}{y} = \frac{1}{0}$$

This means that the slope is undefined; this means the tangent line is *vertical*.

- For the plane curve defined by the equation

$$y^2 = x^3 - x,$$

find the points where the tangent line is horizontal or vertical.

$$\begin{aligned} 2y(y') &= 3x^2 - 1 \\ y' &= \frac{3x^2 - 1}{2y} \end{aligned}$$

horizontal: set the top =0. Get $3x^2 - 1 = 0$, or $x = \pm\sqrt{\frac{1}{3}} = \pm 0.577$. Find the y-value(s) at $x = 0.577$:

$$y^2 = 0.577^3 - 0.577 = -0.385$$

No solutions! For $x = -0.577$:

$$y^2 = (-0.577)^3 - (-0.577) = +0.385$$

So we get two answers: $y = \pm 0.62$. In summary,

$$(-0.577, 0.62) \quad \text{and} \quad (-0.577, -0.62)$$

are places where the tangent line is horizontal.

Vertical: set the denominator =0. Get: $2y = 0$, so $y = 0$. What is x?

$$0 = x^3 - x = x(x^2 - 1)$$

$x = 0$ or $x = \pm 1$. Three points:

$$(0, 0), \quad (1, 0) \quad (-1, 0).$$