Day 3: Derivatives (and Shortcuts)

1 Power Rule

- Discuss what $\frac{dy}{dx}$ means
- We now discuss the derivative *function*.
- Graph $f(x) = x^2$, $f'(x) = 2x$.
- Note: y-values on $f'(x)$ mean slopes on $f(x)$.
- Ex: Draw the graph of f' from the graph of f.
- ### 1.1 Some derivatives we did
- $$\frac{d}{dx}(1) = 0.$$
- $\frac{d}{dx}x = 1$.
- You saw on worksheet: $\frac{d}{dx}x^2 = 2x$.
- $x^{1/2} = \sqrt{x}$. You found $f'(x) = \frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-1/2}$.
- Power Rule: Guess the pattern:

  $$\boxed{\frac{d}{dx}(x^n) = nx^{n-1}.}$$

  Works for any number n. In words, this says that to differentiate x^n, pull the power down in front and decrease the exponent by 1.
- Every shortcut is just that: you can always use the definition, but the shortcut tells you what the answer will be.

2 Constant multiple, sum and difference

- Constant Multiple Rule: $\frac{d}{dx}\Big(cf(x)\Big) = c\frac{d}{dx}f(x)$. In words: constants can scoot past the derivative.
- Intuition: if you multiply a function by $c = 2$, you also double its rate of change.
- Important case to note: minus signs. $\frac{d}{dx}(-f(x)) = -\frac{d}{dx}f(x)$.
- Sum (and difference): $\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$. In other words, the derivative operation distributes over sums (and hence subtractions).
- Intuition: When you add functions, their rates of change (derivatives) add.

3 Exponential Functions

- You can read the discussion in chapter 3.1 on exponential functions. The important results from the discussion are:

  1. For the exponential function $f(x) = a^x$,

     $$f'(x) = f'(0)a^x.$$

     In other words, its rate of change is proportional to itself.

  2. $$f'(0) = \lim_{h\to 0} \frac{a^h - 1}{h}$$

     Then, we define the number $e(\approx 2.71828)$ as the number such that

     $$\lim_{h\to 0} \frac{e^h - 1}{h} = 1.$$

Therefore,

$$\boxed{\frac{d}{dx}e^x = e^x.}$$

3. Later we'll derive shortcuts for more complicated functions.

4 Optimization

- A big application of differentiation is to find maximums and minimums of a function on a *closed* interval (one of the form $[a, b]$.
- Example: Consider $f(x) = \frac{1}{3}x^3 - x + 1$. (Draw Graph; peak at -1, valley at 1) on the interval $[-1.5, 2.5]$.
- To "optimize" a function on a closed interval $[a, b]$:
  1. Find all x values with $f'(x) = 0$.
  2. Plug these values, along with the endpoints a and b, into the original equation $f(x)$.
  3. The largest value is the max, the smallest (most negative) is the minimum.
- For this example: $f'(x) = x^2 - 1$. Setting $f'(x) = 0$, we have the equation

  $$x^2 - 1 = 0.$$

  We have two answers: $x = 1$ and $x = -1$.
- Endpoints: -3 and 3. Plug all these in:

$$f(-1.5) = 1.375$$
$$f(-1) = 1.67$$
$$f(1) = 0.33$$
$$f(2.5) = 3.7$$

So the maximum is 3.7 at $x = 2.5$, and the minimum is 0.33 at $x = 1$.

5 Trigonometric Functions

- Graph derivative of $\sin(x)$.

$$\boxed{\frac{d}{dx}\sin(x) = \cos(x)}$$

- Graph derivative of $\cos(x)$.

$$\boxed{\frac{d}{dx}\cos(x) = -\sin(x)}$$

- Proofs can be read in the book, 3.3.