Exam 2 Practice Problems (solutions)

True/False

Practice justifying your answers to the following problems.

1. An autonomous diffy-Q must have a stable equilibrium. False; there are lots of examples of equations with no stable equilibria.

2. A system of differential equations must have a predator and a prey population. False; a system could be modeling two species that compete for resources, and neither could be hunting each other.

3. A numerical solution to a differential equation can always be computed, no matter what kind of differential equation is given. True; Euler's method works on *any* diffy-Q, not just autonomous equations.

4. An analytic solution to a differential equation can always be found, no matter what kind of differential equation is given. False; "analytic" means we can write down a formula, and there are many diffy-Q's we cannot do this for.

Free Response Problems

1. Use Euler's method with $\Delta t = 0.5$ on the diffy-Q

$$\frac{dS}{dt} = \frac{e^S}{2+t}, \qquad S(0) = 0$$

to estimate $S(1.5)$. Then solve the differential equation exactly.

t	S'	$\hat{S}$
0	0.5	0.25
0.5	0.514	0.507
1	0.553	0.784

So, $\hat{S}(1.5) = 0.784$.
Solving:

$$\begin{aligned}
e^{-S}\,dS &= \frac{1}{2+t}\,dt \\
\int e^{-S}\,dS &= \int \frac{1}{2+t}\,dt \\
-e^{-S} &= \ln|2+t| + C \quad (*) \\
e^{-S} &= -\ln|2+t| - C \\
-S &= \ln\left(\Big| -\ln|2+t| - C \Big|\right) \\
S(t) &= -\ln\left(\Big| -\ln|2+t| - C \Big|\right).
\end{aligned}$$

To find the value of C, I suggest using the equation marked $(*)$.

$$-e^0 = \ln(2) + C \implies C = -\ln(2) - 1 \approx -1.693.$$

So, the overall solution is

$$S(t) = -\ln\left(\left|1.693 - \ln|2+t|\right|\right)$$

2. A population of dingos in Australia grows according to the differential equation

$$\frac{dD}{dt} = 30e^{-0.4t},$$

where D is measured in thousands. Assuming their population is currently five thousand, will their numbers ever reach one hundred thousand?

One way to do this is to solve for $D(t)$ and set it equal to 100. This is painful though, and an easier method is to just calculate the total change in the long run:

$$\int_0^\infty 30e^{-0.4t}\,dt = 75.$$

So, their population will never grow by more than $75,000$. Hence, they will never get to $100,000$.

3. Suppose that two bacteria are growing in the same environment. Colony A has a per-capita growth rate $-\alpha + \beta B$, and colony B has a per-capita growth rate $-\gamma + \eta A$.

   (a) Write down a system of equations that will model the dynamics of these two bacteria populations.
   Remember that (growth rate) = (per capita rate) $\times$ (population). Thus, the equations are

$$\frac{dA}{dt} = -\alpha A + \beta AB$$
$$\frac{dB}{dt} = -\gamma B + \eta AB$$

   (b) Describe the interactions between the bacteria. Is one hunting the other; are they working together; or they competing for resources? Explain.
   In this model, both A and B are dying off without the presence of the other (this is because of the $-\alpha A$ and the $-\gamma B$ terms). However, both of their interaction terms are positive ($+\beta AB, +\eta AB$), which means that the populations are helping each other survive. Thus, these populations are working together.