Guide for Section 8 - Inverse Functions

Section Goals:

- Relate functional inverses to inverses of real numbers
- Interpret the value of an inverse in an applied context
- Determine if a function is invertible
- From a formula, graph, table, or written description find a function's inverse

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Ex 1 a) Let it be midnight on January 1, 2020. Given a particular location, is it guaranteed that we can determine a unique temperature there?

b) Let it be midnight on January 1, 2020. Given a particular temperature, is it guaranteed that we can determine a unique location?

Def A function $f(t)$ is one-to-one if, for each value Q in the image of f, there is exactly one t in the domain of f so that $Q = f(t)$.

In this case, we say that an inverse for f exists on its domain (or that f is invertible).

Ex 2 Consider the function $Q = f(t) = t^3 - 2$. Is it possible to find a unique function $t = g(Q)$? (Hint: try solving for t). Is f one-to-one or not?

Def The inverse of a function $y = f(t)$, if it exists, is the function we'll call $f^{[-1]}$ whose rule is that $t = f^{[-1]}(y)$.

This definition has the effect of implying that as long as t is in the domain of f, we get

$$(f^{[-1]} \circ f)(t) = t$$

and as long as Q is in the domain of $f^{[-1]}$, we get

$$(f \circ f^{[-1]})(Q) = Q.$$

Using the same function diagram for composition as in Section 2.1, we see that the machine first takes an input t, passes it through the function and its inverse, and the net result is back where it began: t.

$$\begin{tikzcd}
t \arrow[r, "\boxed{f}"] \arrow[rr, bend right, "\boxed{f^{[-1]} \circ f}"'] & Q = f(t) \arrow[r, "\boxed{f^{[-1]}}"] & f^{[-1]}(f(t)) = t
\end{tikzcd}$$

Ex 3 Let $C = f(n)$ be the thread count on a set of bed sheets with n lengthwise and n widthwise threads in a square foot of fabric. Thread count is defined to be the sum of the warp (lengthwise) and weft (widthwise) threads in one square inch of fabric.

a) Why should we believe that f is a one-to-one function?

b) Interpret the equation $f^{[-1]}(300) = 1800$ in the context of thread count.

c) Without computing, what should be the value of $(f \circ f^{[-1]})(180)$? Why?

$\boxed{\text{Thm}}$ (Finding an Inverse) The general process for finding the inverse of a function $Q = f(t)$ is to exchange the roles of t and Q: that is, make t the "output" and Q the "input".

Ex 4 Let $P = f(T) = 4T + 7$. Find the inverse function of f.

$\boxed{\text{Thm}}$ (Exponential and Logarithmic Functions are Inverses) For $f(t) = b^t$ we have

$$f^{[-1]}(Q) = \log_b(Q).$$

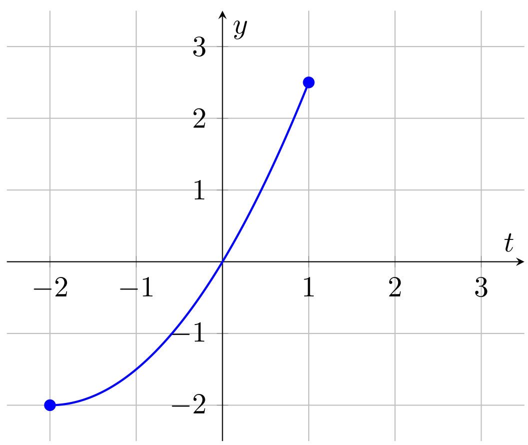
Ex 5 Find the formula for the inverse function of $f(t) = 2 - e^{-t}$.

Thm (Domain and Image of an Inverse Function) For a one-to-one function f with domain D and image C, the inverse function $f^{[-1]}$ has domain C and image D.

Ex 6 Find the domain and image of $f^{[-1]}$, where $f(t) = 2 - e^{-t}$ (from Example 5 above).

Thm (Graph of an Inverse Function) Given a one-to-one function $Q = f(t)$, the graph of $f^{[-1]}(t)$ is the graph of $Q = f(t)$ reflected about the line $Q = t$.

Ex 7 On the same coordinate plane, sketch the graph of $f^{[-1]}$ along with the graph of f provided:



Ex 8 Graph $\log_2(x)$ and $\ln(x)$.