4.1: Differential Equation Basics

Contents

1 Definitions

1.1 State variable

- A differential equation (DE) always has two variables: the *state* variable and the *independent* variable. Usually the independent variable is time.

- For example, in the equation

$$\frac{dP}{dt} = 3P,$$

The variable P is the state variable.

- The state variable is the quantity whose evolution through time we want to understand and predict.
- The study of DE's is about predicting the future.

1.2 Linear and non-linear

- A DE is called *linear* if all terms involving the state variable and its derivatives appear without any extra decorations. This means we don't have anything like y^2, $\left(\frac{dy}{dt}\right)^2$, $\cos(y)$, etc.
- For example:

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} - 10y = 0$$

  is linear, because all of the terms with y and its derivatives have no fancy operations on them.
- This is also linear:

$$\cos(t)\frac{dy}{dt} + t^4 y = e^t$$

  Again, t is allowed to have fancy operations, but the derivatives are not allowed to have fancy operations.
- This example is non-linear:

$$\left(\frac{dy}{dt}\right)^2 = y.$$

  The reason for this is that the derivative term is *squared*.

1.3 Order

- The *order* of a DE is the highest derivative you see in the equation.
- Ex:

$$\frac{dP}{dt} = 3P$$

  is a *first*-order DE.

- Ex: the spring equation,

$$m\frac{d^2y}{dt^2} + ky = 0$$

is a second-order DE, since it involves a second derivative.

- If the equation has more than one derivative appearing, the order is the highest derivative. E.g.

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} - 3y = 0$$

is a second-order equation (and not first-order).

1.4 Pure time, Autonomous

- Most of this class will focus on first-order differential equations. For higher order, there is a separate class, Math 320.
- We can sort-of break down first-order DE's as pure-time, autonomous, or *other*.
- A first-order DE is called a *pure-time* DE if it can be made to look like this:

$$\frac{dy}{dt} = g(t)$$

for some function g depending only on time, t.

- For example:

$$\frac{dy}{dt} = t^2 \qquad \frac{dy}{dt} = 10$$
$$\frac{dx}{du} = \cos(u) \qquad \frac{dg}{ds} = -e^{2s}$$

- A DE is called *autonomous* if it can be made to look like

$$\frac{dy}{dt} = g(y),$$

where the right-hand side is purely a function of the state variable, y.

- Examples:

$$\frac{dy}{dt} = 3y \qquad \frac{dP}{dt} = k(P_0 - P)$$
$$\frac{ds}{du} = s^4 \qquad \frac{dR}{dx} = R(1-R)$$

- Many DE's are *neither* pure-time nor autonomous. For example,

$$\frac{dy}{dt} = t^2 y$$

is neither pure-time nor autonomous.

2 Initial Conditions

- We saw that the differential equation $\frac{dx}{dt} = 10$ had more than one solution: $x(t) = 10t$, and $x(t) = 10t + 1$ were two instances, and the general solution is of the form

$$x(t) = 10t + C$$

for some unknown constant C.

- To match what happens in reality, we also specify an *initial condition* to the differential equation. Usually this looks something like

$$x(0) = 1.$$

- This means "Find the function $x(t)$ whose derivative is 10 and starts at an initial value of 1."

- Example. What are the solutions to the equation

$$\frac{dy}{dt} = t^3 + \sin(t)$$

That satisfy the initial condition $y(0) = 3$?

- A: $y = \frac{1}{4}t^4 - \cos(t) + 4$.

- Conceptually, intial conditions are very similar to intial conditions to discrete dynamical systems, and the DE is analogous to the updating function.

3 More on solutions

Remember, a solution to a DE is a function that satisfies the differential equation upon being inserted to the equation.

3.1 Educated Guessing

- One way to find solutions: use an educated guess.
- Ex: Find a solution to the DE $y' = y$.
- A: In words, this says y is its own derivative. You know one: $y = e^x$!
- You try: Guess a solution to the DE $y' = -6y$. A: $y = e^{-6x}$ is one.
- Note: this method might not get you all the solutions!
- Ex: again with $y' = -6y$, I guess $y = 2e^{-6x}$. Check it works!

3.2 Procedures

- We will learn a few procedures to find (all!) solutions to a few classes of DE's, mainly what are known as *separable* DE's, of which pure-time and autonomous are examples of.
- To come in 4.2: how to solve pure-time DE's and the other side of calculus.
- 5.5: General Separable DE's.

3.3 Numerical Methods

- Euler's method is a way to find a numerical solution to *any* first-order differential equation. More on this after the first midterm.