Final Review Solutions

This review sheet is meant to give you practice with some of the later material in the class; also, this sheet may not contain everything we did in the course, but it is a starting point.

1. Summarize the three types of hearts we saw in class via their cobweb diagrams. What distinguishes their behavior?

   The three are healthy, 2:1 AV block, and Wenkebach phenomenon. Read notes or book for summaries and cobweb diagrams.

2. With the usual parameter values of $V_c = u = 1\,\mathrm{mV}$ and $T = 1\,\mathrm{s}$ and with a decay rate of $\alpha = \ln(2)$, make a cobweb diagram and classify the health of the heart. Does it show any signs of second-degree block?

   This will be a (barely) healthy heart. The diagonal does touch the "top branch" of the updating function, meaning there is an equilibrium on the healthy side of the updating function. Thus, there are no signs of second-degree block.

3. With the same parameter values but $\alpha = \ln(1.2)$, make a cobweb diagram. Analyze this heart's health. With these parameter values, if $V_t = 4\,\mathrm{mV}$, will the heart beat on the next SA signal?

   This heart will display signs of 2:1 AV block based on its cobweb diagram. To answer the second question, check if $e^{-\alpha\tau}V_t \le V_c$: we have

   $$e^{-\ln(1.2)(1)}(5\,\mathrm{mV}) = \frac{5}{6}(4\,\mathrm{mV}) \approx 3.3\,\mathrm{mV} > 1\,\mathrm{mV} = V_c.$$

   Therefore, the heart would skip the next beat.

4. Suppose a bacteria strain grows by 40% per hour, and then a mutation occurs, producing a new strain that triples every hour. (a) Write down the model for the fraction of mutated bacteria. (b) If originally 20% of the population mutated, then what is their fraction after two hours? (c) What happens to both populations in the long run? Explain using the known equilibria and stability.

   (a) The model is

   $$p_{t+1} = \frac{3p_t}{3p_t + 1.4(1-p_t)}.$$

   (b) This condition says $p_0 = 0.20$. Using the above equation, you may calculate

   $$p_1 = 0.3488, \quad \text{and} \quad p_2 = 0.5344.$$

So after two hours, about 53.4% of the population consists of the mutated strain.
(c) From class we know the equilibria are always at $p^* = 0$ and $p^* = 1$. Since the mutated fraction grows more quickly, the calculation done in class shows that $p^* = 1$ is the stable equilibrium, so in the long run the fraction p_t tends to 1. This means that the population of the mutated strain dominates the population in the long run.

5. Find the inflection points of the following functions. Be sure to justify that any points you find are actually inflection points.

   (a) $f(x) = e^{-x^2}$ (b) $g(x) = x\ln(x)$

   (a) $f'(x) = -2xe^{-x^2}$ and $f''(x) = -2e^{-x^2} + 4x^2e^{-x^2}$ (by the product rule!!!). Setting $f''(x) = 0$ we get the equation

$$e^{-x^2}(4x^2 - 2) = 0$$

   which gives $x = \pm\frac{1}{\sqrt{2}}$.
   (b) $g'(x) = x \cdot \frac{1}{x} + \ln(x)$, or

$$g'(x) = \ln(x) + 1.$$

   Then, $g''(x) = \frac{1}{x} = 0$ has no solutions, so this function does not have any inflection points.

6. Suppose a person breathes in 15% of their lung capacity, and that the ambient air contains a concentration of 1.7 mmol/L of nitrogen. If their lungs contain a concentration of 1.3 mmol/L, then what is the concentration in their lungs after three breaths?

   $q = 0.15$, $\gamma = 1.7$, and $c_0 = 1.3$. The discrete dynamical system is

$$c_{t+1} = \gamma q + (1 - q)c_t = 0.255 + 0.85c_t.$$

   Iterating three times gives $c_3 = 1.4544$ mmol/L.

7. Find and classify the critical points of these functions as either local minima, maxima, or neither.

   (a) $f(x) = e^x + 3e^{-2x}$
   (b) $g(t) = t^2\ln(13t)$
   (c) $h(t) = t - \sqrt{t}$.

   (a) $f'(x) = e^x - 6e^{-2x} = 0$. Then,

$$\begin{aligned}
e^x - 6e^{-2x} &= 0 \\
e^x &= 6e^{-2x} \\
e^{3x} &= 6 && \text{(divide by } e^{-2x} \text{ on both sides)} \\
3x &= \ln(6) \\
x &= \frac{1}{3}\ln(6) \approx 0.597.
\end{aligned}$$

Now, $f''(x) = e^x + 12e^{-2x}$, so $f''(0.597) = 5.45 > 0$, which tells us that $x = 0.597$ is a local minimum.
(b) $g'(t) = 2t\ln(13t) + t^2\frac{1}{13t} \cdot 13$ (using chain rule!). This simplifies to

$$\begin{aligned} 2t\ln(13t) + t &= 0 \\ t\,(2\ln(13t) + 1) &= 0. \end{aligned}$$

This gives $t = 0$ and $2\ln(13t) + 1 = 0$. This last equation gives $t = \frac{1}{13}e^{-1/2} \approx 0.047$. Then,

$$g''(t) = 2\ln(13t) + 2t\frac{1}{13t}13 + 1 = 2\ln(13t) + 3.$$

Then,

$$g''(0.047) = 2 \quad \text{and} \quad g''(0) = \text{DNE},$$

so 0 is not really classifiable, and 0.047 is a local minimum. (Don't worry, this weird situation won't happen on the final.)
(c) $h'(t) = 1 - \frac{1}{2\sqrt{t}}$. We solve:

$$\begin{aligned} 1 - \frac{1}{2\sqrt{t}} &= 0 \\ 1 &= \frac{1}{2\sqrt{t}} \\ 2\sqrt{t} &= 1 \\ \sqrt{t} &= \frac{1}{2} \\ t &= \frac{1}{4}. \end{aligned}$$

Then, $h''(t) = \frac{1}{4}t^{-3/2}$, so

$$h''\left(\frac{1}{4}\right) = \frac{1}{4}(1/4)^{-3/2} > 0,$$

so $t = \frac{1}{4}$ is a local minimum agian.

8. Using only the limit definition of the derivative, calculate the following derivatives. Do not use l'Hopital's rule.

   (a) $f(x) = x^2 - x$
   (b) $g(x) = \frac{1}{x}$
   (c) $r(t) = \frac{1}{t^2}$
   (d) $s(t) = \sqrt{t}$

9. Evaluate the following limits.

   (a) $\lim_{x \to \infty} \frac{e^x + x}{x^2}$

(b) $\displaystyle\lim_{x\to 2} \frac{x^3 - 7x^2 + 10x}{x^2 + x - 6}$

(c) $\displaystyle\lim_{w\to -4} \frac{\sin(\pi w)}{w^2 - 16}$

(d) $\displaystyle\lim_{t\to \infty} \frac{\ln(3t)}{t^2}$

(e) $\displaystyle\lim_{t\to 0} \frac{\sin(2z) + 7z^2 - 2z}{z^2(z+1)^2}$

(a) The limit is ∞.

(b) The limit is $-\dfrac{6}{5}$.

(c) The limit is $-\dfrac{\pi}{8}$.

(d) The limit is 0.

(e) The limit is 7.