Guide for Section 8 - Inverse Functions

Section Goals:

- Relate functional inverses to inverses of real numbers
- Interpret the value of an inverse in an applied context
- Determine if a function is invertible
- From a formula, graph, table, or written description find a function's inverse

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Ex 1 a) Let it be midnight on January 1, 2020. Given a particular location, is it guaranteed that we can determine a unique temperature there?

Yes; this is the statement that Temperature is a function of your location.

b) Let it be midnight on January 1, 2020. Given a particular temperature, is it guaranteed that we can determine a unique location?

No; for example, it could be $50°$ in Boston and New York at the same time. This is saying that location is not a function of position.

Def A function $f(t)$ is one-to-one if, for each value Q in the image of f, there is exactly one t in the domain of f so that $Q = f(t)$.

In this case, we say that an inverse for f exists on its domain (or that f is invertible).

Ex 2 Consider the function $Q = f(t) = t^3 - 2$. Is it possible to find a unique function $t = g(Q)$? (Hint: try solving for t). Is f one-to-one or not?

$$\begin{aligned} Q &= t^3 - 2 \\ Q + 2 &= t^3 \\ t &= (Q+2)^{1/3}. \end{aligned}$$

We seem to have solved for t as a function of Q. Indeed, with odd-powers, the cube-root function $(\_)^{1/3}$ has only one answer (no plus or minus, like with $(\_)^{1/2}$). Because of this, given any output Q, we can solve for a unique input t, and hence f is one-to-one.

$\boxed{\text{Def}}$ The <u>inverse of a function</u> $y = f(t)$, if it exists, is the function we'll call $f^{[-1]}$ whose rule is that $t = f^{[-1]}(y)$.

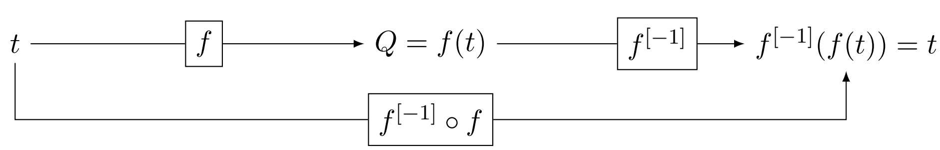
This definition has the effect of implying that as long as t is in the domain of f, we get

$$(f^{[-1]} \circ f)(t) = t$$

and as long as Q is in the domain of $f^{[-1]}$, we get

$$(f \circ f^{[-1]})(Q) = Q.$$

Using the same function diagram for composition as in Section 2.1, we see that the machine first takes an input t, passes it through the function and its inverse, and the net result is back where it began: t.



Ex 3 Let $C = f(n)$ be the thread count on a set of bed sheets with n lengthwise and n widthwise threads in a square foot of fabric. Thread count is defined to be the sum of the warp (lengthwise) and weft (widthwise) threads in one square inch of fabric.

a) Why should we believe that f is a one-to-one function?
Given any thread count C, there should only be one number of threads n giving that thread count.

b) Interpret the equation $f^{[-1]}(300) = 1800$ in the context of thread count.
The input to $f^{[-1])}$ is 300. Remember: outputs and inputs switch roles for the inverse function. So, 300 is an *output* of f. Thus, 300 is a thread count, C. The number 1800, being an output of $f^{[-1]}$, is an *input* of f. So, 1800 is the number of threads length-and-width-wise in one square foot of fabric.

c) Without computing, what should be the value of $(f \circ f^{[-1]})(180)$? Why?
It should be just 180, because of the definition of the inverse function.

Thm (Finding an Inverse) The general process for finding the inverse of a function $Q = f(t)$ is to exchange the roles of t and Q: that is, make t the "output" and Q the "input".

Ex 4 Let $P = f(T) = 4T + 7$. Find the inverse function of f.

Solve for T in the equation $P = 4T + 7$:

$$P - 7 = 4T$$
$$T = \frac{1}{4}(P - 7) = \frac{1}{4}P - \frac{7}{4}.$$

So, we have $T = f^{[-1]}(P) = \frac{1}{4}P - \frac{7}{4}$.

Thm (Exponential and Logarithmic Functions are Inverses) For $f(t) = b^t$ we have

$$f^{[-1]}(Q) = \log_b(Q).$$

Ex 5 Find the formula for the inverse function of $f(t) = 2 - e^{-t}$.

Let $y = 2 - e^{-t}$. Solve for t; the result (in terms of y) will be $f^{[-1]}$.

$$\begin{aligned} y &= 2 - e^{-t} \\ y - 2 &= -e^{-t} \\ 2 - y &= e^{-t} \\ \ln(2 - y) &= -t \\ t &= -\ln(2 - y) = \ln\left(\frac{1}{2 - y}\right) \end{aligned}$$

so $f^{[-1]}(y) = \ln\left(\frac{1}{2 - y}\right)$.

Thm (Domain and Image of an Inverse Function) For a one-to-one function f with domain D and image C, the inverse function $f^{[-1]}$ has domain C and image D.

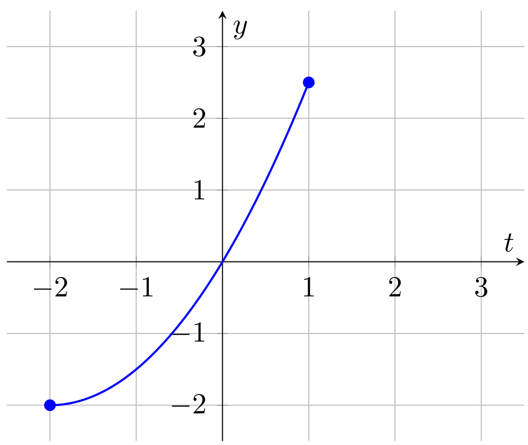
Ex 6 Find the domain and image of $f^{[-1]}$, where $f(t) = 2 - e^{-t}$ (from Example 5 above).

The domain of $f(t)$ is all real numbers: $(-\infty, \infty)$. The image is trickier: by looking at a graph, it is all real numbers less than 2: $(-\infty, 2)$. Thus,

$$\text{Dom}\left(\ln\left(\frac{1}{2-y}\right)\right) = (-\infty, 2), \qquad \text{Range}\left(\ln\left(\frac{1}{2-y}\right)\right) = (-\infty, \infty).$$

Thm (Graph of an Inverse Function) Given a one-to-one function $Q = f(t)$, the graph of $f^{[-1]}(t)$ is the graph of $Q = f(t)$ reflected about the line $Q = t$.

Ex 7 On the same coordinate plane, sketch the graph of $f^{[-1]}$ along with the graph of f provided:



Ex 8 Graph $\log_2(x)$ and $\ln(x)$.