4.3: Integration by substitution

Contents

1 Substitutions

- Sometimes a variable substitution can clarify an integral.
- Example: consider the integral

$$\int 2x\cos(x^2)\,dx.$$

  You might notice that this came from $\sin(x^2) + C$.
- If you didn't know that, you might also notice that some pieces are related: x^2 and its derivative, $2x$, both appear in the integral.
- This observation leads you to guess a substitution of $u = x^2$ in the integral.
- The method is rather strange looking:

$$\frac{du}{dx} = 2x \implies du = 2xdx.$$

Then

$$\begin{aligned}
\int 2x\cos(x^2)\,dx &= \int \cos(x^2)2x\,dx \\
&= \int \cos(u)\,du \\
&= \sin(u) + C \\
&= \sin(x^2) + C.
\end{aligned}$$

- Why does it work?
- The answer is that it has to do with the associated DE.
- Here's what it looks like when you *change variables on the DE*:

$$\frac{dy}{dx} = 2x\cos(x^2)$$

We let $u = x^2$ be the change of variable.

- The chain rule says

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{dy}{du}\cdot(2x)$$

where we got $\frac{du}{dx} = 2x$ from $u = x^2$. Then

$$\begin{aligned}
\frac{dy}{dx} &= 2x\cos(x^2) && \leftrightarrow && \int 2x\cos(x^2)\,dx \\
\frac{dy}{du}\cdot(2x) &= (2x)\cos(u) && && \\
\frac{dy}{du} &= \cos(u) && \leftrightarrow && \int \cos(u)\,du
\end{aligned}$$

- One advantage of integral notation is it gives a quick way of performing variable substitutions.
- You don't need to always go back to the DE side; I recommend just sticking with the integral notation.

2 Examples with guidance

- Find antiderivatives of these functions with these substitutions.

$$
\begin{aligned}
f(x) &= xe^{x^2}, & u &= x^2 \\
g(t) &= \frac{\ln(x)}{x} & u &= \ln(x) \\
f(x) &= \frac{x-1}{x^2-x+17} & u &= x^2-x+17
\end{aligned}
$$

3 Examples without guidance

- Tips: Substitution works well when you can easily spot two pieces related by a derivative.

- compute these integrals:

$$
\begin{aligned}
\int \frac{x^2}{x^3+4}\,dx &= \frac{1}{3}\ln|x^3+4| + C \\
\int x\sqrt{5+2x^2}\,dx &= \frac{1}{6}\left(5+2x^2\right)^{3/2} + C \\
\int (2x-5)^{10}\,dx &= \frac{1}{22}(2x-5)^{11} + C \\
\int \frac{x}{\sqrt{16-x^2}}\,dx &= -\sqrt{16-x^2} + C
\end{aligned}
$$