Day 4: Product and Quotient Rule

We're going to add in tools to differentiate more complex functions.

Product Rule

- Draw the "square" with sides $f(x)$ and $g(x)$. Extend the sides a little bit.
- Motivates the rule:

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x).$$

- Examples:
  - $h(x) = xe^x$:

$$h'(x) = (x)'e^x + x(e^x)' = e^x + xe^x.$$

  - $h(x) = (x^2 + 2)(4x + 1)$

$$h'(x) = (x^2+2)'(4x+1)+(x^2+2)(4x+1)' = (2x)(4x+1)+(x^2+2)(4) = 12x^2+2x+8$$

1 Quotient Rule

- Consider $\frac{f(x)}{g(x)}$.
- Quotient rule:

$$\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$$

- Can be derived by brute force, similar to product rule. Or, can be derived from product rule and chain rule (tomorrow).

- Examples:
  - $h(x) = \frac{e^x}{x}$

$$h'(x) = \frac{(e^x)'x - (x)'e^x}{(x)^2} = \frac{xe^x - e^x}{x^2}.$$

  - $h(x) = \frac{xe^x}{1+x}$

$$\begin{aligned} h'(x) &= \frac{(xe^x)'(1+x) - (xe^x)(1+x)'}{(1+x)^2} \\ &= \frac{\left((x)'e^x + x(e^x)'\right)(1+x) - xe^x}{(1+x)^2} \\ &= \frac{\left(e^x + xe^x\right)(1+x) - xe^x}{(1+x)^2} \\ &= \frac{e^x + xe^x + xe^x + x^2e^x - xe^x}{(1+x)^2} \\ &= e^x \frac{1+x+x^2}{(1+x)^2} \end{aligned}$$

- A note on simplifying: it's not strictly necessary; you only ever *need* an expression to plug x-values into. But if you need to do algebra (e.g. solving $f'(x) = 0$) or take a second derivative, simplifying makes your life easier in the long run.

- Ex: Find all extrema of $h(x) = \frac{x^2 - x}{1+x}$ on the interval $[0, 2]$.
  - Remember: to optimize, find *critical points*, x's where $h'(x) = 0$. Then, plug in these points, with endpoints, back into original

function.

$$\begin{aligned}
h'(x) &= \frac{(x^2-x)'(1+x)-(x^2-x)(1+x)'}{(1+x)^2} \\
&= \frac{(2x-1)(1+x)-(x^2-x)}{(1+x)^2} \\
0 &= 2x^2+2x-x-1-x^2+x \\
0 &= x^2+2x-1 \\
x &= \frac{-2\pm\sqrt{4-4(1)(-1)}}{2} \\
x &= \frac{-2\pm\sqrt{8}}{2} = -1\pm\sqrt{2}.
\end{aligned}$$

– Plug in $0, 2$, and $-1+\sqrt{2}$. (The other one, $-1-\sqrt{2}$, is not inside the interval $[0, 2]$.)

$$\begin{aligned}
h(0) &= 0 \\
h(2) &= \frac{4-2}{1+2} = \frac{2}{3} \\
h(-1+\sqrt{2}) &= h(0.414) \approx -0.172
\end{aligned}$$

So the minimum is -0.172 at $x = 0.414$ and the maximum is $2/3$ at $x = 2$.