4.2: Pure-Time DE's, AKA integration

Contents

1 Pure-Time Equations

- Recall, a pure-time DE is a first-order DE where the state variable does not appear in the DE. These always look like

$$\frac{dy}{dt} = f(t).$$

- Terminology and notation. The solution to such a DE is called an *antiderivative* of $f(t)$, because it's derivatives but in reverse.

- We write

$$y = F(t) = \int f(t)\, dt$$

for a solution of this particular DE, and we call it an *indefinite integral*.

- A note on notation. The curvy symbol is a stylized "S." Historically, integrals were introduced to compute certain kinds of sums, which we will examine later. Think of it as "S" for "sum."

- The "dt" portion of the notation is mostly for us to remember what the independent variable should be. This is helpful in case we have parameters in the equation where it might not be clear which letter was the independent variable.

- We now have two phrases that mean the same thing:

  1. A solution to the pure-time DE $\frac{dy}{dt} = f(t)$.
  2. The antiderivative of the function $f(t)$.

2 Computing Antiderivatives: First steps

- $\int x\, dx = \frac{1}{2}x^2 + C$

- We have the "$+C$" just like when we solved these kinds of DE's: we know that we can add any constant to $y = \frac{1}{2}x^2$ and it will still be a solution to the DE $y' = x$.

- Ex: (a) What is the DE corresponding to the following indefinite integral?
$$\int (\cos(2t) + 1)\ dt$$
(b) Evaluate the integral.

- (a) this is mostly an exercise in translation, not in solving; the associated DE is
$$\frac{dy}{dt} = \cos(2t) + 1.$$
Both the intergal and the DE say "find me a function with $\cos(2t) + 1$ as the derivative."

- (b): Educated guessing: $y = F(t) = \frac{1}{2}\sin(2t) + t + C$.

- Ex: you try: (a) write down the associated pure-time DE for this integral, and (b) evaluate the integral.
$$\int \left(\frac{1}{t} + \frac{1}{t^2}\right)\ dt = \ln(t) - \frac{1}{t} + C$$

$$\int e^{-t/3}\,dt = -3e^{-t/3} + C$$

$$\int \frac{2}{\sqrt[3]{t^5}}\,dt = -3t^{-2/3} + C$$

$$\int 10\,dt$$

3 Summary of properties of integrals

- We know how to do integrals of the usual suspects:

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$$

$$\int \cos(x)\,dx = \sin(x) + C$$

$$\int \sin(x)\,dx = -\cos(x) + C$$

$$\int e^x\,dx = e^x + C$$

- Integrals work very similarly to derivatives, and the following properties are the analogues of the derivative properties.

- Constant multiple rule: since constants can scoot outside the derivative, so too can they scoot out of integrals:

$$\int cf(t)\,dt = c\int f(t)\,dt$$

- Sum rule: derivative of a sum is the sum of derivatives. Same for antiderivatives:

$$\int \left(f(t) + g(t)\right)\,dt = \int f(t)\,dt + \int g(t)\,dt$$

- For more complicated functions, things are not as straight-forward.

- Moral: If you write down an antiderivative, then you can *always* check it by taking a derivative. Just like you can always take a function and check it solves a DE!

4 Application to physics

- We can use integrals to determine how objects fall!
- integrating acceleration produces a velocity function, and integrating velocity gives a positiion function.
- Ex. A person throws a ball up in the air from the top of a building that is 10 meters tall, and they measure the initial velocity to be 0.5 meters per second. (a) find the height of the ball as a function of time. (b) What is the height after one second?
- Solution. (a) The acceleration is $a = -9.8\,\mathrm{m/s^2}$, which is constant in time. So,
$$v(t) = \int a\,dt = \int -9.8\,dt = -9.8t + C.$$
We know $v(0) = 0.5$, giving $C = 0.5$. Then,
$$h(t) = \int v(t)\,dt = \int (-9.8t + 0.5)\,dt = -\frac{9.8}{2}t^2 + 0.5t + C'.$$
But we know $h(0) = 10$, giving $C' = 10$. So, we've built the height function
$$h(t) = -4.9t^2 + 0.5t + 10.$$
(b) $h(1) = -4.9 + 0.5 + 10 = 5.6m$, so it is still 5.6 meters off the ground.