Exam 2

Math 251, Summer 2017

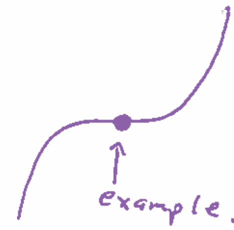
Name: Key

- You have the full class time to work on the exam, but it is designed to be 50 minutes.
- There are $\boxed{33}$ points on this exam.
- **Show all of your work and justification** for each answer.
- In each problem, **draw a box around your final answer.**

1. Decide if each of the following statements is true or false and briefly explain why.

(a) [2 pts] Critical points of a function $f(x)$ must be local maximums or minimums.

False; they can be saddle points:



(b) [2 pts] The function $h(x) = e^x$ has an inflection point.

$h''(x) = e^x = 0$ has no solutions, so false.

2. [2 pts] Find $\lim_{x\to\infty} \frac{x^4 - x^2}{x^3 - 1}$.

$$= \lim_{x\to\infty} \frac{x^4\left(1-\frac{1}{x^2}\right)}{x^3\left(1-\frac{1}{x^3}\right)} = \lim_{x\to\infty} \frac{x^4}{x^3} = \lim_{x\to\infty} \frac{x}{1} = \infty$$

$\frac{1}{x^2} \to 0$, $\frac{1}{x^3} \to 0$

3. [2 pts] Describe the tangent line to the curve defined by $xy + xy^2 = 1$ at the point $(-4, -0.5)$.

$$x'y + xy' + x'y^2 + x(y^2)' = 0$$

$$y + xy' + y^2 + x \cdot 2y \cdot y' = 0$$

$$xy' + 2xy \cdot y' = -y^2 - y$$

$$y'(x + 2xy) = -y^2 - y$$

$$y' = \frac{-y^2 - y}{x + 2xy}$$

$$y' = \frac{-(-.5)^2 - (-.5)}{-4 + 2(-4)(-.5)}$$

$$y' = \frac{-0.25 + 0.5}{-4 + 4}$$

$$\boxed{y' = \frac{0.25}{0}}$$

The tangent line is vertical!

4. [3 pts] Find the linearization to $f(x) = \arcsin(2x)$ at $x = 0$.

$$f'(x) = \frac{1}{\sqrt{1-(2x)^2}} \cdot 2$$

$$f'(0) = \frac{1}{\sqrt{1-0^2}} \cdot 2 = \frac{2}{\sqrt{1}} = 2.$$

$$f(0) = \arcsin(0) = 0$$

$$y = 2x + b$$

$$0 = 2(0) + b$$

$$0 = b$$

Linearization: $y = 2x$

5. [3 pts] Suppose that f is an invertible function. Suppose also that $f(2) = 7$ and that $f'(2) = 3$. Find the value of $(f^{-1})'(7)$.

$$(f^{-1})'(7) = \frac{1}{f'(f^{-1}(7))} = \frac{1}{f'(2)}$$

$$= \boxed{\frac{1}{3}}$$

6. [3 pts] Let $g(x) = 2x^2$. Use the *definition of the derivative* to find the formula for $g'(x)$.

$$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h\to 0} \frac{2(x+h)^2 - 2x^2}{h}$$

$$= \lim_{h\to 0} \frac{2(x^2+2xh+h^2) - 2x^2}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{2x^2} + 4xh + 2h^2 - \cancel{2x^2}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}[4x + 2h]}{\cancel{h}}$$

$$= 4x + 0 \quad (h=0)$$

$$\boxed{g'(x) = 4x}$$

7. Let $f(x) = \ln(x^2 + 1)$.

(a) [3 pts] There is one critical point of $f(x)$. Find it, and decide if it is a local maximum, local minimum, or neither. (Your justification cannot just be based on the graph from your calculator.)

$$f'(x) = \frac{1}{x^2+1} \cdot 2x = 0$$

$$2x = 0$$

$$x = 0$$

0

$$f'(-1) = \frac{-2}{(-1)^2+1} < 0$$

$$f'(1) = \frac{2}{1^2+1} > 0$$

It's a local min by 1st derivative test.

(b) [3 pts] Find all inflection points of $f(x)$.

$$f'(x) = \frac{2x}{x^2+1}$$

$$f''(x) = \frac{2(x^2+1) - 2x(2x)}{(x^2+1)^2}$$

$$= \frac{2x^2 + 2 - 4x^2}{\cancel{(\;)^2}} = 0$$

$$-2x^2 + 2 = 0$$

$$2x^2 = 2$$

$$x^2 = 1$$

$$x = \pm 1$$

f'' ∩ ∪ ∩

-1 1

$$f''(0) = \frac{2}{pos} > 0$$

$$f''(-2) = \frac{(-2)(4)+2}{pos} < 0$$

$$f''(+2) < 0$$

Check that concavity changes (inflection points need that!)

So both really are inflection points.

8. [3 pts] Calculate $f'(x)$ for the function $f(x) = x^{3\sqrt{x}}$.

$$y = x^{3\sqrt{x}}$$

$$\ln(y) = 3\sqrt{x} \ln(x)$$

$$\frac{1}{y} \cdot y' = 3\frac{1}{2}x^{-1/2}\ln(x) + 3\sqrt{x} \cdot \frac{1}{x}$$

$$y' = y \cdot \left[\frac{3}{2}x^{-1/2}\ln(x) + 3\sqrt{x} \cdot \frac{1}{x}\right]$$

$$y' = x^{3\sqrt{x}}\left[\frac{3}{2}x^{-1/2}\ln(x) + 3\sqrt{x} \cdot \frac{1}{x}\right]$$

9. [3 pts] Optimize the function $g(x)$ on the interval $[-2, 4]$.

$$g(x) = \frac{x-4}{x^2+2}$$

$$g'(x) = \frac{1(x^2+2) - (x-4)(2x)}{\cancel{(x^2+2)^2}} = 0$$

$$= x^2 + 2 - 2x^2 + 8x = 0$$

$$= -x^2 + 8x + 2 = 0$$

$$x^2 - 8x - 2 = 0$$

$$x = \frac{8 \pm \sqrt{64+8}}{2}$$

$$x = \frac{8 \pm \sqrt{72}}{2}$$

$x = -0.24$

$x = 8.24$ outside interval!

$g(-2) = -1$

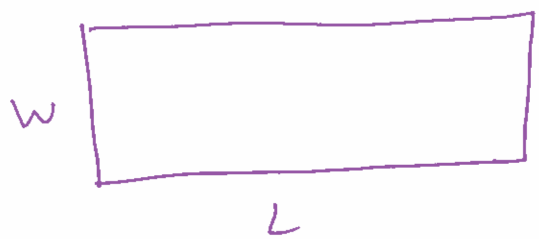
$g(4) = 0$

$g(-0.24) = -2.06$

Min @ $x = -0.24$

Max @ $x = 4$

10. [4 pts] The length of a rectangle is increasing at a rate of 4 cm per second, and its width is increasing at a rate of 5 cm per second. There is a moment at which the length is 2 cm and the width is 7 cm. At that same moment, how fast is the area increasing? Include units.



$\frac{dw}{dt} = 5$

$\frac{dL}{dt} = 4$

Find: $\frac{dA}{dt}$ when $L = 2$, $W = 7$.

$$A = W \cdot L$$

$$\frac{dA}{dt} = \frac{dw}{dt} \cdot L + W \cdot \frac{dL}{dt} \quad \text{(Product rule)}$$

$$= 5 \cdot 2 + 7 \cdot 4$$

$$= 10 + 28 = \boxed{38 \text{ cm}^2/\text{sec.}}$$