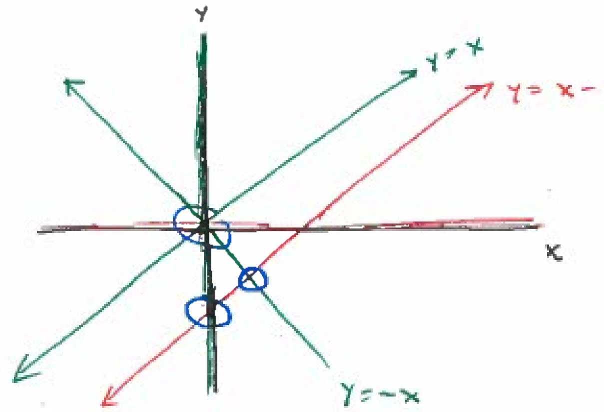
Review Handout

1. For the systems shown below, find and graph the nullclines. Find all equilibria.

(a) $\begin{cases} \frac{dx}{dt} = y^2 - xy + y \\ \frac{dy}{dt} = x^3 - xy^2 \end{cases}$



X-nullclines:

$y(y-x+1)=0$

$y=0$ or $y=x-1$

Y-nullclines

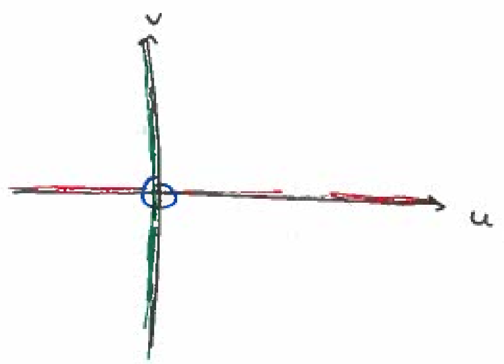
$x(x^2-y^2)=0$

$x=0$ or $y^2=x^2$

$y=+x$ or $y=-x$

Three equilibria: $(0,0)$, $(\frac{1}{2}, \frac{-1}{2})$, $(0,-1)$.

(b) $\begin{cases} \frac{du}{dt} = u \\ \frac{dv}{dt} = v \end{cases}$



u-nullcline:

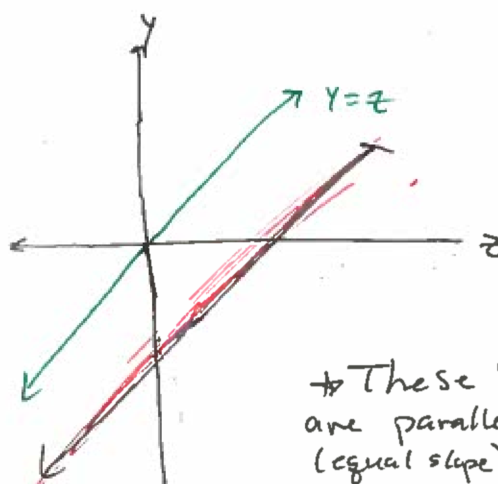
$u=0$

v-nullcline

$v=0$

One equilibrium: $(0,0)$.

(c) $\begin{cases} \frac{dz}{dt} = 13 - z + y \\ \frac{dy}{dt} = z - y \end{cases}$



Z-nullclines:

$13 - z + y = 0$

$y = z + 13$

Y-nullclines:

$z - y = 0$

$y = z$

* These lines are parallel (equal slope) so there are no equilibria.

2. Find the general solutions to the following differential equations.

(a) $\frac{dx}{dt} = \frac{t^2}{x}$

$$\int x\,dx = \int t^2\,dt$$

$$\frac{1}{2}x^2 = \frac{1}{3}t^3 + C$$

$$x = \pm\sqrt{\frac{2}{3}t^3 + C}$$

(b) $\frac{dx}{dt} = \frac{x}{t^2}$

$$\int \frac{dx}{x} = \int \frac{dt}{t^2}$$

$$\ln|x| = -\frac{1}{t} + C$$

$$|x| = e^{-1/t}\,e^{C}$$

$$x = A\,e^{-1/t}$$

(c) $\frac{dx}{dt} = \frac{t}{x^2}$

$$\int x^2\,dx = \int t\,dt$$

$$\frac{1}{3}x^3 = \frac{1}{2}t^2 + C$$

Absorb the 3 into C.

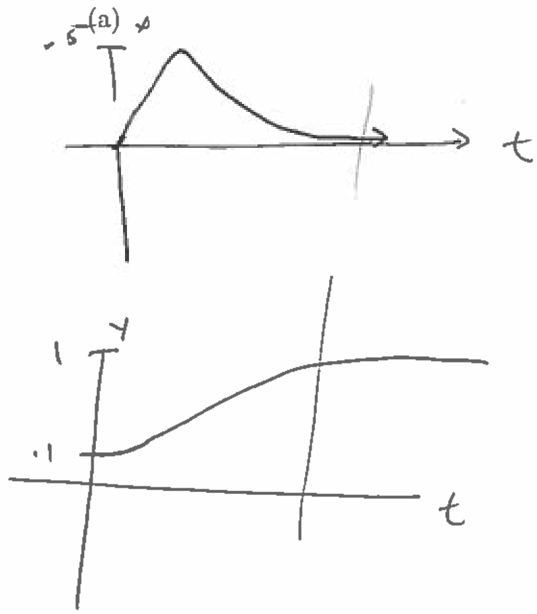
$$x^3 = \frac{3}{2}t^2 + C$$

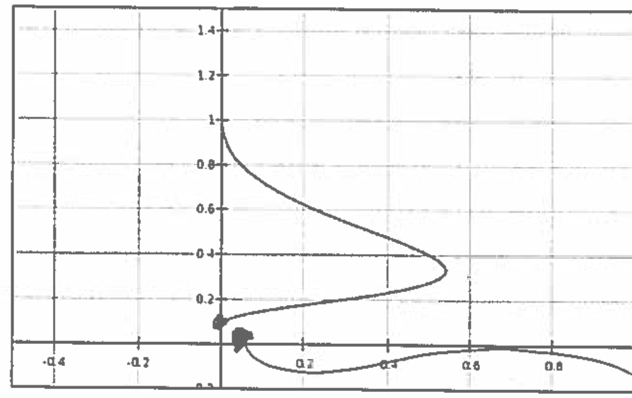
$$x = \sqrt[3]{\frac{3}{2}t^2 + C}$$

* Cannot simplify further!

3. The following graphs represents a trajectory in the phase-plane for a system of differential equations for $x(t)$ and $y(t)$. Sketch possible graphs of $x(t)$ and $y(t)$.

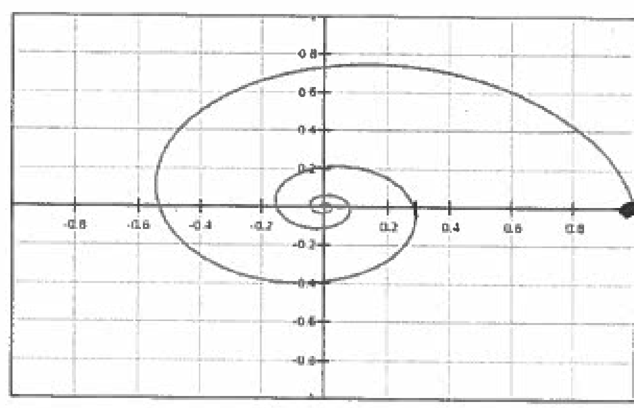
(a)



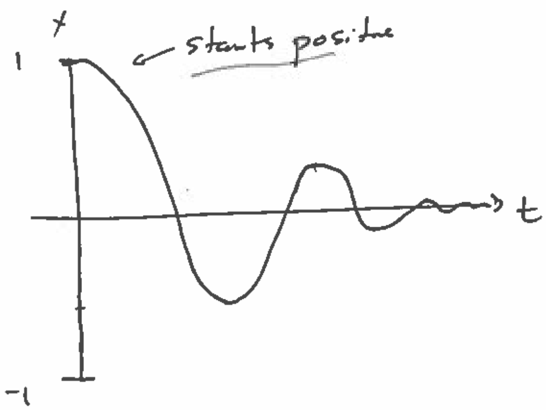


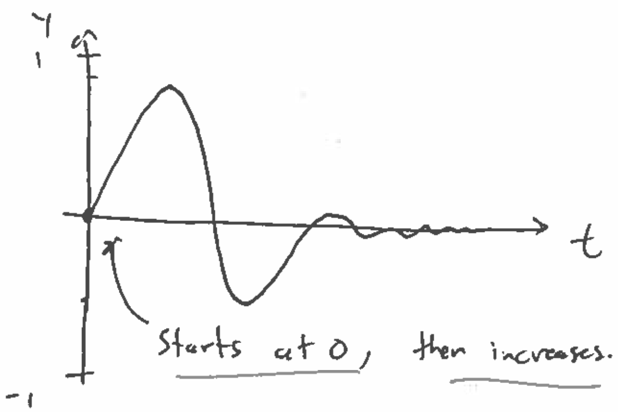
*I assume that t=0 is here.

(b)



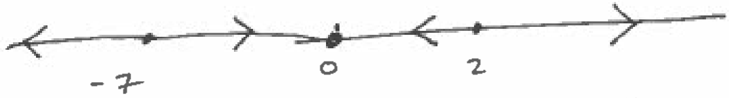
I assume t=0 was here





4. For the following DE, make a phase-line diagram.

(a) $\frac{dx}{dt} = \overset{0}{x}(\overset{2}{x-2})(\overset{-7}{x+7})$



(b) $\frac{ds}{dt} = x^2 - 9x = x(x-9)$

5. Evaluate the following integrals.

(a) $\int \sqrt[5]{x^7}\, dx = \int x^{7/5} dx = \frac{5}{12} x^{12/5} + C$

(b) $\int xe^x\, dx = xe^x - \int e^x dx = xe^x - e^x + C$

$u = x \qquad dv = e^x dx$

$du = dx \qquad v = e^x$

(c) $\int_{-2}^{0} \frac{14}{2x}\, dx = 7 \ln|x| \Big|_{-2}^{0} = 7\left(\ln|0| - \ln|-2|\right) = -\infty.$

bad! This was secretly an improper integral, and it diverged.

(d) $\int_4^{16} \frac{x}{1+x^2}\, dx = \frac{1}{2}\int \frac{1}{u} du = \frac{1}{2}\ln(1+\cancel{x}^2)\Big|_4^{16}$

$u = 1 + x^2$

$\frac{1}{2} du = x\,dx$

$= \frac{1}{2}\ln(1+256) - \frac{1}{2}\ln(17)$

$\approx 1.358.$

6. If a population of wolves is currently at one-hundred thousand, and $\int_0^{20} f(t)\,dt = 3000$, where $f(t)$ is the growth rate in wolves per year, then how many wolves are there after 20 years?

Current + total change = 103,000 wolves

$\underset{100{,}000}{\text{Current}} \;\; + \;\; \underset{3{,}000}{\text{total change}} = 103{,}000$ wolves

7. For the neuron model with a constant applied current, find the nullclines of the system.

$$\frac{dv}{dt} = -v(v-a)(v-1) - w + I$$

$$\frac{dw}{dt} = \varepsilon(v - \gamma w).$$

w-nullcline:

Still:

$$w = \frac{1}{\gamma} v.$$

v-nullcline:

$$w = -v(v-a)(v-1) + I$$

v-nullcline gets shifted up!

8. For the following differential equation, approximate $x(1)$ given $\Delta t = 0.5$ and $x(0) = 1$.

$$\frac{dx}{dt} = \frac{x}{1+t}$$

t	x'	$\hat{x}$
0	$\frac{1}{1+0} = 1$	$1 + 1 \cdot (0.5) = 1.5$ $(\hat{x}(.5))$
0.5	$\frac{1.5}{1+.5} = 1$	$1.5 + 1(.5) = 2$ $(\hat{x}(1))$.

9. Consider our system of DE for the disease model:

$$\frac{dI}{dt} = \alpha IS - \mu I$$
$$\frac{dS}{dt} = -\alpha IS + \mu I$$

(a) Describe exactly what the variables I and S are, as well as the parameters α and μ.

I = frac. of pop. infected.

S = frac. of pop. susceptible to disease.

α = spread factor

μ = recovery factor.

(b) Find the nullclines of this system and describe the equilibria.

I-nullcline:

$\alpha IS - \mu I = 0$

$I(\alpha S - \mu I) = 0$

$I = 0$, $S = \text{any}$ or $I = \frac{\alpha}{\mu} S$

S-nullcline

$-\alpha(IS) + \mu I = 0$

Same nullclines!

$I = 0$, $S = \text{any}$ all equilibria

$I = \frac{\alpha}{\mu} S$ all equil.

Infinite # of equilibria! Makes sense because as soon as $I = 0$, disease can't come back, so the whole S-axis needs to be equilibria.

(c) Describe how one might incorporate birth rates into the model.

add $+ \beta \cdot S$ to $\frac{dS}{dt}$'s eqn.

$$\frac{dS}{dt} = -\alpha IS + \mu I + \beta S$$

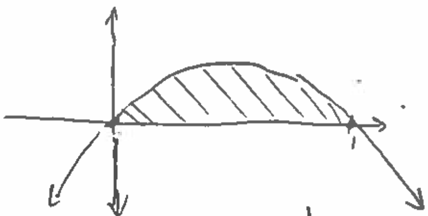
(d) Describe how one might incorporate death from disease into the model.

~~A~~ Add ~~Add~~ $-\delta I$ to the $\frac{dI}{dt}$ eqn.

$$\frac{dI}{dt} = \alpha IS - \mu I - \delta I.$$

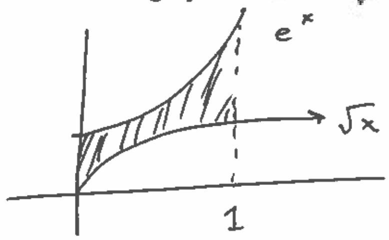
10. Find the area of the following regions.

(a) Between the x-axis and the graph of the function $x - x^2$.



$$A = \int_0^1 (x - x^2)\,dx = \left(\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right)\Big|_0^1 = \tfrac{1}{2} - \tfrac{1}{3} = \tfrac{1}{6}$$

(b) Between the graphs of e^x and $\sqrt{x}$ from $x = 0$ to $x = 1$.



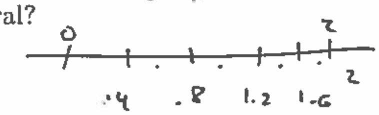
$$\int_0^1 e^x\,dx - \int_0^1 \sqrt{x}\,dx$$

$$= e^x\Big|_0^1 - \left(\tfrac{2}{3}x^{3/2}\right)\Big|_0^1$$

$$= \boxed{e - 1 - \tfrac{2}{3}} \approx 1.052$$

11. Calculate the left and right Riemann sums for the function $g(t) = 10 - t^2$ on the interval $[0, 2]$ with 5 subintervals. How does your answer compare to the exact value for the integral?

$\Delta x = \frac{2-0}{5} = .4$



LRS: $g(0)(0.4) + g(.4)(.4) + g(.8)(.4) + g(1.2)(.4) + g(1.6)(.4)$

$= (45.2)(.4) = 18.08$

RRS: $g(.4)(.4) + g(.8)(.4) + g(1.2)(.4) + g(1.6)(.4) + g(2)(.4)$

$= (41.2)(.4)$ 16.48

Exact: $\int_0^2 (10 - t^2)\,dt = 10t - \tfrac{1}{3}t^3 \Big|_0^2 = 20 - \tfrac{1}{3}(8) \approx 17.33$

12. Below are four differential equations, along with six functions. Identify which function is a solution to which equation (note that some functions may not be solutions to any of the DE's).

$$\frac{dy}{dt} + y = 2$$
$$\frac{dy}{dt} = y^2$$
$$\frac{dy}{dt} = ty^2$$
$$\frac{dy}{dt} = y + t$$

$$y(t) = e^t - t - 1$$
$$y(t) = e^{-t^2}$$
$$y(t) = \frac{e^t}{1+t}$$
$$y(t) = \frac{1}{1-t}$$
$$y(t) = -\frac{2}{1+t^2}$$
$$y(t) = e^{-t} + 2$$

Just gotta calculate a bunch o' derivatives!

13. The mass density (in kg/meter) of a poorly-made construction beam seems to follow the function

$$D(x) = 12.2xe^{-4x^2},$$

where x is a distance in meters along the beam. How much does the bar weigh?

$$\int_0^5 12.2\, x e^{-4x^2}\, dx = \frac{12.2}{8}\int e^{-u}\, du = -\frac{12.2}{8} e^{-u}$$

Let's say the bar is about 5 meters long. It won't actually matter too much because the antideriv. goes to 0 super fast.

$u = 4x^2$

$du = 8x\, dx$

$\frac{1}{8} du = x\, dx$

$$= -\frac{12.2}{8} e^{-4x^2}\Big|_0^5$$

$$= -\frac{12.2}{8}\left(e^{-100} - 1\right) = \frac{12.2}{8} = 1.525 \text{ kg.}$$

14. Use the stability theorem to classify the equilibria of the differential equation

$$\frac{dA}{dt} = (A-2)(13-A^2).$$

$$f(A) = (A-2)(13-A^2) = 13A - 26 - A^3 + 2A^2.$$

$$f'(A) = 13 - 3A^2 + 4A.$$

Eq: $A^* = 2$, $A^* = \pm\sqrt{13}$

$f'(2) = 9$ unstable.

$f'(+\sqrt{13}) = -11.58$ stable

$f'(-\sqrt{13}) = -40.4$ stable