Guide for Section 2 - Linear Functions

Section Goals:

- Compute average rate of change in a function over an interval.
- Identify a function as increasing, decreasing, or constant on an interval.
- Express a formula for a linear function.
- Identify phenomena which lend themselves to a linear relationship.
- Determine whether points are collinear.
- Compute percentage change in a function over an interval.

Def The **average rate of change** in function $Q = f(t)$ on the interval $[a, b]$ is

$$\frac{f(b) - f(a)}{b - a}.$$

This is also commonly expressed as

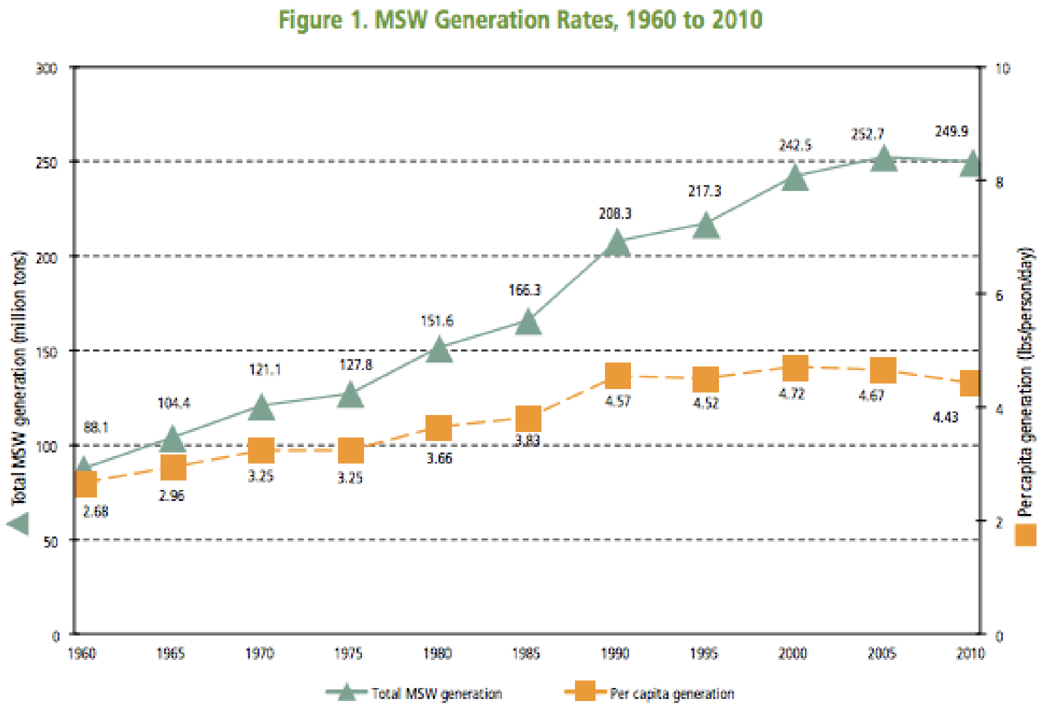
$$\frac{\Delta Q}{\Delta t},$$

where ΔQ is the total change in output Q and Δt is the change in input t over the interval $[a, b]$.

Ex 1 Find the average rate of change in $f(t) = 0.1t^2 + 5$ on the interval $[0, 4]$.

$$\frac{\Delta f}{\Delta t} = \frac{f(4) - f(0)}{4 - 0} = \frac{(0.1(4^2) + 5) - (0.1(0^2) + 5)}{4} = \frac{1.6}{4} = 0.4.$$

Ex 2 The graph below shows two interesting sets of information together by year: The total amount of solid waste generated in the United States (triangles, uses the left vertical axis), and the *per capita* amount of solid waste generated in the United States (squares, uses the right vertical axis).



a) What is the average rate of change in total waste generated between 1990 and 2000? Include units.

$$\text{ARC} = \frac{242.5 - 208.3}{2000 - 1990} = 3.42 \text{ million tons/ year}$$

b) What is the average rate of change in per capita waste generated between 1990 and 2010? Include units.

$$\text{ARC} = \frac{4.43 - 4.57}{2010 - 1990} = -0.007(\text{lbs per person per day})/\text{year}$$

c) During what five-year period was the average rate of change in total waste largest? Smallest? The largest was between 1985 and 1990, while the smallest was between 2005 and 2010. You can tell because the slope is the largest in the first range, while the slope is the smallest in the second range.

d) Identify any periods in which total waste grew but per capita waste fell. Describe what must have happened to cause this change.

This must have happened between 1990-1995 and 2000-2005. This means that while more waste was produced in total, each person (on average) made less waste. So, there must have been a large population increase in these years.

Def A function $f(t)$ is $\begin{bmatrix} \text{strictly increasing} \\ \text{strictly decreasing} \\ \text{constant} \end{bmatrix}$ on the interval (c, d) as long as $\begin{bmatrix} f(b) > f(a) \\ f(b) < f(a) \\ f(a) = f(b) \end{bmatrix}$ for every value $c < d$ on the interval (and where the entire interval is in the domain of f).

In other words: On the graph of a function as you move to the right, if the function consistently grows then it is increasing, if it consistently falls then it is decreasing, and if it stays flat then it is constant.

Ex 3 Revisit the graph provided in Focus 1. Let $T(t)$ be the total waste and $P(t)$ the per capita waste generated in the year t years after 1960, where $0 \le t \le 50$.

a) Is T a strictly increasing function on its domain? Strictly decreasing? Neither? T is neither strictly increasing nor strictly decreasing. It is mostly increasing, except on the last interval from 2005 to 2010.

b) On what interval(s) of t-values is P increasing? Decreasing? Constant?
P is increasing on the intervals $[1960, 1970], [1975, 1990], [1995, 2000]$.
P is decreasing on the intervals $[1990, 1995], [2000, 2010]$.
P is constant on the interval $[1970, 1975]$.

Def A linear function is a function f with the characteristic that for any pair of points x and y in its domain, the average rate of change between those points has the same value, e.g. a. Such a function can be written

$$f(t) = at + b$$

for constant rate of change (often called its slope) a, and constant b.

We can easily compute slope by finding the average rate of change using any two points (t_1, y_1) and (t_2, y_2): $ARC = \dfrac{y_2 - y_1}{t_2 - t_1}$.

Ex 4 Find the rule of the linear function with slope $-1/5$ whose graph contains the point $(20, -6)$.
We are given that $a = -1/5$. With the point $(20, -6)$, we can insert $t = 20$ and $f(20) = -6$:

$$\begin{aligned} f(20) &= -\frac{1}{5}(20) + b \\ -6 &= -4 + b \\ -2 &= b. \end{aligned}$$

So, $f(t) = -\frac{1}{5}t - 2$.

Def The graph of a linear function is a line.

Linear functions with positive slope are strictly increasing, linear functions with negative slope are strictly decreasing, and linear functions with zero slope are constant.

Ex 5 A line contains the points $(2, 7)$ and $(-3, 10)$.

a) Find the rule of the function whose graph is described. Slope: $a = \frac{10-7}{-3-2} = -\frac{3}{5}$.

$$
\begin{aligned}
f(t) &= -\frac{3}{5}t + b \\
7 &= -\frac{3}{5}(2) + b \\
b &= 7 + \frac{6}{5} \\
b &= \frac{35}{5} + \frac{6}{5} = \frac{41}{5}.
\end{aligned}
$$

So, the equation is

$$f(t) = -\frac{3}{5}t + \frac{41}{5}.$$

b) Sketch the graph of the function in the ty-plane.

Ex 6 The estimated wolf population in Idaho[1] was 14 wolves in 1995 and increased at an approximate rate of 43 wolves per year, each year in the next decade. Write a formula for the number of wolves in the state as a function of time, t, in years after 1995. According to the model, what is the predicted wolf population in 2010? Let $W(t) = at + b$ be the function giving the number of wolves at time t. Note that $W(0) = b$, the y-intercept. Also, $t = 0$ corresponds to the year 1995. Hence, $b = W(0) = 14$. Also, we know that the slope a is the ARC, which is given to be 43 wolves per year. So, $a = 43$. The equation is $W(t) = 43t + 14$. In 2010, we have $t = 15$. So, the predicted population is $W(15) = 43(15) + 14 = 659$.

[1] `http://www.thewildlifenews.com/2013/04/02/idaho-year-end-wolf-population-declines-11-to-683-livestock\-losses-increase/`

Def The percentage change in a function f on the interval $[c, d]$ is

$$PC_{[c,d]} = 100 \cdot \frac{f(d) - f(c)}{f(c)} \%$$

Ex 7 Find the percentage change in $f(t) = 3 + 2t$ on the interval $[1, 4]$.

$$\begin{aligned} PC_{[1,4]} &= 100 \cdot \frac{f(4) - f(1)}{f(1)} \% \\ &= 100 \cdot \frac{3 + 2(4) - (3 + 2(1))}{3 + 2(1)} \% \\ &= 100 \cdot \frac{8 - 2}{5} \% \\ &= 120\%. \end{aligned}$$

Ex 8 Find the percentage change in $g(x) = \sqrt{x^2 + 1}$ on the interval $[-2, 3]$.

$$\begin{aligned} PC_{[-2,3]} &= 100 \cdot \frac{g(3) - g(-2)}{g(-2)} \% \\ &= 100 \cdot \frac{\sqrt{10} - \sqrt{5}}{\sqrt{5}} \% \\ &= 100 \cdot (0.4142)\% \\ &= 41.42\%. \end{aligned}$$

You should interpret this as saying "the value of g at -2 compared to the value at 3 increased by 41.42%." To make sense of this, take the value of $g(-2) = \sqrt{5} \approx 2.24$. Find 41.42% of this value, which is $0.4142 * \sqrt{5} \approx 0.926$. That is how much it changed by, so add this to what it started at:

$$2.24 + 0.926 = 3.166.$$

So, the value of $g(3)$ should be about 3.166. Let's see if that works: $g(3) = \sqrt{10} \approx 3.162$, which is exactly what we expected.