4.3: Integration by parts

Contents

1 Integration by parts: What it is

- The idea of integration by parts is "integrating the product rule."
- Product rule:

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x).$$

  Then integrate both sides:

$$\int \frac{d}{dx}(f(x)g(x))\,dx = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx$$

- You get

$$f(x) \cdot g(x) = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx.$$

- We then use this if we can recognize an integral that looks like $\int f(x)g'(x)\,dx$. In essence, this formula lets us "move the derivative" from one function to the other. This gives us the resulting formula:

- Integration by parts formula.

$$\boxed{\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx}$$

- Ex: What is $\int xe^x\,dx$? Well, it looks like a function times the derivative of another.

- You can pattern match it:

$$\int xe^x\,dx \stackrel{?}{=} \int f(x)g'(x)\,dx$$

Seems like we should guess $f(x) = x$, $g'(x) = e^x$. Then, $f'(x) = 1$ and $g(x) = e^x$, so

$$\int xe^x\,dx = xe^x - \int 1 \cdot e^x\,dx.$$

Now, the resulting integral is easier to do: it's just e^x. So,

$$\int xe^x\,dx = xe^x - e^x + C.$$

2 Integration by Parts: examples

- Generally more flexible than subtitution, but often trickier.

- Recommendation: to keep track of your steps and your final goal, start by calling your integral by a letter, like I. Think "I" for "integral."

- Ex:

$$I = \int x\cos(x)\,dx$$

- Choose $f(x) = x$, $g'(x) = \cos(x)$. Gotta choose one to integrate, one to differentiate.

- Get: $f'(x) = 1$, $g(x) = \sin(x)$.

$$I = x\sin(x) - \int 1 \cdot \sin(x)\,dx = x\sin(x) + \cos(x) + C.$$

- Note: The integration by parts theorem has a subtle minus sign! It is crucial to getting the right antiderivative.

- Sometimes a strange choice makes things work really well:

$$I = \int \ln(x)\, dx$$

Try $f(x) = \ln(x)$ and $g'(x) = 1$! after all, there's no harm in writing $\ln(x) \cdot 1$.

Get $f'(x) = \frac{1}{x}$ and $g(x) = x$.

$$I = x\ln(x) - \int \frac{1}{x} \cdot x\, dx = x\ln(x) - x + C$$

- Ex:

$$I = \int x \ln(x)\, dx$$

Try $f'(x) = x$ and $g(x) = \ln(x)$ (this will get rid of the ln!)

So, $f(x) = \frac{1}{2}x^2$ and $g'(x) = \frac{1}{x}$.

$$I = \frac{1}{2}x^2 \ln(x) - \int \frac{1}{2}x^2 \cdot \frac{1}{x}\, dx$$

That last integral is easier: it simplifies to $\frac{1}{2}\int x\, dx = \frac{1}{4}x^2$, so the final answer is

$$I = \frac{1}{2}x^2 \ln(x) - \frac{1}{4}x^2 + C$$

- Sometimes you need to do IBP more than once.

$$I = \int x^2 e^x\, dx$$

Do $f' = e^x$, $g = x^2$, so that

$f = e^x$, $g' = 2x$. You now have

$$I = x^2 e^x - \int 2xe^x\, dx.$$

Here comes a subtle issue: if you use integration by parts again, you need to pay close attention to the minus signs.

We want to do IBP agian with $f' = e^x$, $g = 2x$. Then $f = e^x$, $g' = 2$. To illustrate, I will use enormous parentheses.

$$I = x^2 e^x - \left(2xe^x - \int 2e^x\right).$$

The final integral is now apparent, so we have

$$I = x^2 e^x - (2xe^x - 2e^x) + C$$

which simplifies to

$$I = x^2 e^x - 2xe^x + 2e^x + C.$$

You must distribute minus signs appropriately! If you do not pay attention to this it will cause unnecessary headache, especially on webwork problems.

- Ex: Sometimes you need to analyze and manipulate using algebra to find the antiderivative. I call this the "algebra trick."

$$\int \sin^2(x)\,dx$$

Choose $f(x) = \sin(x)$, $g'(x) = \sin(x)$.

Then $f'(x) = \cos(x)$, $g(x) = -\cos(x)$.

$$\int \sin^2(x)\,dx = -\sin(x)\cos(x) - \int \cos(x)(-\cos(x))\,dx = -\sin(x)\cos(x) + \int \cos^2(x)\,dx$$

Now, $\cos^2(x) = 1 - \sin^2(x)$. So,

$$\begin{aligned}
\int \sin^2(x)\,dx &= -\sin(x)\cos(x) + \int 1 - \sin^2(x)\,dx \\
\int \sin^2(x)\,dx &= -\sin(x)\cos(x) + x - \int \sin^2(x)\,dx \\
2\int \sin^2(x)\,dx &= -\sin(x)\cos(x) + x \\
\int \sin^2(x)\,dx &= \frac{-\sin(x)\cos(x) + x}{2} + C
\end{aligned}$$

- You try:

$$\int x^2 \sin(x)\, dx = -x^2 cos(x) + 2x \sin(x) + 2\cos(x) + C$$

$$\int x^{30} \ln(x)\, dx = \frac{1}{31} x^{31} \ln(x) - \frac{1}{(31)^2} x^{31} + C$$