4.3: Substitution

- Recall: Chain rule
$$\frac{d}{dx}(f(g(x)) = f'(g(x))g'(x).$$
Example:
$$\frac{d}{dx}(\sin(x^2)) = \cos(x^2) \cdot 2x$$

- Features to notice:

  1. the resulting derivative looks like a product (or possibly a fraction, if we had to do a quotient rule somewhere)

  2. you can see a function and its derivative (the $2x$ and the x^2)

- Do:
$$\int xe^{x^2}\,dx = \frac{1}{2}e^{x^2} + C$$
There's a mechanical process to the madness:

- Let $u = x^2$ (most wrapped-up expression).

- Then $\frac{du}{dx} = 2x$, $du = 2xdx$ ...

- Don't forget to go back to x!

- Why does it work? Integrating is solving the diffy-Q
$$\frac{dy}{dx} = xe^{x^2}.$$

- Setting $u = x^2$, we have $\frac{du}{dx} = 2x$. Chain rule says $\frac{dy}{du}\frac{du}{dx} = \frac{dy}{dx}$. Thus,
$$\frac{dy}{du}\frac{du}{dx} = \frac{dy}{dx}$$
So,
$$\frac{dy}{du} = \frac{1}{2}e^u.$$

- The du and the dx are important!

Examples

- Ex: $\int \cos(2x)\,dx = \frac{1}{2}\sin(2x)\,dx$
- Ex: $\int \frac{1}{2x-1}\,dx$
- Ex: $\int e^{4x}\,dx$
- Ex: $\int 2^x\,dx$ again.

$$2^x = (e^{\ln(2)})^x = e^{\ln(2)x},$$

so

$$\int 2^x\,dx = \frac{1}{\ln(2)}e^{\ln(2)x} + C = \frac{1}{\ln(2)}2^x + C.$$