Day 9: Related Rates

1 Balloon Example

- Suppose we fill a spherical balloon so that the volume increases at a rate of $100\text{cm}^3/\text{s}$. How fast is the radius of the balloon increasing when its diameter is 50 cm?
- First step: define variables. $V =$volume, $r =$radius.
- Second step: identify what is known.
- $\frac{dV}{dt} = 100$, $2r = 50$.
- Third step: identify what you're after.
- $\frac{dr}{dt} =$?
- Once you know all of the players, you need to relate them.
- $V = \frac{4}{3}\pi r^3$.
- Differentiate with respect to time (it has the feel of implicit differentiation):

$$\begin{aligned}\frac{dV}{dt} &= \frac{4}{3}\pi 3r^2\frac{dr}{dt}\\ 100 &= 4\pi(25)^2\frac{dr}{dt}\\ \frac{dr}{dt} &= \frac{100}{4\pi(25)^2} \approx 0.013\text{cm/s}.\end{aligned}$$

2 Camera following a person

- A person walks in straight line at a speed of 3 ft/s. A camera, located 15 feet from the person's path, rotates so as to follow the person. How quickly is the camera rotating when the person is 20 feet from the point where the camera is closest to the path?

- Set up picture: Let x be the distance from point P, where P is the closest point on the path to the camera. Let θ be the angle made by the camera.

  - We know: $\frac{dx}{dt} = 3$
  - We want to know: $\frac{d\theta}{dt}$ when $x = 20$.

- Relation: $\tan(\theta) = \frac{x}{15}$ by the geometry.

$$\sec^2(\theta)\frac{d\theta}{dt} = \frac{1}{15}\frac{dx}{dt}$$
$$\sec^2(\theta)\frac{d\theta}{dt} = \frac{1}{15} \cdot 3$$

Still need: θ when $x = 20$!

$$\tan(\theta) = \frac{20}{15} \implies \theta = \arctan(20/15) = 0.923\text{rad}$$

So,
$$\frac{d\theta}{dt} = \frac{3}{15}\frac{1}{\sec^2(0.923)} = 0.072\text{rad/sec} = 4.125\text{deg/sec}.$$

3 Conical Water Tank

- A conical water tank has base radius 3m and height 5m. Water is being pumped into the tank at a rate of 2m^3/min. Find the rate at which the water level is rising when the water is 3 feet deep.

- First: define variables. r =radius of water at time t
- h =height of water at time t.
- V=volume of the water at time t.
- We know: $\frac{dV}{dt} = 2$
- We want: $\frac{dh}{dt}$ when $h = 3$.
- Relate volume to r and h for a cone:

$$V = \frac{1}{3}\pi r^2 h$$

- Problem: two unknowns, r and h. So, relate them.
- Similar triangles:

$$\frac{r}{3} = \frac{h}{5} \implies r = \frac{3}{5}h.$$

So,

$$V = \frac{1}{3}\pi \frac{9}{25}h^3.$$

Differentiate:

$$\frac{dV}{dt} = \frac{3}{25}\pi 3h^2 \frac{dh}{dt}$$

Plug in $\frac{dV}{dt} = 2$ and $h = 3$ to solve for $\frac{dh}{dt} = 0.196$ m/min.