5.5: Systems of DE

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1 Law of Mass Action

- The Law of mass action describes the rate of encounter of individuals in a big collection of things that move randomly. (Inherently left vague because it is very general!)
- What is says is, *the rate for a thing to hit something in a compartment C is proportional to the number of things in C.*
- Ex: If Sodium atoms Na and Potassium, K, atoms are flying around each other, the rate that one Na atom hits *any* K atom is proportional to the number of K atoms. This is very sensible: if you double the number of K, then there is double the chance of collision, meaning double the rate of hitting.
- Or, for example, in modeling two species, say penguins and whales, one penguin's rate of encountering whales is proportional to the number of whales.

- The important thing here is the principle, not any one formula that comes from it. It will be central to understanding our models here on out.

2 Application to Chemistry

- Application of this law: Suppose we have three chemicals, A, B, and C.
- Notation: $[A]$ for the concentration of chemical A, same with $[B]$, $[C]$.
- Chemical A can turn into chemical C, and C can decay back into A and B.
- Another assumption: there is an abundance of chemical B, meaning its concentration $[B]$ doesn't change in time.
- $A + B \rightleftharpoons C$ is the reaction.
- Differential equation for $[A]$:

$$\frac{d[A]}{dt} = (\text{Rate for reaction producing } A) - (\text{rate for reaction consuming } A)$$
$$\frac{d[A]}{dt} = k_1[C] - k_2[A][B].$$

- Since the reactions producing C consume A and vice versa, the equation for $[C]$ is the same, but with minus signs. We get

$$\frac{d[C]}{dt} = -k_1[C] + k_2[A][B].$$

We get a *system of differential equations.* It looks like this:

$$\frac{d[A]}{dt} = k_1[C] - k_2[A][B]$$
$$\frac{d[C]}{dt} = -k_2[C] + k_2[A][B].$$

3 Application to Biology: Predator-Prey

- Suppose we have two interacting species, say penguins and whales. Let P and W be the number of penguins and whales, respectively.
- Assumptions:
  1. The penguins grow according to a simple population model.
  2. Whales die off according to a simple population model.
  3. Whales benefit from eating penguins and are able to produce offspring.
- The principle of mass action comes in to describe how whales eating penguins shows up in the differential equation.
- Mass action says: One penguin experiences a rate proportional to W, the number of whales. Thus, the equation for P is

$$\frac{dP}{dt} = aP - bWP.$$

- The W shows up because of "mass action," and P shows up because we need to multiply the rate for one penguin by all the penguins.
- Similarly, the equation for whales is similar:

$$\frac{dW}{dt} = -cW + dWP.$$

- The simple death model shows up as $(-cW)$, and the positive benefit for whales eating penguins shows up as $(+dWP)$. Again, the fact that we see W times P in this equation is indicative of an interaction, which we modeled with the principle of mass action.
- For this example, the lowercase letters are parameters, and the upper case letters are the state variables. The system is now

$$\begin{aligned}\frac{dP}{dt} &= aP - bWP \\ \frac{dW}{dt} &= -cP + dWP.\end{aligned}$$

- The most general predator-prey model is

$$\frac{dx}{dt} = \lambda x - \epsilon xy$$
$$\frac{dy}{dt} = -\delta y + \eta xy$$

Here, $\lambda, \epsilon, \delta$, and η are parameters. λ is the growth rate for x, ϵ is the eating rate, δ is the death rate for y, and η is the benefit rate for y eating x. This notation aligns more with the book. (However, they use a and b for the state variables. I choose to use x and y here. What you choose to name your variables is unimportant.)

4 Examples

- Ex: Given the differential equations below, identify which species is the predator and which is the prey.

$$\frac{dS}{dt} = -0.2S + 4ST$$
$$\frac{dT}{dt} = 1.6S - 3ST$$

Sol: from these equations, S dies off without T around. So, S must be the predator, and T is the prey.

- Ex: Same question.

$$\frac{dr}{dt} = -0.3rf + 2r$$
$$\frac{df}{dt} = 0.7fr - 0.1f$$

Sol: r is the prey and f is the predator. Again, there are many things that clue you in. Notice that r is negatively affected by interactions, which you see by the $-0.3rf$ term. Alternatively, the $+2r$ says that r grows without the other species, and the $-0.1f$ says that population f dies off without the other species.

5 Variations on Predator-Prey

- By changing the coefficients in a predator-prey type model, we can interpret the biological scenario differently.
- Ex:

$$\frac{dA}{dt} = 0.2A - 4AB$$
$$\frac{dB}{dt} = 0.9B - 3AB.$$

  Interpretation: both A and B populations reproduce, but they both are negatively affected by the other's presence. This is like two species in competition with each other.

- Ex:

$$\frac{dx}{dt} = -2x + 4xy$$
$$\frac{dy}{dt} = -0.5y + 2xy$$

  This describes two populations that would die without (their growth terms are negative) but benefit from each others' presence. This describes two populations that are helping each other survive!