4.6: Applications

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1 Applications

1.1 Areas

- The first application is to areas.
- Note that a Riemann sum is adding things like $f(x)\Delta x$. This can be interpreted as a rectangle of height $f(x)$ and width Δx.
- As Δx shrinks, rectangles do a better job at approximating area between the curve and the x-axis.
- Thus, we have the first result:

$$\boxed{\text{area under the curve of } f(x)} = \int_a^b f(x)\,dx$$

- Ex: Find the area between the curve and the x-axis for the function $f(x) = \sin(x)$ between $x = 0$ and $x = \pi$.
- $\int_0^\pi \sin(x)\,dx = -\cos(\pi) - (-\cos(0)) = 1 - (-1) = 2$. So that area is 2!

- Ex: Find the area underneath the function $f(x) = 4x^3$ between $x = -1$ and $x = 1$.

- A: $\int_{-1}^{1} 4x^3\,dx = x^4\Big|_{-1}^{1} = 1^4 - (-1)^4 = 0$. Why? The answer is that area below the axis is counted as *negative area*.

- Negative area might make you feel queasy. Remember, in applications to biology, $f(t)$ is usually a rate, so negative f values (which lead to negative areas) are interpreted as a negative rate (or a negative net change).

1.2 Averages

- Next, we can interpret a (multiplied) Riemann sum as an average of a function.

- What is an average? Add up values and divide by the number of them.

- Suppose we take $f(x)$ on an interval, like $[1, 4]$.

- Take N values spaced by Δx apart and make a Riemann sum.

$$\text{average} \;=\; \frac{f(1) + f(1.1) + \cdots + f(3.9)}{N}$$

(Note: I'm making a general argument here; the numbers like 1.1, 3.9, etc would change from problem to problem.)

- Now multiply the top and bottom by Δx:

$$\text{average} \;=\; \frac{f(1)\Delta x + f(1.1)\Delta x + \cdots + f(3.9)\Delta x}{N\Delta x}$$

- Now, $N\Delta x$ is just the length of the interval, $4-1 = 3$. The top becomes a Riemann sum. Letting $\Delta x \to 0$, we find that

$$\text{average of } f \text{ over } [1, 4] = \frac{1}{4-1}\int_1^4 f(x)\,dx$$

- In general,

$$\text{avg of } f \text{ on } [a, b] = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

- Ex: Suppose the voltage level in a cell is given by $V(t) = \frac{t}{t^2+1}$ millivolts. Find the average voltage level over the first five seconds.

- $$\text{avg voltage} = \frac{1}{5-0}\int_0^5 V(t)\,dt = \frac{1.629}{5} = 0.326\,\text{mV}.$$

1.3 Geometry

- Any time you notice a Riemann sum you can do an integral.
- Ex: area of a circle. Why is it πR^2?
- draw a circle of radius R.
- draw concentric circles. One of those thin pieces has an area of (circumfrence)$\times(\Delta r)$.
- So, the area should be a sum of the form $\sum 2\pi r \cdot \Delta r$, and letting $\Delta r \to 0$, we see
$$\text{area of a circle} = \int_0^R 2\pi r\,dr = 2\pi \frac{1}{2} r^2 \Big|_0^R = \pi R^2.$$
Notice more relationships: derivative of $\frac{4}{3}\pi R^3$ is $4\pi R^2$, so the derivative of the volume of a sphere is the surface area! There are many more relationships like this.

1.4 Density

- Density: amount of mass per unit length, or area, or volume.
- Measures how compactly matter is distributed. Larger density, more compact. Smaller density, less compact (more spread out).
- Ex: Imagine iron bars. Assuming uniform density, an iron bar of 10kg with a length of 2 meters has a density of $\frac{10\,\text{kg}}{2\,\text{m}} = 5\,\text{kg/m}$. However, if you make a bar with the same weight but longer, say 5 m, then the density would be $\frac{10}{5} = 2\,\text{kg/m}$. In the second scenario, the mass is more spread out, giving a smaller density.
- Can have other kinds of density: e.g. in a strand of DNA, you can ask, how many nucleotides (nuc) are there? How many are there per unit nanometer? Density would be nuc/nm.

- Virtually any static quantity can have a density. kg/m, nuc/m, people/acre, dollars/m^2,...
- Integration comes into play when you have a density that isn't uniform and you want to know the total amount of "stuff" the density measures.
- Ex: Suppose an iron bar that is 3 meters long has non-uniform density $\rho(x) = \frac{1}{1+x}$ in kg/m, where x is the distance from one end of the bar. How much does the bar weigh?
- A: We won't calculate a Riemann sum, but they help us formulate the problem. Imagine cutting the bar into tiny segments. On each segment of length Δx, density is almost constant, so each segment has mass $\rho(x) \cdot \Delta x$. Summing these up over the length of the bar, a Riemann sum would have the form $\sum \rho(x)\Delta x$. So, letting $\Delta x \to 0$, we get a definite integral:

$$\text{total mass} = \int_0^3 \rho(x)\,dx = \int_0^3 \frac{1}{1+x}dx = \ln(1+x)\Big|_0^3 \approx 1.38\,\text{kg}.$$

- In this class we only consider densities along a line. For densities along areas (like population density in a city) you need multiple variables like x and y, and to develop the idea of "multiple integration." Intuitively you just integrate along each direction of the considered area. For example, if $\rho(x,y)$ is the population density in a city, the total population would be

$$\text{total population} = \int_a^b \int_c^d \rho(x,y)dxdy$$

  where a, b, c, d are the physical city limits. We won't do this in this class, but you hopefully see the way to generalize.
- For this class you need to know: if $\rho(x)$ is a density of (stuff)/distance, then

$$\boxed{\text{total stuff} = \int_a^b \rho(x)\,dx}$$

  where the limits are the limits of the relevant physical object.