4.7 Improper Integrals

- A chemical reaction produces Phosphorus (P) at a rate according to

$$\frac{dP}{dt} = e^{-t},$$

where t is measured in nanoseconds.

- If the reaction runs for T seconds, the total chemical produced would be

$$\int_0^T e^{-t}\,dt.$$

- What if the reaction runs forever? That is, what if $T = \infty$? The total amount of chemical produced is

$$\int_0^\infty e^{-t}\,dt = \text{Total amount of P produced after a very long time.}$$

What does "∞" mean in this context? Well, ∞ never happens in the real world; what we usually mean by ∞ is "a really long time."

- Mathematically, sometimes definite integrals with limits of integration $= \infty$ are easier to evaluate. Ex: compare doing

$$\int_0^{10} e^{-t}\,dt = (-e^{-10} - (-e^0)) = 1 - e^{-10} = 0.99995$$

and

$$\int_0^\infty e^{-t}\,dt = \lim_{t\to\infty}(-e^{-t}) - (-e^0) = 1.$$

- Point is, sometimes its easier to say a limit is 0 rather than plug in an annoying number, like e^{-10}.

- To evaluate integrals with infinite limits, we use FTC like normal, and then plug in "infinity" by taking $\lim_{t\to\infty}$.

- Ex: Calculate $\int_1^\infty \frac{1}{x^2}\,dx$.

$$= -\frac{1}{x}\bigg|_1^\infty = \lim_{x\to\infty}\left(-\frac{1}{x}\right) - \left(-\frac{1}{1}\right) = 1.$$

- Ex: Consider a different chemical reaction where a quantity Q is produced at a rate
$$\frac{dQ}{dt} = \frac{1}{1+t}.$$
How much is produced?
$$\int_0^\infty \frac{1}{1+x}\,dx = \ln(1+x)\bigg|_0^\infty = \lim_{x\to\infty}\ln(1+x) - \ln(1) = \infty.$$
The point is, we need to be careful when applying the mathematical technique "infinity = long time."

Convergence

- We say $\int_a^\infty f(x)\,dx$ *converges* if it is a finite number. Otherwise, we say it diverges if it is infinite.

- One class of functions its easy to decide convergence of is power functions: $\frac{1}{x^p}$.

- $\int_1^\infty \frac{1}{x}\,dx$ diverges, $\int_1^\infty \frac{1}{x^2}\,dx$ converges.

- In general, $\int_1^\infty \frac{1}{x^p}\,dx$ converges if $p > 1$, and diverges if $p \le 1$.

- Ex.
$$\int_1^\infty \frac{1}{x^{1.2}}\,dx = \frac{1}{-0.2}x^{-0.2}\bigg|_1^\infty$$
is finite, so it converges.

- We can also look at integrals of the form

$$\int_0^1 \frac{1}{x^p}\,dx.$$

These converge when $p < 1$ and diverge when $p \geq 1$. Ex:

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \int_0^1 \frac{1}{0.5} x^{0.5}\Big|_0^1 = 0.5.$$

$$\int_0^1 \frac{1}{x}\,dx = \ln(x)|_0^1 = -\infty.$$

Summary: at $x = 0$, $\frac{1}{x^p}$ is divergent for $p > 1$. at ∞, $\frac{1}{x^p}$ is bad for $p < 1$. Also, $\frac{1}{x}$ is always bad.

Determining Convergence

In applications, you'll get an improper integral by integrating some rate of change. The procedure for dealing with it then is:

1. Determine if it converges.
2. Plug it into a computer to evaluate if its finite.

- Technique for determining convergence: We can (more or less) determine if $\int_0^\infty f(x)\,dx$ converges by comparing to well-known functions.

- Ex:

$$\int_0^\infty \frac{1}{1+x^2}\,dx$$

Even if we don't know what the number is, we can compare the integrand to $\frac{1}{x^2}$, and so we should expect this integral converges because $\frac{1}{x^2}$ does.

- Ex: $\int_1^\infty \frac{3x}{19x^2-x}\,dx$. This function, as $x \to \infty$, behaves like $\frac{1}{x}$ (you look at the ratio of leading terms of the function). Thus, this integral likely diverges.