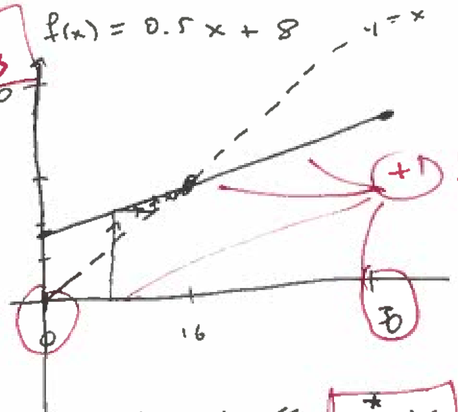
WA 7

$\frac{60}{60}$

with 12 possible points.

1) ⓐ $\boxed{3/3}$ $f(x) = 0.5x + 8$



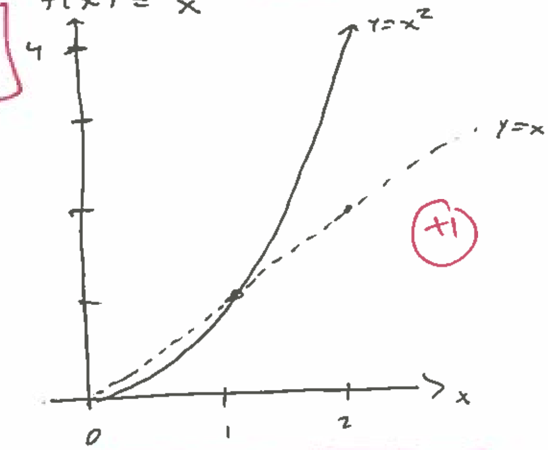
+1 for whole pic

equilibrium: $\boxed{c^* = 16}$ +0.5

$\boxed{|f'(c^*)| = 0.5 < 1}$ +1

so it's $\boxed{\text{stable.}}$ +.5

ⓑ $\boxed{5/5}$ $f(x) = x^2$



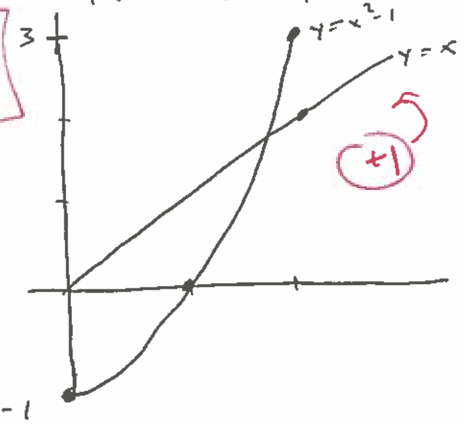
+1

eq: $\boxed{m^* = 1, \quad m^* = 0}$ + .5 each

$f'(x) = 2x$

- $|f'(0)| = 0 < 1$ so $\boxed{m^* = 0 \text{ is stable}}$ +1.5
- $|f'(1)| = |2| = 2 > 1$ so $\boxed{m^* = 1 \text{ is unstable}}$ +1.5

ⓒ $\boxed{3/3}$ $f(x) = x^2 - 1$



+1

$x^2 - 1 = x$

$x^2 - x - 1 = 0$

$x = \frac{-1 \pm \sqrt{5}}{2}$

$x = \frac{1+\sqrt{5}}{2}$

$b^* = \boxed{x = \cancel{\quad}}$ +0.5

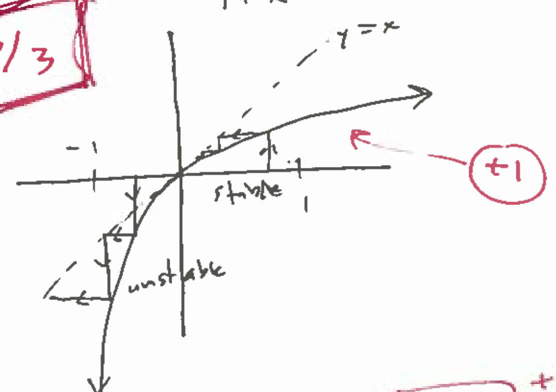
$|f'(b^*)| = |2 \cdot (\frac{-1+\sqrt{5}}{2})|$
$= |-1+\sqrt{5}| > 1$

So $b^* = \frac{+1 + \sqrt{5}}{2}$ is unstable.

Still unstable

+1.5

ⓓ $\boxed{3/3}$ $f(x) = \frac{x}{1+x}$



+1

equilibrium at $\boxed{x = 0.}$ +.5

$f'(x) = \frac{(1+x)(1) - (1)(x)}{(1+x)^2} = \frac{1}{(1+x)^2}$

$|f'(0)| = 1$, so stability theorem is inconclusive.

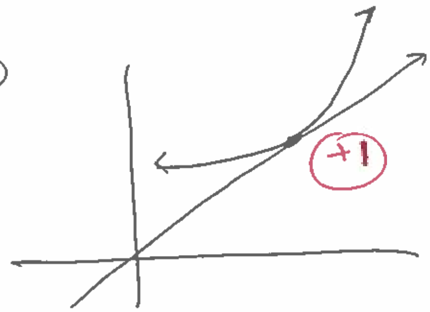
+1.5 cobwebbing shows it's stable from right, unstable to left.

★ give points as long as it is clear that they know stability theorem doesn't apply

2

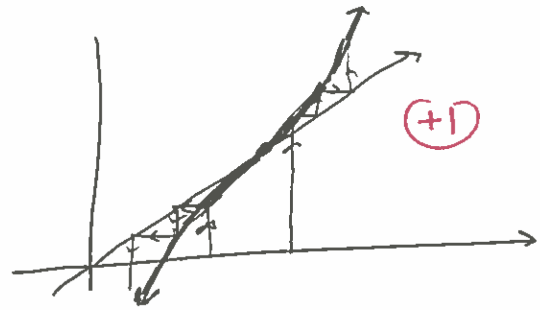
3/3

(a)



+1

(b)



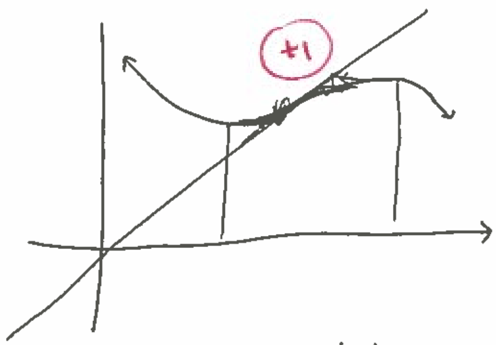
+1

* equilibrium unstable

* second derivative is zero

* if function isn't tangent to the line $y=x$, take off .5.

(c)



+1

* eq. is stable.

* f'' is zero

3

6/6

$$f'(x) = \frac{[rx+s(1-x)]\cdot r - [r-s]\cdot rx}{[rx+s(1-x)]^2}$$

$$f'(x) = \frac{\cancel{r^2x} + sr - \cancel{rsx} - \cancel{r^2x} + \cancel{rsx}}{[(r-s)x+s]^2}$$

$$\boxed{f'(x) = \frac{sr}{[(r-s)x+s]^2}}.$$

or unsimplified is ok. +2

$$|f'(0)| = \left|\frac{sr}{s^2}\right| = \boxed{\frac{r}{s} > 1} \quad \text{since } r>s.$$ +1

so $x=0$ is $\boxed{\text{unstable}}$. +1

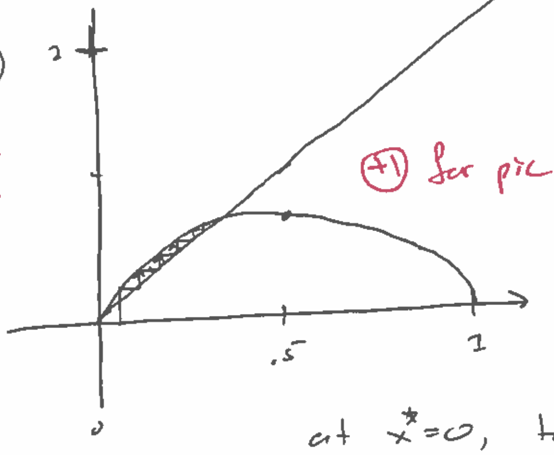
$$|f'(1)| = \left|\frac{sr}{r^2}\right| = \boxed{\frac{s}{r} < 1} \quad \text{since } r>s.$$ +1

so $x=1$ is $\boxed{\text{stable.}}$ +1

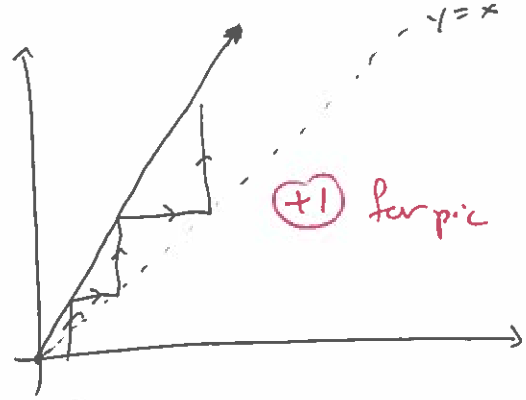
4

$\frac{7}{7}$

(a)



(+1) for pic



(+1) for pic

at $x^* = 0$, tangent line is $y = 1.5x$.

$f'(x) = 1.5 - 3x$

$\boxed{|f'(0)| = |1.5| > 1 \implies \text{unstable}}$ ($x = 0$ unstable) (can see in both pictures)

+2

(b) tangent line at $x^* = 1/3$:

$f'(\frac{1}{3}) = 1.5 - 3(\frac{1}{3}) = 0.5$.

$y = 0.5x + b$

$f(\frac{1}{3}) = 1.5 \cdot \frac{1}{3}(1 - \frac{1}{3})$

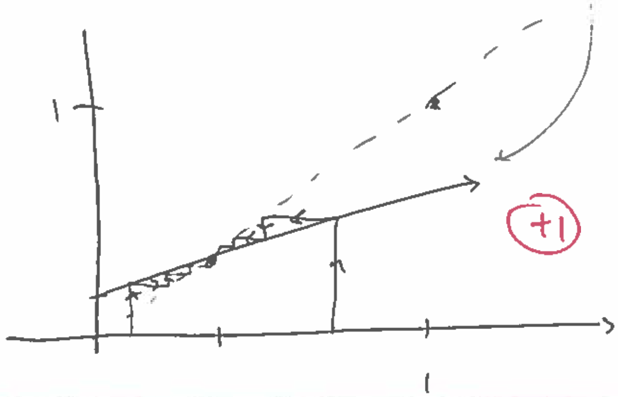
$= \frac{3}{2} \cdot \frac{1}{3}(\frac{2}{3}) = \frac{1}{3}$.

$\frac{1}{3} = 0.5\frac{1}{3} + b$

$\frac{1}{2} \cdot \frac{1}{3} = b$

$\frac{1}{6} = b$

$\boxed{y = 0.5x + \frac{1}{6}}$



(+1)

$\boxed{|f'(\frac{1}{3})| = 0.5 < 1 \text{ so } x^* = \frac{1}{3} \text{ is } \underline{\text{stable}}.}$

(+2)

5

(a) $f(x) = 1+2x-x^2 \quad , \quad 0 \le x \le 2$

6/6

$f'(x) = 2-2x$

$f''(x) = -2.$

Endpoints: $\boxed{f(0) = 1}$ +1

$\boxed{f(2) = 1+4-4 = 1.}$ +1

Crit pts: $f'(x) = 0$

$2-2x = 0$

$\boxed{x = 1}$ +1 $\rightarrow$ $\boxed{f(1) = 1+2-1 = 2}$ +1

Global max: $(1, 2)$ +1

Global min: $(0, 1)$ and $(2, 1).$ +1

(b) $g(t) = \dfrac{t}{1+t^2} \quad , \quad 0 \le t \le 2$

6/6

$g'(t) = \dfrac{(1+t^2) - t(2t)}{(1+t^2)^2} = \dfrac{1-t^2}{(1+t^2)^2}$

Crit pts: $g'(t) = 0$

$\dfrac{1-t^2}{(1+t^2)^2} = 0$

$1-t^2 = 0 \Rightarrow t = \pm 1.$ only $t = 1$ in our interval. +1

$g(1) = \dfrac{1}{1+1^2} = \boxed{\dfrac{1}{2}}$ +1

Endpoints: $\boxed{g(0) = 0}$ +1

$\boxed{g(2)} = \dfrac{2}{1+2^2} = \dfrac{2}{5} = \boxed{0.4}$ +1

Global min: $(0, 0)$ +1

Global max: $(1, \frac{1}{2})$ +1

(c) $h(r) = r^3 e^{-r}$ on $[0, \infty)$.

$\boxed{5/5}$ $h'(r) = 3r^2 e^{-r} - r^3 e^{-r} = (3r^2 - r^3)e^{-r}$

Crit pts: $h'(r) = 0$

$(3r^2 - r^3)\cancel{e^{-r}} = 0$

$3r^2 - r^3 = 0$

$r^2(3-r) = 0$

$\boxed{r = 0 \text{ or } r = 3}$ +1 (.5 each)

$\boxed{h(0) = 0}$ +1

$\boxed{h(3) = 3^3 \cdot e^{-3} = 1.344}$ +1

$\lim_{r\to\infty} r^3 e^{-r} = 0$ (by graph)

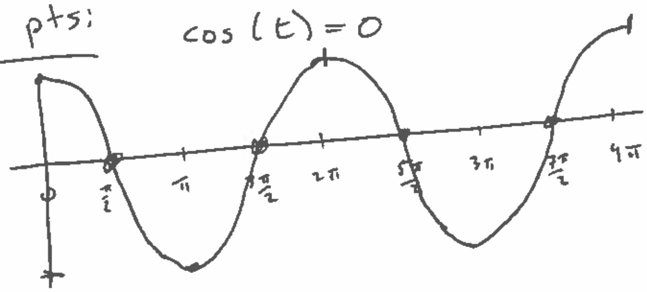
so: global max at $(3, 1.344)$ +1
global min at $(0,0)$ +1

Maybe they arrive at results by other method, so don't grade this part.

(d) $s(t) = \sin(t)$. on $[0, 4\pi]$.

$\boxed{6/6}$ $s'(t) = \cos(t)$.

Crit pts: $\cos(t) = 0$



Four places:

$\boxed{t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}.}$

+0.5 each

Values:

$s(0) = \sin(0) = 0$ } endpoints
$s(4\pi) = \sin(4\pi) = 0$

$s(\frac{\pi}{2}) = 1$

$s(\frac{3\pi}{2}) = -1$

$s(\frac{5\pi}{2}) = 1$

$s(\frac{7\pi}{2}) = -1$

+2 total

Global max: $(\frac{\pi}{2}, 1)$, $(\frac{5\pi}{2}, 1)$ +1

Global min: $(\frac{3\pi}{2}, -1)$, $(\frac{7\pi}{2}, -1)$ +1

$\boxed{6}$ (a) $f(x) = e^{-x^2}$ $\qquad f'(x) = -2xe^{-x^2} = 0$

7/7

$\Rightarrow$ $x = 0$ +1

$$f''(x) = -2e^{-x^2} + 4x^2e^{-x^2}$$
$$= (-2 + 4x^2)\,e^{-x^2}$$

$\boxed{f''(0) = -2 < 0}$ + 1

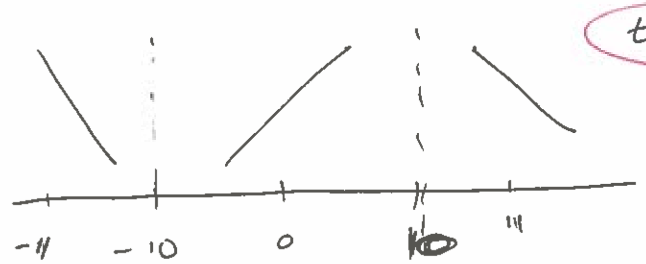
$\Rightarrow$ concave down at crit pt $\Rightarrow$ $\boxed{x=0 \text{ a local max.}}$

+1

(b) $g(t) = 10 + \dfrac{7t}{100 + t^2}$

$$g'(t) = \frac{(100+t^2)\cdot 7 - 7t(2t)}{(100+t^2)^2} = \frac{700 - 7t^2}{(100+t^2)^2} = 0$$

$$700 - 7t^2 = 0$$
$$700 = 7t^2$$
$t = \pm 10.$ + 2



$g'(-11) < 0$ $\qquad g'(0) > 0$ $\qquad g'(11) < 0$

$\Rightarrow x = -10$ a local min +1

$x = +10$ a local max. +1

or by using 2nd derivative.

$\boxed{7}$ EC:

$$b_{t+1} = 2(1-p_t)\, b_t$$
$$m_{t+1} = (1+p_t)\, m_t$$

$\boxed{\text{+12 EC possible}}$

(a) $\boxed{2/2}$ per capita production for slow zombies $= 2\cdot(1-p_t)$ +1

for fast zombies $= (1+p_t)$ +1

(b) $\boxed{2/2}$ If pop. is mainly fast zombies, so $p_t \approx 1$, per capita prod. for fast zombies ≈ 2, while for slow zombies it is about 0. So fast zombies take over and slow zombies stop reproducing (but don't die out). +1

If pop. mainly slow zombies, $p_t \approx 0$, the slow zombies reproduce quickly while fast reproduce slowly. +1

(c) $\boxed{2/2}$

$$p_{t+1} = \frac{m_{t+1}}{b_{t+1} + m_{t+1}} = \frac{(1+p_t)\, m_t}{2(1-p_t)\, b_t + (1+p_t)\, m_t}$$

$$\boxed{p_{t+1} = \frac{(1+p_t)\, p_t}{2(1-p_t)(1-p_t) + (1+p_t)\, p_t}}$$ +2

(d) $f(x) = \dfrac{(1+x)\,x}{2(1-x)^2 + x(1+x)} = x.$

3/3

$x = \boxed{0}$ is one.

+1 $\Rightarrow$ $\dfrac{1+x}{2(1-x)^2 + x + x^2} = 1$

$1 + \cancel{x} = 2(1 - 2x + x^2) + \cancel{x} + x^2$

$1 = 2 - 4x + 2x^2 + x^2$

$0 = 1 - 4x + 3x^2$

$x = \dfrac{4 \pm \sqrt{16 - 4(3)(1)}}{6}$

$x = \dfrac{4 \pm \sqrt{4}}{6} = \dfrac{4 \pm 2}{6} = \boxed{1}$ and $\frac{2}{6} = \boxed{\frac{1}{3}}$

+1 +1

equilibria: $x = 0, \frac{1}{3}, 1.$

weird!

(e) $f(x) = \dfrac{x + x^2}{2 - 3x + 3x^2}$

3/3

$f'(x) = \dfrac{(2 - 3x + 3x^2)(2x+1) - (x+x^2)(6x-3)}{(2 - 3x + 3x^2)^2}$

$\boxed{|f'(0)| = \dfrac{(2)(1) - 0}{2^2} = \frac{1}{2} < 1}$ +1 so $p^* = 0$ is stable.

$\boxed{|f'(\tfrac{1}{3})| = 1.5 > 1}$ +1 so $p^* = \frac{1}{3}$ is unstable.

tedious

$|f'(1)| = \dfrac{(2 - 3 + 3)(3) - (2)(3)}{2^2} = \boxed{0 < 1}$ +1

$p^* = 1$ is very stable