Final Exam Overview Solutions

True/False (Clearly indicate whether each of the following statements is True (T) or False (F).)

1. $\lim_{x\to 1^+} \sqrt{1-x}$ *does not exist.*

   True. For values of x larger than 1, the expression $\sqrt{1-x}$ is not a real number. So with no values larger than 1 to evaluate, the limit does not exist.

2. *If V is a function of y and y is a function of x, then $V'(x) = V'(y) \cdot y'(x)$.*

   True. This is the chain rule for derivatives.

3. *The function $C(x) = \sin(\pi x)$ has a period of π.*

   False. The period of a sine function of the form $\sin(Bx)$ is $\frac{2\pi}{B}$. In this case, $B = \pi$, so the function has period $\frac{2\pi}{\pi} = 2$.

4. $N_{t+1} = a(1 - N_t) + b(N_t + 1)$, *for constants a and b, is a linear discrete-time dynamical system.*

   True. We can write the system as $N_{t+1} = a(1 - N_t) + b(N_t + 1) = a - aN_t + bN_t + b = (b - a)N_t + (a + b)$, which is the form of a generic linear system $N_{t+1} = mN_t + c$.

5. An equilibrium of the system defined by $D_{t+1} = pD_t + q$ is stable exactly when $p < 1$.
   False. This will be unstable if $p < -1$; for example, $p = -2$ is less than 1, but will lead to an unstable equilibrium.

Multiple Choice (Choose the best answer from among the choices given)

6. l'Hôpital's rule can be used to determine the value of the limit as long as it is of the form...

   (a) $\frac{0}{0}$
   (b) $\frac{\infty}{\infty}$
   (c) $\frac{\infty}{0}$
   (d) $\boxed{\text{Only (a) and (b).}}$
   (e) All of the above.

7. The number of predators P of a species of ape is a function of number of male apes a in the group. The number of viable mates M for the apes is also a function of a. Using these functions, the most reasonable expression that the apes would want to maximize is...

   (a) $P(a) \cdot M(a)$
   (b) $P(a) - M(a)$
   (c) $M(a) + P(a)$
   (d) $\dfrac{P(a)}{M(a)}$
   (e) $\boxed{\dfrac{M(a)}{P(a)}}$ This is the only expression that increases when there are more mates and fewer predators, both desirable properties from the perspective of the apes.

8. The units on M are dollars while the units of radiation r are in milli-Curie's (mCu). Then the units on $M'(r)$ would be...

   (a) \$

(b) mCu

(c) $\boxed{\$ \text{ / mCu}}$ The units on the derivative of a function are consistently "units of output divided by units of input".

(d) mCu / $

Free Response

(Write your answers clearly and concisely, including all work. If asked to explain something, use complete sentences. Any numerical answers may be written in approximate form as long an exact solving method is used.)

9. In a Ricker model for population growth, the discrete-time dynamical system is $p_{t+1} = 4p_t e^{-p_t}$. Find any equilibria of this system and classify them as stable or unstable using the slope criterion.

   Start by writing down the updating function:

$$f(x) = 4xe^{-x}.$$

   Equilibria satisfy $f(x) = x$, so $4xe^{-x} = x$ is the equation to solve. One solution is $x = 0$; then, after dividing by x, we see that

$$\begin{aligned} 4e^{-x} &= 1 \\ e^{-x} &= \frac{1}{4} \\ x &= -\ln\left(\frac{1}{4}\right) = \ln(4) \approx 1.386. \end{aligned}$$

   Now, $f'(x) = 4e^{-x} - 4xe^{-x} = (4 - 4x)e^{-x}$. Thus,

$$|f'(0)| = |4e^0| = 4 > 1,$$

   so $p^* = 0$ is an unstable equilibrium. On the other hand,

$$|f'(\ln(4))| = |(4 - 4\ln(4))e^{-\ln(4)}| = |(4 - 4\ln(4))\frac{1}{4}| = |(1 - \ln(4))| \approx |-0.386| = 0.386 < 1,$$

   so $p^* = \ln(4)$ is a stable equilibrium.

10. Let $f(x) = 0.1e^x \sin\left(\frac{\pi}{4}x\right)$ on the interval $[0, 5]$.

(a) Does the extreme value theorem apply to this situation? What do we learn from it? Find the global maxima and minima if they exist.

The EVT does apply in this situation; the function is continuous (no corners, no jumps, and certainly no asymptotes), and we are on a closed and bounded interval. We learn that f must attain its global extrema.

Find critical points:

$$f'(x) = 0.1e^x \frac{\pi}{4} \cos\left(\frac{\pi}{4}x\right) + 0.1e^x \sin\left(\frac{\pi}{4}x\right) = 0$$

You cannot solve this by hand, but using a calculator or computer you can find the zeros to be $x = 0$ and $x \approx 3.152$.

Check y-values:

$$f(0) = 0, \quad f(3.152) \approx 1.445, f(5) = -10.49,$$

so the global minimum is at $(5, -10.49)$ and the global maximum is at $(3.152, 1.445)$.

(b) What if we took our interval to instead be $[1, \infty)$? That is, does the extreme value theorem still apply? Does f have a global maximum on this interval?

The EVT no longer applies, so we may not have a global maximum or minimum. In fact, this function oscillates up and down, and the peaks and valleys get increasingly high and low; this means that f will not have either a global max or a global min. (If you want to graph this and see for yourself, you will probably have to mess around with the window settings because the function's height grows very quickly.)