Section Goals:

- Identify a phenomenon having constant second differences as quadratic relationship
- Compare characteristics of a quadratic and a linear function
- Find the vertex of a parabola and interpret its value as an extremum
- Explore the definition of a higher-order polynomial function
- Classify the long-term behavior of a polynomial function

Examples (Ex) are guided by your instructor.
Focus problems are intended to be attempted at first on your own, and then in collaboration with $1-3$ other students. I will answer questions from groups, but not individuals, in order to encourage you to use one another as resources.

Def A **quadratic function** is a function which can be written in the form

$$Q(t) = at^2 + bt + c,$$

with constants a, b, and c (where a is not zero).

Def Consider a function $f(t)$ with inputs $a < b < c$. The **second difference** between these three values is

$$[f(c) - f(b)] - [f(b) - f(a)].$$

Thm (Quadratic Models) Any phenomenon which exhibits nonzero, constant second differences *between inputs a fixed distance apart* can be fit to a quadratic model.

Ex 1 Consider the table of values describing the height (in feet) of an object t seconds after being launched into the air.

a) Compute the average rate of change between each adjacent point, then fill out the table to the right.

b) Compute the *second differences*, that is, the difference between adjacent rates of change. Then fill out the relevant column in the table. What do you notice about these values?

t	$h(t)$	ARC	2nd Diff
0	5		
		64	
1	69		-32
		32	
2	101		-32
		0	
3	101		-32
		-32	
4	69		-32
		-64	
5	5		

c) What conclusion can you draw about the model for height of the object as a function of time?
The motion is very well approximated by a quadratic model.

$\boxed{\text{Thm}}$ (The Quadratic Formula) An equation of the form $0 = at^2 + bt + c$ has solutions

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The solutions are non-real if the quantity $b^2 - 4ac$ under the radical is negative.

Ex 2 Find all real solutions to the equation $3x - 2x^2 = -7$.

Move everything to one side to get $3x - 2x^2 + 7 = 0$. Then, rearrange so that the quadratic term is first: $-2x^2 + 3x + 7 = 0$. Then,

$$\begin{aligned} t &= \frac{-3 \pm \sqrt{3^2 - 4(-2)(7)}}{2(-2)} \\ &= \frac{-3 \pm \sqrt{65}}{-4} \\ &= \frac{3 \pm \sqrt{65}}{4}, \end{aligned}$$

so the two roots are

$$t = \frac{3 + \sqrt{65}}{4} \approx 2.766 \qquad \text{and} \qquad t = \frac{3 - \sqrt{65}}{4} \approx -1.266$$

Thm (Extremum of a Quadratic Function) A quadratic function $f(t) = at^2 + bt + c$ has either a maximum or minimum located at

$$t_{\text{vert}} = -\frac{b}{2a}.$$

The quadratic function $f(t)$ has a $\begin{bmatrix} \text{maximum} \\ \text{minimum} \end{bmatrix}$ if $\begin{bmatrix} a < 0 \\ a > 0 \end{bmatrix}$.

(*Extremum* is a fancy word for "either a maximum or a minimum")

Def The graph of a quadratic function $f(t)$ is called a **parabola**. It is symmetric about the vertical line through the vertex: $t = -\frac{b}{2a}$.

Ex 3 Find the vertex and axis intercepts of $g(t) = 2t^2 - 6t + 1$, then sketch a graph of the associated parabola.
The y-intercept is when $t = 0$, so $g(0) = 2(0^2) - 6(0) + 1 = 1$ is the y-intercept.

To get the t-intercepts, use the quadratic formula:

$$t = \frac{6 \pm \sqrt{6^2 - 4(2)(1)}}{2(2)} = \frac{6 \pm \sqrt{28}}{4} \approx 0.177, 2.823.$$

Here's a picture:

Thm (Vertex Form of a Quadratic) A quadratic function written in the form

$$Q = f(t) = a(t - t_{\text{vert}})^2 + Q_{\text{vert}}$$

has its maximum or minimum value at the point $(t_{\text{vert}}, Q_{\text{vert}})$.

Ex 4 An object is thrown upwards with an initial velocity v_0 and initial height h_0. It's height h (in feet) as a function of time t (in seconds) will be

$$h(t) = -16t^2 + v_0 t + h_0.$$

(The 16 has to do with gravity; in fact, if we did everything in SI units with meters instead of feet, we would use $9.8m/s^2$, which you may recognize from physics as the acceleration due to gravity.)

Suppose an object is thrown upwards at four feet per second, with an initial height of three feet.

a) When does the object reach its maximum height? We need to find the time when the parabola is maximized; in other words, the vertex. The equation for the parabola is

$$h(t) = -16t^2 + 4t + 3.$$

So,

$$t_{\text{vert}} = -\frac{b}{2a} = -\frac{4}{2(-16)} = \frac{1}{8} = 0.125 \text{ seconds.}$$

Therefore the object reaches its maximum height after 0.125 seconds (or an eigth of a second).

b) How high does the object go? We need to realize that we are asking for the maximum of the parabola. To get this, calculate $h(t_{\text{vert}})$:

$$h(t_{\text{vert}}) = h(0.125) = -16(0.125)^2 + 4(0.125) + 3 = 3.25 \text{ feet.}$$

Not very high.

c) When does the rocket hit the ground?
To do this, we need to find when $h(t) = 0$, as the ground is assumed to be where the height is zero. Well, we have to solve

$$h(t) = -16t^2 + 4t + 3 = 0.$$

Quadratic formula gives

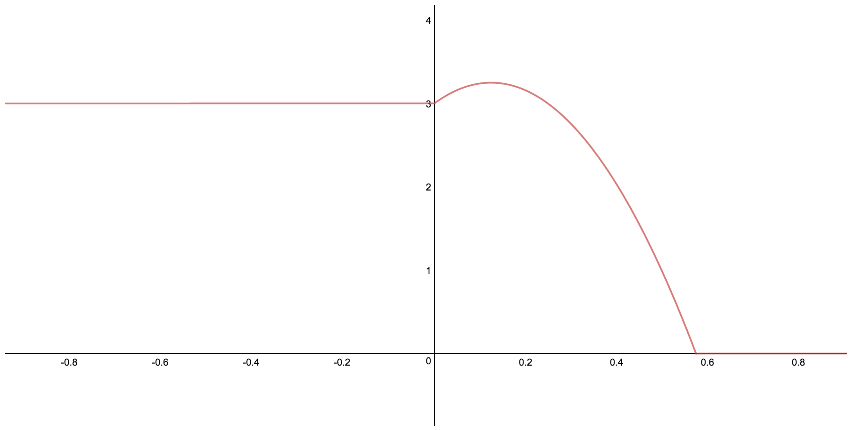
$$\begin{aligned} t &= \frac{-4 \pm \sqrt{4^2 - 4(-16)(3)}}{2(-16)} \\ &= \frac{-4 \pm \sqrt{208}}{-32} \\ &= 0.576 \text{ and } -0.326. \end{aligned}$$

We get two answers because mathematially the quadratic has two roots. But in reality, we only care about the future after we throw the object, so we should take the positive answer.
Thus, the object lands at $t = 0.576$.

d) Write the equation of the object's height for all times. We are going to write a piecewise function. For $t < 0$, the object can be assumed to be at the height we release it at. For $0 \leq t \leq 0.526$, the height follows the quadratic we showed earlier. For $t > 0.526$, the object is resting at 0. We can encode this with a piecewise function:

$$h(t) = \begin{cases} 3 & t < 0 \\ -16t^2 + 4t + 3 & 0 \leq t \leq 0.526 \\ 0 & t > 0.526 \end{cases}$$

Here's a picture of what it looks like:


4
3
2
1
0
-0.8
-0.6
-0.4
-0.2
0.2
0.4
0.6
0.8

Ex 5 A farmer wants to build a rectangular fence. One side costs \$30 per foot, while the other side costs \$50 per foot to build. He has \$1400 to spend. What dimensions should his fence be to maximize the area enclosed?

Let's call the cheap side "x" and the expensive side "y." We can write down two equations, one for the area, and one for the total cost.

$$\begin{aligned} \$1400 &= 2(30)x + 2(50)y \\ A &= xy. \end{aligned}$$

There are 2's in the first equation because we have to build two sides. Our strategy now is to solve for one of the variables in the first equation to use in the second. Let's solve for y.

$$\begin{aligned} 1400 &= 60x + 100y \\ 14 &= 0.6x + y \\ y &= 14 - 0.6x \end{aligned}$$

Plug in to the other:

$$A = xy = x(14 - 0.6x) = -0.6x^2 + 14x.$$

A quadratic emerges! We want to maximize this, so find x_{vert}:

$$x_{\text{vert}} = -\frac{b}{2a} = -\frac{14}{2(-0.6)} = \frac{7}{0.6} = 11.7 \text{ feet.}$$

So, we have that $x = 11.7$ feet. We need both dimensions of the yard; luckily we solved for y already:

$$y = 14 - 0.6x = 14 - 0.6(11.7) = 7.$$

So, the farmer should make a 7-by-11.7 foot yard, where the 7 feet is the expensive side.

Def A **monomial** is an expression of the form at^b, where a is any constant (called its **coefficient**) and b is a non-negative whole number (called its **degree**).

A **polynomial** is a sum or difference of any number of monomials (including just one).

A **polynomial function** is a function whose formula can be written as a polynomial.

The **leading term** of a polynomial is the term containing the highest power on t.

The **leading coefficient** of a polynomial is the coefficient of the leading term.

Ex 6 In each case, identify whether or not the function is polynomial. If so, write its degree, leading term, and leading coefficient.

a) $f(x) = \pi x^6 - 4x^4 - 3x^9$
Yes! Degree: 9
Leading Term: $-3x^9$
Leading Coeff.: -3

b) $g(t) = t^4 + 8t^3 - 3t + t^{1/3}$
Not a polynomial! Cannot have fractional/decimal powers

c) $h(n) = (n-7)(n^2+1)(2n+1)^2$
Yes! After doing algebra, it will be a sum of monomials. Degree 5, Leading term: $4n^5$, Leading Coeff.: 4.

Ex 7 Consider the function

$$P(t) = 6.75t^3 - 1639t^2 + 109732t - 471072,$$

which gives the approximate population of Detroit , Michigan t years after 1900.

a) Use technology to sketch a graph of $Q = P(t)$ on the interval $[0, 110]$.

b) The censused population of Detroit in 2010 was 713,777 individuals. How well does that compare to the model's prediction of the population in that year? $P(120) = 759,168$, which agrees pretty well (but not exactly) with the observed value.

c) In what year(s) does the model predict that nobody lives in Detroit? (You may use technology to find this value) Technology gives $t \approx 4.6$, meaning about halfway through 1906. For this reason, the model seems inaccurate early on, in the 1900's.

Def The **long-term behavior** or **long-run behavior** of a function $Q = f(t)$ is the value that Q approaches when t approaches infinity. That is, t gets larger and larger with no ceiling.

This trend in values of Q could be: a real number, ∞, $-\infty$, or the value could not exist.

Thm (Long-Term Behavior of Polynomials) The leading term of a non-constant polynomial uniquely determines its long-term behavior.

If the leading coefficient is $\begin{bmatrix} \text{positive} \\ \text{negative} \end{bmatrix}$, then the function tends toward $\begin{bmatrix} \infty \\ -\infty \end{bmatrix}$ as t increases without bound.

"Tends toward infinity" or "increases without bound" are equivalent ways of says that the value keeps getting bigger and bigger with no ceiling. Similarly, "tends toward negative infinity" or "decreases without bound" mean that the value keeps getting more and more negative with no bottom.

Ex 8 What is the long-term behavior of $P(t)$ from Example 7? Interpret this in the context of the model.

The leading coefficient is 6.75, which is positive. So, $P(t)$ tends toward infinity as t goes to infinity. In other words, the population grows without bound as time goes on, according to the model.