General Questions/Things You Should be Familiar With

- Finding exponential equations
- Big-Little Principle
- What is a polynomial? What is a rational function? What's the relationship between them?
- algebra related to exponents and fractions (Exponent rules, fraction rules)
- What is a logarithm? What is the defining property of logarithms? How do they relate to exponential functions? How do we use them to solve equations?
- Algebra of logarithms (i.e "log rules")

Practice Problems

1. Let $r(t) = \dfrac{10t^6 + 7t^4 + 9t + 10t^9}{7.3t^2 - 2t - 9}$ and $s(t) = \dfrac{10t^6}{90t - 10t^7}$. Find the domains of each function, and compute the long-run behavior of each as $t \to \infty$ and $t \to -\infty$.

   For $r(t)$, we need to have the denominator be nonzero. Use the quadratic formula to find when this happens:

$$t = \frac{2 \pm \sqrt{4 - 4(7.3)(-9)}}{2(7.3)} \approx -0.982, 1.256.$$

   So, the domain is $(-\infty, -0.982) \cup (-0.982, 1.256) \cup (1.256, \infty)$.

   For $s(t)$, we need the denominator to be nonzero. This happens when $90t - 10t^7 = 0$, which is the equation

$$t(90 - 10t^6) = 0.$$

   So, the denominator will be 0 when $t = 0$ and when $90 - 10t^6 = 0$, which gives the answers $t = \pm 9^{1/6} \approx \pm 1.44$. Hence, the domain is

$$(-\infty, -1.44) \cup (-1.44, 0) \cup (0, 1.44) \cup (1.44, \infty).$$

   Long run behavior of $r(t)$: The leading term up top is $10t^{10}$, the leading term on bottom is $7.3t^2$, so the long run behavior is given by

$$\frac{10t^9}{7.3t^2} = \frac{10t^9}{7.3} \to \infty \text{ as } t \to \infty$$

   and

$$\frac{10t^9}{7.3} \to -\infty \text{ as } t \to -\infty.$$

   Similarly, $s(t)$'s long run behavior is determined by $10t^6/10t^7 = 1/t$, and so $s(t) \to 0$ as $t \to \pm\infty$ (it is the same in either direction).

2. Suppose you have a bank account with an initial value of $\$5,000$, which grows to a value of $\$13,000$ over the course of 10 years.

   (a) If the interest is compounded semiannually, then what was the interest rate attached to the account?
   Set up the equation:

$$\begin{aligned} 13 &= 5\left(1+\frac{r}{2}\right)^{2(10)} \\ \left(\frac{13}{5}\right)^{1/20} &= 1+\frac{r}{2} \\ r &= .09787. \end{aligned}$$

   (b) If the rate is as given in part (a), then what is the account worth after 15 years since the initial deposit?

$$V = 5000\left(1+\frac{.09787}{2}\right)^{2(15)} \approx 20,961.87.$$

   (c) Assuming the same rate from part (a), how long does it take for the account to reach \$30,000?

$$\begin{aligned} 30000 &= 5000\left(1+\frac{.09787}{2}\right)^{2t} \\ 6 &= (1.04894)^{2t} \\ \ln(6) &= \ln\left((1.04894)^{2t}\right) \\ \ln(6) &= 2t\ln(1.04894) \\ t &= \frac{1}{2}\cdot\frac{\ln(6)}{\ln(1.04894)} \approx 18.75005 \approx 18.75 \text{ years.} \end{aligned}$$

3. Find the equation of an exponential function going through the points $(100, 95)$ and $(10, 2)$.
   Set $f(t) = ab^t$. We have two equations:

$$\begin{aligned} ab^{100} &= 95 \\ ab^{10} &= 2. \end{aligned}$$

   Divide:

$$\begin{aligned} \frac{ab^{100}}{ab^{10}} &= \frac{95}{2} \\ b^{90} &= \frac{95}{2} \\ b &= \left(\frac{95}{2}\right)^{1/90} \approx 1.0438 \end{aligned}$$

   Then,

$$a = \frac{2}{b^{10}} = 2\left(\frac{2}{95}\right)^{10/90} \approx 1.302$$

4. Suppose that a bacteria colony grows with a continuous growth rate of 10% an hour. Devise an exponential model that describes such a bacteria colony that initially has 10 bacteria. Then model a bacteria colony that increases by 10% each hour, again with initially 10 bacteria. Which has a higher population after 10 hours? Decide which kind of growth rate (continuous or hourly) describes faster growth.

The keyword that should jump out at you is "continuous." In this case, we must use the number e as the base. We have

$$f(t) = ae^{kt} = 10e^{0.1t}$$

since we have to convert the continuous growth rate k into a decimal (it was given as a percentage). As for the other colony, we know that it increases by 10% an hour. This means that the growth factor is $1 + 0.1 = 1.1$ (if you like, this is using the "formula" $b = 1 + r$). The second colony is then described by

$$g(t) = 10(1.1)^t.$$

After 10 hours, we have

$$\begin{aligned} f(10) &= 10e^{0.1*10} = 27.18, \\ g(10) &= 10(1.1)^{10} = 25.94, \end{aligned}$$

so the first colony has a higher population. So, if we don't change the numbers, a continuous growth rate of 10% is a faster increase than a 10% hourly growth rate.

5. Suppose that you drink a cup of coffee, which has about 150 milligrams of caffeine. Assume that the half-life of caffeine in your blood stream is is about 5 hours. You can fall asleep if the amount of caffeine in your blood stream is less than 50 milligrams. If you want to take a nap after drinking the coffee, how long will you have to wait?

Start with $f(t) = 150b^t$ and find b. A half-life of 5 hours says

$$\begin{aligned} 75 = f(5) &= 150b^5 \\ 75 &= 150b^5 \\ \frac{1}{2} &= b^5 \\ b &= \left(\frac{1}{2}\right)^{1/5}. \end{aligned}$$

So, $f(t) = 150\left(\frac{1}{2}\right)^{t/5}$. Then

$$\begin{aligned} 50 &= 150\left(\frac{1}{2}\right)^{t/5} \\ \frac{1}{3} &= \left(\frac{1}{2}\right)^{t/5} \\ \ln\left(\frac{1}{3}\right) &= \ln\left(\left(\frac{1}{2}\right)^{t/5}\right) \\ \ln\left(\frac{1}{3}\right) &= \frac{t}{5}\ln\left(\frac{1}{2}\right) \\ t &= 5 \cdot \frac{\ln(1/3)}{\ln(1/2)} = 5 \cdot \frac{\ln(3)}{\ln(2)} \approx 7.925 \text{ hours.} \end{aligned}$$

To see the last equality, note that $\ln(1/3) = ln(3^{-1}) = (-1)\ln(3)$. Doing the same to the bottom gives a factor of (-1) on top and bottom, so they cancel.

6. The value of two cars, car A and car B, are depreciating at different rates. Car A depreciates at a constant rate of \$1000 each year, while car B depreciates at a continuous rate of 3% each year. Come up with a model for each car, assuming each car started with a value of \$25,000. To do this, note that one of these is linear and the other is exponential. We can tell that the value of car A is described by a linear model since it has a constant rate of change, which is pointed out by the number "1000 dollars per year." This tells you the slope of the linear function. So, we can write

$$A(t) = -1000t + 25,000$$

since the initial value of the car is \$25,000.

For car B, we note that the continuous rate of growth means that we should use an exponential model. We are then told that $k = -0.03$ where k is the continuous rate of growth, or the number k in $B(t) = ae^{kt}$. So,

$$B(t) = 25,000e^{-0.03t}.$$

Be able to distinguish these two scenarios! There is a very good chance that I ask you something like this.