Guide for Section 5 - Exponential Functions

Section Goals:

- Identify a function or phenomenon as exponential.
- Write a formula for an exponential function.
- Determine an exponential function's continuous growth rate and periodic growth rate.
- Sketch the graph of an exponential function.

Ex 1 The average thickness of a piece of paper is about 0.1 mm.

a) How thick is a piece of paper after you fold it over once? Twice? Three times?

b) Write an equation for the function, T, that gives the thickness (in mm) of a piece of paper after being folded f times (ignoring resistance in the paper).

c) After how many foldings will it take for the paper to be 25.6 mm (a little over 1 inch) thick?

d) How thick is the paper after 50 foldings?

Thm (Exponential Function) If Q is changing at a rate proportional to itself, so that $R(t) = kQ$, where R is the rate of growth in Q and k is the continuous growth rate, then

$$Q = f(t) = ae^{kt},$$

where a is a constant (which also happens to be equal to the value of Q at $t = 0$).

An exponential function changes by a factor of e^k for every unit increase in t. This is referred to as its growth factor.

Def An alternate form for an exponential function which is equivalent to the one given above is

$$f(t) = a \cdot b^t,$$

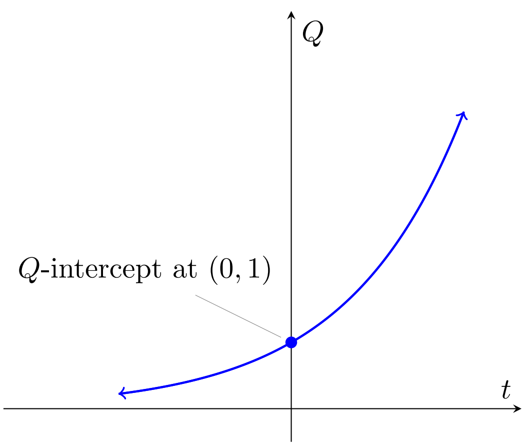
where the constant growth factor is positive value b.

Thm (Basic Exponential Function Graphs)

Exponential Growth

$$Q = f(t) = ae^{kt} = a \cdot b^t$$

$b > 1$ and $k > 0$

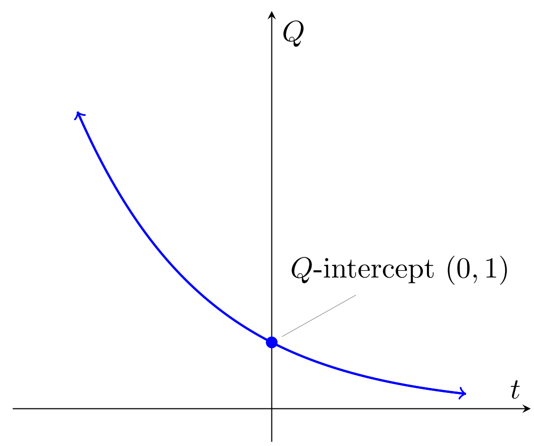


Graph rises dramatically to the right, falls toward a height of 0 to the left

Exponential Decay

$$Q = f(t) = ae^{kt} = a \cdot b^t$$

$0 < b < 1$ and $k < 0$



Graph rises dramatically to the left, falls toward a height of 0 to the right

Thm (Domain of an Exponential Function) For a function $f(t) = ae^{bt}$, with $a > 0$, we have that

Domain	Image
$(-\infty, \infty)$	$(0, \infty)$

Ex 2 Let $f(t) = 3e^{0.2t}$.

a) What is the continuous growth rate of f?

b) What is the constant growth factor of f?

c) For some real number n, $f(n) = 5$. What must be the value of $f(n+1)$?

d) Another function $V = g(t)$ has the property that V is changing at a rate proportional to the value of V, with constant of proportionality -1.4. Write an equation for $g(t)$ assuming that $g(0) = 100$.

Def An exponential function with $\begin{bmatrix} \text{negative continuous growth rate or growth factor} < 1 \\ \text{positive continuous growth rate or growth factor} > 1 \end{bmatrix}$ is $\begin{bmatrix} \text{a decreasing function} \\ \text{an increasing function} \end{bmatrix}$ and is said to exhibit $\begin{bmatrix} \text{exponential decay} \\ \text{exponential growth} \end{bmatrix}$.

Ex 3 Does the function $N(t) = 2(0.9)^t$ exhibit exponential growth or decay? What about $P(t) = 7e^{0.9t}$?

Ex 4 Consider the two functions f and g defined by the table below. What kind of functions are f and g? Write a formula for both f and g.

x	$f(x)$	$g(x)$
1	3	10
2	4.5	25
3	6.75	62.5
4	10.125	156.25

Note The above method only works if the inputs are evenly spaced by 1!

Thm If a quantity experiences a constant yearly percentage growth rate, r, then the growth factor for the exponential function is $b = 1 + r$. If the quantity is *decreasing* by a constant percentage rate, r, then $b = 1 - r$.

Ex 5 The local duck population grows by about 2.02% per year. In 2015, there were about 200,000 ducks in Eugene. What can we predict the population to be in 2020?

Def The value V, of an investment with initial value V_0, which accrues interest compounded n times per year at a (nominal) annual rate of r is worth V at the end of t years, where

$$V = V_0 \left(1 + \frac{r}{n}\right)^{nt}$$

To compare this to the older notation, $a = V_0$ and $b = \left(1 + \frac{r}{n}\right)^n$.

Ex 6 The DeHaven family, tracing lineage back to the American Revolution, claims[1] that in December 1777 their ancestor Jacob DeHaven loaned George Washington $\$450,000$ in gold and supplies which helped turn the tide of the war.

a) In 1989 (as well as several points during the 19th century), the descendants wished to claim compensation for this princely sum and assumed a 6% interest rate compounded monthly. How much did the family request as the value of the loan?

b) Citing their reasonableness, the descendants claimed that $\$100,000,000$ was a sufficient compensation. What interest rate does this amount to over the course of the loan's term from 1777 until 1989?

c) if $\$100,000,000$ was what the DeHaven family was owed "fairly" after 212 years of 6% interest compounded monthly, what does that assume the original loan value to be? (This is called the *present value* of the investment)

[1] http://www.ushistory.org/valleyforge/youasked/069.htm