Guide for Section 1 - Functions

Section Goals:

- Model functions in non-mathematical contexts.
- Determine if a relationship between two variables defines a function.
- Determine a function's input or output, given the other.
- Find the domain and image of a function.
- Evaluate and interpret values of a function defined in a table or graph.

Def A **function** is a pairing of some set of inputs and another set of outputs, along with a rule* that determines how pairing happens. Pairs are commonly written (x, y), where x is an input and y is the output paired with x.

*The only requirement of this rule is that each input yields exactly one output.

For a function named f, we write $f(\text{input}) = \text{output}$.

The set of inputs in a function is called the **domain** and the set of outputs the **image**.

The result of evaluating a function for an input not in its domain is that the function is said to be **undefined** at that value.

Ex 1 Three of my friends are Jeff, Joe, and Patrick. I want to describe a function that assigns their hair color. Jeff and Joe have brown hair, while patrick has black hair. Write the input-output pairs that define this function.

(Jeff, brown)
(Joe, brown)
(Patrick, black)

Describe the domain and image for the relation in this example. Why does the relation described define a function?

Domain: The collection of inputs. In my case, they are {Jeff, Joe, Patrick}, the people showing up in the first slot of the pairs listed above.
Image: This is the collection of outputs, which are the hair colors. The image is {Brown, Black}. Notice that Brown is only counted once when describing the image.
rule: The rule defined takes a person as an input and outputs their hair color.
This rule *does* define a function; in this case, each person (input) has exactly one unique hair color (output).

If we reverse the inputs and outputs, do we still have a function? No; (brown, Jeff) and (brown, Joe) would be two pairs; this shows that the single input brown has two outputs, Joe and Jeff. A function needs to assign a unique output, which it fails to do so here.

Ex 2 Below are recorded the new cases of Hansen's disease (leprosy) in the United States during 2010, by month.

Month	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec
# of cases	9	8	10	11	7	6	11	6	11	11	6	3

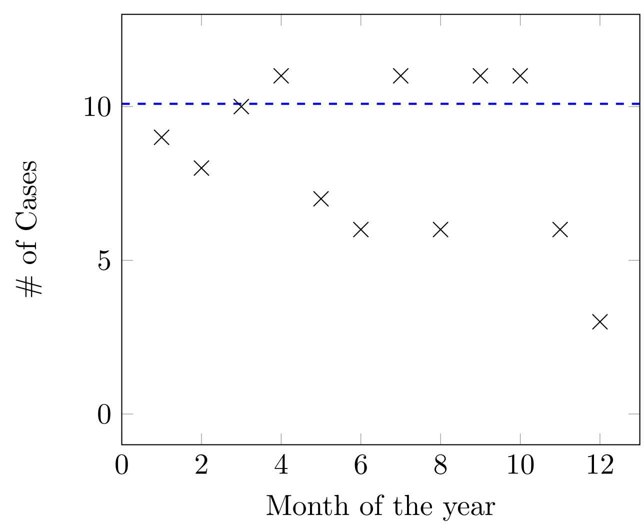
a) Is the number of new cases per month a function of the month? Explain your answer. Yes; for each month (input), there is exactly one unique number of new cases of leprosy (output).

b) Is the month a function of the number of new cases per month? Explain.
No. The number 11 of new cases would be assigned more than one output.

c) Let $N = f(t)$ be the function giving the number of new cases reported during month t of the year 2010 (January is $t = 1$, February is $t = 2$, etc.). Compute and interpret the value of $f(4)$.
Writing $f(4)$ means that $t = 4$, which is the month of April. The table says there were 11 new cases that month, so $f(4) = 11$. We interpret this as saying that "there were eleven new cases of leprosy in April."

d) The same data is presented below in a graphical format. Use the graph to determine all values of t so that $f(t) \geq 10$.

To solve $f(x) \geq 10$, we want to find the coordinates with y-value larger than or equal to 10, since $f(x)$ is the y-coordinate attached to x. The best way to do this is to draw a line at $y = 10$, and then find the data points lying above or on the line. We see four data points lie above the line, and they correspond to $x = 4, 7, 9$, and 10. So, the solutions to $f(x) \geq 10$ are $\{3, 4, 7, 9, 10\}$.



Ex 3 An individual's resting heart rate is 60 beats per minute and at the end of a run 30 minutes later it was 130 beats per minutes.

a) Sketch a plausible graph of the individual's heart rate r as a function of t, in minutes after beginning the run.

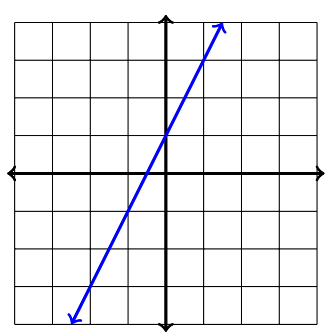
b) What is the value of $r(0)$? What is the only other value of r that we are guaranteed to know?

$r(0)$ means the heart rate at time 0, presumably their resting heart rate. We are told that this is 60 bpm, so $r(0) = 60$. We also know that their heart rate was 130 bpm about half an hour later, so we are guaranteed to know that $r(30) = 130$.

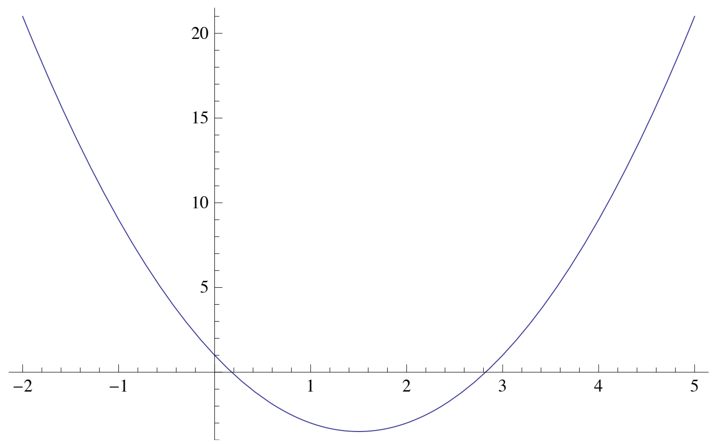
c) How many solutions are there likely to be to $r(t) = 100$? Explain.

We know there should at least be one, since we expect the person's heart rate to gradually increase from 60 to 130. However, there could be many solutions because they could stop and take breaks in the middle of the workout.

Ex 4 Consider the function $f(x) = 2x + 1$. Graph it on the axes provided below.



Ex 5 (a) With the graph of f shown below, evaluate $f(2)$ and solve the equation $f(x) = 4$.



Eyeballing from the graph, $f(2) \approx -3$. Also, there are two values of x that solve the equation $f(x) = 4$, namely $x \approx -2$ and $x \approx 3.5$.

(b) Assuming the domain of f is just $[-2, 5]$, what is the image (or range) of f? Write your answer in set notation and interval notation. To find the range, we just need to identify all outputs. Imagine squishing the graph like a pancake onto the y-axis. What values do we hit? Looks like from $y = -4$ to $y = 21$, so the range is

$$\{y \mid -4 \leq y \leq 21\} = [-4, 21].$$

Ex 6 For the function $G(x) = \sqrt{x + 10}$, find and simplify $G(a + 12)$ along with $G(t^2 - 10)$.

$$G(a + 12) = \sqrt{(a + 12) + 10} = \sqrt{a + 22}.$$

This is simplified. Note: we cannot write this answer as $\sqrt{a} + \sqrt{22}$. Square roots do not distribute over addition.

$$G(t^2 - 10) = \sqrt{(t^2 - 10) + 10} = \sqrt{t^2} = |t|.$$

The absolute value appears because if t were negative, t^2 is positive, and square-rooting gives a positive number.

$\boxed{\text{Def}}$ When the domain of a function is not specified, use the largest subset of the real numbers for which the function produces a real number.

Ex 7 Find the largest possible domain of each of the following functions. Write your answer in both set and interval notation.

a) $f(x) = \sqrt{x-7}$

$$\text{Dom}(f) = \{x \mid x \geq 7\} = [7, \infty)$$

Note: $x = 7$ is a valid input: $f(7) = \sqrt{0} = 0$. You can take a square root of 0!

b) $g(t) = \dfrac{1}{t}$

In this case, every number is a valid input except $t = 0$. So,

$$\text{Dom}(f) = \{t \mid t \neq 0\} = (-\infty, 0) \cup (0, \infty).$$

c) $h(r) = \dfrac{2^r}{r^2 - 16}$

Don't be thrown off by the numerator here! We must exclude points where the denominator is zero. This happens when $r^2 = 16$, or when $r = \pm 4$. So,

$$\text{Dom}(f) = \{r \mid r \neq -4 \text{ and } r \neq 4\} = (-\infty, -4) \cup (-4, 4) \cup (4, \infty)$$

$\boxed{\textit{Note}}$ If there is a non-mathematical context applied to the function, the practical domain of the function is the portion of the domain above which also applies to the context.

Unless otherwise stated, the image of a function will correspond to the outputs from the *practical* domain of the function.

Ex 8 What is the practical domain of $B(t) = 20 - 4\sqrt{2t+1}$, where B is approximating the quantity of burgers sold as a function of time, t (in hours)? Assume $t = 0$ is opening time.

The left endpoint of our domain should be 0, since we only care about time after the burger store opens. As for the right endpoint, we should notice that as t gets bigger, the value of B will eventually become negative. So, we should solve for the time where it becomes zero.

$$\begin{aligned} B(t) &= 0 \\ 20 - 4\sqrt{2t+1} &= 0 \\ 5 &= \sqrt{2t+1} \\ 25 &= 2t+1 \\ t &= 12. \end{aligned}$$

So, we need to have $0 \leq t \leq 12$. Therefore, the practical domain is

$$\text{Dom}(B) = \{t \mid 0 \leq t \leq 12\} = [0, 12].$$

Def A piecewise-defined function is a function defined from several pieces of functions, each of which is defined only on a specific domain.

Ex 9 Let $D(t) = \begin{cases} 2+7t & \text{, if } t < 2 \\ t^2 & \text{, if } t \geq 2 \end{cases}$

a) Evaluate $D(1)$ and $D(3)$.
To evaluate $D(1)$, we will input $t = 1$ in the formula above. Since $1 < 2$, we use the first equation: $D(1) = 2 + 7(1) = 9$. For $D(3)$, we have $t = 3$, and $3 \geq 2$. So, $D(3) = 3^2 = 9$.

b) Find all real values of t such that $D(t) = 16$. Check the first formula to see if it can be equal to 16:

$$\begin{aligned} 16 &= 2 + 7t \\ 14 &= 7t \\ 2 &= t. \end{aligned}$$

So, an input of $t = 2$ will give an output of 16 in the first formula. However, if we try to evaluate $D(2)$, we notice that $2 \not< 2$, and we are forced to use the second formula. We would get $D(2) = 2^2 = 4 \neq 16$, so 2 is *not* a solution, even though we thought it could be.
Let's try the second equation. We have $t^2 = 16$, so $t = \pm 4$ (two solutions!). Again, we have to check the constraints. With $D(-4)$, note $-4 < 2$, so we use the top equation: $D(-4) = 2 + 7(-4) = -26 \neq 16$. So, $t = -4$ is not a solution. Finally, $D(4) = 4^2 = 16$ since $t = 4$ satisfies the constraint $t \geq 2$. Hence, there is only one solution, namely 4, to the equation $D(t) = 16$.

Def Two variables a and b are proportional if there is a constant k so that $a = k \cdot b$.

Two variables a and b are inversely proportional if there is a constant k so that $a = \frac{k}{b}$.

In each case, k is called the constant of proportionality.

Ex 10 Write a formula for G as a function of P assuming that G is proportional to the product of P and $1 - P$, with constant of proportionality 1.5.

We should have $G(P) = kP(1 - P)$ since we said "proportional." We are also told that the constant of proportionality is 1.5, so $k = 1.5$. Our final result is $G(P) = 1.5P(1 - P)$.

Ex 11 A guitar string changes pitch when it becomes tighter. It is known by physicists that the frequency of the guitar string is proportional to the square root of the tension strength in the string.

a) Set up a functional model for how the frequency relates to the tension.

Let T be the tension, and let F be the frequency of the guitar string. Using the definition of proportionality, we can write $F = k\sqrt{T}$, where k is a yet-to-be determined number.

b) After messing around one afternoon, someone figured out that if you play a string with a tension of 30 Newtons (unit of force) produced a nice sounding A (a note of 440 Hz). Determine the unknown constant k.

Plug in $T = 30$ and $F = 440$ to figure out k:

$$\begin{aligned} F &= k\sqrt{T} \\ 440 &= k\sqrt{30} \\ k &= \frac{440}{\sqrt{30}} \\ k &= 80.33 \end{aligned}$$

c) Say you want to make your string play a G, which has a frequency of 784 Hz. How tight must you tune the string?

We use what we found to solve for the tension, T:

$$\begin{aligned} F &= 80.33\sqrt{T} \\ 784 &= 80.33\sqrt{T} \\ T &= \left(\frac{784}{80.33}\right)^2 \approx 95.25 \text{ Newtons.} \end{aligned}$$

Ex 12 Let monthly revenue, $R = f(n)$ (in thousands of dollars), be proportional to the number of units sold during that month, n, in thousands. Last month, the company sold 8,500 units for a revenue of \$106,250.

a) Fill out the table of values for R as a function of n.

n	0	8.5	12	16	22
R	0	106.250	150	200	275

Let $R = f(n) = kn$. We have $R = 106.250$ when $n = 8.5$. We get

$$106.250 \text{ thousand dollars} = k(8.5 \text{ thousand units})$$

$$k = \frac{106.250 \text{ thousand dollars}}{8.5 \text{ thousand units}} = 12.5 \text{ dollars/ unit.}$$

Now, we can use the formula $R = 12.5n$ to fill in the table.

b) What is the value of the constant of proportionality? What is its significance in the context of revenue?

We found the constant of proportionality above to be $k = 12.5$ dollars/ units. Notice that keeping track of the units in our calculation makes it easier to interpret this number: it means that for every unit sold, the company made \$12.50 in revenue.