5.6: Equilibria and the Phase Plane

Contents

1 Equilibria

- As you can imagine, an equilibrium for a *system* of DE's is one where each state variable does not change.
- This leads to systems of equations!
- Ex: Consider this population model:

$$\frac{dx}{dt} = 2x - 3xy$$
$$\frac{dy}{dt} = y - xy$$

- Note: Both of these are hindered by the others' presence.
- Equilibrium is found when both $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$.
- This gives you a *system of equations*.
- Factor these:

$$x(2 - 3y) = 0$$
$$y(1 - x) = 0.$$

Which pair of numbers, (x, y), produces zero in both at the same time?

- The first equation has $x = 0$ and $y = 2/3$. The second has $y = 0$ and $x = 1$.

- *you need to use factoring in these methods.*

- We get two pairs: $(0, 0)$ and $(1, 2/3)$ as equilibria. Note now that equilibrium is not one number, but a list of numbers: one for each state variable.

- We can plot these in the xy- plane. (Our first glimpse of the phase plane!)

- Note: even though our final answers said many x's and y's, not every combination is an equilibrium. For instance, $(0, 2/3)$ is not an equilibrium, because when you plug these values in, the second equation does not give 0!

- Ex: find the equilibrium for these equations:

$$\frac{dx}{dt} = 2x - 3y - 2$$
$$\frac{dy}{dt} = x + y$$

- This is not one of our models, but it is one that we can find equilibria for anyway.

- First, recall the geometry: $2x - 3y - 2 = 0$ is a line: $y = \frac{2}{3}x - \frac{2}{3}$. Similarly, $x + y = 0$ is the line $y = -x$.

- Graph these. The intersection of these two lines is the equilibrium.

- solving: the second says $y = -x$, plugging into the first gives $2x - 3(-x) = 2$, or $5x = 2$, giving $x = 2.5$. Therefore, $y = -2.5$, so $(2.5, -2.5)$ is the equilibrium.

- Ex: Find the equilibrium for the following system:

$$\frac{dx}{dt} = 2x - y + 1$$
$$\frac{dy}{dt} = y - x + 9.$$

- We have $y = x - 9$, so we have $2x - x + 9 + 1 = 0$, or $x + 10 = 0$, giving $x = -10$. Finally, $y = -10 - 9 = -19$.
- We will see more examples later.