3.3: Optimization, pt 2

1 Optimization Problems

- Very common use of calculus: find when a function is maximum or minimum.
- Terminology: "extremum," "extrema," or "optima" refer to extreme or optimum values, meaning maximum or minimum.
- Definition. A global maximum is a y-value on the graph of $f(t)$ that is the largest on the entire graph. Similarly, a global minimum is a y-value that is the smallest on the graph.
- smallest means most negative, not necessarily closest to 0.
- Example: What is the global maximum of the function $f(x) = -x^2 + 1$ on the whole number line? Global minimum? A: global max: $y = 1$ at $t = 0$. No global minimum.
- Notice: the maximum occurred at a *critical point*.
- Ex: Same $f(x)$, but what if we change the interval to $[-2, -1]$? A: global max is $y = 0$, global min is at $y = -3$.
- Notice: the global optima occurred at endpoints of our interval.
- Caution: a global maximum or minimum is sensitive to the choice of endpoints for x or t.

1.1 Algorithm for determining global optima

When $f(x)$ is defined on a closed interval $[a, b]$ (meaning it doesn't go off forever in one direction), the following steps find the global maxima and minima.

1. Find all critical points of $f(t)$. Remember, these are t-values.
2. Throw out any critical points not in between the endpoints.
3. Plug in the t-values found in step 1, as well as the t-values of the endpoints. You now have a list of y-values.
4. select the largest y-value from this list, and the smallest. These are the global maxima and minima.

2 Examples

- Ex: Optimize $f(x) = x^3 - 3x$ on the interval $0 \le x \le 2$.
    1. critical points: $f'(x) = 3x^2 - 3 = 0$ gives $x = -1$ and $x = +1$.
    2. we throw out $x = -1$ since it is not between 0 and 2.

3. we now plug in $x = 0, 1$, and 2 into $f(x)$ (NOT the derivative).
   The y-values are: $0, -2$, and 2. So, global maximum is $y = 2$ at $t = 2$, global minimum is at $y = -2$ with $t = 1$.

- Ex: Find optima of $g(x) = \frac{x}{1+x}$ on the interval $0 \leq x \leq 1$.

1. critical points: this is where $g'(x) = 0$ or $g'(x)$ is undefined.
$$g'(x) = \frac{1(1+x) - x(1)}{(1+x)^2} = \frac{1}{(1+x)^2}$$
   This is never equal to 0. But it is undefined at $x = -1$, so -1 is a critical point.
2. Throw out $x = -1$ since -1 is not between 0 and 1.
3. plug in critical points and endpoints: $g(0) = 0$, $g(1) = 1/2$.
4. global maximum is at $y = 1/2$, $x = 1$, global minimum is at $y = 0$, $x = 0$.

- Extreme Value Theorem: Any continuous function $f(x)$ on a closed and bounded interval will have a global maximum and minimum.

- Closed means that the interval contains its endpoints. For example, $0 \leq x \leq 2$ is closed, but $0 \leq x < 2$ is not closed, nor is $0 < x < 2$.

- bounded means the interval doesn't go on forever. For example, $0 \leq x \leq 2$ is bounded, but $0 \leq x < \infty$ is not bounded.

- Ex: Find optima of $H(x) = xe^{-x}$ on the interval $0 \leq x \leq 2$. Also, explain why the Extereme Value Theorem applies.

1. $H(x)$ is a product of two continuous functions, x and e^{-x}, and so it is continuous. The
2. critical points.
$$H'(x) = e^{-x} - xe^{-x} = e^{-x}(1-x) = 0$$
   gives only one answer: $x = 1$.
3. Throw out any points? no, $x = 1$ is inside the interval, so we're good.
4. check y-values at endpoints and critical points:
$$H(0) = 0e^0 = 0, \quad H(2) = 2e^{-2} \approx 0.271, \quad H(1) = 1e^{-1} \approx 0.369$$
5. Global maximum is $y \approx 0.369$ at $x = 1$, global minimum is $y = 0$ at $x = 0$.