Day 5: Chain Rule

1 Motivation

- Impossible right now: Differentiate $F(x) = \sqrt{1 - x^2}$. No tools let us do it yet.
- Note that $F(x)$ is a composition of two functions: $\sqrt{x}$ after $1 - x^2$, both of which we know how to differentiate.

2 Review: Composition

- Recall: Given two functions $f(x)$ and $g(x)$, we can form the *composition* f after g, notated by $f \circ g$. The definition is

$$(f \circ g)(x) := f(g(x)).$$

Remember that *order matters*. $f \circ g \neq g \circ f$.

- Example: $f(x) = e^x$, $g(x) = 4x + 2$.

$$(f \circ g)(x) = f(g(x)) = f(4x + 2) = e^{4x+2}$$

$$(g \circ f)(x) = g(f(x)) = g(e^x) = 4e^x + 2$$

- Many functions naturally *are* compositions of simpler functions.

3 Chain Rule

- Says how to differentiate a composition of functions.
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$$\boxed{(f \circ g)(x) = f'(g(x))g'(x)}.$$

- In differential notation, where we let $y = f(u)$ and $u = g(x)$,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

4 Examples

- Work outside to inside.

- $F(x) = \sqrt{1 - x^2} = (1 - x^2)^{1/2}$.

$$F'(x) = \frac{1}{2}\left(1 - x^2\right)^{-1/2}(-2x) = -\frac{x}{\sqrt{1 - x}}$$

- If you want to do differential notation: set $u = 1 - x^2$, so $y = F(u) = u^{1/2}$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ &= \frac{1}{2}u^{-1/2} \cdot (-2x) \\ &= \frac{1}{2}(1 - x^2)^{-1/2} \cdot (-2x),\end{aligned}$$

same as before.

- $G(x) = \frac{1}{1+x} = (1 + x)^{-1}$

$$G'(x) = -1\,(1 + x)^{-2}\,(1 + x)' = -\frac{1}{(1 + x)^2}$$

- $H(x) = e^{\sin(x)}$.

$$H'(x) = e^{\sin(x)}\,(\sin(x))' = \cos(x)e^{\sin(x)}$$

- Most common mistake in applying chain rule: incorrectly assuming what f and g are.

- Sometimes its convenient to "chain" a bunch of chain rules together. For example, if $y = f(u)$, $u = g(x)$, and $x = h(t)$, so we're trying to differentiate $f(g(h(t)))$,

$$\frac{dy}{dt} = \frac{dy}{du}\frac{du}{dx}\frac{dx}{dt}.$$

- $F(x) = \cos(\sin(x^2))$.