Final Exam Review Packet

Math 251, Summer 2017

Name: Key

This packet consists of problems from the whole class. Not everything on here will be on the exam, and there may be problems on the exam that do not appear on here. Nonetheless this packet will be very helpful in studying for the final.

1. Find the following limits.

(a) $\lim_{x\to\infty} \frac{x^7 + x^2 + 13x^9 - 2\sqrt{x}}{16x^9 + 47x^3 - 10000}$ $= \frac{13}{16}$.

(b) $\lim_{x\to 0} x^2 \ln(x)$ $= \lim_{x\to 0} \frac{\ln(x)}{x^{-2}} \to \boxed{\frac{-\infty}{\infty}}$

$\overset{L}{=} \lim_{x\to 0} \frac{1/x}{-2x^{-3}} = \lim_{x\to 0} \frac{1}{x} \cdot \frac{1}{(-2)} \cdot x^3$

$= \lim_{x\to 0} \frac{1}{(-2)} x^2$

$= \frac{0}{(-2)} = \boxed{0}$

(c) $\lim_{t\to\infty} e^{-t^2}$

$= \lim_{t\to\infty} \frac{1}{e^{(t^2)}} = "\frac{1}{\infty}" = 0$.

(d) $\lim_{x\to 0} \frac{\sin(2x)}{\tan(3x)}$ $= "\frac{0}{0}"$

$\overset{L}{=} \lim_{x\to 0} \frac{\cos(2x)\cdot 2}{\sec^2(3x)\cdot 3} = \boxed{\frac{2}{3}}$

2. Find $\frac{dy}{dx}$.

(a) $y = x^x$

$\ln(y) = \ln(x^x)$ log rule

Calculus starts here

$\underbrace{\ln(y) = x\ln(x)}$

$\frac{1}{y} \cdot y' = \ln(x) + x \cdot \frac{1}{x}$

$\frac{dy}{dx} = \boxed{y' = x^x(\ln(x) + 1)}$

(b) $y = \arcsin(x^2 - 1)$

it's in the numerator.

$\frac{dy}{dx} = y' = \frac{1}{\sqrt{1-(x^2-1)^2}} \bullet \frac{2x}{1}$

$\boxed{y' = \frac{2x}{\sqrt{1-(x^2-1)^2}}}$

(c) $x^2 + 2xy + y^2 = \frac{x}{y}$

$2x + 2y + 2xy' + 2yy' = \frac{y - xy'}{y^2} = \frac{1}{y} - \frac{x}{y^2}y'$

$2xy' + 2yy' + \frac{x}{y^2}y' = \frac{1}{y} - 2x - 2y$

$y'\left(2x + 2y + \frac{x}{y^2}\right) = \frac{1}{y} - 2x - 2y$

$\boxed{y' = \frac{\frac{1}{y} - 2x - 2y}{2x + 2y + x/y^2}}$

Note: You will arrive at a different looking answer if you simplify before (or after) differentiating. They are equal by doing algebra. Other answer you may have gotten

$\boxed{y' = \frac{1 - 2y^2 - 2xy}{x^2 + 4xy + 3y^2}}$

(d) $y = 4x + x^2 + e^3$

$y' = 4 + 2x$

3. Find all inflection points of the following functions.

(a) $g(x) = x^4 + x - 1$

$g'(x) = 4x^3 + 1$

$g''(x) = 12x^2 = 0$

$x^2 = 0$

$x = 0$ – "pre-inflection" point

0

Test Concavity:

$g''(-1) = 12(-1)^2 = 12$

$g''(1) = 12$

★ Concavity never changes, so this function has no inflection points.

(b) $T(x) = x^2 e^{-x}$

$T'(x) = 2xe^{-x} - e^{-x} \cdot x^2$

$T'(x) = (2x - x^2)e^{-x}$

$T''(x) = (2-2x)e^{-x} + (2x - x^2)(-1)e^{-x}$

$= (2 - 2x - 2x + x^2)e^{-x}$

$= (2 - 4x + x^2)e^{-x}$

Set $2 - 4x + x^2 = 0$:

$x = \frac{4 \pm \sqrt{16-8}}{2} = 2 \pm \sqrt{2}$

up 0.59 down 3.41 up

$T''(0) = 2$ $T''(1) = -0.37$ $T''(4) = 0.0$

both are inflection pts

Find all intervals where f is increasing, decreasing, concave up, and concave down. [Bonus: use this info to construct a graph of f.]

4. (a) $f(x) = xe^x$

$f'(x) = e^x + xe^x$

$f''(x) = e^x + e^x + xe^x$

$= (2+x)e^x$

$f'(x) = 0$:

$x = -1$

decr. -1 incr.

$f'(-2) = -0.135$

$f'(0) = 1$

$f''(x) = 0$

$x = -2$

down -2 up

$f''(-3) = -0.05$ $f''(0) = 2$

Increasing: $(-1, \infty)$

Decreasing: $(-\infty, -1)$

Concave up: $(-\infty, -2)$

Concave down: $(-2, \infty)$

(b) $g(x) = \arctan(x)$

$g'(x) = \frac{1}{1+x^2}$

$g''(x) = \frac{0(1+x^2) - (1+x^2)'}{(1+x^2)^2}$

$g''(x) = \frac{-2x}{(1+x^2)^2}$

$g'(x) = 0$

$\frac{1}{1+x^2} = 0$

$\Rightarrow 1 = 0$

no sol's!

$\Rightarrow$ No crit pts!

$g'(0) = \frac{1}{1} > 0$

or any number!

$g''(x) = \frac{-2x}{(1+x^2)^2} = 0$

$-2x = 0$

$x = 0$

0

$g''(-1) = \frac{2}{pos} > 0$

$g''(1) = \frac{-2}{} < 0$

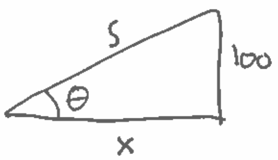
Incr: $(-\infty, \infty)$

Decr: Empty set; $\{\ \}$.

Con. up: $(-\infty, 0)$

Con. down: $(0, \infty)$.

5. A kite 100 ft above the ground moves horizontally at a speed of 8 ft/s. At what rate is the angle between the string and the horizontal decreasing when 200 ft of string has been let out?



Know: $\frac{dx}{dt} = 8$

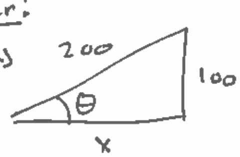
Want: $\frac{d\theta}{dt}$ when $S = 200$.

Relate: $\tan\theta = \frac{100}{x}$

$\frac{d}{dt}$: $(\sec^2\theta)\cdot\frac{d\theta}{dt} = \frac{-100}{x^2}\cdot\frac{dx}{dt}$

* Still need: x and θ when S = 200.

Later!



$\sin(\theta) = \frac{100}{200} = \frac{1}{2}$

$\theta = \arcsin(1/2) = \boxed{.523}$

$200^2 = 100^2 + x^2$

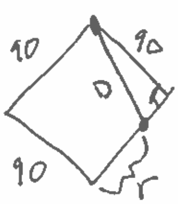
$x = \boxed{173.2}$

$\sec^2(.523)\frac{d\theta}{dt} = \frac{-100}{(173.2)^2}\cdot 8$

$(1.333)\frac{d\theta}{dt} = -0.026668$

$\boxed{\frac{d\theta}{dt} = -0.02 \text{ rad/sec}}$

6. A baseball diamond is a square with side 90 ft. A batter hits the ball and runs toward first base with a speed of 24 ft/s. At what rate is his distance from second base decreasing when he is halfway to first base? What about third base?



r = runner's dist. from home.

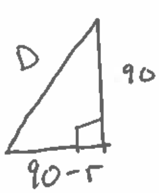
D = dist. to Second base

T = dist. to Third base.

Know: $\frac{dr}{dt} = 24$.

Want: $\frac{dD}{dt}$ when $r = 45$.

2ND BASE



$\boxed{D^2 = 90^2 + (90-r)^2}$

$2D\frac{dD}{dt} = 2(90-r)\cdot(-1)\frac{dr}{dt}$

Need D: when r = 45:

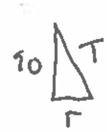
$D^2 = 90^2 + (45)^2$

$D = 100.6$

$2(100.6)\frac{dD}{dt} = -2(45)(24)$

$\boxed{\frac{dD}{dt} = -10.7 \text{ ft/sec}}$

3RD BASE



$T^2 = 90^2 + r^2 \rightsquigarrow T^2 = 90^2 + (45)^2$

$T = 100.6$!

$2T\frac{dT}{dt} = 2r\frac{dr}{dt}$

$2(100.6)\frac{dT}{dt} = 2(45)(24)$

$\boxed{\frac{dT}{dt} = 10.7 \text{ ft/sec}}$

Weird: Right in the middle, dist. to 2nd base and 3rd base change at equal but opposite rates

7. Find the linearization of $\sqrt[3]{x}$ near $x = 1$. Use this to approximate the value of $\sqrt[3]{1.1}$.

$y = mx + b$

$f(x) = \sqrt[3]{x} = x^{1/3}$

$f'(x) = \frac{1}{3}x^{-2/3}$

$f(1) = \sqrt[3]{1} = 1$

$f'(1) = \frac{1}{3} = m$

$y = \frac{1}{3}x + b.$

$1 = \frac{1}{3}(1) + b$

$\frac{2}{3} = b$

$y = \frac{1}{3}x + \frac{2}{3},$

$\sqrt[3]{1.1} \approx \frac{1}{3}(1.1) + \frac{2}{3} = \boxed{1.0333}$

Actual value: 1.03228, pretty close! wow!

8. Find the linearization of $\ln(x)$ near $x = 1$, and use it to approximate the value of $\ln(1.1)$.

$\underline{y = mx + b}$

$f'(x) = \frac{1}{x}$

$f'(1) = \frac{1}{1} = 1 = m$

$f(1) = \ln(1) = 0.$

$0 = 1 \cdot (1) + b$

$b = -1$

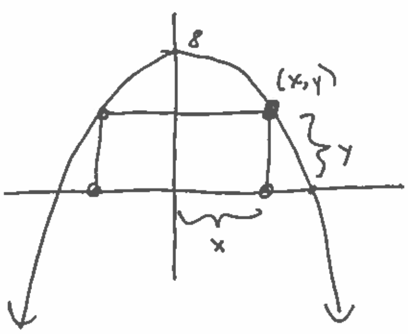
$y = x - 1$ ← Tangent Line.

$\ln(1.1) \approx (1.1) - 1 = \boxed{0.1}$

Actual: 0.0953

Pretty close! neat.

9. Find the dimensions of the rectangle of largest area that has its base on the x-axis and has the other two corners sitting on the parabola $y = 8 - x^2$.



$A = 2xy$ ← Endpoints: $x = 0$ and $x = \sqrt{8}$

get by having $y = 0$

$0 = 8 - x^2$

$A = 2x(8 - x^2)$

$A = 16x - 2x^3$

$A' = 16 - 6x^2 = 0$

$16 = 6x^2$

$\frac{16}{6} = x^2$

$x = 1.63$ ⇝ $y = 8 - \frac{16}{6}$

$y = 5.33$

Check if Max:

$A(0) = 0$

$A(\sqrt{8}) = 2\sqrt{8}(8 - (\sqrt{8})^2) = 0$

$A(1.63) = 17.4$ ✓

10. True or false time! Explain your responses.

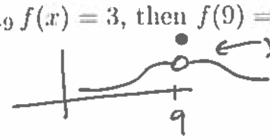
(a) A function $f(x)$ can have a vertical tangent line. (Vertical means the slope comes out as $\frac{1}{0}$.)

T $f(x) = x^{1/3}$ is an example. $f'(x) = \frac{1}{3}\frac{1}{x^{2/3}}$,

$f'(0) = \frac{1}{3} \cdot \frac{1}{0}$ ← vertical.

(b) If $f(9)$ exists and $\lim_{x \to 9} f(x) = 3$, then $f(9) = 3$.

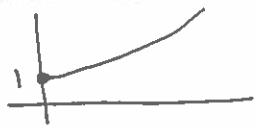
F:



← that could happen.

(c) If $f'(x) > 0$ for $x > 0$ and $f(0) = 1$, then $f(x) > 0$ for all $x > 0$.

T



always increasing!

(d) A horizontal asymptote of $y = 3$ means that either $\lim_{x \to \infty} f(x) = 3$ or $\lim_{x \to -\infty} f(x) = 3$.

T. This is actually the def. of a horizontal asymptote.

(e) $\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$.

T; derivatives add.

(f) An equation for the tangent line to $y = x^2$ at $(-2, 4)$ is $y = 2x(x + 2) + 4$.

F; need to plug in $x = -2$ into y' first.

(g) If f has an absolute minimum at c, then $f'(c) = 0$.

F; the absolute min could happen at an endpoint.

(h) If $f''(2) = 0$, then $(2, f(2))$ is an inflection point of $f(x)$.

F; You also need concavity to change.

(i) Two functions $f(x)$ and $g(x)$ with $f'(x) = g'(x)$ must be equal; that is, $f(x) = g(x)$.

F; counterexample: $f = 2x + 1$

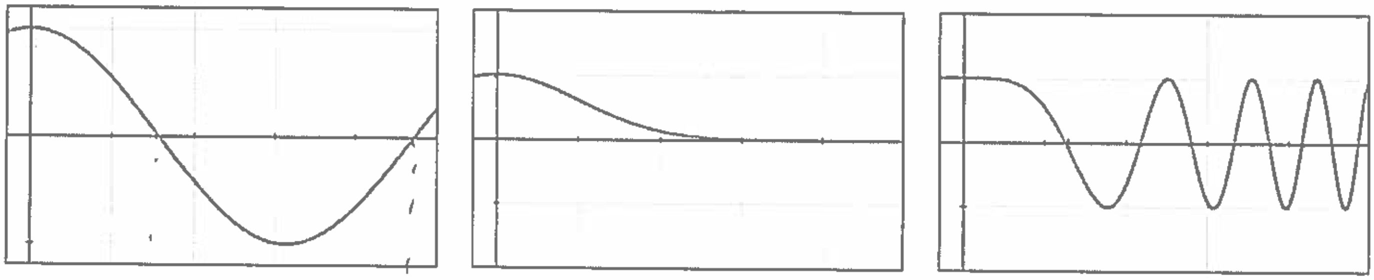
$g = 2x$.

(j) $\lim_{x \to 0} \frac{x}{e^x} = 1$.

F; L'Hopital does not apply, so it's just $\frac{0}{e^0} = \frac{0}{1} = 0$.

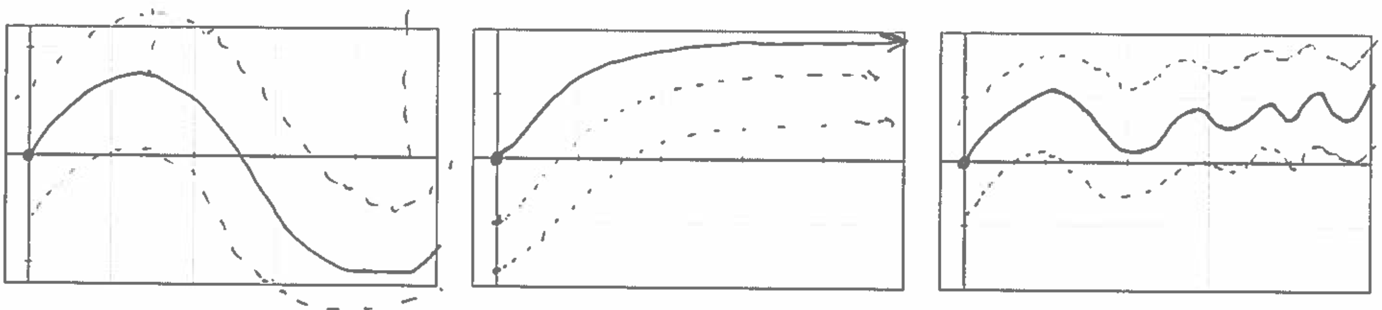
11. Given below are graphs of $f'(x)$ for some function $f(x)$. Sketch both f and f'' (you can choose a point on the graph of f).

f'

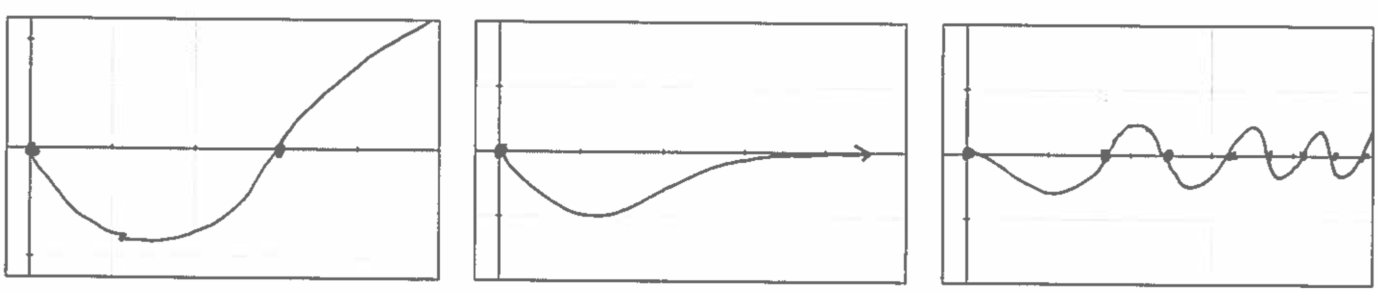


f

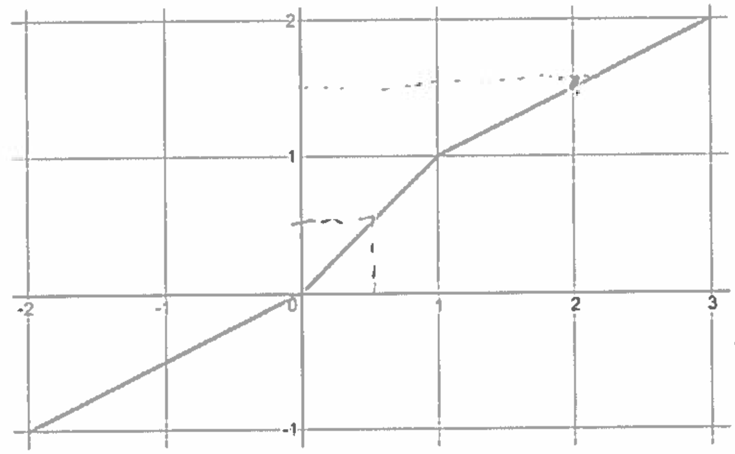
any are correct



f''



12. Below is the graph of the function $f(x)$. Find each of the following values or state why they do not exist.



$f^{-1}(1.5) = x$

$\rightsquigarrow 1.5 = f(x)$

$x = 2$

(a) $f'(1)$ DNE; two different slopes from left and right.

(d) $(f^{-1})'(1.5)$. $= \dfrac{1}{f'(f^{-1}(1.5))} = \dfrac{1}{f'(2)}$

$= \dfrac{1}{1/2}$

$= \boxed{2}$

(b) $\dfrac{d}{dx}\left(\dfrac{x^4}{f(x)}\right)$ at $x = 2$.

$\rightarrow \dfrac{4x^3 \cdot f(x) - x^4 f'(x)}{(f(x))^2} = \dfrac{4(8)f(2) - 16f'(2)}{(f(2))^2}$

$= \dfrac{32(1.5) - 16(1/2)}{(1.5)^2}$

$= \boxed{17.8}$

(e) $(f^{-1})'(1)$.

$= \dfrac{1}{f'(f^{-1}(1))} = \dfrac{1}{f'(1)}$

$=$ DNE

(c) $(f \circ f)'(2)$

$= f'(f(2)) \cdot f'(2)$

$= \underbrace{f'(1.5)}_{\frac{1}{2}} \cdot \underbrace{f'(2)}_{\frac{1}{2}}$

$= \frac{1}{2} \cdot \frac{1}{2} = \boxed{\frac{1}{4}}$

(f) $(f^{-1})'(0.5)$

$= \dfrac{1}{f'(f^{-1}(.5))} = \dfrac{1}{f'(0.5)} = \dfrac{1}{1}$

$= \boxed{1}$

(g) Assuming that $g'(x) = f(x)$ (in which case we might call g the *antiderivative* of f), and that $g(2) = 3$, what is the equation of the tangent line for $g(x)$ at $x = 2$?

$g'(2) = f(2) = 1.5 = m$

$y = mx + b$

$y = 1.5x + b$

$3 = (1.5)(2) + b$

$3 = 3 + b \rightarrow b = 0$

$y = 1.5x$

13. Evaluate the following limits.

(a) $\lim_{x\to\infty} \frac{x^2}{e^x}$ $\overset{L}{=}$ $\lim_{x\to\infty} \frac{2x}{e^x}$ $\overset{L}{=}$ $\lim_{x\to\infty} \frac{2}{e^x} = \frac{2}{\infty} = \boxed{0}$

$\frac{\infty}{\infty}$ $\frac{\infty}{\infty}$

(b) $\lim_{t\to-\infty} \frac{x^2 \to \infty}{e^x \to 0}$ $\longrightarrow \frac{\infty}{0} = \frac{\text{big}}{\text{small}} \Rightarrow \boxed{\infty}$

(c) $\lim_{x\to 0} \frac{\cos(x)}{1-\sin(x)}$ $\frac{1}{1-0} = \boxed{1}$

(d) $\lim_{x\to 0} \frac{x^4}{\sin(x)}$ $\frac{0}{0}$ ✓

$\overset{L}{=} \lim_{x\to 0} \frac{4x^3}{\cos(x)}$ $\frac{0}{0}$

$\overset{L}{=} \lim_{x\to 0} \frac{12x^2}{-\sin(x)}$ $\frac{0}{0}$

$\overset{L}{=} \lim_{x\to 0} \frac{24x}{-\cos(x)}$ $\frac{0}{0}$

$\overset{L}{=} \lim_{x\to 0} \frac{24}{\sin(x)} = \frac{24}{0} = $ "∞"

technically, $\frac{1}{\sin(x)}$ approaches $+\infty$ from right $(x\to 0^+)$ but $-\infty$ from the left $(x\to 0^-)$, so it DNE.