4.4: Definite Integrals, Riemann Sums

1 Speed Example

- Example: $v(t)$ -velocity of a car driving north on I-5. Suppose $v(t)$ is a constant number, like 60 mi/hr. You travel for 2 hr. How far? 120 miles.
- Graph of $v(t)$ is a flat line.
- Let's say we start in Eugene, and let $y = 0$ mean Eugene. Then $y = 120$ is Portland, so in this case $y(t) = 60t$.
- Observe: the *change* in distance is 120 miles.

2 Definite Integrals

- Consider a positive function $f(x)$. Look at it between two numbers, $x = a$ and $x = b$. Pretend that $f(x)$ represents a rate of some kind (speed, growth rate, etc)
- Goal: compute the total change in the original quantity (displacement, mass, etc).
- Ex. Let $\left(\frac{dy}{dx} =\right) f(x) = x^2$, and let's say you want the total change in the original quantity between $x = 1$ and $x = 3$. One way to approximate is choose discrete time steps, and use the values of $f(x)$ as actual rates. Say we pick 4 subdivisions. Then each step has

$$\Delta x = \text{width} = \frac{3-1}{4} = \frac{1}{2}.$$

- An *estimate* of this total change would then be

$$f(1)\frac{1}{2} + f(1.5)\frac{1}{2} + f(2)\frac{1}{2} + f(2.5)\frac{1}{2} \approx 6.75.$$

- (Draw picture of what's happening)
- But you could do better: choose smaller and smaller intervals! Say you do *ten* intervals now. It gets harder to compute by hand, but computers are good at it.

  In general,

$$\Delta x = \frac{\text{right end} - \text{left end}}{\text{number of subintervals}} = \frac{b-a}{n}.$$

$$\begin{aligned}\sum_{i=0}^{5}(i+1) &= (0+1)+(1+1)+(2+1)+(3+1)+(4+1)+(5+1)\\ &= 1+2+3+4+5+6\\ &= 21.\end{aligned}$$

- There are two ways to do such a sum: use left endpoints, or right endpoints. These are called *Riemann sums.* Use graphic to demonstate.
- Left Riemann sum:

$$\sum_{i=0}^{n-1} f(x_i)\Delta x$$

  Right Riemann sum:

$$\sum_{i=1}^{n} f(x_i)\Delta x$$

  (with corresponding pictures). Be sure to pay attention to the summation sign! Main point: left Riemann sum uses the left endpoint (and not the right endpoint), while the right Riemann sum uses the right endpoint (and not the left).

- Use calculators/computers to make these estimates; but all that's going on is adding a bunch of rate $\times$ time's.

3 Riemann Integral

- Take a limit as $\Delta x \to 0$ and $n \to \infty$:

  $$\int_a^b f(x)\,dx := \lim_{\substack{n\to\infty \\ \Delta x \to 0}} \sum_{i=1}^{n} f(x_i)\Delta x.$$

  The result is a *number*, and it's called the Riemann integral, or sometimes the *definite* integral.

- This computes the total change in the antiderivative of $f(t)$. So, if $f(t)$ represented speed, say, then this integral represents the total displacement of whatever is moving.