Guide for Section 5 - Exponential Functions

Section Goals:

- Identify a function or phenomenon as exponential.
- Write a formula for an exponential function.
- Determine an exponential function's continuous growth rate and periodic growth rate.
- Sketch the graph of an exponential function.

Ex 1 The average thickness of a piece of paper is about 0.1 mm.

a) How thick is a piece of paper after you fold it over once? Twice? Three times?
Once doubles the tickness, so $2 \cdot (0.1) = 0.2$ mm. Doing it again, you double what you previously had, so $2 \cdot 2 \cdot (0.1) = 2^2(0.1) = 0.4$ mm. Three times, double it again; you get $2^3(0.1) = 0.8$ mm.

b) Write an equation for the function, T, that gives the thickness (in mm) of a piece of paper after being folded f times (ignoring resistance in the paper).
$T(f) = (0.1) \cdot 2^f$ works just by generalizing the above pattern.

c) After how many foldings will it take for the paper to be 25.6 mm (a little over 1 inch) thick? Keep trying folds: at four folds, $T(4) = 0.1 \cdot 2^4 = 1.6mm$;
$T(5) = 0.1 \cdot 2^5 = 3.2$ mm;
$T(6) = 0.1 \cdot 2^6 = 6.4$ mm;
$T(7) = 0.1 \cdot 2^7 = 12.8$ mm;
$T(8) = 0.1 \cdot 2^8 = 25.6$ mm;
So, we need 8 folds to make it about 1 inch thick.

d) How thick is the paper after 50 foldings?
Compute $T(50) = 0.1 * 2^{50} \approx 122.6$ million kilometers! That's about one-third the distance to Mars.

Thm (Exponential Function) If Q is changing at a rate proportional to itself, so that $R(t) = kQ$, where R is the rate of growth in Q and k is the continuous growth rate, then

$$Q = f(t) = ae^{kt},$$

where a is a constant (which also happens to be equal to the value of Q at $t = 0$).

An exponential function changes by a factor of e^k for every unit increase in t. This is referred to as its growth factor.

Def An alternate form for an exponential function which is equivalent to the one given above is

$$f(t) = a \cdot b^t,$$

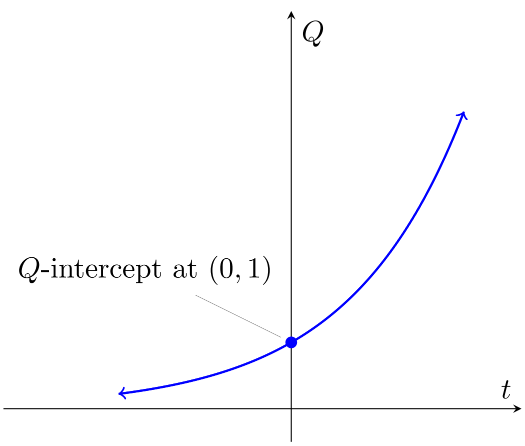
where the constant growth factor is positive value b.

Thm (Basic Exponential Function Graphs)

Exponential Growth

$$Q = f(t) = ae^{kt} = a \cdot b^t$$

$b > 1$ and $k > 0$

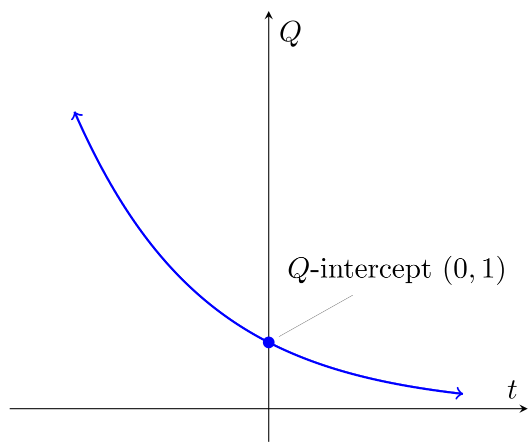


Graph rises dramatically to the right, falls toward a height of 0 to the left

Exponential Decay

$$Q = f(t) = ae^{kt} = a \cdot b^t$$

$0 < b < 1$ and $k < 0$



Graph rises dramatically to the left, falls toward a height of 0 to the right

Thm (Domain of an Exponential Function) For a function $f(t) = ae^{bt}$, with $a > 0$, we have that

Domain	Image
$(-\infty, \infty)$	$(0, \infty)$

Ex 2 Let $f(t) = 3e^{0.2t}$.

a) What is the continuous growth rate of f?
$k = 0.2$ per unit time.

b) What is the constant growth factor of f?
$b = e^{0.2} \approx 1.22$, so the quantity grows by a factor of 1.22 for each unit of time.

c) For some real number n, $f(n) = 5$. What must be the value of $f(n+1)$?
Going forward one unit in time multiplies by the constant growth factor $b = 1.22$. So, $f(n+1) = 5(1.22) = 6.1$.

d) Another function $V = g(t)$ has the property that V is changing at a rate proportional to the value of V, with constant of proportionality -1.4. Write an equation for $g(t)$ assuming that $g(0) = 100$.
We get $k = -1.4$, so $g(t) = ae^{-1.4t}$. Then $100 = g(0) = ae^0 = a$, so $g(t) = 100e^{-1.4t}$.

$\boxed{\text{Def}}$ An exponential function with $\begin{bmatrix} \text{negative continuous growth rate or growth factor} < 1 \\ \text{positive continuous growth rate or growth factor} > 1 \end{bmatrix}$ is $\begin{bmatrix} \text{a decreasing function} \\ \text{an increasing function} \end{bmatrix}$ and is said to exhibit $\begin{bmatrix} \text{exponential decay} \\ \text{exponential growth} \end{bmatrix}$.

Ex 3 Does the function $N(t) = 2(0.9)^t$ exhibit exponential growth or decay? What about $P(t) = 7e^{0.9t}$? $N(t)$ exhibits exponential decay (think about what happens after multiplying 0.9 by itself many times). $P(t)$ has exponential growth since $k = 0.9$ is a positive continuous growth rate.

Ex 4 Consider the two functions f and g defined by the table below. What kind of functions are f and g? Write a formula for both f and g.

x	$f(x)$	$g(x)$
1	3	10
2	4.5	25
3	6.75	62.5
4	10.125	156.25

Note that dividing successive outputs gives a constant growth factor $b = 1.5$. Then, to get a, we need to use a point:

$$\begin{aligned} f(1) &= 3 \\ a(1.5)^1 &= 3 \\ a &= 2 \end{aligned}$$

So, $f(x) = 2(1.5)^x$. Similarly, for g we get a constant growth factor of $b = 2.5$, and

$$\begin{aligned} g(1) &= 10 \\ a(2.5)^1 &= 10 \\ a &= 4. \end{aligned}$$

So, $g(x) = 4(2.5)^x$.

$\boxed{\text{Note}}$ The above method only works if the inputs are evenly spaced by 1!

$\boxed{\text{Thm}}$ If a quantity experiences a constant yearly percentage growth rate, r, then the growth factor for the exponential function is $b = 1 + r$. If the quantity is *decreasing* by a constant percentage rate, r, then $b = 1 - r$.

Ex 5 The local duck population grows by about 2.02% per year. In 2015, there were about 200,000 ducks in Eugene. What can we predict the population to be in 2020?

This is an exponential model. What is b? We know $a = 200$ (assuming we measure population in thousands). Well, one year after 2015, we would have

$$200 + 200 * (0.0202) = 204.04 \text{ thousand ducks.}$$

Note that this looks like

$$200(1 + 0.0202),$$

so the growth factor is $b = 1.0202$. This illustrates the above fact. The model for ducks is then

$$D(t) = 200 * (1.0202)^t.$$

So, in 2020, when $t = 10$, we have

$$D(10) = 200(1.0202)^{10} = 244.3 \text{ thousand ducks.}$$

Try calculating that by hand!

Def The value V, of an investment with initial value V_0, which accrues interest compounded n times per year at a (nominal) annual rate of r is worth V at the end of t years, where

$$V = V_0 \left(1 + \frac{r}{n}\right)^{nt}$$

To compare this to the older notation, $a = V_0$ and $b = \left(1 + \frac{r}{n}\right)^n$.

Ex 6 The DeHaven family, tracing lineage back to the American Revolution, claims[1] that in December 1777 their ancestor Jacob DeHaven loaned George Washington $450,000 in gold and supplies which helped turn the tide of the war.

a) In 1989 (as well as several points during the 19th century), the descendants wished to claim compensation for this princely sum and assumed a 6% interest rate compounded monthly. How much did the family request as the value of the loan? We have

$$V = (450)\left(1 + \frac{0.06}{12}\right)^{12(212)} = 1.457 \times 10^8 \text{ thousand dollars,}$$

which is about $146 billion dollars!

b) Citing their reasonableness, the descendants claimed that $100,000,000 was a sufficient compensation. What interest rate does this amount to over the course of the loan's term from 1777 until 1989? Solve for r:

$$\begin{aligned} 100,000 &= 450\left(1 + \frac{r}{12}\right)^{12(212)} \\ \left(\frac{10000}{45}\right)^{1/(12\times 212)} &= 1 + \frac{r}{12} \\ r &= 12 \times \left[\left(\frac{10000}{45}\right)^{1/(12\times 212)} - 1\right] \approx 0.0255 \end{aligned}$$

which means they would have an interest rate of 2.55%.

c) if $100,000,000 was what the DeHaven family was owed "fairly" after 212 years of 6% interest compounded monthly, what does that assume the original loan value to be? (This is called the *present value* of the investment) Solve for V_0:

$$\begin{aligned} 100,000 &= V_0\left(1 + \frac{0.06}{12}\right)^{12(212)} \\ V_0 &= \frac{100,000}{\left(1 + \frac{0.06}{12}\right)^{12(212)}} \approx .3087, \end{aligned}$$

which means the original loan amount would have to be $308,700 or so.

[1] `http://www.ushistory.org/valleyforge/youasked/069.htm`