Guide for Section 4 - Rational Functions

Section Goals:

- Use inverse proportionality to express a function with a negative exponent
- Model a rational function as a ratio of polynomials in mathematical and non-mathematical contexts
- Use the principle of ratios of small and large numbers to infer long-term behavior of basic rational functions
- Use long-term behavior of polynomials to infer long-term behavior of non-basic rational functions

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Ex 1 Write the rule for a function which doubles the reciprocal of its input.

$$f(t) = \frac{2}{t}$$

Def The **reciprocal function** of t is defined to be

$$Q = f(t) = \frac{1}{t}.$$

The **reciprocal square function** of t is defined to be

$$Q = f(t) = \frac{1}{t^2}.$$

Def A **rational function** is a function which can be written in the form

$$f(t) = \frac{p(t)}{q(t)}$$

where $p(t)$ and $q(t)$ are each polynomial functions (and $q(t)$ isn't always 0).

The mathematical domain of a rational function is all values of t such that $q(t) \neq 0$. As such, a rational function is only undefined at a finite list of inputs.

Ex 2 In each case, (i) identify the domain of the function $f(t)$, (ii) identify whether or not the function is rational, and then if the function *is* rational, then (iii) write possible polynomials $p(t)$ and $q(t)$ so that $f(t) = \frac{p(t)}{q(t)}$.

a) $f(t) = \dfrac{3+t^4}{t^5} - \dfrac{1}{t-1}$

(i) The domain is all real numbers except 0 and 1.

(ii) The function is a rational function. To see this, add the fractions:

$$\begin{aligned} f(t) &= \frac{(3+t^4)(t-1)}{t^5(t-1)} - \frac{t^5}{t^5(t-1)} \\ &= \frac{(t^5 - t^4 + 3t - 3) - t^5}{t^5(t-1)} \\ &= \frac{-t^4 + 3t - 3}{t^6 - t^5} \end{aligned}$$

which is a ratio of two polynomials, i.e. the definition of rational function.

b) $f(t) = \dfrac{1}{t} - \dfrac{5}{t^2}$

(i) We only need to exclude 0 to make sure $f(t)$ makes sense. So, the domain is all real numbers except 0, which can be written as $(-\infty, 0) \cup (0, \infty)$.

(ii) The sum of any two rational functions is rational, so f is rational.

(iii) To find p and q (and to perhaps believe part (ii)), rewrite f by adding the fractions:

$$f(t) = \frac{t}{t^2} - \frac{5}{t^2} = \frac{t-5}{t^2}.$$

We can take $p(t) = t - 5$ and $q(t) = t^2$ in the definition of rational function.
(Another choice is $p(t) = t^2 - 5t$ and $q(t) = t^3$, can you see why?)

$\boxed{\text{Thm}}$ (Big-Little Principle)

- For any constant k and $p > 0$, we write

$$\text{As } t \to \infty, \text{ then } \frac{k}{t^p} \to 0$$

In other words, if you make the bottom of a fraction bigger and bigger (as either a large positive or large negative number), the whole thing gets closer and closer to zero.

- For any constant k and $p > 0$, we write

$$\text{As } t \to 0, \text{ then } \frac{k}{t^p} \to \pm\infty$$

In other words, if you make the bottom of a fraction a tiny number, the whole thing gets larger and larger (either in the positive or negative direction).

Ex 3 In each part, fill in the blank.

a) As $t \to \infty$, $\dfrac{10}{t^2} \to$ __0__.

b) As $t \to -\infty$, $\dfrac{-1.2}{t^{0.1}} \to$ __0__.

c) As t approaches 0 with $t > 0$, $\dfrac{6}{t^3} \to$ __∞__.

d) As t approaches 0 with $t < 0$, $\dfrac{6}{t^3} \to$ __$-\infty$__.

Ex 4 A data mining company uses 36 supercomputers equally in order to search 9000 terabytes of data.

a) Through how much data is each computer responsible for searching? $9000/36 = 250$ TB/computer

b) Through how much data is each computer responsible for searching if there are n supercomputers? $\frac{9000}{n}$ TB/computer

c) What happens to the amount of data searched by each computer as the number of supercomputers increases? Using the Big-Little principle, we see that the amount of data searched by each individual computer goes to 0 as $n \to \infty$.

Ex 5 The relative growth rate of a microorganism can be modeled by the so-called Monod function, which can be given in the form

$$R(S) = \frac{1.35S}{0.004 + S},$$

where S is the concentration of solution (in grams per liter) available for growth of the microorganism.

a) Does the Big-Little Principle inform us about the behavior of $R(S)$ in the long term? Explain. The Big-Little Principle does not tell us anything directly, because $R(S)$ is not of the form $\frac{constant}{S^p}$.

b) Use the formula for $R(S)$ to compute $R(0.1)$, $R(1)$, $R(10)$, and $R(100)$. Use these computations and what you know about the significance of a to guess the value these computations approach as the input grows larger. $R(0.1) = \frac{1.35(0.1)}{0.004+0.1} = 1.298$

$R(1) = \frac{1.35(1)}{0.004+1} = 1.3446$

$R(10) = \frac{1.35(10)}{0.004+10} = 1.3495$

$R(100) = \frac{1.35(100)}{0.004+100} = 1.34995$

Looks like it's getting close to 1.35!

You could interpret the number it approaches, 1.35, as the "ideal growth rate." Think of S as how much food the organisms have. By giving them an unlimited amount of food (i.e. letting $S \to \infty$), the organisms' growth rate stabilizes; they can only grow so quickly!

Thm (Long-Term Behavior of a General Rational Function) Let a rational function be $f(t) = \dfrac{p(t)}{q(t)}$, where p and q are polynomial functions with leading terms $P(t)$ and $Q(t)$, respectively. Then the long-term behavior of $f(t)$ is the long-term behavior of the simplified function $\dfrac{P(t)}{Q(t)}$.

Ex 6 Identify the long-term behaviors of $f(t) = \dfrac{2t+1}{3t^3+1}$ and $g(t) = 5t + \dfrac{4t}{t+2}$.

For f, we have $p(t) = 2t+1$ and $P(t) = 2t$; $q(t) = 3t^2+1$ and $Q(t) = 3t^2$. So, the long-run behavior of f is determined by

$$\frac{P(t)}{Q(t)} = \frac{2t}{3t^2} = \frac{2}{3t} \to 0 \text{ as } t \to \infty.$$

For g, write $5t$ as a fraction and add. We have

$$g(t) = \frac{5t(t+2)}{t+2} + \frac{4t}{t+2} = \frac{5t^2+14t}{t+2}.$$

So, $p(t) = 5t^2 + 10t$ and $q(t) = t+2$. Thus, $P(t) = 5t^2$ and $Q(t) = t$. The long run behavior of g is given by

$$\frac{P(t)}{Q(t)} = \frac{5t^2}{t} = 5t \to \infty \text{ as } t \to \infty.$$

Ex 7 Revisit Example 5. Compare degrees of the numerator and denominator in order to determine the long-term behavior of $R(S)$. Interpret this result in the context of the model.

The top and bottom degrees of the polynomials determining $R(S)$ are both 1. Hence, the long-run behavior of $R(S)$ is a constant. This means that as S, the amount of solution available for growth, gets really large, the rate that the microorganism grows at approaches a fixed constant. This is intuitive; as S goes to infinity, there is nothing stopping the microorganism from growing however it pleases, so it reaches its maximum rate of growth.