Worksheet 8

Math 251, Summer 2017

Name: Key

1. Using the technique of log differentiation, compute the following derivatives.

(a) x^{4x}

$$y = x^{4x}$$
$$\ln y = \ln\left(x^{4x}\right)$$
$$\ln y = 4x\ln(x)$$
$$\frac{1}{y}\cdot y' = 4\ln(x) + 4x\cdot\frac{1}{x}$$
$$y' = y\left(4\ln(x)+4\right)$$
$$\boxed{y' = x^{4x}\left(4\ln(x)+4\right)}$$

(b) $\dfrac{4x-1}{\sin(x)}$ $= y$

$$\ln(y) = \ln\left(\frac{4x-1}{\sin(x)}\right)$$
$$\ln(y) = \ln(4x-1) - \ln(\sin(x))$$
$$\frac{1}{y}\cdot y' = \frac{1}{4x-1}\cdot 4 - \frac{1}{\sin(x)}\cdot\cos(x)$$
$$y' = y\cdot\left[\frac{4}{4x-1} - \cot(x)\right]$$
$$\boxed{y' = \frac{4x-1}{\sin(x)}\left[\frac{4}{4x-1} - \cot(x)\right]}$$

(c) $x^{\cos(x)}$

$$y = x^{\cos(x)}$$
$$\ln(y) = \ln\left(x^{\cos(x)}\right)$$
$$\ln(y) = \cos(x)\cdot\ln(x)$$
$$\frac{1}{y}\cdot y' = -\sin(x)\ln(x) + \cos(x)\cdot\frac{1}{x}$$
$$y' = y\cdot\left[-\sin(x)\ln(x) + \frac{\cos(x)}{x}\right]$$
$$\boxed{y' = x^{\cos(x)}\left[-\sin(x)\ln(x) + \frac{\cos(x)}{x}\right]}$$

(d) $\cos(x)^x$

$$y = \cos(x)^x$$
$$\ln(y) = x\ln(\cos(x))$$
$$\frac{1}{y}\cdot y' = \ln(\cos(x)) + x\cdot\frac{1}{\cos(x)}\cdot(-\sin(x))$$
$$y' = y\cdot\left(\ln(\cos(x)) - x\cdot\tan(x)\right)$$
$$\boxed{y' = \cos(x)^x\cdot\left(\ln(\cos(x)) - x\tan(x)\right)}$$

(e) $(\sqrt{x})^x$ $= y$

$$\ln(y) = \ln\left(\sqrt{x}^{\,x}\right)$$
$$\ln(y) = x\cdot\ln(\sqrt{x})$$
$$\ln(y) = \frac{1}{2}x\ln(x)$$
$$\frac{1}{y}y' = \frac{1}{2}\ln(x) + \frac{1}{2}x\cdot\frac{1}{x}$$
$$y' = y\left[\frac{1}{2}\ln(x) + \frac{1}{2}\right]$$
$$\boxed{y' = (\sqrt{x})^x\left[\frac{1}{2}\ln(x) + \frac{1}{2}\right]}$$

(f) $(2x+5)^{10}(x^4-3)^6$ $= y$

$$\ln(y) = \ln\left[(2x+5)^{10}(x^4-3)^6\right]$$
$$\ln(y) = \ln\left((2x+5)^{10}\right) + \ln\left((x^4-3)^6\right)$$
$$\ln(y) = 10\ln(2x+5) + 6\ln(x^4-3)$$
$$\frac{1}{y}\cdot y' = 10\cdot\frac{1}{(2x+5)}\cdot 2 + 6\cdot\frac{1}{x^4-3}\cdot 4x^3$$
$$y' = y\left[\frac{20}{2x+5} + \frac{24x^3}{x^4-3}\right]$$
$$\boxed{y' = (2x+5)^{10}(x^4-3)^6\left[\frac{20}{2x+5} + \frac{24x^3}{x^4-3}\right]}$$

2. Use implicit differentiation (along with logarithmic differentiation) to find y', given the equation $x^y = y^x$.

$$x^y = y^x$$

$$\ln(x^y) = \ln(y^x)$$

$$y\ln(x) = x\ln(y)$$

$$y' \cdot \ln(x) + y \cdot \frac{1}{x} = x \cdot \frac{1}{y} \cdot y' + \ln(y)$$

$$y' \cdot \left[\ln(x) - \frac{x}{y}\right] = -\frac{y}{x} + \ln(y)$$

$$\boxed{y' = \frac{\ln(y) - \frac{y}{x}}{\ln(x) - \frac{x}{y}}}$$

3. Find the equation of the tangent line to the function $\tan(x)^x$ at $x = 4$

$$y = \tan(x)^x$$

$$\ln(y) = x\ln(\tan(x))$$

$$\frac{1}{y} \cdot y' = \ln(\tan(x)) + x \cdot \frac{1}{\tan(x)} \cdot \underbrace{\sec^2(x)}_{\text{derivative of } \tan(x)}$$

$$y' = \tan(x)^x \cdot \left[\ln(\tan(x)) + \frac{x \cdot \sec^2(x)}{\tan(x)}\right]$$

$$y'(4) = \tan(4)^4 \left[\ln(\tan(4)) + \frac{4\sec^2(4)}{\tan(4)}\right]$$

$$= 14.79$$

$$y = 14.79x + b \qquad y(4) = \tan(4)^4 = 1.797$$

$$1.797 = 14.79(4) + b$$

$$b = -57.36$$

$$\boxed{y = 14.79x - 57.36}$$