Day 7: More Derivatives and Linearization

We add another neat trick.

1 Derivatives of Inverse functions: Preview

- Recall: an Inverse Function to $f(x)$, notated by $f^{-1}(x)$, is such that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$. Graphically, f^{-1} is the graph of f reflected along $y = x$.
- Ex: $f(x) = e^x$. The inverse is $\ln(x)$.
- Ex: x^2 has inverse function $\sqrt{x}$.
- Ex: x^3 has inverse $x^{1/3}$.
- Any time you need to find the derivative of an inverse function, set up $f(f^{-1}(x)) = x$ and use the chain rule.
- Example: Find the derivative of $\ln(x)$. $f(x) = e^x$, $f^{-1}(x) = \ln(x)$.

$$\begin{aligned}
e^{\ln(x)} &= x \\
\left(e^{\ln(x)}\right)' &= (x)' \\
e^{\ln(x)} \cdot (\ln(x))' &= 1 \\
x \cdot (\ln(x))' &= 1 \\
(\ln(x))' &= \frac{1}{x}
\end{aligned}$$

This actually derived for us a new rule:

$$\boxed{\frac{d}{dx}\ln(x) = \frac{1}{x}}$$

- This procedure generalizes:

$$\begin{aligned} f(f^{-1}(x)) &= x \\ f'(f^{-1}(x))\frac{d}{dx}f^{-1}(x) &= 1 \\ \frac{d}{dx}f^{-1}(x) &= \frac{1}{f'(f^{-1}(x))} \end{aligned}$$

The general rule is then

$$\boxed{\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}.}$$

- Ex: Find $\frac{d}{dx}\arcsin(x)$:

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\sin'(\arcsin(x))} = \frac{1}{\cos(\arcsin(x))}$$

- Simplify $\cos(\arcsin(x)) = \sqrt{1-x^2}$ using a triangle.

- You'll get:

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}.$$

- Similar calculations will show:
  - $\frac{d}{dx}\arccos(x) = \frac{-1}{\sqrt{1-x^2}}$
  - $\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$.

2 Other Derivatives

- Other things we can differentiate now: a^x.

$$a^x = (a)^x = (e^{\ln(a)})^x = e^{\ln(a)x}.$$

Take the derivative with the chain rule:

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{\ln(a)x} = \ln(a)e^{\ln(a)x} = \ln(a)a^x$$

So,

$$\boxed{\frac{d}{dx}a^x = \ln(a)\,a^x.}$$

- Ex: Derivative of $f(x) = 2^x$:

$$f'(x) = \ln(2) \cdot 2^x.$$

- Ex: Derivative of e^x:

$$f'(x) = \ln(e)e^x = e^x$$

- We can also take derivatives of $\log_b(x)$: recall the change of base formula:

$$\log_b(x) = \frac{\ln(x)}{\ln(b)}.$$

Use it to take a derivative:

$$\begin{aligned}
\frac{d}{dx}\log_b(x) &= \frac{d}{dx}\left(\frac{\ln(x)}{\ln(b)}\right) \\
&= \frac{1}{\ln(b)}\frac{d}{dx}\ln(x) \\
&= \frac{1}{\ln(b)}\frac{1}{x} \\
&= \frac{1}{x\ln(b)}.
\end{aligned}$$

This derives a new rule:

$$\boxed{\frac{d}{dx}\log_b(x) = \frac{1}{x\ln(b)}.}$$

I recommend not memorizing this, but rather the change of base formula and apply it whenever you need to take such a derivative.