2.3 Continuity

1 Why continuity?

- You need to distinguish continuous and non-continuous functions in order to apply calculus
- Intuitive definition: a function $f(t)$ is *continuous* if you can draw its graph without lifting your pencil up.
- This is not scientific.
- Scientific (i.e. Mathematical) definition: $f(t)$ is continuous at t_0 if every single one of these are true:
  1. $f(t_0)$ is defined.
  2. $\lim_{t \to t_0} f(t)$ exists.
  3. $\lim_{t \to t_0} f(t) = f(t_0)$.
- Here's what they actually say:
  1. says f doesn't have bad stuff at t_0, like asymptotes, holes, or other weird stuff.
  2. says $f(t)$ doesn't suddenly "jump" like the graph of $|t|/t$.
  3. This says that the output matches what the limit tells us to expect. In other words, the function didn't drop part of itself.
- Ex: where are the following functions discontinuous?
  (a) $f(t) = \frac{t^2-t-2}{t-2}$: Note the problem at $t = 2$. Since $f(2)$ isn't defined at 2, it fails property 1, so f is discontinuous there. We won't go into details, but this function is continuous everywhere else.
  (b) Consider
  $$g(t) = \begin{cases} \frac{|t|}{t} & \text{if } t \neq 0 \\ 17 & \text{if } t = 0 \end{cases}$$
  Only problematic point will be $t = 0$, so let's investigate.
    - $g(0) = 17$ is defined, so $g(t)$ passes step 1.
    - Notice $\lim_{t \to 0} g(t) = \lim_{t \to 0} \frac{|t|}{t}$ which does not exist. So, it fails step 2.

  (c)
  $$f(t) = \begin{cases} \frac{t^2-t-2}{t-2} & \text{if } t \neq 2 \\ 1 & \text{if } t = 2 \end{cases}$$
    - Well, the piecewise stuff forces us to compute $f(2) = 1$, so $f(t)$ passes step 1.

- Step 2: Does $\lim_{t\to t_0}$ exist? Well, let's try doing algebra (graph would be hard): Note that $t^2 - t - 2 = (t-2)(t+1)$, do the limit stuff, see that

$$\lim_{t\to 2} \frac{t^2 - t - 2}{t - 2} = \lim_{t\to 2} t + 1 = 3.$$

  This means $f(t)$ passes step 2.

- notice that $f(2) = 1 \neq 3$, so it fails step 3.

- Ex: from biology. Often when modeling a neuron, researchers will use some kind of discontinuity to model the voltage level of the neuron. The discontinuity can be seen as the "turning on" of the neron. (More in math 247!)

- Ex: from physics. Superconductors are a huge area of study. People measure how easily electricity can flow in an object (resistivity, ρ) as a function of tempterature, T. $\rho = f(T)$. There is a critical temperature, T_C, for which the resistivity drops to *zero*, which let's you do cool stuff with floating magnets. The feature of suddenly dropping to zero is a discontinuity of the function $\rho(T)$.

2 Properties of Continuous Functions

- General rule: combinations using arithmetic operations of continuous functions are again continuous.

- Suppose $f(x)$ and $g(x)$ are continuous. Then the following list of functions are again continuous:
  - Sum: $f(x) + g(x)$
  - Difference: $f(x) - g(x)$
  - Product: $f(x) \cdot g(x)$
  - Composition: $(f \circ g)(x)$

- (Perhaps a quick review of what composition is)

- One operation needs care, as we've seen: Quotient: $\frac{f(x)}{g(x)}$. This will be continuous, provided that $g(x) \neq 0$. Points where $g(x) = 0$ will most likely be discontinuities.

- Most examples of functions you know are continuous: powers of x, e^x, $\sin(x)$, $\cos(x)$, ...

- Functions that need a little care: $\ln(x)$, $\sqrt{x}$, $\tan(x)$.

- Places to look for discontinuities:
  - zeros of denominators
  - $\ln(x)$ at $x = 0$
  - points where the definition of a function changes

- Evaluating Limits, returned: We saw in chapter 2.2 that we can evaluate $\lim_{t\to a} f(t)$ by plugging in $t = a$. I said "this works under nice situations." I can now be more exact: you can plug in and evaluate when $f(t)$ is continuous.

- Ex: $f(t) = \sin(\frac{1}{t})$