5.5: Separation of Variables

Contents

1 Separable Equations

- We are now moving on from pure-time equations. Realistically, many more problems are autonomous.

- At the same time we can develop a technique to solve a generalized class of first order equations.

- Def: A *separable* first-order DE is one of the form

$$y' = f(y)g(t)$$

  meaning that on the right-hand side, the state-variable and the independent variable can be "separated" into two things being multiplied.

- Which of these are separable?

  1. $y' = y$
  2. $y' = \cos(y)t^2$
  3. $y' = y + t$
  4. $y' = \frac{t}{y}$

5. $y'' = 2y$.

The first two are separable, the third is not, the fourth is, and the last one is not even a first order equation! So separability makes no sense there.

- Autonomous: $y' = f(y)$ is *always* separable.

2 Solving separable equations

- Ex: $y' = y$. What are the solutions? Guessing gives $y = Ce^t$ for a constant C, but let's see it another way.

- Method: separation of variables.

$$\frac{dy}{dt} = y$$
$$\frac{1}{y}\, dy = dt$$
$$\int \frac{1}{y}\, dy = \int 1\, dt$$
$$\ln|y| = t + C$$
$$|y| = e^{t+C}$$
$$y = \pm e^C e^t.$$

Since C was arbitrary anyway, $\pm e^C$ can be any nonzero number. We just rename C as $\pm e^C$, so the general solution is

$$y = Ce^t.$$

- Don't be afraid to rename the arbitrary constants. Remember, you'll find the correct *function* at the end of the day by solving for C with an initial condition.

- Ex: Solve the DE with initial condition below.

$$y' = y \cdot t, \qquad y(0) = 1.$$

Part (b): find the solution with $y(0) = -2$. Sol: separate variables.

$$\frac{1}{y}dy = t\,dt$$

$$\int \frac{1}{y}dy = \int t\,dt$$

$$\ln|y| = \frac{1}{2}t^2 + C$$

$$|y| = e^C e^{t^2/2}$$

$$y = \pm e^C e^{t^2/2}$$

now, use that $y(0) = 1$, meaning $t = 0$ and $y = 1$:

$$1 = \pm e^C e^0 = \pm e^C.$$

So, we need the $+$ out of the $\pm$, and then $C = 0$. The function is

$$y(t) = e^{t^2/2}.$$

For part (b), same steps give the same general solution. This time, though, we have

$$-2 = \pm e^C,$$

so first we need to choose the negative sign for it to make sense. Then solve $2 = e^C$ by taking ln: $C = \ln(2)$. The solution now is

$$y(t) = -e^{\ln(2)}e^{t^2/2} = -2e^{t^2/2}.$$

3 Closing Remarks

- You can *only* use separation of variables on *separable* DE's. If you try this on $y' = y + t$, for example, you will get the wrong answer because you will have forced an algebra mistake upon yourself.

- Why does it work: secretly a variable substitution happens!

$$y'(t) = f(y)g(t)$$
$$\frac{1}{f(y)}y'(t) = g(t)$$
$$\int \frac{1}{f(y)}y'(t)\,dt = \int g(t)\,dt$$
$$\int \frac{1}{f(u)}\,du = \int g(t)\,dt.$$

This is the equation you would have ended up with had you just "multiplied by dt."

- This technique quickly becomes hard to use if the RHS is complicated.

- Ex: Logistic model, DE version.

$$\frac{dy}{dt} = ry(1 - y)$$

Compare with $x_{t+1} = rx_t(1 - x_t)$.

- This is more realistic; negative rates instead of negative population values.

- You can solve this by separation of variables. Extra credit: find the solution. Hint: You'll need this integral:

$$\int \frac{1}{y - y^2}\,dy = \ln\left|\frac{y}{2(1 - y)}\right|$$

This integral can be done with *partial fraction decomposition,* which is a technique we are skipping in this course.