Written Assignment 7

Due Friday, November 30th by **5 pm.**

For this assignment you may use the cobweb and solution plotter (link on the calendar) to make all of your graphs.

1. (3.1 # 13, 14, 15) The Stability Theorem for (un)stability of an equilibrium does not tell us anything if we get a slope of 1, meaning the updating function is tangent to the diagonal. In this problem you will explore different outcomes when the Stability Theorem is inconclusive.

   (a) Graph an updating function that lies above the diagonal both to the left and to the right of an equilibrium. Is this equilibrium stable or unstable or neither?

   (b) Graph an updating function that is tangent to the diagonal at an equilibrium but crosses from below to above. Use cobwebbing to decide if the equilibrium is stable or unstable.

   (c) Graph an updating function that is tangent to the diagonal at an equilibrium but crosses from above to below. Use cobwebbing to decide if the equilibrium is stable or unstable.

2. In class we derived the discrete dynamical system for the fraction p_t in the selection model. We had

$$p_{t+1} = \frac{sp_t}{sp_t + r(1 - p_t)}.$$

   (a) Explain what the quantities s and r represent in this model.

   (b) Assume that $r > s$. What does this mean about how the two populations grow?

   (c) From class we discovered that $p^* = 0$ and $p^* = 1$ are the only two equilibria for this DDS. Again assuming that $r > s$, use the stability theorem to analyze the stability of both equilibria.

   (d) Based on your answer for part (c), what happens to the mutated population in the long run?

3. For the selection model as above, make a cobweb diagram (using the online cobwebber) for the parameter values of $r = 1.2$, $s = 3$ with an initial condition of $p_0 = 0.01$. Also make a graph of the solution. What happens to the fraction p_t in the long run? Does either population die out?

4. Make a cobweb and solution plot for the logistic model $x_{t+1} = rx_t(1 - x_t)$ with the initial seed $x_0 = 0.3$ for the values of $r = 1.5$, 2.5, and 3.5. (Show the first 10 steps in each graph.) Give one biological reason why the $r = 3.5$ graph looks much different than the others.

5. Chapter 1.10, # 15 - 18 (draw the graphs).

6. Chapter 1.9, # 14, 16.

7. Suppose that a patient's heart's voltage drops by a factor of $\frac{1}{2}$ between their SA node's signals. If the voltage right after a beat is $5\,\text{mV}$ and the critical threshold is $V_c = 2\,\text{mV}$, will the heart be able to beat on the next signal from the SA node?

8. In these problems, compute the value of the decayed voltage ($V_t e^{-\alpha T}$) and the new voltage level (V_{t+1}). Also decide if the heart will beat or not. Assume $T = 1$ millisecond.

   (a) $V_c = 20\text{mV}$, $u = 10\text{mV}$, $\alpha = 0.69$, $V_t = 30\text{mV}$.
   (b) $V_c = 20\text{mV}$, $u = 10\text{mV}$, $\alpha = 0.5$, $V_t = 30\text{mV}$.
   (c) $V_c = 20\text{mV}$, $u = 10\text{mV}$, $\alpha = 0.35$, $V_t = 30\text{mV}$.
   (d) $V_c = 20\text{mV}$, $u = 10\text{mV}$, $\alpha = 0.2$, $V_t = 30\text{mV}$.

9. For each of the cases in the previous problem, make a cobweb diagram of the model and decide if the heart is healthy or if it shows any signs of 2:1 AV block or Wenkebach phenomenon.

   (a) The case in Problem 8(a).
   (b) The case in Problem 8(b).
   (c) The case in Problem 8(c).
   (d) The case in Problem 8(d).

Extra Credit

Complete these problems for extra credit, up to 10 points. You may turn in this extra credit portion on the day of the final exam. The more effort you put in the more credit you receive.

1. Ricker Model. This problem extends the logistic model to a more realistic population model. When we derived the logistic model, the growth factor R as a function of P was assumed to be
$$R = r\left(1 - \frac{P}{N}\right)$$
   where N is the carrying capacity. This has issues, however, because it leads to negative population values. To fix this, we instead try a different decreasing function:
$$R = e^{r(1-P/N)}$$
   which leads to the following DDS:
$$P_{t+1} = P_t e^{r(1-P_t/N)}.$$
   Like before, we can define $x = P/N$, the fraction of carrying capacity. This gives the equation
$$x_{t+1} = x_t e^{r(1-x_t)}.$$
   This last equation is known as the <u>Ricker model</u>.

(a) Describe the growth factor, R, for different sizes of P (think of small, medium, and large values of P).

(b) Make a cobweb diagram with $r = 1.8$ and initial seed $x_0 = 2$ and plot the solution [feel free to use the online web app] out to the first 10 steps.

(c) Find the equilibria of the model (leaving r as a parameter and not 2).

(d) Determine the stability in the case where $r = 1.8$. For extra bonus points, determine the range of r's where the nonzero equilibrium is stable and unstable.

(e) In the logistic model $x_{t+1} = rx_t(1 - x_t)$, one can get to negative populations when x is too big (try $x_t = 1.3$). Can you get negative populations from the Ricker model if x is too positive? Based on this, how does a population following the Ricker model respond if its population starts at a large value?

2. Breathing with absorption. We now assume that a fraction, α, of the chemical is absorbed by the blood stream before you exhale. That is, your body absorbs αc_t of the chemical, leaving behind $c_t - \alpha c_t = (1 - \alpha)c_t$ of the chemical in the lungs. To model this, we employ the weighted average: we average the ambient concentration, γ, with the remaining amount of chemical, $(1 - \alpha)c_t$, where the weighting is again by q, the fraction of your lung capacity breathed in. This gives the DDS

$$c_{t+1} = (1 - q)(1 - \alpha)c_t + q\gamma.$$

(a) With $q = 0.20$, $\alpha = 0.3$, and $\gamma = 3$, make a cobweb with initial value $c_0 = 0$. Then explain what these numbers mean in context of your breathing.

(b) Without substituting any numbers for α or q, find the equilibrium value c^* of this new model. Do you expect it to be bigger or smaller than the case of no absorption? Compare your answer with the case of no absorption ($\alpha = 0$).

(c) Does adding in absorption affect stability of the equilibrium?

3. Selection model with interactions. Suppose we want a more realistic selection model where the two subpopulations interact with each other. Let's assume that population A grows with an ordinary population model, and that population B grows on its own but is hunted by population A. This can be modeled with the system of equations

$$\begin{aligned} a_{t+1} &= ra_t \\ b_{t+1} &= sb_t - a_t, \end{aligned}$$

where the hunting of population B is modeled by putting a term with a_t in the equation for b. (We subtract because a larger A population means there will be fewer of the b population.)

(a) Let p be the proportion of population B. Repeat the derivation of the selection model to find a discrete dynamical system for this proportion.

(b) Find the equilibria of the model with the parameter values $r = 1.5$ and $s = 2$. Are all of them biologically realistic?

(c) Determine the stability of the biologically realistic equilibria, again in the case of $r = 1.5$ and $s = 2$.

(d) How much can you say in general about the equilibria? Explore different values of r and s.

(e) Can you find parameter values that lead to a negative proportion? What would this mean in context of the two populations?

4. A continuous version of the heart model. When we derived the heart model we made the assumption that the spike in voltage, u, was either "on or off," depending on the critical threshold. This lead us to needing a discontinuous updating function (i.e. the piecewise formula). We can instead take the approach of modeling the spike u to depend on the voltage level, giving a continuous function instead of a piecewise function. We now write

$$V_{t+1} = cV_t + u,$$

where now the spike in voltage, u, depends on V_t:

$$u = \frac{2(1-c)}{1+V_t}.$$

This leads us to the continuous version of the discrete dynamical system for the heart model:

$$V_{t+1} = cV_t + \frac{2(1-c)}{1+V_t}.$$

(a) Does this heart ever skip a beat? Why or why not?

(b) Find the equilibrium for this model in terms of the parameter c. (Feel free to use $c = 0.5$ and generalize your answer.)

(c) Is the equilibrium stable or unstable for $c = 0.5$? Are there values of c where it becomes unstable? What might this mean for the person's heart?