Intro to differential equations

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1 What are they?

- A differential equation is an equation relating a function's outputs to its derivatives. Remember, a derivative means a function's rate of change.
- Example: consider the following scenario: the speed of a car is a constant $10\,\mathrm{m/s}$. Let x be the position of the car at time t. We have

$$\frac{dx}{dt} = 10$$

- This is a rather simple differential equation (DE). In particular, it says that the funciton $x(t)$ has a constant rate of change.
- Diffusion across a membrane. Consider a cell that absorbs and emits potassium. Let P be the concentration of potassium in the cell, and let P_0 be the concentration of potassium outside the cell.
- Sensible model: *the amount of potassium entering the cell is proportional to the difference between the cell's potassium level and the ambient potassium level.*

- This gives us the following:

$$\frac{dP}{dt} = k(P - P_0).$$

- k is called a *constant of proportionality.*

- Let's pretend that $P_0 = 10\,\mathrm{mmol/ml}$ and $k = 0.1$. Then this equation reads

$$\frac{dP}{dt} = 0.1P - 1$$

- This equation relates the amount of potassium to how much potassium enters the cell.

2 In real life

- *You do not need to memorize or know any of these examples. They are just to illustrate how common and applicable they are to science.*

- Many examples of DE's from real life.

- Newton's Second law, $F_{\text{net}} = ma$, gives rise to DE's for the position, y, of an object:

$$F = m\frac{d^2y}{dt^2}$$

- As a simple example, consider the force of gravity on a falling object. Near the surface of the Earth, $F = -mg$, giving the differential equation

$$m\frac{d^2y}{dt^2} = -mg,$$

or

$$\frac{d^2y}{dt^2} = -9.8\,\mathrm{m/s}^2.$$

In fact, starting from Newton's second law, all of classical physics can be derived from DE's to understand everything we know about how objects move!

- A different example from physics: using Hooke's law that $F = kx$ and x is the displacement of a spring from equilibrium, gives an equation

$$m\frac{d^2x}{dt^2} = kx$$

- In Quantum Mechanics (Physics), the location of a particle is described probablistically by a function, $\psi(x)$, that obeys a DE known as the *Schrödinger equation*:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi$$

  which forms the basis of study of many phenomena, from optics, to particles, to electronics and more.

- Chemistry and Chemical Reactions. Consider a chemical reaction $X + Y \xrightarrow{k} Z$, where X, Y, Z are some chemicals. Write $[X]$ for the concentration of X. The *law of mass action* leads one to consider a *system* of DE's for the concentrations:

$$\begin{aligned}\frac{d[X]}{dt} &= -k[X][Y]\\ \frac{d[Y]}{dt} &= -k[X][Y]\\ \frac{d[Z]}{dt} &= +k[X][Y]\end{aligned}$$

  (disclaimer: I am not an expert on chemistry, so this example is foreign to me. But it highlights an application of DE's to chemistry.)

- To biology: The Lotka-Volterra equations describe how two populations change in time with interspecies interactions, like a predator population and a prey population:

$$\begin{aligned}\frac{dx}{dt} &= \alpha x + \beta xy\\ \frac{dy}{dt} &= \gamma y + \delta xy\end{aligned}$$

- Neurons: we will study the funcitoning of a single neuron, which will be modeled with a system of two equations, called the *Fitzhugh-Nagumo* equations:

$$\frac{dv}{dt} = -v(v-a)(v-1) - w$$
$$\frac{dw}{dt} = \varepsilon(v - \gamma w).$$

  These are actually a simplification of a more complex neuron model, called the Hodgkin-Huxley equations.

3 Solutions to DEs

- A *solution* to a differential equation is a *whole function.*
- Contrast this with an algebraic equation.
- Compare: $14y'' + y = x$ and $14t^2 + t = 9$.
- For an algebraic equation, the solution is a particular number, or a couple of numbers.
- For a differential equation, the solution is an entire function.

3.1 Verifying Solutions

- We verify that a function is a solution by instering it into the DE and checking equality.
- Example: verify that $f(x) = e^{2x}$ is a solution to the differential equation $f' = 2f$.
- One side: $f' = 2e^{2x}$
- Other side: $2f = 2e^{2x}$.
- These are equal! Hence $f(x) = e^{2x}$ is a solution to this DE.
- Ex: Verify that $g(x) = x^5$ is a solution to the differential equation

$$\frac{dg}{dx} = 5x^4.$$

- Since $g'(x) = 5x^4$, we have verified $g(x)$ as a solution.
- Ex: Verify that $\sin(x)$ is a solution to the equation

$$\frac{d^2y}{dx^2} = -y$$