Day 2: The Derivative

1 Definition

- Secant Line: Given $x = a$ and $x = b$, it's the line from $f(a)$ to $f(b)$.
- Tangent Line: Given $x = a$, it's the line that just barely comes into contact with $f(x)$ at $x = a$. (This definition is absolutely essential to remember and understand throughout this course.)
- Important Concept: We can find the tangent line by limiting the secant lines in a way where $x = b$ approaches $x = a$.
- Derivative: Just another word for "slope of the tangent line."
- The derivative of $f(x)$ at a point x is rigorously defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- The quantity $\dfrac{f(x+h) - f(x)}{h}$ is called the *difference quotient*.
- Be aware: the limit is with respect to h, not x.
- Alternative notations: $\dfrac{df}{dx}, \dfrac{dy}{dx}$.

2 Interpretations and related definitions

- Suppose $f(t)$ represents the position of a particle moving in a straight line. Then $f'(t)$ represents the particle's *instantaneous velocity* at time t. (Positive = moving forward, negative = moving backward.)

  Contrast this with:

- average velocity: on an interval $[a,b]$, the average velocity is just

$$\frac{f(b)-f(a)}{b-a}.$$

  (This is just the slope of the secant line on this interval).

- If $f(t)$ represents the velocity of a particle, then $f'(t)$ represents the *acceleration* of the particle.

- When $f(x)$ isn't a position or velocity, we usually say "average rate of change" or "instantaneous rate of change."

Calculations

- $f(x) = x + 2$. Calculate $f'(2)$.

$$\begin{aligned} f'(2) &= \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} \\ &= \lim_{h\to 0} \frac{(2+h)+2-(2+2)}{h} \\ &= \lim_{h\to 0} \frac{h}{h} \\ &= \lim_{h\to 0} 1 = 1. \end{aligned}$$

  So, the slope of the tangent line to $f(x)$ at $x = 2$ is 1. Expected, because tangent line to a line is the line itself.

- $f(x) = \frac{1}{x}$. Find $f'(4)$.

$$
\begin{aligned}
f'(4) &= \lim_{h \to 0} \frac{f(4+h) - f(4)}{h} \\
&= \lim_{h \to 0} \frac{\frac{1}{4+h} - \frac{1}{4}}{h} \\
&= \lim_{h \to 0} \frac{1}{h} \left(\frac{1}{4+h} - \frac{1}{4} \right) \\
&= \lim_{h \to 0} \frac{1}{h} \frac{4 - (4+h)}{4(4+h)} \\
&= \lim_{h \to 0} \frac{1}{h} \frac{-h}{4(4+h)} \\
&= -\frac{1}{16}.
\end{aligned}
$$

So, the slope of the tangent line to $\frac{1}{x}$ at $x = 4$ is $-\frac{1}{16}$.

- Calculations often involve using algebra to manipulate the difference quotient in order to cancel a factor of h, making the limit calculatable.