5.1: Analyzing Diffy-Q's

Setting up Diffy-Q's from a description

- Shifting focus: we'll learn to set up differential equations from word descriptions.
- Also pay attention to constants and their units.
- Use Euler's method to numerically solve a differential equation. (Most often done in practice: you plug in the derived diffy-Q into a computer to analyze its numerically found solutions.)
- Ex: Newton's Law of Cooling. This states that

  The rate at which heat is lost is proportional to the difference between the temperature of the object and the ambient temperature.

  Let H denote the temperature of the object.
- **Rate at which heat is lost**: $\frac{dH}{dt}$
- **is**: $=$
- **proportional to**: we multiply one side by a constant α.
- **difference between temperature and ambient temperature**: $(A - H)$, letting A be the ambient temp.
- Equation:
$$\frac{dH}{dt} = \alpha(A - H).$$
- Units of α: inverse seconds, or 1/second.
- Let them do some examples.

- Diffusion across membrane. Consider a cell that both absorbs and spits out a chemical. We'll keep track of the concentration, C (mmol/L), of the chemical in the cell. Let

$$\Gamma = \text{ ambient concentration (mmol)/L}$$

$$\beta = \text{ transfer coefficient across membrane (mmol/L/sec).}$$

$$\frac{dC}{dt} = \text{ entering rate} - \text{ leaving rate}$$

$$\frac{dC}{dt} = \beta(\Gamma - C).$$

Selection Model Revisited

- Selection Model revisited. Two populations (say one represents humans, the other represents giant telepathic spiders), a and b. Interested in the relative fraction of one of them, call it p. By definition, $p = \frac{a}{a+b}$. Also, $1 - p = \frac{b}{a+b}$.
- Say that two pop's have

$$\frac{da}{dt} = \mu a, \qquad \frac{db}{dt} = \lambda b.$$

- Then you can derive a diffy-Q for the function p. This is done in the book, section 5.1.4.
- Get:

$$\frac{dp}{dt} = (\mu - \lambda)p(1 - p)$$

- Looks very different than the discrete dynamical system you found last quarter:

$$p_{t+1} = \frac{sp_t}{sp_t + r(1 - p_t)}$$

- Moral: Never try to turn a DDS into a Diffy-Q by just changing letters around. Always derive one from first principles.

Euler's Method

- Euler's method is a way to approximate solutions to differential equations.
- Bonus: It works for *any* (first order) diffy-Q.
- Downside: you only get lists of numbers, not actual formulas.
- Algorithm: for a diffy-Q $\frac{df}{dt} = ...$,
  1. Choose a Δt step size.
  2. Use the diffy-Q to get a slope at time t.
  3. The *predicted* value $\hat{f}$ at the next time step $t + \Delta t$ will then be $\hat{f} = f(t) + \Delta t \times \text{(slope)}$.
  4. Repeat ad infinitum.
- Ex. Take the simple population model $\frac{dP}{dt} = 3P$ with initial condition $P(0) = 2$. Let's choose a step size $\Delta t = 0.5$. We'll do three steps of Euler's method.
- First step: $P'(0) = 3P(0) = 3(2) = 6$. So,

$$\hat{P}(0.5) = 2 + (6)\Delta t = 2 + 6(0.5) = 5.$$

- Second step: using the diffy-Q again: $P'(0.5) = 3P(0.5) = 3(5) = 15$. So,

$$\hat{P}(1) = 5 + 15(0.5) = 5 + 7.5 = 12.5$$

- Third step: use the diffy-Q: $P'(1) = 3P(1) = 3(12.5) = 37.5$. Then

$$\hat{P}(1.5) = 12.5 + (37.5)\Delta t = 12.5 + (37.5)(0.5) = 31.25.$$