Limits, Part 3: At infinity and l'Hopital's rule

This follows roughly chapters 3.5 and 3.6 in the textbook.

1 Limits at infinity

- The idea of a limit as $x \to \infty$ is that $\lim_{x\to\infty} f(x)$ is the *horizontal asymptote* of $f(x)$.
- Put another way, it is the *long-run behavior* of $f(x)$.
- Useful trick for evaluating: factor out the largest power of x.
- Ex:
$$\lim_{x\to\infty} \frac{x}{x+1} = \lim_{x\to\infty} \frac{x}{x(1+\frac{1}{x})} = \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = 1$$
because $\frac{1}{x} \to 0$ as $x \to \infty$.
- Review: Little-Big principle: $\lim_{x\to\infty} \frac{1}{x^p} = 0$ as $x \to \infty$, if p is a positive power.
- Ex:
$$\begin{aligned}\lim_{x\to\infty} \frac{x^3+x^2+17}{6x^3+15x+5000} &= \lim_{x\to\infty} \frac{x^3(1+\frac{1}{x}+\frac{17}{x^2})}{x^3(6+\frac{15}{x^2}+\frac{5000}{x^3})}\\ &= \frac{1}{6}\end{aligned}$$
because all the terms with x's in their denominators all approach 0 by the Little-Big principle.
- Ex:
$$\lim_{x\to\infty} \frac{x^2+1}{3x+7} = \lim_{x\to\infty} \frac{x^2(1+\frac{1}{x^2})}{x(3+\frac{7}{x})} = \lim_{x\to\infty} \frac{x}{3} = \infty.$$

- When you see $\lim_{x\to\infty} f(x) = \infty$, remember that ∞ is not a number. What this equation means is really that $f(x)$ just keeps getting larger as x goes off to ∞.

- Exponentials: $\lim_{x\to\infty} b^x = 0$ or ∞, depending on which direction the exponential goes. Always use a graph to decide.

- Ex:
$$\lim_{x\to\infty} \frac{1-(0.5)^x}{1-(0.5)^{10}} = \frac{1}{1-(0.5)^{10}}$$
because $(0.5)^x$ is a decaying exponential function, so it goes to 0 as $x \to \infty$.

- Factoring also works with exponentials:

- Ex:
$$\begin{aligned}\lim_{x\to\infty} \frac{e^{2x}-e^{-x}}{e^{2x}+e^{-x}} &= \lim_{x\to\infty} \frac{e^{2x}(1-e^{-3x})}{e^{2x}(1+e^{-3x})}\\ &= \lim_{x\to\infty} \frac{1-e^{-3x}}{1+e^{-3x}}\end{aligned}$$
and now e^{-3x} is a decaying exponential, so $e^{-3x} \to 0$. The final result is then 1, so
$$\lim_{x\to\infty} \frac{e^{2x}-e^{-x}}{e^{2x}+e^{-x}} = 1.$$

l'Hoptial's Rule

- If both the numerator and denominator of a fraction go off to infinity, then we can study the ratio of *growth rates* instead. This is the idea behind l'Hoptial's rule:

- If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{\infty}{\infty}$, then
$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}.$$

- Ex:
$$\lim_{x\to\infty} \frac{x^2+1}{3x+7} = \frac{\infty}{\infty},$$

so use l'Hopital's rule:

$$\lim_{x\to\infty} \frac{x^2+1}{3x+7} = \lim_{x\to\infty} \frac{2x}{3} = \infty.$$

- Ex:

  $$\lim_{x\to\infty} \frac{x^3}{e^x}.$$

  Use three times to get $\lim_{x\to\infty} \frac{6}{e^x} = 0$.

- Note: You cannot apply l'Hopital's rule without having ∞/∞.

- Note: l'Hopital also works for other limits. Also, you can also have $0/0$ as one of the "allowed" indeterminate forms.

- Ex:

  $$\lim_{x\to 0^+} \frac{\ln(x)}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} \frac{-x}{1} = 0.$$