3.3: Optimization, Part 1

1 Local extrema

- Bee Model: The bee wants to maximize the amount of food it collects *per visit*, or *per second* it spends at each flower.
- $F(t) = \frac{t}{t+1}$ is the amount of food it collects (Draw picture) as a function of time
- $R(t) = F(t)/(t + \tau)$, where τ is a travel time.
- Analyze: How long should the bee spend at the flower if it takes 2 seconds to travel between flowers (on average)?
- Ans: about 1.4 seconds.
- Def: A *local* maximum or minimum is where the graph reaches a high or a low point (but it need not be the *highest* or *lowest*).
- We find local optima by only looking at *critical points*.
- We use the second derivative to tell if the critical point is a max or min. (Called the Second Derivative Test.)
- Algorithm:
  1. find critical points by solving $f'(x) = 0$. You now have x-values.
  2. Plug these x-values into $f''(x)$.
  3. if positive comes out, then that was a local *minimum*.
     If negative comes out, then it was a local *maximum*.

2 Examples

- Find all local optima for the function $f(x) = x^3 - 3x$.
  1. critical points: $f'(x) = 3x^2 - 3 = 0$ gives $x = \pm 1$. (Two answers!)
  2. $f''(x) = 6x$. Note: $f''(-1) = 6(-1) = -6$, so $x = -1$ is a local maximum!
  3. $f''(1) = 6(1) = 6$, so $x = 1$ is a local minimum.
- Find all local optima for the function $G(t) = te^{-t}$.
  1. critical points: $G'(t) = e^{-t} - te^{-t} = e^{-t}(1 - t) = 0$ gives $t = 1$. Only one critical point.
  2. $G''(t) = (-1)e^{-t} + (1 - t)(-1)e^{-t}$. So,

     $$G''(1) = -1e^{-1} + (1 - 1)(\text{stuff}) = -1e^{-1} < 0$$

     meaning that $t = 1$ is a local maximum.

- Find all local optima for the function $H(x) = \frac{1}{x^2+1}$.

  1. critical points: use chain rule!

  $$H'(x) = -(x^2+1)^{-2}2x = \frac{-2x}{(x^2+1)^2} = 0.$$

  only the numerator can make a fraction 0! so we must have $x = 0$ is the only critical point.

  2. Need $H''(x)$! yikes, this seems really ugly. Do it anyway!

  $$H''(x) = \frac{-2(x^2+1)^2 - (-2x)(2(x^2+1)2x)}{(x^2+1)^4}.$$

  Plug in $x = 0$:

  $$H''(0) = \frac{-2(1^2) - 0}{1^4} = -2.$$

  So, $x = 0$ must be a local maximum!