4.7: Improper Integrals

Contents

1 Total Change revisted

- Definite integrals are net change. If $\frac{dy}{dt} = f(t)$, this says in English that the function $f(t)$ is a rate, and integrating a rate produces a net change in that quantity.
- $\int_0^b f(t)\,dt$ is the net change from the initial value of y.
- What if we want to know the rate of change in the long run? Meaning, if we go nearly forever?
- Ex: Suppose that a bunch of bacteria are given some food, but that they end up eating most of the food and become static afterwards. Let's model this by saying that the population $P(t)$ satisfies the differential equation
$$\frac{dP}{dt} = \frac{3.5}{(1+t)^2}$$
- population measured in millions.
- Question: given the initial drop of food, how much can the population grow by?

- No obvious end in time. One option, use a huge time value.
- $$\text{net change} = \int_0^{\text{big}} \frac{3.5}{(1+t)^2}\,dt = \left.\frac{-3.5}{(1+t)}\right|_0^{\text{big}} = \frac{-3.5}{(\text{big})} + \frac{3.5}{1}.$$
- A useful mathematical trick is to replace something (big) with ∞.
- Any time we write ∞, we technically mean "take the limit as the upper bound goes to ∞.
- Then
$$\int_0^{\infty} \frac{3.5}{(1+t)^2}\,dt = \lim_{N\to\infty}\left(\frac{-3.5}{(1+N)} - (-3.5)\right) = 3.5$$
- The trick gives us a nicer round number, like 3.5, instead of something awful, like 3.4698.

2 Improper Integrals

2.1 Infinite Bounds

- We can interpret infinite bounds as saying "the net change in the long run."
$$\boxed{\text{net change in the long run} = \int_0^{\infty} f(t)\,dt}$$
- Caveat: This only works if the function being integrated goes to zero fast enough. Otherwise we're adding up infinite area, and the integral is said to *diverge*.
- Ex: Suppose that our bacteria colony instead grew according to the rate
$$\frac{dy}{dt} = \frac{3.5}{1+t}.$$
Then you get that the net change in the long run is
$$\int_0^{\infty} \frac{3.5}{1+t}\,dt = 3.5\ln|x+1|\Big|_0^{\infty} = \lim_{N\to\infty} 3.5\ln(N+1) - 3.5\ln(0+1) = \infty.$$

We call this integral *Divergent*. We can still get meaning from this: it just means that in this case our bacteria will never stop growing with the food supply it received.

2.2 Systematic study of cases

- Some cases we know really well:

$$\int_a^\infty \frac{1}{x^p}\,dx$$

- This converges if $p > 1$, and diverges if $p \leq 1$. Here, a is any positive number.

- The way to tell is just by integrating, and seeing if you get a sensible limit or not.

- Ex: $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx = 1$. here, $p = 2$, illustrating the above point.

- Ex: $\displaystyle\int_1^\infty \frac{1}{x}\,dx = \infty$, and here $p = 1$. So this again illustrates the above point; it diverges.

- Ex: $\displaystyle\int_1^\infty x\,dx = \infty$ certainly (just look at a picture!). Here, you can think of this as $p = -1$, so it still diverges. The above fact still stands.

- $\displaystyle\int_2^\infty \frac{17x}{x^2+1}\,dx$ is not exactly, but similar to, $\frac{17}{x}$, because in the long run, the $+1$ doesn't matter. So we expect this integral to diverge. You can check this by doing a sub $u = x^2 + 1$. It will turn into something like $\int \frac{1}{u}\,du$, which sits in the case $p = 1$. So it diverges.

- The bizarre thing here is that some functions go to 0 quickly enough, and some don't. $\frac{1}{x}$ goes to 0, but it does so too slowly, and integrating it gives you an infinite amount of area!

- Another one that is well known:

$$\int_0^\infty e^{ax}\,dx$$

This integral converges for $a < 0$, and diverges for $a \geq 0$. Again, just do some examples to get a feel for it.

- Ex: $\displaystyle\int_0^\infty e^{-2x}\,dx = -\frac{1}{2}e^{-2x}\Big|_0^\infty = -\frac{1}{2}\left(\lim_{N\to\infty} e^{-2N}\right) + \frac{1}{2}e^0 = \frac{1}{2}.$

- Moral of the story: if you aren't sure or you can't remember the general rule of thumb, just integrate it and evaluate limits.