DKE Numerical Mathematics
Exam Wednesday 7 June 2017

You may *only* use the formula sheet provided, and an departmental-approved electronic calculator. You may *not* use a textbook or your own notes.

1. **[16 points]** Sketch the data below.

x	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5	6.0
y	0.9829	0.3338	0.0942	0.2320	0.3700	0.6575	0.8575	0.9964	0.5956	−0.1273	?	−1.1185	−0.9737

Comment on the following statements on estimates of the missing value of y when $x = 5.0$, sketching graphs of the approximating function:

7 (a) The linear least-squares approximation $g_1(x) = -0.22711x + 0.88518$ has a root-mean-square error of 0.52, and yields an estimate of $y \approx -0.25$.

(b) The interpolating polynomial yields an estimate of $y \approx 0.7118$.

(c) The cubic spline interpolation yields an estimate of $y \approx -0.774515$.

For each part, write down Matlab commands to compute the approximation, given arrays `xs`, `ys` of the data.

2. **[12 points]** Use the secant method to estimate the positive solution of $\sin(x) - x + 1 = 0$ to an accuracy of 10^{-2}, starting with $p_0 = 0$ and $p_1 = 2$. What is the best bracket for the root you obtain? What would happen if you tried to use Newton's method starting at $p_0 = 0$?

1

3. **[12 points]** Use divided differences to compute the cubic polynomial interpolating following data:

2

x	4.0	4.5	5.0	5.5	6.0
y	0.60	−0.13	?	−1.12	−0.97

Hence estimate the value of y when $x = 5.0$.

4. **[10 points]** Estimate $f'(2.0)$, $f'(2.1)$ and $f''(2.0)$ using the most accurate three-point formula available from the following data:

3

x	1.8	1.9	2.0	2.1
$f(x)$	0.2516	0.2534	0.2558	0.2589

Compute the absolute and relative errors in your answers, and comment on the magnitude of the errors and main sources of error in each case.

What would happen to the error if you used $f(1.99)$ and $f(2.01)$ to estimate $f'(2.0)$ and $f''(2.0)$?

Note: The data points are the exact values of $f(x)$, rounded to 4 decimal places. The exact values of the derivatives (to 6dp) are $f'(2.0) = 0.027230$, $f'(2.1) = 0.033827$, and $f''(2.0) = 0.065296$.

5. **[12 points]** When using the three-stage Adams-Bashforth method to solve a differential equation, which would you prefer to use as a bootstrapping method: Ralston's second order method, Heun's third-order method, or Fehlberg's fourth-order method? Explain your answer.

4

Solve the differential equation $\dot{y} = 1 - t/y$ over $[0, 1]$ with initial condition $y(0) = 2$ using the three-stage Adams-Bashforth method with $h = 0.25$. You should use the following values to bootstrap your method:

i	t_i	w_i	$f(t_i, w_i)$
0	0.0	2.00000	1.00000
1	0.25	2.76515	0.90959
2	0.5	3.47882	0.85627

6. **[10 points]** The least-squares appoximation of degree 6 to a function f has Legendre coeffients c_k given by

5

k	0	1	2	3	4	5	6
c_k	0.182	−0.672	−0.937	−0.497	0.786	0.275	−0.019

Using the recurrence relation for Legendre polynomials, estimate the value of $f(0.2)$.

By how much would the total square error $E = \int_{-1}^{+1}(f(x) - g(x))^2 dx$ decrease by taking a degree 7 approximation with $c_7 = -0.153$?

7. **[16 points]** Use (i) the trapezoid rule, and (ii) Simpson's rule with $n = 8$ to estimate the Fourier coefficient b_1 of the function $f(x) = 1/(3 + \sin(x))$, which takes the following values:

6

x	$-\pi$	$-\frac{3}{4}\pi$	$-\frac{1}{2}\pi$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
y	0.3333	0.4361	0.5000	0.4361	0.3333	0.26975	0.2500	0.26975	0.3333
$\sin(x)$	0	−0.7071	−1	−0.7071	0	0.7071	1	0.7071	0

Use the error estimate formula for the adaptive trapezoid rule to estimate the error of (i).

Given that $-0.25 \leq f''(\xi) \leq 0.08$ for $\xi \in [-\pi, +\pi]$, compute a bound for the error of (i).

8. **[12 points]** The *forced Duffing equation* is given by

$$\ddot{x} + \delta\dot{x} + \alpha x + \beta x^3 = \rho\cos(\omega t),$$

where $\alpha, \beta, \delta, \rho, \omega$ are parameters. Explain how to simulate this system using Matlab, with initial condition $x(0) = 2$ and $\dot{x}(0) = -3$, including giving the code you would write.