

**Math 275 Notes**  
**Calculus For Engineers and Scientists**  
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## 1 Derivatives

### Derivative of a Function:

A function  $f$  is said to have a derivative at a real number  $c$  if

$$\lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h} \text{ exists}$$

### An Alternative Definition of the Derivative:

$$f'(x) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$$

### Differentiable Function:

A function  $f$  is said to be differentiable at  $c$  if  $f'(x)$  exists. However if  $f'(c)$  does not exist, one says  $f$  is not differentiable at  $c$ .

### Other Notations for Derivatives:

Given  $y = f(x)$ , the derivative may be denoted by

1.  $f'(x)$
2.  $y'$
3.  $\frac{dy}{dx}$
4.  $\frac{d}{dx} \{f(x)\}$

### Function Notation:

Let  $f$  and  $g$  be given functions

1. Sum:  $(f + g)(x) = f(x) + g(x)$
2. Difference:  $(f - g)(x) = f(x) - g(x)$
3. Constant Multiple:  $(kf)(x) = k f(x)$
4. Product:  $(fg)(x) = f(x)g(x)$
5. Quotient:  $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$
6. Composition:  $(f \circ g)(x) = f(g(x))$

### The Power Rule:

$$\frac{d}{dx} (x^n) = n x^{n-1}$$

### Derivative Rules:

Let  $f$ ,  $g$ ,  $u$  and  $v$  be differentiable functions and  $k$  be a constant:

1. Sum/Difference Rules:  $\frac{d}{dx} (f(x) \pm g(x)) = \frac{d}{dx} (f(x)) \pm \frac{d}{dx} (g(x))$
2. Constant Multiple:  $\frac{d}{dx} (k f(x)) = k \frac{d}{dx} (f(x))$

3. Product Rule:  $\frac{d}{dx}(uv) = v \frac{d}{dx}(u) + u \frac{d}{dx}(v) = v u' + u v'$

4. Quotient Rule:  $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \frac{d}{dx}(u) - u \frac{d}{dx}(v)}{v^2} = \frac{v u' - u v'}{v^2}$

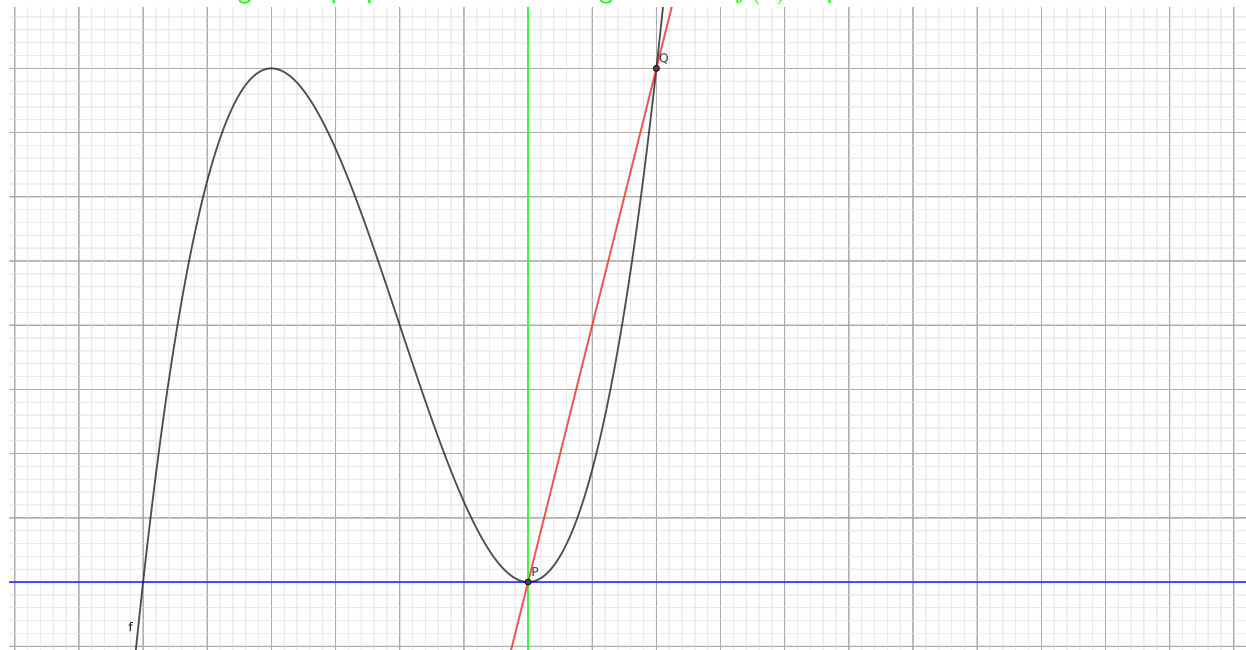
5. Chain Rule:  $\frac{d}{dx}(f(u)) = f'(u) \frac{du}{dx}$

**Geometric Interpretation of the Derivative:** Let  $y = f(x)$  be a differentiable function at  $c$ , and let  $P(c, f(c))$ , and  $Q(x, y) = Q(x, f(x))$  be points on its graph as shown:

**Secant Line:** A straight line joining any 2 points  $P$ , and  $Q$  on  $f(x)$

**Tangent Line:** A straight line that touches  $f(x)$  at point  $P$

**Normal Line:** A straight line perpendicular to the tangent line of  $f(x)$  at point  $P$



### The Slope of a Tangent Line:

Let  $y = f(x)$  be a differentiable function at point  $c$ . Then the slope of the tangent line of  $f$  at  $c$  is

$$m_{tan}|_{x=c} = f'(c)$$

### Equation of a Straight Line:

Point Slope Form: An equation of the straight line passing through the point  $P(x, y)$ , and has slope  $m$  is of the form

$$y - y_0 = m(x - x_0)$$

### Higher Order Derivatives

Let  $f$  be a differentiable function, that  $f'(x)$  exists. From now on we may call  $f'(x)$  the first order derivative of  $f$ . Assume  $f$  is still differentiable. The second order derivative is defined as:

$$\frac{d}{dx} \{f'(x)\} = f''(x)$$

### Continuity at a Point

A function  $f$  is said to be continuous at  $c$  if the following the conditions hold:

1.  $f'(c)$  is defined and real
2.  $\lim_{x \rightarrow c} f(x) = L$  where  $L$  is a real, non infinite number.

$$3. \lim_{x \rightarrow c} f(x) = f(c)$$

If any of the above conditions are not satisfied then  $f$  is not continuous at  $c$ .

**Left and Right Hand Derivatives:** Let  $f(x)$  be a given function.

1. The left-hand derivative of  $f$  at  $c$  is denoted and defined by

$$f'_-(c) = \lim_{x \rightarrow c^-} \frac{f(x) - f(c)}{x - c}$$

provided the limit exists.

2. The right-hand derivative of  $f$  at  $c$  is denoted and defined by

$$f'_+(c) = \lim_{x \rightarrow c^+} \frac{f(x) - f(c)}{x - c}$$

provided the limit exists.

Both the left and right hand derivatives may not exist.

**Differentiability and Left and Right hand derivatives:**

Let  $f$  be a given function

1. If left and right hand derivatives of  $f$  at  $c$  exist and are equal then  $f$  is differentiable at  $c$ , and that  $f'(c) = f'_-(c) = f'_+(c)$ .
2. If either the left or right hand derivatives of  $f$  at  $c$  do not exist or both exist but are not equal, then  $f$  is not differentiable at  $c$  and  $f'(c)$  does not exist.

**Relationship between Differentiability and Continuity:** Let  $f$  be a given function. If  $f$  is differentiable at  $c$ , then  $f$  is necessarily continuous at  $c$ .

- If  $f$  is differentiable at  $c$  then  $f$  must be continuous
- If  $f$  is discontinuous at  $c$  then  $f$  is not differentiable
- The converse of the theorem is not true.

**Easy way to calculate The left and Right Hand Derivative:**

Let  $f$  be a given function continuous at  $c$ .

$$1. f'_-(c) = \lim_{x \rightarrow c^-} f(x)$$

$$2. f'_+(c) = \lim_{x \rightarrow c^+} f(x)$$

## 2 Special Functions

**The Derivatives of the Six Trigonometric Functions:**

$$1. (a) \lim_{h \rightarrow 0} \frac{\sin(h)}{h} = 1$$

$$(b) \lim_{h \rightarrow 0} \frac{\cos(h)}{h} = 0$$

$$2. (a) \frac{d}{dx} (\sin(x)) = \cos(x)$$

$$(b) \frac{d}{dx} (\cos(x)) = -\sin(x)$$

3. (a)  $\frac{d}{dx}(\tan(x)) = \sec^2(x)$   
 (b)  $\frac{d}{dx}(\cot(x)) = -\csc^2(x)$
4. (a)  $\frac{d}{dx}(\sec(x)) = \sec(x)\tan(x)$   
 (b)  $\frac{d}{dx}(\csc(x)) = -\csc(x)\cot(x)$

## 2.1 Inverse Functions

### A Function of a Single Real Number:

A function  $f$  is a rule that assigns to each permissible real number  $x$ , one and only one real number  $y$ .

$$y = f(x)$$

### Vertical Line Test for the Graph of a Function:

Every vertical line cuts the graph of a function at most once.

### Properties of the Graph of a Function:

The graph of a function can be used as a tool which enables us to obtain  $y$  from a given  $x$ .

It is possible that  $f(x_1) = f(x_2) = y$  for  $x_1 \neq x_2$ . If the function  $f(x)$  has no points  $x_1 \neq x_2$  such that  $f(x_1) = f(x_2)$  it is said to be an invertible function. The inverse of  $f$  is denoted  $g$  or  $f^{-1}$ .  $f^{-1}$  takes the  $y$  value back to  $x$ .

$f(x) = y$  and  $f^{-1}(y) = x$  are equivalent.

### Conversion Rules:

To convert one statement to the other simply move  $f$  from one side to the other as  $f^{-1}$  and vice versa.

$$f(x) = y \Leftrightarrow f^{-1}(y) = x$$

### One To One Functions:

Let  $f$  be a given function on  $[a, b]$

The function  $f$  is said to be one to one on  $[a, b]$  if every horizontal and vertical line cuts the graph of the function at most once.

- A function may not be one to one on  $[a, b]$ , however  $[a, b]$  may be restricted so that the function is one to one.
- Important Examples of one to one functions
  1. Strictly Increasing functions ( $f'(x) > 0, x \in [a, b]$ )
  2. Strictly Decreasing function ( $f'(x) < 0, x \in [a, b]$ )

### One to One Functions and the Inverse

Let  $f$  be a given function defined on  $[a, b]$ . If  $f$  is one to one on  $[a, b]$ , then  $f$  is invertible on  $[a, b]$ , meaning  $f^{-1}$  exists.

### A Formula for the Inverse Function:

Let  $f$  be a given function and assume  $f$  has an inverse. To find a formula for  $y = f^{-1}(x)$ :

1. Interchange  $x$  and  $y$ .  $x \leftrightarrow y$
2. Solve for  $y$  as a function of  $x$

If an explicit inverse does not exist, leave in an implicit form.

### The Derivative of the Inverse Function:

Let  $y = f(x)$  be a given function. Assume  $f$  has an inverse. Then

$$\frac{d}{dx}(f^{-1}(c)) = \frac{1}{f'(f^{-1}(c))}$$

### Properties of Inverse Functions:

1. The domain of  $f$  coincide with the range of  $f^{-1}$  and vice versa.
2. Cancellation Properties
  - (a)  $f^{-1}(f(x)) = x$  for all  $x$  in the range of  $f$ .
  - (b)  $f(f^{-1}(x)) = x$  for all  $x$  in the range of  $f$ .
3. The graph  $y = f^{-1}(x)$  is the reflection in the line  $y = x$  of the graph  $y = f(x)$

## 2.2 Exponential and Logarithmic Functions

**The Natural Number  $e$ :** The natural number  $e$  is denoted and defined by,

$$e = \lim_{x \rightarrow 0} (1 + x)^{\frac{1}{x}} \text{ or } e = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x$$

An estimate for  $e \approx 2.718$

### Special Exponential Function:

Consider the exponential function  $y = a^x$ .

If  $a = e$ , then  $y = e^x$ , which is called the natural exponential function.

### Two Special Logarithms:

Consider the logarithmic function  $y = \log(x)$ .

If base  $a = 10$ , then  $y = \log_{10}(x)$ , which is called the common logarithm.

If base  $a = e$ , then  $y = \log_e(x)$ , which is called the natural logarithm and is denoted  $y = \ln(x)$ .

### Properties of Logarithms:

$$\begin{array}{ll} \text{L1: } \log_a(x) + \log_a(y) = \log_a(xy) & \text{L2: } \log_a(x) - \log_a(y) = \log_a\left(\frac{x}{y}\right) \\ \text{L3: } \log_a(x^n) = n \log_a(x) & \text{L4: } \log_a(1) = 0 \\ \text{L5: } \log_a(a) = 1 \leftrightarrow a^1 = a & \text{L6: } \log_a(a^x) = x \\ \text{L7: } \log_b(x) = \frac{\log_a(x)}{\log_a(b)} \end{array}$$

### Derivatives of Exponential and Logarithmic Functions:

- $\frac{d}{dx} e^x = e^x$
- $\frac{d}{dx} \ln(x) = \frac{1}{x}$
- $\frac{d}{dx} a^x = a^x \ln(a)$

- $\frac{d}{dx} \log_a(x) = \frac{1}{x \ln(a)}$

### Logarithmic Differentiation:

- Take natural Logarithm of both sides of an equation and simplify using the properties of logarithms.
- Take the derivative of both sides of the equation with respect to  $x$

$$\begin{aligned} y &= f(x) \\ \ln(y) &= \ln(f(x)) \\ \frac{1}{y} \frac{dy}{dx} &= \frac{f'(x)}{f(x)} \\ \frac{dy}{dx} &= f'(x) \end{aligned}$$

## 2.3 Inverse Trigonometric Functions

Inverse Sine Function:  $\sin^{-1}(x)$  or  $\arcsin(x)$   $D \in (-1, 1)$   $R \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Inverse Cosine Function:  $\cos^{-1}(x)$  or  $\arccos(x)$   $D \in (-1, 1)$   $R \in (0, \pi)$

Inverse Tangent Function:  $\tan^{-1}(x)$  or  $\arctan(x)$   $D \in (-\infty, \infty)$   $R \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Inverse Cosecant Function:  $\csc^{-1}(x)$

Inverse Secant Function:  $\sec^{-1}(x)$

Inverse Cotangent Function:  $\cot^{-1}(x)$

### Derivatives of Inverse Trigonometric Functions:

$$\begin{aligned} \text{i. } \frac{d}{dx} \sin^{-1}(x) &= \frac{1}{\sqrt{1-x^2}} & \text{ii. } \frac{d}{dx} \cos^{-1}(x) &= -\frac{1}{\sqrt{1-x^2}} \\ \text{iii. } \frac{d}{dx} \tan^{-1}(x) &= \frac{1}{1+x^2} & \text{iv. } \frac{d}{dx} \cot^{-1}(x) &= -\frac{1}{1+x^2} \\ \text{v. } \frac{d}{dx} \sec^{-1}(x) &= \frac{1}{|x|\sqrt{x^2-1}} & \text{vi. } \frac{d}{dx} \csc^{-1}(x) &= -\frac{1}{|x|\sqrt{x^2-1}} \end{aligned}$$

### Properties of Inverse Trigonometric Functions:

#### Group A

$$\begin{array}{ll} \text{(i) } \sin(-x) = -\sin(x) & \left| \begin{array}{l} \sin^{-1}(-x) = -\sin^{-1}(x) \\ \tan^{-1}(-x) = -\tan^{-1}(x) \\ \cos^{-1}(-x) = \pi - \cos^{-1}(x) \end{array} \right. \\ \text{(ii) } \tan(-x) = -\tan(x) & \\ \text{(ii) } \cos(-x) = \cos(x) & \end{array}$$

#### Cancellation Properties

$$\begin{array}{ll} \text{(i) } \sin(\sin^{-1}(x)) = x, x \in [-1, 1] & \left| \begin{array}{l} \sin^{-1}(\sin(y)) = y, y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ \tan^{-1}(\tan(y)) = y, y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ \cos^{-1}(\cos(y)) = y, y \in [0, \pi] \end{array} \right. \\ \text{(ii) } \tan(\tan^{-1}(x)) = x, x \in (-\infty, \infty) & \\ \text{(iii) } \cos(\cos^{-1}(x)) = x, x \in [-1, 1] & \end{array}$$

The function  $\sin(x)$  and  $\cos(x)$  are periodic with periods of  $2\pi$

The function  $\tan(x)$  is periodic with period  $\pi$ .

## 2.4 Hyperbolic Functions

The hyperbolic functions are combinations of the exponential function, and have properties very similar to that of the trigonometric functions.

### The 6 Hyperbolic Functions:

1. The Hyperbolic Sine Function:

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

2. The Hyperbolic Cosine Function:

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

3. The Hyperbolic Tangent Function:

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

4. The Hyperbolic Cosecant Function:

$$\operatorname{csch}(x) = \frac{1}{\sinh(x)} = \frac{2}{e^x - e^{-x}}$$

5. The Hyperbolic Secant Function:

$$\operatorname{sech}(x) = \frac{1}{\cosh(x)} = \frac{2}{e^x + e^{-x}}$$

6. The Hyperbolic Cotangent Function:

$$\coth(x) = \frac{1}{\tanh(x)} = \frac{\cosh(x)}{\sinh(x)} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

### The Hyperbolic Identities and Properties:

1.  $\cosh^2(x) - \sinh^2(x) = 1$
2.  $\cosh(x) + \sinh(x) = e^x$
3.  $\cosh(x) - \sinh(x) = e^{-x}$

### Derivative of the Hyperbolic Functions:

1.  $\frac{d}{dx} \sinh(x) = \cosh(x)$
2.  $\frac{d}{dx} \cosh(x) = \sinh(x)$
3.  $\frac{d}{dx} \tanh(x) = \operatorname{sech}^2(x)$
4.  $\frac{d}{dx} \coth(x) = -\operatorname{csch}^2(x)$
5.  $\frac{d}{dx} \operatorname{sech}(x) = -\operatorname{sech}(x) \tanh(x)$
6.  $\frac{d}{dx} \operatorname{csch}(x) = -\operatorname{csch}(x) \coth(x)$

### Inverse Hyperbolic Functions:

1. Inverse Hyperbolic Sine Function:  $D \in (-\infty, \infty)$   $R \in (-\infty, \infty)$

$$\sinh^{-1}(x) = \ln(x + \sqrt{x^2 + 1})$$

2. Inverse Hyperbolic Cosine Function:  $D \in [1, \infty)$   $R \in [0, \infty)$

$$\cosh^{-1}(x) = \ln(x + \sqrt{x^2 - 1})$$

3. Inverse Hyperbolic Tangent Function:  $D \in (-1, 1)$   $R \in (-\infty, \infty)$

$$\tanh^{-1}(x) = \frac{1}{2} \ln \left( \frac{1+x}{1-x} \right)$$

4. Inverse Hyperbolic Cotangent Function:  $D \in (-\infty, -1) \cup (1, \infty)$   $R \in (-\infty, 0) \cup (0, \infty)$

$$\coth^{-1}(x) = \frac{1}{2} \ln \left( \frac{x+1}{x-1} \right)$$

5. Inverse Hyperbolic Secant Function:  $D \in (0, 1]$   $R \in [0, \infty)$

$$\operatorname{sech}^{-1}(x) = \ln \left( \frac{1}{x} + \sqrt{\frac{1}{x^2} - 1} \right)$$

6. Inverse Hyperbolic Cosecant Function:  $D \in (-\infty, 0) \cup (0, \infty)$   $R \in (-\infty, 0) \cup (0, \infty)$

$$\operatorname{csch}^{-1}(x) = \ln \left( \frac{1}{x} + \sqrt{\frac{1}{x^2} + 1} \right)$$

#### Derivatives of Inverse Hyperbolic Functions:

$$\begin{array}{ll} \text{i) } \frac{d}{dx} \sinh^{-1}(x) = \frac{1}{\sqrt{x^2 + 1}} & \text{ii) } \frac{d}{dx} \cosh^{-1}(x) = \frac{1}{\sqrt{x^2 - 1}} \\ \text{iii) } \frac{d}{dx} \tanh^{-1}(x) = \frac{1}{1 - x^2} & \text{iv) } \frac{d}{dx} \coth^{-1}(x) = \frac{1}{1 - x^2} \\ \text{v) } \frac{d}{dx} \operatorname{sech}^{-1}(x) = \frac{-1}{x\sqrt{1 - x^2}} & \text{vi) } \frac{d}{dx} \operatorname{csch}^{-1}(x) = \frac{-1}{|x|\sqrt{1 + x^2}} \end{array}$$

## 3 Applications of Derivatives

### 3.1 Error Estimation

#### Incremental Change of Independent and Dependent Variables:

Let  $y = f(x)$ . Assume that independent variable has changed from  $x_0$  to  $x$

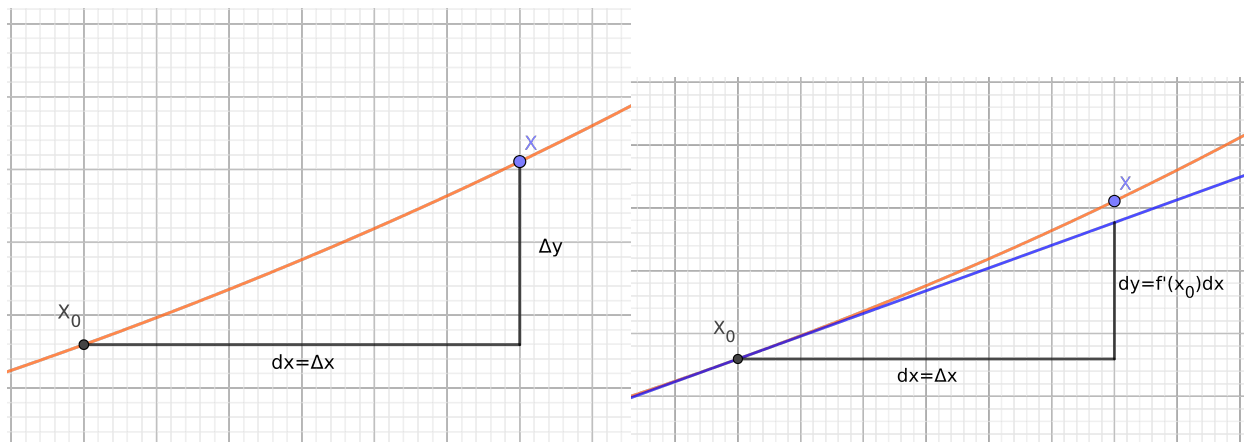
1. The change of the independent variable is defined by  $\Delta x = x - x_0$
2. The change of the dependent variable is defined by  $\Delta y = f(x) - f(x_0)$

#### Differentials of Independent and Dependent Variables:

Let  $y = f(x)$ . Assume that independent variable has changed from  $x_0$  to  $x$

1. The differential of the independent variable is denoted and defined by  $dx = \Delta x$
2. The differential of the dependent variable is denoted and defined by  $dy = f'(x_0) dx$





If the change in  $x$  or  $\Delta x$  is very small then  $\Delta x$  may be thought of as the error in the measurement of  $x$ . Accordingly  $\Delta y$  may be thought of as the corresponding error in the measurement of  $y$ .

#### Relationship between $\Delta y$ and $dy$ :

First observe  $\Delta y$  is much more complicated than  $dy$  to compute. If  $\Delta x$  is small then

$$\frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x) - f(x_0)}{x - x_0}$$

That is  $\frac{\Delta y}{\Delta x} \approx \frac{dy}{dx}$  or  $\Delta y \approx dy$

#### Classifying Error Types:

Assume a certain quantity has changed from  $P_0$  to  $P$ . Then  $\Delta P = P - P_0$ .

1. Error:  $\Delta P = P - P_0$
2. Absolute Error:  $|\Delta P|$
3. Relative Error:  $\frac{\Delta P}{P_0}$
4. Percentage Error:  $\frac{\Delta P}{P_0} \times 100\%$

## 3.2 Implicit Differentiation

#### A Relation:

An equation with the independent variable  $x$  and the dependent variable  $y$  is a relation.

#### Explicit and Implicit Relations:

A relation is explicit, simply if and only if  $y$  is expressed in terms of  $x$ . If a relation is not explicit then it is implicit.

#### Steps for Implicit Differentiation:

1. Take the derivative of both sides of the relation with respect to  $x$
2. Group all terms containing  $\frac{dy}{dx}$  on one side of the equation.
3. Factor out  $\frac{dy}{dx}$ .
4. Solve for  $\frac{dy}{dx}$ . (Often by division)

### 3.3 Related Rates

Let  $P$  be a physical quantity and assume that  $P$  varies as time,  $t$ , advances. That is  $P = P(t)$ . The average change in  $P$  over the time interval  $[t, t + \Delta t]$  is

$$P_{ave} = \frac{P(t + \Delta t) - P(t)}{\Delta t}$$

The instantaneous rate of change of  $P$  is given by:

$$\lim_{\Delta t \rightarrow 0} \frac{P(t + \Delta t) - P(t)}{\Delta t}$$

The rate of change of  $P$  is  $\frac{dP}{dt}$

**Units of Rate of Change:**

The rate of change  $\frac{dP}{dt}$  has units  $P/t$ .

**Positive and Negative Rates:**

The rate of a change  $\frac{dP}{dt}$  is considered positive if  $\frac{dP}{dt} \geq 0$  and is considered negative if  $\frac{dP}{dt} < 0$

**Important Rates:**

- velocity: The rate of change of position over time.
- Acceleration: The rate of change of velocity over time.

**Strategy for Related Rates:**

- Read the problem and find every value. Draw a diagram!
- Find a relationship between the values which have known rates and the values which have unknown rates.
- Take the derivative of the expression with respect to time.
- Substitute given data.

### 3.4 L'Hôpital's Rule

let  $f(x)$  and  $g(x)$  be differentiable functions on  $(a, b)$  with a point  $c$  on the interval. If  $\lim_{x \rightarrow c} \frac{f(x)}{g(x)}$  is of an indeterminate form then,

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}$$

**The 7 Indeterminate Forms:**

1.  $\frac{0}{0}$
2.  $\pm \frac{\infty}{\infty}$
3.  $0 \times \infty$
4.  $0^0$
5.  $\infty^0$
6.  $1^{\pm\infty}$
7.  $\infty - \infty$

## 4 Integrals

### Antiderivative:

A function  $F(x)$  is called an antiderivative of  $f(x)$  if  $F'(x) = f(x)$

### Indefinite Integral:

Let  $F(x)$  be the most general antiderivative of  $f(x)$ .  $F(x)$  is called the indefinite integral of  $f$  with respect to  $x$ .

$$F(x) = \int f(x) dx$$

### Techniques of Integration:

#### 1. Integration by the Table of Standard Basic Integrals

#### 2. Integration by Parts:

let  $u$  and  $v$  be two differentiable functions.

$$\int u dv = uv - \int v du$$

#### 3. Integration by Special Trigonometric Substitution:

Let  $F(x) = \int f(x) dx$

(a) If integrand  $f(x)$  contains  $a^2 - b^2x^2$ , substitute  $x = \frac{a}{b} \sin(\theta)$

(b) If integrand  $f(x)$  contains  $a^2 + b^2x^2$ , substitute  $x = \frac{a}{b} \tan(\theta)$

(c) If integrand  $f(x)$  contains  $b^2x^2 - a^2$ , substitute  $x = \frac{a}{b} \sec(\theta)$

#### 4. Integration By Completing the Square:

Consider  $f(x) = ax^2 + bx + c$  when  $a \neq 0$ , and  $b \neq 0$ , then

$$f(x) = a \left( x + \frac{b}{2a} \right)^2 + \frac{4ac - b^2}{4a} = \left( \sqrt{a}x + \frac{b}{2\sqrt{a}} \right)^2 + c - \frac{b^2}{4a}$$

Given  $F(X) = \int f(x) dx$ . If  $f(x)$  contains  $ax^2 + bx + c$ , complete the standard Substitution  $t = \sqrt{a}x + \frac{b}{2\sqrt{a}}$ .

Used for

$$\int \frac{\alpha x + \beta}{ax^2 + bx + c} dx \text{ or } \int \frac{\alpha x + \beta}{\sqrt{ax^2 + bx + c}} dx$$

#### 5. Integration by Partial Fractional Decomposition:

Given  $F(x) = \int f(x) dx$  if  $f$  is a proper rational function, decompose  $f$  into partial fractions to integrate.

#### 6. Integration by General Substitution:

Given  $F(x) = \int f(x) dx$ . Assume that the integral can not be completed by other methods. In Such a case attempt a substitution.

$$u = u(x)$$

choose a substitution so that its derivative  $\frac{du}{dx}$  is a multiplicative constant of the integrand.

### Table of Standard Basic Integrals:

1. The Power Rule:

$$(a) \int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$$

$$(b) \int dx = \int 1 dx = x + c$$

2. Trigonometric Functions:

$$(a) \int \sin(ax) dx = -\frac{1}{a} \cos(ax) + C$$

$$(b) \int \cos(ax) dx = \frac{1}{a} \sin(ax) + C$$

$$(c) \int \tan(ax) dx = \frac{1}{a} \ln |\sec(ax)| + C$$

$$(d) \int \cot(ax) dx = \frac{1}{a} \ln |\sin(ax)| + C$$

$$(e) \int \sec^2(ax) dx = \frac{1}{a} \tan(ax) + C$$

$$(f) \int \csc^2(ax) dx = -\frac{1}{a} \cot(ax) + C$$

$$(g) \int \sec(ax) \tan(ax) dx = \frac{1}{a} \sec(ax) + C$$

$$(h) \int \csc(ax) \cot(ax) dx = -\frac{1}{a} \csc(ax) \cot(ax) + C$$

3. Exponential Functions:

$$(a) \int \frac{u'}{u} = \ln |u| + C$$

$$(b) \int e^{ax} dx = \frac{1}{a} e^{ax} + C$$

4. Inverse Functions:

$$(a) \int \frac{1}{\sqrt{\beta^2 - x^2}} dx = \sin^{-1} \left( \frac{x}{\beta} \right) + C$$

$$(b) \int \frac{1}{\beta^2 + x^2} dx = \frac{1}{\beta} \tan^{-1} \left( \frac{x}{\beta} \right) + C$$

$$(c) \int \frac{1}{\sqrt{\beta^2 + x^2}} dx = \sinh^{-1} \left( \frac{x}{\beta} \right) + C$$

$$(d) \int \frac{1}{\sqrt{x^2 - \beta^2}} dx = \cosh^{-1} \left( \frac{x}{\beta} \right) + C$$

## 5 Vertical and Horizontal Asymptotes

### Horizontal Asymptotes:

A function  $f$  is said to have a right horizontal asymptotes  $y = L_1$  if  $\lim_{x \rightarrow \infty} f(x) = L_1$ ,  $L_1 \in \mathbb{R}$ .

A function  $f$  is said to have a left horizontal asymptotes  $y = L_2$  if  $\lim_{x \rightarrow -\infty} f(x) = L_2$ ,  $L_2 \in \mathbb{R}$ .

A function can have at most 2 horizontal asymptotes. If  $L_1 = L_2$  then  $f$  has a horizontal asymptotes  $y = L$ , where  $L_1 = L_2 = L$ . If  $\lim_{x \rightarrow \infty} f(x) = \pm\infty$ , then  $f$  has no right horizontal asymptote. Likewise if  $\lim_{x \rightarrow -\infty} f(x) = \pm\infty$  then  $f$  has no left horizontal asymptote.

**Vertical Asymptotes:**

A function  $f$  is said to have a vertical asymptote  $x = c$  if either  $\lim_{x \rightarrow c^-} f(x) = \pm\infty$  or  $\lim_{x \rightarrow c^+} f(x) = \pm\infty$ . If  $f$  has a vertical asymptote at  $x = c$  then  $c$  is not in the domain of  $f$ . For a rational function the possible vertical asymptotes occur where the denominator is 0.

## 6 Taylor Series

**Taylor Formula with Remainder:**

Let  $f(x)$  be a given function. Assume  $f$  has derivatives of all order up to and including  $n + 1$  at  $x = c$ . Taylor's Formula States,

$$f(x) = P_n(x) + R_n(x)$$

where  $P_n(x)$  is the Taylor polynomial of  $f$  of degree  $n$  about  $c$

$$P_n(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

and  $R_n(x)$  is the remainder and is given by

$$R_n(x) = \frac{f^{(n+1)}(s)}{(n+1)!}(x-c)^{(n+1)}$$

Where  $s$  is some number between  $x$  and  $c$ .

**The use of Taylor Polynomials:**

If  $x$  is close to  $c$  we may approximate  $f(x)$  as  $P_n(x)$  or

$$f(x) \approx P_n(x)$$

Then  $R_n(x)$  is the error in the approximation.

**Two Special Cases of Taylor Polynomials:**

1. The Taylor polynomial at  $c=0$  is known as the Maclaurin polynomial.
2. If  $n = 1$ , the Taylor polynomial of degree 1 about  $c$  is referred to as the local linearization of  $f(x)$  at  $c$ .

$$L(x) = f(c) + f'(c)(x-c)$$

**Taylor and Maclaurin Series:**

Let  $f$  be a function with derivatives of all orders at  $x = c$ , then  $\lim_{n \rightarrow \infty} R_n(x) = 0$

$$\therefore f(x) = \lim_{n \rightarrow \infty} P_n(x)$$

That is

$$f(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

The series is referred to as the Taylor series of  $f$  about  $c$ . If  $c = 0$  then it is referred to as the Maclaurin series of  $f$ .

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

**Table Of Standard Maclaurin Series:**

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n, \quad x \in (-1, 1)$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + (-x)^n + \cdots = \sum_{n=0}^{\infty} (-x)^n, \quad x \in (-1, 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad x \in (-\infty, \infty)$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \quad x \in (-\infty, \infty)$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \quad x \in (-\infty, \infty)$$

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n-1} x^n}{n} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \quad x \in (-1, 1]$$

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \quad x \in [-1, 1]$$