Math 277
Multivariable Calculus for Engineers and Scientists

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1 Vector Functions in Two and Three Space

1.1 Vector Function:

A vector function $\vec{r}$ is a rule that assigns to each real number t, one and only one vector

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$

or using the unit vectors $\hat{i} = \left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right)$, $\hat{j} = \left(\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right)$ and $\hat{k} = \left(\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right)$,

$$x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$

and is written

$$\begin{aligned} \vec{r}(t) &= \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} \\ \text{or } &= x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k} \end{aligned}$$

1.1.1 Geometric Interpretation of a Vector Function:

Given a vector function

$$\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$

assuming that the functions $x(t)$, $y(t)$, and $z(t)$ are continuous for some interval I. The vector function $\vec{r}(t)$, $t \in I$ may be thought of as the position of a moving particle at time t in three-space. As time t varies, the terminal point of the position vector traces a space curve C.

The space curve C is said to be given parametrically by the vector function $\vec{r}(t) = \left(\begin{smallmatrix} x(t) \\ y(t) \\ z(t) \end{smallmatrix}\right)$ or is given by the three equations

$$\begin{cases} x(t) & \\ y(t) & t \in I \\ z(t) & \end{cases}$$

1.1.2 Endpoints and Orientation of a space or Plane Curve:

Let C be the space curve given parametrically by the vector function $\vec{r}(t)$, where t is in the closed interval $[a, b]$. The initial and terminal points of the curve C are defined respectively by $\vec{r}(a)$, and $\vec{r}(b)$. Note that if the endpoints coincide the curve C is closed. The orientation of curve C is the direction from the initial point P toward the terminal point Q and is usually denoted with one or two arrow heads.

1.1.3 Derivative Rules for Vector Functions:

let $\vec{u}(t)$ and $\vec{v}(t)$ be vector functions with differentiable components and $f(t)$ be a scalar function.

1. The sum and difference rule: $\dfrac{d}{dt}\Big\{\vec{u}(t) \pm \vec{v}(t)\Big\} = \vec{u}\,'(t) \pm \vec{v}\,'(t)$
2. The scalar Multiple Rule: $\dfrac{d}{dt}\Big\{f(t)\vec{u}(t)\Big\} = f'(t)\vec{u}(t) + f(t)\vec{u}\,'(t)$
3. The dot Product Rule: $\dfrac{d}{dt}\Big\{\vec{u}(t) \cdot \vec{v}(t)\Big\} = \vec{u}\,'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}\,'(t)$
4. The Cross Product Rule: $\dfrac{d}{dt}\Big\{\vec{u}(t) \times \vec{v}(t)\Big\} = \vec{u}\,'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}\,'(t)$
5. The Chain Rule: $\dfrac{d}{dt}\Big\{\vec{u}(f(t))\Big\} = \vec{u}\,'(f(t))f'(t)$

1.2 Motion of a particle in Two and Three-Space

Position: $\vec{r}(t)$
By definition the position of a moving particle at time t is $\vec{r}(t)$.

Velocity: $\vec{v}(t)$
By definition the average velocity is given by $\vec{v}_{ave} = \dfrac{\Delta\vec{r}(t)}{\Delta t}$. Let P and Q be the position of a particle at time t and $t + \Delta t$ where Δt is small. The the velocity of a particle between P and Q is defined,

$$\vec{v}_{P\to Q} = \frac{Q - P}{t + \Delta t - t} = \frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t}$$

If the limit is taken as $\Delta t \to 0$ then,

$$\vec{v}(t) = \lim_{\Delta t \to 0} \frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t} = \frac{d\vec{r}}{dt} = \vec{r}\,'(t)$$

It follows that the tangent line to the curve C at P is in the direction of the velocity vector at P.

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

Acceleration: $\vec{a}(t)$
By definition the acceleration is the derivative of velocity with respect to time,

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

Speed: $v(t)$
By definition the speed is the magnitude or norm of the velocity,

$$v(t) = ||\vec{v}(t)||$$

Distance Traveled: L
The distance traveled or arc length of a curve on the interval $[a, b]$ is denoted and defined

$$L = \int_a^b v\,dt = \int_a^b \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2 + \frac{dz}{dt}^2}\,dt$$

1.3 Special Parametric Curves:

The Straight Line Segment

Recall the parametric Vector equation of a Straight Line

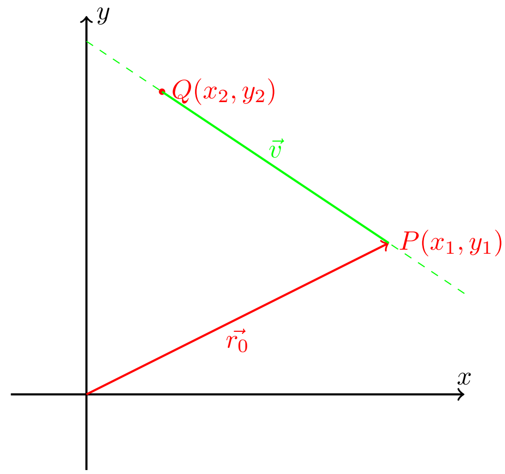
$$\vec{r}(t) = \vec{r_0} = t\vec{v}, \ t \in \mathbb{R}$$

Here $\vec{r_0}$ is equivilant to the point $P = (x_1, y_1)$ and $\vec{v} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}$.

$$\therefore \vec{r}(t) = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + t \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}, \ t \in \mathbb{R}$$

If follows that the parametric vector equation of the straight line segment with initial point $P = (x_1, y_1)$ and terminal point $Q = (x_2, y_2)$ is given by

$$\vec{r}(t) = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + t \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}, \ t \in [0, 1]$$



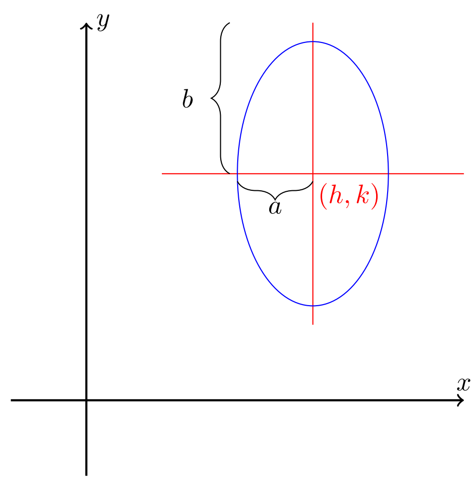
The Ellipse

The parametric vector equation of an ellipse with center at (h, k) and with semi-axis length of a, b is given by

$$\vec{r}(t) = (h + a\cos(t))\hat{i} + (k + b\sin(t))\hat{j}, \ t \in [0, 2\pi]$$

or parametrically as

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$



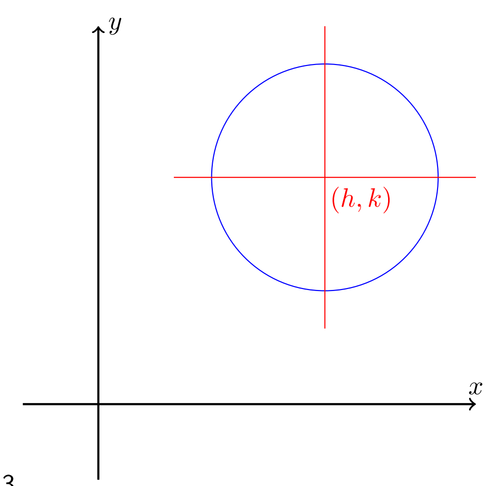
The Circle

The parametric vector equation of a circle centered at (h, k) with radius a is given by

$$\vec{r}(t) = (h + a\cos(t))\hat{i} + (k + b\sin(t))\hat{j}$$

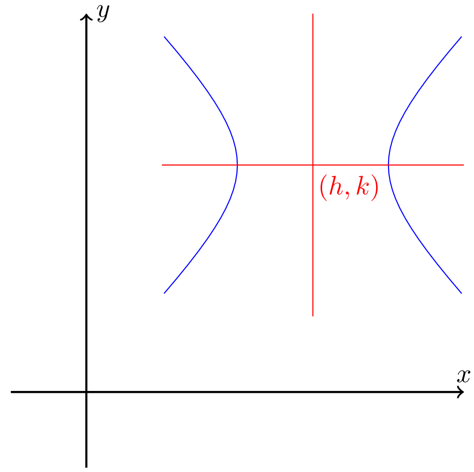
or parametrically as

$$(x-h)^2 + (y-k)^2 = a^2$$



The Hyperbola

The parametric vector equation of the right hand branch of a hyperbola with center at (h, k), semi transverse axis of length a and semi conjugate axis of length b is given by



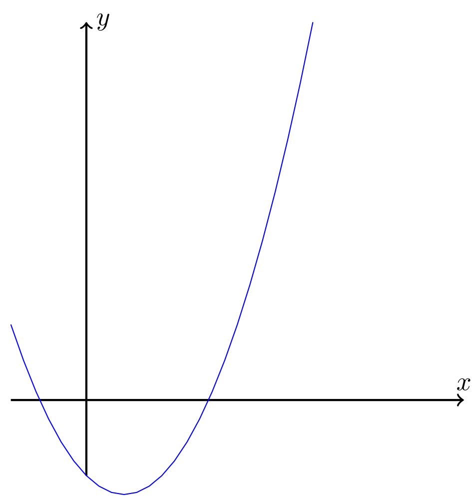
$$\vec{r}(t) = (h + a\cosh(t))\hat{i} + (k + b\sinh(t))\hat{j}, t \in \mathbb{R}$$

or parametrically as

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

The Parabola

There is no standard parametric vector equation for a parabola. There instead are infinitely many parametric vector equations for a parabola. To find a parametric vector equation for a parabola simply assign any value for x, or y asa long as the values of x and y are not restricted. The cartesian equations of a parabola are



$$\begin{aligned} y &= ax^2 + bx + c, \qquad && a \neq 0 \\ \text{or } x &= ay^2 + by + c, \qquad && a \neq 0 \end{aligned}$$

1.4 Special Parametric Curves in Space:

The Straight Line Segment:
The parametric Vector equation of the straight line segment given by the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is given by

$$\vec{r} = \vec{r_0} + t\vec{v}, \ t \in [0, 1]$$

Where $\vec{r_0} = P$ and $\vec{v} = \vec{PQ}$ or parametrically as

$$\vec{r}(t) = \begin{cases} x = x_1 + (x_2 - x_1)t \\ y = y_1 + (y_2 - y_1)t \\ z = z_1 + (z_2 - z_1)t \end{cases}$$

The Helix:
A Helix is a wire wrapped around the surface of a cylinder. If the cylinder is a right circular cylinder with radius

a and height b, then the parametric vector equation of the helix is given by

$$\vec{r}(t) = \begin{pmatrix} a\cos(t) \\ a\sin(t) \\ bt \end{pmatrix}, \ t \in [0, 2\pi]$$

1.5 General Curves in Space:

A space curve C is the intersection of 2 surfaces, say S_1 and S_2. For instance

1. The intersection of 2 planes is a straight line
2. The intersection of a cone and a plane generates a conic section (circle, an ellipse, a parabola, a hyperbola, or a pair of lines.)

Let C be the curve of intersection of S_1 and S_2 where

$$S_1 : f(x, y, z) = 0 \tag{1}$$

$$S_2 : g(x, y, z) = 0 \tag{2}$$

To find the parametric vector equation of the space curve C, attempt to use equations (1) and (2) to obtain a third equation consisting of only two of the three variables. When this equation is viewed in $\mathbb{R}^2$ (the xy-plane, xz-plane or yz-plane), it can easily be parametrized.

1.6 The $\vec{T}$, $\vec{N}$, and $\vec{B}$ Frame

First Recall the definition of a unit vector. If If $\vec{v}$ is a vector in $\mathbb{R}^n$, then $\vec{v}$ is a unit vector if and only if the length of v is 1. Let C be a plane or space curve given by the vector function $\vec{r}(t), \ t \in I$ and let P be a point on curve C.

1.6.1 The Unit Tangent Vector: $\vec{T}(t)$

The unit tangent vector to curve C at P is denoted and defined by

$$\vec{T}(t) = \frac{\vec{v}(t)}{||\vec{v}(t)||}$$

$\vec{T}(t)$ is a unit vector in the direction of the velocity and hence is tangent to the curve C at P and points in the orientation of C.

1.6.2 The Principle Unit Normal: $\vec{N}(t)$

The principle unit vector to the curve C at P is denoted and defined by

$$\vec{N}(t) = \frac{\vec{T}(t)}{||\vec{T}(t)||}$$

1.6.3 The Curvature: κ

Given $\vec{r}(t)$ the rate of turn is given by

$$\kappa = \left|\left|\frac{d\vec{T}}{ds}\right|\right|$$

where s denotes the arc length. This is the scalar quantity representing the change in $\vec{T}$ with respect to distance travelled. This is called the curvature κ. The curvature κ can also be defined as

$$\kappa = \frac{\vec{v} \times \vec{a}}{v^3}$$

1.6.4 The Radius of Curvature ρ

At a point P on a curve C we define the radius of curvature by

$$\rho = \frac{1}{\kappa}$$

The circle of radius ρ tangent to curve C at P on the concave side is called the circle of curvature.

1.6.5 The Unit Binormal Vector: $\vec{B}$

The cross product of $\vec{T}$ and $\vec{N}$ is a vector orthogonal to both $\vec{T}$ and $\vec{N}$. This vector is denoted $\vec{B}(t)$ and is defined

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

The vector $\vec{B}$ is called the Unit Binormal Vector. Geometrically the $\vec{T}$, $\vec{N}$, and $\vec{B}$ vector determine the spacial properties of direction of travel, turn, and twist respectively of the curve C.

1.6.6 The Torsion: τ

The torsion τ of a space curve C is denoted and defined by

$$\tau = -\frac{d\vec{B}}{ds} \cdot \vec{N}$$

Geometrically the torsion provides a measure of the degree of twisting of a space curve. Given a curve $C = \vec{r}(t)$ the torsion can also be defined

$$\tau = \frac{(\vec{v} \times \vec{a}) \cdot \vec{a}\,'}{||\vec{v} \times \vec{a}||^2}$$

1.7 Tangential and Normal Components of Acceleration

Let $a_T = \frac{dv}{dt}$ and $a_N = \kappa v^2$

$$\begin{aligned} \therefore \vec{a}(t) &= a_T\vec{T} + a_N\vec{N} \\ &= \frac{dv}{dt}\vec{T} + \kappa v^2 \vec{N} \end{aligned}$$

The scalars a_T and a_N are respectively called the tangent component and normal component of acceleration.

1.7.1 Alternative formula for the Tangent and Normal components of Acceleration

The normal component of acceleration can be given by

$$a_N = \frac{||\vec{v} \times \vec{a}||}{v}$$

The tangential component of acceleration can be given by

$$a_T = \vec{T} \cdot \vec{a} = \frac{\vec{v} \cdot \vec{a}}{v}$$

1.8 Summary and alternative Formula of $\vec{T}$, $\vec{N}$, $\vec{B}$, κ, ρ, τ, a_T, and a_N

$$\vec{T} = \frac{\vec{v}}{v} \tag{1}$$
$$\vec{B} = \frac{\vec{v} \times \vec{a}}{||\vec{v} \times \vec{a}||} \tag{2}$$
$$\vec{N} = \vec{B} \times \vec{T} \tag{3}$$
$$\kappa = \frac{||\vec{v} \times \vec{a}||}{v^3} \tag{4}$$
$$\rho = \frac{1}{\kappa} \tag{5}$$
$$\tau = \frac{(\vec{v} \times \vec{a}) \cdot \vec{a}\,'}{||\vec{v} \times \vec{a}||^2} \tag{6}$$
$$a_N = \frac{||\vec{v} \times \vec{a}||}{v} \tag{7}$$
$$a_T = \vec{T} \cdot \vec{a} = \frac{\vec{v} \cdot \vec{a}}{v} \tag{8}$$
$$\vec{T} = \vec{N} \times \vec{B} \tag{9}$$
$$\vec{B} = \vec{B} \times N \tag{10}$$

1.9 Applications of Vector Functions

1.9.1 The Rocket Equation

A rocket moves forward by the backward expulsion of a mass of gas formed by burning its onboard fuel.

1. M: The total initial mass of the rocket including its fuel.

2. $m = m(t)$: total mass of the rocket at time t. Hence $m + \Delta m$ is the total mass of the rocket at time $t + \Delta t$. It follows that $\Delta m < 0$ and $-\Delta m > 0$. Therefore the amount of fuel burnet over a time interval Δt is $-\Delta m$.

3. $\vec{v} = \vec{v}(t)$: The velocity of the rocket at time t relative to the earth. Hence $\vec{v} + \Delta\vec{v}$ is the velocity at time $t + \Delta t$

4. $-\vec{v_e}$: The velocity of the ejected gas (assume constant). It follows that $\vec{v} + \vec{v_e}$ is the velocity of the ejected gas relative to the earth.

5. α: The rate at which the fuel mixture is burned in the rocket (assume constant).

$$\therefore -\alpha = \frac{dm}{dt} \Rightarrow m = \int \alpha \, dt = -\alpha t + M$$

   or

$$m(t) = M - \alpha t \tag{1}$$

6. $\vec{F}$: The net force acting on the rocket

7. $\vec{p}(t)$ The momentum of the rocket. $\vec{p}(t) = m\vec{v}$. Hence the change in momentum over time is thus given by

$$\begin{aligned}
\Delta\vec{p} &= \vec{p}(t+\Delta t) - \vec{p}(t) \\
&= \big[(m+\Delta m)(\vec{v}+\Delta\vec{v}) + (-\Delta m)(\vec{v}+\vec{v_e})\big] - m\vec{v} \\
&= \big[m\vec{v} + m\Delta\vec{v} + \vec{v}\Delta m + \Delta m\Delta\vec{v} - \vec{v}\Delta m - \vec{v_e}\Delta m\big] - m\vec{v}\ {}^{1} \\
&= m\Delta\vec{v} - \Delta m\,\vec{v}_e \\
\frac{\Delta\vec{p}}{\Delta t} &= m\frac{\Delta\vec{v}}{\Delta t} - \vec{v_e}\frac{\Delta m}{\Delta t} \\
\lim_{\Delta\to 0}\frac{\Delta\vec{p}}{\Delta t} &= m\lim_{\Delta\to 0}\frac{\Delta\vec{v}}{\Delta t} - \vec{v_e}\lim_{\Delta\to 0}\frac{\Delta m}{\Delta t}
\end{aligned}$$

$$\frac{d\vec{p}}{dt} = m\frac{d\vec{v}}{dt} - \vec{v_e}\frac{dm}{dt} \tag{2}$$

Apply Newtons second law of motion, the derivative of momentum with respect to time is $\vec{F} = \frac{d\vec{p}}{dt}$, to (2) to obtain

$$\vec{F} = m\frac{d\vec{v}}{dt} - \vec{v_e}\frac{dm}{dt} \tag{3}$$

Assumptions

1. Assume the rocket moves in a straight line vertically upward. Hence $\vec{F} = 0 \to F = 0\hat{k}$, $\vec{v} = v\hat{k}$, v being the speed of the rocket relative to the earth, and $\vec{v_e} = -v_e\hat{k}$, v_e being the speed of the ejected gas relative to the rocket.

2. The rocket is initially at rest. Hence $M = m(t)$ when $v =$.

Substitute for v, v_e, and F from the above assumptions into (3)

$$\begin{aligned}
0 &= m\frac{dv}{dt} - v_e\frac{dm}{dt} \\
m\,\frac{dv}{dt} &= -v_e\frac{dm}{dt} \\
\frac{dv}{dt} &= -\frac{v_e}{m}\frac{dm}{dt}
\end{aligned} \tag{4}$$

Integrate both sides of (4)

$$\begin{aligned}
\int_0^t \frac{dv}{dt}\,dt &= \int_0^t -\frac{v_e}{m}\frac{dm}{dt}\,dt \\
v(t) - v(0) &= -v_e\ln(m(t)) + v_e\ln(m(0))
\end{aligned}$$

Therefore the velocity of a rocket at time t is given by

$$v(t) = v_e\ln\left(\frac{M}{m(t)}\right) \tag{5}$$

Subbing in (1) into (5) then gives

$$v(t) = v_e\ln\left(\frac{M}{M-\alpha t}\right)$$

[1] Note $\Delta m\Delta v$ is very small and is therefore omitted

1.9.2 Banking of a Road Turn:

If a road is straight, its design is horizontal. However, when on a sharp turn it becomes angled. This design is referred to as banking of a road turn. Banked road turns have a rated speed limit that must be followed in order to use the road safely. Here we shall only look at frictionless roads. If a curve is banking at an angle θ, with a radius of curvature ρ and a rated speed of v, then the quantities are related by

$$\tan\theta = \frac{v^2}{g\rho}$$

2 Functions of Several Variables

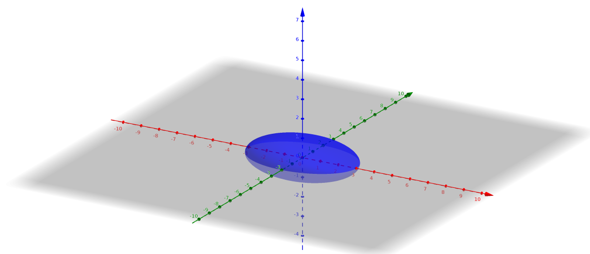
2.1 Quadric Surfaces:

A quadratic equation in x, y, and z is called a quadric surface. Quadric surfaces may be thought of as three dimensional versions of conic sections. Let a, b, and c be positive

2.1.1 The Ellipsoid Family:

The Ellipsoid:

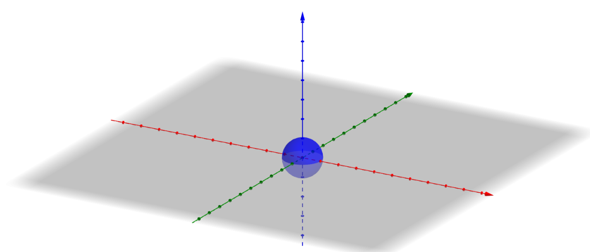
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{1}$$



The ellipsoid is centered at $(0, 0, 0)$ and has semi axis a, b, and c.

The Sphere:

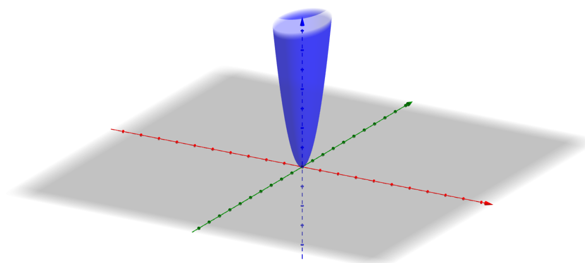
$$x^2 + y^2 + z^2 = a^2 \tag{2}$$



The sphere is centered at $(0, 0, 0)$ and has radius a

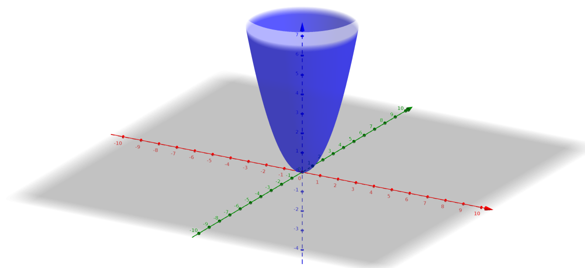
2.1.2 The Paraboloid Family:

The Elliptic Paraboloid:



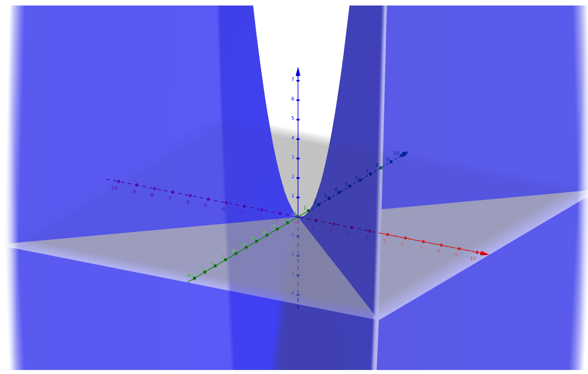
$$z = \pm\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) \tag{3}$$

The Circular Paraboloid:



$$z = \pm\left(\frac{x^2}{a^2} + \frac{y^2}{a^2}\right) \tag{4}$$

The Hyperbolic Paraboloid:

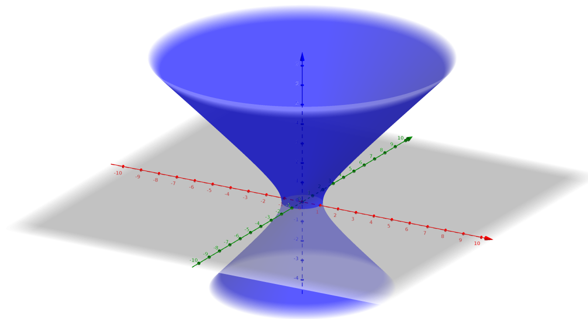


$$z = \pm\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right) \tag{5}$$

Each Paraboloid has vertex at the origin $(0, 0, 0)$ and axis of symmetry about the z-axis.

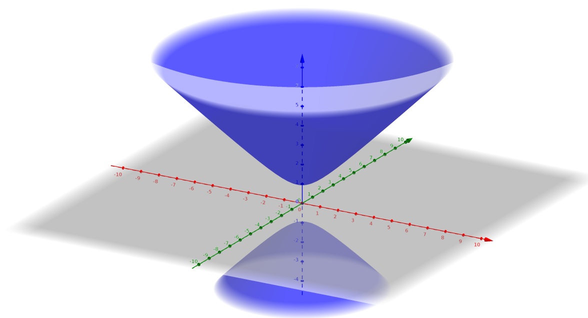
2.1.3 The Hyperboloid Family:

The Hyperboloid of One Sheet:



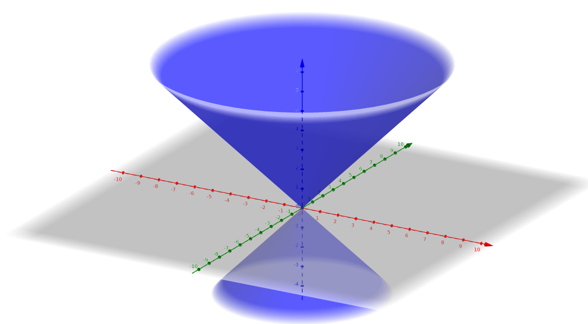
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \tag{6}$$

The Hyperboloid of Two Sheets:



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1 \tag{7}$$

The Cone:



$$z^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2} \tag{8}$$

Hyperboloids of one and two sheets have centers about the origin $(0, 0, 0)$ and a axis of symmetry about the z-axis. If the cone has $a = b$ then it is a circular cone. If we solve the equation of the cone for z.

$$z = \pm\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2}}$$

With the positive being the upper half and the negative being the lower half.

If in all eight equations we replace x, y, and z respectively by $x - h$, $y - k$ and $z - l$, we obtain a translated quadric surface with center or vertex at (h, k, l). The equation of a quadric surface with axis of symmetry being or parallel to the x or y-axis is similar to the z axis.

2.2 Special Surfaces:

2.2.1 The Plane:

$$Ax + By + Cz = D$$

where A, B, C, D are real and not all 0. There are 6 special planes.

1. $z = 0$ The equation of the xy plane.
2. $z = l$ A plane parallel to the xy plane l units apart.
3. $y = 0$ The equation of the xz plane.
4. $y = k$ A plane parallel to the xz plane k units apart.
5. $x = 0$ The equation of the yz plane.
6. $x = h$ A plane parallel to the yz plane h units apart.

2.2.2 Special Cylinders

An equation in $\mathbb{R}^3$ containing only 2 variables is an equation of a cylinder with generators parallel to the missing variable axis. The function $F(x, y) = 0$ is a cylinder parallel to the z-axis where $F(x, y) = 0$ is the boundary of the base of the cylinder.

2.3 Functions of two and three independent Variables:

2.3.1 A Function of Two Independent Variables:

A function f of two independent variables is a rule that assigns to each permissible ordered pair from a set D in the xy-plane, one and only one real number z and is denoted

$$z = f(x, y)$$

A function of three or more independent variables is defined similarly.

2.3.2 Domain of a Function of Two Independent Variables:

The domain of $z = f(x, y)$ is the set (collection) of all ordered pairs (x, y) such that f is defined and real. The domain of f may be denoted D or dmf. The domain for functions of three or more variables is defined similarly.

2.3.3 Graph of a Function of Two Independent Variables:

First recall the graph of a function of a single variable $y = f(x)$ is: The set of ordered pairs of $(x, f(x))$. The graph of a function of a single variable is referred to as a curve in $\mathbb{R}^2$. Likewise the graph of a function of two independent variables $z = f(x, y)$ is the set of all ordered triples,

$$(x, y, z) = (x, y, z(f(x, y)))$$

The graph of $z = f(x, y)$ is referred to as a surface in $\mathbb{R}^3$. Likewise for a function of n variables its graph generates a hypersurface in $\mathbb{R}^{n+1}$

2.3.4 Level Curves and Surfaces of a function of 2 or 3 variables:

Let S be the surface given by $z = f(x, y)$. If z is fixed to a constant $z = c$, then the curve $c = f(x, y)$, is a cross section or level curve of f at $z = c$. Likewise if we have a hypersurface $w = f(x, y, z)$, w can be fixed as a constant $w = c$ and then $c = f(x, y, z)$ is a level surface in $\mathbb{R}^3$. A collection of level curves is known as a contour map.

2.4 Partial Derivatives for functions of Several Variables

2.4.1 Partial Derivatives of a function of Two Independent Variables

Let $z = f(x, y)$. The partial derivative of z with respect to x is denoted and defined by

$$\frac{\partial z}{\partial x} = f_x = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

provided the limit exists.
The partial derivative of z with respect to y is denoted and defined by

$$\frac{\partial z}{\partial y} = f_y = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}$$

provided the limit exists. It is evident from the definition that to compute f_x treat y as a constant and differentiate. The partial derivative f_y is computed similarly by holding x constant.

2.4.2 Other Notation for Partial Derivatives:

Let $z = f(x, y)$. The partial derivative of f may be denoted by

1. $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$
2. $f_x(x, y),\ f_y(x, y)$
3. $f_1(x, y),\ f_2(x, y)$

2.4.3 Partial Derivatives of functions of n independent variables

Let $f(x_1, x_2, \cdots, x_n)$ be a function of n independent variables. The partial derivative of f with respect to x_i where $i \in \mathbb{N},\ i \leq n$ is given by

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x_1, x_2, \cdots, x_i + h, \cdots, x_n) - f(x_1, x_2, \cdots, x_n)}{h}$$

provided the limit exists.

2.4.4 Higher Order Derivatives:

Let $z = f(x, y)$. $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are called the first order partial derivatives of f. The second order partial derivatives are given by

$$\frac{\partial^2 z}{\partial x^2} = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$$

$$\frac{\partial^2 z}{\partial y^2} = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

Mixed Partials

$$\frac{\partial^2 z}{\partial x \partial y} = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

$$\frac{\partial^2 z}{\partial y \partial x} = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

The mixed partials are not necessarily equal. Let $z = f(x, y)$. If f_x, f_y, f_{xy}, f_{yx} are all continuous at some point P, then the mixed partials exist and are equal at P.

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

Second order derivatives are defined similarly for functions of more variables. In general for a function of m variables there is m^n n^{th} order derivatives.

2.4.5 The Chain Rule for functions of Several Variables:

Let us first recall the chain rule for a single variable function. Let $y = f(x)$ where x is a function of t or $x = x(t)$. Hence y is indirectly a function of t, That is $y = y(t)$. To find $\frac{dy}{dt}$ compute,

$$\frac{dy}{dt} = \frac{df}{dx}\frac{dx}{dt}$$

Likewise $z = f(x, y)$ where x and y are functions of t, or $x = x(t)$, $y = y(t)$. Obviously z is a function of t, $z = z(t)$. It can be shown that,

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

There are endless formula for the chain rule for functions of several variables. Another example is let $z = f(x, y)$ where $x = x(u, v)$ and $y = y(u, v)$. Now z is a function fo u and v and has derivatives

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

2.5 Tangent Planes and Normal Line to Surface:

2.5.1 Gradient of a Function of Several Variables

Let $f(x, y, z)$ be a function of three independent variables x, y, and z, and let P be the point (x_0, y_0, z_0). The gradient of F at P is denoted and defined by

$$\vec{\nabla} F = \begin{pmatrix} \frac{\partial F}{\partial x} \\ \frac{\partial F}{\partial y} \\ \frac{\partial F}{\partial z} \end{pmatrix}$$

2.5.2 Geometric Interpretation of Gradient

Let S be the surface given by the equation $F(x, y, z) = 0$ and let $P(x_0, y_0, z_0)$ be a point on the surface. It can be easily verified that any vector orthogonal to the surface at P is $N = \vec{\nabla} F(P)$. The line through point P orthogonal to the surface is called the normal line to the surface at P

2.5.3 The Point Normal Form of a Plane

The equation of a plane passing through a point $P(x_0, y_0, z_0)$ with a normal vector $\vec{N}$ has the equation

$$\vec{N} \cdot \vec{r} = 0$$

where the vector $\vec{r}$ is given by

$$\vec{r} = \begin{pmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{pmatrix}$$

This equation is the point normal form of a plane.

2.6 Increments and Decrements:

Let $F(x, y)$ be a function of two independent variables x, and y. Assume that (x, y) has changed from the initial value $(x_0, y_0$ to (x_1, y_1).

2.6.1 Increment or Change in Independent Variables

The increment or change in independent variables x, y are respectively denoted and defined by

$$\Delta x = x_1 - x_0 \text{ and } \Delta y = y_1 - y_0$$

The relative change in the dependent variable is

$$\Delta = z_1 - z_0 = F(x_1, y_1) - F(x_0, y_0)$$

2.6.2 Differentials of Independent and Dependent Variables

The differentials of independent variables x and y are respectively denoted and defined by

$$\partial x = \Delta x \Rightarrow \partial x = x_1 - x_0$$

$$\partial y = \Delta y \Rightarrow \partial y = y_1 - y_0$$

The differential of the dependent variable is denoted and defined by

$$\begin{aligned} \partial z = \partial F &= \frac{\partial F}{\partial x} \partial x + \frac{\partial F}{\partial x} \partial y \\ \text{or } &= \frac{\partial F}{\partial x} \Delta x + \frac{\partial F}{\partial x} \Delta y \end{aligned}$$

provided $\dfrac{\partial F}{\partial x}$ and $\dfrac{\partial F}{\partial y}$ are continuous.

We may think of Δx and Δy as the error in x, and y respectively. Δz or $F(x_1, y_1) - F(x_0, y_0)$ is not easy to calculate. However ∂z is much easier to calculate. If (x_1, y_1) is close to (x_0, y_0) then

$$\Delta F(x_1, y_1) \approx \partial F(x_0, y_0)$$

2.6.3 Error Types

Assume a certain quantity has changed from P_0 to P. Then $\Delta P = P - P_0$.

1. Error: $\Delta P = P - P_0$
2. Absolute Error: $|\Delta P|$
3. Relative Error: $\dfrac{\Delta P}{P_0}$
4. Percentage Error: $\dfrac{\Delta P}{P_0} \times 100\%$

2.7 The Laplace Equation in $\mathbb{R}^2$ and $\mathbb{R}^3$

The laplace equation in $\mathbb{R}^3$ is given by

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

An equation u satisfying the laplace equation is called a harmonic function.

2.8 Linearization of a function of several variables

Let $z = f(x, y)$ and let $P(x_0, y_0)$ be a given point. The linearization of $f(x, y)$ at a point $P(x_0, y_0)$ is denoted and defined by

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

2.9 Directional Derivatives of functions of several variables

Let $f(x, y, z)$ be a function of three independent variables x, y, and z, and let $\vec{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be a unit vector in the direction from the point $P(x_0, y_0, z_0)$ to an arbitrary point $Q(x, y, z)$. The vector $\vec{u}$ is defined

$$\vec{u} = \frac{\vec{PQ}}{||\vec{PQ}||} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Let $s = ||\vec{PQ}||$. Therefore the components of $\vec{u}$ can be written

$$as = x - x_0 \tag{1}$$
$$bs = y - y_0 \tag{2}$$
$$cs = z - z_0 \tag{3}$$

Now let $w = f(x, y, z)$ where $x = x_0 + as$, $y = y_0 + bs$, $z = z_0 + cs$. Hence w is a function of s. By the chain rule

$$\frac{dw}{ds} = f_x(x, y, z)a + f_y(x, y, z)b + f_z(x, y, z)c$$

and evaluate $\frac{dw}{ds}$ at $s = 0$. Recall (1), (2), and (3). If $s = 0$ is Substituted into (1), (2), and (3) it is easily shown that $x = x_0$, $y = y_0$, and $z = z_0$. Therefore

$$\left.\frac{dw}{ds}\right|_{s=0} = f_x(x_0, y_0, z_0)a + f_y(x_0, y_0, z_0)b + f_z(x_0, y_0, z_0)c \tag{4}$$

The result (4) can berepresented as a dot product of two vectors,

$$\begin{aligned} \frac{dw}{ds} &= \begin{pmatrix} f_x(x_0, y_0, z_0) \\ f_y(x_0, y_0, z_0) \\ f_z(x_0, y_0, z_0) \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} \\ &= \vec{\nabla} f(P) \cdot \vec{u} \end{aligned}$$

The above result is the directional derivative. The directional derivative of the function $f(x, y, z)$ at the point $P(x_0, y_0, z_0)$ in the direction of the unit vector $\vec{u}$ is denoted and defined by

$$D_{\vec{u}} f(P) = \left.\frac{dw}{ds}\right|_{s=0}$$
$$D_{\vec{u}} f(P) = \vec{\nabla} f(P) \cdot \vec{u}$$

The directional derivative $D_{\vec{u}} f(P)$ is the rate of change of the function f at the point P in the direction of the unit vector $\vec{u}$.

2.9.1 Maximum and Minimum Rates

Recall the angle θ between the vectors $\vec{a}$ and $\vec{b}$ is given by

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{||\vec{a}||\,||\vec{b}||}$$

Now let θ be the angle between $\vec{\nabla} f(P)$ and the unit vector $\vec{u}$

$$\therefore \vec{\nabla} f(P)\cdot\vec{u} = ||\vec{\nabla} f(P)||\,||\vec{u}||\cos\theta$$

but $\vec{u}$ is a unit vector and $||\vec{u}|| = 1$

$$\vec{\nabla} f(P)\cdot\vec{u} = ||\vec{\nabla} f(P)||\cos\theta$$

$$\therefore D_{\vec{u}} f(P) = ||\vec{\nabla} f(P)||\cos\theta$$

The directional derivative has extreme values when $\cos\theta = \pm 1$. $D_{\vec{u}} f(P)$ has an absolute maximum $||\vec{\nabla} f(P)||$ when $\theta = 0$, and $\vec{u}$ must be in the direction of $\vec{n}_1 = \dfrac{\vec{\nabla} f(P)}{||\vec{\nabla} f(P)||}$. $D_{\vec{u}}$ has an absolute minimum $-||\vec{\nabla} f(P)||$ when $\theta = \pi$ and must be in the direction of $\vec{n}_2 = -\dfrac{\vec{\nabla} f(P)}{||\vec{\nabla} f(P)||}$.

2.10 Implicit Differentiation

2.10.1 The Jacobian Determinant

The jacobian of 2 functions F and G with respect to the variables x and y is denoted and defined by

$$J = \frac{\partial(F,G)}{\partial(x,y)} = \begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix}$$

Likewise the Jacobian of n functions $f_1,\ f_2,\ f_3,\cdots f_n$ with respect to the n variables $x_1,\ x_2,\cdots x_n$ is given by

$$J = \frac{\partial(f_1, f_2, \cdots, f_n)}{\partial(x_1, x_2, \cdots, x_n)} = \begin{vmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} & \cdots & \dfrac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial f_n}{\partial x_1} & \dfrac{\partial f_n}{\partial x_2} & \cdots & \dfrac{\partial f_n}{\partial x_n} \end{vmatrix}$$

The Jacobian of one function f with respect to the variable x is simply $\dfrac{df}{dx}$

2.10.2 Implicit Differentiation

Consider a non-linear system that consists of m equations in n variables, where $m \leq n$. Under certain conditions, we may be able to solve for m variables as functions of the remaining $n - m$ variables. For instance consider a system of one non-linear equation in three variables,

$$F(x,y,z) = 0$$

There are three possible ways to solve this system. It can be solved with x as a function of y, and z; y as a function of x, and z; or z as a function of x and y. Consider a system of two equations in 5 variables say

$$\begin{cases} F(x,y,z,u,v) = 0 \\ G(x,y,z,u,v) = 0 \end{cases}$$

There are 10 possible choices of systems to solve for. If we wanted to find $\dfrac{\partial u}{\partial z}$ where $u = u(x,y,z)$ then it is denoted

$$\left(\frac{\partial u}{\partial z}\right)_{x,y}$$

This denotes that x,y,z are the independent variables and that u, and v depend on x, y, and z.

2.10.3 A formula for Implicit Integration:

Consider a non-linear system of two equations in four variables

$$\begin{cases} F(x,y,u,v) = 0 \\ G(x,y,u,v) = 0 \end{cases}$$

Assume that u and v depend on x, and y. The system has a condition of solvability and solutions $\dfrac{\partial u}{\partial x}$, $\dfrac{\partial u}{\partial y}$, $\dfrac{\partial v}{\partial x}$, and $\dfrac{\partial v}{\partial y}$. Let $w = F(x,y,u,v) = 0$ where $u = u(x,y)$, $v = v(x,y)$ hence $w = w(x,y)$. By the chain rule

$$\frac{\partial w}{\partial x} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x}$$

but with $w = 0$ the following is obtained

$$-\frac{\partial F}{\partial x} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x} \tag{1}$$

similarly for G

$$-\frac{\partial G}{\partial x} = \frac{\partial G}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial G}{\partial v}\frac{\partial v}{\partial x} \tag{2}$$

The equation (1) and (2) are linear systems in $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial v}{\partial x}$. By Cramer's rule

$$\frac{\partial u}{\partial x} = \frac{\begin{vmatrix} -\frac{\partial F}{\partial x} & \frac{\partial F}{\partial v} \\ -\frac{\partial G}{\partial x} & \frac{\partial G}{\partial v} \end{vmatrix}}{\begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} \end{vmatrix}} \qquad \frac{\partial v}{\partial x} = \frac{\begin{vmatrix} \frac{\partial F}{\partial u} & -\frac{\partial F}{\partial x} \\ \frac{\partial G}{\partial u} & -\frac{\partial G}{\partial x} \end{vmatrix}}{\begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} \end{vmatrix}}$$

Using the Jacobian the following is obtained,

$$\frac{\partial u}{\partial x} = -\frac{\frac{\partial(F,G)}{\partial(x,v)}}{\frac{\partial(F,G)}{\partial(u,v)}} \qquad \frac{\partial v}{\partial x} = -\frac{\frac{\partial(F,G)}{\partial(u,x)}}{\frac{\partial(F,G)}{\partial(u,v)}}$$

provided the denominator $\dfrac{\partial(F,G)}{\partial(u,v)} \neq 0$. $\dfrac{\partial u}{\partial y}$ and $\dfrac{\partial v}{\partial y}$. If there is a non linear system

$$F(x,y,z) = 0$$

that can be solved for y as a function of x and z, then

$$\frac{\partial y}{\partial x} = -\frac{F_x}{F_y} \qquad \frac{\partial y}{\partial z} = -\frac{F_z}{F_y}$$

provided $F_y \neq 0$.

3 Double and Triple Integration

3.1 Notation for a Double Integral

The double integral of a function $f(x, y)$ over a closed region D in the xy-plane (or $\mathbb{R}^2$) is denoted

$$\iint_D f(x, y)\, dA$$

where dA is an element of area and is given by

$$dA = dx\, dy = dy\, dx$$

3.2 Notation for a Triple Integral

The triple integral of $f(x, y, z)$ over a closed region E in xyz-space (or $\mathbb{R}^3$) is denoted by

$$\iiint_E f(x, y, z)\, dV$$

where dV is an element of volume and is given by

$$dV = dx\, dy\, dz$$

3.3 Types of regions in $\mathbb{R}^2$

3.3.1 The y-simple Region:

A region D is called a y-simple region if its bounded from the bottom and top by the continuous curves $y = g(x)$ and $y = h(x)$ respectively and is bound from the left and right by the vertical lines $x = a$ and $x = b$ respectively as shown. A y-simple region may be sliced vertically and hence may be described by the pair of inequalities,

$$D = \begin{cases} a \leq x \leq b \\ g(x) \leq y \leq h(x) \end{cases}$$

3.3.2 The x-simple Region:

A region D is called a x-simple region if its bounded from the bottom and top by the horizontal lines $y = c$ and $y = d$ respectively and is bound from the left and right by the continuous curves $x = p(y)$ and $x = q(y)$ respectively as shown. A x-simple region may be sliced vertically and hence may be described by the pair of inequalities,

$$D = \begin{cases} p(y) \leq x \leq q(y) \\ c \leq y \leq d \end{cases}$$

Some regions in $\mathbb{R}^2$ are both x and y-simple. However some regions in $\mathbb{R}^2$ are neither x OR Y-simple. In such a case the region may be subdivided into m non-overlapping regions each of which is x-simple, y-simple, or both. Let D be a planar region. Assume D is subdivided into m non-overlapping regions where

$$D = D_1 \cup D_2 \cup \cdots \cup D_m$$

then if $f(x, y)$ is a function on D then

$$\iint_D f(x, y)\, dA = \iint_{D_1} f(x, y)\, dA + \iint_{D_2} f(x, y)\, dA + \cdots + \iint_{D_m} f(x, y)\, dA$$

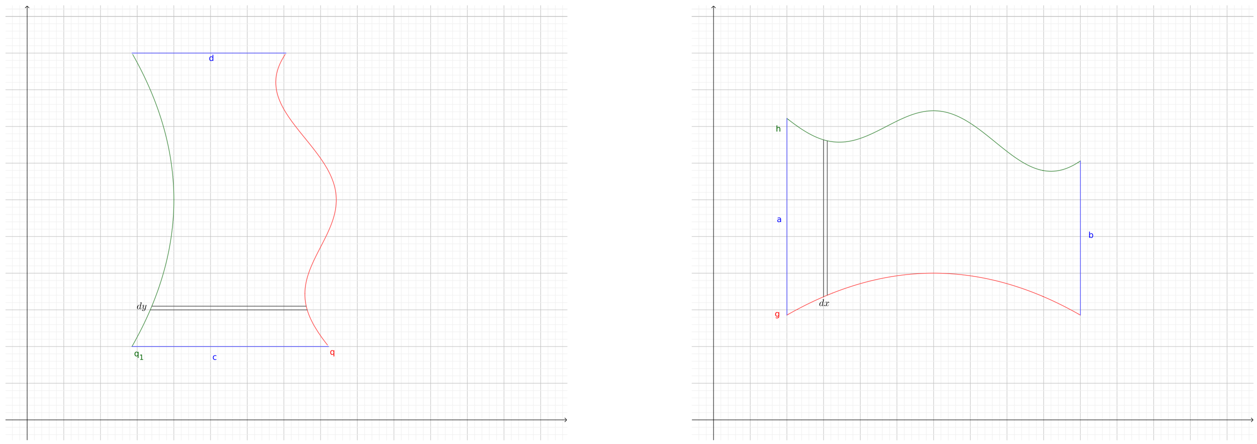


Figure 1: An x (left) and y-simple (right) region

3.4 Types of regions in $\mathbb{R}^3$

The description of a z-simple region in $\mathbb{R}^3$ is given. The description for x and y-simple regions is similar. A region in three space is called z simple if it is bounded from the bottom and top by the continuous surfaces $z = g(x, y)$ and $z = h(x, y)$ respectively. A z-simple region may be sliced vertically and hence is described by

$$E = \begin{cases} g(x,y) \leq z \leq h(x,y) \\ (x,y) \in B \end{cases}$$

Where B is a region in $\mathbb{R}^2$.

3.5 A definite partial integral

A definite integral of the form

$$\int\limits_{x=g(y)}^{x=h(x)} f(x,y)\,\partial x \quad \text{or} \quad \int\limits_{g(y)}^{h(y)} f(x,y)\,dx$$

is a definite partial integral with respect to x. To compute the definite partial integral, integrate $f(x, y)$ with respect to x but treating y as constant. Definite partial integrals of three or more variables are defined similarly.

3.6 An Iterated Integral

An iterated integral consists of two or more definite partial integrals. For instance

$$\int\limits_{c}^{d}\int\limits_{a}^{b} f(x,y)\,dx\,dy$$

is an example of an iterated integral. To compute an iterated integral and evaluate outward.

3.7 Setting Up Limits for a Double Integral:

Given the double integral

$$\iint\limits_{D} f(x,y)\,dA$$

If region D is a y-simple region. For a y-simple region, the region is sliced vertically, and hence to integrate over a y-simple region dA is written,

$$dA = dy\,dx$$

It follows that the integral becomes

$$\int_a^b \int_{g(x)}^{h(x)} f(x,y)\,dy\,dx$$

If region D is a x-simple region, the region is sliced horizontally, and hence to integrate over a x-simple region dA is written,

$$dA = dx\,dy$$

It follows that the integral becomes

$$\int_c^d \int_{p(x)}^{q(x)} f(x,y)\,dx\,dy$$

3.8 Setting up Limits for a Triple Integral

Consider the triple integral

$$\iiint_E f(x,y,z)\,dV$$

Here assume that the region E is z-simple. Setting up the limits for and x-simple or y-simple region is similar. Recall for z-simple regions they are sliced vertically and hence integrate with respect to z first. That is

$$dV = dz\,dA$$

Once the inner most integral is computed, the triple integral is reduced to a double integral.

3.9 Geometric Interpretation of the Double Integral:

Consider the double integral

$$\iint_D f(x,y)\,dA$$

for simplicity sake, we shall assume $f(x,y) \geq 0$ for $(x,y) \in D$. Let S be the surface given by the equation $z = f(x,y)$, and let V be the volume which lies vertically bellow S and above the xy plane on the region D. The slice $dA = dx\,dy$ is a small area on D. Therefore it is implied that

$$V = \iint_D f(x,y)\,dA$$

In general where $f(x,y)$ is not necessarily greater than or equal to zero. In general the triple integral,

$$\iiint_E f(x,y,z)\,dV$$

is the signed hyper volume in four dimensional space. If $f(x,y) = 1$ on D then the integral reduces to

$$A = \iint_D dA$$

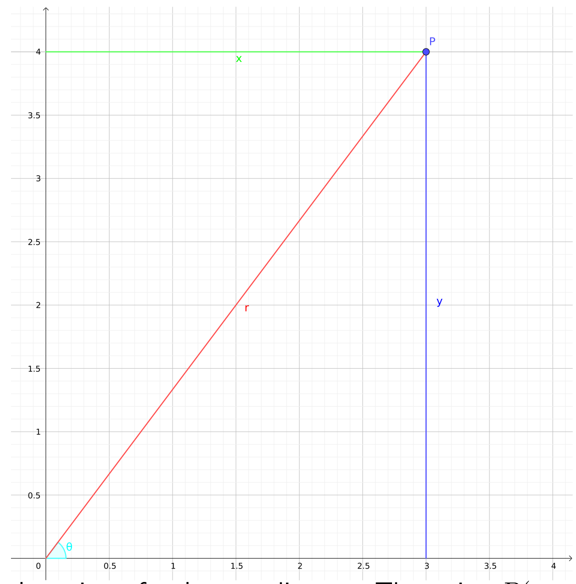
and gives the area of D. If $f(x,y,z) = 1$ in the region E then the integral reduces to

$$V = \iiint_E dV$$

and gives the volume of the region E. This implies that a volume can be computed by either a double or triple integral.

3.10 Polar, Cylindrical and Spherical Coordinate Systems:

3.10.1 Polar Coordinates System:



Let $P(x, y)$ be a point in the xy-plane. The polar coordinates of P are r, and θ where r is the distance between the origin and the point P and $r \in [0, \infty)$ and θ is the angle between $\vec{OP}$ and the positive x-axis and $\theta \in [0, 2\pi]$. The cartesian and polar coordinates are usually displayed on the same figure as shown and have the following relationships,

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ r^2 &= x^2 + y^2 \\ \frac{y}{x} &= \tan\theta \\ dA &= r\,dr\,d\theta \end{aligned}$$

3.10.2 Cylindrical Coordinate System:

The cylindrical coordinate system is a three dimensional version of polar coordinates. The point $P(x, y, z)$ is (r, θ, z) in cylindrical coordinates.

3.10.3 Spherical Coordinate System

The spherical coordinate system is closely related to the geographical longitudes and latitudes. Let $P(x, y, z)$ be a point of the surface of a sphere. The spherical coordinates of P are ρ, ϕ, and θ where where ρ is the distance from the origin to P with $\rho \in [0, \infty)$, ϕ is the angle made between $\vec{OP}$ and the positive z-axis with $\phi \in [0, \pi]$ and θ is the angle made between $\vec{OQ}$ and th positive x-axis with $\theta \in [0, 2\pi]$.

$$x = r\cos\theta \tag{1}$$
$$y = r\sin\theta \tag{2}$$
$$r^2 = x^2 + y^2 \tag{3}$$
$$r = \rho\sin\phi \tag{4}$$
$$z = \rho\cos\phi \tag{5}$$
$$\rho^2 = z^2 + r^2 \tag{6}$$

Substitute (4) into (1), (2), (3), and (6), to obtain,

$$\begin{aligned} x &= \rho\,\sin\phi\,\cos\theta \\ y &= \rho\,\sin\phi\,\sin\theta \\ z &= \rho\cos\phi \\ \rho^2 &= x^2 + y^2 + z^2 \\ x^2 + y^2 &= \rho^2\sin^2\phi \\ dV &= \rho^2\sin\phi\,d\rho\,d\phi\,d\theta \end{aligned}$$

When the equations describing a region D contain $x^2 + y^2$, substitute $x^2 + y^2 = r^2$ and use polar or cylindrical coordinates. When the equation describing a region E contains $x^2 + y^2 + z^2$, substitute $x^2 + y^2 + z^2 = \rho^2$, and use spherical coordinates.

3.10.4 A special Curve in Polar Coordinates:

The equation of a circle centered at $(0,0)$ and with radius a is given by

$$x^2 + y^2 = a^2$$

in polar coordinates $r^2 = x^2 + y^2$ so therefore $r^2 = a^2$, or

$$r = a$$

3.10.5 A special Surface in cylindrical Coordinates

Recall that a cylinder can be given by any curve in $\mathbb{R}^2$ projected into $\mathbb{R}^3$. Therefore the right circular cylinder is given by the equation of a circle. From above it can be seen that a circular cylinder in cylindrical coordinates, centered at $(0,0)$, and with radius a can be given by,

$$r = a$$

3.10.6 Three Special Surfaces in Spherical Coordinates:

1. The equation of a sphere of radius a centered at $(0,0,0)$ is given by

   $$x^2 + y^2 + z^2 = a^2$$

   in spherical coordinates $x^2 + y^2 + z^2 = \rho^2$

   $$\therefore \rho = a$$

2. The equation of a sphere of radius a with center $(0,0,a)$ is given by

   $$\begin{aligned} x^2 + y^2 + (z-a)^2 &= a^2 \\ x^2 + y^2 + z^2 &= 2az \end{aligned}$$

   In spherical coordinates $z = \rho\cos\phi$ and $x^2 + y^2 + z^2 = \rho^2$,

   $$\therefore \rho^2 = 2a\rho\cos\phi$$
   $$\rho = 2a\cos\phi$$

3. The equation of a cone is given by

   $$z = \alpha\sqrt{x^2 + y^2}$$

   in spherical coordinates $z = \rho\cos\phi$ and $x^2 + y^2 = \rho^2\sin^2\phi$, so

   $$\begin{aligned} \rho\cos\phi &= \alpha\sqrt{\rho^2\sin^2\phi} \\ \therefore \phi &= \tan^{-1}\left(\frac{1}{\alpha}\right) \end{aligned}$$

3.10.7 Setting Up Limits of Integration in Polar Coordinates:

Given the double integral

$$\iint_D f(x,y)\,dA$$

To compute using polar coordinates, apply the following three steps,

1. In the expression of $f(x,y)$, replace x by $r\cos\theta$, y by $r\sin\theta$, and $x^2 + y^2$ by r^2.
2. Replace dA by $r\,dr\,d\theta$
3. Express D in polar coordinates.

3.11 Application of Double and Triple Integrals in Calculating Mass, Moments, Centers of Mass, and Centroids

Let D be the planar region occupied by a thin plate or lamina. Assume that the plate is not uniform and that the area density is given by the function

$$\delta(x, y)$$

The mass is given by definition as the sum of the elements of mass

$$dm = \delta(x, y)\, dA$$

The mass is given by

$$m = \iint\limits_{D} \delta(x, y)\, dA$$

The moment about the y axis may be denoted $M_{x=0}$. By definition the element of moment about the y-axis is

$$dM_{x=0} = x\, dm$$

The total moment is then given by,

$$M_{x=0} = \iint\limits_{D} x\, dm$$

likewise the moment about the x-axis is

$$M_{y=0} = \iint\limits_{D} y\, dm$$

The center of mass $(\bar{x}, \bar{y})$ is an imaginary point where the entire mass is assumed to be concentrated. By definition

$$M_{x=0} = \bar{x}m \Rightarrow \bar{x} = \frac{M_{x=0}}{m}$$
$$M_{y=0} = \bar{y}m \Rightarrow \bar{y} = \frac{M_{y=0}}{m}$$

If the lamina is uniform then its density is constant[2]

$$\delta(x, y) = C$$

In such a case, the center is mass is referred to as a centroid. Likewise let E be the region in three space occupied by a solid. Assume the solid is not uniform and that the density function is

$$\delta(x, y, z)$$

The mass is given by

$$m = \iiint\limits_{E} dm$$

The moment about the yz-plane is

$$M_{x=0} = \iiint\limits_{E} x\, dm$$

The moment about the xz-plane is

$$M_{y=0} = \iiint\limits_{E} y\, dm$$

The moment about the xy-plane is

$$M_{z=0} = \iiint\limits_{E} z\, dm$$

The center of mass $(\bar{x}, \bar{y}, \bar{z})$ is given by

$$\bar{x} = \frac{M_{x=0}}{m} \qquad \bar{y} = \frac{M_{y=0}}{m} \qquad \bar{z} = \frac{M_{z=0}}{m}$$

if $\delta(x, y, z)$ is constant[2] then the center of mass is the centroid. In all equations above $dm = \delta(x, y, z)\, dV$.

4 Extreme Values for functions of Several Variables

4.1 Critical Points

Let $f(x, y)$ be a function of the two independent variables x, and y. The critical points of $f(x, y)$ occur where

$$\vec{\nabla} f(x, y) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Therefore the critical points occur at the solutions to the non-linear system of equations,

$$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases}$$

4.2 The Hessian Matrix and the Second Derivative Test

4.2.1 Quadric Forms of a matrix

Let $\vec{x}$ be a $n \times 1$ vector and A be a $n \times n$ symmetric matrix. Consider the expression

$$\vec{x} \cdot A\vec{x}$$

A is referred to as positive definite if $\vec{x} \cdot A\vec{x} > 0$ for all non-zero vectors $\vec{x}$. A is referred to as negative definite if $\vec{x} \cdot A\vec{x} < 0$ for all non-zero vectors $\vec{x}$. A is referred to as positive semi definite if $\vec{x} \cdot A\vec{x} \geq 0$ for all non-zero vectors $\vec{x}$. A is referred to as negative semi definite if $\vec{x} \cdot A\vec{x} \leq 0$ for all non-zero vectors $\vec{x}$. If there are vectors $\vec{x}$, and $\vec{y}$ such that $\vec{x} \cdot A\vec{x} > 0$ and $\vec{y} \cdot A\vec{y} < 0$ then A is indefinite.
Alternatively let D_i be the determinant of the upper left $i \times i$ block of A.

1. If $D_i > 0$ for all i then A is positive definite.
2. If $D_i > 0$ for all even values of i and $D_i < 0$ for all odd values of i then A is negative definite.
3. If $D_n = |A| \neq 0$ and neither 1 nor 2 hold then A is indefinite.
4. If $|A| = 0$ then A could be positive or negative semi definite or indefinite but not positive nor negative definite.

4.2.2 The Hessian Matrix

Let $f(x_1, x_2, \cdots, x_n)$ be a function such that all second order partials are continuous. The Hessian matrix of f is denoted and defined by

$$H = \begin{pmatrix} f_{x_1x_1} & f_{x_1x_2} & \cdots & f_{x_1x_n} \\ f_{x_2x_1} & f_{x_2x_2} & \cdots & f_{x_2x_n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{x_nx_1} & f_{x_nx_2} & \cdots & f_{x_nx_n} \end{pmatrix}$$

Note that as the second order partials are continuous H is a symmetric matrix.

[2] It is generally assumed in this case that the density is equal to 1

4.2.3 The second Derivative Test

Suppose (a, b) is a critical point for f : $\vec{\nabla} f(a, b) = 0$

1. If $H(a, b)$ is positive definite, then f has a local min at (a, b)
2. If $H(a, b)$ is negative definite, then f has a local max at (a, b)
3. If $H(a, b)$ is indefinite, then f has a saddle point at (a, b)
4. If $H(a, b)$ is positive or negative semi definite then the test is inconclusive.

4.3 The Discriminant and Second Derivative Test for a function of Several Variables

4.3.1 The Discriminant

Let $f(x, y)$ be a function of two independent variables. The second order partials of f are f_{xx}, f_{xy}, f_{yy}, and f_{yx}. Let the discriminant of f be denoted and defined by

$$D(x, y) = (f_{xy})^2 - f_{xx} f_{yy}$$

4.3.2 The second derivative test

Let $f(x, y)$ be a function of the two independent variables x, and y, and let $P(x_0, y_0)$ be a critical point for the function f.

1. If $D(x_0, y_0) < 0$ and $f_{xx} > 0$, f has a local minimum at P.
2. If $D(x_0, y_0) < 0$ and $f_{xx} < 0$, f has a local maximum at P.
3. If $D(x_0, y_0) > 0$, f has a neither a local minimum or maximum at P. Such a point is referred to as a saddle point and occurs when f is at a local minimum in x and a local maximum in y or vice versa.
4. If $D(x_0, y_0) = 0$ the test is inconclusive.

4.4 Extreme Values for functions of several variables

Let $f(x, y)$ be a function of two independent variables and let D be a closed region in the xy-plane. To find the extreme values of f over the region D, first calculate all critical points in the interior of the region D. Next find all critical points on the boundary of the region D, and compute f at these critical points as well as at endpoints of the region. Finally compare all the obtained values if f. The largest value of f computed is the absolute maximum of f on D. Likewise the smallest computed value of f is the absolute maximum of f on D.

4.5 Method of Lagrange Multipliers

Suppose that $f(x, y)$ is bound by the constraint $g(x, y) = C$. Then the maximum and minimum points of f on g are solutions to the following non-linear system of equations,

$$\begin{cases} \vec{\nabla} f = \lambda \vec{\nabla} g \\ g(x, y) = C \end{cases}$$

where λ is a constant referred to as a Lagrange Multiplier.