Complete and Easy Bidirectional Typechecking for Higher-Rank Polymorphism

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Abstract

Bidirectional typechecking, in which terms either synthesize a type or are checked against a known type, has become popular for its scalability (unlike Damas-Milner type inference, bidirectional typing remains decidable even for very expressive type systems), its error reporting, and its relative ease of implementation. Following design principles from proof theory, bidirectional typing can be applied to many type constructs. The principles underlying a bidirectional approach to polymorphism, however, are less obvious. We give a declarative, bidirectional account of higher-rank polymorphism, grounded in proof theory; this calculus enjoys many properties such as η -reduction and predictability of annotations. We give an algorithm for implementing the declarative system; our algorithm is remarkably simple and well-behaved, despite being both sound and complete.

1. Introduction

Bidirectional typechecking (Pierce and Turner 2000) has become one of the most popular techniques for implementing typecheckers in new languages. This technique has been used for dependent types (Coquand 1996; Abel et al. 2008; Löh et al. 2008; Asperti et al. 2012); subtyping (Pierce and Turner 2000); intersection, union, indexed and refinement types (Xi 1998; Davies and Pfenning 2000; Dunfield and Pfenning 2004); termination checking (Abel 2004); higher-rank polymorphism (Peyton Jones et al. 2007; Dunfield 2009); refinement types for LF (Lovas 2010); contextual modal types (Pientka 2008); compiler intermediate representations (Chlipala et al. 2005); and object-oriented languages including C[‡] (Bierman et al. 2007) and Scala (Odersky et al. 2001). As can be seen, it scales well to advanced type systems; moreover, it is easy to implement, and yields relatively high-quality error messages (Peyton Jones et al. 2007).

However, the theoretical foundation of bidirectional typechecking has lagged behind its application. As shown by Watkins et al. (2004), bidirectional typechecking can be understood in terms of the normalization of intuitionistic type theory, in which normal forms correspond to the checking mode of bidirectional typechecking, and neutral (or atomic) terms correspond to the synthesis mode. This enables a proof of the elegant property that type anno-

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tations are only necessary at reducible expressions, and that normal forms need no annotations at all. The benefit of the proof-theoretic view is that it gives a simple and easy-to-understand declarative account of where type annotations are necessary, without reference to the details of the typechecking algorithm.

While the proof-theoretic account of bidirectional typechecking has been scaled up as far as type refinements and intersection and union types (Pfenning 2008), as yet there has been no completely satisfactory account of how to extend the proof-theoretic approach to handle polymorphism. This is especially vexing, since the ability of bidirectional *algorithms* to gracefully accommodate polymorphism (even higher-rank polymorphism) has been one of their chief attractions.

In this paper, we extend the proof-theoretic account of bidirectional typechecking to full higher-rank polymorphism (i.e., predicative System F), and consequently show that bidirectional typechecking is not merely sound with respect to the declarative semantics, but also that it is *complete*. Better still, the algorithm we give for doing so is extraordinarily *simple*.

First, as a specification of type checking, we give a declarative bidirectional type system which guesses all quantifier instantiations. This calculus is a small but significant contribution of this paper, since it possesses desirable properties, such as the preservation of typability under η -reduction, that are missing from the type assignment version of System F. Furthermore, we can use the bidirectional character of our declarative calculus to show a number of refactoring theorems, enabling us to precisely characterize what sorts of substitutions (and reverse substitutions) preserve typability, where type annotations are needed, and when programmers may safely delete type annotations.

Then, we give a bidirectional algorithm that always finds corresponding instantiations. As a consequence of completeness, we can show that our algorithm never needs explicit type applications, and that type annotations are only required for polymorphic, reducible expressions—which, in practice, means that only let-bindings of functions at polymorphic type need type annotations; no other expressions need annotations.

Our algorithm is both simple to understand and simple to implement. The key data structure is an ordered context containing all bindings, including type variables, term variables, and existential variables denoting partial type information. By maintaining order, we are able to easily manage scope information, which is particularly important in higher-rank systems, where different quantifiers may be instantiated with different sets of free variables. Furthermore, ordered contexts admit a notion of *extension* or *information increase*, which organizes and simplifies the soundness and completeness proofs of the algorithmic system with respect to the declarative one.

Terms
$$e ::= x \mid () \mid \lambda x. e \mid e \mid e \mid (e : A)$$

Figure 1. Source expressions

$$\begin{array}{lll} \text{Types} & A,B,C & ::= & 1 \mid \alpha \mid \forall \alpha.A \mid A \rightarrow B \\ \text{Monotypes} & \tau,\sigma & ::= & 1 \mid \alpha \mid \tau \rightarrow \sigma \\ \text{Contexts} & \Psi & ::= & \cdot \mid \Psi,\alpha \mid \Psi,x:A \end{array}$$

Figure 2. Syntax of declarative types and contexts

Contributions. We make the following contributions:

 We give a declarative, bidirectional account of higher-rank polymorphism, grounded strongly in proof theory. This calculus has important properties (such as η-reduction) that the type assignment variant of System F lacks, yet is sound and complete (up to βη-equivalence) with respect to System F.

As a result, we can explain where type annotations are needed, where they may be deleted, and why important code transformations are sound, all without reference to the implementation.

- We give a very simple algorithm for implementing the declarative system. Our algorithm does not need any data structure more sophisticated than a list, but can still solve all of the problems which arise in typechecking higher-rank polymorphism without any need for search or backtracking.
- We prove that our algorithm is both sound and complete with respect to our declarative specification of typing. This proof is cleanly structured around context extension, a relational notion of information increase, corresponding to the intuition that our algorithm progressively resolves type constraints.

As a result of completeness, programmers may safely "pay no attention to the implementor behind the curtain", and ignore all the algorithmic details of unification and type inference: the algorithm does exactly what the declarative specification says, no more and no less.

Lemmas and proofs. Proofs of the main results, as well as statements of all lemmas (and their proofs), can be found in the appendix, available from www.cs.cmu.edu/~joshuad/bidir/.

2. Declarative Type System

In order to show that our algorithm is sound and complete, we need to give a declarative type system to serve as the specification for our algorithm. Surprisingly, it turns out that finding the correct declarative system to use as a specification is itself an interesting problem!

Much work on type inference for higher-rank polymorphism takes the type assignment variant of System F as a specification of type inference. Unfortunately, under these rules typing is not stable under η -reductions. For example, suppose f is a variable of type $1 \to \forall \alpha$. α . Then the term λx . f x can be ascribed the type $1 \to 1$, since the polymorphic quantifier can be instantiated to 1 between the f and the x. But the η -reduct f cannot be ascribed the type $1 \to 1$, because the quantifier cannot be instantiated until after f has been applied. This is especially unfortunate in pure languages like Haskell, where the η law is a valid program equality.

Therefore, we do not use the type assignment version of System F as our declarative specification of type checking and inference. Instead, we give a declarative, bidirectional system as the specification. Traditionally, bidirectional systems are given in terms of a *checking* judgment $\Psi \vdash e \Leftarrow A$, which takes a type A as input and

$$\begin{array}{c|c} \hline \Psi \vdash A & \text{Under context } \Psi \text{, type } A \text{ is well-formed} \\ \hline \frac{\alpha \in \Psi}{\Psi \vdash \alpha} \text{ DeclUvarWF} & \overline{\Psi \vdash 1} \text{ DeclUnitWF} \\ \hline \frac{\Psi \vdash A & \Psi \vdash B}{\Psi \vdash A \to B} \text{ DeclArrowWF} & \frac{\Psi, \alpha \vdash A}{\Psi \vdash \forall \alpha. \, A} \text{ DeclForalIWF} \\ \hline \hline \Psi \vdash A \leq B & \text{Under context } \Psi \text{, type } A \text{ is a subtype of } B \\ \hline \frac{\alpha \in \Psi}{\Psi \vdash \alpha \leq \alpha} \leq \text{Var} & \overline{\Psi \vdash 1 \leq 1} \leq \text{Unit} \\ \hline \frac{\Psi \vdash B_1 \leq A_1 & \Psi \vdash A_2 \leq B_2}{\Psi \vdash A_1 \to A_2 \leq B_1 \to B_2} \leq \to \\ \hline \frac{\Psi \vdash \tau & \Psi \vdash [\tau/\alpha]A \leq B}{\Psi \vdash \forall \alpha. \, A \leq B} \leq \forall \text{L} & \frac{\Psi, \beta \vdash A \leq B}{\Psi \vdash A \leq \forall \beta. \, B} \leq \forall \text{R} \\ \hline \end{array}$$

Figure 3. Well-formedness of types and subtyping in the declarative system

ensures that the term e checks against that type, and a synthesis judgment $\Psi \vdash e \Rightarrow A$, which takes a term e and produces a type A. This two-judgment formulation is satisfactory for simple types, but breaks down in the presence of polymorphism.

The essential problem is as follows: the standard bidirectional rule for checking applications e_1 e_2 in non-polymorphic systems is to synthesize type $A \to B$ for e_1 , and then check e_2 against A, returning B as the type. With polymorphism, however, we may have an application e_1 e_2 in which e_1 synthesizes a term of polymorphic type (e.g., $\forall \alpha. \alpha \to \alpha$). Furthermore, we do not know *a priori* how many quantifiers we need to instantiate.

To solve this problem, we turn to *focalization* (Andreoli 1992), the proof-theoretic foundation of bidirectional typechecking. In focused sequent calculi, it is natural to give terms in *spine form* (Cervesato and Pfenning 2003; Simmons 2012), sequences of applications to a head. So we view every application as really being a spine consisting of a series of type applications followed by a term application, and introduce an *application judgment* $\Psi \vdash A \bullet e \Rightarrow C$, which says that if a term of type A is applied to argument e, the result has type C. Consequently, quantifiers will be instantiated exactly when needed to reveal a function type.

The application judgment lets us suppress explicit type applications, but to get the η law, we need more. Recall the example with $f:1\to\forall\alpha.$ $\alpha.$ In $\eta\text{-reducing }\lambda x.$ f x to f, we reduce the number of applications in the term. That is, we no longer have a syntactic position at which we can (implicitly) instantiate polymorphic quantifiers. To handle this, we follow Odersky and Läufer (1996) in modeling type instantiation using subtyping, where subtyping is defined as a "more-polymorphic-than" relation that guesses type instantiations arbitrarily deeply within types. As a result, $1\to\forall\alpha.$ α is a subtype of $1\to1$, and the η law holds.

Happily, subtyping *does* fit naturally into bidirectional systems (Davies and Pfenning 2000; Dunfield 2007; Lovas 2010), so we can give a declarative, bidirectional type system that guesses type instantiations, but is otherwise entirely syntax-directed. In particular, subsumption is confined to a single rule (which switches from checking to synthesis), and our use of an application judgment determines when to instantiate quantifiers. The resulting system is very well-behaved, and ensures that the expected typability results (such as typability being preserved by η -reductions) continue to hold. Furthermore, our declarative formulation makes it clear that

$$\begin{array}{c} \underline{\Psi \vdash e \Leftarrow A} \\ \underline{\Psi \vdash e \Rightarrow A} \end{array} \text{ Under context } \Psi, \text{ e checks against input type A} \\ \underline{\Psi \vdash e \Rightarrow A} \end{array} \text{ Under context } \Psi, \text{ e synthesizes output type A} \\ \underline{\Psi \vdash A \bullet e \Rightarrow C} \end{array} \text{ Under context } \Psi, \text{ applying a function of type A to e synthesizes type C} \\ \\ \underline{\frac{(x:A) \in \Psi}{\Psi \vdash x \Rightarrow A}} \text{ DeclVar} \qquad \frac{\Psi \vdash e \Rightarrow A}{\Psi \vdash e \Leftarrow B} \qquad \frac{\Psi \vdash A \leq B}{\Psi \vdash e \Leftarrow B} \text{ DeclSub} \qquad \frac{\Psi \vdash A}{\Psi \vdash (e:A) \Rightarrow A} \text{ DeclAnno} \\ \\ \underline{\Psi \vdash (e:A) \Rightarrow A} \end{array} \text{ DeclAnno} \\ \\ \underline{\Psi \vdash (e:A) \Rightarrow A} \text{ DeclAnno} \\ \\ \underline{\Psi \vdash (e:A) \Rightarrow A} \text{ DeclAnno} \\ \\ \underline{\Psi \vdash (e:A) \Rightarrow A} \text{ DeclAnno} \\ \\ \underline{\Psi \vdash (e:A) \Rightarrow A} \text{ DeclAnno} \\ \\ \underline{\Psi \vdash (e:A) \Rightarrow A} \text{ DeclAnno} \\ \\ \underline{\Psi \vdash (e:A) \Rightarrow A} \text{ DeclAnno} \\ \\ \underline{\Psi \vdash (e:A) \Rightarrow A} \text{ DeclAnno} \\ \\ \underline{\Psi \vdash (e:A) \Rightarrow A} \text{ DeclAnno} \\ \\ \underline{\Psi \vdash (e:A) \Rightarrow A} \text{ DeclAnno} \\ \\ \underline{\Psi \vdash (e:A) \Rightarrow A} \text{ DeclAnno} \\ \\ \underline{\Psi \vdash (e:A) \Rightarrow A} \text{ DeclAnno} \\ \\ \underline{\Psi \vdash (e:A) \Rightarrow A} \text{ DeclAnno} \\ \\ \underline{\Psi \vdash (e:A) \Rightarrow A} \text{ DeclAnno} \\ \\ \underline{\Psi \vdash (e:A) \Rightarrow A} \text{ DeclAnno} \\ \\ \underline{\Psi \vdash (e:A) \Rightarrow A} \text{ DeclAnno} \\ \\ \underline{\Psi \vdash (e:A) \Rightarrow A} \text{ DeclAnno} \\ \\ \underline{\Psi \vdash (e:A) \Rightarrow A} \text{ DeclAnno} \\ \\ \underline{\Psi \vdash (e:A) \Rightarrow A} \text{ DeclAnno} \\ \\ \underline{\Psi \vdash (e:A) \Rightarrow A} \text{ DeclAnno} \\ \\ \underline{\Psi \vdash (e:A) \Rightarrow A} \text{ DeclAnno} \\ \\ \underline{\Psi \vdash (e:A) \Rightarrow A} \text{ DeclAnno} \\ \\ \underline{\Psi \vdash (e:A) \Rightarrow A} \text{ DeclAnno} \\ \\ \underline{\Psi \vdash (e:A) \Rightarrow A} \text{ DeclAnno} \\ \\ \underline{\Psi \vdash (e:A) \Rightarrow A} \text{ DeclAnno} \\ \\ \underline{\Psi \vdash (e:A) \Rightarrow A} \text{ DeclAnno} \\ \\ \underline{\Psi \vdash (e:A) \Rightarrow A} \text{ DeclAnno} \\ \\ \underline{\Psi \vdash (e:A) \Rightarrow A} \text{ DeclAnno} \\ \\ \underline{\Psi \vdash (e:A) \Rightarrow A} \text{ DeclAnno} \\ \\ \underline{\Psi \vdash (e:A) \Rightarrow A} \text{ DeclAnno} \\ \\ \underline{\Psi \vdash (e:A) \Rightarrow A} \text{ DeclAnno} \\ \\ \underline{\Psi \vdash (e:A) \Rightarrow A} \text{ DeclAnno} \\ \\ \underline{\Psi \vdash (e:A) \Rightarrow A} \text{ DeclAnno} \\ \\ \underline{\Psi \vdash (e:A) \Rightarrow A} \text{ DeclAnno} \\ \\ \underline{\Psi \vdash (e:A) \Rightarrow A} \text{ DeclAnno} \\ \\ \underline{\Psi \vdash (e:A) \Rightarrow A} \text{ DeclAnno} \\ \\ \underline{\Psi \vdash (e:A) \Rightarrow A} \text{ DeclAnno} \\ \\ \underline{\Psi \vdash (e:A) \Rightarrow A} \text{ DeclAnno} \\ \\ \underline{\Psi \vdash (e:A) \Rightarrow A} \text{ DeclAnno} \\ \\ \underline{\Psi \vdash (e:A) \Rightarrow A} \text{ DeclAnno} \\ \\ \underline{\Psi \vdash (e:A) \Rightarrow A} \text{ DeclAnno} \\ \\ \underline{\Psi \vdash (e:A) \Rightarrow A} \text{ DeclAnno} \\ \\ \underline{\Psi \vdash (e:A) \Rightarrow A} \text{ DeclAnno} \\ \\ \underline{\Psi \vdash (e:A) \Rightarrow A} \text{ DeclAnno} \\ \\ \underline{\Psi \vdash (e:A) \Rightarrow A} \text{ DeclAnno} \\ \\ \underline{\Psi \vdash (e:A) \Rightarrow A} \text{ DeclAnno} \\ \\ \underline{\Psi \vdash (e:A) \Rightarrow A} \text{ DeclAnno} \\ \\ \underline{\Psi \vdash (e:A) \Rightarrow A} \text{ DeclA$$

Figure 4. Declarative typing

the fundamental algorithmic problem in extending bidirectional typechecking to polymorphism is precisely the problem of figuring out what the missing type applications are.

Preserving the η-rule for functions comes at a cost. The subtyping relation induced by instantiation is undecidable for *impredicative* polymorphism (Tiuryn and Urzyczyn 1996; Chrząszcz 1998). Since we want a complete typechecking algorithm, we consequently restrict our system to predicative polymorphism, where polymorphic quantifiers can be instantiated only with monomorphic types. We discuss alternatives in Section 9.

2.1 Typing in Detail

Language overview. In Figure 1, we give the grammar for our language. We have a unit term (), variables x, lambda-abstraction λx . e, application e_1 e_2 , and type annotation (e: A). We write A, B, C for types (Figure 2): types are the unit type 1, type variables α , universal quantification $\forall \alpha$. A, and functions $A \to B$. Monotypes τ and σ are the same, less the universal quantifier. Contexts Ψ are lists of type variable declarations, and term variables x: A, with the expected well-formedness condition. (We give a single-context formulation mixing type and term hypotheses to simplify the presentation.)

Checking, synthesis, and application. Our type system has three main judgments, given in Figure 4. The checking judgment $\Psi \vdash e \Leftarrow A$ asserts that e checks against the type A in the context Ψ . The synthesis judgment $\Psi \vdash e \Rightarrow A$ says that we can synthesize the type A for e in the context Ψ . Finally, an *application judgment* $\Psi \vdash A \bullet e \Rightarrow C$ says that if a (possibly polymorphic) function of type A is applied to argument e, then the whole application synthesizes C for the whole application.

As is standard in the proof-theoretic presentations of bidirectional typechecking, each of the introduction forms in our calculus has a corresponding checking rule. The Decl11 rule says that () checks against the unit type 1. The Decl \rightarrow I rule says that $\lambda x.e$ checks against the function type $A \rightarrow B$ if e checks against B with the additional hypothesis that x has type A. The Decl \forall I rule says that e has type e A if e has type e in a context extended with a fresh e A. Sums, products and recursive types can be added similarly (we leave them out for simplicity). Rule DeclSub mediates between

synthesis and checking: it says that *e* can be checked against B, if *e* synthesizes A and A is a subtype of B (that is, A is at least as polymorphic as B).

As expected, we can infer a type for a variable (the DeclVar rule) just by looking it up in the context. Likewise, the DeclAnno rule says that we can synthesize a type A for a term with a type annotation (e:A) just by returning that type (after checking that the term does actually check against A).

Application is a little more complex: we have to eliminate universals until we reach an arrow type. To do this, we use an application judgment $\Psi \vdash A \bullet e \Rightarrow C$, which says that if we apply a term of type A to an argument e, we get something of type C. This judgment works by guessing instantiations of polymorphic quantifiers in rule $Decl \forall App$. Once we have instantiated enough quantifiers to expose an arrow $A \rightarrow C$, we check e against A and return C in rule $Decl \rightarrow App$.

In the following example, where we are applying some function polymorphic in α , Decl \forall App instantiates the outer quantifier (to the unit type 1; we elide the premise $\Psi \vdash 1$), but leaves the inner quantifier over β alone.

$$\frac{ \begin{array}{c} \Psi \vdash x \Leftarrow (\forall \beta. \ \beta \rightarrow \beta) \\ \hline \Psi \vdash (\forall \beta. \ \beta \rightarrow \beta) \rightarrow 1 \rightarrow 1 \bullet x \Longrightarrow 1 \rightarrow 1 \end{array} \mathsf{Decl} \rightarrow \mathsf{App}}{ \Psi \vdash (\forall \alpha. \ (\forall \beta. \ \beta \rightarrow \beta) \rightarrow \alpha \rightarrow \alpha) \bullet x \Longrightarrow 1 \rightarrow 1} \; \mathsf{Decl} \forall \mathsf{App} \\ \end{array}}$$

In the minimal proof-theoretic formulation of bidirectionality (Davies and Pfenning 2000; Dunfield and Pfenning 2004), introduction forms are checked and elimination forms synthesize, full stop. Even () cannot synthesize its type! Actual bidirectional typecheckers tend to take a more liberal view, adding synthesis rules for at least some introduction forms. To show that our system can accommodate these kinds of extensions, we add the Decl11 \Rightarrow and Decl \rightarrow I \Rightarrow rules, which synthesize a unit type for () and a monomorphic function type for lambda expressions $\lambda x.e.$ We examine other variations, including a purist bidirectional no-inference alternative, and a liberal Damas-Milner alternative, in Section 8.

Instantiating types. We express the fact that one type is a polymorphic generalization of another by means of the subtyping judgment given in Figure 3. One important aspect of the judgment is that types are compared relative to a context of free variables. This simplifies our rules, by letting us eliminate the awkward side conditions on sets of free variables that plague many presentations. Most of the subtyping judgment is typical: it proceeds structurally on types, with a contravariant twist for the arrow; all the real ac-

¹ Note that we do not require an explicit type abstraction operation. As a result, an implementation needs to use some technique like scoped type variables (Peyton Jones and Shields 2004) to mention bound type variables in annotations. This point does not matter to the abstract syntax, though.

tion is contained within the two subtyping rules for the universal

The left rule, $\leq \forall L$, says that a type $\forall \alpha. A$ is a subtype of B, if some instance $[\tau/\alpha]A$ is a subtype of B. This is what makes these rules only a declarative specification: $\forall L$ guesses the instantiation τ "out of thin air", and so the rules do not directly yield an algorithm.

The right rule $\leq \forall R$ is a little more subtle. It says that A is a subtype of $\forall \beta$. B if we can show that A is a subtype of B in a context extended with β . There are two intuitions for this rule, one semantic, the other proof-theoretic. The semantic intuition is that since $\forall \beta$. B is a subtype of $[\tau/\beta]B$ for any τ , we need A to be a subtype of $[\tau/\beta]B$ for any τ . Then, if we can show that A is a subtype of B, with a free variable β , we can appeal to a substitution principle for subtyping to conclude that for all τ , type A is a subtype of $[\tau/\beta]B$.

The proof-theoretic intuition is simpler. The rules $\leq \forall L$ and ≤∀R are just the left and right rules for universal quantification in the sequent calculus. Type inference is a form of theorem proving, and our subtype relation gives some of the inference rules a theorem prover may use. Following good proof-theoretic hygiene enables us to leave the reflexivity and transitivity rules out of the subtype relation, since they are admissible properties (in sequent calculus terms, they are the identity and cut-admissibility properties). The absence of these rules (particularly, the absence of transitivity), in turn, simplifies a number of proofs. In fact, the rules are practically syntax-directed: the only exception is when both types are quantifiers, and either $\forall VL$ or $\forall R$ could be tried. Since $\forall R$ is invertible, however, in practice one can apply it eagerly.

Let-generalization. In many accounts of type inference, letbindings are treated specially. For example, traditional Damas-Milner type inference only does polymorphic generalization at letbindings. Instead, we have sought to avoid a special treatment of let-bindings. In logical terms, let-bindings internalize the cut rule, and so special treatment puts the cut-elimination property of the calculus at risk—that is, typability may not be preserved when a let-binding is substituted away. To make let-generalization safe, additional properties like the principal types property are needed, a property endangered by rich type system features like higher-rank polymorphism, refinement types (Dunfield 2007) and GADTs (Vytiniotis et al. 2010).

To emphasize this point, we have omitted let-binding from our formal development. But since cut is admissible—i.e., the substitution theorem holds—restoring let-bindings is easy, as long as they get no special treatment incompatible with substitution. For example, the standard bidirectional rule for let-bindings is suitable:

$$\frac{\Psi \vdash e \Rightarrow A \qquad \Psi, x : A \vdash e' \Leftarrow C}{\Psi \vdash \text{let } x = e \text{ in } e' \Leftarrow C}$$

Note the absence of generalization in this rule.

2.2 Bidirectional Typing and Type Assignment System F

Since our declarative specification is (quite consciously) not the usual type-assignment presentation of System F, a natural question is to ask what the relationship is. Luckily, the two systems are quite closely related: we can show that if a term is well-typed in our type assignment system, it is always possible to add type annotations to make the term well-typed in the bidirectional system; conversely, if the bidirectional system types a term, then some βη-equal term is well-typed under the type assignment system.

We formalize these properties with the following theorems, taking |e| to be the erasure of all type annotations from a term. We give the rules for our type assignment System F in Figure 5.

$$\begin{array}{c} \boxed{\Psi \vdash e : A} \quad \text{Under context } \Psi, \, e \text{ has type } A \\ \\ \frac{(x : A) \in \Psi}{\Psi \vdash x : A} \quad \text{AVar} \qquad \qquad \overline{\Psi \vdash () : 1} \quad \text{AUnit} \\ \\ \frac{\Psi, x : A \vdash e : B}{\Psi \vdash \lambda x. \, e : A \to B} \quad \text{A} \to \text{I} \quad \frac{\Psi \vdash e_1 : A \to B}{\Psi \vdash e_1 \: e_2 : B} \quad \text{A} \to \text{E} \\ \\ \frac{\Psi, \alpha \vdash e : A}{\Psi \vdash e : \forall \alpha. \, A} \quad \text{A} \to \text{I} \quad \frac{\Psi \vdash e : \forall \alpha. \, A}{\Psi \vdash e : [\tau/\alpha] A} \quad \text{A} \to \text{E} \end{array}$$

Figure 5. Type assignment rules for predicative System F

Theorem 1 (Completeness of Bidirectional Typing). If $\Psi \vdash e : A$ then there exists e' such that $\Psi \vdash e' \Rightarrow A$ and |e'| = e.

Theorem 2 (Soundness of Bidirectional Typing). If $\Psi \vdash e \Leftarrow A$ then there exists e' such that $\Psi \vdash e'$: A and $e' =_{\beta \eta} |e|$.

Note that in the soundness theorem, the equality is up to β and η . We may need to η -expand bidirectionally-typed terms to make them typecheck under the type assignment system, and within the proof of soundness, we β -reduce identity coercions.

2.3 Robustness of Typing

Type annotations are an essential part of the bidirectional approach: they mediate between type checking and type synthesis. However, we want to relieve programmers from having to write redundant type annotations, and even more importantly, enable programmers to easily predict where type annotations are needed.

Since our declarative system is bidirectional, the basic property is that type annotations are required only at redexes. Additionally, these typing rules can infer (actually, guess) all monomorphic types, so the answer to the question of where annotations are needed is: only on bindings of polymorphic type.² Where bidirectional typing really stands out is in its robustness under substitution. We can freely substitute and "unsubstitute" terms:

Theorem 3 (Substitution). Assume $\Psi \vdash e \Rightarrow A$.

- If $\Psi, x : A \vdash e' \Leftarrow C$ then $\Psi \vdash [e/x]e' \Leftarrow C$.
- If $\Psi, \chi : A \vdash e' \Rightarrow C$ then $\Psi \vdash [e/\chi]e' \Rightarrow C$.

Theorem 4 (Inverse Substitution). Assume $\Psi \vdash e \Leftarrow A$.

- If $\Psi \vdash [(e:A)/x]e' \Leftarrow C$ then $\Psi, x:A \vdash e' \Leftarrow C$. If $\Psi \vdash [(e:A)/x]e' \Rightarrow C$ then $\Psi, x:A \vdash e' \Rightarrow C$.

Substitution is stated in terms of synthesizing expressions, since any checking term can be turned into a synthesizing term by adding an annotation. Dually, inverse substitution allows extracting any checking term into a let-binding with a type annotation.³ However, doing so indiscriminately can lead to a term with many redundant annotations, and so we also characterize when annotations can safely be removed:

Theorem 5 (Annotation Removal). We have that:

- If $\Psi \vdash ((\lambda x. e) : A) \Leftarrow C$ then $\Psi \vdash \lambda x. e \Leftarrow C$.
- If $\Psi \vdash (() : A) \Leftarrow C$ then $\Psi \vdash () \Leftarrow C$.
- If $\Psi \vdash e_1 (e_2 : A) \Rightarrow C$ then $\Psi \vdash e_1 e_2 \Rightarrow C$.
- If $\Psi \vdash (x : A) \Rightarrow A$ then $\Psi \vdash x \Rightarrow B$ and $\Psi \vdash B < A$.

²The number of annotations can be reduced still further; see Section 8 for how to infer the types of all terms typable under Damas-Milner.

³ The generalization of Theorem 4 to any synthesizing term—not just (e: A)—does not hold. For example, given $e=\lambda y.y$ and e'=x and $\Psi\vdash\lambda y.y\Rightarrow 1\to 1$ and $\Psi\vdash\lambda y.y\Leftarrow C_1\to C_2$, we cannot derive $\Psi, x: 1 \to 1 \vdash x \Leftarrow C_1 \to C_2$ unless C_1 and C_2 happen to be 1.

Types
$$A, B, C ::= 1 \mid \alpha \mid \hat{\alpha} \mid \forall \alpha. A \mid A \rightarrow B$$

Monotypes $\tau, \sigma ::= 1 \mid \alpha \mid \hat{\alpha} \mid \tau \rightarrow \sigma$
Contexts $\Gamma, \Delta, \Theta ::= \cdot \mid \Gamma, \alpha \mid \Gamma, x : A$
 $\mid \Gamma, \hat{\alpha} \mid \Gamma, \hat{\alpha} = \tau \mid \Gamma, \blacktriangleright_{\hat{\alpha}}$
Complete Contexts $\Omega ::= \cdot \mid \Omega, \alpha \mid \Omega, x : A$
 $\mid \Omega, \hat{\alpha} = \tau \mid \Omega, \blacktriangleright_{\hat{\alpha}}$

Figure 6. Syntax of types, monotypes, and contexts in the algorithmic system

```
    If Ψ ⊢ ((e₁ e₂): A) ⇒ A
        then Ψ ⊢ e₁ e₂ ⇒ B and Ψ ⊢ B ≤ A.
    If Ψ ⊢ ((e: B): A) ⇒ A
        then Ψ ⊢ (e: B) ⇒ B and Ψ ⊢ B ≤ A.
    If Ψ ⊢ ((λx. e): σ → τ) ⇒ σ → τ then Ψ ⊢ λx. e ⇒ σ → τ.
```

We can also show that the expected η -laws hold:

Theorem 6 (Soundness of Eta). If $\Psi \vdash \lambda x$, $e \ x \Leftarrow A$ and $x \notin FV(e)$, then $\Psi \vdash e \Leftarrow A$.

3. Algorithmic Type System

Our declarative bidirectional system is a fine specification of how typing should behave, but it enjoys guessing entirely too much: the typing rules $Decl \forall App$ and $Decl \rightarrow I \Rightarrow$ could only be implemented with the help of an oracle. The declarative subtyping rule $\leq \forall L$ has the same problem.

The first step in building our *algorithmic* bidirectional system will be to modify the three oracular rules so that, instead of guessing a type, they defer the choice by creating an existential type variable, to be solved later. However, our existential variables are not exactly unification variables; they are organized into *ordered algorithmic contexts* (Section 3.1), which define the variables' scope and controls the free variables of their solutions.

The algorithmic type system consists of subtyping rules (Figure 9, discussed in Section 3.2), instantiation rules (Figure 10, discussed in Section 3.3), and typing rules (Figure 11, discussed in Section 3.4). All of the rules manipulate the contexts in a way consistent with *context extension*, a metatheoretic notion described in Section 4; context extension is key in stating and proving decidability, soundness and completeness.

3.1 Algorithmic Contexts

A notion of (ordered) algorithmic context is central to our approach. Like declarative contexts Ψ , algorithmic contexts Γ (see Figure 6; we also use the letters Δ and Θ) contain declarations of universal type variables α and term variable typings x: A. Unlike declarative contexts, algorithmic contexts also contain declarations of existential type variables $\hat{\alpha}$, which are either unsolved (and we simply write $\hat{\alpha}$) or solved to some monotype ($\hat{\alpha} = \tau$). Finally, for scoping reasons that will become clear when we examine the rules, algorithmic contexts also include a $marker \blacktriangleright_{\hat{\alpha}}$.

Complete contexts Ω are the same as contexts, except that they cannot have unsolved variables.

The well-formedness rules for contexts (Figure 7, bottom) do not only prohibit duplicate declarations, but also enforce order: if $\Gamma = (\Gamma_L, x: A, \Gamma_R)$, the type A must be well-formed under Γ_L ; it cannot refer to variables α or $\hat{\alpha}$ in Γ_R . Similarly, if $\Gamma = (\Gamma_L, \hat{\alpha} = \tau, \Gamma_R)$, the solution type τ must be well-formed under Γ_L . Consequently, circularity is ruled out: $(\hat{\alpha} = \hat{\beta}, \hat{\beta} = \hat{\alpha})$ is not well-formed.

Figure 7. Well-formedness of types and contexts in the algorithmic system

$$\begin{array}{lll} [\Gamma]\alpha & = & \alpha \\ [\Gamma]1 & = & 1 \\ \left[\Gamma[\hat{\alpha} = \tau]\right]\hat{\alpha} & = & \left[\Gamma[\hat{\alpha} = \tau]\right]\tau \\ \left[\Gamma[\hat{\alpha}]\right]\hat{\alpha} & = & \hat{\alpha} \\ [\Gamma](A \to B) & = & ([\Gamma]A) \to ([\Gamma]B) \\ [\Gamma](\forall \alpha, A) & = & \forall \alpha, [\Gamma]A \end{array}$$

Figure 8. Applying a context, as a substitution, to a type

Contexts as substitutions on types. An algorithmic context can be viewed as a substitution for its solved existential variables. For example, $\hat{\alpha}=1, \hat{\beta}=\hat{\alpha}\to 1$ can be applied as if it were the substitution $1/\hat{\alpha}, (\hat{\alpha}\to 1)/\hat{\beta}$ (applied right to left), or the simultaneous substitution $1/\hat{\alpha}, (1\to 1)/\hat{\beta}$. We write $[\Gamma]A$ for Γ applied as a substitution to type A; this operation is defined in Figure 8.

Complete contexts. Complete contexts Ω (Figure 6) have no unsolved variables. Therefore, applying such a context to a type A (provided it is well-formed: $\Omega \vdash A$) yields a type $[\Omega]A$ with no existentials. Complete contexts are essential for stating and proving soundness and completeness, but are not explicitly distinguished in any of our rules.

Hole notation. Since we will manipulate contexts not only by appending declarations (as in the declarative system) but by inserting and replacing declarations in the middle, a notation for contexts with a hole is useful:

$$\Gamma = \Gamma_0[\Theta]$$
 means Γ has the form $(\Gamma_L, \Theta, \Gamma_R)$

For example, if $\Gamma = \Gamma_0[\hat{\beta}] = (\hat{\alpha}, \hat{\beta}, x : \hat{\beta})$, then $\Gamma_0[\hat{\beta} = \hat{\alpha}] = (\hat{\alpha}, \hat{\beta} = \hat{\alpha}, x : \hat{\beta})$. Since this notation is concise, we use it even

in rules that do not replace declarations, such as the rules for type well-formedness in Figure 7.

Occasionally, we also need contexts with two ordered holes:

$$\Gamma = \Gamma_0[\Theta_1][\Theta_2]$$
 means Γ has the form $(\Gamma_L, \Theta_1, \Gamma_M, \Theta_2, \Gamma_R)$

Input and output contexts. Our declarative system used a subtyping judgment and three typing judgments: checking, synthesis, and application. Our algorithmic system includes similar judgment forms, except that we replace the declarative context Ψ with an algorithmic context Γ (the *input context*), and add an *output* context Δ written after a backwards turnstile: $\Gamma \vdash A <: B \dashv \Delta$ for subtyping, $\Gamma \vdash e \Leftarrow A \dashv \Delta$ for checking, and so on. Unsolved existential variables get solved when they are compared against a type. For example, $\hat{\alpha} <: \beta$ would lead to replacing the unsolved declaration $\hat{\alpha}$ with $\hat{\alpha} = \beta$ in the context (provided β is declared to the left of $\hat{\alpha}$). Input contexts thus evolve into output contexts that are "more solved".

The differences between the declarative and algorithmic systems, particularly manipulations of existential variables, are most prominent in the subtyping rules, so we discuss those first.

3.2 Algorithmic Subtyping

The first four subtyping rules in Figure 9 do not directly manipulate the context, but do illustrate how contexts are propagated.

Rules <: Var and <: Unit are reflexive rules; neither involves existential variables, so the output context in the conclusion is the same as the input context Γ . Rule <: Exvar concludes that any unsolved existential variable is a subtype of itself, but this gives no clue as to how to solve that existential, so the output context is similarly unchanged.

Rule $\langle : \rightarrow$ is a bit more interesting: it has two premises, where the first premise has an output context Θ , which is used as the input context to the second (subtyping) premise; the second premise has output context Δ , which is the output of the conclusion. Note that in $\langle : \rightarrow \rangle$'s second premise, we do not simply check that $A_2 \leq B_2$, but apply the first premise's output Θ to those types:

$$\Theta \vdash [\Theta]A_2 <: [\Theta]B_2 \dashv \Delta$$

This maintains a general invariant: whenever we try to derive $\Gamma \vdash A \mathrel{<:} B \dashv \Delta$, the types A and B are already fully applied under Γ . That is, they contain no existential variables already solved in Γ . On balance, this invariant simplifies the system: the extra applications of Θ in $\mathrel{<:}\to$ avoid the need for extra rules for replacing solved variables with their solutions.

All the rules discussed so far have been natural extensions of the declarative rules, with <: Exvar being a logical way to extend reflexivity to types containing existentials. Rule <: ∀L diverges significantly from the corresponding declarative rule $\leq \forall L$. Instead of replacing the type variable α with a guessed τ , rule <: \forall L replaces α with a new existential variable $\hat{\alpha}$, which it adds to the premise's input context: $\Gamma, \triangleright_{\hat{\alpha}}, \hat{\alpha} \vdash [\hat{\alpha}/\alpha]A <: B \dashv \Delta, \triangleright_{\hat{\alpha}}, \Theta$. The peculiarlooking $\triangleright_{\hat{\alpha}}$ is a *scope marker*, pronounced "marker $\hat{\alpha}$ ", which will delineate existentials created by *articulation* (the step of solving $\hat{\alpha}$ to $\hat{\alpha}_1 \rightarrow \hat{\alpha}_2$, discussed in the next subsection). The output context $(\Delta, \blacktriangleright_{\hat{\alpha}}, \Theta)$ allows for some additional (existential) variables to appear after $\triangleright_{\hat{\alpha}}$, in a trailing context Θ . These existential variables could mention $\hat{\alpha}$, or (if they appear between $\triangleright_{\hat{\alpha}}$ and $\hat{\alpha}$) could be mentioned by $\hat{\alpha}$; since $\hat{\alpha}$ goes out of scope in the conclusion, we drop such "trailing existentials" from the concluding output context, which is simply Δ .⁵

Rule $<: \forall R$ is fairly close to the declarative version, but for scoping reasons similar to $<: \forall L$, it also drops Θ , the part of the context to the right of the universal type variable α . (Articulation makes no sense for universal variables, so α can act as its own marker.)

The last two rules are essential: they derive subtypings with an unsolved existential on one side, and an arbitrary type on the other. Rule <:InstantiateL derives $\hat{\alpha} <: A$, and <:InstantiateR derives $A <: \hat{\alpha}$. These rules do not directly change the output context; they just do an "occurs check" $\hat{\alpha} \notin FV(A)$ to avoid circularity, and leave all the real work to the instantiation judgment.

3.3 Instantiation

Two almost-symmetric judgments instantiate unsolved existential variables: $\Gamma \vdash \hat{\alpha} : \leq A \dashv \Delta$ and $\Gamma \vdash A \leq : \hat{\alpha} \dashv \Delta$. The symbol $:\leq$ suggests assignment of the variable to its left, but also subtyping: the subtyping rule <: InstantiateL moves from instantiation $\hat{\alpha} : \leq A$, read "instantiate $\hat{\alpha}$ to a subtype of A", to subtyping $\hat{\alpha} <: A$. The symmetric judgment $A \leq : \hat{\alpha}$ can be read "instantiate $\hat{\alpha}$ to a supertype of A".

The first instantiation rule in Figure 10, InstLSolve, sets $\hat{\alpha}$ to τ in the output context: its conclusion is $\Gamma,\hat{\alpha},\Gamma'\vdash\hat{\alpha}:\stackrel{<}{=}\tau\dashv\Gamma,\hat{\alpha}=\tau,\Gamma'.$ The premise $\Gamma\vdash\tau$ checks that the monotype τ is well-formed under the prefix context $\Gamma.$ To check the soundness of this rule, we can take the conclusion $\hat{\alpha}:\stackrel{<}{=}\tau,$ substitute our new solution for $\hat{\alpha},$ and check that the resulting subtyping makes sense. Since $[\Gamma,\hat{\alpha}=\tau,\Gamma']\hat{\alpha}=\tau,$ we ask whether $\tau<:\tau$ makes sense, and of course it does through reflexivity.

Rule InstLArr can be applied when the type A in $\hat{\alpha} \stackrel{\leq}{:} A$ has the form $A_1 \to A_2$. It follows that $\hat{\alpha}$'s solution must have the form $\cdots \to \cdots$, so we "articulate" $\hat{\alpha}$, giving it the solution $\hat{\alpha}_1 \to \hat{\alpha}_2$ where the $\hat{\alpha}_k$ are fresh existentials. We insert their declarations just before $\hat{\alpha}$ —they must be to the left of $\hat{\alpha}$ so they can be mentioned in its solution, but they must be close enough to $\hat{\alpha}$ that they appear to the *right* of the marker $\blacktriangleright_{\hat{\alpha}}$ introduced by $<: \forall L$. Note that the first premise $A_1 \stackrel{\leq}{=} \hat{\alpha}_1$ switches to the other instantiation judgment. Also, the second premise $\Theta \vdash \hat{\alpha}_2 \stackrel{\leq}{=} [\Theta]A_2 \dashv \Delta$ applies Θ to A_2 , to apply any solutions found in the first premise.

The other rules are somewhat subtle. Rule InstLReach derives

$$\Gamma[\hat{\alpha}][\hat{\beta}] \vdash \hat{\alpha} := \hat{\beta} \dashv \Gamma[\hat{\alpha}][\hat{\beta} = \hat{\alpha}]$$

where, as explained in Section 3.1, $\Gamma[\widehat{\alpha}][\widehat{\beta}]$ denotes a context where some unsolved existential variable $\widehat{\alpha}$ is declared to the left of $\widehat{\beta}$. In this situation, we cannot use InstLSolve to set $\widehat{\alpha}$ to $\widehat{\beta}$ because $\widehat{\beta}$ is not well-formed under the part of the context to the left of $\widehat{\alpha}$. Instead, we set $\widehat{\beta}$ to $\widehat{\alpha}$.

Rule InstLAllR is the instantiation version of $\langle : \forall R$. Since our polymorphism is predicative, we can't assign $\forall \beta$. B to $\hat{\alpha}$, but we can decompose the quantifier in the same way that subtyping does.

The rules for the second judgment $A \leq : \hat{\alpha}$ are similar: InstRSolve, InstRReach and InstRArr are direct analogues of the first three $\hat{\alpha} : \leq A$ rules, and InstRAIL is the instantiation version of $<: \forall L$.

Example. The interplay between instantiation and quantifiers is delicate. For example, consider the problem of instantiating $\hat{\beta}$ to a supertype of $\forall \alpha$. α . In this case, the type $\forall \alpha$. α is so polymorphic that it places no constraints at all on $\hat{\beta}$. Therefore, it seems we are at risk of being forced to make a necessarily incomplete choice—but the instantiation judgment's ability to "change its mind" about

 $^{^4}$ Rule <: → enforces that the function domains B_1 , A_1 are compared first: Θ is an input to the second premise. But this is an arbitrary choice; the system would behave the same if we chose to check the codomains first.

⁵ In our setting, it is safe to drop trailing existentials that are unsolved: such variables are unconstrained, and we can treat them as having been

instantiated to any well-formed type, such as 1. In a dependently typed setting, we would need to check that at least one solution exists.

 $\Gamma \vdash A <: B \dashv \Delta$ Under input context Γ , type A is a subtype of B, with output context Δ

$$\frac{\Gamma[\alpha] \vdash \alpha \mathrel{<:} \alpha \dashv \Gamma[\alpha]}{\Gamma[\alpha] \vdash \alpha \mathrel{<:} \alpha \dashv \Gamma[\alpha]} \mathrel{<:} \mathsf{Var} \qquad \frac{\Gamma \vdash 1 \mathrel{<:} 1 \dashv \Gamma}{\Gamma \vdash 1 \mathrel{<:} 1 \dashv \Gamma} \mathrel{<:} \mathsf{Unit} \qquad \frac{\Gamma[\widehat{\alpha}] \vdash \widehat{\alpha} \mathrel{<:} \widehat{\alpha} \dashv \Gamma[\widehat{\alpha}]}{\Gamma[\widehat{\alpha}] \vdash \widehat{\alpha} \mathrel{<:} \widehat{\alpha} \dashv \Gamma[\widehat{\alpha}]} \mathrel{<:} \mathsf{Exvar} \qquad \frac{\Gamma \vdash B_1 \mathrel{<:} A_1 \dashv \Theta \qquad \Theta \vdash [\Theta] A_2 \mathrel{<:} [\Theta] B_2 \dashv \Delta}{\Gamma \vdash A_1 \to A_2 \mathrel{<:} B_1 \to B_2 \dashv \Delta} \mathrel{<:} \to \qquad \frac{\Gamma, \alpha \vdash A \mathrel{<:} B \dashv \Delta, \alpha, \Theta}{\Gamma \vdash A \mathrel{<:} \Theta \land A \mathrel{<:} B \dashv \Delta} \mathrel{<:} \forall \mathsf{R} \qquad \frac{\Gamma, \alpha \vdash A \mathrel{<:} B \dashv \Delta, \alpha, \Theta}{\Gamma \vdash A \mathrel{<:} \Theta \land A \mathrel{<:} B \dashv \Delta} \mathrel{<:} \forall \mathsf{R} \qquad \frac{\widehat{\alpha} \not\in \mathsf{FV}(A) \qquad \Gamma[\widehat{\alpha}] \vdash A \mathrel{\leq:} \widehat{\alpha} \dashv \Delta}{\Gamma[\widehat{\alpha}] \vdash A \mathrel{<:} \widehat{\alpha} \dashv \Delta} \mathrel{<:} \mathsf{InstantiateR} \qquad \frac{\widehat{\alpha} \not\in \mathsf{FV}(A) \qquad \Gamma[\widehat{\alpha}] \vdash A \mathrel{\leq:} \widehat{\alpha} \dashv \Delta}{\Gamma[\widehat{\alpha}] \vdash A \mathrel{<:} \widehat{\alpha} \dashv \Delta} \mathrel{<:} \mathsf{InstantiateR} \qquad \frac{\widehat{\alpha} \not\in \mathsf{FV}(A) \qquad \Gamma[\widehat{\alpha}] \vdash A \mathrel{\leq:} \widehat{\alpha} \dashv \Delta}{\Gamma[\widehat{\alpha}] \vdash A \mathrel{<:} \widehat{\alpha} \dashv \Delta} \mathrel{<:} \mathsf{InstantiateR} \qquad \frac{\widehat{\alpha} \not\in \mathsf{FV}(A) \qquad \Gamma[\widehat{\alpha}] \vdash A \mathrel{\leq:} \widehat{\alpha} \dashv \Delta}{\Gamma[\widehat{\alpha}] \vdash A \mathrel{<:} \widehat{\alpha} \dashv \Delta} \mathrel{<:} \mathsf{InstantiateR} \qquad \frac{\widehat{\alpha} \not\in \mathsf{FV}(A) \qquad \Gamma[\widehat{\alpha}] \vdash A \mathrel{\leq:} \widehat{\alpha} \dashv \Delta}{\Gamma[\widehat{\alpha}] \vdash A \mathrel{<:} \widehat{\alpha} \dashv \Delta} \mathrel{<:} \mathsf{InstantiateR} \qquad \frac{\widehat{\alpha} \not\in \mathsf{FV}(A) \qquad \Gamma[\widehat{\alpha}] \vdash A \mathrel{\leq:} \widehat{\alpha} \dashv \Delta}{\Gamma[\widehat{\alpha}] \vdash A \mathrel{<:} \widehat{\alpha} \dashv \Delta} \mathrel{<:} \mathsf{InstantiateR} \qquad \frac{\widehat{\alpha} \not\in \mathsf{FV}(A) \qquad \Gamma[\widehat{\alpha}] \vdash A \mathrel{<:} \widehat{\alpha} \dashv \Delta}{\Gamma[\widehat{\alpha}] \vdash A \mathrel{<:} \widehat{\alpha} \dashv \Delta} \mathrel{<:} \mathsf{InstantiateR} \qquad \frac{\widehat{\alpha} \not\in \mathsf{FV}(A) \qquad \Gamma[\widehat{\alpha}] \vdash A \mathrel{<:} \widehat{\alpha} \dashv \Delta}{\Gamma[\widehat{\alpha}] \vdash A \mathrel{<:} \widehat{\alpha} \dashv \Delta} \mathrel{<:} \mathsf{InstantiateR} \qquad \frac{\widehat{\alpha} \not\in \mathsf{FV}(A) \qquad \Gamma[\widehat{\alpha}] \vdash A \mathrel{<:} \widehat{\alpha} \dashv \Delta}{\Gamma[\widehat{\alpha}] \vdash A \mathrel{<:} \widehat{\alpha} \dashv \Delta} \mathrel{<:} \mathsf{InstantiateR} \qquad \frac{\widehat{\alpha} \not\in \mathsf{FV}(A) \qquad \Gamma[\widehat{\alpha}] \vdash A \mathrel{<:} \widehat{\alpha} \dashv \Delta}{\Gamma[\widehat{\alpha}] \vdash A \mathrel{<:} \widehat{\alpha} \dashv \Delta} \mathrel{<:} \mathsf{InstantiateR} \qquad \frac{\widehat{\alpha} \not\in \mathsf{FV}(A) \qquad \Gamma[\widehat{\alpha}] \vdash A \mathrel{<:} \widehat{\alpha} \dashv \Delta}{\Gamma[\widehat{\alpha}] \vdash A \mathrel{<:} \widehat{\alpha} \dashv \Delta} \mathrel{<:} \mathsf{InstantiateR} \qquad \frac{\widehat{\alpha} \not\in \mathsf{FV}(A) \qquad \Gamma[\widehat{\alpha}] \vdash A \mathrel{<:} \widehat{\alpha} \dashv \Delta}{\Gamma[\widehat{\alpha}] \vdash A \mathrel{<:} \widehat{\alpha} \dashv \Delta} \mathrel{<:} \mathsf{FV}(A) \qquad \Gamma[\widehat{\alpha}] \vdash A \mathrel{<:} \widehat{\alpha} \dashv \Delta} \mathrel{<:} \mathsf{A} \mathrel{<:} \mathsf{A}$$

Figure 9. Algorithmic subtyping

 $\Gamma \vdash \widehat{\alpha} : \stackrel{\leq}{=} A \dashv \Delta$ Under input context Γ , instantiate $\widehat{\alpha}$ such that $\widehat{\alpha} <: A$, with output context Δ

$$\frac{\Gamma \vdash \tau}{\Gamma, \hat{\alpha}, \Gamma' \vdash \hat{\alpha} : \stackrel{\leq}{=} \tau \dashv \Gamma, \hat{\alpha} = \tau, \Gamma'} \text{ InstLSolve } \frac{\Gamma[\hat{\alpha}][\hat{\beta}] \vdash \hat{\alpha} : \stackrel{\leq}{=} \hat{\beta} \dashv \Gamma[\hat{\alpha}][\hat{\beta} = \hat{\alpha}]}{\Gamma[\hat{\alpha}], \hat{\beta} \vdash \hat{\alpha} : \stackrel{\leq}{=} \hat{\beta} \dashv \Gamma[\hat{\alpha}][\hat{\beta} = \hat{\alpha}]} \text{ InstLReach}} \frac{\Gamma[\hat{\alpha}_{2}, \hat{\alpha}_{1}, \hat{\alpha} = \hat{\alpha}_{1} \to \hat{\alpha}_{2}] \vdash A_{1} \stackrel{\leq}{=} : \hat{\alpha}_{1} \dashv \Theta}{\Gamma[\hat{\alpha}] \vdash \hat{\alpha} : \stackrel{\leq}{=} A_{1} \to A_{2} \dashv \Delta}}{\Gamma[\hat{\alpha}] \vdash \hat{\alpha} : \stackrel{\leq}{=} A_{1} \to A_{2} \dashv \Delta} \text{ InstLAIR}} \frac{\Gamma[\hat{\alpha}], \beta \vdash \hat{\alpha} : \stackrel{\leq}{=} B \dashv \Delta, \beta, \Delta'}{\Gamma[\hat{\alpha}] \vdash \hat{\alpha} : \stackrel{\leq}{=} \forall \beta, B \dashv \Delta}} \text{ InstLAIR}}$$

 $\Gamma \vdash A \stackrel{\leq}{=} : \hat{\alpha} \dashv \Delta$ Under input context Γ , instantiate $\hat{\alpha}$ such that $A <: \hat{\alpha}$, with output context Δ

$$\frac{\Gamma \vdash \tau}{\Gamma, \hat{\alpha}, \Gamma' \vdash \tau \stackrel{\leq}{=} : \hat{\alpha} \dashv \Gamma, \hat{\alpha} = \tau, \Gamma'} \text{ InstRSolve } \frac{\Gamma[\hat{\alpha}][\hat{\beta}] \vdash \hat{\beta} \stackrel{\leq}{=} : \hat{\alpha} \dashv \Gamma[\hat{\alpha}][\hat{\beta} = \hat{\alpha}]}{\Gamma[\hat{\alpha}] \vdash \Lambda_1 \rightarrow \Lambda_2 \stackrel{\leq}{=} : \hat{\alpha} \dashv \Delta} \text{ InstRArr } \frac{\Gamma[\hat{\alpha}], \blacktriangleright_{\hat{\beta}}, \hat{\beta} \vdash [\hat{\beta}/\beta]B \stackrel{\leq}{=} : \hat{\alpha} \dashv \Delta, \blacktriangleright_{\hat{\beta}}, \Delta'}{\Gamma[\hat{\alpha}] \vdash \Lambda_1 \rightarrow \Lambda_2 \stackrel{\leq}{=} : \hat{\alpha} \dashv \Delta} \text{ InstRAIL}$$

Figure 10. Instantiation

 $\Gamma \vdash e \Leftarrow A \dashv \Delta$ Under input context Γ , e checks against input type A, with output context Δ

 $\Gamma \vdash e \Rightarrow A \dashv \Delta$ Under input context Γ , e synthesizes output type A, with output context Δ

 $\Gamma \vdash A \bullet e \Rightarrow C \dashv \Delta$ Under input context Γ , applying a function of type A to e synthesizes type C, with output context Δ

$$\frac{(x:A) \in \Gamma}{\Gamma \vdash x \Rightarrow A \dashv \Gamma} \text{ Var } \qquad \frac{\Gamma \vdash e \Rightarrow A \dashv \Theta}{\Gamma \vdash e \Leftarrow B \dashv \Delta} \xrightarrow{\Theta \vdash [\Theta]A <: [\Theta]B \dashv \Delta} \text{ Sub } \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash e \Leftarrow A \dashv \Delta}{\Gamma \vdash (e:A) \Rightarrow A \dashv \Delta} \text{ Anno}$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash e \Leftarrow A \dashv \Delta}{\Gamma \vdash (e:A) \Rightarrow A \dashv \Delta} \xrightarrow{\Gamma \vdash (e:A) \Rightarrow A \dashv \Delta} \text{ Anno}$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash e \Leftarrow A \dashv \Delta}{\Gamma \vdash (e:A) \Rightarrow A \dashv \Delta} \xrightarrow{\Gamma \vdash (e:A) \Rightarrow A \dashv \Delta} \text{ Anno}$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash e \Leftarrow A \dashv \Delta}{\Gamma \vdash (e:A) \Rightarrow A \dashv \Delta} \xrightarrow{\Gamma \vdash (e:A) \Rightarrow A \dashv \Delta} \text{ Anno}$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash e \Leftarrow A \dashv \Delta}{\Gamma \vdash (e:A) \Rightarrow A \dashv \Delta} \xrightarrow{\Gamma \vdash (e:A) \Rightarrow A \dashv \Delta} \xrightarrow{\Gamma \vdash (e:A) \Rightarrow A \dashv \Delta} \text{ Anno}$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash e \Leftarrow A \dashv \Delta}{\Gamma \vdash A \Rightarrow A \Rightarrow B \dashv \Delta} \xrightarrow{\Gamma \vdash (e:A) \Rightarrow A \vdash \Delta} \xrightarrow{$$

Figure 11. Algorithmic typing

which variable to instantiate saves the day:

$$\frac{\overline{\Gamma[\hat{\beta}], \blacktriangleright_{\hat{\alpha}}, \hat{\alpha} \vdash \hat{\alpha} \stackrel{\leq}{=} : \hat{\beta} \dashv \Gamma[\hat{\beta}], \blacktriangleright_{\hat{\alpha}}, \hat{\alpha} = \hat{\beta}}}{\Gamma[\hat{\beta}] \vdash \forall \alpha. \; \alpha \stackrel{\leq}{=} : \hat{\beta} \dashv \Gamma[\hat{\beta}]} \xrightarrow{\mathsf{InstRReach}} \mathsf{InstRAIIL}$$

Here, we introduce a new variable $\hat{\alpha}$ to go under the universal quantifier; then, instantiation applies InstRReach to set $\hat{\alpha}$, not $\hat{\beta}$. Hence, $\hat{\beta}$ is, correctly, *not constrained* by this subtyping problem.

Thus, instantiation does not necessarily solve *any* existential variable. However, instantiation to any monotype τ will solve an existential variable—that is, for input context Γ and output Δ , we have $\operatorname{unsolved}(\Delta) < \operatorname{unsolved}(\Gamma)$. This will be critical for decidability of subtyping (Section 5.2).

Another example. In Figure 12 we show a derivation that uses quantifier instantiation (InstRAIIL), articulation (InstRArr) and "reaching" (InstLReach), as well as InstRSolve. In the output context $\Delta = \Gamma[\hat{\beta}_2, \hat{\beta}_1 = \hat{\beta}_2, \hat{\alpha} = \hat{\beta}_1 \rightarrow \hat{\beta}_2]$ note that $\hat{\alpha}$ is solved to $\hat{\beta}_1 \rightarrow \hat{\beta}_2$, and $\hat{\beta}_2$ is solved to $\hat{\beta}_1$. Thus, $[\Delta]\hat{\alpha} = \hat{\beta}_1 \rightarrow \hat{\beta}_1$, which is a monomorphic approximation of $\forall \beta.\beta \rightarrow \beta$.

3.4 Algorithmic Typing

We now turn to the typing rules in Figure 11. Many of these rules follow the declarative rules, with extra context machinery. Rule Var uses an assumption x:A without generating any new information, so the output context in its conclusion $\Gamma \vdash x \Rightarrow A \dashv \Gamma$ is just the input context. Rule Sub's first premise has an output context Θ , used as the input context to the second (subtyping) premise, which has output context Δ , the output of the conclusion. Rule Anno does not directly change the context, but the derivation of its premise might include the use of some rule that does, so we propagate the premise's output context Δ to the conclusion.

Unit and \forall . In the second row of typing rules, 11 and $11 \Rightarrow$ generate no new information and simply propagate the input context.

 $\forall I$ is more interesting: Like the declarative rule $\mathsf{Decl} \forall I$, it adds a universal type variable α to the (input) context. The output context of the premise $\Gamma, \alpha \vdash e \Leftarrow A \dashv \Delta, \alpha, \Theta$ allows for some additional (existential) variables to appear after α , in a trailing context Θ . These existential variables could depend on α ; since α goes out of scope in the conclusion, we must drop them from the concluding output context, which is just Δ : the part of the premise's output context that cannot depend on α .

The application-judgment rule $\forall App$ serves a similar purpose to the subtyping rule $<: \forall L$, but does *not* place a marker before $\hat{\alpha}$: the variable $\hat{\alpha}$ may appear in the output type C, so $\hat{\alpha}$ must survive in the output context Δ .

Functions. In the third row of typing rules, rule \rightarrow I follows the same scheme: the declarations Θ following x:A are dropped in the conclusion's output context.

Rule \rightarrow I \Rightarrow corresponds to Decl \rightarrow I \Rightarrow , one of the guessing rules, so we create new existential variables $\hat{\alpha}$ (for the function domain) and $\hat{\beta}$ (for the codomain) and check the function body against $\hat{\beta}$. As in \forall App, we do not place a marker before $\hat{\alpha}$, because $\hat{\alpha}$ and $\hat{\beta}$ appear in the output type (λx . $e \Rightarrow \hat{\alpha} \rightarrow \hat{\beta}$).

Rule \rightarrow E is the expected analogue of Decl \rightarrow E; like other rules with two premises, it applies the intermediate context Θ .

On the last row of typing rules, $\hat{\alpha}$ App derives $\hat{\alpha} \cdot e \implies \hat{\alpha}_2$ where $\hat{\alpha}$ is unsolved in the input context. Here we have an application judgment, which is supposed to synthesize a type for an application e_1 e where e_1 has type $\hat{\alpha}$. We know that e_1 should have function type; similarly to InstLArr/InstRArr, we introduce $\hat{\alpha}_1$ and $\hat{\alpha}_2$ and add $\hat{\alpha} = \hat{\alpha}_1 \rightarrow \hat{\alpha}_2$ to the context. (Rule $\hat{\alpha}$ App is the only algorithmic typing rule that does not correspond to a declarative rule.)

Finally, rule $\rightarrow App$ is analogous to Decl $\rightarrow App$.

4. Context Extension

We motivated the algorithmic rules by saying that they evolved input contexts to output contexts that were "more solved". To state and prove the metatheoretic results of decidability, soundness and completeness (Sections 5–7), we introduce a context extension judgment $\Gamma \longrightarrow \Delta$. This judgment captures a notion of information increase from an input context Γ to an output context Δ , and relates algorithmic contexts Γ and Δ to completely solved extensions Ω , which correspond—via the context application described in Section 4.1—to declarative contexts Ψ .

The judgment $\Gamma \longrightarrow \Delta$ is read " Γ is extended by Δ " (or Δ extends Γ). Another reading is that Δ carries at least as much information as Γ . A third reading is that $\Gamma \longrightarrow \Delta$ means that Γ is *entailed by* Δ : all positive information derivable from Γ (say, that existential variable $\hat{\alpha}$ is in scope) can also be derived from Δ (which may have more information, say, that $\hat{\alpha}$ is equal to a particular type). This reading is realized by several key lemmas; for instance, extension preserves well-formedness: if $\Gamma \vdash A$ and $\Gamma \longrightarrow \Delta$, then $\Delta \vdash A$.

The rules deriving the context extension judgment (Figure 13) say that the empty context extends the empty context (\longrightarrow ID); a term variable typing x:A' extends x:A if applying the extending context Δ to A and A' yields the same type (\longrightarrow Var); universal type variables must match (\longrightarrow Uvar); scope markers must match (\longrightarrow Marker); and, existential variables may:

- appear unsolved in both contexts (—>Unsolved),
- appear solved in both contexts, if applying the extending context Δ makes the solutions τ and τ' equal (—>Solved),
- get solved by the extending context (—>Solve),
- be added by the extending context, either without a solution (→Add) or with a solution (→AddSolved);

Extension does *not* allow solutions to disappear: information must increase. It *does* allow solutions to change, but only if the change preserves or increases information. The extension

$$(\hat{\alpha}, \hat{\beta} = \hat{\alpha}) \longrightarrow (\hat{\alpha} = 1, \hat{\beta} = \hat{\alpha})$$

directly increases information about $\hat{\alpha}$, and indirectly increases information about $\hat{\beta}$. Perhaps more interestingly, the extension

$$\underbrace{(\hat{\alpha}=1,\hat{\beta}=\hat{\alpha})}_{\Delta} \longrightarrow \underbrace{(\hat{\alpha}=1,\hat{\beta}=1)}_{\Omega}$$

also holds: while the solution of $\hat{\beta}$ in Ω is different, in the sense that Ω contains $\hat{\beta}=1$ while Δ contains $\hat{\beta}=\hat{\alpha}$, applying Ω to the two solutions gives the same thing: applying Ω to Δ 's solution of $\hat{\beta}$ gives $[\Omega]\hat{\alpha}=[\Omega]1=1$, while applying Ω to Ω 's own solution for $\hat{\beta}$ also gives 1, because $[\Omega]1=1$.

Extension is quite rigid, however, in two senses. First, if a declaration appears in Γ , it appears in all extensions of Γ . Second, *extension preserves order*. For example, if $\hat{\beta}$ is declared after $\hat{\alpha}$ in Γ , then $\hat{\beta}$ will also be declared after $\hat{\alpha}$ in every extension of Γ . This holds for every variety of declaration. This rigidity aids in enforcing type variable scoping and dependencies, which are nontrivial in a setting with higher-rank polymorphism.

This combination of rigidity (in demanding that the order of declarations be preserved) with flexibility (in how existential type variable solutions are expressed) manages to satisfy scoping and dependency relations *and* give enough room to maneuver in the algorithm and metatheory.

4.1 Context Application

A complete context Ω (Figure 6) has no unsolved variables, so applying it to a (well-formed) type yields a type $[\Omega]A$ with no existen-

$$\begin{array}{c} \text{context to the left of } \hat{\beta} \\ \hline \Gamma', \blacktriangleright_{\hat{\beta}} \vdash \hat{\beta}_1 \\ \hline \Gamma', \blacktriangleright_{\hat{\beta}}, \hat{\beta} \vdash \hat{\beta}_2 : \leq \hat{\beta} \dashv \Gamma', \blacktriangleright_{\hat{\beta}}, \hat{\beta} = \hat{\beta}_1 \\ \hline \Gamma', \blacktriangleright_{\hat{\beta}}, \hat{\beta} \vdash \hat{\beta}_2 : \leq \hat{\beta} \dashv \Gamma', \blacktriangleright_{\hat{\beta}}, \hat{\beta} = \hat{\beta}_1 \\ \hline \Gamma', \blacktriangleright_{\hat{\beta}}, \hat{\beta} \vdash \hat{\beta}_2 : \leq \hat{\beta} \dashv \Gamma', \blacktriangleright_{\hat{\beta}}, \hat{\beta} = \hat{\beta}_1 \\ \hline \Gamma', \blacktriangleright_{\hat{\beta}}, \hat{\beta} = \hat{\beta}_1 \vdash \hat{\beta}_2 \leq : \hat{\beta}_1 \dashv \Delta, \blacktriangleright_{\hat{\beta}}, \hat{\beta} = \hat{\beta}_1 \\ \hline \Gamma[\hat{\alpha}], \blacktriangleright_{\hat{\beta}}, \hat{\beta} \vdash \hat{\beta} \rightarrow \hat{\beta} \leq : \hat{\alpha} \dashv \Delta, \blacktriangleright_{\hat{\beta}}, \hat{\beta} = \hat{\beta}_1 \\ \hline \Gamma[\hat{\alpha}] \vdash (\forall \beta, \beta \rightarrow \beta) \leq : \hat{\alpha} \dashv \Delta \\ \hline \Gamma[\hat{\alpha}] \vdash (\forall \beta, \beta \rightarrow \beta) \leq : \hat{\alpha} \dashv \Delta \\ \hline \end{array} \begin{array}{c} \text{InstRAIIL} \\ \hline \end{array}$$

Figure 12. Example of instantiation

Figure 13. Context extension

$$\begin{array}{lll} [\cdot] \cdot & = & \cdot \\ [\Omega, x : A](\Gamma, x : A_{\Gamma}) & = & [\Omega]\Gamma, \ x : [\Omega]A & \text{if } [\Omega]A = [\Omega]A_{\Gamma} \\ [\Omega, \alpha](\Gamma, \alpha) & = & [\Omega]\Gamma, \ \alpha \\ [\Omega, \hat{\alpha} = \tau](\Gamma, \hat{\alpha}) & = & [\Omega]\Gamma \\ [\Omega, \hat{\alpha} = \tau](\Gamma, \hat{\alpha} = \tau_{\Gamma}) & = & [\Omega]\Gamma & \text{if } [\Omega]\tau = [\Omega]\tau_{\Gamma} \\ [\Omega, \hat{\alpha} = \tau]\Gamma & = & [\Omega]\Gamma & \text{if } \hat{\alpha} \notin \mathsf{dom}(\Gamma) \\ [\Omega, \blacktriangleright_{\hat{\alpha}}](\Gamma, \blacktriangleright_{\hat{\alpha}}) & = & [\Omega]\Gamma \end{array}$$

Figure 14. Applying a complete context Ω to a context

tials. Such a type is well-formed under a *declarative* context—with just α and x:A declarations—obtained by dropping all the existential declarations and applying Ω to declarations x:A (to yield $x:[\Omega]A$). We can think of this context as the result of applying Ω to itself: $[\Omega]\Omega$.

More generally, we can apply Ω to any context Γ that it extends. This operation of context application $[\Omega]\Gamma$ is given in Figure 14. The application $[\Omega]\Gamma$ is defined if and only if $\Gamma \longrightarrow \Omega$, and applying Ω to any such Γ yields the same declarative context $[\Omega]\Omega$:

Lemma (Stability of Complete Contexts). If $\Gamma \longrightarrow \Omega$ then $[\Omega]\Gamma = [\Omega]\Omega$.

5. Decidability

Our algorithmic type system is decidable. Since the typing rules (Figure 11) depend on the subtyping rules (Figure 9), which in turn depend on the instantiation rules (Figure 10), showing that the typing judgments (checking, synthesis and application) are decidable requires that we show that the instantiation and subtyping judgments are decidable.

5.1 Decidability of Instantiation

As discussed in Section 3.3, deriving $\Gamma \vdash \hat{\alpha} :\stackrel{\leq}{=} A \dashv \Delta$ does not necessarily instantiate any existential variable (unless A is a monotype). However, the instantiation rules do preserve the size of (substituted) types:

Lemma (Instantiation Size Preservation). If
$$\Gamma = (\Gamma_0, \hat{\alpha}, \Gamma_1)$$
 and $\Gamma \vdash \hat{\alpha} : \stackrel{\checkmark}{=} A \dashv \Delta$ or $\Gamma \vdash A \stackrel{\checkmark}{=} : \hat{\alpha} \dashv \Delta$,

and $\Gamma \vdash B$ and $\hat{\alpha} \notin FV([\Gamma]B)$, then $|[\Gamma]B| = |[\Delta]B|$, where |C| is the plain size of C.

Using this lemma, we can show that the type A in the instantiation judgment always get smaller, even in rule InstLArr: the second premise applies the intermediate context Θ to A_2 , but the lemma tells us that this application cannot make A_2 larger, and A_2 is smaller than the conclusion's type $(A_1 \rightarrow A_2)$.

Now we can prove decidability of instantiation, assuming that $\hat{\alpha}$ is unsolved in the input context Γ , that A is well-formed under Γ , that A is fully applied ($[\Gamma]A = A$), and that $\hat{\alpha}$ does not occur in A. These conditions are guaranteed when instantiation is invoked, because the typing rule Sub applies the input substitution, and the subtyping rules apply the substitution where needed—in exactly one place: the second premise of $<:\to$. The proof is based on the (substituted) types in the premises being smaller than the (substituted) type in the conclusion.

Theorem 7 (Decidability of Instantiation).

If $\Gamma = \Gamma_0[\hat{\alpha}]$ and $\Gamma \vdash A$ such that $[\Gamma]A = A$ and $\hat{\alpha} \notin FV(A)$, then:

- (1) Either there exists Δ such that $\Gamma_0[\hat{\alpha}] \vdash \hat{\alpha} : \leq A \dashv \Delta$, or not.
- (2) Either there exists Δ such that $\Gamma_0[\hat{\alpha}] \vdash A \stackrel{\leq}{=} \hat{\alpha} \dashv \Delta$, or not.

5.2 Decidability of Algorithmic Subtyping

To prove decidability of the subtyping system in Figure 9, measure judgments $\Gamma \vdash A <: B \dashv \Delta$ lexicographically by

- (S1) the number of \forall quantifiers in A and B;
- (S2) $|unsolved(\Gamma)|$, the number of unsolved existentials in Γ ;
- (S3) $|\Gamma \vdash A| + |\Gamma \vdash B|$.

Part (S3) uses *contextual size*, which penalizes solved variables (*):

Definition (Contextual Size).

$$\begin{array}{lll} |\Gamma \vdash \alpha| & = & 1 \\ |\Gamma[\hat{\alpha}] \vdash \hat{\alpha}| & = & 1 \\ |\Gamma[\hat{\alpha} = \tau] \vdash \hat{\alpha}| & = & 1 + |\Gamma[\hat{\alpha} = \tau] \vdash \tau| \\ |\Gamma \vdash \forall \alpha. \ A| & = & 1 + |\Gamma, \alpha \vdash A| \\ |\Gamma \vdash A \rightarrow B| & = & 1 + |\Gamma \vdash A| + |\Gamma \vdash B| \end{array}$$

For example, if $\Gamma = (\beta, \hat{\alpha} = \beta)$ then $|\Gamma \vdash \hat{\alpha}| = 1 + |\Gamma \vdash \beta| = 1 + 1 = 2$, whereas the plain size of $\hat{\alpha}$ is simply 1.

The connection between (S1) and (S2) may be clarified by examining rule $\langle : \rightarrow \rangle$, whose conclusion says that $A_1 \rightarrow A_2$ is a subtype of $B_1 \rightarrow B_2$. If A_2 or B_2 is polymorphic, then the first premise on $A_1 \rightarrow A_2$ is smaller by (S1). Otherwise, the first premise has the same input context as the conclusion, so it has the same (S2), but is smaller by (S3). If B_1 or A_1 is polymorphic, then the second premise is smaller by (S1). Otherwise, we use the property that instantiating a monotype always solves an existential:

Lemma (Monotypes Solve Variables). If $\Gamma \vdash \hat{\alpha} := \tau \dashv \Delta$ or $\Gamma \vdash \tau := \hat{\alpha} \dashv \Delta$, then if $[\Gamma]\tau = \tau$ and $\hat{\alpha} \notin FV([\Gamma]\tau)$, we have $|unsolved(\Gamma)| = |unsolved(\Delta)| + 1$.

A couple of other lemmas are worth mentioning: subtyping on two monotypes cannot increase the number of unsolved existentials, and applying a substitution Γ to a type does not increase the type's size with respect to Γ .

Lemma (Monotype Monotonicity).

If $\Gamma \vdash \tau_1 <: \tau_2 \dashv \Delta$ then $|\mathsf{unsolved}(\Delta)| \leq |\mathsf{unsolved}(\Gamma)|$.

Lemma (Substitution Decreases Size). *If* $\Gamma \vdash A$ *then* $|\Gamma \vdash [\Gamma]A| \leq |\Gamma \vdash A|$.

Theorem 8 (Decidability of Subtyping).

Given a context Γ and types A, B such that $\Gamma \vdash A$ and $\Gamma \vdash B$ and $[\Gamma]A = A$ and $[\Gamma]B = B$, it is decidable whether there exists Δ such that $\Gamma \vdash A <: B \dashv \Delta$.

5.3 Decidability of Algorithmic Typing

Theorem 9 (Decidability of Typing).

- (i) Synthesis: Given a context Γ and a term e, it is decidable whether there exist a type A and a context Δ such that Γ ⊢ e ⇒ A ⊢ Δ.
- (ii) Checking: Given a context Γ, a term e, and a type B such that Γ ⊢ B, it is decidable whether there is a context Δ such that Γ ⊢ e ← B ⊢ Δ.
- (iii) Application: Given a context Γ , a term e, and a type A such that $\Gamma \vdash A$, it is decidable whether there exist a type C and a context Δ such that

$$\Gamma \vdash A \bullet e \Longrightarrow C \dashv \Delta$$
.

The following induction measure suffices to prove decidability:

$$\left\langle e, \begin{array}{cc} \Rightarrow \\ \left\langle e, & \leftarrow, & |\Gamma \vdash B| \\ \Rightarrow & |\Gamma \vdash A| \end{array} \right\rangle$$

where $\langle \dots \rangle$ denotes lexicographic order, and where (when comparing two judgments typing the same term e) the synthesis judgment (top line) is considered smaller than the checking judgment (second line), which in turn is considered smaller than the application judgment (bottom line). That is, $\Rightarrow \prec \Leftarrow \prec \Rightarrow >$. In Sub, this makes the synthesis premise smaller than the checking conclusion; in \rightarrow App and $\hat{\alpha}$ App, this makes the checking premise smaller than the application conclusion.

Since we have no explicit introduction form for polymorphism, the rule $\forall I$ has the same term e in its premise and conclusion, and both the premise and conclusion are the same kind of judgment (checking). The rule $\forall App$ is similar (with application judgments in premise and conclusion). Therefore, given two judgments on the same term, and that are both checking judgments or both application judgments, we use the size of the input type expression—which *does* get smaller in $\forall I$ and $\forall App$.

6. Soundness

We want the algorithmic specifications of subtyping and typing to be sound with respect to the declarative specifications. Roughly, given a derivation of an algorithmic judgment with input context Γ and output context Δ , and some complete context Ω that extends Δ (which therefore extends Γ), applying Ω throughout the given algorithmic judgment should yield a derivable declarative judgment. Let's make that rough outline concrete for instantiation, showing that the action of the instantiation rules is consistent with declarative subtyping:

Theorem 10 (Instantiation Soundness).

Given $\Delta \longrightarrow \Omega$ and $[\Gamma]B = B$ and $\hat{\alpha} \notin FV(B)$:

- (1) If $\Gamma \vdash \hat{\alpha} := B \dashv \Delta$ then $[\Omega]\Delta \vdash [\Omega]\hat{\alpha} \leq [\Omega]B$.
- (2) If $\Gamma \vdash B \stackrel{\leq}{=} : \hat{\alpha} \dashv \Delta \text{ then } [\Omega] \Delta \vdash [\Omega] B \leq [\Omega] \hat{\alpha}.$

Note that the declarative derivation is under $[\Omega]\Delta$, which is Ω applied to the algorithmic output context Δ .

With instantiation soundness, we can prove the expected soundness property for subtyping:

Theorem 11 (Soundness of Algorithmic Subtyping).

If
$$\Gamma \vdash A <: B \dashv \Delta$$
 where $[\Gamma]A = A$ and $[\Gamma]B = B$ and $\Delta \longrightarrow \Omega$ then $[\Omega]\Delta \vdash [\Omega]A \leq [\Omega]B$.

Finally, knowing that subtyping is sound, we can prove that typing is sound:

Theorem 12 (Soundness of Algorithmic Typing). *Given* $\Delta \longrightarrow \Omega$:

- (i) If $\Gamma \vdash e \Leftarrow A \dashv \Delta$ then $[\Omega] \Delta \vdash e \Leftarrow [\Omega] A$.
- (ii) If $\Gamma \vdash e \Rightarrow A \dashv \Delta$ then $[\Omega]\Delta \vdash e \Rightarrow [\Omega]A$.
- (iii) If $\Gamma \vdash A \bullet e \Rightarrow C \dashv \Delta$ then $[\Omega]\Delta \vdash [\Omega]A \bullet e \Rightarrow [\Omega]C$.

The proofs need several lemmas, including this one:

Lemma (Typing Extension).

If $\Gamma \vdash e \Leftarrow A \dashv \Delta \text{ or } \Gamma \vdash e \Rightarrow A \dashv \Delta \text{ or } \Gamma \vdash A \bullet e \Rightarrow C \dashv \Delta$ then $\Gamma \longrightarrow \Delta$.

7. Completeness

Completeness of the algorithmic system is something like soundness in reverse: given a declarative derivation of $[\Omega]\Gamma \vdash [\Omega]\cdots$, we want to get an algorithmic derivation of $\Gamma \vdash \cdots \dashv \Delta$.

For soundness, the output context Δ such that $\Delta \longrightarrow \Omega$ was given; $\Gamma \longrightarrow \Omega$ followed from Typing Extension (the above lemma) and transitivity of extension. For completeness, only Γ is given, so we have $\Gamma \longrightarrow \Omega$ in the antecedent. Then we might expect to show, along with $\Gamma \vdash \cdots \dashv \Delta$, that $\Delta \longrightarrow \Omega$. But this is not general enough: the algorithmic rules generate fresh existential variables, so Δ may have existentials that are not found in Γ , nor in Ω . In completeness, we are given a declarative derivation, which contains no existentials; the completeness proof must build up the completing context Ω along with the algorithmic derivation. Thus, completeness will produce an Ω' which extends both the given Ω and the output context of the algorithmic derivation: $\Omega \longrightarrow \Omega'$ and $\Delta \longrightarrow \Omega'$. (By transitivity, we also get $\Gamma \longrightarrow \Omega'$.)

As with soundness, we have three main completeness results, for instantiation, subtyping and typing.

Theorem 13 (Instantiation Completeness). Given $\Gamma \longrightarrow \Omega$ and $A = [\Gamma]A$ and $\hat{\alpha} \in \mathsf{unsolved}(\Gamma)$ and $\hat{\alpha} \notin \mathsf{FV}(A)$:

- (1) If $[\Omega]\Gamma \vdash [\Omega]\hat{\alpha} \leq [\Omega]A$ then there are Δ , Ω' such that $\Omega \longrightarrow \Omega'$ and $\Delta \longrightarrow \Omega'$ and $\Gamma \vdash \hat{\alpha} : \leq A \dashv \Delta$.
- (2) If $[\Omega]\Gamma \vdash [\Omega]A \leq [\Omega]\hat{\alpha}$ then there are Δ , Ω' such that $\Omega \longrightarrow \Omega'$ and $\Delta \longrightarrow \Omega'$ and $\Gamma \vdash A \leq : \hat{\alpha} \dashv \Delta$.

Theorem 14 (Generalized Completeness of Subtyping). If $\Gamma \longrightarrow \Omega$ and $\Gamma \vdash A$ and $\Gamma \vdash B$ and $[\Omega]\Gamma \vdash [\Omega]A \leq [\Omega]B$ then there exist Δ and Ω' such that $\Delta \longrightarrow \Omega'$ and $\Omega \longrightarrow \Omega'$ and $\Gamma \vdash [\Gamma]A <: [\Gamma]B \dashv \Delta$.

Theorem 15 (Completeness of Algorithmic Typing). Given $\Gamma \longrightarrow \Omega$ and $\Gamma \vdash A$:

- (i) If $[\Omega]\Gamma \vdash e \Leftarrow [\Omega]A$ then there exist Δ and Ω' such that $\Delta \longrightarrow \Omega'$ and $\Omega \longrightarrow \Omega'$ and $\Gamma \vdash e \Leftarrow [\Gamma]A \dashv \Delta$.
- (ii) If $[\Omega]\Gamma \vdash e \Rightarrow A$ then there exist Δ , Ω' , and A'such that $\Delta \longrightarrow \Omega'$ and $\Omega \longrightarrow \Omega'$ and $\Gamma \vdash e \Rightarrow A' \dashv \Delta$ and $A = [\Omega']A'$.
- (iii) If $[\Omega]\Gamma \vdash [\Omega]A \bullet e \Rightarrow C$ then there exist Δ , Ω' , and C'such that $\Delta \longrightarrow \Omega'$ and $\Omega \longrightarrow \Omega'$ and $\Gamma \vdash [\Gamma]A \bullet e \Rightarrow C' \dashv \Delta$ and $C = [\Omega']C'$.

8. Design Variations

The rules we give infer monomorphic types, but require annotations for all polymorphic bindings. In this section, we consider alternatives to this choice.

Eliminating type inference. To eliminate type inference from the declarative system, it suffices to drop the $Decl \rightarrow l \Rightarrow$ and $Decl 1l \Rightarrow$ rules. The corresponding alterations to the algorithmic system are a little more delicate: simply deleting the $\rightarrow l \Rightarrow$ and $1l \Rightarrow$ rules breaks completeness. To see why, suppose that we have a variable f of type $\forall \alpha. \alpha \rightarrow \alpha$, and consider the application f (). Our algorithm will introduce a new existential variable $\hat{\alpha}$ for α , and then check () against $\hat{\alpha}$. Without the $1l \Rightarrow$ rule, typechecking will fail. To restore completeness, we need to modify these two rules. Instead of being *synthesis* rules, we will change them to *checking* rules that check values against an unknown existential variable.

$$\overline{\Gamma[\hat{\alpha}] \vdash \text{ ()} \Leftarrow \hat{\alpha} \dashv \Gamma[\hat{\alpha}=1]} \ ^{1}\hat{\alpha}$$

$$\frac{\Gamma[\hat{\alpha}_{2},\hat{\alpha}_{1},\hat{\alpha}=\hat{\alpha}_{1}\rightarrow\hat{\alpha}_{2}],x:\hat{\alpha}_{1}\vdash e\Leftarrow\hat{\alpha}_{2}\dashv\Delta,x:\hat{\alpha}_{1},\Delta'}{\Gamma[\hat{\alpha}]\vdash\lambda x,e\Leftarrow\hat{\alpha}\dashv\Delta}\rightarrow l\hat{\alpha}$$

With these two rules replacing $1 \Rightarrow$ and $\rightarrow 1 \Rightarrow$, we have a complete algorithm for the no-inference bidirectional system.

Full Damas-Milner type inference. Another alternative is to increase the amount of type inference done. For instance, a natural question is whether we can extend the bidirectional approach to subsume the inference done by the algorithm of Damas and Milner (1982). This appears feasible: we can alter the \rightarrow I \Rightarrow rule to support ML-style type inference.

$$\begin{split} \frac{\Gamma, \blacktriangleright_{\hat{\alpha}}, \hat{\alpha}, \hat{\beta}, \chi : \hat{\alpha} \vdash e \Leftarrow \hat{\beta} \dashv \Delta, \blacktriangleright_{\hat{\alpha}}, \Delta'}{\tau = [\Delta'](\hat{\alpha} \to \hat{\beta}) \qquad \vec{\hat{\alpha}} = \mathsf{unsolved}(\Delta')}{\Gamma \vdash \lambda x. \, e \Rightarrow \forall \vec{\alpha}. \, [\vec{\alpha}/\vec{\hat{\alpha}}]\tau \dashv \Delta} \to \mathsf{I} \Rightarrow' \end{split}$$

In this rule, we introduce a marker $\triangleright_{\hat{\alpha}}$ into the context, and then check the function body against the type $\hat{\beta}$. Then, our output type substitutes away all the solved existential variables to the right of the marker $\triangleright_{\hat{\alpha}}$, and generalizes over all of the unsolved variables to the right of the marker. Using an ordered context gives precise control over the scope of the existential variables, making it easy to express polymorphic generalization.

The above is only a sketch; we have not defined the corresponding declarative system, nor proved completeness.

9. Related Work and Discussion

9.1 Type Inference for System F

Because type inference for System F is undecidable (Wells 1999), designing type inference algorithms for first-class polymorphism

inherently involves navigating a variety of design tradeoffs. As a result, there have been a wide variety of proposals for extending type systems beyond the Damas-Milner "sweet spot". The main tradeoff appears to be a "two-out-of-three" choice: language designers can keep any two of: (1) the η -law for functions, (2) impredicative instantiation, and (3) the standard type language of System F.

As discussed in Section 2, for typability under η -reductions, it is necessary for subtyping to instantiate deeply: that is, we must allow instantiation of quantifiers to the right of an arrow. However, Tiuryn and Urzyczyn (1996) and Chrząszcz (1998) showed that the subtyping relation for impredicative System F is undecidable. As a result, if we want η and a complete algorithm, then either the polymorphic instantiations must be predicative, or a different language of types must be used.

Figure 15 summarizes the different choices made by the designers of this and related systems.

Impredicativity and the η -law. The designers of ML^F (Le Botlan and Rémy 2003; Rémy and Yakobowski 2008; Le Botlan and Rémy 2009) chose to chose to use a different language of types, one with a form of bounded quantification. This increases the expressivity of types enough to ensure principal types, which means that (1) required annotations are few and predictable, and (2) their system is very robust in the face of program transformations, including η . However, the richness of the ML^F type structure requires a sophisticated metatheory and correspondingly intricate implementation techniques.

Impredicativity and System F types. Much of the other work on higher-rank polymorphism avoids changing the language of types.

The HML system of Leijen (2009) and the FPH system of Vytiniotis et al. (2008) both retain the type language of (impredicative) System F. Each of these systems gives as a specification a slightly different extension to the declarative Damas-Milner type system, and handle the issue of inference in slightly different ways. HML is essentially a restriction of ML^F, in which the external language of types is limited to System F, but which uses the technology of ML^F internally, as part of type inference. FPH, on the other hand, extends and generalizes work on boxy types (Vytiniotis et al. 2006) to control type inference. The differences in expressive power between these two systems are subtle—roughly speaking, FPH requires slightly more annotations, but has a less complicated specification. However, in both systems, the same heuristic guidance to the programmer applies: place explicit annotations on binders with fancy types.

The n-law and System F types. Peyton Jones et al. (2007) developed an approach for typechecking higher-rank predicative polymorphism that is closely related to ours. They define a bidirectional declarative system similar to our own, but which lacks an application judgment. This complicates the presentation of their system, forcing them to introduce an additional grammatical category of types beyond monotypes and polytypes, and requires many rules to carry an additional subtyping premise. Next, they enrich the subtyping rules of Odersky and Läufer (1996) with the distributivity axiom of Mitchell (1988), which we rejected on ideological grounds: it is a valid coercion, but is not orthogonal (it is a single rule mixing two different type connectives) and does not correspond to a rule in the sequent calculus. They do not prove the soundness and completeness of their Haskell reference implementation, but it appears to implement behavior close to our application judgment.

History of our approach. Several of the ideas used in the present paper descend from Dunfield (2009), an approach to first-class polymorphism (including impredicativity) also based on ordered contexts with existential variables instantiated via subtyping. In

System	η-laws?	Impredicative?	System F type language?
ML ^F	yes	yes	no
FPH	no	yes	yes
HML	no	yes	yes
Peyton Jones et al. (2007)	yes	no	yes
This paper	yes	no	yes

Figure 15. Comparison of type inference algorithms

fact, the present work began as an attempt to extend Dunfield (2009) with type-level computation. During that attempt, we found several shortcomings and problems. The most serious is that the decidability and completeness arguments were not valid. These problems may be fixable, but instead we started over, reusing several of the high-level ideas in different technical forms.

9.2 Other Type Systems

Pierce and Turner (2000) developed bidirectional typechecking for rich subtyping, with specific techniques for instantiating polymorphism within function application (hence, *local* type inference). Their declarative specification is more complex than ours, and their algorithm depends on computing approximations of upper and lower bounds on types. *Colored local type inference* (Odersky et al. 2001) allows different *parts* of type expressions to be propagated in different directions. Our approach gets a similar effect by manipulating type expressions with existential variables.

9.3 Our Algorithm

One of our main contributions is our new algorithm for type inference, which is remarkable in its simplicity. Three key ideas underpin our algorithm.

Ordered contexts. We move away from the traditional "bag of constraints" model of type inference, and instead embed existential variables and their values directly into an ordered context. Thus, straightforward scoping rules control the free variables of the types each existential variable may be instantiated with, without any need for model-theoretic techniques like skolemization, which fit awkwardly into a type-theoretic discipline. Using an ordered context permits handling quantifiers in a manner resembling the level-based generalization mechanism of Rémy (1992), used also in ML^F (Le Botlan and Rémy 2009).

The instantiation judgment. The original inspiration for instantiation comes from the "greedy" algorithm of Cardelli (1993), which eagerly uses type information to solve existential constraints. In that setting—a language with rather ambitious subtyping—the greedy algorithm was incomplete: consider a function of type $\forall \alpha.\ \alpha \to \alpha \to \alpha$ applied to a *Cat* and an *Animal*; the cat will be checked against an existential $\hat{\alpha}$, which instantiates $\hat{\alpha}$ to *Cat*, but checking the second argument, *Animal* <: *Cat*, fails. (Reversing the order of arguments makes typing succeed!)

In our setting, where subtyping represents the specialization order induced by quantifier instantiation, it is possible to get a complete algorithm, by slightly relaxing the pure greedy strategy. Rather than eagerly setting constraints, we first look under quantifiers (in the InstLAllR and InstRAllL rules) to see if there is a feasible monotype instantiation, and we also use the the InstLReach and InstRReach to set the "wrong" existential variable in case we need to equate an existential variable with one to its right in the context. Looking under quantifiers seems forced by our restriction to predicative polymorphism, and "reaching" seems forced by our use of an ordered context, but the combination of these mechanisms fortuitously enables our algorithm to find good upper and lower monomorphic approximations of polymorphic types.

This is surprising, since it is quite contrary to the implementation strategy of ML^F (also used by HML and FPH). There, the language of type constraints supports bounds on fully quantified types, and the algorithm incrementally refines these constraints. In contrast, we only ever create equational constraints on existentials (bounds are not needed), and once we have a solution for an existential, our algorithm never needs to revisit its decision.

Distinguishing instantiation as a separate judgment is new in this paper, and beneficial: Dunfield (2009) baked instantiation into the subtyping rules, resulting in a system whose direct implementation required substantial backtracking—over a set of rules including arbitrary application of substitutions. We, instead, maintain an invariant in subtyping and instantiation that the types are always fully applied with respect to an input context, obviating the need for explicit rules to apply substitutions.

Context extension. Finally, we introduce a context-extension judgment as the central invariant in our correctness proofs. This permits us to state many properties important to our algorithm abstractly, without reference to the details of our algorithm.

We are not the only ones to study context-based approaches to type inference. Recently, Gundry et al. (2010) recast the classic Damas-Milner algorithm, which manipulates unstructured sets of equality constraints, as structured constraint solving under ordered contexts. A (semantic) notion of information increase is central to their development, as (syntactic) context extension is to ours. While their formulation supports only ML-style prenex polymorphism, the ultimate goal is a foundation for type inference for dependent types. To some extent, both our algorithm and theirs can be understood in terms of the proof system of Miller (1992) for mixed-prefix unification. We each restrict the unification problem, and then give a proof search algorithm to solve the type inference problem.

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References

Andreas Abel. Termination checking with types. *RAIRO—Theoretical Informatics and Applications*, 38(4):277–319, 2004. Special Issue: Fixed Points in Computer Science (FICS'03).

Andreas Abel, Thierry Coquand, and Peter Dybjer. Verifying a semantic βη-conversion test for Martin-Löf type theory. In *Mathematics of Program Construction (MPC'08)*, volume 5133 of *LNCS*, pages 29–56, 2008.

Jean-Marc Andreoli. Logic programming with focusing proofs in linear logic. *J. Logic and Computation*, 2(3):297–347, 1992.

Andrea Asperti, Wilmer Ricciotti, Claudio Sacerdoti Coen, and Enrico Tassi. A bi-directional refinement algorithm for the calculus of (co)inductive constructions. *Logical Methods in Computer Science*, 8(1), 2012.

- Gavin M. Bierman, Erik Meijer, and Mads Torgersen. Lost in translation: formalizing proposed extensions to C[‡]. In OOPSLA, 2007.
- Luca Cardelli. An implementation of F_{<1}. Research report 97, DEC/Compaq Systems Research Center, February 1993.
- Iliano Cervesato and Frank Pfenning. A linear spine calculus. *J. Logic and Computation*, 13(5):639–688, 2003.
- Adam Chlipala, Leaf Petersen, and Robert Harper. Strict bidirectional type checking. In *Workshop on Types in Language Design and Impl. (TLDI '05)*, pages 71–78, 2005.
- Jacek Chrząszcz. Polymorphic subtyping without distributivity. In Mathematical Foundations of Computer Science, volume 1450 of LNCS, pages 346–355. Springer, 1998.
- Thierry Coquand. An algorithm for type-checking dependent types. *Science of Computer Programming*, 26(1–3):167–177, 1996.
- Luis Damas and Robin Milner. Principal type-schemes for functional programs. In *POPL*, pages 207–212. ACM, 1982.
- Rowan Davies and Frank Pfenning. Intersection types and computational effects. In *ICFP*, pages 198–208, 2000.
- Joshua Dunfield. *A Unified System of Type Refinements*. PhD thesis, Carnegie Mellon University, 2007. CMU-CS-07-129.
- Joshua Dunfield. Greedy bidirectional polymorphism. In ML Workshop, pages 15–26, 2009. http://www.cs.cmu.edu/~joshuad/papers/poly/.
- Joshua Dunfield and Frank Pfenning. Tridirectional typechecking. In POPL, pages 281–292, January 2004.
- Adam Gundry, Conor McBride, and James McKinna. Type inference in context. In *Mathematically Structured Functional Programming (MSFP)*, 2010.
- Didier Le Botlan and Didier Rémy. MLF: raising ML to the power of System F. In *ICFP*, pages 27–38, 2003.
- Didier Le Botlan and Didier Rémy. Recasting ML^F. *Information and Computation*, 207:726–785, 2009.
- Daan Leijen. Flexible types: robust type inference for first-class polymorphism. In *POPL*, pages 66–77, 2009.
- Andres Löh, Conor McBride, and Wouter Swierstra. A tutorial implementation of a dependently typed lambda calculus. Unpublished draft, http://people.cs.uu.nl/andres/LambdaPi/index.html, 2008.
- William Lovas. *Refinement Types for Logical Frameworks*. PhD thesis, Carnegie Mellon University, 2010. CMU-CS-10-138.
- Dale Miller. Unification under a mixed prefix. *J. Symbolic Computation*, 14:321–358, 1992.
- John C. Mitchell. Polymorphic type inference and containment. Information and Computation, 76:211–249, 1988.
- Martin Odersky and Konstantin Läufer. Putting type annotations to work. In *POPL*, 1996.
- Martin Odersky, Matthias Zenger, and Christoph Zenger. Colored local type inference. In *POPL*, pages 41–53, 2001.
- Simon Peyton Jones and Mark Shields. Lexically scoped type variables. Technical report, Microsoft Research, 2004.
- Simon Peyton Jones, Dimitrios Vytiniotis, Stephanie Weirich, and Mark Shields. Practical type inference for arbitrary-rank types. *J. Functional Programming*, 17(1):1–82, 2007.
- Frank Pfenning. Church and Curry: Combining intrinsic and extrinsic typing. In *Reasoning in Simple Type Theory: Festschrift*

- in Honor of Peter B. Andrews on His 70th Birthday. College Publications, 2008.
- Brigitte Pientka. A type-theoretic foundation for programming with higher-order abstract syntax and first-class substitutions. In *POPL*, pages 371–382, 2008.
- Benjamin C. Pierce and David N. Turner. Local type inference. *ACM Trans. Prog. Lang. Sys.*, 22:1–44, 2000.
- Didier Rémy. Extension of ML type system with a sorted equational theory on types. Research Report 1766, INRIA, 1992.
- Didier Rémy and Boris Yakobowski. From ML to ML^F: graphic type constraints with efficient type inference. In *ICFP*, pages 63–74, 2008.
- Robert J. Simmons. Structural focalization. arXiv:1109.6273v4 [cs.L0], 2012.
- Jerzy Tiuryn and Paweł Urzyczyn. The subtyping problem for second-order types is undecidable. In *LICS*, 1996.
- Dimitrios Vytiniotis, Stephanie Weirich, and Simon L. Peyton Jones. Boxy types: inference for higher-rank types and impredicativity. In *ICFP*, pages 251–262, 2006.
- Dimitrios Vytiniotis, Stephanie Weirich, and Simon L. Peyton Jones. FPH: First-class polymorphism for Haskell. In *ICFP*, pages 295–306, 2008.
- Dimitrios Vytiniotis, Simon Peyton Jones, and Tom Schrijvers. Let should not be generalised. In *Workshop on Types in Language Design and Impl. (TLDI '10)*, pages 39–50, 2010.
- Kevin Watkins, Iliano Cervesato, Frank Pfenning, and David Walker. A concurrent logical framework: The propositional fragment. In *Types for Proofs and Programs*, pages 355–377. Springer-Verlag LNCS 3085, 2004.
- J. B. Wells. Typability and type checking in System F are equivalent and undecidable. *Annals of Pure and Applied Logic*, 98: 111–156, 1999.
- Hongwei Xi. Dependent Types in Practical Programming. PhD thesis, Carnegie Mellon University, 1998.

Lemmas and Proofs for "Complete and Easy Bidirectional Typechecking for Higher-Rank Polymorphism"

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A Declarative Subtyping

A.1 Properties of Well-Formedness

Proposition 1 (Weakening). If $\Psi \vdash A$ then $\Psi, \Psi' \vdash A$ by a derivation of the same size.

Proposition 2 (Substitution). *If* $\Psi \vdash A$ *and* Ψ , α , $\Psi' \vdash B$ *then* Ψ , $\Psi' \vdash [A/\alpha]B$.

A.2 Reflexivity

Lemma 3 (Reflexivity of Declarative Subtyping). *Subtyping is reflexive:* if $\Psi \vdash A$ then $\Psi \vdash A \leq A$.

A.3 Subtyping Implies Well-Formedness

Lemma 4 (Well-Formedness). *If* $\Psi \vdash A \leq B$ *then* $\Psi \vdash A$ *and* $\Psi \vdash B$.

A.4 Substitution

Lemma 5 (Substitution). If $\Psi \vdash \tau$ and Ψ , α , $\Psi' \vdash A \leq B$ then Ψ , $[\tau/\alpha]\Psi' \vdash [\tau/\alpha]A \leq [\tau/\alpha]B$.

A.5 Transitivity

Lemma 6 (Transitivity of Declarative Subtyping). If $\Psi \vdash A \leq B$ and $\Psi \vdash B \leq C$ then $\Psi \vdash A \leq C$.

A.6 Invertibility of $\leq \forall R$

Lemma 7 (Invertibility).

If \mathcal{D} derives $\Psi \vdash A \leq \forall \beta$. B then \mathcal{D}' derives $\Psi, \beta \vdash A \leq B$ where $\mathcal{D}' < \mathcal{D}$.

A.7 Non-Circularity and Equality

Definition 1 (Subterm Occurrence).

Let $A \prec B$ iff A is a subterm of B.

Let $A \prec B$ iff A is a proper subterm of B (that is, $A \leq B$ and $A \neq B$).

Let $A \supseteq B$ iff A occurs in B inside an arrow, that is, there exist B_1 , B_2 such that $(B_1 \rightarrow B_2) \subseteq B$ and $A \subseteq B_k$ for some $k \in \{1, 2\}$.

Lemma 8 (Occurrence).

- (i) If $\Psi \vdash A \leq \tau$ then $\tau \not\supseteq A$.
- (ii) If $\Psi \vdash \tau \leq B$ then $\tau \not\supseteq B$.

Lemma 9 (Monotype Equality). *If* $\Psi \vdash \sigma \leq \tau$ *then* $\sigma = \tau$.

Definition 2 (Contextual Size). The size of A with respect to a context Γ , written $|\Gamma \vdash A|$, is defined by

$$\begin{array}{lll} |\Gamma \vdash \alpha| & = & 1 \\ |\Gamma[\widehat{\alpha}] \vdash \widehat{\alpha}| & = & 1 \\ |\Gamma[\widehat{\alpha} = \tau] \vdash \widehat{\alpha}| & = & 1 + |\Gamma[\widehat{\alpha} = \tau] \vdash \tau| \\ |\Gamma \vdash \forall \alpha. \ A| & = & 1 + |\Gamma, \alpha \vdash A| \\ |\Gamma \vdash A \rightarrow B| & = & 1 + |\Gamma \vdash A| + |\Gamma \vdash B| \end{array}$$

B Type Assignment

Lemma 10 (Well-Formedness).

If $\Psi \vdash e \Leftarrow A$ or $\Psi \vdash e \Rightarrow A$ or $\Psi \vdash A \bullet e \Rightarrow C$ then $\Psi \vdash A$ (and in the last case, $\Psi \vdash C$).

Theorem 1 (Completeness of Bidirectional Typing).

If $\Psi \vdash e : A$ then there exists e' such that $\Psi \vdash e' \Rightarrow A$ and |e'| = e.

Lemma 11 (Subtyping Coercion). *If* $\Psi \vdash A \leq B$ *then there exists* f *which is* $\beta \eta$ -equal to the identity such that $\Psi \vdash f : A \rightarrow B$.

Lemma 12 (Application Subtyping). If $\Psi \vdash A \bullet e \Rightarrow C$ then there exists B such that $\Psi \vdash A \leq B \rightarrow C$ and $\Psi \vdash e \Leftarrow B$ by a smaller derivation.

Theorem 2 (Soundness of Bidirectional Typing). We have that:

- If $\Psi \vdash e \Leftarrow A$, then there is an e' such that $\Psi \vdash e'$: A and $e' =_{\beta \eta} |e|$.
- If $\Psi \vdash e \Rightarrow A$, then there is an e' such that $\Psi \vdash e'$: A and $e' =_{\beta \eta} |e|$.

C Robustness of Typing

Lemma 13 (Type Substitution).

Assume $\Psi \vdash \tau$.

- If Ψ , α , $\Psi' \vdash e' \Leftarrow C$ then Ψ , $[\tau/\alpha]\Psi' \vdash [\tau/\alpha]e' \Leftarrow [\tau/\alpha]C$.
- If $\Psi, \alpha, \Psi' \vdash e' \Rightarrow C$ then $\Psi, [\tau/\alpha]\Psi' \vdash [\tau/\alpha]e' \Rightarrow [\tau/\alpha]C$.
- If Ψ , α , $\Psi' \vdash B \bullet e' \Rightarrow C$ then Ψ , $[\tau/\alpha]\Psi' \vdash [\tau/\alpha]B \bullet [\tau/\alpha]e' \Rightarrow [A/\alpha]C$.

Moreover, the resulting derivation contains no more applications of typing rules than the given one. (Internal subtyping derivations, however, may grow.)

Definition 3 (Context Subtyping). We define the judgment $\Psi' \leq \Psi$ with the following rules:

$$\frac{\Psi' \leq \Psi}{\Psi', \alpha \leq \Psi, \alpha} \ \mathsf{CtxSubUvar} \qquad \qquad \frac{\Psi' \leq \Psi}{\Psi', \alpha : A' \leq \Psi, x : A} \ \mathsf{CtxSubVar}$$

Lemma 14 (Subsumption). *Suppose* $\Psi' \leq \Psi$. *Then:*

- (i) If $\Psi \vdash e \Leftarrow A$ and $\Psi \vdash A < A'$ then $\Psi' \vdash e \Leftarrow A'$.
- (ii) If $\Psi \vdash e \Rightarrow A$ then there exists A' such that $\Psi \vdash A' \leq A$ and $\Psi' \vdash e \Rightarrow A'$.
- (iii) If $\Psi \vdash C \bullet e \Rightarrow A$ and $\Psi \vdash C' \leq C$ then there exists A' such that $\Psi \vdash A' \leq A$ and $\Psi' \vdash C' \bullet e \Rightarrow A'$.

Theorem 3 (Substitution).

Assume $\Psi \vdash e \Rightarrow A$.

- (i) If $\Psi, x : A \vdash e' \Leftarrow C$ then $\Psi \vdash [e/x]e' \Leftarrow C$.
- (ii) If $\Psi, x : A \vdash e' \Rightarrow C$ then $\Psi \vdash [e/x]e' \Rightarrow C$.
- (iii) If $\Psi, \chi : A \vdash B \bullet e' \Rightarrow C$ then $\Psi \vdash B \bullet [e/\chi]e' \Rightarrow C$.

Theorem 4 (Inverse Substitution).

Assume $\Psi \vdash e \Leftarrow A$.

(i) If $\Psi \vdash [(e : A)/x]e' \Leftarrow C$ then $\Psi, x : A \vdash e' \Leftarrow C$.

- (ii) If $\Psi \vdash [(e : A)/x]e' \Rightarrow C$ then $\Psi, x : A \vdash e' \Rightarrow C$.
- (iii) If $\Psi \vdash B \bullet [(e : A)/x]e' \Rightarrow C$ then $\Psi, x : A \vdash B \bullet e' \Rightarrow C$.

Theorem 5 (Annotation Removal). We have that:

- If $\Psi \vdash ((\lambda x. e) : A) \Leftarrow C$ then $\Psi \vdash \lambda x. e \Leftarrow C$.
- If $\Psi \vdash (() : A) \Leftarrow C$ then $\Psi \vdash () \Leftarrow C$.
- If $\Psi \vdash e_1 (e_2 : A) \Rightarrow C$ then $\Psi \vdash e_1 e_2 \Rightarrow C$.
- If $\Psi \vdash (x : A) \Rightarrow A$ then $\Psi \vdash x \Rightarrow B$ and $\Psi \vdash B \leq A$.
- If $\Psi \vdash ((e_1 e_2) : A) \Rightarrow A$ then $\Psi \vdash e_1 e_2 \Rightarrow B$ and $\Psi \vdash B \leq A$.
- If $\Psi \vdash ((e : B) : A) \Rightarrow A$ then $\Psi \vdash (e : B) \Rightarrow B$ and $\Psi \vdash B \leq A$.
- If $\Psi \vdash ((\lambda x. e) : \sigma \rightarrow \tau) \Rightarrow \sigma \rightarrow \tau$ then $\Psi \vdash \lambda x. e \Rightarrow \sigma \rightarrow \tau$.

Theorem 6 (Soundness of Eta).

If $\Psi \vdash \lambda x$. $e \ x \Leftarrow A$ and $x \notin FV(e)$, then $\Psi \vdash e \Leftarrow A$.

D Properties of Context Extension

D.1 Syntactic Properties

Lemma 15 (Declaration Preservation). *If* $\Gamma \longrightarrow \Delta$, *and* $\mathfrak u$ *is a variable or marker* $\blacktriangleright_{\hat{\alpha}}$ *declared in* Γ , *then* $\mathfrak u$ *is declared in* Δ .

Lemma 16 (Declaration Order Preservation). If $\Gamma \longrightarrow \Delta$ and $\mathfrak u$ is declared to the left of $\mathfrak v$ in Γ , then $\mathfrak u$ is declared to the left of $\mathfrak v$ in Δ .

Lemma 17 (Reverse Declaration Order Preservation). *If* $\Gamma \longrightarrow \Delta$ *and* $\mathfrak u$ *and* $\mathfrak v$ *are both declared in* Γ *and* $\mathfrak u$ *is declared to the left of* $\mathfrak v$ *in* Δ *, then* $\mathfrak u$ *is declared to the left of* $\mathfrak v$ *in* Γ .

Lemma 18 (Substitution Extension Invariance). If $\Theta \vdash A$ and $\Theta \longrightarrow \Gamma$ then $[\Gamma]A = [\Gamma]([\Theta]A)$ and $[\Gamma]A = [\Theta]([\Gamma]A)$.

Lemma 19 (Extension Equality Preservation).

If $\Gamma \vdash A$ and $\Gamma \vdash B$ and $[\widehat{\Gamma}]A = [\Gamma]B$ and $\Gamma \longrightarrow \Delta$, then $[\Delta]A = [\Delta]B$.

Lemma 20 (Reflexivity). *If* Γ *is well-formed, then* $\Gamma \longrightarrow \Gamma$.

Lemma 21 (Transitivity). *If* $\Gamma \longrightarrow \Delta$ *and* $\Delta \longrightarrow \Theta$, *then* $\Gamma \longrightarrow \Theta$.

Definition 4 (Softness). A context Θ is soft iff it consists only of $\hat{\alpha}$ and $\hat{\alpha} = \tau$ declarations.

Lemma 22 (Right Softness). *If* $\Gamma \longrightarrow \Delta$ *and* Θ *is soft (and* (Δ, Θ) *is well-formed) then* $\Gamma \longrightarrow \Delta, \Theta$.

Lemma 23 (Evar Input).

If Γ , $\hat{\alpha} \longrightarrow \Delta$ then $\Delta = (\Delta_0, \Delta_{\hat{\alpha}}, \Theta)$ where $\Gamma \longrightarrow \Delta_0$, and $\Delta_{\hat{\alpha}}$ is either $\hat{\alpha}$ or $\hat{\alpha} = \tau$, and Θ is soft.

Lemma 24 (Extension Order).

- (i) If Γ_L , α , $\Gamma_R \longrightarrow \Delta$ then $\Delta = (\Delta_L, \alpha, \Delta_R)$ where $\Gamma_L \longrightarrow \Delta_L$. Moreover, if Γ_R is soft then Δ_R is soft.
- (ii) If Γ_L , $\blacktriangleright_{\hat{\alpha}}$, $\Gamma_R \longrightarrow \Delta$ then $\Delta = (\Delta_L$, $\blacktriangleright_{\hat{\alpha}}$, Δ_R) where $\Gamma_L \longrightarrow \Delta_L$. Moreover, if Γ_R is soft then Δ_R is soft.
- (iii) If Γ_L , $\hat{\alpha}$, $\Gamma_R \longrightarrow \Delta$ then $\Delta = \Delta_L$, Θ , Δ_R where $\Gamma_L \longrightarrow \Delta_L$ and Θ is either $\hat{\alpha}$ or $\hat{\alpha} = \tau$ for some τ .
- (iv) If Γ_L , $\hat{\alpha} = \tau$, $\Gamma_R \longrightarrow \Delta$ then $\Delta = \Delta_L$, $\hat{\alpha} = \tau'$, Δ_R where $\Gamma_L \longrightarrow \Delta_L$ and $[\Delta_L]\tau = [\Delta_L]\tau'$.

(v) If $\Gamma_L, x : A, \Gamma_R \longrightarrow \Delta$ then $\Delta = (\Delta_L, x : A', \Delta_R)$ where $\Gamma_L \longrightarrow \Delta_L$ and $[\Delta_L]A = [\Delta_L]A'$. Moreover, Γ_R is soft if and only if Δ_R is soft.

Lemma 25 (Extension Weakening). *If* $\Gamma \vdash A$ *and* $\Gamma \longrightarrow \Delta$ *then* $\Delta \vdash A$.

Lemma 26 (Solution Admissibility for Extension). *If* $\Gamma_L \vdash \tau$ *then* $\Gamma_L, \hat{\alpha}, \Gamma_R \longrightarrow \Gamma_L, \hat{\alpha} = \tau, \Gamma_R$.

Lemma 27 (Solved Variable Addition for Extension). *If* $\Gamma_L \vdash \tau$ *then* $\Gamma_L, \Gamma_R \longrightarrow \Gamma_L, \hat{\alpha} = \tau, \Gamma_R$.

Lemma 28 (Unsolved Variable Addition for Extension). We have that $\Gamma_L, \Gamma_R \longrightarrow \Gamma_L, \hat{\alpha}, \Gamma_R$.

Lemma 29 (Parallel Admissibility).

If $\Gamma_L \longrightarrow \Delta_L$ and $\Gamma_L, \Gamma_R \longrightarrow \Delta_L, \Delta_R$ then:

- (i) Γ_L , $\hat{\alpha}$, $\Gamma_R \longrightarrow \Delta_L$, $\hat{\alpha}$, Δ_R
- (ii) If $\Delta_{\rm I} \vdash \tau'$ then $\Gamma_{\rm I}$, $\hat{\alpha}$, $\Gamma_{\rm R} \longrightarrow \Delta_{\rm I}$, $\hat{\alpha} = \tau'$, $\Delta_{\rm R}$.
- (iii) If $\Gamma_I \vdash \tau$ and $\Delta_I \vdash \tau'$ and $[\Delta_I]\tau = [\Delta_I]\tau'$, then Γ_I , $\hat{\alpha} = \tau$, $\Gamma_R \longrightarrow \Delta_I$, $\hat{\alpha} = \tau'$, Δ_R .

Lemma 30 (Parallel Extension Solution).

$$\textit{If } \Gamma_L, \hat{\alpha}, \Gamma_R \longrightarrow \Delta_L, \hat{\alpha} = \tau', \Delta_R \textit{ and } \Gamma_L \vdash \tau \textit{ and } [\Delta_L] \tau = [\Delta_L] \tau' \textit{ then } \Gamma_L, \hat{\alpha} = \tau, \Gamma_R \longrightarrow \Delta_L, \hat{\alpha} = \tau', \Delta_R.$$

Lemma 31 (Parallel Variable Update).

If
$$\Gamma_L, \hat{\alpha}, \Gamma_R \longrightarrow \Delta_L, \hat{\alpha} = \tau_0, \Delta_R$$
 and $\Gamma_L \vdash \tau_1$ and $\Delta_L \vdash \tau_2$ and $[\Delta_L]\tau_0 = [\Delta_L]\tau_1 = [\Delta_L]\tau_2$ then $\Gamma_L, \hat{\alpha} = \tau_1, \Gamma_R \longrightarrow \Delta_L, \hat{\alpha} = \tau_2, \Delta_R$.

D.2 Instantiation Extends

Lemma 32 (Instantiation Extension).

If
$$\Gamma \vdash \hat{\alpha} : \stackrel{\leq}{=} \tau \dashv \Delta \text{ or } \Gamma \vdash \tau \stackrel{\leq}{=} : \hat{\alpha} \dashv \Delta \text{ then } \Gamma \longrightarrow \Delta.$$

D.3 Subtyping Extends

Lemma 33 (Subtyping Extension).

If $\Gamma \vdash A \lt : B \dashv \Delta$ *then* $\Gamma \longrightarrow \Delta$.

E Decidability of Instantiation

Lemma 34 (Left Unsolvedness Preservation).

$$\textit{If} \ \underbrace{\Gamma_0, \hat{\alpha}, \Gamma_1}_{\Gamma} \vdash \ \hat{\alpha} : \stackrel{\leq}{=} A \ \dashv \Delta \ \textit{or} \ \underbrace{\Gamma_0, \hat{\alpha}, \Gamma_1}_{\Gamma} \vdash \ A \stackrel{\leq}{=} : \hat{\alpha} \ \dashv \Delta, \ \textit{and} \ \hat{\beta} \in \mathsf{unsolved}(\Gamma_0), \ \textit{then} \ \hat{\beta} \in \mathsf{unsolved}(\Delta).$$

Lemma 35 (Left Free Variable Preservation). If Γ_0 , $\hat{\alpha}$, $\Gamma_1 \vdash \hat{\alpha} : \leq A \dashv \Delta$ or Γ_0 , $\hat{\alpha}$, $\Gamma_1 \vdash A \leq \hat{\alpha} \dashv \Delta$, and $\Gamma \vdash B$ and $\hat{\alpha} \notin FV([\Gamma]B)$ and $\hat{\beta} \in unsolved(\Gamma_0)$ and $\hat{\beta} \notin FV([\Gamma]B)$, then $\hat{\beta} \notin FV([\Delta]B)$.

Lemma 36 (Instantiation Size Preservation). If Γ_0 , $\hat{\alpha}$, $\Gamma_1 \vdash \hat{\alpha} : \leq A \dashv \Delta$ or Γ_0 , $\hat{\alpha}$, $\Gamma_1 \vdash A \leq \hat{\alpha} \dashv \Delta$, and $\Gamma \vdash B$ and $\hat{\alpha} \notin FV([\Gamma]B)$, then $|[\Gamma]B| = |[\Delta]B|$, where |C| is the plain size of the term C.

This lemma lets us show decidability by taking the size of the type argument as the induction metric. **Theorem 7** (Decidability of Instantiation). If $\Gamma = \Gamma_0[\hat{\alpha}]$ and $\Gamma \vdash A$ such that $[\Gamma]A = A$ and $\hat{\alpha} \notin FV(A)$, then:

- (1) Either there exists Δ such that $\Gamma_0[\hat{\alpha}] \vdash \hat{\alpha} := A \dashv \Delta$, or not.
- (2) Either there exists Δ such that $\Gamma_0[\hat{\alpha}] \vdash A \stackrel{\leq}{=} \hat{\alpha} \dashv \Delta$, or not.

F Decidability of Algorithmic Subtyping

F.1 Lemmas for Decidability of Subtyping

Lemma 37 (Monotypes Solve Variables). *If* $\Gamma \vdash \hat{\alpha} := \tau \dashv \Delta$ *or* $\Gamma \vdash \tau = \hat{\alpha} := \hat{\alpha} \dashv \Delta$, *then if* $[\Gamma]\tau = \tau$ *and* $\hat{\alpha} \notin FV([\Gamma]\tau)$, *then* $[\text{unsolved}(\Gamma)] = [\text{unsolved}(\Delta)] + 1$.

Lemma 38 (Monotype Monotonicity). *If* $\Gamma \vdash \tau_1 <: \tau_2 \dashv \Delta \ then \ |unsolved(\Delta)| \leq |unsolved(\Gamma)|$.

Lemma 39 (Substitution Decreases Size). *If* $\Gamma \vdash A$ *then* $|\Gamma \vdash [\Gamma]A| < |\Gamma \vdash A|$.

Lemma 40 (Monotype Context Invariance).

If $\Gamma \vdash \tau <: \tau' \dashv \Delta$ where $[\Gamma]\tau = \tau$ and $[\Gamma]\tau' = \tau'$ and $|\text{unsolved}(\Gamma)| = |\text{unsolved}(\Delta)|$ then $\Gamma = \Delta$.

F.2 Decidability of Subtyping

Theorem 8 (Decidability of Subtyping).

Given a context Γ and types A, B such that $\Gamma \vdash A$ and $\Gamma \vdash B$ and $[\Gamma]A = A$ and $[\Gamma]B = B$, it is decidable whether there exists Δ such that $\Gamma \vdash A <: B \dashv \Delta$.

G Decidability of Typing

Theorem 9 (Decidability of Typing).

- (i) Synthesis: Given a context Γ and a term e,
 it is decidable whether there exist a type A and a context Δ such that Γ ⊢ e ⇒ A ⊢ Δ.
- (ii) Checking: Given a context Γ , a term e, and a type B such that $\Gamma \vdash B$, it is decidable whether there is a context Δ such that $\Gamma \vdash e \Leftarrow B \dashv \Delta$.
- (iii) Application: Given a context Γ , a term e, and a type A such that $\Gamma \vdash A$, it is decidable whether there exist a type C and a context Δ such that $\Gamma \vdash A \bullet e \Rightarrow C \dashv \Delta$.

H Soundness of Subtyping

Definition 5 (Filling). The filling of a context $|\Gamma|$ solves all unsolved variables:

$$\begin{array}{lll} |\cdot| & = & \cdot \\ |\Gamma, x : A| & = & |\Gamma|, x : A \\ |\Gamma, \alpha| & = & |\Gamma|, \alpha \\ |\Gamma, \hat{\alpha} = \tau| & = & |\Gamma|, \hat{\alpha} = \tau \\ |\Gamma, \blacktriangleright_{\hat{\alpha}}| & = & |\Gamma|, \blacktriangleright_{\hat{\alpha}} \\ |\Gamma, \hat{\alpha}| & = & |\Gamma|, \hat{\alpha} = 1 \end{array}$$

H.1 Lemmas for Soundness

Lemma 41 (Uvar Preservation).

If $\alpha \in \Omega$ and $\Delta \longrightarrow \Omega$ then $\alpha \in [\Omega]\Delta$.

Proof. By induction on Ω , following the definition of context application.

Lemma 42 (Variable Preservation).

If $(x : A) \in \Delta$ or $(x : A) \in \Omega$ and $\Delta \longrightarrow \Omega$ then $(x : [\Omega]A) \in [\Omega]\Delta$.

Lemma 43 (Substitution Typing). *If* $\Gamma \vdash A$ *then* $\Gamma \vdash [\Gamma]A$.

Lemma 44 (Substitution for Well-Formedness). *If* $\Omega \vdash A$ *then* $[\Omega]\Omega \vdash [\Omega]A$.

Lemma 45 (Substitution Stability).

For any well-formed complete context (Ω, Ω_Z) , if $\Omega \vdash A$ then $[\Omega]A = [\Omega, \Omega_Z]A$.

Lemma 46 (Context Partitioning).

If $\Delta, \triangleright_{\hat{\alpha}}, \Theta \longrightarrow \Omega, \triangleright_{\hat{\alpha}}, \Omega_Z$ then there is a Ψ such that $[\Omega, \triangleright_{\hat{\alpha}}, \Omega_Z](\Delta, \triangleright_{\hat{\alpha}}, \Theta) = [\Omega]\Delta, \Psi$.

Lemma 47 (Softness Goes Away).

If $\Delta, \Theta \longrightarrow \Omega, \Omega_Z$ where $\Delta \longrightarrow \Omega$ and Θ is soft, then $[\Omega, \Omega_Z](\Delta, \Theta) = [\Omega]\Delta$.

Proof. By induction on Θ , following the definition of $[\Omega]\Gamma$.

Lemma 48 (Filling Completes). If $\Gamma \longrightarrow \Omega$ and (Γ, Θ) is well-formed, then $\Gamma, \Theta \longrightarrow \Omega, |\Theta|$.

Proof. By induction on Θ , following the definition of |-| and applying the rules for \longrightarrow .

Lemma 49 (Stability of Complete Contexts).

If $\Gamma \longrightarrow \Omega$ *then* $[\Omega]\Gamma = [\Omega]\Omega$.

Lemma 50 (Finishing Types).

If $\Omega \vdash A$ and $\Omega \longrightarrow \Omega'$ then $[\Omega]A = [\Omega']A$.

Lemma 51 (Finishing Completions).

If $\Omega \longrightarrow \Omega'$ then $[\Omega]\Omega = [\Omega']\Omega'$.

Lemma 52 (Confluence of Completeness).

If $\Delta_1 \longrightarrow \Omega$ and $\Delta_2 \longrightarrow \Omega$ then $[\Omega]\Delta_1 = [\Omega]\Delta_2$.

H.2 Instantiation Soundness

Theorem 10 (Instantiation Soundness).

Given $\Delta \longrightarrow \Omega$ and $[\Gamma]B = B$ and $\hat{\alpha} \notin FV(B)$:

- (1) If $\Gamma \vdash \hat{\alpha} := B \dashv \Delta$ then $[\Omega]\Delta \vdash [\Omega]\hat{\alpha} < [\Omega]B$.
- (2) If $\Gamma \vdash B \leq : \hat{\alpha} \dashv \Delta \text{ then } [\Omega] \Delta \vdash [\Omega] B \leq [\Omega] \hat{\alpha}.$

H.3 Soundness of Subtyping

Theorem 11 (Soundness of Algorithmic Subtyping).

If $\Gamma \vdash A \lt : B \dashv \Delta$ where $[\Gamma]A = A$ and $[\Gamma]B = B$ and $\Delta \longrightarrow \Omega$ then $[\Omega]\Delta \vdash [\Omega]A \le [\Omega]B$.

Corollary 53 (Soundness, Pretty Version). *If* $\Psi \vdash A \lt : B \dashv \Delta$, *then* $\Psi \vdash A \leq B$.

I Typing Extension

Lemma 54 (Typing Extension).

If $\Gamma \vdash e \Leftarrow A \dashv \Delta$ or $\Gamma \vdash e \Rightarrow A \dashv \Delta$ or $\Gamma \vdash A \bullet e \Rightarrow C \dashv \Delta$ then $\Gamma \longrightarrow \Delta$.

J Soundness of Typing

Theorem 12 (Soundness of Algorithmic Typing). *Given* $\Delta \longrightarrow \Omega$:

- (i) If $\Gamma \vdash e \Leftarrow A \dashv \Delta$ then $[\Omega]\Delta \vdash e \Leftarrow [\Omega]A$.
- (ii) If $\Gamma \vdash e \Rightarrow A \dashv \Delta$ then $[\Omega] \Delta \vdash e \Rightarrow [\Omega] A$.
- (iii) If $\Gamma \vdash A \bullet e \Rightarrow C \dashv \Delta$ then $[\Omega]\Delta \vdash [\Omega]A \bullet e \Rightarrow [\Omega]C$.

K Completeness of Subtyping

K.1 Instantiation Completeness

Theorem 13 (Instantiation Completeness). Given $\Gamma \longrightarrow \Omega$ and $A = [\Gamma]A$ and $\hat{\alpha} \in \text{unsolved}(\Gamma)$ and $\hat{\alpha} \notin FV(A)$:

- (1) If $[\Omega]\Gamma \vdash [\Omega] \hat{\alpha} \leq [\Omega]A$ then there are Δ , Ω' such that $\Omega \longrightarrow \Omega'$ and $\Delta \longrightarrow \Omega'$ and $\Gamma \vdash \hat{\alpha} : \leq A \dashv \Delta$.
- (2) If $[\Omega]\Gamma \vdash [\Omega]A \leq [\Omega]\hat{\alpha}$ then there are Δ , Ω' such that $\Omega \longrightarrow \Omega'$ and $\Delta \longrightarrow \Omega'$ and $\Gamma \vdash A \stackrel{\leq}{=}: \hat{\alpha} \dashv \Delta$.

K.2 Completeness of Subtyping

Theorem 14 (Generalized Completeness of Subtyping). If $\Gamma \longrightarrow \Omega$ and $\Gamma \vdash A$ and $\Gamma \vdash B$ and $[\Omega]\Gamma \vdash [\Omega]A \leq [\Omega]B$ then there exist Δ and Ω' such that $\Delta \longrightarrow \Omega'$ and $\Omega \longrightarrow \Omega'$ and $\Gamma \vdash [\Gamma]A <: [\Gamma]B \dashv \Delta$.

Corollary 55 (Completeness of Subtyping). *If* $\Psi \vdash A \leq B$ *then there is a* Δ *such that* $\Psi \vdash A \leq B \dashv \Delta$.

L Completeness of Typing

Theorem 15 (Completeness of Algorithmic Typing). *Given* $\Gamma \longrightarrow \Omega$ *and* $\Gamma \vdash A$:

- (i) If $[\Omega]\Gamma \vdash e \Leftarrow [\Omega]A$ then there exist Δ and Ω' such that $\Delta \longrightarrow \Omega'$ and $\Omega \longrightarrow \Omega'$ and $\Gamma \vdash e \Leftarrow [\Gamma]A \dashv \Delta$.
- (ii) If $[\Omega]\Gamma \vdash e \Rightarrow A$ then there exist Δ , Ω' , and A' such that $\Delta \longrightarrow \Omega'$ and $\Omega \longrightarrow \Omega'$ and $\Gamma \vdash e \Rightarrow A' \dashv \Delta$ and $A = [\Omega']A'$.
- (iii) If $[\Omega]\Gamma \vdash [\Omega]A \bullet e \Rightarrow C$ then there exist Δ , Ω' , and C'such that $\Delta \longrightarrow \Omega'$ and $\Omega \longrightarrow \Omega'$ and $\Gamma \vdash [\Gamma]A \bullet e \Rightarrow C' \dashv \Delta$ and $C = [\Omega']C'$.

Proofs

In the rest of this document, we prove the results stated above, with the same sectioning.

A' Declarative Subtyping

Proposition 1 (Weakening). If $\Psi \vdash A$ then $\Psi, \Psi' \vdash A$ by a derivation of the same size.

Proposition 2 (Substitution). *If* $\Psi \vdash A$ *and* $\Psi, \alpha, \Psi' \vdash B$ *then* $\Psi, \Psi' \vdash [A/\alpha]B$.

The proofs of these two propositions are routine inductions.

A'.1 Properties of Well-Formedness

A'.2 Reflexivity

Lemma 3 (Reflexivity of Declarative Subtyping). *Subtyping is reflexive:* if $\Psi \vdash A$ then $\Psi \vdash A \leq A$.

Proof. By induction on A.

- Case A = 1: Apply rule \leq Unit.
- Case $A = \alpha$: Apply rule $\leq Var$.
- Case $A = A_1 \to A_2$:

$$\begin{array}{ll} \Psi \vdash A_1 \leq A_1 & \text{By i.h.} \\ \Psi \vdash A_2 \leq A_2 & \text{By i.h.} \\ \Psi \vdash A_1 \rightarrow A_2 \leq A_1 \rightarrow A_2 & \text{By } \leq \rightarrow \end{array}$$

• Case $A = \forall \alpha$. A_0 :

$$\begin{array}{ll} \Psi, \alpha \vdash A_0 \leq A_0 & \text{By i.h.} \\ \Psi, \alpha \vdash \alpha & \text{By DeclUvarWF} \\ \Psi, \alpha \vdash [\alpha/\alpha]A_0 \leq A_0 & \text{By def. of substitution} \\ \Psi, \alpha \vdash \forall \alpha. \ A_0 \leq A_0 & \text{By } \leq \forall L \\ \Psi \vdash \forall \alpha. \ A_0 \leq \forall \alpha. \ A_0 & \text{By } \leq \forall R \end{array}$$

A'.3 Subtyping Implies Well-Formedness

Lemma 4 (Well-Formedness). *If* $\Psi \vdash A \leq B$ *then* $\Psi \vdash A$ *and* $\Psi \vdash B$.

Proof. By induction on the given derivation. All 5 cases are straightforward.

A'.4 Substitution

Lemma 5 (Substitution). If $\Psi \vdash \tau$ and Ψ , α , $\Psi' \vdash A \leq B$ then Ψ , $[\tau/\alpha]\Psi' \vdash [\tau/\alpha]A \leq [\tau/\alpha]B$.

Proof. By induction on the given derivation.

$$\begin{tabular}{ll} \bullet & \textbf{Case} \\ \hline & \frac{\beta \in (\Psi,\alpha,\Psi')}{\Psi,\alpha,\Psi' \vdash \ \beta \leq \beta} \leq \!\!\! \mathsf{Var} \\ \hline \end{tabular}$$

It is given that $\Psi \vdash \tau$.

Either $\beta=\alpha$ or $\beta\neq\alpha$. In the former case: We need to show $\Psi,\Psi'\vdash [\tau/\alpha]\alpha\leq [\tau/\alpha]\alpha$, that is, $\Psi,\Psi'\vdash \tau\leq\tau$, which follows by Lemma 3 (Reflexivity of Declarative Subtyping). In the latter case: We need to show $\Psi,\Psi'\vdash [\tau/\alpha]\beta\leq [\tau/\alpha]\beta$, that is, $\Psi,\Psi'\vdash \beta\leq\beta$. Since $\beta\in(\Psi,\alpha,\Psi')$ and $\beta\neq\alpha$, we have $\beta\in(\Psi,\Psi')$, so applying \leq Var gives the result.

Case

$$\frac{}{\Psi,\alpha,\Psi'\vdash 1<1}\leq Unit$$

For all τ , substituting $[\tau/\alpha]1=1$, and applying \leq Unit gives the result.

$$\bullet \ \, \textbf{Case} \ \, \frac{\Psi, \alpha, \Psi' \vdash \ \, B_1 \leq A_1 \qquad \Psi, \alpha, \Psi' \vdash \ \, A_2 \leq B_2}{\Psi, \alpha, \Psi' \vdash \ \, A_1 \rightarrow A_2 \leq B_1 \rightarrow B_2} \leq \rightarrow$$

$$\Psi, \alpha, \Psi' \vdash B_1 \le A_1$$
 Subderivation $\Psi, \Psi' \vdash [\tau/\alpha]B_1 < [\tau/\alpha]A_1$ By i.h.

$$\Psi, \alpha, \Psi' \vdash A_2 \leq B_2$$
 Subderivation

$$\Psi, \Psi' \vdash [\tau/\alpha] A_2 \leq [\tau/\alpha] B_2 \qquad \qquad \text{By i.h.}$$

$$\begin{array}{ll} \Psi, \Psi' \vdash ([\tau/\alpha]A_1) \to ([\tau/\alpha]A_2) \leq ([\tau/\alpha]B_1) \to ([\tau/\alpha]B_2) & \text{By} \leq \to \\ \Psi, \Psi' \vdash [\tau/\alpha](A_1 \to A_2) \leq [\tau/\alpha](B_1 \to B_2) & \text{By definition of subst.} \end{array}$$

$$\bullet \ \, \textbf{Case} \ \, \frac{\Psi, \alpha, \Psi' \vdash \sigma \quad \Psi, \alpha, \Psi' \vdash [\sigma/\beta] A_0 \leq B}{\Psi, \alpha, \Psi' \vdash \forall \beta. \, A_0 \leq B} \leq \forall \mathsf{L}$$

$$\Psi, \alpha, \Psi' \vdash [\sigma/\beta] A_0 \le B$$
 Subderivation

$$\Psi, \Psi' \vdash [\tau/\alpha][\sigma/\beta] A_0 \leq [\tau/\alpha] B \qquad \qquad \text{By i.h.}$$

$$\Psi, \Psi' \vdash \big[[\tau/\alpha] \sigma / \beta \big] [\tau/\alpha] A_0 \leq [\tau/\alpha] B \quad \text{By distributivity of substitution}$$

$$\begin{array}{ccc} \Psi, \alpha, \Psi' \vdash \sigma & & \text{Premise} \\ \Psi \vdash \tau & & & \text{Given} \end{array}$$

$$\Psi, \Psi' \vdash [\tau/\alpha]\sigma$$
 By Proposition 2

$$\Psi, \Psi' \vdash \forall \beta. [\tau/\alpha] A_0 \le [\tau/\alpha] B$$
 By $\le \forall L$

$$\Psi, \Psi' \vdash [\tau/\alpha] \ \big(\forall \beta. \ A_0 \big) \le [\tau/\alpha] B \qquad \qquad \text{By definition of substitution}$$

• Case
$$\frac{\Psi, \alpha, \Psi', \beta \vdash A \leq B_0}{\Psi, \alpha, \Psi' \vdash A \leq \forall \beta. \ B_0} \leq \forall R$$

$$\begin{array}{ll} \Psi,\alpha,\Psi',\beta\vdash A\leq B_0 & \text{Subderivation}\\ \Psi,\Psi',\beta\vdash [\tau/\alpha]A\leq [\tau/\alpha]B_0 & \text{By i.h.}\\ \Psi,\Psi'\vdash [\tau/\alpha]A\leq \forall\beta.\, [\tau/\alpha]B_0 & \text{By}\leq \forall R \end{array}$$

$$\begin{array}{ll} \Psi, \Psi' \vdash [\tau/\alpha] A \leq \forall \beta. \, [\tau/\alpha] B_0 & \text{By } \leq \forall R \\ \Psi, \Psi' \vdash [\tau/\alpha] A \leq [\tau/\alpha] (\forall \beta. \, B_0) & \text{By definition of substitution} \end{array}$$

A'.5 Transitivity

To prove transitivity, we use a metric that adapts ideas from a proof of cut elimination by Pfenning (1995).

Lemma 6 (Transitivity of Declarative Subtyping). If $\Psi \vdash A \leq B$ and $\Psi \vdash B \leq C$ then $\Psi \vdash A \leq C$.

Proof. By induction with the following metric:

$$\langle \# \forall (B), \mathcal{D}_1 + \mathcal{D}_2 \rangle$$

where $\langle ... \rangle$ denotes lexicographic order, the first part $\#\forall (B)$ is the number of quantifiers in B, and the second part is the (simultaneous) size of the derivations $\mathcal{D}_1 :: \Psi \vdash A \leq B$ and $\mathcal{D}_2 :: \Psi \vdash B \leq C$. We need to consider the number of quantifiers first in one case: when $\leq \forall R$ concluded \mathcal{D}_1 and $\leq \forall L$ concluded \mathcal{D}_2 , because in that case, the derivations on which the i.h. must be applied are not necessarily smaller.

$$\bullet \ \, \textbf{Case} \ \, \frac{\alpha \in \Psi}{\Psi \vdash \ \, \alpha \leq \alpha} \leq \! \text{Var} \quad \, \frac{\alpha \in \Psi}{\Psi \vdash \ \, \alpha \leq \alpha} \leq \! \text{Var}$$

Apply rule $\leq Var$.

• Case \leq Unit $/ \leq$ Unit: Similar to the \leq Var $/ \leq$ Var case.

By i.h. on the 3rd and 1st subderivations, $\Psi \vdash C_1 < A_1$.

By i.h. on the 2nd and 4th subderivations, $\Psi \vdash A_2 \leq C_2$.

By
$$\leq \rightarrow$$
, $\Psi \vdash A_1 \rightarrow A_2 \leq C_1 \rightarrow C_2$.

If $\leq \forall L$ concluded \mathcal{D}_1 :

$$\begin{array}{cccc} \bullet & \textbf{Case} & \underline{\Psi \vdash \tau} & \underline{\Psi \vdash [\tau/\alpha]A_0 \leq B} \\ & \underline{\Psi \vdash \tau} & \underline{\Psi \vdash [\tau/\alpha]A_0 \leq B} \\ & \underline{\Psi \vdash \tau} & \underline{Premise} \\ & \underline{\Psi \vdash [\tau/\alpha]A_0 \leq B} & \underline{Subderivation} \\ & \underline{\Psi \vdash B \leq C} & \underline{Given} \ (\mathcal{D}_2) \\ & \underline{\Psi \vdash [\tau/\alpha]A_0 \leq C} & \underline{By} \ i.h. \\ & \underline{\blacksquare} & \underline{\Psi \vdash \forall \alpha. \ A_0 < C} & \underline{By} \ \forall L \\ \end{array}$$

If $\forall R$ concluded \mathcal{D}_2 :

$$\begin{array}{ll} \bullet \ \, \textbf{Case} & \frac{\Psi,\,\beta \vdash \, B \leq C}{\Psi \vdash \, B \leq \forall \beta . \, C} \leq \forall R \\ & \qquad \qquad \Psi \vdash \tau \qquad \qquad \text{Premise} \\ \Psi,\,\beta \vdash \, B \leq C \qquad \qquad \text{Subderivation} \\ \Psi \vdash \, A \leq B \qquad \qquad \text{Given } (\mathcal{D}_1) \\ \Psi,\,\beta \vdash \, A \leq B \qquad \qquad \text{By Proposition 1} \\ \Psi,\,\beta \vdash \, A \leq C \qquad \qquad \text{By i.h.} \\ \blacksquare \\ \blacksquare \\ \Psi \vdash \, A \leq \forall \beta . \, C \qquad \text{By } \leq \forall L \\ \end{array}$$

The only remaining possible case is $\leq \forall R / \leq \forall L$.

$$\begin{array}{lll} \bullet \ \textbf{Case} & \dfrac{\Psi, \, \beta \vdash A \leq B_0}{\Psi \vdash A \leq \forall \beta. \, B_0} \leq \forall R & \dfrac{\Psi \vdash \tau \quad \Psi \vdash [\tau/\beta] B_0 \leq C}{\Psi \vdash \forall \beta. \, B_0 \leq C} \leq \forall L \\ & \dfrac{\Psi, \, \beta \vdash A \leq B_0}{\Psi \vdash \tau} & \textrm{Subderivation of } \mathcal{D}_1 \\ & \dfrac{\Psi \vdash \tau}{\Psi \vdash [\tau/\beta] A} \leq [\tau/\beta] B_0 & \textrm{By Lemma 5 (Substitution)} \\ & [\tau/\beta] A = A & \beta \ cannot \ appear \ in \ A \\ & \dfrac{\Psi \vdash A \leq [\tau/\beta] B_0}{\Psi \vdash [\tau/\beta] B_0 \leq C} & \textrm{Subderivation of } \mathcal{D}_2 \\ & \dfrac{\Psi \vdash [\tau/\beta] B_0 \leq C}{\Psi \vdash A \leq C} & \textrm{Subderivation of } \mathcal{D}_2 \\ & \dfrac{\Psi \vdash A \leq C}{\Psi \vdash A \leq C} & \textrm{By i.h. (one less } \forall \ quantifier \ in \ B) \\ \end{array}$$

A'.6 Invertibility of $\leq \forall R$

Lemma 7 (Invertibility).

If \mathcal{D} derives $\Psi \vdash A \leq \forall \beta$. B then \mathcal{D}' derives $\Psi, \beta \vdash A \leq B$ where $\mathcal{D}' < \mathcal{D}$.

Proof. By induction on the given derivation \mathcal{D} .

• Cases \leq Var, \leq Unit, \leq \rightarrow : Impossible: the supertype cannot have the form $\forall \beta$. B.

$$\bullet \ \, \textbf{Case} \ \, \frac{\Psi, \beta \vdash A \leq B}{\Psi \vdash A \leq \forall \beta . \, B} \leq \forall \mathsf{R}$$

The subderivation is exactly what we need, and is strictly smaller than \mathcal{D} .

Case

$$\frac{\mathcal{D}_0}{\Psi \vdash \tau \qquad \Psi \vdash [\tau/\alpha] A_0 \leq \forall \beta. B} \leq \forall L$$

$$\frac{\Psi \vdash \forall \alpha. A_0 < \forall \beta. B}{\Psi \vdash \forall \alpha. A_0 \leq \forall \beta. B} \leq \forall L$$

By i.h., \mathcal{D}_0' derives $\Psi, \beta \vdash [\tau/\alpha] A_0 \leq B$ where $\mathcal{D}_0' < \mathcal{D}_0$. By $\leq \forall L, \mathcal{D}'$ derives $\Psi, \beta \vdash \forall \alpha. A_0 \leq B$; since $\mathcal{D}_0' < \mathcal{D}_0$, we have $\mathcal{D}' < \mathcal{D}$.

A'.7 Non-Circularity and Equality

Lemma 8 (Occurrence).

(i) If
$$\Psi \vdash A \leq \tau$$
 then $\tau \not\subset A$.

(ii) If
$$\Psi \vdash \tau \leq B$$
 then $\tau \not \supseteq B$.

Proof. By induction on the given derivation.

• Cases ≤Var, ≤Unit: (i), (ii): Here A and B have no subterms at all, so the result is immediate.

$$\bullet \ \, \textbf{Case} \ \, \frac{\Psi \vdash \ \, B_1 \leq A_1 \qquad \Psi \vdash \ \, A_2 \leq B_2}{\Psi \vdash \ \, A_1 \longrightarrow A_2 \leq B_1 \longrightarrow B_2} \leq \rightarrow$$

(i) Here, $A = A_1 \rightarrow A_2$ and $\tau = B_1 \rightarrow B_2$.

$$B_1 \not\subset A_1$$
 By i.h. (ii)

 $B_1 \to B_2 \not\preceq A_1$ Suppose $B_1 \to B_2 \preceq A_1$. Then $B_1 \rightrightarrows A_1$: contradiction.

$$B_2 \not \subset A_2$$
 By i.h. (i)

$$B_1 \to B_2 \not \preceq A_2 \quad \text{Similar}$$

Suppose (for a contradiction) that $B_1 \to B_2 \rightleftarrows A_1 \to A_2$.

Now
$$B_1 \rightarrow B_2 \leq A_1$$
 or $B_1 \rightarrow B_2 \leq A_2$.

But above, we showed that both were false: contradiction.

Therefore,
$$B_1 \rightarrow B_2 \not\prec A_1 \rightarrow A_2$$

Therefore,
$$B_1 \rightarrow B_2 \not\prec A_1 \rightarrow A_2$$
. Therefore, $B_1 \rightarrow B_2 \not \subset A_1 \rightarrow A_2$.

(ii) Here, $A = \tau$ and $B = B_1 \rightarrow B_2$.

Symmetric to the previous case.

$$\begin{tabular}{ll} \bullet \ \ \pmb{\mathsf{Case}} \\ \hline & \frac{\Psi \vdash \ \tau' \qquad \Psi \vdash \ [\tau'/\alpha] A_0 \leq \tau}{\Psi \vdash \ \, \forall \alpha. \ \, A_0 \leq \tau} \leq \forall \mathsf{L} \\ \hline \end{aligned}$$

In part (ii), this case cannot arise, so we prove part (i).

By i.h. (i),
$$\tau \not \supseteq [\tau'/\alpha] A_0$$
.

It follows from the definition of \vec{z} that $\tau \not \vec{z} \forall \alpha$. A_0 .

$$\bullet \ \, \textbf{Case} \ \, \frac{\Psi,\, \beta \vdash \, \tau \leq B_0}{\Psi \vdash \, \tau \leq \forall \beta. \; B_0} \leq \forall \mathsf{R}$$

In part (i), this case cannot arise, so we prove part (ii).

Similar to the $\leq \forall L$ case.

Lemma 9 (Monotype Equality). *If* $\Psi \vdash \sigma \leq \tau$ *then* $\sigma = \tau$.

Proof. By induction on the given derivation.

• **Case** ≤ Var: Immediate.

• Case ≤Unit: Immediate.

$$\bullet \ \, \textbf{Case} \ \, \frac{\Psi \vdash \ \, B_1 \leq A_1 \qquad \Psi \vdash \ \, A_2 \leq B_2}{\Psi \vdash \ \, A_1 \longrightarrow A_2 \leq B_1 \longrightarrow B_2} \leq \rightarrow$$

By i.h. on each subderivation, $B_1 = A_1$ and $A_2 = B_2$. Therefore $A_1 \rightarrow A_2 = B_1 \rightarrow B_2$.

- Case $\leq \forall L$: Here $\sigma = \forall \alpha$. A_0 , which is not a monotype, so this case is impossible.
- Case $\leq \forall R$: Here $\tau = \forall \beta$. B₀, which is not a monotype, so this case is impossible.

B' Type Assignment

Lemma 10 (Well-Formedness).

If $\Psi \vdash e \Leftarrow A$ or $\Psi \vdash e \Rightarrow A$ or $\Psi \vdash A \bullet e \Rightarrow C$ then $\Psi \vdash A$ (and in the last case, $\Psi \vdash C$).

Proof. By induction on the given derivation.

In all cases, we apply the induction hypothesis to all subderivations.

• In the DeclVar and Decl→l cases, we use our standard assumption that every context appearing in a derivation is well-formed.

- In the Decl \rightarrow l \Rightarrow case, we use inversion on the $\Psi \vdash \sigma \rightarrow \tau$ premise.
- In the Decl \forall App case, we use the property that if $\Psi \vdash [\tau/\alpha]A_0$ then $\Psi \vdash \forall \alpha. A_0$.
- In the DeclAnno case, we use its premise.

Theorem 1 (Completeness of Bidirectional Typing).

If $\Psi \vdash e : A$ then there exists e' such that $\Psi \vdash e' \Rightarrow A$ and |e'| = e.

Proof. By induction on the derivation of $\Psi \vdash e : A$.

$$\bullet \ \ \, \textbf{Case} \ \ \, \frac{x:A\in \Psi}{\Psi \vdash x:A} \ \, \textbf{AVar}$$

Immediate, by rule DeclVar.

• Case
$$\frac{\Psi, \chi: A \vdash e: B}{\Psi \vdash \lambda x. \, e: A \longrightarrow B} \ A {\longrightarrow} I$$

By inversion, we have $\Psi, \chi : A \vdash e : B$.

By induction, we have $\Psi, x : A \vdash e' \Rightarrow B$, where |e'| = e.

By Lemma 3 (Reflexivity of Declarative Subtyping), $\Psi \vdash B \leq B$.

By rule DeclSub, $\Psi, x : A \vdash e' \Leftarrow B$.

By rule Decl \rightarrow I, $\Psi \vdash \lambda x. e' \Leftarrow A \rightarrow B$.

By Lemma 10 (Well-Formedness), $\Psi \vdash A \rightarrow B$.

By rule DeclAnno, $\Psi \vdash ((\lambda x. e') : A \rightarrow B) \Rightarrow A \rightarrow B$.

By definition, $|((\lambda x. e') : A \rightarrow B)| = |\lambda x. e'| = \lambda x. |e'| = \lambda x. e$.

$$\bullet \ \, \textbf{Case} \ \, \frac{\Psi \vdash e_1 : A \to B \qquad \Psi \vdash e_2 : A}{\Psi \vdash e_1 \; e_2 : B} \; \mathsf{A} \!\to\! \mathsf{E}$$

By induction, $\Psi \vdash e_1' \Rightarrow A \rightarrow B$ and $|e_1'| = e_1$. By induction, $\Psi \vdash e_2' \Rightarrow A$ and $|e_2'| = e_2$.

By Lemma 3 (Reflexivity of Declarative Subtyping), $\Psi \vdash A \leq A$.

By rule DeclSub, $\Psi \vdash e_2' \Leftarrow A$. By rule Decl \rightarrow App, $\Psi \vdash A \rightarrow B \bullet e_2' \Longrightarrow B$.

By rule $Decl \rightarrow E$, $\Psi \vdash e_1' e_2' \Rightarrow B$. By definition, $|e_1' e_2'| = |e_1'| |e_2'| = e_1 e_2$.

• Case $\frac{\Psi, \alpha \vdash e : A}{\Psi \vdash e : \forall \alpha. \ A} \ \mathsf{A} \forall \mathsf{I}$

By induction, Ψ , $\alpha \vdash e' \Rightarrow A$ where |e'| = e.

By Lemma 3 (Reflexivity of Declarative Subtyping), Ψ , $\alpha \vdash A \leq A$.

By rule DeclSub, Ψ , $\alpha \vdash e' \Leftarrow A$.

By rule Decl \forall I, $\Psi \vdash e' \Leftarrow \forall \alpha$. A.

By Lemma 10 (Well-Formedness), $\Psi \vdash \forall \alpha$. A.

By rule DeclAnno, $\Psi \vdash (e' : \forall \alpha. A) \Rightarrow \forall \alpha. A$.

By definition, $|e': \forall \alpha. A| = |e'| = e$.

$$\bullet \ \, \textbf{Case} \ \, \frac{\Psi \vdash e : \forall \alpha. \ A \qquad \Psi \vdash \tau}{\Psi \vdash e : [\tau/\alpha]A} \ \, \mathsf{A} \forall \mathsf{E}$$

By induction, $\Psi \vdash e' \Rightarrow \forall \alpha$. A where |e'| = e.

By Lemma 3 (Reflexivity of Declarative Subtyping), $\Psi \vdash [\tau/\alpha]A \leq [\tau/\alpha]A$.

By $\leq \forall L$, $\Psi \vdash \forall \alpha$. $A \leq [\tau/\alpha]A$.

By rule DeclSub, $\Psi \vdash e' \leftarrow [\tau/\alpha]A$.

By Lemma 10 (Well-Formedness), $\Psi \vdash [\tau/\alpha]A$.

By rule DeclAnno, $\Psi \vdash (e' : [\tau/\alpha]A) \Leftarrow [\tau/\alpha]A$.

By definition, $|e': [\tau/\alpha]A| = |e'| = e$.

Lemma 11 (Subtyping Coercion). If $\Psi \vdash A \leq B$ then there exists f which is $\beta \eta$ -equal to the identity such that $\Psi \vdash f : A \rightarrow B$.

Proof. By induction on the derivation of $\Psi \vdash A \leq B$.

$$\bullet \ \ \, \text{Case} \ \ \, \frac{\alpha \in \Psi}{\Psi \vdash \ \, \alpha \leq \alpha} \leq \text{Var}$$

Choose $f = \lambda x. x$.

Clearly $\Psi \vdash \lambda x. x : \alpha \rightarrow \alpha$.

Case

$$\frac{}{\Psi \vdash 1 < 1} \le Unit$$

Choose $f = \lambda x \cdot x$.

Clearly $\Psi \vdash \lambda x. x : 1 \rightarrow 1$.

$$\bullet \ \, \textbf{Case} \ \, \frac{\Psi \vdash \ \, B_1 \leq A_1 \qquad \Psi \vdash \ \, A_2 \leq B_2}{\Psi \vdash \ \, A_1 \rightarrow A_2 \leq B_1 \rightarrow B_2} \leq \rightarrow$$

By induction, we have $g:B_1\to A_1,$ which is $\beta\eta\text{-equal}$ to the identity.

By induction, we have $k: A_2 \to B_2$, which is $\beta\eta$ -equal to the identity.

Let f be $\lambda h. k \circ h \circ g$.

It is easy to verify that $\Psi \vdash f : (A_1 \rightarrow A_2) \rightarrow (B_1 \rightarrow B_2)$.

Since k and g are identities, $f =_{\beta \eta} \lambda h$. h.

$$\bullet \ \, \textbf{Case} \ \, \frac{\Psi \vdash \tau \qquad \Psi \vdash [\tau/\alpha]A \leq B}{\Psi \vdash \forall \alpha.\, A \leq B} \leq \forall \textbf{L}$$

By induction, $g: [\tau/\alpha]A \to B$.

Let $f \triangleq \lambda x$. g x.

f is an eta-expansion of g, which is $\beta\eta$ -equal to the identity. Hence f is too.

Also, $\lambda x. g \ x : (\forall \alpha. A) \rightarrow B$, using the Decl $\forall E$ rule on x.

$$\begin{array}{c} \bullet \ \, \textbf{Case} \\ \frac{\Psi,\, \beta \vdash \, A \, \leq \, B}{\Psi \vdash \, A \, \leq \, \forall \beta . \, B} \leq \forall \mathsf{R} \end{array}$$

By induction, we have g such that Ψ , $\beta \vdash g : A \rightarrow B$.

Let $f \triangleq \lambda x$. g x.

Use the following derivation:

WEAKEN
$$\frac{\vdots}{\Psi, \beta \vdash g : A \to B}$$

$$\frac{\Psi, x : A, \beta \vdash g : A \to B}{\Psi, x : A, \beta \vdash g : A \to B} \qquad \Psi, x : A, \beta \vdash x : A$$

$$\frac{\Psi, x : A, \beta \vdash g : X : B}{\Psi, x : A \vdash g : X : \forall \beta . B}$$

$$\frac{\Psi, x : A \vdash g : X : A \to \forall \beta . B}{\Psi \vdash \lambda x . g : X : A \to \forall \beta . B}$$

Lemma 12 (Application Subtyping). If $\Psi \vdash A \bullet e \Rightarrow C$ then there exists B such that $\Psi \vdash A \leq B \rightarrow C$ and $\Psi \vdash e \Leftarrow B$ by a smaller derivation.

Proof. By induction on the given derivation \mathcal{D} .

$$\begin{array}{c} \bullet \text{ Case} & \underline{\Psi \vdash e \Leftarrow B} \\ \hline \Psi \vdash B \to C \bullet e \Longrightarrow C \end{array} \text{ Decl} \to \mathsf{App} \\ \blacksquare & \mathcal{D}' :: \Psi \vdash e \Leftarrow B \qquad \qquad \text{Subderivation} \\ \blacksquare & \mathcal{D}' < \mathcal{D} \qquad \qquad \mathcal{D}' \text{ is a subderivation of } \mathcal{D} \\ \blacksquare & \Psi \vdash \underline{B \to C} \leq B \to C \qquad \text{By Lemma 3 (Reflexivity of Declarative Subtyping)} \end{array}$$

$$\begin{array}{lll} \bullet \ \, \textbf{Case} & \underline{\Psi \vdash \tau} & \underline{\Psi \vdash [\tau/\alpha]A_0 \bullet e \Rightarrow C} \\ & \underline{\Psi \vdash \forall \alpha. A_0 \bullet e \Rightarrow C} \\ & \underline{\Psi \vdash \tau} & \text{Subderivation} \\ & \underline{\Psi \vdash [\tau/\alpha]A_0 \bullet e \Rightarrow C} & \text{Subderivation} \\ & \underline{\Psi \vdash [\tau/\alpha]A_0 \bullet e \Rightarrow C} & \text{Subderivation} \\ & \underline{\Psi \vdash [\tau/\alpha]A_0 \leq B \rightarrow C} & \text{By i.h.} \\ \\ \blacksquare & \mathcal{D}' :: \underline{\Psi \vdash e \Leftarrow B} & \text{"} \\ & \underline{\blacksquare} & \mathcal{D}' < \mathcal{D} & \text{"} \\ & \underline{\blacksquare} & \underline{\Psi \vdash \forall \alpha. A_0 \leq B \rightarrow C} & \text{By } \underline{\leq \forall L} \\ \end{array}$$

Theorem 2 (Soundness of Bidirectional Typing). We have that:

- If $\Psi \vdash e \Leftarrow A$, then there is an e' such that $\Psi \vdash e'$: A and $e' =_{\beta \eta} |e|$.
- If $\Psi \vdash e \Rightarrow A$, then there is an e' such that $\Psi \vdash e' : A$ and $e' =_{\beta \eta} |e|$.

Proof.
 • Case
$$\frac{(x:A) \in \Psi}{\Psi \vdash x \Rightarrow A} \text{ DeclVar}$$

By rule AVar, $\Psi \vdash x : A$.

Note $x =_{\beta \eta} x$.

$$\bullet \ \, \textbf{Case} \ \, \frac{\Psi \vdash \ e \Rightarrow A \qquad \Psi \vdash A \leq B}{\Psi \vdash \ e \Leftarrow B} \ \, \textbf{DeclSub}$$

By induction, $\Psi \vdash e' : A$ and $e' =_{\beta \eta} |e|$.

By Lemma 11 (Subtyping Coercion), $f: A \to B$ such that $f =_{\beta n} id$.

By $A \rightarrow E$, $\Psi \vdash f e' : B$.

Note $f e' =_{\beta \eta} id e' =_{\beta \eta} e' =_{\beta \eta} |e|$.

• Case
$$\frac{\Psi \vdash A \qquad \Psi \vdash e \Leftarrow A}{\Psi \vdash (e : A) \Rightarrow A}$$
 DeclAnno

By induction, $\Psi \vdash e' : A$ such that $e' =_{\beta \eta} |e|$.

Note $e' =_{\beta \eta} |e| = |e : A|$.

Case

$$\frac{}{\Psi \vdash () \leftarrow 1}$$
 Decl1I

By AUnit, $\Psi \vdash$ (): 1.

Note () $=_{\beta\eta}$ ().

Case

$$\frac{}{\Psi \vdash () \Rightarrow 1} \text{ Decl1I} \Rightarrow$$

By AUnit, $\Psi \vdash$ (): 1.

Note () $=_{\beta\eta}$ ().

$$\bullet \ \, \textbf{Case} \ \, \frac{\Psi, \alpha \vdash \, e \Leftarrow A}{\Psi \vdash \, e \Leftarrow \forall \alpha. \, A} \, \, \mathsf{Decl} \forall \mathsf{I}$$

By induction, Ψ , $\alpha \vdash e'$: A such that $e' =_{\beta \eta} |e|$.

By rule $A \forall I, \Psi \vdash e' : \forall \alpha. A$.

$$\begin{array}{c} \bullet \ \, \textbf{Case} \\ \hline \frac{\Psi, \chi : A \vdash e \Leftarrow B}{\Psi \vdash \lambda x. \, e \Leftarrow A \to B} \ \, \textbf{Decl} \to \textbf{I} \end{array}$$

By induction, Ψ , $x : A \vdash e' : B$ such that $e' =_{\beta n} |e|$.

By $A \rightarrow I$, $\Psi \vdash \lambda x$. $e' : A \rightarrow B$.

Note $\lambda x. e' =_{\beta \eta} \lambda x. |e| = |\lambda x. e|$.

$$\bullet \ \, \textbf{Case} \ \, \underbrace{ \begin{array}{ccc} \Psi \vdash \ \sigma \rightarrow \tau & \Psi, x : \sigma \vdash \ e \Leftarrow \tau \\ \hline \Psi \vdash \ \lambda x. \ e \Rightarrow \sigma \rightarrow \tau \end{array} }_{ \ \, \text{Decl} \rightarrow \text{I} \Rightarrow }$$

By induction, $\Psi, x : \sigma \vdash e' : \tau$ such that $e' =_{\beta \eta} |e|$.

By $A \rightarrow I$, $\Psi \vdash \lambda x$. $e' : \sigma \rightarrow \tau$.

Note $\lambda x. e' =_{\beta \eta} \lambda x. |e| = |\lambda x. e|$.

$$\bullet \ \, \textbf{Case} \ \, \underbrace{ \Psi \vdash \, e_1 \Rightarrow A \qquad \Psi \vdash A \bullet e_2 \Rightarrow C}_{ \ \, \Psi \vdash \, e_1 \, e_2 \Rightarrow \, C} \, \, \text{Decl} \rightarrow \text{E}$$

By induction, $\Psi \vdash e'_1$: A such that $e'_1 =_{\beta \eta} |e_1|$.

By Lemma 12 (Application Subtyping), there is a B such that

1. $\Psi \vdash A \leq B \rightarrow C$, and

2. $\Psi \vdash e_2 \Leftarrow B$, which is no bigger than $\Psi \vdash A \bullet e_2 \Rightarrow C$.

By Lemma 11 (Subtyping Coercion), we have f such that $\Psi \vdash f : A \to B \to C$ and $f =_{\beta \eta} id$.

By induction, we get $\Psi \vdash e_2'$: B and $e_2' =_{\beta \eta} |e_2|$.

By $A \rightarrow E$ twice, $\Psi \vdash f e'_1 e'_2 : C$.

Note $f e_1' e_2' =_{\beta \eta} id e_1' e_2' =_{\beta \eta} e_1' e_2' =_{\beta \eta} |e_1| e_2' =_{\beta \eta} |e_1| |e_2| = |e_1| e_2|.$

C' Robustness of Typing

Lemma 13 (Type Substitution).

Assume $\Psi \vdash \tau$.

- If $\Psi, \alpha, \Psi' \vdash e' \Leftarrow C$ then $\Psi, [\tau/\alpha]\Psi' \vdash [\tau/\alpha]e' \Leftarrow [\tau/\alpha]C$.
- If $\Psi, \alpha, \Psi' \vdash e' \Rightarrow C$ then $\Psi, [\tau/\alpha]\Psi' \vdash [\tau/\alpha]e' \Rightarrow [\tau/\alpha]C$.
- If $\Psi, \alpha, \Psi' \vdash B \bullet e' \Rightarrow C$ then $\Psi, [\tau/\alpha]\Psi' \vdash [\tau/\alpha]B \bullet [\tau/\alpha]e' \Rightarrow [A/\alpha]C$.

Moreover, the resulting derivation contains no more applications of typing rules than the given one. (Internal subtyping derivations, however, may grow.)

Proof. By induction on the given derivation.

In the DeclVar case, split on whether the variable being typed is in Ψ or Ψ' ; the former is immediate, and in the latter, use the fact that $(x : C) \in \Psi'$ implies $(x : [\tau/\alpha]C) \in [\tau/\alpha]\Psi'$.

In the DeclSub case, use the i.h. and Lemma 5 (Substitution).

In the DeclAnno case, we are substituting in the annotation in the term, as well as in the type; we also need Proposition 2.

In $\mathsf{Decl} \to \mathsf{I}$, $\mathsf{Decl} \to \mathsf{I} \Rightarrow$ and $\mathsf{Decl} \forall \mathsf{I}$, we add to the context in the premise, which is why the statement is generalized for nonempty Ψ' .

Lemma 14 (Subsumption). *Suppose* $\Psi' \leq \Psi$. *Then:*

- (i) If $\Psi \vdash e \Leftarrow A$ and $\Psi \vdash A < A'$ then $\Psi' \vdash e \Leftarrow A'$.
- (ii) If $\Psi \vdash e \Rightarrow A$ then there exists A' such that $\Psi \vdash A' < A$ and $\Psi' \vdash e \Rightarrow A'$.
- (iii) If $\Psi \vdash C \bullet e \Rightarrow A$ and $\Psi \vdash C' \leq C$ then there exists A' such that $\Psi \vdash A' \leq A$ and $\Psi' \vdash C' \bullet e \Rightarrow A'$.

Proof. By mutual induction: in (i), by lexicographic induction on the derivation of the checking judgment, then of the subtyping judgment; in (ii), by induction on the derivation of the synthesis judgment; in (iii), by lexicographic induction on the derivation of the application judgment, then of the subtyping judgment.

For part (i), checking:

• Case
$$\frac{\Psi \vdash e \Rightarrow B \qquad \Psi \vdash B \leq A}{\Psi \vdash e \Leftarrow A} \text{ DeclSub}$$

$$\Psi \vdash e \Rightarrow B \qquad \text{Subderivation}$$

$$\Psi' \vdash e \Rightarrow B' \qquad \text{By i.h.}$$

$$\Psi \vdash B' \leq B \qquad ''$$

$$\Psi \vdash B \leq A \qquad \text{Subderivation}$$

$$\Psi \vdash A \leq A' \qquad \text{Given}$$

$$\Psi \vdash B' \leq A' \qquad \text{By Lemma 6 (Transitivity of Declarative Subtyping) (twice)}$$

$$\Psi' \vdash B' \leq A' \qquad \text{By weakening}$$

$$\Psi' \vdash e \Leftarrow A' \qquad \text{By DeclSub}$$

Case

$$\begin{array}{ccc} \overline{\Psi \vdash \ () \Leftarrow 1} & \mathsf{Decl1I} \\ \Psi' \vdash \ () \Rightarrow 1 & \mathsf{By} \ \mathsf{Decl1I} \Rightarrow \\ \Psi \vdash 1 \leq \mathsf{A'} & \mathsf{Given} \\ \Psi' \vdash 1 \leq \mathsf{A'} & \mathsf{By} \ \mathsf{weakening} \\ \hline \blacksquare & \Psi' \vdash () \Leftarrow \mathsf{A'} & \mathsf{By} \ \mathsf{DeclSub} \\ \end{array}$$

$$\bullet \ \, \textbf{Case} \ \, \frac{\Psi, \alpha \vdash \, e \Leftarrow A_0}{\Psi \vdash \, e \Leftarrow \forall \alpha.\, A_0} \, \, \textbf{Decl} \forall \textbf{I}$$

We consider cases of $\Psi \vdash \forall \alpha. A_0 \leq A'$:

$$\begin{array}{ll} \textbf{- Case} & \dfrac{\Psi,\,\beta \vdash \, \forall \alpha.\, A_0 \leq B}{\Psi \vdash \, \forall \alpha.\, A_0 \leq \forall \beta.\, B} \leq \forall \mathsf{R} \\ \\ \Psi,\,\beta \vdash \, \forall \alpha.\, A_0 \leq \mathsf{B} & \text{Subderivation} \\ \Psi \vdash e \Leftarrow \forall \alpha.\, A_0 & \text{Given} \\ \Psi' \vdash e \Leftarrow \mathsf{B} & \text{By i.h. (i)} \\ \Psi' \vdash e \Leftarrow \forall \beta.\, \mathsf{B} & \text{By Decl} \forall \mathsf{I} \\ \end{array}$$

- Case
$$\frac{\Psi \vdash \tau \quad \Psi \vdash [\tau/\alpha]A_0 \leq A'}{\Psi \vdash \forall \alpha. \ A_0 \leq A'} \leq \forall L$$

$$\Psi, \alpha \vdash e \Leftarrow A_0 \qquad \text{Subderivation}$$

$$\Psi \vdash e \Leftarrow [\tau/\alpha]A_0 \qquad \text{By Lemma 13 (Type Substitution)}$$

$$\Psi \vdash [\tau/\alpha]A_0 \leq A' \qquad \text{Subderivation}$$

By i.h. (i)

• Case
$$\frac{\Psi, \chi: A_1 \vdash e_0 \Leftarrow A_2}{\Psi \vdash \lambda \chi. e_0 \Leftarrow A_1 \to A_2} \; \mathsf{Decl} \to \mathsf{I}$$

 $\Psi' \vdash e \Leftarrow A'$

We consider cases of $\Psi \vdash A_1 \rightarrow A_2 \leq A'$:

$$\begin{array}{ccccc} \textbf{- Case} & \underline{\Psi \vdash B_1 \leq A_1} & \underline{\Psi \vdash A_2 \leq B_2} \\ & \underline{\Psi \vdash A_1 \rightarrow A_2 \leq B_1 \rightarrow B_2} & \leq \rightarrow \\ & \underline{\Psi \leq \Psi'} & Given \\ & \underline{\Psi \vdash B_1 \leq A_1} & Subderivation \\ & \underline{\Psi', x : B_1 \leq \Psi, x : A_1} & By CtxSubVar \\ & \underline{\Psi', x : B_1 \vdash e_0 \Leftarrow B_2} & By i.h. \ (i) \\ & \underline{\Psi'} \vdash \lambda x. \ e_0 \Leftarrow B_1 \rightarrow B_2 & By \ Decl \rightarrow I \\ \\ \textbf{- Case} & \underline{\Psi, \beta \vdash A_1 \rightarrow A_2 \leq B'} \end{array}$$

$$\begin{array}{ll} \textbf{- Case} & \underline{\Psi, \beta \vdash A_1 \to A_2 \leq B'} \\ \underline{\Psi \vdash A_1 \to A_2 \leq \forall \beta. \, B'} \leq \forall \mathsf{R} \\ & \underline{\Psi, \beta \vdash A_1 \to A_2 \leq B'} & \text{Subderivation} \\ & \underline{\Psi, \beta \vdash \lambda x. \, e_0 \Leftarrow A_1 \to A_2} & \text{By weakening} \\ & \underline{\Psi', \beta \vdash \lambda x. \, e_0 \Leftarrow B'} & \text{By i.h. (i)} \\ & \underline{\blacksquare} & \underline{\Psi' \vdash \lambda x. \, e_0 \Leftarrow \forall \beta. \, B'} & \text{By Decl} \forall \mathsf{I} \\ \end{array}$$

For part (ii), synthesis:

$$\bullet \ \, \textbf{Case} \ \, \frac{(x:A) \in \Psi}{\Psi \vdash x \Rightarrow A} \ \, \textbf{DeclVar}$$

By inversion on $\Psi' \leq \Psi$, we have $(x : A') \in \Psi'$ where $\Psi \vdash A' \leq A$. By DeclVar, $\Psi' \vdash x \Rightarrow A'$.

$$\bullet \ \, \textbf{Case} \ \, \frac{\Psi \vdash A \qquad \Psi \vdash \, e_0 \Leftarrow A}{\Psi \vdash \, (e_0 : A) \Rightarrow A} \, \, \textbf{DeclAnno}$$

• Case
$$\frac{\Psi \vdash e \Leftarrow C_0}{\Psi \vdash C_0 \to A \bullet e \Rightarrow A} \text{ Decl} \to \text{App}$$

$$\Psi \vdash C' \leq C_0 \to A \quad \text{ Given}$$
- Case
$$\frac{\Psi \vdash C_0 \leq C_1' \quad \Psi \vdash C_2' \leq A}{\Psi \vdash C_1' \to C_2' \leq C_0 \to A} \leq \to$$

$$\text{Let } A' = C_2'.$$

$$\Psi \vdash e \Leftarrow C_0 \qquad \text{Subderivation}$$

$$\Psi \vdash C_0 \leq C_1' \qquad \text{Subderivation}$$

$$\Psi' \vdash e \Leftarrow C_1' \qquad \text{By i.h.}$$

$$\Psi' \vdash C_1' \to C_2' \bullet e \Rightarrow C_2' \quad \text{By Decl} \to \text{App}$$

$$\Psi' \vdash C_1' \to A' \bullet e \Rightarrow A' \quad A' = C_2'$$

$$\Psi \vdash C_2' \leq A \quad \text{Subderivation}$$

$$\Psi \vdash A' \leq A \quad A' = C_2'$$
- Case
$$\frac{\Psi \vdash \tau \quad \Psi \vdash [\tau/\beta]B \leq C_0 \to A}{\Psi \vdash [\tau/\beta]B \leq C_0 \to A} \leq \forall L$$

$$\Psi \vdash [\tau/\beta]B \leq C_0 \to A \quad \text{Subderivation}$$

$$\Psi' \vdash [\tau/\beta]B \bullet e \Rightarrow A' \quad \text{By i.h. (iii)}$$

$$\Psi' \vdash A' \leq A \quad H \vdash T \quad \text{Subderivation}$$

$$\Psi' \vdash T \quad \text{Subderivation}$$

Theorem 3 (Substitution).

Assume $\Psi \vdash e \Rightarrow A$.

(i) If
$$\Psi, x : A \vdash e' \Leftarrow C$$
 then $\Psi \vdash [e/x]e' \Leftarrow C$.

(ii) If
$$\Psi, x : A \vdash e' \Rightarrow C$$
 then $\Psi \vdash [e/x]e' \Rightarrow C$.

(iii) If
$$\Psi, x : A \vdash B \bullet e' \Rightarrow C$$
 then $\Psi \vdash B \bullet [e/x]e' \Rightarrow C$.

Proof. By a straightforward mutual induction on the given derivation.

Theorem 4 (Inverse Substitution).

Assume $\Psi \vdash e \Leftarrow A$.

(i) If
$$\Psi \vdash [(e : A)/x]e' \Leftarrow C$$
 then $\Psi, x : A \vdash e' \Leftarrow C$.

(ii) If
$$\Psi \vdash [(e : A)/x]e' \Rightarrow C$$
 then $\Psi, x : A \vdash e' \Rightarrow C$.

(iii) If
$$\Psi \vdash B \bullet [(e : A)/x]e' \Rightarrow C$$
 then $\Psi, x : A \vdash B \bullet e' \Rightarrow C$.

Proof. By mutual induction on the given derivation.

(i) We have $\Psi \vdash [(e : A)/x]e' \Leftarrow C$.

$$\begin{array}{cccc} \bullet & \textbf{Case} & \underline{\Psi \vdash [(e:A)/x]e' \Rightarrow B} & \underline{\Psi \vdash B \leq C} \\ & \underline{\Psi \vdash [(e:A)/x]e' \Leftarrow C} \end{array} \text{ DeclSub} \\ \text{By i.h. (ii), } \Psi, x:A \vdash e' \Rightarrow B. \\ \text{By DeclSub, } \Psi, x:A \vdash e' \Leftarrow C. \end{array}$$

Case

$$\overline{\Psi \vdash \text{ ()} \Leftarrow \underbrace{1}_{C}} \text{ Decl1I}$$

We have [(e:A)/x]e'=(). Therefore e'=(), and the result follows by Decl11.

- $\begin{array}{l} \bullet \ \, \textbf{Case} \\ \frac{\Psi, \alpha \vdash \, [(e:A)/x]e' \Leftarrow C'}{\Psi \vdash \, [(e:A)/x]e' \Leftarrow \forall \alpha.\,C'} \ \, \textbf{Decl} \forall \textbf{I} \\ \end{array}$
 - By i.h. (i), Ψ , α , $x : A \vdash e' \Leftarrow C'$.
 - By exchange, Ψ , x : A, $\alpha \vdash e' \Leftarrow C'$.
 - By Decl \forall I, Ψ , $x : A \vdash e' \Leftarrow \forall \alpha$. C'.
- $\begin{array}{c} \bullet \ \, \textbf{Case} \\ \frac{\Psi, y: C_1 \vdash e'' \Leftarrow C_2}{\Psi \vdash \lambda y. \, e'' \Leftarrow C_1 \rightarrow C_2} \ \, \textbf{Decl} \rightarrow \textbf{I} \end{array}$
 - We have $[(e : A)/x]e' = \lambda y. e''$.

By the definition of substitution, $e' = \lambda y$. e_0 and $e'' = [(e : A)/x]e_0$.

$$\begin{array}{c} \Psi,y:C_1\vdash e'' \Leftarrow C_2 & \text{Subderivation} \\ \Psi,y:C_1\vdash [(e:A)/x]e_0 \Leftarrow C_2 & \text{By above equality} \\ \Psi,y:C_1,x:A\vdash e_0 \Leftarrow C_2 & \text{By i.h. (i)} \\ \Psi,x:A,y:C_1\vdash e_0 \Leftarrow C_2 & \text{By exchange} \\ \Psi,x:A\vdash \underbrace{\lambda y.e_0}_{e'} \Leftarrow \underbrace{C_1\to C_2}_{C} & \text{By Decl}\to I \end{array}$$

- (ii) We have $\Psi \vdash [(e : A)/x]e' \Rightarrow C$.
 - Case e' = x:
 - Note [(e : A)/x]x = (e : A).
 - Hence $\Psi \vdash (e : A) \Rightarrow C$; by inversion, C = A.
 - By Lemma 10 (Well-Formedness), $\Psi \vdash C$, which is $\Psi \vdash A$.
 - By DeclAnno, $\Psi \vdash (e : A) \Rightarrow A$.
 - By DeclVar, $\Psi, x : A \vdash \underbrace{x}_{e'} \Rightarrow A$.
 - Case $e' \neq x$:

We now proceed by cases on the derivation of $\Psi \vdash [(e : A)/x]e' \Rightarrow C$.

- Case $\frac{(y:C) \in \Psi}{\Psi \vdash y \Rightarrow C} \; \mathsf{DeclVar}$

Since [(e:A)/x]e' = y, we know that e' = y.

By DeclVar, Ψ , $x : A \vdash y \Rightarrow C$.

- Case

$$\frac{\Psi \vdash e'' \Leftarrow C}{\Psi \vdash \underbrace{(e'' : C)}_{[(e:A)/x]e'} \Rightarrow C} \text{ DeclAnno}$$

We know [(e:A)/x]e' = (e'':C) and $e' \neq x$.

Hence there is e_0 such that $e' = (e_0 : C)$ and $[(e : A)/x]e_0 = e''$.

$$\begin{array}{cccc} \Psi \vdash e'' \Leftarrow C & \text{Subderivation} \\ \Psi \vdash [(e:A)/x]e_0 \Leftarrow C & \text{By above equality} \\ \Psi, x:A \vdash e_0 \Leftarrow C & \text{By i.h. (i)} \\ \Psi, x:A \vdash C & \text{By Lemma 10 (Well-Formedness)} \\ \Psi, x:A \vdash (e_0:C) \Rightarrow C & \text{By DeclAnno} \\ & \Psi, x:A \vdash e' \Rightarrow C & \text{By above equality} \end{array}$$

- Case

$$\frac{}{\Psi \vdash () \Rightarrow 1} \mathsf{Decl}1 \mathrel{\Rightarrow}$$

Since [(e:A)/x]e' = (), it follows that e' = ().

By Decl1I \Rightarrow , Ψ , $x : A \vdash () \Rightarrow 1$.

We have $[(e : A)/x]e' = \lambda y \cdot e''$.

By definition of substitution, there exists e_0 such that $e' = \lambda y$. e_0 and $e'' = [(e : A)/x]e_0$.

So Ψ , $y : \sigma \vdash [(e : A)/x]e_0 \Leftarrow \tau$.

By i.h. (i), Ψ , y : σ , x : $A \vdash e_0 \Leftarrow \tau$.

By exchange and Decl \rightarrow I, Ψ , $x : A \vdash \lambda y$. $e_0 \leftarrow \sigma \rightarrow \tau$.

Hence $Decl \rightarrow l \Rightarrow$, $\Psi, \chi : A \vdash e' \Rightarrow \sigma \rightarrow \tau$.

- Case
$$\Psi \vdash e_1 \Rightarrow B \qquad \Psi \vdash B \bullet e_2 \Rightarrow C$$
 $\Psi \vdash e_1 e_2 \Rightarrow C$ $\Psi \vdash e_1 e_2 \Rightarrow C$ Decl $\rightarrow E$

Note that $[(e : A)/x]e' = e_1 e_2$.

So there exist e_1' , e_2' such that $e'=e_1'$ e_2' and $[(e:A)/x]e_k'=e_k$ for $k\in\{1,2\}$.

Applying these equalities to each subderivation gives

By i.h. (ii) and (iii),
$$\Psi, x : A \vdash e'_1 \Rightarrow B$$
 and $\Psi, x : A \vdash B \bullet e'_2 \Rightarrow C$.
By Decl \rightarrow E, $\Psi, x : A \vdash e'_1 e'_2 \Rightarrow C$, which is $\Psi, x : A \vdash e' \Rightarrow C$.

- (iii) We have $\Psi \vdash [(e : A)/x]e' \bullet A \Rightarrow\Rightarrow C$.

Follows by i.h. (iii) and Decl∀App.

• Case
$$\frac{\Psi \vdash [(e:A)/x]e' \Leftarrow B}{\Psi \vdash B \to C \bullet [(e:A)/x]e' \Rightarrow C} \text{ Decl} \to \mathsf{App}$$

Follows by i.h. (i) and Decl→App.

Theorem 5 (Annotation Removal). We have that:

- If $\Psi \vdash ((\lambda x. e) : A) \Leftarrow C$ then $\Psi \vdash \lambda x. e \Leftarrow C$.
- If $\Psi \vdash (() : A) \Leftarrow C$ then $\Psi \vdash () \Leftarrow C$.
- If $\Psi \vdash e_1 (e_2 : A) \Rightarrow C$ then $\Psi \vdash e_1 e_2 \Rightarrow C$.
- If $\Psi \vdash (x : A) \Rightarrow A$ then $\Psi \vdash x \Rightarrow B$ and $\Psi \vdash B \leq A$.
- If $\Psi \vdash ((e_1 e_2) : A) \Rightarrow A$ then $\Psi \vdash e_1 e_2 \Rightarrow B$ and $\Psi \vdash B \leq A$.
- If $\Psi \vdash ((e : B) : A) \Rightarrow A$ then $\Psi \vdash (e : B) \Rightarrow B$ and $\Psi \vdash B \leq A$.
- If $\Psi \vdash ((\lambda x. e) : \sigma \rightarrow \tau) \Rightarrow \sigma \rightarrow \tau \text{ then } \Psi \vdash \lambda x. e \Rightarrow \sigma \rightarrow \tau.$

Proof. All of these follow directly from inversion and Lemma 14 (Subsumption). The one exception is the third, which additionally requires a small induction on the application judgment. \Box

Theorem 6 (Soundness of Eta).

If $\Psi \vdash \lambda x$. $e \ x \Leftarrow A$ and $x \notin FV(e)$, then $\Psi \vdash e \Leftarrow A$.

Proof. By induction on the derivation of $\Psi \vdash \lambda x$. $e x \Leftarrow A$. There are three non-impossible cases:

$$\begin{array}{c} \bullet \ \, \textbf{Case} \\ \frac{\Psi, x: B \vdash \ e \ x \Leftarrow C}{\Psi \vdash \lambda x. \, e \ x \Leftarrow B \to C} \ \, \textbf{Decl} \to \textbf{I} \end{array}$$

We have $\Psi, x : B \vdash e x \Leftarrow C$.

By inversion on DeclSub, we get $\Psi, x : B \vdash e x \Rightarrow C'$ and $\Psi \vdash C' \leq C$.

By inversion on Decl \rightarrow E, we get $\Psi, \chi : B \vdash e \Rightarrow A'$ and $\Psi, \chi : B \vdash A' \bullet \chi \Rightarrow C'$.

By thinning, we know that $\Psi \vdash e \Rightarrow A'$.

By Lemma 12 (Application Subtyping), we get B' so $\Psi, \chi : B \vdash A' \leq B' \to C'$ and $\Psi, \chi : B \vdash \chi \Leftarrow B'$.

By inversion, we know that $\Psi, x : B \vdash x \Rightarrow B$ and $\Psi \vdash B \leq B'$.

By
$$\leq \rightarrow$$
, Ψ , χ : $B \vdash B' \rightarrow C' \leq B \rightarrow C$.

Hence by Lemma 6 (Transitivity of Declarative Subtyping), $\Psi, x : B \vdash A' \leq B \rightarrow C$.

Hence $\Psi \vdash A' \leq B \rightarrow C$.

By DeclSub, $\Psi \vdash e \Leftarrow B \rightarrow C$.

$\begin{array}{c} \bullet \ \, \textbf{Case} \\ \frac{\Psi,\,\alpha \vdash \, \lambda x.\,e \; x \Leftarrow B}{\Psi \vdash \, \lambda x.\,e \; x \Leftarrow \forall \alpha.\,B} \ \, \textbf{Decl} \forall \textbf{I} \end{array}$

By induction, Ψ , $\alpha \vdash \lambda x$. $e \ x \Leftarrow B$. By $\mathsf{Decl} \forall \mathsf{I}$, $\Psi \vdash \lambda x$. $e \ x \Leftarrow \forall \alpha$. B.

$$\bullet \ \, \textbf{Case} \ \, \frac{\Psi \vdash \ \, \lambda x.\, e \,\, x \Rightarrow B \qquad \Psi \vdash \, B \leq A}{\Psi \vdash \ \, \lambda x.\, e \,\, x \Leftarrow A} \,\, \textbf{DeclSub}$$

We have $\Psi \vdash \lambda x$. $e x \Rightarrow B$ and $\Psi \vdash B \leq A$.

By inversion on Decl \rightarrow I \Rightarrow , Ψ , $x : \sigma \vdash e x \Leftarrow \tau$ and $B = \sigma \rightarrow \tau$.

By inversion on DeclSub, we get $\Psi, x : \sigma \vdash e x \Rightarrow C_2$ and $\Psi \vdash C_2 \leq \tau$.

By inversion on Decl \rightarrow E, we get $\Psi, x : \sigma \vdash e \Rightarrow C$ and $\Psi, x : \sigma \vdash C \bullet x \Rightarrow C_2$.

By thinning, we know that $\Psi \vdash e \Rightarrow C$.

By Lemma 12 (Application Subtyping), we get C_1 such that $\Psi, x : \sigma \vdash C \leq C_1 \rightarrow C_2$ and $\Psi, x : \sigma \vdash x \Leftarrow C_1$.

By inversion on DeclSub, $\Psi, x : \sigma \vdash x \Rightarrow \sigma$ and $\Psi \vdash \sigma \leq C_1$.

By $\leq \rightarrow$, Ψ , $x : \sigma \vdash C_1 \rightarrow C_2 \leq \sigma \rightarrow \tau$.

Hence by Lemma 6 (Transitivity of Declarative Subtyping), $\Psi, \chi : \sigma \vdash C \leq \sigma \rightarrow \tau$.

Hence $\Psi \vdash C \leq \sigma \rightarrow \tau$.

Hence by Lemma 6 (Transitivity of Declarative Subtyping), $\Psi \vdash C \leq A$.

By DeclSub, $\Psi \vdash e \Leftarrow A$.

D' Properties of Context Extension

D'.1 Syntactic Properties

Lemma 15 (Declaration Preservation). *If* $\Gamma \longrightarrow \Delta$, *and* $\mathfrak u$ *is a variable or marker* $\blacktriangleright_{\hat{\alpha}}$ *declared in* Γ , *then* $\mathfrak u$ *is declared in* Δ .

Proof. By a routine induction on $\Gamma \longrightarrow \Delta$.

Lemma 16 (Declaration Order Preservation). *If* $\Gamma \longrightarrow \Delta$ *and* $\mathfrak u$ *is declared to the left of* $\mathfrak v$ *in* Γ , *then* $\mathfrak u$ *is declared to the left of* $\mathfrak v$ *in* Δ .

Proof. By induction on the derivation of $\Gamma \longrightarrow \Delta$.

• Case
$$\xrightarrow{\cdot \longrightarrow \cdot} \longrightarrow ID$$

This case is impossible.

• Case
$$\frac{\Gamma \longrightarrow \Delta}{\Gamma, x: A \longrightarrow \Delta, x: A} \longrightarrow \mathsf{Var}$$

There are two cases, depending on whether or not v = x.

- Case v = x:
 Since u is declared to the left of v, u is declared in Γ.
 By Lemma 15 (Declaration Preservation), u is declared in Δ.
 Hence u is declared to the left of x in Δ, x : A.
- Case v ≠ x:
 Then v is declared in Γ, and u is declared to the left of v in Γ.
 By induction, u is declared to the left of v in Δ.
 Hence u is declared to the left of v in Δ, x : A.

• Case
$$\frac{\Gamma \longrightarrow \Delta}{\Gamma, \alpha \longrightarrow \Delta, \alpha} \longrightarrow \mathsf{Uvar}$$

This case is similar to the \longrightarrow Var case.

$$\bullet \ \ \, \textbf{Case} \ \ \, \frac{\Gamma \longrightarrow \Delta}{\Gamma, \, \widehat{\alpha} \longrightarrow \Delta, \, \widehat{\alpha}} \longrightarrow \textbf{Unsolved}$$

This case is similar to the \longrightarrow Var case.

$$\begin{array}{cccc} \bullet & \textbf{Case} & \underline{\Gamma \longrightarrow \Delta} & [\Delta]\tau = [\Delta]\tau' \\ \hline \Gamma, \hat{\alpha} = \tau \longrightarrow \Delta, \hat{\alpha} = \tau' \end{array} \longrightarrow \\ \textbf{Solved}$$

This case is similar to the \longrightarrow Var case.

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma \longrightarrow \Delta}{\Gamma, \blacktriangleright_{\hat{\alpha}} \longrightarrow \Delta, \blacktriangleright_{\hat{\alpha}}} \longrightarrow \mathsf{Marker}$$

This case is similar to the \longrightarrow Var case.

• Case
$$\frac{\Gamma \longrightarrow \Delta}{\Gamma, \widehat{\alpha} \longrightarrow \Delta, \widehat{\alpha} = \tau} \longrightarrow \mathsf{Solve}$$

This case is similar to the → Var case.

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma \longrightarrow \Delta}{\Gamma \longrightarrow \Delta, \hat{\alpha}} \longrightarrow \! \mathsf{Add}$$

By induction, u is declared to the left of v in Δ . Therefore u is declared to the left of v in Δ , $\hat{\alpha}$.

• Case
$$\frac{\Gamma \longrightarrow \Delta}{\Gamma \longrightarrow \Delta, \ \widehat{\alpha} = \tau} \longrightarrow \mathsf{AddSolved}$$

By induction, $\mathfrak u$ is declared to the left of $\mathfrak v$ in Δ . Therefore $\mathfrak u$ is declared to the left of $\mathfrak v$ in $\Delta, \hat \alpha = \tau$.

Lemma 17 (Reverse Declaration Order Preservation). *If* $\Gamma \longrightarrow \Delta$ *and* $\mathfrak u$ *and* $\mathfrak v$ *are both declared in* Γ *and* $\mathfrak u$ *is declared to the left of* $\mathfrak v$ *in* Δ , *then* $\mathfrak u$ *is declared to the left of* $\mathfrak v$ *in* Γ .

Proof. It is given that $\mathfrak u$ and $\mathfrak v$ are declared in Γ . Either $\mathfrak u$ is declared to the left of $\mathfrak v$ in Γ , or $\mathfrak v$ is declared to the left of $\mathfrak u$. Suppose the latter (for a contradiction). By Lemma 16 (Declaration Order Preservation), $\mathfrak v$ is declared to the left of $\mathfrak u$ in Δ . But we know that $\mathfrak u$ is declared to the left of $\mathfrak v$ in Δ : contradiction. Therefore $\mathfrak u$ is declared to the left of $\mathfrak v$ in Γ .

Lemma 18 (Substitution Extension Invariance). If $\Theta \vdash A$ and $\Theta \longrightarrow \Gamma$ then $[\Gamma]A = [\Gamma]([\Theta]A)$ and $[\Gamma]A = [\Theta]([\Gamma]A)$.

Proof. To show that $[\Gamma]A = [\Theta][\Gamma]A$, observe that $\Theta \vdash A$, and that by definition of $\Theta \longrightarrow \Gamma$, every solved variable in Θ is solved in Γ . Therefore $[\Theta]([\Gamma]A) = [\Gamma]A$, since unsolved($[\Gamma]A$) contains no variables that Θ solves.

To show that $[\Gamma]A = [\Gamma][\Theta]A$, we proceed by induction on $|\Gamma \vdash A|$.

• Case $\alpha \in \Theta$

Note that $[\Gamma]\alpha = \alpha = [\Theta]\alpha$, so $[\Gamma]\alpha = [\Gamma][\Theta]\alpha$.

 $\bullet \ \, \textbf{Case} \ \, \underline{\Theta \vdash A \qquad \Theta \vdash B} \\ \overline{\Theta \vdash A \to B}$

By induction, $[\Gamma]A = [\Gamma][\Theta]A$.

By induction, $[\Gamma]B = [\Gamma][\Theta]B$.

Then

 $\begin{array}{lll} [\Gamma](A \to B) & = & [\Gamma]A \to [\Gamma]B & \text{By definition of substitution} \\ & = & [\Gamma][\Theta]A \to [\Gamma][\Theta]B & \text{By induction hypothesis (twice)} \\ & = & [\Gamma]([\Theta]A \to [\Theta]B) & \text{By definition of substitution} \\ & = & [\Gamma][\Theta](A \to B) & \text{By definition of substitution} \end{array}$

• Case $\Theta, \alpha \vdash A$ $\Theta \vdash \forall \alpha. A$

By inversion, we have Θ , $\alpha \vdash A$.

By rule \longrightarrow Uvar, Θ , $\alpha \longrightarrow \Gamma$, α .

By induction, $[\Gamma, \alpha]A = [\Gamma, \alpha][\Theta, \alpha]A$.

By definition, $[\Gamma]A = [\Gamma][\Theta]A$.

Then

 $\begin{array}{lll} [\Gamma]\forall\alpha.\,A & = & \forall\alpha.\,[\Gamma]A & \text{By definition} \\ & = & \forall\alpha.\,[\Gamma][\Theta]A & \text{By conclusion above} \\ & = & [\Gamma](\forall\alpha.\,[\Theta]A) & \text{By definition} \\ & = & [\Gamma][\Theta](\forall\alpha.\,A) & \text{By definition} \\ & = & [\Gamma,\alpha][\Theta,\alpha](\forall\alpha.\,A) & \text{By definition} \end{array}$

• Case

$$\underbrace{\Theta_0, \hat{\alpha}, \Theta_1}_{\Theta} \vdash \hat{\alpha}$$

Note that $[\Theta] \hat{\alpha} = \hat{\alpha}$. Hence $[\Gamma][\Theta] \hat{\alpha} = [\Gamma] \hat{\alpha}$.

• Case

$$\Theta_0, \hat{\alpha} = \tau, \Theta_1 \vdash \hat{\alpha}$$

From $\Theta \longrightarrow \Gamma$, By a nested induction we get $\Gamma = \Gamma_0$, $\hat{\alpha} = \tau'$, Γ_1 , and $\Gamma = \Gamma_0$.

Note that $|\Theta \vdash \tau| < |\Theta \vdash \hat{\alpha}|$.

By induction, $[\Gamma]\tau = [\Gamma][\Theta]\tau$.

Hence

$$\begin{split} [\Gamma] \widehat{\alpha} &= [\Gamma] \tau' & \text{By definition} \\ &= [\Gamma] \tau & \text{From the extension judgment} \\ &= [\Gamma] [\Theta] \tau & \text{From the induction hypothesis} \\ &= [\Gamma] [\Theta] \widehat{\alpha} & \text{By definition} \end{split}$$

Lemma 19 (Extension Equality Preservation).

If $\Gamma \vdash A$ and $\Gamma \vdash B$ and $[\Gamma]A = [\Gamma]B$ and $\Gamma \longrightarrow \Delta$, then $[\Delta]A = [\Delta]B$.

Proof. By induction on the derivation of $\Gamma \longrightarrow \Delta$.

Case

$$\underbrace{\frac{\cdot}{\Gamma} \longrightarrow \underbrace{\cdot}_{\Delta}} \longrightarrow \mathsf{ID}$$

We have $[\Gamma]A = [\Gamma]B$, but $\Gamma = \Delta$, so $[\Delta]A = [\Delta]B$.

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma' \longrightarrow \Delta'}{\Gamma', x: C \longrightarrow \Delta', x: C} \longrightarrow \text{Var}$$

We have $[\Gamma', x : C]A = [\Gamma', x : C]B$.

By definition of substitution, $[\Gamma']A = [\Gamma']B$.

By i.h., $[\Delta']A = [\Delta']B$.

By definition of substitution, $[\Delta', x : C]A = [\Delta', x : C]B$.

Case

$$\frac{\Gamma' \longrightarrow \Delta'}{\Gamma', \alpha \longrightarrow \Delta', \alpha} \longrightarrow \mathsf{Uvar}$$

We have $[\Gamma', \alpha]A = [\Gamma', \alpha]B$.

By definition of substitution, $[\Gamma']A = [\Gamma']B$.

By i.h., $[\Delta']A = [\Delta']B$.

By definition of substitution, $[\Delta', \alpha]A = [\Delta', \alpha]B$.

$$\frac{\Gamma' \longrightarrow \Delta'}{\Gamma', \hat{\alpha} \longrightarrow \Delta', \hat{\alpha}} \longrightarrow \mathsf{Unsolved}$$

Similar to the \longrightarrow Uvar case.

Case

$$\frac{\Gamma' \longrightarrow \Delta'}{\Gamma', \blacktriangleright_{\hat{\alpha}} \longrightarrow \Delta', \blacktriangleright_{\hat{\alpha}}} \longrightarrow \mathsf{Marker}$$

Similar to the \longrightarrow Uvar case.

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma \longrightarrow \Delta'}{\Gamma \longrightarrow \Delta', \hat{\alpha}} \longrightarrow \! \mathsf{Add}$$

We have $[\Gamma]A = [\Gamma]B$.

By i.h., $[\Delta']A = [\Delta']B$.

By definition of substitution, $[\Delta', \hat{\alpha}]A = [\Delta', \hat{\alpha}]B$.

Case

$$\frac{\Gamma \longrightarrow \Delta'}{\Gamma \longrightarrow \Delta', \hat{\alpha} = \tau} \longrightarrow \mathsf{AddSolved}$$

We have $[\Gamma]A = [\Gamma]B$.

By i.h.,
$$[\Delta']A = [\Delta']B$$
.

We implicitly assume that Δ is well-formed, so $\hat{\alpha} \notin dom(\Delta')$.

Since $\Gamma \longrightarrow \Delta'$ and $\hat{\alpha} \notin dom(\Delta')$, it follows that $\hat{\alpha} \notin dom(\Gamma)$.

We have $\Gamma \vdash A$ and $\Gamma \vdash B$, so $\hat{\alpha} \notin (FV(A) \cup FV(B))$.

Therefore, by definition of substitution, $[\Delta', \hat{\alpha} = \tau]A = [\Delta', \hat{\alpha} = \tau]B$.

We have $[\Gamma', \hat{\alpha} = \tau]A = [\Gamma', \hat{\alpha} = \tau]B$.

By definition, $[\Gamma', \hat{\alpha} = \tau]A = [\Gamma', \hat{\alpha} = \tau]\tau$, but we implicitly assume that Γ is well-formed, so $\hat{\alpha} \notin FV(\tau)$, so actually $[\Gamma', \hat{\alpha} = \tau]A = [\Gamma']\tau$.

Combined with similar reasoning for B, we get

$$[\Gamma'][\tau/\hat{\alpha}]A = [\Gamma'][\tau/\hat{\alpha}]B$$

By i.h., $[\Delta'][\tau/\hat{\alpha}]A = [\Delta'][\tau/\hat{\alpha}]B$.

By distributivity of substitution, $\left[[\Delta'] \tau / \hat{\alpha} \right] [\Delta'] A = \left[[\Delta'] \tau / \hat{\alpha} \right] [\Delta'] B$.

Using the premise $[\Delta']\tau = [\Delta']\tau'$, we get $[[\Delta']\tau'/\hat{\alpha}][\Delta']A = [[\Delta']\tau'/\hat{\alpha}][\Delta']B$.

By distributivity of substitution (in the other direction), $[\Delta'][\tau'/\hat{\alpha}]A = [\Delta'][\tau'/\hat{\alpha}]B$.

It follows from the definition of substitution that $[\Delta', \hat{\alpha} = \tau']A = [\Delta', \hat{\alpha} = \tau']B$.

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma' \longrightarrow \Delta'}{\Gamma', \hat{\alpha} \longrightarrow \Delta', \hat{\alpha} = \tau} \longrightarrow \\ \textbf{Solve}$$

We have $[\Gamma', \widehat{\alpha}]A = [\Gamma', \widehat{\alpha}]B$.

By definition of substitution, $[\Gamma']A = [\Gamma']B$.

By i.h., $[\Delta'][\tau/\hat{\alpha}]A = [\Delta'][\tau/\hat{\alpha}]B$.

It follows from the definition of substitution that $[\Delta', \hat{\alpha} = \tau]A = [\Delta', \hat{\alpha} = \tau]B$.

Lemma 20 (Reflexivity). *If* Γ *is well-formed, then* $\Gamma \longrightarrow \Gamma$.

Proof. By induction on the structure of Γ .

- Case $\Gamma = \cdot$: Apply rule \longrightarrow ID.
- Case $\Gamma = (\Gamma', \alpha)$: By i.h., $\Gamma' \longrightarrow \Gamma'$. By rule \longrightarrow Uvar, we get $\Gamma', \alpha \longrightarrow \Gamma', \alpha$.
- Case $\Gamma = (\Gamma', \hat{\alpha})$: By i.h., $\Gamma' \longrightarrow \Gamma'$. By rule \longrightarrow Unsolved, we get $\Gamma', \hat{\alpha} \longrightarrow \Gamma', \hat{\alpha}$.
- Case $\Gamma = (\Gamma', \hat{\alpha} = \tau)$:

By i.h., $\Gamma' \longrightarrow \Gamma'$.

Clearly, $[\Gamma']\tau = [\Gamma']\tau$, so we can apply \longrightarrow Solved to get Γ' , $\hat{\alpha} = \tau \longrightarrow \Gamma'$, $\hat{\alpha} = \tau$.

• Case $\Gamma = (\Gamma', \blacktriangleright_{\hat{\alpha}})$: By i.h., $\Gamma' \longrightarrow \Gamma'$. By rule \longrightarrow Marker, we get $\Gamma', \blacktriangleright_{\hat{\alpha}} \longrightarrow \Gamma', \blacktriangleright_{\hat{\alpha}}$.

Lemma 21 (Transitivity). *If* $\Gamma \longrightarrow \Delta$ *and* $\Delta \longrightarrow \Theta$, *then* $\Gamma \longrightarrow \Theta$.

Proof. By induction on the derivation of $\Delta \longrightarrow \Theta$.

• **Case** → ID:

In this case $\Theta = \Delta$.

Hence $\Gamma \longrightarrow \Delta$ suffices.

• Case $\Delta' \longrightarrow \Theta' \over \Delta', \alpha \longrightarrow \Theta', \alpha \longrightarrow \mathsf{Uvar}$

We have $\Delta = (\Delta', \alpha)$ and $\Theta = (\Theta', \alpha)$.

By inversion on $\Gamma \longrightarrow \Delta$, we have $\Gamma = (\Gamma', \alpha)$ and $\Gamma' \longrightarrow \Delta'$.

By i.h., $\Gamma' \longrightarrow \Theta'$.

Applying rule \longrightarrow Uvar gives Γ' , $\alpha \longrightarrow \Theta'$, α .

• Case
$$\Delta' \longrightarrow \Theta' \over \Delta', \hat{\alpha} \longrightarrow \Theta', \hat{\alpha} \longrightarrow \mathsf{Uvar}$$

We have $\Delta = (\Delta', \hat{\alpha})$ and $\Theta = (\Theta', \hat{\alpha})$. Either of two rules could have derived $\Gamma \longrightarrow \Delta$:

$$\begin{array}{ccc} \textbf{- Case} & & & & & \\ & & & & & \\ \hline \Gamma', \widehat{\alpha} & \longrightarrow \Delta', \widehat{\alpha} & \longrightarrow & \\ \hline \end{array} \\ \end{array} \\ \text{Unsolved}$$

Here we have $\Gamma = (\Gamma', \hat{\alpha})$ and $\Gamma' \longrightarrow \Delta'$.

By i.h., $\Gamma' \longrightarrow \Theta'$.

Applying rule \longrightarrow Unsolved gives Γ' , $\hat{\alpha} \longrightarrow \Theta'$, $\hat{\alpha}$.

– Case
$$\frac{\Gamma \longrightarrow \Delta'}{\Gamma \longrightarrow \Delta', \hat{\alpha}} \longrightarrow \mathsf{Add}$$

By i.h., $\Gamma \longrightarrow \Theta'$.

By rule \longrightarrow Add, we get $\Gamma \longrightarrow \Theta'$, $\hat{\alpha}$.

In this case $\Delta = (\Delta', \hat{\alpha} = \tau_1)$ and $\Theta = (\Theta', \hat{\alpha} = \tau_2)$. One of three rules must have derived $\Gamma \longrightarrow \Delta', \hat{\alpha} = \tau$:

Here, $\Gamma = (\Gamma', \hat{\alpha} = \tau_0)$ and $\Delta = (\Delta', \hat{\alpha} = \tau_1)$.

By i.h., we have $\Gamma' \longrightarrow \Theta'$.

The premises of the respective \longrightarrow derivations give us $[\Delta']\tau_0 = [\Delta']\tau_1$ and $[\Theta']\tau_1 = [\Theta']\tau_2$.

We know that $\Gamma' \vdash \tau_0$ and $\Delta' \vdash \tau_1$ and $\Theta' \vdash \tau_2$.

By extension weakening (Lemma 25 (Extension Weakening)), $\Theta' \vdash \tau_0$.

By extension weakening (Lemma 25 (Extension Weakening)), $\Theta' \vdash \tau_1$.

Since $[\Delta']\tau_0 = [\Delta']\tau_1$, we know that $[\Theta'][\Delta']\tau_0 = [\Theta'][\Delta']\tau_1$.

By Lemma 18 (Substitution Extension Invariance), $[\Theta'][\Delta']\tau_0 = [\Theta']\tau_0$.

By Lemma 18 (Substitution Extension Invariance), $[\Theta'][\Delta']\tau_1 = [\Theta']\tau_1$.

So $[\Theta']\tau_0 = [\Theta']\tau_1$.

Hence by transitivity of equality, $[\Theta']\tau_0 = [\Theta']\tau_1 = [\Theta']\tau_2$. By rule \longrightarrow Solved, Γ' , $\hat{\alpha} = \tau \longrightarrow \Theta'$, $\hat{\alpha} = \tau_2$.

– Case
$$\frac{\Gamma \longrightarrow \Delta'}{\Gamma \longrightarrow \Delta', \hat{\alpha} = \tau_1} \longrightarrow \mathsf{AddSolved}$$

By induction, we have $\Gamma \longrightarrow \Theta'$.

By rule \longrightarrow AddSolved, we get $\Gamma \longrightarrow \Theta'$, $\hat{\alpha} = \tau_2$.

- Case
$$\frac{\Gamma' \longrightarrow \Delta'}{\Gamma', \hat{\alpha} \longrightarrow \Delta', \hat{\alpha} = \tau_1} \longrightarrow \mathsf{Solve}$$

We have $\Gamma = (\Gamma', \hat{\alpha})$.

By induction, $\Gamma' \longrightarrow \Theta'$.

By rule \longrightarrow Solve, we get Γ' , $\hat{\alpha} \longrightarrow \Theta'$, $\hat{\alpha} = \tau_2$.

• Case
$$\Delta' \longrightarrow \Theta' \longrightarrow \Delta'$$
 $A' \longrightarrow A' \longrightarrow A'$ $A' \longrightarrow A' \longrightarrow A'$

In this case we know $\Delta = (\Delta', \blacktriangleright_{\hat{\alpha}})$ and $\Theta = (\Theta', \blacktriangleright_{\hat{\alpha}})$.

Since $\Delta = (\Delta', \blacktriangleright_{\hat{\alpha}})$, only \longrightarrow Marker could derive $\Gamma \longrightarrow \Delta$, so by inversion, $\Gamma = (\Gamma', \blacktriangleright_{\hat{\alpha}})$ and $\Gamma' \longrightarrow \Delta'$.

By induction, we have $\Gamma' \longrightarrow \Theta'$.

Applying rule \longrightarrow Marker gives $\Gamma', \triangleright_{\hat{\alpha}} \longrightarrow \Theta', \triangleright_{\hat{\alpha}}$.

$\bullet \ \, \textbf{Case} \ \, \frac{\Delta \longrightarrow \Theta'}{\Delta \longrightarrow \Theta', \hat{\alpha}} \longrightarrow \textbf{Add}$

In this case, we have $\Theta = (\Theta', \hat{\alpha})$.

By induction, we get $\Gamma \longrightarrow \Theta'$.

By rule \longrightarrow Add, we get $\Gamma \longrightarrow \Theta'$, $\hat{\alpha}$.

$$\bullet \ \, \textbf{Case} \ \, \frac{\Delta \longrightarrow \Theta'}{\Delta \longrightarrow \Theta', \hat{\alpha} = \tau} \longrightarrow \text{AddSolved}$$

In this case, we have $\Theta = (\Theta', \hat{\alpha} = \tau)$.

By induction, we get $\Gamma \longrightarrow \Theta'$.

By rule \longrightarrow AddSolved, we get $\Gamma \longrightarrow \Theta'$, $\hat{\alpha} = \tau$.

• Case
$$\Delta' \longrightarrow \Theta' \over \Delta', \hat{\alpha} \longrightarrow \Theta', \hat{\alpha} = \tau \longrightarrow \mathsf{Solve}$$

In this case, we have $\Delta = (\Delta', \hat{\alpha})$ and $\Theta = (\Theta', \hat{\alpha} = \tau)$.

One of two rules could have derived $\Gamma \longrightarrow \Delta'$, $\hat{\alpha}$:

- Case
$$\frac{\Gamma' \longrightarrow \Delta'}{\Gamma', \hat{\alpha} \longrightarrow \Delta', \hat{\alpha}} \longrightarrow \text{Unsolved}$$

In this case, we have $\Gamma = (\Gamma', \hat{\alpha})$ and $\Gamma' \longrightarrow \Delta'$ and $\Delta' \longrightarrow \Theta'$.

By induction, we have $\Gamma' \longrightarrow \Theta'$.

By rule \longrightarrow Solve, we get Γ' , $\hat{\alpha} \longrightarrow \Theta'$, $\hat{\alpha} = \tau$.

$$\begin{array}{ccc} \textbf{- Case} & & & & \\ & \frac{\Gamma \longrightarrow \Delta'}{\Gamma \longrightarrow \Delta', \hat{\alpha}} \longrightarrow & \text{Add} \end{array}$$

In this case, we have $\Gamma \longrightarrow \Delta'$ and $\Delta' \longrightarrow \Theta'$.

By induction, we have $\Gamma \longrightarrow \Theta'$.

By rule \longrightarrow Solve, we get $\Gamma \longrightarrow \Theta'$, $\hat{\alpha} = \tau$.

Lemma 22 (Right Softness). If $\Gamma \longrightarrow \Delta$ and Θ is soft (and (Δ, Θ) is well-formed) then $\Gamma \longrightarrow \Delta, \Theta$.

Proof. By induction on Θ , applying rules \longrightarrow Add and \longrightarrow AddSolved as needed.

Lemma 23 (Evar Input).

If Γ , $\hat{\alpha} \longrightarrow \Delta$ then $\Delta = (\Delta_0, \Delta_{\hat{\alpha}}, \Theta)$ where $\Gamma \longrightarrow \Delta_0$, and $\Delta_{\hat{\alpha}}$ is either $\hat{\alpha}$ or $\hat{\alpha} = \tau$, and Θ is soft.

Proof. By induction on the given derivation.

• Cases \longrightarrow ID, \longrightarrow Var, \longrightarrow Uvar, \longrightarrow Solved, \longrightarrow Marker: Impossible: the left-hand context cannot have the form Γ , $\hat{\alpha}$.

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma \longrightarrow \Delta_0}{\Gamma, \hat{\alpha} \longrightarrow \underbrace{\Delta_0, \hat{\alpha}}_{\Delta}} \longrightarrow \textbf{Unsolved}$$

Let $\Theta = \cdot$, which is vacuously soft. Therefore $\Delta = (\Delta_0, \hat{\alpha}) = (\Delta_0, \hat{\alpha}, \Theta)$; the subderivation is the rest of the result.

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma \longrightarrow \Delta_0}{\Gamma, \hat{\alpha} \longrightarrow \underbrace{\Delta_0, \hat{\alpha} = \tau}_{\Lambda}} \longrightarrow \mathsf{Solve}$$

Let $\Theta = \cdot$, which is vacuously soft. Therefore $\Delta = (\Delta_0, \hat{\alpha}) = (\Delta_0, \hat{\alpha} = \tau, \Theta)$; the subderivation is the rest of the result.

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma, \, \widehat{\alpha} \longrightarrow \Delta_0}{\Gamma, \, \widehat{\alpha} \longrightarrow \underbrace{\Delta_0, \, \widehat{\beta}}_{\Delta}} \longrightarrow \mathsf{Add}$$

Suppose $\hat{\beta} = \hat{\alpha}$.

We have $\Gamma, \hat{\alpha} \longrightarrow \Delta_0$. By Lemma 15 (Declaration Preservation), $\hat{\alpha}$ is declared in Δ_0 .

But then $(\Delta_0, \hat{\beta}) = (\Delta_0, \hat{\alpha})$ with multiple $\hat{\alpha}$ declarations,

which violates the implicit assumption that Δ is well-formed. Contradiction.

Therefore $\hat{\beta} \neq \hat{\alpha}$.

By i.h., $\Delta' = (\Delta_0, \Delta_{\hat{\alpha}}, \Theta')$ where $\Gamma \longrightarrow \Delta_0$ and Θ' is soft.

Let $\Theta = (\Theta', \hat{\beta})$. Therefore $(\Delta', \hat{\beta}) = (\Delta_0, \Delta_{\hat{\alpha}}, \Theta', \hat{\beta})$. As Θ' is soft, $(\Theta', \hat{\beta})$ is soft. Since $\Delta = (\Delta', \hat{\beta})$, this gives $\Delta = (\Delta_0, \Delta_{\hat{\alpha}}, \Theta)$.

• Case → AddSolved: Similar to the case for → Add.

Lemma 24 (Extension Order).

- (i) If Γ_L , α , $\Gamma_R \longrightarrow \Delta$ then $\Delta = (\Delta_L, \alpha, \Delta_R)$ where $\Gamma_L \longrightarrow \Delta_L$. Moreover, if Γ_R is soft then Δ_R is soft.
- (ii) If Γ_L , $\blacktriangleright_{\hat{\alpha}}$, $\Gamma_R \longrightarrow \Delta$ then $\Delta = (\Delta_L$, $\blacktriangleright_{\hat{\alpha}}$, Δ_R) where $\Gamma_L \longrightarrow \Delta_L$. Moreover, if Γ_R is soft then Δ_R is soft.
- (iii) If Γ_L , $\hat{\alpha}$, $\Gamma_R \longrightarrow \Delta$ then $\Delta = \Delta_L$, Θ , Δ_R where $\Gamma_L \longrightarrow \Delta_L$ and Θ is either $\hat{\alpha}$ or $\hat{\alpha} = \tau$ for some τ .
- (iv) If Γ_L , $\hat{\alpha} = \tau$, $\Gamma_R \longrightarrow \Delta$ then $\Delta = \Delta_L$, $\hat{\alpha} = \tau'$, Δ_R where $\Gamma_L \longrightarrow \Delta_L$ and $[\Delta_L]\tau = [\Delta_L]\tau'$.
- (v) If $\Gamma_L, x : A, \Gamma_R \longrightarrow \Delta$ then $\Delta = (\Delta_L, x : A', \Delta_R)$ where $\Gamma_L \longrightarrow \Delta_L$ and $[\Delta_L]A = [\Delta_L]A'$. Moreover, Γ_R is soft if and only if Δ_R is soft.

Proof. (i) By induction on the derivation of Γ_L , α , $\Gamma_R \longrightarrow \Delta$.

$$\bullet \ \, \textbf{Case} \\ \hline \\ \hline \\ \hline \\ \cdot \longrightarrow \cdot \\ \\ \\ \square \\ D$$

This case is impossible since $(\Gamma_L, \alpha, \Gamma_R)$ cannot have the form \cdot .

• Cases —→Uvar:

We have two cases, depending on whether or not the rightmost variable is α .

$$\begin{array}{c} \textbf{- Case} \\ \hline \Gamma & \longrightarrow \Delta' \\ \hline \Gamma, \alpha & \longrightarrow \Delta', \alpha \end{array} \longrightarrow \text{Uvar} \\ \text{Let } \Delta_L = \Delta', \text{ and let } \Delta_R = \cdot \text{ (which is soft)}. \\ \text{We have } \Gamma & \longrightarrow \Delta', \text{ which is } \Gamma_L & \longrightarrow \Delta_L. \end{array}$$

- Case
$$\frac{\Gamma_{L}, \alpha, \Gamma_{R}' \longrightarrow \Delta'}{\Gamma_{L}, \alpha, \underbrace{\Gamma_{R}', \beta}_{\Gamma_{R}} \longrightarrow \underbrace{\Delta', \beta}_{\Delta}} \longrightarrow \mathsf{Uvar}$$

By i.h., $\Delta' = (\mathring{\Delta}_L, \alpha, \Delta'_R)$ where $\Gamma_L \longrightarrow \Delta_L$.

Hence $\Delta = (\Delta_L, \alpha, \Delta'_R, \beta)$.

(Since $\beta \in \Gamma_R$, it cannot be the case that Γ_R is soft.)

Case

$$\frac{\Gamma_{L},\alpha,\Gamma_{R}'\longrightarrow\Delta'}{\Gamma_{L},\alpha,\underbrace{\Gamma_{R}',x:A}_{\Gamma_{R}}\longrightarrow\underbrace{\Delta',x:A}_{\Delta}}\longrightarrow \mathsf{Var}$$

By i.h., $\Delta' = (\Delta_L, \alpha, \Delta_R')$ where $\Gamma_L \longrightarrow \Delta_L.$

Hence $\Delta = (\Delta_L, \alpha, \Delta'_R, x : A)$.

(Since $x : A \in \Gamma_R$, it cannot be the case that Γ_R is soft.)

Case

$$\frac{\Gamma_L,\alpha,\Gamma_R'\longrightarrow\Delta'}{\Gamma_L,\alpha,\underbrace{\Gamma_R',\hat{\alpha}}_{\Gamma_R}\longrightarrow\underbrace{\Delta',\hat{\alpha}}_{\Delta}}\longrightarrow \mathsf{Unsolved}$$

By i.h., $\Delta' = (\Delta_L, \alpha, \Delta'_R)$ where $\Gamma_L \longrightarrow \Delta_L$.

Hence $\Delta = (\Delta_L, \alpha, \Delta'_R, \hat{\alpha}).$

(If Γ_R is soft, by i.h. Δ_R' is soft, so $\Delta_R = (\Delta_R', \hat{\alpha})$ is soft.)

Case

$$\frac{\Gamma_{L},\alpha,\Gamma_{R}'\longrightarrow\Delta'}{\Gamma_{L},\alpha,\underbrace{\Gamma_{R}',\blacktriangleright_{\hat{\beta}}}_{\Gamma_{R}'}\longrightarrow\underbrace{\Delta',\blacktriangleright_{\hat{\beta}}}_{\Delta}}\longrightarrow\mathsf{Marker}$$

By i.h., $\Delta' = (\Delta_L, \alpha, \Delta_R')$ where $\Gamma_L \longrightarrow \Delta_L.$

Hence $\Delta = (\Delta_L, \alpha, \Delta'_R, \blacktriangleright_{\widehat{B}}).$

(Since $\triangleright_{\hat{\beta}} \in \Gamma_R$, it cannot be the case that Γ_R is soft.)

Case

$$\frac{\Gamma_L,\alpha,\Gamma_R'\longrightarrow\Delta'}{\Gamma_L,\alpha,\underbrace{\Gamma_R',\hat{\alpha}=\tau}_{\Gamma_R}\longrightarrow\underbrace{\Delta',\hat{\alpha}=\tau'}_{\Delta'}}\longrightarrow \mathsf{Solved}$$

By i.h., $\Delta' = (\Delta_L, \alpha, \Delta_R')$ where $\Gamma_L \longrightarrow \Delta_L$.

Hence $\Delta = (\Delta_L, \alpha, \Delta'_R, \hat{\alpha} = \tau')$.

(If Γ_R is soft, by i.h. Δ_R' is soft, so $\Delta_R=(\Delta_R',\hat{\alpha}=\tau)$ is soft.)

• Case

$$\frac{\Gamma_L,\alpha,\Gamma_R'\longrightarrow\Delta'}{\Gamma_L,\alpha,\underbrace{\Gamma_R',\hat{\alpha}}_{\Gamma_R}\longrightarrow\underbrace{\Delta',\hat{\alpha}=\tau'}_{\Delta}}\longrightarrow \mathsf{Solve}$$

By i.h., $\Delta' = (\Delta_L, \alpha, \Delta'_R)$ where $\Gamma_L \longrightarrow \Delta_L$.

Therefore $\Delta=(\Delta_L,\alpha,\Delta_R,\hat{\alpha}=\tau).$

(If Γ_R is soft, by i.h. Δ_R' is soft, so $\Delta_R=(\Delta_R',\hat{\alpha}=\tau)$ is soft.)

Case

$$\frac{\Gamma_L, \alpha, \Gamma_R \longrightarrow \Delta'}{\Gamma_L, \alpha, \Gamma_R \longrightarrow \underbrace{\Delta', \hat{\alpha}}_{\Delta}} \longrightarrow \mathsf{Add}$$

By i.h., $\Delta' = (\Delta_L, \alpha, \Delta'_R)$ where $\Gamma_L \longrightarrow \Delta_L$.

Therefore $\Delta = (\Delta_L, \alpha, \Delta'_R, \hat{\alpha})$.

(If Γ_R is soft, by i.h. Δ_R' is soft, so $\Delta_R = (\Delta_R', \hat{\alpha})$ is soft.)

Case

$$\frac{\Gamma_L,\alpha,\Gamma_R\longrightarrow\Delta'}{\Gamma_L,\alpha,\Gamma_R\longrightarrow\Delta',\hat{\alpha}=\tau}\longrightarrow \mathsf{AddSolved}$$

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In this case, we know that \Delta=(\Delta',\hat{\alpha}=\tau). By i.h., \Delta'=(\Delta_L,\alpha,\Delta_R') where \Gamma_L\longrightarrow\Delta_L. Hence \Delta=(\Delta_L,\alpha,\Delta_R',\hat{\alpha}=\tau). (If \Gamma_R is soft, by i.h. \Delta_R' is soft, so \Delta_R=(\Delta_R',\hat{\alpha}=\tau) is soft.)
```

- (ii) Similar to the proof of (i), except that the \longrightarrow Marker and \longrightarrow Uvar cases are swapped.
- (iii) Similar to (i), with $\Theta = \hat{\alpha}$ in the Unsolved case and $\Theta = (\hat{\alpha} = \tau)$ in the Solve case.
- (iv) Similar to (iii).
- (v) Similar to (i), but using the equality premise of —→Var.

Lemma 25 (Extension Weakening). *If* $\Gamma \vdash A$ *and* $\Gamma \longrightarrow \Delta$ *then* $\Delta \vdash A$.

Proof. By a straightforward induction on $\Gamma \vdash A$.

In the UvarWF case, we use Lemma 24 (Extension Order) (i). In the EvarWF case, use Lemma 24 (Extension Order) (iii). In the SolvedEvarWF case, use Lemma 24 (Extension Order) (iv).

In the other cases, apply the i.h. to all subderivations, then apply the rule.

Lemma 26 (Solution Admissibility for Extension). *If* $\Gamma_L \vdash \tau$ *then* Γ_L , $\hat{\alpha}$, $\Gamma_R \longrightarrow \Gamma_L$, $\hat{\alpha} = \tau$, Γ_R .

Proof. By induction on Γ_R .

- Case $\Gamma_R = \cdot$: By Lemma 20 (Reflexivity) (reflexivity), $\Gamma_L \longrightarrow \Gamma_L$. Applying rule \longrightarrow Solve gives $\Gamma_L, \hat{\alpha} \longrightarrow \Gamma_L, \hat{\alpha} = \tau$.
- Case $\Gamma_R = (\Gamma_R', x : A)$: By i.h., $\Gamma_L, \hat{\alpha}, \Gamma_R' \longrightarrow \Gamma_L, \hat{\alpha} = \tau, \Gamma_R'$. Applying rule \longrightarrow Var gives $\Gamma_L, \hat{\alpha}, \Gamma_R', x : A <math>\longrightarrow \Gamma_L, \hat{\alpha} = \tau, \Gamma_R', x : A$.
- Case $\Gamma_R = (\Gamma_R', \alpha)$: By i.h. and rule \longrightarrow Uvar.
- Case $\Gamma_R = (\Gamma_R', \widehat{\beta})$: By i.h. and rule $\longrightarrow Add$.
- Case $\Gamma_R = (\Gamma_R', \hat{\beta} = \tau')$: By i.h. and rule \longrightarrow AddSolved.
- Case $\Gamma_R = (\Gamma_R', \blacktriangleright_{\widehat{B}})$: By i.h. and rule \longrightarrow Marker.

Lemma 27 (Solved Variable Addition for Extension). If $\Gamma_L \vdash \tau$ then $\Gamma_L, \Gamma_R \longrightarrow \Gamma_L, \hat{\alpha} = \tau, \Gamma_R$.

Proof. By induction on Γ_R . The proof is exactly the same as the proof of Lemma 26 (Solution Admissibility for Extension), except that in the $\Gamma_R = \cdot$, we apply rule \longrightarrow AddSolved instead of \longrightarrow Solve.

Lemma 28 (Unsolved Variable Addition for Extension). We have that Γ_L , $\Gamma_R \longrightarrow \Gamma_L$, $\hat{\alpha}$, Γ_R .

Proof. By induction on Γ_R . The proof is exactly the same as the proof of Lemma 26 (Solution Admissibility for Extension), except that in the $\Gamma_R = \cdot$ case, we apply rule \longrightarrow Add instead of \longrightarrow Solve.

Lemma 29 (Parallel Admissibility).

If $\Gamma_L \longrightarrow \Delta_L$ and $\Gamma_L, \Gamma_R \longrightarrow \Delta_L, \Delta_R$ then:

- (i) Γ_L , $\hat{\alpha}$, $\Gamma_R \longrightarrow \Delta_L$, $\hat{\alpha}$, Δ_R
- (ii) If $\Delta_I \vdash \tau'$ then Γ_I , $\hat{\alpha}$, $\Gamma_R \longrightarrow \Delta_I$, $\hat{\alpha} = \tau'$, Δ_R .
- (iii) If $\Gamma_I \vdash \tau$ and $\Delta_I \vdash \tau'$ and $[\Delta_I]\tau = [\Delta_I]\tau'$, then Γ_I , $\hat{\alpha} = \tau$, $\Gamma_R \longrightarrow \Delta_I$, $\hat{\alpha} = \tau'$, Δ_R .

Proof. By induction on Δ_R . As always, we assume that all contexts mentioned in the statement of the lemma are well-formed. Hence, $\hat{\alpha} \notin \mathsf{dom}(\Gamma_L) \cup \mathsf{dom}(\Gamma_R) \cup \mathsf{dom}(\Delta_L) \cup \mathsf{dom}(\Delta_R)$.

(i) We proceed by cases of Δ_R . Observe that in all the extension rules, the right-hand context gets smaller, so as we enter subderivations of Γ_L , $\Gamma_R \longrightarrow \Delta_L$, Δ_R , the context Δ_R becomes smaller.

The only tricky part of the proof is that to apply the i.h., we need $\Gamma_L \longrightarrow \Delta_L$. So we need to make sure that as we drop items from the right of Γ_R and Δ_R , we don't go too far and start decomposing Γ_L or Δ_L ! It's easy to avoid decomposing Δ_L : when $\Delta_R = \cdot$, we don't need to apply the i.h. anyway. To avoid decomposing Γ_L , we need to reason by contradiction, using Lemma 15 (Declaration Preservation).

- Case $\Delta_R = \cdot$:
 We have $\Gamma_L \longrightarrow \Delta_L$. Applying \longrightarrow Unsolved to that derivation gives the result.
- Case $\Delta_R = (\Delta_R', \hat{\beta})$: We have $\hat{\beta} \neq \hat{\alpha}$ by the well-formedness assumption. The concluding rule of $\Gamma_L, \Gamma_R \longrightarrow \Delta_L, \Delta_R', \hat{\beta}$ must have been \longrightarrow Unsolved or \longrightarrow Add. In both cases, the result follows by i.h. and applying \longrightarrow Unsolved or \longrightarrow Add. Note: In \longrightarrow Add, the left-hand context doesn't change, so we clearly maintain $\Gamma_L \longrightarrow \Delta_L$. In \longrightarrow Unsolved, we can correctly apply the i.h. because $\Gamma_R \neq \cdot$. Suppose, for a contradiction, that $\Gamma_R = \cdot$. Then $\Gamma_L = (\Gamma_L', \hat{\beta})$. It was given that $\Gamma_L \longrightarrow \Delta_L$, that is, $\Gamma_L', \hat{\beta} \longrightarrow \Delta_L$. By Lemma 15 (Declaration Preservation), Δ_L has a declaration of $\hat{\beta}$. But then $\Delta = (\Delta_L, \Delta_R', \hat{\beta})$ is not well-formed: contradiction. Therefore $\Gamma_R \neq \cdot$.
- Case $\Delta_R = (\Delta_R', \hat{\beta} = \tau)$: We have $\hat{\beta} \neq \hat{\alpha}$ by the well-formedness assumption. The concluding rule must have been \longrightarrow Solved, \longrightarrow Solve or \longrightarrow AddSolved. In each case, apply the i.h. and then the corresponding rule. (In \longrightarrow Solved and \longrightarrow Solve, use Lemma 15 (Declaration Preservation) to show $\Gamma_R \neq \cdot$.)
- Case $\Delta_R = (\Delta_R', \alpha)$: The concluding rule must have been —>Uvar. The result follows by i.h. and applying —>Uvar.
- Case $\Delta_R = (\Delta_R', \blacktriangleright_{\hat{B}})$: Similar to the previous case, with rule \longrightarrow Marker.
- Case $\Delta_R = (\Delta_R', x : A)$: Similar to the previous case, with rule \longrightarrow Var.
- (ii) Similar to part (i), except that when $\Delta_R = \cdot$, apply rule \longrightarrow Solve.
- (iii) Similar to part (i), except that when $\Delta_R = \cdot$, apply rule \longrightarrow Solved, using the given equality to satisfy the second premise.

Lemma 30 (Parallel Extension Solution).

If
$$\Gamma_L$$
, $\hat{\alpha}$, $\Gamma_R \longrightarrow \Delta_L$, $\hat{\alpha} = \tau'$, Δ_R and $\Gamma_L \vdash \tau$ and $[\Delta_L]\tau = [\Delta_L]\tau'$ then Γ_L , $\hat{\alpha} = \tau$, $\Gamma_R \longrightarrow \Delta_L$, $\hat{\alpha} = \tau'$, Δ_R .

Proof. By induction on Δ_R .

In the case where $\Delta_R = (\Delta_R', \hat{\alpha} = \tau')$, we know that rule —>Solve must have concluded the derivation (we can use Lemma 15 (Declaration Preservation) to get a contradiction that rules out —>AddSolved); then we have a subderivation $\Gamma_L \longrightarrow \Delta_L$, to which we can apply —>Solved.

Lemma 31 (Parallel Variable Update).

If
$$\Gamma_L$$
, $\hat{\alpha}$, $\Gamma_R \longrightarrow \Delta_L$, $\hat{\alpha} = \tau_0$, Δ_R and $\Gamma_L \vdash \tau_1$ and $\Delta_L \vdash \tau_2$ and $[\Delta_L]\tau_0 = [\Delta_L]\tau_1 = [\Delta_L]\tau_2$ then Γ_L , $\hat{\alpha} = \tau_1$, $\Gamma_R \longrightarrow \Delta_L$, $\hat{\alpha} = \tau_2$, Δ_R .

Proof. By induction on Δ_R . Similar to the proof of Lemma 30 (Parallel Extension Solution), but applying \longrightarrow Solved at the end.

D'.2 Instantiation Extends

Lemma 32 (Instantiation Extension).

If
$$\Gamma \vdash \hat{\alpha} : \stackrel{\leq}{=} \tau \dashv \Delta \text{ or } \Gamma \vdash \tau \stackrel{\leq}{=} : \hat{\alpha} \dashv \Delta \text{ then } \Gamma \longrightarrow \Delta.$$

Proof. By induction on the given instantiation derivation.

• Case
$$\frac{\Gamma \vdash \tau}{\Gamma, \hat{\alpha}, \Gamma' \vdash \hat{\alpha} : \stackrel{\checkmark}{=} \tau \dashv \Gamma, \hat{\alpha} = \tau, \Gamma'} \text{InstLSolve}$$

By Lemma 26 (Solution Admissibility for Extension), Γ , $\hat{\alpha}$, $\Gamma' \longrightarrow \Gamma$, $\hat{\alpha} = \tau$, Γ' .

Case

$$\frac{}{\Gamma[\hat{\alpha}][\hat{\beta}] \vdash \; \hat{\alpha} : \stackrel{\leq}{=} \hat{\beta} \; \dashv \Gamma[\hat{\alpha}][\hat{\beta} = \hat{\alpha}]} \; \mathsf{InstLReach}$$

 $\Gamma[\hat{\alpha}][\hat{\beta}] = \Gamma_0, \hat{\alpha}, \Gamma_1, \hat{\beta}, \Gamma_2 \text{ for some } \Gamma_0, \Gamma_1, \Gamma_2.$

By the definition of well-formedness, Γ_0 , $\hat{\alpha}$, $\Gamma_1 \vdash \hat{\alpha}$.

Therefore, by Lemma 26 (Solution Admissibility for Extension), Γ_0 , $\hat{\alpha}$, Γ_1 , $\hat{\beta}$, $\Gamma_2 \longrightarrow \Gamma_0$, $\hat{\alpha}$, Γ_1 , $\hat{\beta} = \hat{\alpha}$, Γ_2 .

 $\frac{\Gamma[\hat{\alpha}_2,\hat{\alpha}_1,\hat{\alpha}=\hat{\alpha}_1\rightarrow\hat{\alpha}_2]\vdash A_1\stackrel{\leq}{=}:\hat{\alpha}_1\dashv\Gamma'\qquad \Gamma'\vdash\hat{\alpha}_2:\stackrel{\leq}{=}[\Gamma']A_2\dashv\Delta}{\Gamma[\hat{\alpha}]\vdash\hat{\alpha}:\stackrel{\leq}{=}A_1\rightarrow A_2\dashv\Delta} \text{ InstLArr}$

By Lemma 28 (Unsolved Variable Addition for Extension), we can insert an (unsolved) $\hat{\alpha}_2$, giving $\Gamma[\hat{\alpha}] \longrightarrow \Gamma[\hat{\alpha}_2, \hat{\alpha}].$

By Lemma 28 (Unsolved Variable Addition for Extension) again, $\Gamma[\hat{\alpha}_2, \hat{\alpha}] \longrightarrow \Gamma[\hat{\alpha}_2, \hat{\alpha}_1, \hat{\alpha}]$.

By Lemma 26 (Solution Admissibility for Extension), we can solve $\hat{\alpha}$, giving $\Gamma[\hat{\alpha}_2, \hat{\alpha}_1, \hat{\alpha}] \longrightarrow \Gamma[\hat{\alpha}_2, \hat{\alpha}_1, \hat{\alpha} = \hat{\alpha}_1 \to \hat{\alpha}_2]$

Then by transitivity (Lemma 21 (Transitivity)), $\Gamma[\hat{\alpha}] \longrightarrow \Gamma[\hat{\alpha}_2, \hat{\alpha}_1, \hat{\alpha} = \hat{\alpha}_1 \to \hat{\alpha}_2]$.

By i.h. on the first subderivation, $\Gamma[\hat{\alpha}_2, \hat{\alpha}_1, \hat{\alpha} = \hat{\alpha}_1 \to \hat{\alpha}_2] \longrightarrow \Gamma'$. By i.h. on the second subderivation, $\Gamma' \longrightarrow \Delta$.

By transitivity (Lemma 21 (Transitivity)), $\Gamma[\hat{\alpha}_2, \hat{\alpha}_1, \hat{\alpha} = \hat{\alpha}_1 \to \hat{\alpha}_2] \longrightarrow \Delta$.

By transitivity (Lemma 21 (Transitivity)), $\Gamma[\hat{\alpha}] \longrightarrow \Delta$.

 $\bullet \ \, \textbf{Case} \ \, \frac{\Gamma[\hat{\alpha}], \, \beta \vdash \, \hat{\alpha} : \stackrel{\leq}{=} B \ \, \dashv \Delta, \, \beta, \Delta'}{\Gamma[\hat{\alpha}] \vdash \, \hat{\alpha} : \stackrel{\leq}{=} \forall \beta. \, B \ \, \dashv \Delta} \ \, \textbf{InstLAIIR}$

By induction, $\Gamma[\hat{\alpha}]$, $\beta \longrightarrow \Delta$, β , Δ' .

By Lemma 24 (Extension Order) (i), we have $\Gamma[\hat{\alpha}] \longrightarrow \Delta$.

Case $\frac{\Gamma \vdash \tau}{\Gamma, \hat{\alpha}, \Gamma' \vdash \tau \stackrel{\leq}{=} \hat{\alpha} \dashv \Gamma, \hat{\alpha} = \tau, \Gamma'} \text{ InstRSolve}$

By Lemma 26 (Solution Admissibility for Extension), we can solve $\hat{\alpha}$, giving Γ , $\hat{\alpha}$, $\Gamma' \longrightarrow \Gamma$, $\hat{\alpha} = \tau$, Γ' .

Case

$$\frac{1}{\Gamma[\hat{\alpha}][\hat{\beta}] \vdash \hat{\beta} \stackrel{\leq}{=} \hat{\alpha} \dashv \Gamma[\hat{\alpha}][\hat{\beta} = \hat{\alpha}]} \text{ InstRReach}$$

 $\Gamma[\hat{\alpha}][\hat{\beta}] = \Gamma_0, \hat{\alpha}, \Gamma_1, \hat{\beta}, \Gamma_2 \text{ for some } \Gamma_0, \Gamma_1, \Gamma_2.$

By the definition of well-formedness, Γ_0 , $\hat{\alpha}$, $\Gamma_1 \vdash \hat{\alpha}$.

Hence by Lemma 26 (Solution Admissibility for Extension), we can solve $\hat{\beta}$, giving Γ_0 , $\hat{\alpha}$, Γ_1 , $\hat{\beta}$, Γ_2 \longrightarrow $\Gamma_0, \hat{\alpha}, \Gamma_1, \hat{\beta} = \hat{\alpha}, \Gamma_2.$

 $\frac{\Gamma[\hat{\alpha}_2,\hat{\alpha}_1,\hat{\alpha}=\hat{\alpha}_1\rightarrow\hat{\alpha}_2]\vdash\hat{\alpha}_1:\stackrel{\leq}{=}A_1\dashv\Gamma'\qquad\Gamma'\vdash[\Gamma']A_2\stackrel{\leq}{=}:\hat{\alpha}_2\dashv\Delta}{\Gamma[\hat{\alpha}]\vdash A_1\rightarrow A_2\stackrel{\leq}{=}:\hat{\alpha}\dashv\Delta} \text{ InstRArr}$

Because the contexts here are the same as in InstLArr, this is the same as the InstLArr case.

 $\bullet \ \, \textbf{Case} \ \, \frac{\Gamma[\hat{\alpha}], \blacktriangleright_{\hat{\beta}}, \hat{\beta} \vdash [\hat{\beta}/\beta] B \mathrel{\stackrel{<}{=}:} \hat{\alpha} \dashv \Delta, \blacktriangleright_{\hat{\beta}}, \Delta'}{\Gamma[\hat{\alpha}] \vdash \forall \beta. \, B \mathrel{\stackrel{<}{=}:} \hat{\alpha} \dashv \Delta} \ \, \textbf{InstRAIIL}$

By i.h., $\Gamma[\hat{\alpha}], \blacktriangleright_{\hat{\beta}}, \hat{\beta} \longrightarrow \Delta, \blacktriangleright_{\hat{\beta}}, \Delta'$.

By Lemma 24 (Extension Order) (ii), $\Gamma[\hat{\alpha}] \longrightarrow \Delta$.

Subtyping Extends

Lemma 33 (Subtyping Extension).

If
$$\Gamma \vdash A \lt : B \dashv \Delta$$
 then $\Gamma \longrightarrow \Delta$.

Proof. By induction on the given derivation.

For cases <: Var, <: Unit, <: Exvar, we have $\Delta = \Gamma$, so Lemma 20 (Reflexivity) suffices.

By IH on each subderivation, $\Gamma \longrightarrow \Theta$ and $\Theta \longrightarrow \Delta$.

By Lemma 21 (Transitivity) (transitivity), $\Gamma \longrightarrow \Delta$, which was to be shown.

• Case
$$\frac{\Gamma, \blacktriangleright_{\hat{\alpha}}, \hat{\alpha} \vdash [\hat{\alpha}/\alpha]A <: B \dashv \Delta, \blacktriangleright_{\hat{\alpha}}, \Theta}{\Gamma \vdash \forall \alpha. \ A <: B \dashv \Delta} <: \forall L$$

By IH, $\Gamma, \triangleright_{\hat{\alpha}}, \hat{\alpha} \longrightarrow \Delta, \triangleright_{\hat{\alpha}}, \Theta$.

By Lemma 24 (Extension Order) (ii) with $\Gamma_L = \Gamma$ and $\Gamma_L' = \Delta$ and $\Gamma_R = \hat{\alpha}$ and $\Gamma_R' = \Theta$, we obtain

$$\Gamma \longrightarrow \Delta$$

• Case
$$\frac{\Gamma, \beta \vdash A <: B \dashv \Delta, \beta, \Theta}{\Gamma \vdash A <: \forall \beta. B \dashv \Delta} <: \forall R$$

By IH, we have $\Gamma, \beta \longrightarrow \Delta, \beta, \Theta$.

By Lemma 24 (Extension Order) (i), we obtain $\Gamma \longrightarrow \Delta$, which was to be shown.

• Cases <: InstantiateL, <: InstantiateR: In each of these rules, the premise has the same input and output contexts as the conclusion, so Lemma 32 (Instantiation Extension) suffices.

\mathbf{E}' **Decidability of Instantiation**

Lemma 34 (Left Unsolvedness Preservation).

Lemma 34 (Left Unsolvedness Preservation). If
$$\Gamma_0$$
, $\hat{\alpha}$, Γ_1 $\vdash \hat{\alpha} := A \dashv \Delta$ or Γ_0 , $\hat{\alpha}$, Γ_1 $\vdash A \stackrel{\text{def}}{=} : \hat{\alpha} \dashv \Delta$, and $\hat{\beta} \in \text{unsolved}(\Gamma_0)$, then $\hat{\beta} \in \text{unsolved}(\Delta)$.

Proof. By induction on the given derivation.

$$\bullet \ \, \textbf{Case} \ \, \underbrace{\frac{\Gamma_0 \vdash \tau}{\Gamma_0, \hat{\alpha}, \Gamma_1 \vdash \hat{\alpha} : \stackrel{\leq}{=} \tau \dashv \Gamma_0, \hat{\alpha} = \tau, \Gamma_1}_{\Gamma} \ \, \textbf{InstLSolve}}$$

Immediate, since to the left of $\hat{\alpha}$, the contexts Δ and Γ are the same.

Case

$$\frac{}{\Gamma[\hat{\alpha}][\hat{\beta}] \vdash \hat{\alpha} : \stackrel{\leq}{=} \hat{\beta} \dashv \Gamma[\hat{\alpha}][\hat{\beta} = \hat{\alpha}]} \text{ InstLReach}$$

Immediate, since to the left of $\hat{\alpha}$, the contexts Δ and Γ are the same.

$$\begin{array}{c} \bullet \ \, \textbf{Case} \\ \frac{\Gamma[\hat{\alpha}_2,\hat{\alpha}_1,\hat{\alpha}=\hat{\alpha}_1\rightarrow\hat{\alpha}_2] \vdash A_1 \stackrel{\leq}{=} \hat{\alpha}_1 \dashv \Gamma' \qquad \Gamma' \vdash \hat{\alpha}_2 : \stackrel{\leq}{=} [\Gamma'] A_2 \dashv \Delta}{\Gamma[\hat{\alpha}] \vdash \hat{\alpha} : \stackrel{\leq}{=} A_1 \rightarrow A_2 \dashv \Delta} \ \, \textbf{InstLArr} \\ \end{array}$$

We have $\hat{\beta} \in \mathsf{unsolved}(\Gamma_0)$. Therefore $\hat{\beta} \in \mathsf{unsolved}(\Gamma_0, \hat{\alpha}_2)$.

Clearly, $\hat{\alpha}_2 \in \mathsf{unsolved}(\Gamma_0, \hat{\alpha}_2)$.

We have two subderivations:

$$\Gamma_0, \hat{\alpha}_2, \hat{\alpha}_1, \hat{\alpha} = \hat{\alpha}_1 \to \hat{\alpha}_2, \Gamma_1 \vdash A_1 \stackrel{\leq}{=}: \hat{\alpha}_1 \dashv \Gamma' \tag{1}$$

$$\Gamma' \vdash \hat{\alpha}_2 : \leq [\Gamma'] A_2 \dashv \Delta \tag{2}$$

By induction on (1), $\hat{\beta} \in \mathsf{unsolved}(\Gamma')$.

Also by induction on (1), with $\hat{\alpha}_2$ playing the role of $\hat{\beta}$, we get $\hat{\alpha}_2 \in \mathsf{unsolved}(\Gamma')$.

Since $\hat{\beta} \in \Gamma_0$, it is declared to the left of $\hat{\alpha}_2$ in $\Gamma_0, \hat{\alpha}_2, \hat{\alpha}_1, \hat{\alpha} = \hat{\alpha}_1 \to \hat{\alpha}_2, \Gamma_1$.

Hence by Lemma 16 (Declaration Order Preservation), $\hat{\beta}$ is declared to the left of $\hat{\alpha}_2$ in Γ' . That is, $\Gamma' = (\Gamma'_0, \hat{\alpha}_2, \Gamma'_1)$, where $\hat{\beta} \in \mathsf{unsolved}(\Gamma'_0)$.

By induction on (2), $\hat{\beta} \in \mathsf{unsolved}(\Delta)$.

$$\frac{\Gamma_{0}, \hat{\alpha}, \Gamma_{1}, \beta \vdash \hat{\alpha} : \stackrel{\leq}{=} B \dashv \Delta, \beta, \Delta'}{\Gamma_{0}, \hat{\alpha}, \Gamma_{1} \vdash \hat{\alpha} : \stackrel{\leq}{=} \forall \beta, B \dashv \Delta} \text{InstLAIIR}$$

We have $\hat{\beta} \in \mathsf{unsolved}(\Gamma_0)$.

By induction, $\hat{\beta} \in \mathsf{unsolved}(\Delta, \beta, \Delta')$.

Note that $\hat{\beta}$ is declared to the left of β in Γ_0 , $\hat{\alpha}$, Γ_1 , β .

By Lemma 16 (Declaration Order Preservation), $\hat{\beta}$ is declared to the left of β in (Δ, β, Δ') , that is, in Δ . Since $\hat{\beta} \in \text{unsolved}(\Delta, \beta, \Delta')$, we have $\hat{\beta} \in \text{unsolved}(\Delta)$.

• Cases InstRSolve, InstRReach: Similar to the InstLSolve and InstLReach cases.

$$\begin{array}{c} \bullet \ \, \textbf{Case} \\ \frac{\Gamma[\hat{\alpha}_2,\hat{\alpha}_1,\hat{\alpha}=\hat{\alpha}_1\rightarrow\hat{\alpha}_2]\vdash \hat{\alpha}_1: \stackrel{\leq}{=} A_1\dashv \Gamma' \qquad \Gamma'\vdash [\Gamma']A_2\stackrel{\leq}{=}: \hat{\alpha}_2\dashv \Delta}{\Gamma[\hat{\alpha}]\vdash A_1\rightarrow A_2\stackrel{\leq}{=}: \hat{\alpha}\dashv \Delta} \ \, \textbf{InstRArr} \\ \end{array}$$

Similar to the InstLArr case.

$$\frac{\mathcal{E}}{\Gamma[\hat{\alpha}], \blacktriangleright_{\hat{\gamma}}, \hat{\gamma} \vdash [\hat{\gamma}/\beta]B \stackrel{\leq}{=} : \hat{\alpha} \dashv \Delta, \blacktriangleright_{\hat{\gamma}}, \Delta'} \text{InstRAIIL}$$

$$\Gamma[\hat{\alpha}] \vdash \forall \beta, B \stackrel{\leq}{=} : \hat{\alpha} \dashv \Delta$$

We have $\hat{\beta} \in \mathsf{unsolved}(\Gamma_0)$.

By induction, $\hat{\beta} \in \mathsf{unsolved}(\Delta, \blacktriangleright_{\hat{\gamma}}, \Delta')$.

Note that $\hat{\beta}$ is declared to the left of $\triangleright_{\hat{\gamma}}$ in Γ_0 , $\hat{\alpha}$, Γ_1 , $\triangleright_{\hat{\gamma}}$, $\hat{\gamma}$.

By Lemma 16 (Declaration Order Preservation), $\hat{\beta}$ is declared to the left of $\triangleright_{\hat{\gamma}}$ in Δ , $\triangleright_{\hat{\gamma}}$, Δ' .

Hence $\hat{\beta}$ is declared in Δ , and we know it is in unsolved $(\Delta, \blacktriangleright_{\hat{\gamma}}, \Delta')$, so $\hat{\beta} \in \text{unsolved}(\Delta)$.

Lemma 35 (Left Free Variable Preservation). If $\widehat{\Gamma_0}$, $\widehat{\alpha}$, $\widehat{\Gamma_1}$ $\vdash \widehat{\alpha} : \stackrel{\leq}{=} A \dashv \Delta$ or $\widehat{\Gamma_0}$, $\widehat{\alpha}$, $\widehat{\Gamma_1}$ $\vdash A \stackrel{\leq}{=} : \widehat{\alpha} \dashv \Delta$, and $\Gamma \vdash B \text{ and } \hat{\alpha} \notin FV([\Gamma]B) \text{ and } \hat{\beta} \in unsolved(\Gamma_0) \text{ and } \hat{\beta} \notin FV([\Gamma]B), \text{ then } \hat{\beta} \notin FV([\Delta]B).$

Proof. By induction on the given instantiation derivation.

Case

$$\underbrace{\frac{\Gamma_0 \vdash \tau}{\Gamma_0, \hat{\alpha}, \Gamma_1 \vdash \hat{\alpha} : \stackrel{\leq}{=} \tau \dashv \underbrace{\Gamma_0, \hat{\alpha} = \tau, \Gamma_1}_{\Delta}}_{\Gamma} \text{ InstLSolve}$$

We have $\hat{\alpha} \notin FV([\Gamma]B)$. Since Δ differs from Γ only in $\hat{\alpha}$, it must be the case that $[\Gamma]B = [\Delta]B$. It is given that $\hat{\beta} \notin FV([\Gamma]B)$, so $\hat{\beta} \notin FV([\Delta]B)$.

Case

$$\underbrace{\underline{\Gamma'[\hat{\alpha}][\hat{\gamma}]} \vdash \hat{\alpha} : \stackrel{\leq}{=} \hat{\gamma} \dashv \underbrace{\Gamma'[\hat{\alpha}][\hat{\gamma} = \hat{\alpha}]}_{\Delta} \text{ InstLReach}}$$

Since Δ differs from Γ only in solving $\hat{\gamma}$ to $\hat{\alpha}$, applying Δ to a type will not introduce a $\hat{\beta}$. We have $\hat{\beta} \notin FV([\Gamma]B)$, so $\hat{\beta} \notin FV([\Delta]B)$.

Case

$$\frac{\Gamma_0 \vdash \tau}{\Gamma_0, \hat{\alpha}, \Gamma_1 \vdash \tau \leqq: \hat{\alpha} \dashv \Gamma_0, \hat{\alpha} = \tau, \Gamma_1} \text{ InstRSolve}$$

Similar to the InstLSolve case.

Case

$$\frac{}{\Gamma'[\hat{\alpha}][\hat{\gamma}]\vdash\hat{\gamma}\stackrel{\leq}{=}:\hat{\alpha}\dashv\Gamma'[\hat{\alpha}][\hat{\gamma}=\hat{\alpha}]}\;\mathsf{InstRReach}$$

Similar to the InstLReach case.

Case

$$\underbrace{\frac{\Gamma'}{\Gamma_0, \hat{\alpha}_2, \hat{\alpha}_1, \hat{\alpha} = \hat{\alpha}_1 \to \hat{\alpha}_2, \Gamma_1 \vdash A_1 \stackrel{\leq}{=} \hat{\alpha}_1 \dashv \Delta}_{\Gamma_0, \hat{\alpha}, \Gamma_1 \vdash \hat{\alpha} : \stackrel{\leq}{=} A_1 \to A_2 \dashv \Delta} = \underbrace{\Gamma_0, \hat{\alpha}, \Gamma_1 \vdash \hat{\alpha} : \stackrel{\leq}{=} A_1 \to A_2 \dashv \Delta}_{\Gamma_0, \hat{\alpha}, \Gamma_1 \vdash \hat{\alpha} : \stackrel{\leq}{=} A_1 \to A_2 \dashv \Delta} = \underbrace{\Gamma_0, \hat{\alpha}, \Gamma_1 \vdash \hat{\alpha} : \stackrel{\leq}{=} A_1 \to A_2 \dashv \Delta}_{\Gamma_0, \hat{\alpha}, \Gamma_1 \vdash \hat{\alpha} : \stackrel{\leq}{=} A_1 \to A_2 \dashv \Delta} = \underbrace{\Gamma_0, \hat{\alpha}, \Gamma_1 \vdash \hat{\alpha} : \stackrel{\leq}{=} A_1 \to A_2 \dashv \Delta}_{\Gamma_0, \hat{\alpha}, \Gamma_1 \vdash \hat{\alpha} : \stackrel{\leq}{=} A_1 \to A_2 \dashv \Delta}_{\Gamma_0, \hat{\alpha}, \Gamma_1 \vdash \hat{\alpha} : \stackrel{\tilde{\alpha}}{=} A_1 \to A_2 \dashv \Delta}$$

We have $\Gamma \vdash B$ and $\hat{\alpha} \notin FV([\Gamma]B)$ and $\hat{\beta} \notin FV([\Gamma]B)$.

By weakening, we get $\Gamma' \vdash B$; since $\hat{\alpha} \notin FV([\Gamma]B)$ and Γ' only adds a solution for $\hat{\alpha}$, it follows that $[\Gamma']B = [\Gamma]B$.

Therefore $\hat{\alpha}_1 \notin FV([\Gamma']B)$ and $\hat{\alpha}_2 \notin FV([\Gamma']B)$ and $\hat{\beta} \notin FV([\Gamma']B)$.

Since we have $\hat{\beta} \in \Gamma_0$, we also have $\hat{\beta} \in (\Gamma_0, \hat{\alpha}_2)$.

By induction on the first premise, $\hat{\beta} \notin FV([\Delta]B)$.

Also by induction on the first premise, with $\hat{\alpha}_2$ playing the role of $\hat{\beta}$, we have $\hat{\alpha}_2 \notin FV([\Delta]B)$.

Note that $\hat{\alpha}_2 \in \mathsf{unsolved}(\Gamma_0, \hat{\alpha}_2)$.

By Lemma 34 (Left Unsolvedness Preservation), $\hat{\alpha}_2 \in \mathsf{unsolved}(\Delta)$.

Therefore Δ has the form $(\Delta_0, \hat{\alpha}_2, \Delta_1)$.

Since $\hat{\beta} \neq \hat{\alpha}_2$, we know that $\hat{\beta}$ is declared to the left of $\hat{\alpha}_2$ in Γ_0 , $\hat{\alpha}_2$, so by Lemma 16 (Declaration Order Preservation $\hat{\beta}$ is declared to the left of $\hat{\alpha}_2$ in Δ . Hence $\hat{\beta} \in \Delta_0$.

Furthermore, by Lemma 32 (Instantiation Extension), we have $\Gamma' \longrightarrow \Delta$.

Then by Lemma 25 (Extension Weakening), we have $\Delta \vdash B$. Using induction on the second premise, $\hat{\beta} \notin FV([\Delta]B)$.

Case

$$\frac{\Gamma_{0}, \hat{\alpha}, \Gamma_{1}, \gamma \vdash \hat{\alpha} : \stackrel{\leq}{=} C \dashv \Delta, \gamma, \Delta'}{\underbrace{\Gamma_{0}, \hat{\alpha}, \Gamma_{1}}_{\Gamma} \vdash \hat{\alpha} : \stackrel{\leq}{=} \forall \gamma. \ C \dashv \Delta} \text{ InstLAIIR}$$

We have $\Gamma \vdash B$ and $\hat{\alpha} \notin FV([\Gamma]B)$ and $\hat{\beta} \in \Gamma_0$ and $\hat{\beta} \notin FV([\Gamma]B)$.

By weakening, $\Gamma, \gamma \vdash B$; by the definition of substitution, $[\Gamma, \gamma]B = [\Gamma]B$.

Substituting equals for equals, $\hat{\alpha} \notin FV([\Gamma, \gamma]B)$ and $\hat{\beta} \notin FV([\Gamma, \gamma]B)$.

By induction, $\hat{\beta} \notin FV([\Delta, \gamma, \Delta']B)$.

Since $\hat{\beta}$ is declared to the left of γ in (Γ, γ) , we can use Lemma 16 (Declaration Order Preservation) to show that $\hat{\beta}$ is declared to the left of γ in $(\Delta, \gamma, \Delta')$, that is, in Δ .

We have $\Gamma \vdash B$, so $\gamma \notin FV(B)$. Thus each free variable u in B is in Γ , to the left of γ in (Γ, γ) .

Therefore, by Lemma 16 (Declaration Order Preservation), each free variable u in B is in Δ .

Therefore $[\Delta, \gamma, \Delta']B = [\Delta]B$.

Earlier, we obtained $\hat{\beta} \notin FV([\Delta, \gamma, \Delta']B)$, so substituting equals for equals, $\hat{\beta} \notin FV([\Delta]B)$.

Case

$$\frac{\Gamma_0, \hat{\alpha}_2, \hat{\alpha}_1, \hat{\alpha} = \hat{\alpha}_1 \rightarrow \hat{\alpha}_2, \Gamma_1 \vdash \hat{\alpha}_1 : \stackrel{\leq}{=} A_1 \dashv \Delta \qquad \Gamma' \vdash [\Delta] A_2 \stackrel{\leq}{=} : \hat{\alpha}_2 \dashv \Delta}{\Gamma_0, \hat{\alpha}, \Gamma_1 \vdash A_1 \rightarrow A_2 \stackrel{\leq}{=} : \hat{\alpha} \dashv \Delta} \text{InstRArr}$$

Similar to the InstLArr case.

• Case $\frac{\Gamma[\hat{\alpha}], \blacktriangleright_{\hat{\gamma}}, \hat{\gamma} \vdash [\hat{\gamma}/\gamma]C \stackrel{\leq}{=} : \hat{\alpha} \dashv \Delta, \blacktriangleright_{\hat{\gamma}}, \Delta'}{\Gamma[\hat{\alpha}] \vdash \forall \gamma, C \stackrel{\leq}{=} : \hat{\alpha} \dashv \Delta} \text{InstRAIIL}$

We have $\Gamma \vdash B$ and $\hat{\alpha} \notin FV([\Gamma]B)$ and $\hat{\beta} \in \Gamma_0$ and $\hat{\beta} \notin FV([\Gamma]B)$.

By weakening, $\Gamma, \blacktriangleright_{\hat{\gamma}}, \hat{\gamma} \vdash B$; by the definition of substitution, $[\Gamma, \blacktriangleright_{\hat{\gamma}}, \hat{\gamma}]B = [\Gamma]B$.

Substituting equals for equals, $\hat{\alpha} \notin FV([\Gamma, \blacktriangleright_{\hat{\gamma}}, \hat{\gamma}]B)$ and $\hat{\beta} \notin FV([\Gamma, \blacktriangleright_{\hat{\gamma}}, \hat{\gamma}]B)$.

By induction, $\hat{\beta} \notin FV([\Delta, \blacktriangleright_{\hat{\gamma}}, \Delta']B)$.

Note that $\hat{\beta}$ is declared to the left of $\triangleright_{\hat{\gamma}}$ in $\Gamma, \triangleright_{\hat{\gamma}}, \hat{\gamma}$.

By Lemma 16 (Declaration Order Preservation), $\hat{\beta}$ is declared to the left of $\triangleright_{\hat{\gamma}}$ in Δ , $\triangleright_{\hat{\gamma}}$, Δ' .

So $\hat{\beta}$ is declared in Δ .

Now, note that each free variable $\mathfrak u$ in B is in Γ , which is to the left of $\blacktriangleright \gamma$ in $\Gamma, \blacktriangleright \gamma, \hat{\gamma}$.

Therefore, by Lemma 16 (Declaration Order Preservation), each free variable u in B is in Δ .

Therefore $[\Delta, \blacktriangleright_{\hat{\gamma}}, \Delta']B = [\Delta]B$.

Earlier, we obtained $\hat{\beta} \notin FV([\Delta, \blacktriangleright_{\hat{\gamma}}, \Delta']B)$, so substituting equals for equals, $\hat{\beta} \notin FV([\Delta]B)$.

Lemma 36 (Instantiation Size Preservation). If $\overbrace{\Gamma_0, \hat{\alpha}, \Gamma_1}^{\Gamma} \vdash \hat{\alpha} : \stackrel{\leq}{=} A \dashv \Delta \text{ or } \overbrace{\Gamma_0, \hat{\alpha}, \Gamma_1}^{\Gamma} \vdash A \stackrel{\leq}{=} \hat{\alpha} \dashv \Delta$, and $\Gamma \vdash B$ and $\hat{\alpha} \notin FV([\Gamma]B)$, then $|[\Gamma]B| = |[\Delta]B|$, where |C| is the plain size of the term C.

Proof. By induction on the given derivation.

$\bullet \ \ \textbf{Case} \ \ \underbrace{\frac{\Gamma_0 \vdash \tau}{\Gamma_0, \hat{\alpha}, \Gamma_1 \vdash \hat{\alpha} : \stackrel{\leq}{=} \tau \dashv \Gamma_0, \hat{\alpha} = \tau, \Gamma_1}_{\text{InstLSolve}}}_{}_{\text{InstLSolve}}$

Since Δ differs from Γ only in solving $\hat{\alpha}$, and we know $\hat{\alpha} \notin FV([\Gamma]B)$, we have $[\Delta]B = [\Gamma]B$; therefore $[\Delta]B = [\Gamma]B|$.

Case

$$\frac{1}{\Gamma[\hat{\alpha}][\hat{\beta}] \vdash \hat{\alpha} : \stackrel{\leq}{=} \hat{\beta} \dashv \Gamma[\hat{\alpha}][\hat{\beta} = \hat{\alpha}]} \text{ InstLReach}$$

Here, Δ differs from Γ only in solving $\hat{\beta}$ to $\hat{\alpha}$. However, $\hat{\alpha}$ has the same size as $\hat{\beta}$, so even if $\hat{\beta} \in FV([\Gamma]B)$, we have $|[\Delta]B = [\Gamma]B|$.

• Case

$$\frac{\overbrace{\Gamma_{0}, \hat{\alpha}_{2}, \hat{\alpha}_{1}, \hat{\alpha} = \hat{\alpha}_{1} \rightarrow \hat{\alpha}_{2}, \Gamma_{1}}^{\Gamma'} \vdash A_{1} \stackrel{\leq}{=} \hat{\alpha}_{1} \rightarrow \Theta}{\underbrace{\Gamma_{0}, \hat{\alpha}, \Gamma_{1}}} \vdash \hat{\alpha} \stackrel{\leq}{=} A_{1} \rightarrow A_{2} \rightarrow \Delta} = \frac{\left[\Theta\right] A_{2} \rightarrow \Delta}{\left[\Phi\right] A_{2} \rightarrow \Delta} = \frac{\left[\Phi\right] A_{2} \rightarrow \Delta}{\left[\Phi$$

We have $\Gamma \vdash B$ and $\hat{\alpha} \notin FV([\Gamma]B)$. Since $\hat{\alpha}_1, \hat{\alpha}_2 \notin dom(\Gamma)$, we have $\hat{\alpha}, \hat{\alpha}_1, \hat{\alpha}_2 \notin FV([\Gamma]B)$. It follows that $[\Gamma']B = [\Gamma]B$.

By weakening, $\Gamma' \vdash B$.

By induction on the first premise, $|[\Gamma']B| = |[\Theta]B|$.

By Lemma 16 (Declaration Order Preservation), since $\hat{\alpha}_2$ is declared to the left of $\hat{\alpha}_1$ in Γ' , we have that $\hat{\alpha}_2$ is declared to the left of $\hat{\alpha}_1$ in Θ .

By Lemma 34 (Left Unsolvedness Preservation), since $\hat{\alpha}_2 \in \mathsf{unsolved}(\Gamma')$, it is unsolved in Θ : that is, $\Theta = (\Theta_0, \hat{\alpha}_2, \Theta_1)$.

By Lemma 32 (Instantiation Extension), we have $\Gamma' \longrightarrow \Theta$.

By Lemma 25 (Extension Weakening), $\Theta \vdash B$.

Since $\hat{\alpha}_2 \notin FV([\Gamma']B)$, Lemma 35 (Left Free Variable Preservation) gives $\hat{\alpha}_2 \notin FV([\Theta]B)$.

By induction on the second premise, $|[\Theta]B| = |[\Delta]B|$, and by transitivity of equality, $|[\Gamma]B| = |[\Delta]B|$.

• Case

$$\frac{\Gamma_{0}, \hat{\alpha}, \Gamma_{1}, \beta \vdash \hat{\alpha} : \stackrel{\leq}{=} A_{0} \dashv \Delta, \beta, \Delta'}{\underbrace{\Gamma_{0}, \hat{\alpha}, \Gamma_{1}}_{\Gamma} \vdash \hat{\alpha} : \stackrel{\leq}{=} \forall \beta. A_{0} \dashv \Delta} \text{ InstLAIIR}$$

We have $\Gamma \vdash B$ and $\hat{\alpha} \notin FV([\Gamma]B)$.

By weakening, Γ , $\beta \vdash B$.

From the definition of substitution, $[\Gamma]B = [\Gamma, \beta]B$. Hence $\hat{\alpha} \notin FV([\Gamma, \beta]B)$.

The input context of the premise is $(\Gamma_0, \hat{\alpha}, \Gamma_1, \beta)$, which is (Γ, β) , so by induction, $|[\Gamma, \beta]B| = |[\Delta, \beta, \Delta']B|$. Suppose u is a free variable in B. Then u is declared in Γ , and so occurs before β in Γ, β .

By Lemma 16 (Declaration Order Preservation), u is declared before β in Δ , β , Δ' .

So every free variable u in B is declared in Δ .

Hence $[\Delta, \beta, \Delta']B = [\Delta]B$.

We have $[\Gamma]B = [\Gamma, \beta]B$, so $|[\Gamma]B| = |[\Gamma, \beta]B|$; by transitivity of equality, $|[\Gamma]B| = |[\Delta]B|$.

Case

$$\frac{\Gamma_0 \vdash \tau}{\Gamma_0, \hat{\alpha}, \Gamma_1 \vdash \tau \stackrel{\leq}{=}: \hat{\alpha} \dashv \Gamma_0, \hat{\alpha} = \tau, \Gamma_1} \text{ InstRSolve}$$

Similar to the InstLSolve case.

Case

$$\frac{}{\Gamma[\hat{\alpha}][\hat{\beta}] \vdash \hat{\beta} \stackrel{\leq}{=} \hat{\alpha} \dashv \Gamma[\hat{\alpha}][\hat{\beta} = \hat{\alpha}]} \mathsf{InstRReach}$$

Similar to the InstLReach case.

Case

$$\frac{\overbrace{\Gamma_{0}, \hat{\alpha}_{2}, \hat{\alpha}_{1}, \hat{\alpha} = \hat{\alpha}_{1} \rightarrow \hat{\alpha}_{2}, \Gamma_{1}}}{\underbrace{\Gamma_{0}, \hat{\alpha}_{1}, \hat{\alpha}_{1} = \hat{\alpha}_{1} \rightarrow \hat{\alpha}_{2}, \Gamma_{1}}} \vdash \hat{\alpha}_{1} : \stackrel{\leq}{=} A_{1} \dashv \Theta \qquad \Theta \vdash [\Theta] A_{2} \stackrel{\leq}{=} : \hat{\alpha}_{2} \dashv \Delta} \\
\underbrace{\Gamma_{0}, \hat{\alpha}_{1}, \hat{\alpha}_{1} = \hat{\alpha}_{1} \rightarrow \hat{\alpha}_{2}, \Gamma_{1}}}_{\Gamma} \vdash A_{1} \rightarrow A_{2} \stackrel{\leq}{=} : \hat{\alpha} \dashv \Delta} \quad \text{InstRArr}$$

Similar to the InstLArr case.

 $\bullet \ \, \textbf{Case} \ \, \frac{\Gamma'[\hat{\alpha}], \blacktriangleright_{\hat{\beta}}, \hat{\beta} \vdash [\hat{\beta}/\beta] A_0 \mathrel{\stackrel{\scriptstyle \leq}{\scriptscriptstyle :}} \hat{\alpha} \dashv \Delta, \blacktriangleright_{\hat{\beta}}, \Delta'}{\Gamma'[\hat{\alpha}] \vdash \forall \beta. \, A_0 \mathrel{\stackrel{\scriptstyle \leq}{\scriptscriptstyle :}} \hat{\alpha} \dashv \Delta} \, \text{InstRAIIL}$

We have $\Gamma \vdash B$ and $\hat{\alpha} \notin FV([\Gamma]B)$.

By weakening, $\Gamma, \triangleright_{\widehat{\beta}}, \widehat{\beta} \vdash B$.

From the definition of substitution, $[\Gamma]B = [\Gamma, \blacktriangleright_{\hat{G}}, \hat{\beta}]B$. Hence $\hat{\alpha} \notin FV([\Gamma, \blacktriangleright_{\hat{G}}, \hat{\beta}]B)$.

By induction, $|[\Gamma, \blacktriangleright_{\widehat{\beta}}, \widehat{\beta}]B| = |[\Delta, \blacktriangleright_{\widehat{\beta}}, \Delta']B|$.

Suppose u is a free variable in B.

Then \mathfrak{u} is declared in Γ , and so occurs before $\blacktriangleright_{\hat{\beta}}$ in $\Gamma, \blacktriangleright_{\hat{\beta}}, \hat{\beta}$.

By Lemma 16 (Declaration Order Preservation), \mathfrak{u} is declared before $\blacktriangleright_{\hat{\mathfrak{h}}}$ in $\Delta, \blacktriangleright_{\hat{\mathfrak{h}}}, \Delta'$.

So every free variable u in B is declared in Δ .

Hence $[\Delta, \blacktriangleright_{\widehat{B}}, \Delta']B = [\Delta]B$.

Since $[\Gamma]B = [\Gamma, \blacktriangleright_{\hat{G}}, \hat{\beta}]B$, we have $|[\Gamma]B| = |[\Gamma, \blacktriangleright_{\hat{G}}, \hat{\beta}]B|$; by transitivity of equality, $|[\Gamma]B| = |[\Delta]B|$.

Theorem 7 (Decidability of Instantiation). If $\Gamma = \Gamma_0[\hat{\alpha}]$ and $\Gamma \vdash A$ such that $[\Gamma]A = A$ and $\hat{\alpha} \notin FV(A)$,

- (1) Either there exists Δ such that $\Gamma_0[\hat{\alpha}] \vdash \hat{\alpha} := A \dashv \Delta$, or not.
- (2) Either there exists Δ such that $\Gamma_0[\hat{\alpha}] \vdash A \leq : \hat{\alpha} \dashv \Delta$, or not.

Proof. By induction on the derivation of $\Gamma \vdash A$.

- (1) $\Gamma \vdash \hat{\alpha} := A \dashv \Delta$ is decidable.
 - Case

$$\underbrace{\overline{\Gamma_L, \hat{\alpha}, \Gamma_R} \vdash \alpha}_{\Gamma'[\alpha]} \mathsf{UvarWF}$$

If $\alpha \in \Gamma_L$, then by UvarWF we have $\Gamma_L \vdash \alpha$, and by rule InstLSolve we have a derivation. Otherwise no rule matches, and so no derivation exists.

• Case UnitWF: By rule InstLSolve.

Case

$$\underbrace{\overline{\Gamma_L, \hat{\alpha}, \Gamma_R} \vdash \ \hat{\beta}}_{\Gamma} \text{ EvarWF}$$

By inversion, we have $\hat{\beta} \in \Gamma$, and $[\Gamma]\hat{\beta} = \hat{\beta}$. Since $\hat{\alpha} \notin FV([\Gamma]\hat{\beta}) = FV(\hat{\beta}) = \{\hat{\beta}\}$, it follows that $\hat{\alpha} \neq \hat{\beta}$: Either $\hat{\beta} \in \Gamma_L$ or $\hat{\beta} \in \Gamma_R$.

If $\hat{\beta} \in \Gamma_L$, then we have a derivation by InstLSolve.

If $\hat{\beta} \in \Gamma_R$, then we have a derivation by InstLReach.

Case

$$\underbrace{\overline{\Gamma'[\hat{\beta}=\tau]}\vdash \hat{\beta}}_{\Gamma} \ \mathsf{SolvedEvarWF}$$

It is given that $[\Gamma]\hat{\beta} = \hat{\beta}$, so this case is impossible.

$$\bullet \ \, \textbf{Case} \ \, \underbrace{\frac{\Gamma \vdash A_1 \qquad \Gamma \vdash A_2}{\prod_L, \hat{\alpha}, \Gamma_R} \vdash A_1 \to A_2}_{\Gamma} \ \, \text{ArrowWF}$$

By assumption, $[\Gamma](A_1 \to A_2) = A_1 \to A_2$ and $\hat{\alpha} \notin FV([\Gamma](A_1 \to A_2))$.

If $A_1 \to A_2$ is a monotype and is well-formed under Γ_L , we can apply InstLSolve.

Otherwise, the only rule with a conclusion matching $A_1 \rightarrow A_2$ is InstLArr.

First, consider whether Γ_L , $\hat{\alpha}_2$, $\hat{\alpha}_1$, $\hat{\alpha} = \hat{\alpha}_1 \rightarrow \hat{\alpha}_2$, $\Gamma_R \vdash A \stackrel{\leq}{=}: \hat{\alpha}_1 \dashv -$ is decidable.

By definition of substitution, $[\Gamma](A_1 \to A_2) = ([\Gamma]A_1) \to ([\Gamma]A_2)$. Since $[\Gamma](A_1 \to A_2) = A_1 \to A_2$ A_2 , we have $[\Gamma]A_1 = A_1$ and $[\Gamma]A_2 = A_2$.

By weakening, Γ_L , $\hat{\alpha}_2$, $\hat{\alpha}_1$, $\hat{\alpha} = \hat{\alpha}_1 \rightarrow \hat{\alpha}_2$, $\Gamma_R \vdash A_1 \rightarrow A_2$.

Since $\Gamma \vdash A_1$ and $\Gamma \vdash A_2$, we have $\hat{\alpha}_1, \hat{\alpha}_2 \notin FV(A_1) \cup FV(A_2)$.

Since $\hat{\alpha} \notin FV(A) \supseteq FV(A_1)$, it follows that $[\Gamma']A_1 = A_1$.

By i.h., either there exists Θ such that Γ_L , $\hat{\alpha}_2$, $\hat{\alpha}_1$, $\hat{\alpha}=\hat{\alpha}_1\to\hat{\alpha}_2$, $\Gamma_R\vdash A_1\stackrel{\leq}{=}:\hat{\alpha}_1\dashv\Theta$, or not.

If not, then no derivation by InstLArr exists.

If so, then we have Γ_L , $\hat{\alpha}_2$, $\hat{\alpha}_1$, $\hat{\alpha} = \hat{\alpha}_1 \rightarrow \hat{\alpha}_2$, $\Gamma_R \vdash \hat{\alpha}_1 := A_1 \dashv \Theta$.

By Lemma 34 (Left Unsolvedness Preservation), we know that $\hat{\alpha}_2 \in \mathsf{unsolved}(\Theta)$.

By Lemma 35 (Left Free Variable Preservation), we know that $\hat{\alpha}_2 \notin FV([\Theta]A_2)$.

Clearly, $[\Theta]([\Theta]A_2) = [\Theta]A_2$.

Hence by i.h., either there exists Δ such that $\Theta \vdash \hat{\alpha}_2 := [\Theta]A_2 \dashv \Delta$, or not.

If not, then no derivation by InstLArr exists.

If it does, then by rule InstLArr, we have $\Gamma \vdash \hat{\alpha} : \leq A \dashv \Delta$.

• Case
$$\frac{\Gamma, \alpha \vdash A_0}{\Gamma \vdash \forall \alpha. A_0} \text{ For all WF}$$

We have $\forall \alpha$. $A_0 = [\Gamma](\forall \alpha$. $A_0)$. By definition of substitution, $[\Gamma](\forall \alpha$. $A_0) = \forall \alpha$. $[\Gamma]A_0$, so $A_0 = [\Gamma]A_0$.

By definition of substitution, $[\Gamma, \alpha]A_0 = [\Gamma]A_0$.

We have $\hat{\alpha} \notin FV([\Gamma](\forall \alpha. A_0))$. Therefore $\hat{\alpha} \notin FV([\Gamma]A_0) = FV([\Gamma, \alpha]A_0)$.

By i.h., either there exists Θ such that $\Gamma, \alpha \vdash \hat{\alpha} : \stackrel{\leq}{=} A_0 \dashv \Theta$, or not.

Suppose $\Gamma, \alpha \vdash \hat{\alpha} : \leq A_0 \dashv \Theta$.

By Lemma 32 (Instantiation Extension), $\Gamma \longrightarrow \Theta$;

by Lemma 24 (Extension Order) (i), $\Theta = \Delta, \alpha, \Delta'$.

Hence by rule InstLAIIR, $\Gamma \vdash \hat{\alpha} : \leq \forall \alpha. A_0 \dashv \Delta$.

Then there is no derivation, since InstLAllR is the only rule matching $\forall \alpha$. A_0 .

(2) $\Gamma \vdash A \stackrel{\leq}{=} : \hat{\alpha} \dashv \Delta \text{ is decidable.}$

• Case UvarWF:

Similar to the UvarWF case in part (1), but applying rule InstRSolve instead of InstLSolve.

• Case UnitWF: Apply InstRSolve.

Case

$$\underbrace{\overline{\Gamma_L, \hat{\alpha}, \Gamma_R} \vdash \ \hat{\beta}}_{\Gamma} \text{ EvarWF}$$

Similar to the EvarWF case in part (1), but applying InstRSolve/InstRReach instead of InstLSolve/InstLReach.

Case SolvedEvarWF:

Impossible, for exactly the same reasons as in the SolvedEvarWF case of part (1).

 $\bullet \ \, \textbf{Case} \ \, \underbrace{\frac{\Gamma \vdash A_1 \qquad \Gamma \vdash A_2}{\Gamma_L, \hat{\alpha}, \Gamma_R \vdash A_1 \to A_2}}_{} \ \, \textbf{ArrowWF}$

As the ArrowWF case of part (1), except applying InstRArr instead of InstLArr.

Case

$$\underbrace{\frac{\Gamma, \beta \vdash B}{\Gamma_{L}, \hat{\alpha}, \Gamma_{R} \vdash \forall \beta. B}}_{\Gamma} \text{ For all WF}$$

By assumption, $[\Gamma](\forall \beta. B) = \forall \beta. B$. With the definition of substitution, we get $[\Gamma]B = B$. Hence $[\Gamma]B = B$.

Hence $[\hat{\beta}/\beta][\Gamma]B = [\hat{\beta}/\beta]B$. Since $\hat{\beta}$ is fresh, $[\hat{\beta}/\beta][\Gamma]B = [\Gamma][\hat{\beta}/\beta]B$.

By definition of substitution, $[\Gamma, \blacktriangleright_{\hat{\beta}}, \hat{\beta}][\hat{\beta}/\beta]B = [\Gamma][\hat{\beta}/\beta]B$, which by transitivity of equality is $[\hat{\beta}/\beta]B$.

We have $\hat{\alpha} \notin FV([\Gamma](\forall \beta. B))$, so $\hat{\alpha} \notin FV([\Gamma, \blacktriangleright_{\hat{\beta}}, \hat{\beta}][\hat{\beta}/\beta]B)$.

Therefore, by induction, either $\Gamma, \blacktriangleright_{\hat{\beta}}, \hat{\beta} \vdash [\hat{\beta}/\beta]B \stackrel{\leq}{=}: \hat{\alpha} \dashv \Theta$ or not.

Suppose $\Gamma, \blacktriangleright_{\widehat{\beta}}, \widehat{\beta} \vdash [\widehat{\beta}/\beta]B \stackrel{\leq}{=} \widehat{\alpha} \dashv \Theta$.

By Lemma 32 (Instantiation Extension), $\Gamma, \blacktriangleright_{\hat{\beta}}, \hat{\beta} \longrightarrow \Theta;$

by Lemma 24 (Extension Order) (ii), $\Theta = \Delta, \triangleright_{\hat{G}}, \Delta'$.

Hence by rule InstRAIIL, $\Gamma \vdash \forall \beta$. B \leq : $\hat{\alpha} \dashv \Delta$.

Suppose not.

Then there is no derivation, since InstRAIL is the only rule matching $\forall \beta$. B.

F' Decidability of Algorithmic Subtyping

F'.1 Lemmas for Decidability of Subtyping

Lemma 37 (Monotypes Solve Variables). *If* $\Gamma \vdash \hat{\alpha} := \tau \dashv \Delta$ *or* $\Gamma \vdash \tau = \hat{\alpha} \dashv \Delta$, *then if* $[\Gamma]\tau = \tau$ *and* $\hat{\alpha} \notin FV([\Gamma]\tau)$, *then* $[\text{unsolved}(\Gamma)] = [\text{unsolved}(\Delta)] + 1$.

Proof. By induction on the given derivation.

Case

$$\frac{\Gamma_L \vdash \tau}{\Gamma_L, \hat{\alpha}, \Gamma_R \vdash \hat{\alpha} : \stackrel{\leq}{=} \tau \dashv \underbrace{\Gamma_L, \hat{\alpha} = \tau, \Gamma_R}_{\Delta}} \text{ InstLSolve}$$

It is evident that $|\mathsf{unsolved}(\Gamma_L, \hat{\alpha}, \Gamma_R)| = |\mathsf{unsolved}(\Gamma_L, \hat{\alpha} = \tau, \Gamma_R)| + 1$.

Case

$$\frac{}{\Gamma[\hat{\alpha}][\hat{\beta}] \vdash \; \hat{\alpha} : \stackrel{\leq}{=} \hat{\beta} \; \dashv \Gamma[\hat{\alpha}][\hat{\beta} = \hat{\alpha}]} \; \mathsf{InstLReach}$$

Similar to the previous case.

$$\begin{array}{c} \bullet \; \text{Case} \\ & \frac{\Gamma_0[\hat{\alpha}_2,\hat{\alpha}_1,\hat{\alpha}=\hat{\alpha}_1 \to \hat{\alpha}_2] \vdash \tau_1 \stackrel{\leq}{=}: \hat{\alpha}_1 \dashv \Theta \qquad \Theta \vdash \hat{\alpha}_2 : \stackrel{\leq}{=} [\Theta] \tau_2 \dashv \Delta}{\Gamma_0[\hat{\alpha}] \vdash \hat{\alpha} : \stackrel{\leq}{=} \tau_1 \to \tau_2 \dashv \Delta} \; \text{InstLArr} \\ & | \text{unsolved}(\Gamma_0[\hat{\alpha}_2,\hat{\alpha}_1,\hat{\alpha}=\hat{\alpha}_1 \to \hat{\alpha}_2])| = | \text{unsolved}(\Gamma_0[\hat{\alpha}])| + 1 \quad \text{Immediate} \\ & | \text{unsolved}(\Gamma_0[\hat{\alpha}_2,\hat{\alpha}_1,\hat{\alpha}=\hat{\alpha}_1 \to \hat{\alpha}_2])| = | \text{unsolved}(\Theta)| + 1 \quad \text{By i.h.} \\ & | \text{unsolved}(\Gamma)| = | \text{unsolved}(\Theta)| & \text{Subtracting 1} \\ \end{array}$$

 $= |\mathsf{unsolved}(\Delta)| + 1$

By i.h.

• Case
$$\frac{\Gamma, \beta \vdash \hat{\alpha} : \stackrel{\leq}{=} B \dashv \Delta, \beta, \Delta'}{\Gamma \vdash \hat{\alpha} : \stackrel{\leq}{=} \forall \beta. \ B \dashv \Delta} \text{ InstLAIIR}$$

This case is impossible, since a monotype cannot have the form $\forall \beta$. B.

- Cases InstRSolve, InstRReach: Similar to the InstLSolve and InstLReach cases.
- Case InstRArr: Similar to the InstLArr case.

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma[\hat{\alpha}], \, \beta \vdash \, B \, \stackrel{\leq}{=} : \, \hat{\alpha} \, \dashv \Delta, \, \beta, \Delta'}{\Gamma[\hat{\alpha}] \vdash \, \forall \beta . \, B \, \stackrel{\leq}{=} : \, \hat{\alpha} \, \dashv \Delta} \ \, \textbf{InstRAIIL}$$

This case is impossible, since a monotype cannot have the form $\forall \beta$. B.

Lemma 38 (Monotype Monotonicity). *If* $\Gamma \vdash \tau_1 \mathrel{<:} \tau_2 \dashv \Delta \textit{ then } |\mathsf{unsolved}(\Delta)| \leq |\mathsf{unsolved}(\Gamma)|$.

Proof. By induction on the given derivation.

- Cases <: Var, <: Exvar: In these rules, $\Delta = \Gamma$, so unsolved(Δ) = unsolved(Γ); therefore |unsolved(Δ)| \leq |unsolved(Γ)|.
- Case $\langle : \rightarrow :$ We have an intermediate context Θ .

By inversion, $\tau_1 = \tau_{11} \to \tau_{12}$ and $\tau_2 = \tau_{21} \to \tau_{22}$. Therefore, we have monotypes in the first and second premises.

By induction on the first premise, $|\mathsf{unsolved}(\Theta)| \leq |\mathsf{unsolved}(\Gamma)|$. By induction on the second premise, $|\mathsf{unsolved}(\Delta)| \leq |\mathsf{unsolved}(\Theta)|$. By transitivity of \leq , $|\mathsf{unsolved}(\Delta)| \leq |\mathsf{unsolved}(\Gamma)|$, which was to be shown.

- Cases <:∀L, <:∀R: We are given a derivation of subtyping on monotypes, so these cases are impossible.
- Cases <: InstantiateL, <: InstantiateR: The input and output contexts in the premise exactly match the conclusion, so the result follows by Lemma 37 (Monotypes Solve Variables). □

Lemma 39 (Substitution Decreases Size). *If* $\Gamma \vdash A$ *then* $|\Gamma \vdash [\Gamma]A| \leq |\Gamma \vdash A|$.

Proof. By induction on $|\Gamma \vdash A|$. If A = 1 or $A = \alpha$, or $A = \hat{\alpha}$ and $\hat{\alpha} \in \mathsf{unsolved}(\Gamma)$ then $[\Gamma]A = A$. Therefore, $|\Gamma \vdash [\Gamma]A| = |\Gamma \vdash A|$.

If $A=\hat{\alpha}$ and $(\hat{\alpha}=\tau)\in \Gamma$, then by induction hypothesis, $|\Gamma\vdash [\Gamma]\tau|\leq |\Gamma\vdash \tau|$. Of course $|\Gamma\vdash \tau|\leq |\Gamma\vdash \tau|+1$. By definition of substitution, $[\Gamma]\tau=[\Gamma]\hat{\alpha}$, so

$$|\Gamma \vdash [\Gamma] \widehat{\alpha}| \leq |\Gamma \vdash \tau| + 1$$

By the definition of type size, $|\Gamma \vdash \hat{\alpha}| = |\Gamma \vdash \tau| + 1$, so

$$|\Gamma \vdash [\Gamma] \hat{\alpha}| < |\Gamma \vdash \hat{\alpha}|$$

which was to be shown.

If $A = A_1 \rightarrow A_2$, the result follows via the induction hypothesis (twice).

If $A = \forall \alpha$. A_0 , the result follows via the induction hypothesis.

Lemma 40 (Monotype Context Invariance).

If
$$\Gamma \vdash \tau \lt : \tau' \dashv \Delta$$
 where $[\Gamma]\tau = \tau$ and $[\Gamma]\tau' = \tau'$ and $[\tau]\tau' = \tau'$

Proof. By induction on the derivation of $\Gamma \vdash \tau <: \tau' \dashv \Delta$.

• Cases <: Var, <: Unit, <: Exvar:

In these rules, the output context is the same as the input context, so the result is immediate.

We have that $[\Gamma](\tau_1 \to \tau_2) = \tau_1 \to \tau_2$. By definition of substitution, $[\Gamma]\tau_1 = \tau_1$ and $[\Gamma]\tau_2 = \tau_2$. Similarly, $[\Gamma]\tau_1 = \tau_1'$ and $[\Gamma]\tau_2 = \tau_2'$.

By i.h.,
$$\Theta = \Gamma$$
.

Since Θ is predicative, $[\Theta]\tau_2$ and $[\Theta]\tau_2'$ are monotypes.

Substitution is idempotent: $[\Theta][\Theta]\tau_2 = [\Theta]\tau_2$ and $[\Theta][\Theta]\tau_2' = [\Theta]\tau_2'$.

By i.h., $\Delta = \Theta$. Hence $\Delta = \Gamma$.

• Cases $\langle : \forall L, \langle : \forall R : T \rangle$ Impossible, since τ and τ' are monotypes.

$$\bullet \ \, \textbf{Case} \ \, \frac{\hat{\alpha} \notin FV(A) \qquad \Gamma_0[\hat{\alpha}] \vdash \, \hat{\alpha} : \stackrel{\leq}{=} A \ \, \dashv \Delta }{\Gamma_0[\hat{\alpha}] \vdash \, \hat{\alpha} <: \, A \ \, \dashv \Delta } <: \mathsf{InstantiateL}$$

By Lemma 37 (Monotypes Solve Variables), $|\mathsf{unsolved}(\Delta)| < |\mathsf{unsolved}(\Gamma_0[\hat{\alpha}])|$, but it is given that $|\mathsf{unsolved}(\Gamma_0[\hat{\alpha}])| = |\mathsf{unsolved}(\Delta)|$, so this case is impossible.

• Case <: InstantiateR: Impossible, as for the <: InstantiateL case.

F'.2 Decidability of Subtyping

Theorem 8 (Decidability of Subtyping).

Given a context Γ and types A, B such that $\Gamma \vdash A$ and $\Gamma \vdash B$ and $[\Gamma]A = A$ and $[\Gamma]B = B$, it is decidable whether there exists Δ such that $\Gamma \vdash A <: B \dashv \Delta$.

Proof. Let the judgment $\Gamma \vdash A \lt: B \dashv \Delta$ be measured lexicographically by

- (S1) the number of \forall quantifiers in A and B;
- (S2) $|unsolved(\Gamma)|$, the number of unsolved existential variables in Γ ;
- (S3) $|\Gamma \vdash A| + |\Gamma \vdash B|$.

For each subtyping rule, we show that every premise is smaller than the conclusion. The condition that $[\Gamma]A = A$ and $[\Gamma]B = B$ is easily satisfied at each inductive step, using the definition of substitution.

• Rules <: Var, <: Unit and <: Exvar have no premises.

If A_2 or B_2 has a quantifier, then the first premise is smaller by (S1). Otherwise, the first premise shares an input context with the conclusion, so it has the same (S2). The types B_1 and A_1 are subterms of the conclusion's types, so the first premise is smaller by (S3).

If B_1 or A_1 has a quantifier, then the second premise is smaller by (S1). Otherwise, by Lemma 38 (Monotype Monotonicity) on the first premise, $|\mathsf{unsolved}(\Theta)| \leq |\mathsf{unsolved}(\Gamma)|$.

– If $|unsolved(\Theta)| < |unsolved(\Gamma)|$, then the second premise is smaller by (S2).

- If $|unsolved(\Theta)| = |unsolved(\Gamma)|$, we have the same (S2). However, by Lemma 40 (Monotype Context Invariance), $\Theta = \Gamma$, so $|\Theta \vdash [\Theta]A_2| = |\Gamma \vdash [\Gamma]A_2|$, which by Lemma 39 (Substitution Decreases Size) is less than or equal to $|\Gamma \vdash A_2|$. By the same logic, $|\Theta \vdash [\Theta]B_2| \leq |\Gamma \vdash B_2|$. Therefore.

$$|\Theta \vdash [\Theta]A_2| \ + \ |\Theta \vdash [\Theta]B_2| \quad \leq \quad |\Gamma \vdash (A_1 \to A_2)| \ + \ |\Gamma \vdash (B_1 \to B_2)|$$

and the second premise is smaller by (S3).

- Cases $<: \forall L, <: \forall R:$ In each of these rules, the premise has one less quantifier than the conclusion, so the premise is smaller by (S1).
- Cases <: InstantiateL, <: InstantiateR: Follows from Theorem 7.

G' Decidability of Typing

Theorem 9 (Decidability of Typing).

- (i) Synthesis: Given a context Γ and a term e,
 it is decidable whether there exist a type A and a context Δ such that
 Γ ⊢ e ⇒ A ⊢ Δ.
- (ii) Checking: Given a context Γ , a term e, and a type B such that $\Gamma \vdash B$, it is decidable whether there is a context Δ such that $\Gamma \vdash e \Leftarrow B \dashv \Delta$.
- (iii) Application: Given a context Γ , a term e, and a type A such that $\Gamma \vdash A$, it is decidable whether there exist a type C and a context Δ such that $\Gamma \vdash A \bullet e \Rightarrow C \dashv \Delta$.

Proof. For rules deriving judgments of the form

$$\Gamma \vdash e \Rightarrow - \dashv -
\Gamma \vdash e \Leftarrow B \dashv -
\Gamma \vdash A \bullet e \Longrightarrow - \dashv -$$

(where we write "—" for parts of the judgments that are outputs), the following induction measure on such judgments is adequate to prove decidability:

$$\left\langle e, \begin{array}{cc} \Rightarrow \\ \left\langle e, & \leftarrow, & |\Gamma \vdash B| \\ \Rightarrow & |\Gamma \vdash A| \end{array} \right\rangle$$

where $\langle ... \rangle$ denotes lexicographic order, and where (when comparing two judgments typing terms of the same size) the synthesis judgment (top line) is considered smaller than the checking judgment (second line), which in turn is considered smaller than the application judgment (bottom line). That is,

$$\Rightarrow$$
 \prec \Leftarrow \prec \Rightarrow

Note that this measure only uses the input parts of the judgments, leading to a straightforward decidability argument.

We will show that in each rule, every synthesis/checking/application premise is smaller than the conclusion.

• Case Var: No premises.

• Case Sub: The first premise has the same subject term *e* as the conclusion, but the judgment is smaller because the measure considers a synthesis judgment to be smaller than a checking judgment.

The second premise is a subtyping judgment, which by Theorem 8 is decidable.

• Case Anno:

It is easy to show that the judgment $\Gamma \vdash A$ is decidable.

The second premise types e, but the conclusion types (e : A), so the first part of the measure gets smaller.

- Case 11: No premises.
- Case \rightarrow I: In the premise, the term is smaller.
- Case \rightarrow E: In both premises, the term is smaller.
- Case $\forall I$: Both the premise and conclusion type e, and both are checking; however, $|\Gamma, \alpha \vdash A| < |\Gamma \vdash \forall \alpha$. A|, so the premise is smaller.
- **Case** → App: Both the premise and conclusion type *e*, but the premise is a checking judgment, so the premise is smaller.
- Case Subst \Leftarrow : Both the premise and conclusion type e, and both are checking; however, since we can apply this rule only when Γ has a solution for $\hat{\alpha}$ —that is, when $\Gamma = \Gamma_0[\hat{\alpha} = \tau]$ —we have that $|\Gamma \vdash [\Gamma]\hat{\alpha}| < |\Gamma \vdash \hat{\alpha}|$, making the last part of the measure smaller.
- Case SubstApp: Similar to Subst←.
- Case \forall App: Both the premise and conclusion type e, and both are application judgments; however, by the definition of $|\Gamma \vdash -|$, the size of the type in the premise $[\hat{\alpha}/\alpha]A$ is smaller than $\forall \alpha$. A.
- Case $\hat{\alpha}$ App: Both the premise and conclusion type e, but we switch to checking in the premise, so the premise is smaller.

- Case 11⇒: No premises.
- Case $\rightarrow l \Rightarrow$: In the premise, the term is smaller.

H' Soundness of Subtyping

H'.1 Lemmas for Soundness

Lemma 42 (Variable Preservation).

If $(x : A) \in \Delta$ or $(x : A) \in \Omega$ and $\Delta \longrightarrow \Omega$ then $(x : [\Omega]A) \in [\Omega]\Delta$.

Proof. By mutual induction on Δ and Ω .

Suppose $(x:A) \in \Delta$. In the case where $\Delta = (\Delta', x:A)$ and $\Omega = (\Omega', x:A_{\Omega})$, inversion on $\Delta \longrightarrow \Omega$ gives $[\Omega']A = [\Omega']A_{\Omega}$; by the definition of context application, $[\Omega', x:A_{\Omega}](\Delta', x:A) = [\Omega']\Delta', x:[\Omega']A_{\Omega}$, which contains $x:[\Omega']A_{\Omega}$, which is equal to $x:[\Omega']A$. By well-formedness of Ω , we know that $[\Omega']A = [\Omega]A$.

Suppose $(x : A) \in \Omega$. The reasoning is similar, because equality is symmetric.

Lemma 43 (Substitution Typing). *If* $\Gamma \vdash A$ *then* $\Gamma \vdash [\Gamma]A$.

Proof. By induction on $|\Gamma \vdash A|$ (the size of A under Γ).

• Cases UvarWF, UnitWF: Here $A = \alpha$ or A = 1, so applying Γ to A does not change it: $A = [\Gamma]A$. Since $\Gamma \vdash A$, we have $\Gamma \vdash [\Gamma]A$, which was to be shown.

- Case EvarWF: In this case $A = \hat{\alpha}$, but $\Gamma = \Gamma_0[\hat{\alpha}]$, so applying Γ to A does not change it, and we proceed as in the UnitWF case above.
- Case SolvedEvarWF: In this case $A=\hat{\alpha}$ and $\Gamma=\Gamma_L, \hat{\alpha}=\tau, \Gamma_R$. Thus $[\Gamma]A=[\Gamma]\alpha=[\Gamma_L]\tau$. We assume contexts are well-formed, so all free variables in τ are declared in Γ_L . Consequently, $|\Gamma_L \vdash \tau| = |\Gamma \vdash \tau|$, which is less than $|\Gamma \vdash \hat{\alpha}|$. We can therefore apply the i.h. to τ , yielding $\Gamma \vdash [\Gamma]\tau$. By the definition of substitution, $[\Gamma]\tau=[\Gamma]\hat{\alpha}$, so we have $\Gamma \vdash [\Gamma]\hat{\alpha}$.
- Case ArrowWF: In this case $A = A_1 \to A_2$. By i.h., $\Gamma \vdash [\Gamma]A_1$ and $\Gamma \vdash [\Gamma]A_2$. By ArrowWF, $\Gamma \vdash ([\Gamma]A_1) \to ([\Gamma]A_2)$, which by the definition of substitution is $\Gamma \vdash [\Gamma](A_1 \to A_2)$.
- **Case** ForallWF: In this case $A = \forall \alpha$. A_0 . By i.h., Γ , $\alpha \vdash [\Gamma, \alpha]A_0$. By the definition of substitution, $[\Gamma, \alpha]A_0 = [\Gamma]A_0$, so by ForallWF, $\Gamma \vdash \forall \alpha$. $[\Gamma]A_0$, which by the definition of substitution is $\Gamma \vdash [\Gamma](\forall \alpha. A_0)$.

Lemma 44 (Substitution for Well-Formedness). *If* $\Omega \vdash A$ *then* $[\Omega]\Omega \vdash [\Omega]A$.

Proof. By induction on $|\Omega \vdash A|$, the size of A under Ω (Definition 2).

We consider cases of the well-formedness rule concluding the derivation of $\Omega \vdash A$.

Case

$$\frac{}{\Omega \vdash 1} \ \, \text{UnitWF}$$

$$[\Omega]\Omega \vdash 1 \qquad \text{By DeclUnitWF}$$

$$[\Omega]\Omega \vdash [\Omega]1 \qquad \text{By definition of substitution}$$

Case

Case

• Case

$$\underbrace{\underline{\Omega'[\hat{\alpha}]} \vdash \hat{\alpha}}_{\mathsf{C}} \mathsf{EvarWF}$$

Impossible: the grammar for Ω does not allow unsolved declarations.

$$\bullet \ \, \textbf{Case} \ \, \frac{\Omega \vdash A_1 \qquad \Omega \vdash A_2}{\Omega \vdash A_1 \to A_2} \ \, \textbf{ArrowWF}$$

$$\begin{array}{ll} \Omega \vdash A_1 & \text{Subderivation} \\ |\Omega \vdash A_1| < |\Omega \vdash A_1 \to A_2| & \text{Follows from definition of type size} \\ [\Omega]\Omega \vdash [\Omega]A_1 & \text{By i.h.} \\ \\ [\Omega]\Omega \vdash [\Omega]A_2 & \text{By similar reasoning on 2nd subderivation} \\ [\Omega]\Omega \vdash [\Omega]A_1 \to [\Omega]A_2 & \text{By DeclArrowWF} \end{array}$$

• Case
$$\frac{\Omega, \alpha \vdash A_0}{\Omega \vdash \forall \alpha. A_0}$$
 ForallWF

 $[\Omega]\Omega \vdash [\Omega](A_1 \rightarrow A_2)$

$$\Omega, \alpha \vdash A_0$$
 Subderivation

Let $\Omega' = (\Omega, \alpha)$.

$$|\Omega' \vdash A_0| < |\Omega \vdash \forall \alpha. A_0|$$
 Follows from definition of type size

 $[\Omega'](\Omega, \alpha) \vdash [\Omega']A_0$ By i.h.

$$[\Omega]\Omega, \alpha \vdash [\Omega']A_0$$
 By definition of context application

$$[\Omega]\Omega, \alpha \vdash [\Omega]A_0$$
 By definition of substitution

$$[\Omega]\Omega \vdash \forall \alpha. [\Omega]A_0$$
 By DeclForallWF

$$[\Omega]\Omega \vdash [\Omega](\forall \alpha. A_0)$$
 By definition of substitution

Lemma 45 (Substitution Stability).

For any well-formed complete context (Ω, Ω_Z) , if $\Omega \vdash A$ then $[\Omega]A = [\Omega, \Omega_Z]A$.

Proof. By induction on Ω_Z . If $\Omega_Z = \cdot$, the result is immediate. Otherwise, use the i.h. and the fact that $\Omega \vdash A$ implies $FV(A) \cap dom(\Omega_Z) = \emptyset$.

By definition of substitution

Lemma 46 (Context Partitioning).

If $\Delta, \blacktriangleright_{\hat{\alpha}}, \Theta \longrightarrow \Omega, \blacktriangleright_{\hat{\alpha}}, \Omega_Z$ then there is a Ψ such that $[\Omega, \blacktriangleright_{\hat{\alpha}}, \Omega_Z](\Delta, \blacktriangleright_{\hat{\alpha}}, \Theta) = [\Omega]\Delta, \Psi$.

Proof. By induction on the given derivation.

- Case \longrightarrow ID: Impossible: Δ , $\triangleright_{\hat{\alpha}}$, Θ cannot have the form \cdot .
- Case \longrightarrow Var: We have $\Omega_Z = (\Omega'_Z, x : A)$ and $\Theta = (\Theta', x : A')$. By i.h., there is Ψ' such that $[\Omega, \blacktriangleright_{\hat{\alpha}}, \Omega'_Z](\Delta, \blacktriangleright_{\hat{\alpha}}, \Theta') = [\Omega]\Delta, \Psi'$. Then by the definition of context application, $[\Omega, \blacktriangleright_{\hat{\alpha}}, \Omega'_Z, x : A](\Delta, \blacktriangleright_{\hat{\alpha}}, \Theta', x : A') = [\Omega]\Delta, \Psi', x : [\Omega']A$. Let $\Psi = (\Psi', x : [\Omega']A)$.
- Case \longrightarrow Uvar: Similar to the \longrightarrow Var case, with $\Psi = (\Psi', \alpha)$.
- Cases \longrightarrow Unsolved, \longrightarrow Solve, \longrightarrow Marker, \longrightarrow Add, \longrightarrow AddSolved: Broadly similar to the \longrightarrow Uvar case, but since the rightmost context element is soft it disappears in context application, so we let $\Psi = \Psi'$.

Lemma 49 (Stability of Complete Contexts).

If
$$\Gamma \longrightarrow \Omega$$
 then $[\Omega]\Gamma = [\Omega]\Omega$.

Proof. By induction on the derivation of $\Gamma \longrightarrow \Omega$.

Case

$$\overline{\cdot \longrightarrow \cdot} \longrightarrow ID$$

In this case, $\Omega = \Gamma = \cdot$.

By definition, $[\cdot] \cdot = \cdot$, which gives us the conclusion.

$$\begin{array}{ll} \bullet \ \, \textbf{Case} & \underline{\Gamma' \longrightarrow \Omega'} & [\Omega']A_{\Gamma} = [\Omega']A \\ \hline & \Gamma', x: A_{\Gamma} \longrightarrow \Omega', x: A \\ \hline & [\Omega']\Gamma' = [\Omega']\Omega' & \text{By i.h.} \\ \hline & [\Omega']A_{\Gamma} = [\Omega']A & \text{Premise} \\ \hline & [\Omega]\Gamma = [\Omega', x: A](\Gamma', x: A_{\Gamma}) & \text{Expanding Ω and Γ} \\ & = [\Omega']\Gamma', x: [\Omega']A_{\Gamma} & \text{By definition of context application} \\ & & (using $[\Omega']A_{\Gamma} = [\Omega']A$) \\ \end{array}$$

$$= [\Omega]\Omega \qquad \qquad \text{By definition of context application}$$

 • Case
$$\frac{\Gamma' \longrightarrow \Omega'}{\Gamma', \alpha \longrightarrow \Omega', \alpha} \longrightarrow \text{Uvar}$$

$$[\Omega]\Gamma = [\Omega', \alpha](\Gamma', \alpha) \qquad \text{Expanding } \Omega \text{ and } \Gamma$$

By above equalities

$$\Omega | \Gamma = [\Omega', \alpha](\Gamma', \alpha)$$
 Expanding Ω and Γ

$$= [\Omega']\Gamma', \alpha$$
 By definition of context application
$$= [\Omega']\Omega', \alpha$$
 By i.h.
$$= \Omega', \alpha$$
 By definition of context application
$$= [\Omega]\Omega$$
 By $\Omega = (\Omega', \alpha)$

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma' \longrightarrow \Omega'}{\Gamma', \blacktriangleright_{\hat{\alpha}} \longrightarrow \Omega', \blacktriangleright_{\hat{\alpha}}} \longrightarrow \mathsf{Marker}$$

 $= [\Omega']\Omega', x : [\Omega']A$

Similar to the \longrightarrow Uvar case.

$$\bullet \ \ \, \textbf{Case} \ \ \, \frac{\Gamma \longrightarrow \Omega'}{\Gamma \longrightarrow \Omega', \hat{\alpha} = \tau} \longrightarrow \text{AddSolved}$$

$$\begin{split} [\Omega]\Gamma &= [\Omega', \hat{\alpha} = \tau]\Gamma & \text{Expanding } \Omega \\ &= [\Omega']\Gamma & \text{By } \hat{\alpha} \notin \mathsf{dom}(\Gamma) \\ &= [\Omega']\Omega' & \text{By i.h.} \\ &= \Omega', \hat{\alpha} = \tau & \text{By definition of context application} \\ &= [\Omega]\Omega & \text{By } \Omega = (\Omega', \hat{\alpha} = \tau) \end{split}$$

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma' \longrightarrow \Omega' \qquad [\Omega'] \tau_{\Gamma} = [\Omega'] \tau}{\Gamma', \hat{\alpha} = \tau_{\Gamma} \longrightarrow \Omega', \hat{\alpha} = \tau} \longrightarrow \\ \textbf{Solved}$$

$$\begin{split} [\Omega]\Gamma &= [\Omega', \hat{\alpha} = \tau](\Gamma', \hat{\alpha} = \tau_\Gamma) & \text{ Expanding } \Omega \text{ and } \Gamma \\ &= [\Omega']\Gamma' & \text{ By definition of context application} \\ &= [\Omega']\Omega' & \text{ By i.h.} \\ &= \Omega', \hat{\alpha} = \tau & \text{ By definition of context application} \\ &= [\Omega]\Omega & \text{ By } \Omega = (\Omega', \hat{\alpha} = \tau) \end{split}$$

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma' \longrightarrow \Omega'}{\Gamma', \widehat{\alpha} \longrightarrow \Omega', \widehat{\alpha} = \tau} \longrightarrow \\ \textbf{Solve}$$

$$\begin{split} [\Omega]\Gamma &= [\Omega', \hat{\alpha} = \tau](\Gamma', \hat{\alpha}) & \text{Expanding } \Omega \text{ and } \Gamma \\ &= [\Omega']\Gamma' & \text{By definition of context application} \\ &= [\Omega']\Omega' & \text{By i.h.} \\ &= \Omega', \hat{\alpha} = \tau & \text{By definition of context application} \\ &= [\Omega]\Omega & \text{By } \Omega = (\Omega', \hat{\alpha} = \tau) \end{split}$$

$$\bullet \ \ \, \textbf{Case} \ \ \, \frac{\Gamma \longrightarrow \Delta}{\Gamma, \, \widehat{\alpha} \longrightarrow \Delta, \, \widehat{\alpha}} \longrightarrow \textbf{Unsolved}$$

Impossible: Ω cannot have the form Δ , $\hat{\alpha}$.

$$\bullet \ \ \, \textbf{Case} \ \ \, \frac{\Gamma \longrightarrow \Delta}{\Gamma \longrightarrow \Delta, \hat{\alpha}} \longrightarrow \mathsf{Add}$$

Impossible: Ω cannot have the form Δ , $\hat{\alpha}$.

Lemma 50 (Finishing Types).

If $\Omega \vdash A$ and $\Omega \longrightarrow \Omega'$ then $[\Omega]A = [\Omega']A$.

Proof. By Lemma 18 (Substitution Extension Invariance), $[\Omega']A = [\Omega'][\Omega]A$.

If $FEV(C) = \emptyset$ then $[\Omega']C = C$.

Since Ω is complete and $\Omega \vdash A$, we have $\mathsf{FEV}([\Omega]A) = \emptyset$. Therefore $[\Omega'][\Omega]A = [\Omega]A$.

Lemma 51 (Finishing Completions).

If $\Omega \longrightarrow \Omega'$ then $[\Omega]\Omega = [\Omega']\Omega'$.

Proof. By induction on the given derivation of $\Omega \longrightarrow \Omega'$.

Only cases \longrightarrow ID, \longrightarrow Var, \longrightarrow Uvar, \longrightarrow Solved, \longrightarrow Marker and \longrightarrow AddSolved are possible. In all of these cases, we use the i.h. and the definition of context application; in cases \longrightarrow Var and \longrightarrow Solved, we also use the equality in the premise of the respective rule.

Lemma 52 (Confluence of Completeness).

If $\Delta_1 \longrightarrow \Omega$ and $\Delta_2 \longrightarrow \Omega$ then $[\Omega]\Delta_1 = [\Omega]\Delta_2$.

Proof.

 $\Delta_1 \longrightarrow \Omega$ Given

 $[\Omega]\Delta_1 = [\Omega]\Omega$ By Lemma 49 (Stability of Complete Contexts)

 $\Delta_2 \longrightarrow \Omega$ Given

 $[\Omega]\Delta_2 = [\Omega]\Omega$ By Lemma 49 (Stability of Complete Contexts)

 $[\Omega]\Delta_1 = [\Omega]\Delta_2$ By transitivity of equality

H'.2 Instantiation Soundness

Theorem 10 (Instantiation Soundness).

Given $\Delta \longrightarrow \Omega$ and $[\Gamma]B = B$ and $\hat{\alpha} \notin FV(B)$:

- (1) If $\Gamma \vdash \hat{\alpha} := B \dashv \Delta$ then $[\Omega]\Delta \vdash [\Omega]\hat{\alpha} < [\Omega]B$.
- (2) If $\Gamma \vdash B \stackrel{\leq}{=} : \hat{\alpha} \dashv \Delta \text{ then } [\Omega] \Delta \vdash [\Omega] B < [\Omega] \hat{\alpha}$.

Proof. By induction on the given instantiation derivation.

(1) • Case

$$\underbrace{\frac{\Gamma_0 \vdash \tau}{\Gamma_0, \hat{\alpha}, \Gamma_1} \vdash \hat{\alpha} : \stackrel{\leq}{=} \tau \dashv \underbrace{\Gamma_0, \hat{\alpha} = \tau, \Gamma_1}_{\Lambda}}_{\text{InstLSolve}} \text{InstLSolve}$$

In this case $[\Delta] \hat{\alpha} = [\Delta] \tau$. By reflexivity of subtyping (Lemma 3 (Reflexivity of Declarative Subtyping)), $[\Omega] \Delta \vdash [\Delta] \hat{\alpha} \leq [\Delta] \tau$.

Case

$$\frac{1}{\Gamma[\hat{\alpha}][\hat{\beta}] \vdash \hat{\alpha} : \stackrel{\leq}{=} \hat{\beta} \dashv \underbrace{\Gamma[\hat{\alpha}][\hat{\beta} = \hat{\alpha}]}_{\Lambda} \text{ InstLReach}}$$

We have $\Delta = \Gamma[\hat{\alpha}][\hat{\beta} = \hat{\alpha}]$. Therefore $[\Delta]\hat{\alpha} = \hat{\alpha} = [\Delta]\hat{\beta}$.

By reflexivity of subtyping (Lemma 3 (Reflexivity of Declarative Subtyping)), $[\Omega]\Delta \vdash [\Delta]\hat{\alpha} \leq [\Delta]\hat{\beta}$.

 $[\Omega]\Delta \vdash [\Omega]\hat{\alpha}_2 < [\Omega]A_2$ By Lemma 18 (Substitution Extension Invariance)

 $[\Omega]\Delta \vdash [\Omega](\hat{\alpha}_1 \to \hat{\alpha}_2) \leq [\Omega]A_1 \to [\Omega]A_2 \quad \text{ By } \leq \to \text{ and definition of substitution}$

Since $(\hat{\alpha} = \hat{\alpha}_1 \to \hat{\alpha}_2) \in \Gamma_1$ and $\Gamma_1 \longrightarrow \Delta$, we know that $[\Omega] \hat{\alpha} = [\Omega](\hat{\alpha}_1 \to \hat{\alpha}_2)$. Therefore $[\Omega]\Delta \vdash [\Omega]\hat{\alpha} \leq [\Omega](A_1 \rightarrow A_2)$.

$\frac{\Gamma[\hat{\alpha}], \beta \vdash \hat{\alpha} : \stackrel{\leq}{=} B_0 \dashv \Delta, \beta, \Delta'}{\Gamma[\hat{\alpha}] \vdash \hat{\alpha} : \stackrel{\leq}{=} \forall \beta, B_0 \dashv \Delta} \text{ InstLAIIR}$

We have $\Delta \longrightarrow \Omega$ and $[\Gamma[\hat{\alpha}]](\forall \beta. B_0) = \forall \beta. B_0$ and $\hat{\alpha} \notin FV(\forall \beta. B_0)$.

Hence $\hat{\alpha} \notin FV(B_0)$ and by definition, $[\Gamma[\hat{\alpha}], \beta]B_0 = B_0$.

By Lemma 48 (Filling Completes), Δ , β , $\Delta' \longrightarrow \Omega$, β , $|\Delta'|$.

By induction, $[\Omega, \beta, |\Delta'|](\Delta, \beta, \Delta') \vdash [\Omega, \beta, |\Delta'|] \hat{\alpha} \leq [\Omega, \beta, |\Delta'|] B_0$.

Each free variable in $\hat{\alpha}$ and B_0 is declared in (Ω, β) , so $\Omega, \beta, |\Delta'|$ behaves as $[\Omega, \beta]$ on $\hat{\alpha}$ and on B_0 , yielding $[\Omega, \beta, |\Delta'|](\Delta, \beta, \Delta') \vdash [\Omega, \beta] \hat{\alpha} \leq [\Omega, \beta] B_0$.

By Lemma 46 (Context Partitioning) and thinning, $[\Omega, \beta](\Delta, \beta) \vdash [\Omega, \beta] \hat{\alpha} \leq [\Omega, \beta] B_0$.

By the definition of context application, $[\Omega]\Delta, \beta \vdash [\Omega, \beta]\hat{\alpha} < [\Omega, \beta]B_0$.

By the definition of substitution, $[\Omega]\Delta, \beta \vdash [\Omega]\hat{\alpha} \leq [\Omega]B_0$.

Since $\hat{\alpha}$ is declared to the left of β , we have $\beta \notin FV([\Omega]\hat{\alpha})$.

Applying rule $\leq \forall L$ gives $[\Omega]\Delta \vdash [\Omega]\hat{\alpha} \leq \forall \beta$. $[\Omega]B_0$.

(2)Case

$$\underbrace{\frac{\Gamma_0 \vdash \tau}{\Gamma_0, \hat{\alpha}, \Gamma_1} \vdash \tau \stackrel{\text{$\stackrel{<}{=}$}}{:} \hat{\alpha} \dashv \underbrace{\Gamma_0, \hat{\alpha} = \tau, \Gamma_1}_{\Gamma'}}_{\Gamma'} \text{ InstRSolve}$$

Similar to the InstLSolve case.

Case

$$\overline{\Gamma[\hat{\alpha}][\hat{\beta}] \vdash \ \hat{\beta} \stackrel{\leq}{=} \ \hat{\alpha} \ \dashv \underbrace{\Gamma[\hat{\alpha}][\hat{\beta} = \hat{\alpha}]}_{\Gamma'} \ \mathsf{InstRReach}$$

Similar to the InstLReach case

• Case
$$\frac{\Gamma[\hat{\alpha}_2,\hat{\alpha}_1,\hat{\alpha}=\hat{\alpha}_1\rightarrow\hat{\alpha}_2]\vdash\hat{\alpha}_1:\stackrel{\leq}{=}A_1\dashv\Gamma'\qquad\Gamma'\vdash[\Gamma']A_2\stackrel{\leq}{=}:\hat{\alpha}_2\dashv\Delta}{\Gamma[\hat{\alpha}]\vdash A_1\rightarrow A_2\stackrel{\leq}{=}:\hat{\alpha}\dashv\Delta} \text{InstRArr}$$

Similar to the InstLArr case.

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma[\hat{\alpha}], \blacktriangleright_{\hat{\beta}}, \hat{\beta} \vdash [\hat{\beta}/\beta] B_0 \stackrel{\leq}{=}: \hat{\alpha} \ \, \dashv \Delta, \blacktriangleright_{\hat{\beta}}, \Delta'}{\Gamma[\hat{\alpha}] \vdash \forall \beta. \ \, B_0 \stackrel{\leq}{=}: \hat{\alpha} \ \, \dashv \Delta} \ \, \textbf{InstRAIIL}$$

$$\begin{split} & [\Gamma[\hat{\alpha}]](\forall \beta. \, B_0) = \forall \beta. \, B_0 \qquad \text{Given} \\ & [\Gamma[\hat{\alpha}]] \, B_0 = B_0 \\ & [\Gamma[\hat{\alpha}], \blacktriangleright_{\hat{\beta}}, \hat{\beta}] [\hat{\beta}/\beta] B_0 = [\hat{\beta}/\beta] B_0 \\ & \Delta \longrightarrow \Omega \qquad \text{Given} \\ & \Delta, \blacktriangleright_{\hat{\beta}}, \Delta' \longrightarrow \Omega, \blacktriangleright_{\hat{\beta}}, |\Delta'| \qquad \text{By Lemma 48 (Filling Completes)} \\ & \hat{\alpha} \notin FV(\forall \beta. \, B_0) \qquad \text{Given} \\ & \hat{\alpha} \notin FV(B_0) \qquad \text{By definition of } FV(-) \\ & \Gamma[\hat{\alpha}], \blacktriangleright_{\hat{\beta}}, \hat{\beta} \vdash [\hat{\beta}/\beta] B_0 \stackrel{\leq}{=} : \hat{\alpha} \dashv \Delta, \blacktriangleright_{\hat{\beta}}, \Delta' \qquad \text{Subderivation} \\ & [\Omega, \blacktriangleright_{\hat{\alpha}}, |\Delta'] [(A, \blacktriangleright_{\hat{\alpha}}, |\Delta')] \vdash [\Omega, \blacktriangleright_{\hat{\alpha}}, |\Delta'] [(\hat{\beta}/\beta] B_0 \stackrel{\leq}{=} : [\Omega, \blacktriangleright_{\hat{\alpha}}, |\Delta']] \hat{\alpha} \qquad \text{By i.b.} \end{split}$$

 $[\Omega, \blacktriangleright_{\hat{\beta}}, |\Delta'|](\Delta, \blacktriangleright_{\hat{\beta}}, \Delta') \vdash [\Omega, \blacktriangleright_{\hat{\beta}}, |\Delta'|][\hat{\beta}/\beta] B_0 \leq [\Omega, \blacktriangleright_{\hat{\beta}}, |\Delta'|] \hat{\alpha} \quad \text{By i.h.}$ $\Gamma[\hat{\alpha}], \blacktriangleright_{\hat{\beta}}, \hat{\beta} \longrightarrow \Delta, \blacktriangleright_{\hat{\beta}}, \Delta'$ By Lemma 32 (Instantiation Extension)

By Lemma 16 (Declaration Order Preservation), $\hat{\alpha}$ is declared before $\triangleright_{\hat{\alpha}}$, that is, in Ω .

Thus, $\left[\Omega, \blacktriangleright_{\hat{\alpha}}, |\Delta'|\right] \hat{\alpha} = [\Omega] \hat{\alpha}$.

By Lemma 23 (Evar Input), we know that Δ' is soft, so by Lemma 47 (Softness Goes Away), $[\Omega, \blacktriangleright_{\hat{\beta}}, |\Delta'|](\Delta, \blacktriangleright_{\hat{\beta}}, \Delta') = [\Omega, \blacktriangleright_{\hat{\beta}}](\Delta, \blacktriangleright_{\hat{\beta}}) = [\Omega]\Delta.$

Applying these equalities to the derivation above gives

$$[\Omega]\Delta \vdash [\Omega, \blacktriangleright_{\hat{\beta}}, |\Delta'|][\hat{\beta}/\beta]B_0 \leq [\Omega]\hat{\alpha}$$

By distributivity of substitution,

$$[\Omega]\Delta \vdash \left[[\Omega, \blacktriangleright_{\hat{\beta}}, |\Delta'|] \hat{\beta} / \beta \right] \left[\Omega, \blacktriangleright_{\hat{\beta}}, |\Delta'| \right] B_0 \leq [\Omega] \hat{\alpha}$$

Furthermore, $[\Omega, \blacktriangleright_{\hat{\beta}}, |\Delta'|]B_0 = [\Omega]B_0$, since B_0 's free variables are either β or in Ω , giving

$$[\Omega]\Delta \vdash \left[[\Omega, \blacktriangleright_{\widehat{\beta}}, |\Delta'|] \widehat{\beta}/\beta \right] [\Omega] B_0 \leq [\Omega] \widehat{\alpha}$$

Now apply $\leq \forall L$ and the definition of substitution to get $[\Omega]\Delta \vdash [\Omega](\forall \beta. B_0) \leq [\Omega]\hat{\alpha}$.

Soundness of Subtyping

Theorem 11 (Soundness of Algorithmic Subtyping).

If $\Gamma \vdash A <: B \dashv \Delta$ where $[\Gamma]A = A$ and $[\Gamma]B = B$ and $\Delta \longrightarrow \Omega$ then $[\Omega]\Delta \vdash [\Omega]A \leq [\Omega]B$.

Proof. By induction on the derivation of $\Gamma \vdash A \lt : B \dashv \Delta$.

Case

- Case <: Unit: Similar to the <: Var case, applying rule \le Unit instead of \le Var.
- Case

$$\frac{}{\Gamma_L,\hat{\alpha},\Gamma_R\vdash\,\hat{\alpha}<:\,\hat{\alpha}\,\dashv\Gamma_L,\hat{\alpha},\Gamma_R}<:\mathsf{Exvar}}{[\Omega]\hat{\alpha}\;\mathsf{defined}\quad\mathsf{Follows\;from\;definition\;of\;context\;application}}$$

Assumption that $[\Omega]\Delta$ is well-formed $[\Omega]\Delta \vdash [\Omega]\hat{\alpha}$

 $[\Omega]\Delta \vdash [\Omega]\hat{\alpha} \leq [\Omega]\hat{\alpha}$ By Lemma 3 (Reflexivity of Declarative Subtyping)

$$\begin{array}{c} \bullet \ \, \textbf{Case} \\ \frac{\Gamma, \, \alpha \vdash \ A \mathrel{<:} \, B_0 \ \dashv \Delta, \, \alpha, \Theta}{\Gamma \vdash \ A \mathrel{<:} \, \forall \alpha. \, B_0 \ \dashv \Delta} \mathrel{<:} \forall \mathsf{R} \\ \end{array}$$

$$\begin{array}{ll} \Gamma,\alpha \vdash A <: B_0 \dashv \Delta,\alpha,\Theta & Subderivation \\ \text{Let } \Omega_Z = |\Theta| \,. \\ \text{Let } \Omega' = (\Omega,\alpha,\Omega_Z). \\ (\Delta,\alpha,\Theta) \longrightarrow \Omega' & \text{By Lemma 48 (Filling Completes)} \\ [\Omega'](\Delta,\alpha,\Theta) \vdash [\Omega']A \leq [\Omega']B_0 & \text{By i.h.} \\ [\Omega,\alpha](\Delta,\alpha) \vdash [\Omega,\alpha]A \leq [\Omega,\alpha]B_0 & \text{By Lemma 45 (Substitution Stability)} \\ [\Omega,\alpha](\Delta,\alpha) \vdash [\Omega]A \leq [\Omega]B_0 & \text{By def. of substitution} \\ [\Omega]\Delta \vdash [\Omega]A \leq \forall \alpha. \ [\Omega]B_0 & \text{By } \leq \forall \mathsf{R} \\ \end{array}$$

By def. of substitution

 $[\Omega]\Delta \vdash [\Omega]A \leq [\Omega](\forall \alpha. B_0)$

• Case <: InstantiateR: Similar to the case for <: InstantiateL.

Corollary 53 (Soundness, Pretty Version). *If* $\Psi \vdash A \lt : B \dashv \Delta$, *then* $\Psi \vdash A \leq B$.

Proof. By reflexivity (Lemma 20 (Reflexivity)), $\Psi \longrightarrow \Psi$.

 $[\Omega]\Delta \vdash [\Omega]\hat{\alpha} \leq [\Omega]B$ By Theorem 10

Since Ψ has no existential variables, it is a complete context Ω .

By Theorem 11, $[\Psi]\Psi \vdash [\Psi]A \leq [\Psi]B$.

Since Ψ has no existential variables, $[\Psi]\Psi = \Psi$, and $[\Psi]A = A$, and $[\Psi]B = B$.

Therefore $\Psi \vdash A \leq B$.

I' Typing Extension

Lemma 54 (Typing Extension).

If
$$\Gamma \vdash e \Leftarrow A \dashv \Delta$$
 or $\Gamma \vdash e \Rightarrow A \dashv \Delta$ or $\Gamma \vdash A \bullet e \Rightarrow C \dashv \Delta$ then $\Gamma \longrightarrow \Delta$.

Proof. By induction on the given derivation.

- Cases Var, 1I, 1I \Rightarrow : Since $\Delta = \Gamma$, the result follows by Lemma 20 (Reflexivity).
- Case $\frac{\Gamma \vdash e \Rightarrow B \dashv \Theta \qquad \Theta \vdash [\Theta]B \mathrel{<:} [\Theta]A \dashv \Delta}{\Gamma \vdash e \Leftarrow A \dashv \Delta} \operatorname{Sub}$

 $\Gamma \longrightarrow \Theta$ By i.h.

 $\Theta \longrightarrow \Delta$ By Lemma 33 (Subtyping Extension)

- $\Gamma \longrightarrow \Delta$ By Lemma 21 (Transitivity)
- $\bullet \ \, \textbf{Case} \ \, \frac{\Gamma \vdash A \qquad \Gamma \vdash e \Leftarrow A \dashv \Delta}{\Gamma \vdash (e : A) \Rightarrow A \dashv \Delta} \ \, \textbf{Anno}$
 - $\quad \ \ \, \Gamma \longrightarrow \Delta \quad \text{ By i.h.}$
- Case $\frac{\Gamma, \alpha \vdash e \Leftarrow A_0 \dashv \Delta, \alpha, \Theta}{\Gamma \vdash e \Leftarrow \forall \alpha. A_0 \dashv \Delta} \forall I$

 $\Gamma, \alpha \longrightarrow \Delta, \alpha, \Theta$ By i.h.

 $\Gamma \longrightarrow \Delta$ By Lemma 24 (Extension Order) (i)

• Case
$$\frac{\Gamma, \hat{\alpha} \vdash [\hat{\alpha}/\alpha] A_0 \bullet e \Longrightarrow C \dashv \Delta}{\Gamma \vdash \forall \alpha. \ A_0 \bullet e \Longrightarrow C \dashv \Delta} \forall \mathsf{App}$$

$$\Gamma, \hat{\alpha} \longrightarrow \Delta \qquad \mathsf{By i.h.}$$

$$\Gamma \longrightarrow \Gamma, \hat{\alpha} \qquad \mathsf{By} \longrightarrow \mathsf{Add}$$

$$\Gamma \longrightarrow \Delta \qquad \mathsf{By Lemma 21 (Transitivity)}$$

• Case
$$\frac{\Gamma, x: A_1 \vdash e \Leftarrow A_2 \dashv \Delta, x: A_1, \Theta}{\Gamma \vdash \lambda x. \, e \Leftarrow A_1 \rightarrow A_2 \dashv \Delta} \rightarrow \mathsf{I}$$

$$\Gamma, x: A_1 \longrightarrow \Delta, x: A_1, \Theta \quad \text{By i.h.}$$

$$\Gamma \longrightarrow \Delta \qquad \qquad \text{By Lemma 24 (Extension Order) (v)}$$

• Case
$$\frac{\Gamma \vdash e_1 \Rightarrow B \dashv \Theta \qquad \Theta \vdash [\Theta]B \bullet e_2 \Longrightarrow A \dashv \Delta}{\Gamma \vdash e_1 e_2 \Longrightarrow A \dashv \Delta} \rightarrow \mathsf{E}$$

By the i.h. on each premise, then Lemma 21 (Transitivity).

$$\begin{array}{c} \bullet \ \, \textbf{Case} \\ \hline \begin{array}{c} \Gamma, \hat{\alpha}, \hat{\beta}, x: \hat{\alpha} \vdash e \Leftarrow \hat{\beta} \ \, \dashv \Delta, x: \hat{\alpha}, \Theta \\ \hline \Gamma \vdash \lambda x. \ e \Rightarrow \hat{\alpha} \rightarrow \hat{\beta} \ \, \dashv \Delta \\ \hline \Gamma, \hat{\alpha}, \hat{\beta}, x: \hat{\alpha} \longrightarrow \Delta, x: \hat{\alpha}, \Theta \quad \text{By i.h.} \\ \hline \Gamma, \hat{\alpha}, \hat{\beta} \longrightarrow \Delta \quad \qquad \text{By Lemma 24 (Extension Order) (v)} \\ \hline \Gamma \longrightarrow \Gamma, \hat{\alpha}, \hat{\beta} \quad \quad \text{By } \longrightarrow \text{Add (twice)} \\ \hline \Gamma \longrightarrow \Delta \quad \qquad \text{By Lemma 21 (Transitivity)} \end{array}$$

• Case
$$\frac{\Gamma \vdash e \Leftarrow A \dashv \Delta}{\Gamma \vdash A \to C \bullet e \Rightarrow C \dashv \Delta} \to \mathsf{App}$$
•• T \to \Delta \text{ By i.h.}

• Case
$$\frac{\Gamma[\hat{\alpha}_2,\hat{\alpha}_1,\hat{\alpha}=\hat{\alpha}_1\to\hat{\alpha}_2]\vdash e\Leftarrow\hat{\alpha}_1\dashv\Delta}{\Gamma[\hat{\alpha}]\vdash\hat{\alpha}\bullet e\Rightarrow\hat{\alpha}_2\dashv\Delta}\,\hat{\alpha}\mathsf{App}$$

$$\Gamma[\hat{\alpha}_2,\hat{\alpha}_1,\hat{\alpha}=\hat{\alpha}_1\to\hat{\alpha}_2]\longrightarrow\Delta\quad\mathsf{By i.h.}$$

$$\Gamma[\hat{\alpha}]\longrightarrow\Gamma[\hat{\alpha}_2,\hat{\alpha}_1,\hat{\alpha}=\hat{\alpha}_1\to\hat{\alpha}_2]\quad\mathsf{By Lemma 27 (Solved Variable Addition for Extension)}$$
 then Lemma 29 (Parallel Admissibility) (ii)
$$\Gamma\longrightarrow\Delta\quad\mathsf{By Lemma 21 (Transitivity)}\qquad\square$$

J' Soundness of Typing

Theorem 12 (Soundness of Algorithmic Typing). *Given* $\Delta \longrightarrow \Omega$:

(i) If
$$\Gamma \vdash e \Leftarrow A \dashv \Delta$$
 then $[\Omega]\Delta \vdash e \Leftarrow [\Omega]A$.

(ii) If
$$\Gamma \vdash e \Rightarrow A \dashv \Delta$$
 then $[\Omega]\Delta \vdash e \Rightarrow [\Omega]A$.

(iii) If
$$\Gamma \vdash A \bullet e \Rightarrow C \dashv \Delta$$
 then $[\Omega] \Delta \vdash [\Omega] A \bullet e \Rightarrow [\Omega] C$.

Proof. By induction on the given algorithmic typing derivation.

• Case
$$\frac{(\kappa:A) \in \Gamma}{\Gamma \vdash \kappa \Rightarrow A \dashv \Gamma} \, \mathsf{Var}$$

```
(x:A) \in \Gamma
                                                      Premise
                                                      By \Gamma = \Delta
                   (x : A) \in \Delta
                       \Delta \longrightarrow \Omega
                                                      Given
              (x : [\Omega]A) \in [\Omega]\Gamma
                                                      By Lemma 42 (Variable Preservation)
                      [\Omega]\Gamma \vdash \chi \Rightarrow [\Omega]A By DeclVar
• Case \Gamma \vdash e \Rightarrow A \dashv \Theta \Theta \vdash \Theta \land A : \Theta \land B \dashv \Delta Sub
                         \Gamma \vdash e \Rightarrow A \dashv \Theta
                                                                   Subderivation
                        \Theta \vdash [\Theta]A <: [\Theta]B \dashv \Delta
                                                                  Subderivation
                    \Theta \longrightarrow \Delta
                                                                  By Lemma 54 (Typing Extension)
                     \Delta \longrightarrow \Omega
                                                                   Given
                    \Theta \longrightarrow \Omega
                                                                   By Lemma 21 (Transitivity)
                   [\Omega]\Theta \vdash e \Rightarrow [\Omega]A
                                                                   By i.h.
                                                                   By Lemma 52 (Confluence of Completeness)
                   [\Omega]\Theta = [\Omega]\Delta
                   [\Omega]\Delta \vdash e \Rightarrow [\Omega]A
                                                                   By above equalities
                                                                   Subderivation
                        \Theta \vdash [\Theta]A <: [\Theta]B \dashv \Delta
                   [\Omega]\Delta \vdash [\Omega][\Theta]A \leq [\Omega][\Theta]B
                                                                  By Theorem 11
              [\Omega][\Theta]A = [\Omega]A
                                                                   By Lemma 18 (Substitution Extension Invariance)
              [\Omega][\Theta]B = [\Omega]B
                                                                   By Lemma 18 (Substitution Extension Invariance)
                   [\Omega]\Delta \vdash [\Omega]A \leq [\Omega]B
                                                                   By above equalities
                   [\Omega]\Delta \vdash e \Leftarrow [\Omega]B
                                                                   By DeclSub
      ₽
• Case \frac{\Gamma \vdash A \qquad \Gamma \vdash e_0 \Leftarrow A \dashv \Delta}{\Gamma \vdash (e_0 : A) \Rightarrow A \dashv \Delta} \text{ Anno}
                    \Gamma \vdash e_0 \Leftarrow A \dashv \Delta
                                                                                       Subderivation
              [\Omega]\Delta \vdash e_0 \Leftarrow [\Omega]A
                                                                                       By i.h.
                    \Gamma \vdash A
                                                                                       Subderivation
                \Gamma \longrightarrow \Delta
                                                                                       By Lemma 54 (Typing Extension)
                \Delta \longrightarrow \Omega
                                                                                       Given
                \Gamma \longrightarrow \Omega
                                                                                       By Lemma 21 (Transitivity)
                   \Omega \vdash A
                                                                                       By Lemma 25 (Extension Weakening)
              [\Omega]\Omega \vdash [\Omega]A
                                                                                       By Lemma 44 (Substitution for Well-Formedness)
              [\Omega]\Delta = [\Omega]\Omega
                                                                                       By Lemma 49 (Stability of Complete Contexts)
               [\Omega]\Delta \vdash [\Omega]A
                                                                                       By above equality
               [\Omega]\Delta \vdash (e_0 : [\Omega]A) \Rightarrow [\Omega]A
                                                                                       By DeclAnno
                           A contains no existential variables
                                                                                       Assumption about source programs
                                                                                       From definition of substitution
              [\Omega]A = A
             [\Omega]\Delta \vdash (e_0 : A) \Rightarrow [\Omega]A
                                                                                       By above equality
Case
             \overline{\Gamma \vdash \ () \Leftarrow 1 \dashv \underbrace{\Gamma}_{\Delta}} \ ^{1\text{I}}
             [\Omega]\Delta \vdash () \Leftarrow 1 By Decl11
      [\Omega]\Delta \vdash () \leftarrow [\Omega]1 By definition of substitution
```

• Case $\frac{\Gamma, x : A_1 \vdash e_0 \Leftarrow A_2 \dashv \Delta, x : A_1, \Theta}{\Gamma \vdash \lambda x. e \Leftarrow A_1 \rightarrow A_2 \dashv \Delta} \rightarrow I$

(Similar to \rightarrow I, using a different subpart of Lemma 24 (Extension Order) and applying Decl \forall I; written out anyway.)

```
\Delta \longrightarrow \Omega
                                                                                  Given
                       \Delta, \alpha \longrightarrow \Omega, \alpha
                                                                                  By \longrightarrow Uvar
                        \Gamma, \alpha \longrightarrow \Delta, \alpha, \Theta
                                                                                  By Lemma 54 (Typing Extension)
                                          \boldsymbol{\Theta} is soft
                                                                                  By Lemma 24 (Extension Order) (i) (with \Gamma_R = \cdot, which is soft)
                 \underbrace{\Delta,\alpha,\Theta}_{\Delta'}\longrightarrow\underbrace{\Omega,\alpha,|\Theta|}_{\Omega'}
                                                                                  By Lemma 48 (Filling Completes)
                              \Gamma, \alpha \vdash e \Leftarrow A_0 \dashv \Delta'
                                                                                  Subderivation
                       [\Omega']\Delta' \vdash e \Leftarrow [\Omega']A_0
                                                                                  By i.h.
                     [\Omega']A_0 = [\Omega]A_0
                                                                                  By Lemma 45 (Substitution Stability)
                      [\Omega']\Delta' \vdash e \Leftarrow [\Omega]A_0
                                                                                  By above equality
                                                                                  Above
                                                                                  Above
                      [\Omega']\Delta' = [\Omega]\Delta, \alpha
                                                                                  By Lemma 47 (Softness Goes Away)
                     [\Omega]\Delta, \alpha \vdash e \Leftarrow [\Omega]A_0
                                                                                  By above equality
                          [\Omega]\Delta \vdash e \Leftarrow \forall \alpha. [\Omega]A_0
                                                                                  By Decl∀I
                          [\Omega]\Delta \vdash e \Leftarrow [\Omega](\forall \alpha. A_0)
                                                                                 By definition of substitution
       37
 \bullet \  \, \textbf{Case} \  \, \frac{\Gamma, \hat{\alpha} \vdash [\hat{\alpha}/\alpha] A_0 \bullet e \Longrightarrow C \dashv \Delta}{\Gamma \vdash \forall \alpha. \, A_0 \bullet e \Longrightarrow C \dashv \Delta} \, \forall \mathsf{App} 
                     \Gamma, \hat{\alpha} \vdash [\hat{\alpha}/\alpha] A_0 \bullet e \Longrightarrow C \dashv \Delta
                                                                                                     Subderivation
                    \Delta \longrightarrow \Omega
                                                                                                     Given
                   [\Omega]\Delta \vdash [\Omega][\hat{\alpha}/\alpha]A_0 \bullet e \Rightarrow [\Omega]C
                                                                                                     By i.h.
                   [\Omega]\Delta \vdash [[\Omega]\hat{\alpha} / \alpha] [\Omega]A_0 \bullet e \Rightarrow [\Omega]C
                                                                                                     By distributivity of substitution
                  \Gamma, \widehat{\alpha} \longrightarrow \Delta
                                                                                                     By Lemma 54 (Typing Extension)
                  \Gamma, \hat{\alpha} \longrightarrow \Omega
                                                                                                     By Lemma 21 (Transitivity)
                      \Gamma, \hat{\alpha} \vdash \hat{\alpha}
                                                                                                     By EvarWF
                         \Omega \vdash \hat{\alpha}
                                                                                                     By Lemma 25 (Extension Weakening)
                  [\Omega]\Omega \vdash [\Omega]\hat{\alpha}
                                                                                                     By Lemma 44 (Substitution for Well-Formedness)
                  [\Omega]\Omega = [\Omega]\Delta
                                                                                                     By Lemma 49 (Stability of Complete Contexts)
                   [\Omega]\Delta \vdash [\Omega]\hat{\alpha}
                                                                                                     By above equality
                   [\Omega]\Delta \vdash \forall \alpha. [\Omega]A_0 \bullet e \Rightarrow [\Omega]C
                                                                                                     By Decl∀App
                   [\Omega]\Delta \vdash [\Omega](\forall \alpha. A_0) \bullet e \Rightarrow [\Omega]C
                                                                                                     By definition of substitution
Case
                  \Gamma \vdash e \Leftarrow B \dashv \Delta
                                                                                                    Subderivation
                   \Delta \longrightarrow \Omega
                                                                                                    Given
                                                                                                    By i.h.
                  [\Omega]\Delta \vdash e \Leftarrow [\Omega]B
                  [\Omega]\Delta \vdash ([\Omega]B) \rightarrow ([\Omega]C) \bullet e \Longrightarrow [\Omega]C
                                                                                                    By Decl→App
                [\Omega]\Delta \vdash [\Omega](B \to C) \bullet e \Longrightarrow [\Omega]C
                                                                                                    By definition of substitution
 \bullet \  \, \textbf{Case} \  \, \underbrace{\frac{\Gamma_0[\hat{\alpha}_2,\hat{\alpha}_1,\hat{\alpha}=\hat{\alpha}_1\rightarrow\hat{\alpha}_2]\vdash e \Leftarrow \hat{\alpha}_1 \dashv \Delta}{\Gamma_0[\hat{\alpha}]\vdash \hat{\alpha}\bullet e \Rightarrow \hat{\alpha}_2 \dashv \Delta}}_{\Gamma} \, \hat{\alpha} \mathsf{App}
```

```
\overbrace{\Gamma_0[\hat{\alpha}_2,\hat{\alpha}_1,\hat{\alpha}=\hat{\alpha}_1\rightarrow\hat{\alpha}_2]}^{\Gamma'}\vdash e \Leftarrow \hat{\alpha}_1 \dashv \Delta
                                                                                                                                                 Subderivation
                                                                                                                                                 Given
                                                  [\Omega]\Delta \vdash e \Leftarrow [\Omega]\hat{\alpha}_1
                                                                                                                                                 By i.h.
                                                  [\Omega]\Delta \vdash ([\Omega]\hat{\alpha}_1) \rightarrow ([\Omega]\hat{\alpha}_2) \bullet e \Longrightarrow [\Omega]\hat{\alpha}_2 By Decl\rightarrowApp
                            \Gamma' \longrightarrow \Delta
                                                                                          By Lemma 54 (Typing Extension)
                            \Delta \longrightarrow \Omega
                                                                                          Given
                           \Gamma' \longrightarrow \Omega
                                                                                          By Lemma 21 (Transitivity)
                          [\Gamma'] {\hat{\alpha}} = [\Gamma'] ({\hat{\alpha}}_1 \to {\hat{\alpha}}_2)
                                                                                          By definition of [\Gamma'](-)
                   [\Omega][\Gamma']\hat{\alpha} = [\Omega][\Gamma'](\hat{\alpha}_1 \to \hat{\alpha}_2)
                                                                                          Applying \Omega to both sides
                          [\Omega]\hat{\alpha} = [\Omega](\hat{\alpha}_1 \to \hat{\alpha}_2)
                                                                                          By Lemma 18 (Substitution Extension Invariance), twice
                                      =([\Omega]\hat{\alpha}_1) \rightarrow ([\Omega]\hat{\alpha}_2)
                                                                                          By definition of substitution
                          [\Omega]\Delta \vdash [\Omega]\hat{\alpha} \bullet e \Rightarrow [\Omega]\hat{\alpha}_2 By above equality
        13
 Case
                   \overline{\Gamma \vdash () \Rightarrow 1 \dashv \Gamma} \stackrel{1 \mid \Rightarrow}{}
                  [\Omega]\Delta \vdash () \Rightarrow [\Omega]1 By Decl11\Rightarrow and definition of substitution
 \bullet \  \, \textbf{Case} \  \, \frac{\Gamma, \hat{\alpha}, \hat{\beta}, x: \hat{\alpha} \vdash \, e_0 \Leftarrow \hat{\beta} \, \dashv \Delta, x: \hat{\alpha}, \Theta}{\Gamma \vdash \, \lambda x. \, e_0 \Rightarrow \hat{\alpha} \rightarrow \hat{\beta} \, \dashv \Delta} \rightarrow \mathsf{I} \Rightarrow 
         \Gamma, \hat{\alpha}, \hat{\beta}, x : \hat{\alpha} \longrightarrow \Delta, x : \hat{\alpha}, \Theta
                                                                                  By Lemma 54 (Typing Extension)
                                         Θ is soft
                                                                                  By Lemma 24 (Extension Order) (v) (with \Gamma_R = \cdot, which is soft)
                     \Gamma, \hat{\alpha}, \hat{\beta} \longrightarrow \Delta
                              \Delta \longrightarrow \Omega
                                                                                  Given
                  \Delta, x : \hat{\alpha} \longrightarrow \Omega, x : [\Omega] \hat{\alpha}
                                                                                  By \longrightarrow Var
            \underbrace{\Delta, x : \hat{\alpha}, \Theta}_{\Delta'} \longrightarrow \underbrace{\Omega, x : [\Omega] \hat{\alpha}, |\Theta|}_{\Omega'} \quad \text{By Lemma 48 (Filling Completes)}
              \Gamma, \hat{\alpha}, \hat{\beta}, x : \hat{\alpha} \vdash e \Leftarrow \hat{\beta} \dashv \Delta, x : \hat{\alpha}, \Theta
                                                                                                   Subderivation
                         [\Omega']\Delta' \vdash e_0 \Leftarrow [\Omega']\hat{\beta}
                                                                                                   By i.h.
                         [\Omega']\hat{\beta} = \left[\Omega, x : [\Omega]\hat{\alpha}\right]\hat{\beta}
                                                                                                   By Lemma 45 (Substitution Stability)
                                        = [\Omega] \hat{\beta}
                                                                                                   By definition of substitution
                        [\Omega']\Delta' = [\Omega, x : [\Omega]\hat{\alpha}](\Delta, x : \hat{\alpha})
                                                                                                   By Lemma 47 (Softness Goes Away)
                                        = [\Omega]\Delta, x : [\Omega]\hat{\alpha}
                                                                                                   By definition of context substitution
         [\Omega]\Delta, x : [\Omega]\hat{\alpha} \vdash e_0 \Leftarrow [\Omega]\hat{\beta}
                                                                                                   By above equalities
                     \Gamma, \hat{\alpha}, \hat{\beta} \longrightarrow \Delta
                                                                                                   Above
                     \Gamma, \hat{\alpha}, \hat{\beta} \longrightarrow \Omega
                                                                                                   By Lemma 21 (Transitivity)
                          \Gamma, \hat{\alpha}, \hat{\beta} \vdash \hat{\alpha}
                                                                                                   By EvarWF
                                  \Omega \vdash \hat{\alpha}
                                                                                                   By Lemma 25 (Extension Weakening)
                            [\Omega]\Delta \vdash [\Omega]\hat{\alpha}
                                                                                                   By Lemma 44 (Substitution for Well-Formedness)
                                                                                                      and Lemma 49 (Stability of Complete Contexts)
                            [\Omega]\Delta \vdash [\Omega]\hat{\beta}
                                                                                                   By similar reasoning
                   [\Omega]\Delta \vdash ([\Omega]\hat{\alpha}) \rightarrow ([\Omega]\hat{\beta})
                                                                                                  By DeclArrowWF
                                   [\Omega]\hat{\alpha}, [\Omega]\hat{\beta} monotypes
                                                                                                  \Omega predicative
                   [\Omega]\Delta \vdash \lambda x. e_0 \Rightarrow ([\Omega]\hat{\alpha}) \rightarrow ([\Omega]\hat{\beta})
                                                                                                  By Decl\rightarrowI\Rightarrow
        \square \square \square \Delta \vdash \lambda x. e_0 \Rightarrow [\Omega](\widehat{\alpha} \rightarrow \widehat{\beta})
                                                                                                  By definition of substitution \Box
```

\mathbf{K}' Completeness

K'.1 Instantiation Completeness

Theorem 13 (Instantiation Completeness).

Given $\Gamma \longrightarrow \Omega$ and $A = [\Gamma]A$ and $\hat{\alpha} \in \mathsf{unsolved}(\Gamma)$ and $\hat{\alpha} \notin \mathsf{FV}(A)$:

- (1) If $[\Omega]\Gamma \vdash [\Omega]\hat{\alpha} < [\Omega]A$ then there are Δ , Ω' such that $\Omega \longrightarrow \Omega'$ and $\Delta \longrightarrow \Omega'$ and $\Gamma \vdash \hat{\alpha} : \stackrel{\leq}{=} A \dashv \Delta$.
- (2) If $[\Omega]\Gamma \vdash [\Omega]A < [\Omega]\hat{\alpha}$ then there are Δ , Ω' such that $\Omega \longrightarrow \Omega'$ and $\Delta \longrightarrow \Omega'$ and $\Gamma \vdash A \stackrel{\leq}{=} : \hat{\alpha} \dashv \Delta$.

Proof. By mutual induction on the given declarative subtyping derivation.

- (1) We have $[\Omega]\Gamma \vdash [\Omega]\hat{\alpha} \leq [\Omega]A$. We now case-analyze the shape of A.
 - Case $A = \hat{\beta}$:

It is given that $\hat{\alpha} \notin FV(\hat{\beta})$, so $\hat{\alpha} \neq \hat{\beta}$.

Since $A = \hat{\beta}$, we have $[\Omega]\Gamma \vdash [\Omega]\hat{\alpha} \leq [\Omega]\hat{\beta}$.

Since Ω is predicative, $[\Omega]\hat{\alpha} = \tau_1$ and $[\Omega]\hat{\beta} = \tau_2$, so we have $[\Omega]\Gamma \vdash \tau_1 \leq \tau_2$.

By Lemma 9 (Monotype Equality), $\tau_1 = \tau_2$.

We have $A = \hat{\beta}$ and $[\Gamma]A = A$, so $[\Gamma]\hat{\beta} = \hat{\beta}$. Thus $\hat{\beta} \in \mathsf{unsolved}(\Gamma)$.

Let Ω' be Ω . By Lemma 20 (Reflexivity), $\Omega \longrightarrow \Omega$.

Now consider whether $\hat{\alpha}$ is declared to the left of $\hat{\beta}$, or vice versa.

- Case $\Gamma = (\Gamma_0, \hat{\alpha}, \Gamma_1, \hat{\beta}, \Gamma_2)$:

Let Δ be Γ_0 , $\hat{\alpha}$, Γ_1 , $\hat{\beta} = \hat{\alpha}$, Γ_2 . By rule InstLReach, $\Gamma \vdash \hat{\alpha} := \hat{\beta} \dashv \Delta$.

It remains to show that $\Delta \longrightarrow \Omega$.

We have $[\Omega]\hat{\alpha} = [\Omega]\hat{\beta}$. Then by Lemma 30 (Parallel Extension Solution), $\Delta \longrightarrow \Omega$.

- Case $(\Gamma = \Gamma_0, \hat{\beta}, \Gamma_1, \hat{\alpha}, \Gamma_2)$:

Let Δ be Γ_0 , $\hat{\beta}$, Γ_1 , $\hat{\alpha} = \hat{\beta}$, Γ_2 .

By rule InstLSolve, $\Gamma \vdash \hat{\alpha} := \hat{\beta} \dashv \Delta$.

It remains to show that $\Delta \longrightarrow \Omega$.

We have $[\Omega]\hat{\beta} = [\Omega]\hat{\alpha}$. Then by Lemma 30 (Parallel Extension Solution), $\Delta \longrightarrow \Omega$.

• Case $A = \alpha$:

Since $A = \alpha$, we have $[\Omega]\Gamma \vdash [\Omega]\hat{\alpha} \leq [\Omega]\alpha$.

Since $[\Omega]\alpha = \alpha$, we have $[\Omega]\Gamma \vdash [\Omega]\hat{\alpha} \leq \alpha$.

By inversion, \leq Var was used, so $[\Omega]\hat{\alpha} = \alpha$; therefore, since Ω is well-formed, α is declared to the left of $\hat{\alpha}$ in Ω .

We have $\Gamma \longrightarrow \Omega$.

By Lemma 17 (Reverse Declaration Order Preservation), we know that α is declared to the left of $\hat{\alpha}$ in Γ ; that is, $\Gamma = \Gamma_0[\alpha][\hat{\alpha}]$.

Let $\Delta = \Gamma_0[\alpha][\hat{\alpha} = \alpha]$ and $\Omega' = \Omega$.

By InstLSolve, $\Gamma_0[\alpha][\hat{\alpha}] \vdash \hat{\alpha} := \alpha \dashv \Delta$.

By Lemma 30 (Parallel Extension Solution), $\Gamma_0[\alpha][\hat{\alpha} = \alpha] \longrightarrow \Omega$.

• Case $A = A_1 \to A_2$:

By the definition of substitution, $[\Omega]A = ([\Omega]A_1) \rightarrow ([\Omega]A_2)$.

Therefore $[\Omega]\Gamma \vdash [\Omega]\hat{\alpha} \leq ([\Omega]A_1) \rightarrow ([\Omega]A_2)$.

Since we have an arrow as the supertype, only $\leq \forall L$ or $\leq \rightarrow$ could have been used, and the subtype $[\Omega]\hat{\alpha}$ must be either a quantifier or an arrow. But Ω is predicative, so $[\Omega]\hat{\alpha}$ cannot be a quantifier. Therefore, it is an arrow: $[\Omega]\hat{\alpha} = \tau_1 \to \tau_2$, and $\leq \to$ concluded the derivation. Inverting $\leq \rightarrow$ gives $[\Omega]\Gamma \vdash [\Omega]A_2 \leq \tau_2$ and $[\Omega]\Gamma \vdash \tau_1 \leq [\Omega]A_1$.

Since $\hat{\alpha} \in \text{unsolved}(\Gamma)$, we know that Γ has the form $\Gamma_0[\hat{\alpha}]$.

By Lemma 28 (Unsolved Variable Addition for Extension) twice, inserting unsolved variables

 $\hat{\alpha}_2$ and $\hat{\alpha}_1$ into the middle of the context extends it, that is: $\Gamma_0[\hat{\alpha}] \longrightarrow \Gamma_0[\hat{\alpha}_2,\hat{\alpha}_1,\hat{\alpha}]$.

Clearly, $\hat{\alpha}_1 \to \hat{\alpha}_2$ is well-formed in $(\dots, \hat{\alpha}_2, \hat{\alpha}_1)$, so by Lemma 26 (Solution Admissibility for Extension), solving $\hat{\alpha}$ extends the context: $\Gamma_0[\hat{\alpha}_2, \hat{\alpha}_1, \hat{\alpha}] \longrightarrow \Gamma_0[\hat{\alpha}_2, \hat{\alpha}_1, \hat{\alpha} = \hat{\alpha}_1 \to \hat{\alpha}_2]$. Then by Lemma 21 (Transitivity), $\Gamma_0[\hat{\alpha}] \longrightarrow \Gamma_0[\hat{\alpha}_2, \hat{\alpha}_1, \hat{\alpha} = \hat{\alpha}_1 \to \hat{\alpha}_2]$.

Since $\hat{\alpha} \in \text{unsolved}(\Gamma)$ and $\Gamma \longrightarrow \Omega$, we know that Ω has the form $\Omega_0[\hat{\alpha} = \tau_0]$. To show that we can extend this context, we apply Lemma 27 (Solved Variable Addition for Extension) twice to introduce $\hat{\alpha}_2 = \tau_2$ and $\hat{\alpha}_1 = \tau_1$, and then Lemma 26 (Solution Admissibility for Extension) to overwrite τ_0 :

$$\underbrace{\Omega_0[\hat{\alpha}=\tau_0]}_{\Omega} \longrightarrow \Omega_0[\hat{\alpha}_2=\tau_2,\hat{\alpha}_1=\tau_1,\hat{\alpha}=\hat{\alpha}_1 \rightarrow \hat{\alpha}_2]$$

We have $\Gamma \longrightarrow \Omega$, that is,

$$\Gamma_0[\boldsymbol{\hat{\alpha}}] \longrightarrow \Omega_0[\boldsymbol{\hat{\alpha}} = \tau_0]$$

By Lemma 29 (Parallel Admissibility) (i) twice, inserting unsolved variables $\hat{\alpha}_2$ and $\hat{\alpha}_1$ on both contexts in the above extension preserves extension:

$$\underbrace{\frac{\Gamma_0[\hat{\alpha}_2,\hat{\alpha}_1,\hat{\alpha}]}{\Gamma_0[\hat{\alpha}_2,\hat{\alpha}_1,\hat{\alpha}] \longrightarrow \Omega_0[\hat{\alpha}_2=\tau_2,\hat{\alpha}_1=\tau_1,\hat{\alpha}=\tau_0]}{\Omega_0[\hat{\alpha}_2=\tau_2,\hat{\alpha}_1=\tau_1,\hat{\alpha}=\hat{\alpha}_1\to\hat{\alpha}_2]}}_{\Omega_1} \quad \text{By Lemma 31 (Parallel Variable Update)}$$

Since $\hat{\alpha} \notin FV(A)$, it follows that $[\Gamma_1]A = [\Gamma]A = A$.

Therefore $\hat{\alpha}_1 \notin FV(A_1)$ and $\hat{\alpha}_1, \hat{\alpha}_2 \notin FV(A_2)$.

By Lemma 51 (Finishing Completions) and Lemma 50 (Finishing Types), $[\Omega_1]\Gamma_1 = [\Omega]\Gamma$ and $[\Omega_1]\hat{\alpha}_1 = \tau_1$.

By i.h., there are Δ_2 and Ω_2 such that $\Gamma_1 \vdash A_1 \stackrel{\leq}{=} : \hat{\alpha}_1 \dashv \Delta_2$ and $\Delta_2 \longrightarrow \Omega_2$ and $\Omega_1 \longrightarrow \Omega_2$.

Next, note that $[\Delta_2][\Delta_2]A_2 = [\Delta_2]A_2$.

By Lemma 34 (Left Unsolvedness Preservation), we know that $\hat{\alpha}_2 \in \mathsf{unsolved}(\Delta_2)$.

By Lemma 35 (Left Free Variable Preservation), we know that $\hat{\alpha}_2 \notin FV([\Delta_2]A_2)$.

By Lemma 21 (Transitivity), $\Omega \longrightarrow \Omega_2$.

We know $[\Omega_2]\Delta_2 = [\Omega]\Gamma$ because:

$$[\Omega_2]\Delta_2 = [\Omega_2]\Omega_2$$
 By Lemma 49 (Stability of Complete Contexts)
= $[\Omega]\Omega$ By Lemma 51 (Finishing Completions)
= $[\Omega]\Gamma$ By Lemma 49 (Stability of Complete Contexts)

By Lemma 50 (Finishing Types), we know that $[\Omega_2]\hat{\alpha}_2 = [\Omega_1]\hat{\alpha}_2 = \tau_2$.

By Lemma 50 (Finishing Types), we know that $[\Omega_2]A_2 = [\Omega]A_2$.

Hence we know that $[\Omega_2]\Delta_2 \vdash [\Omega_2]\hat{\alpha}_2 \leq [\Omega_2]A_2$.

By i.h., we have Δ and Ω' such that $\Delta_2 \vdash \hat{\alpha}_2 := [\Delta_2]A_2 \dashv \Delta$ and $\Omega_2 \longrightarrow \Omega'$ and $\Delta \longrightarrow \Omega'$.

By rule InstLArr, $\Gamma \vdash \hat{\alpha} : \stackrel{\leq}{=} A \dashv \Delta$.

By Lemma 21 (Transitivity), $\Omega \longrightarrow \Omega'$.

• Case A = 1:

We have A = 1, so $[\Omega]\Gamma \vdash [\Omega]\hat{\alpha} \leq [\Omega]1$.

Since $[\Omega]1 = 1$, we have $[\Omega]\Gamma \vdash [\Omega]\hat{\alpha} < 1$.

The only declarative subtyping rules that can have 1 as the supertype in the conclusion are $\leq \forall L$ and $\leq U$ nit. However, since Ω is predicative, $[\Omega] \hat{\alpha}$ cannot be a quantifier, so $\leq \forall L$ cannot have been used. Hence $\leq U$ nit was used and $[\Omega] \hat{\alpha} = 1$.

Let $\Delta = \Gamma[\hat{\alpha} = 1]$ and $\Omega' = \Omega$.

By InstLSolve, $\Gamma[\hat{\alpha}] \vdash \hat{\alpha} : \leq 1 \dashv \Delta$.

By Lemma 30 (Parallel Extension Solution), $\Gamma[\hat{\alpha}=1] \longrightarrow \Omega$.

• Case $A = \forall \beta$. B:

We have $[\Omega]\Gamma \vdash [\Omega]\hat{\alpha} < [\Omega](\forall \beta. B)$.

By definition of substitution, $[\Omega](\forall \beta. B) = \forall \beta. [\Omega]B$, so we have $[\Omega]\Gamma \vdash [\Omega]\hat{\alpha} \leq \forall \beta. [\Omega]B$. The only declarative subtyping rules that can have a quantifier as supertype are $\leq \forall L$ and $\leq \forall R$. However, since Ω is predicative, $[\Omega]\hat{\alpha}$ cannot be a quantifier, so $\leq \forall L$ cannot have been used. Hence $\leq \forall R$ was used, and we have a subderivation of $[\Omega]\Gamma, \beta \vdash [\Omega]\hat{\alpha} \leq [\Omega]B$.

```
Let \Omega_1 = (\Omega, \beta) and \Gamma_1 = (\Gamma, \beta).
By \longrightarrow Uvar, \Gamma_1 \longrightarrow \Omega_1.
By the definition of substitution, [\Omega_1]B = [\Omega]B and [\Omega_1]\hat{\alpha} = [\Omega]\hat{\alpha}.
Note that [\Omega_1]\Gamma_1 = [\Omega]\Gamma, \beta.
Since \hat{\alpha} \in \mathsf{unsolved}(\Gamma), we have \hat{\alpha} \in \mathsf{unsolved}(\Gamma_1).
Since \hat{\alpha} \notin FV(A) and A = \forall \beta. B, we have \hat{\alpha} \notin FV(B).
By i.h., there are \Omega_2 and \Delta_2 such that \Gamma, \beta \vdash \hat{\alpha} := B \dashv \Delta_2 and \Delta_2 \longrightarrow \Omega_2 and \Omega_1 \longrightarrow \Omega_2.
By Lemma 32 (Instantiation Extension), \Gamma_1 \longrightarrow \Delta_2, that is, \Gamma, \beta \longrightarrow \Delta_2.
Therefore by Lemma 24 (Extension Order), \Delta_2 = (\Delta', \beta, \Omega'') where \Gamma \longrightarrow \Delta'.
By equality, we know \Delta', \beta, \Delta'' \longrightarrow \Omega_2.
By Lemma 24 (Extension Order), \Omega_2=(\Omega',\beta,\Omega'') where \quad \Delta' \longrightarrow \Omega'.
We have \Omega_1 \longrightarrow \Omega_2, that is, \Omega, \beta \longrightarrow \Omega', \beta, \Omega'', so Lemma 24 (Extension Order) gives
\Omega \longrightarrow \Omega'.
By rule InstLAIIR, \Gamma \vdash \hat{\alpha} : \stackrel{\leq}{=} \forall \beta. B \dashv \Delta'.
```

(2) $[\Omega]\Gamma \vdash [\Omega]A \leq [\Omega]\hat{\alpha}$

These cases are mostly symmetric. The one exception is the one connective that is not treated symmetrically in the declarative subtyping rules:

• Case $A = \forall \alpha$. B:

Since $A = \forall \alpha$. B, we have $[\Omega]\Gamma \vdash [\Omega]\forall \beta$. $B \leq [\Omega]\hat{\alpha}$.

By symmetric reasoning to the previous case (the last case of part (1) above), ≤∀L must have been used, with a subderivation of $[\Omega]\Gamma \vdash [\Omega]\hat{\alpha} \leq [\tau/\beta][\Omega]B$.

Since $[\Omega]\Gamma \vdash \tau$, the type τ has no existential variables and is therefore invariant under substitution: $\tau = [\Omega]\tau$. Therefore $[\tau/\beta][\Omega]B = [[\Omega]\tau/\beta][\Omega]B$.

By distributivity of substitution, this is $[\Omega][\tau/\beta]B$. Interposing $\hat{\beta}$, this is equal to $[\Omega][\tau/\hat{\beta}][\hat{\beta}/\beta]B$. Therefore $[\Omega]\Gamma \vdash [\Omega]\hat{\alpha} \leq [\Omega][\tau/\hat{\beta}][\hat{\beta}/\beta]B$.

Let Ω_1 be $\Omega, \triangleright_{\hat{\alpha}}, \hat{\beta} = \tau$ and let Γ_1 be $\Gamma, \triangleright_{\hat{\alpha}}, \hat{\beta}$.

- By the definition of context application, $[\Omega_1]\Gamma_1 = [\Omega]\Gamma$.
- From the definition of substitution, $[\Omega_1]\hat{\alpha} = [\Omega]\hat{\alpha}$.
- It follows from the definition of substitution that $[\Omega][\tau/\hat{\beta}]C = [\Omega_1]C$ for all C. Therefore $[\Omega][\tau/\beta][\beta/\beta]B = [\Omega_1][\beta/\beta]B.$

Applying these three equalities, $[\Omega_1]\Gamma_1 \vdash [\Omega_1]\hat{\alpha} < [\Omega_1][\hat{\beta}/\beta]B$. By the definition of substitution, $[\Gamma, \blacktriangleright_{\hat{B}}, \hat{\beta}]B = [\Gamma]B = B$, so $\hat{\alpha} \notin FV([\Gamma_1]B)$.

Since $\hat{\alpha} \in \text{unsolved}(\Gamma)$, we have $\hat{\alpha} \in \text{unsolved}(\Gamma_1)$.

By i.h., there exist Δ_2 and Ω_2 such that $\Gamma_1 \vdash B \stackrel{\leq}{=} : \hat{\alpha} \dashv \Delta_2$ and $\Omega_1 \longrightarrow \Omega_2$ and $\Delta_2 \longrightarrow \Omega_2$.

By Lemma 32 (Instantiation Extension), $\Gamma_1 \longrightarrow \Delta_2$, which is, $\Gamma, \blacktriangleright_{\hat{\beta}}, \hat{\beta} \longrightarrow \Delta_2$.

By Lemma 24 (Extension Order), $\Delta_2 = (\Delta', \blacktriangleright_{\hat{\beta}}, \Delta'')$ and $\Gamma \longrightarrow \Delta'$.

By equality, $\Delta', \blacktriangleright_{\widehat{\beta}}, \Delta'' \longrightarrow \Omega_2$.

By Lemma 24 (Extension Order), $\Omega_2 = (\Omega', \blacktriangleright_{\hat{B}}, \Omega'')$ and $\square \square \Delta' \longrightarrow \Omega'$.

By equality, $\Omega, \blacktriangleright_{\hat{\beta}}, \hat{\beta} = \tau \longrightarrow \Omega', \blacktriangleright_{\hat{\beta}}, \Omega''.$ By Lemma 24 (Extension Order), $\Omega \longrightarrow \Omega'.$

By InstRAIIL, $\Gamma \vdash \forall \beta$. $B \leq : \hat{\alpha} \dashv \Delta'$.

K'.2 Completeness of Subtyping

Theorem 14 (Generalized Completeness of Subtyping). *If* $\Gamma \longrightarrow \Omega$ *and* $\Gamma \vdash A$ *and* $\Gamma \vdash B$ *and* $[\Omega]\Gamma \vdash [\Omega]A \leq [\Omega]B$ *then there exist* Δ *and* Ω' *such that* $\Delta \longrightarrow \Omega'$ *and* $\Omega \longrightarrow \Omega'$ *and* $\Gamma \vdash [\Gamma]A <: [\Gamma]B \dashv \Delta$.

Proof. By induction on the derivation of $[\Omega]\Gamma \vdash [\Omega]A \leq [\Omega]B$.

We distinguish cases of $[\Gamma]B$ and $[\Gamma]A$ that are *impossible*, **fully written out**, and similar to fully-written-out cases.

				[Γ]B		
		∀β. B′	1	α	β	$B_1 \to B_2$
	$\forall \alpha. A'$	1 (B poly)	2.Poly	2.Poly	2.Poly	2.Poly
	1	1 (B poly)	2.Units	impossible	2.BEx.Unit	impossible
$[\Gamma]A$	α	1 (B poly)	impossible	2.Uvars	2.BEx.Uvar	impossible
	â	1 (B poly)	2.AEx.Unit	2.AEx.Uvar	2.AEx.SameEx 2.AEx.OtherEx	2.AEx.Arrow
	$A_1 \to A_2$	1 (B poly)	impossible	impossible	2.BEx.Arrow	2.Arrows

The impossibility of the "impossible" entries follows from inspection of the declarative subtyping rules.

We first split on $[\Gamma]B$.

• Case 1 (B poly): $[\Gamma]B$ polymorphic: $[\Gamma]B = \forall \beta. B'$:

$$B = \forall \beta. \, B_0 \\ B' = [\Gamma]B_0 \\ [\Omega]B = [\Omega](\forall \beta. \, B_0) \\ = \forall \beta. \, [\Omega]B_0 \\ = \forall \beta. \,$$

• Cases 2.*: [Γ]B not polymorphic:

We split on the form of $[\Gamma]A$.

- Case 2.Poly: $[\Gamma]A$ is polymorphic: $[\Gamma]A = \forall \alpha. A'$:

$$A = \forall \alpha. A_0 \qquad \Gamma \text{ predicative } \\ A' = [\Gamma]A_0 \qquad \Gamma \text{ predicative } \\ [\Omega]A = [\Omega](\forall \alpha. A_0) \qquad \text{Applying } \Omega \text{ to both sides } \\ [\Omega]A = \forall \alpha. [\Omega]A_0 \qquad \text{By definition of substitution } \\ [\Omega]\Gamma \vdash [\Omega]A \leq [\Omega]B \qquad \text{Given } \\ [\Omega]\Gamma \vdash [\alpha]A \leq [\Omega]B \qquad \text{By above equality } \\ [\Gamma]B \neq (\forall \beta. \dots) \qquad \Gamma \text{ predicative } \\ [B \neq (\forall \beta. \dots) \qquad \Gamma \text{ predicative } \\ [C]\Gamma \vdash [\tau/\alpha][\Omega]A_0 \leq [\Omega]B \qquad \text{By inversion on } \leq \forall L \\ [C]\Gamma \vdash [\tau/\alpha][\Omega]A_0 \leq [\Omega]B \qquad \text{By inversion on } \leq \forall L \\ [C]\Gamma \vdash \tau \qquad G \qquad \text{Given } \\ [C]\Gamma \vdash \tau \qquad G \qquad \text{By } \longrightarrow \text{Solve } \\ [C]\Gamma \vdash [\tau/\alpha][\Omega]A_0 \leq [\Omega]B \qquad \text{By } \longrightarrow \text{Solve } \\ [C]\Gamma \vdash [\tau/\alpha][\Omega]A_0 \leq [\Omega]B \qquad \text{By above equality } \\ [C]\Gamma \vdash [\tau/\alpha][\Omega]A_0 \leq [\Omega]B \qquad \text{By above equality } \\ [C]\Gamma \vdash [\tau/\alpha][\Omega]A_0 \leq [\Omega]B \qquad \text{By above equality } \\ [C]\Gamma \vdash [\tau/\alpha][\Omega]A_0 \leq [\Omega]B \qquad \text{By above equality } \\ [C]\Gamma \vdash [\tau/\alpha][\Omega]A_0 \leq [\Omega]B \qquad \text{By definition of substitution } \\ [C]\Gamma \vdash [\tau/\alpha][\Omega]A_0 \leq [\Omega]B \qquad \text{By definition of substitution } \\ [C]\Gamma \vdash [\tau/\alpha][\Omega]A_0 \leq [\Omega]B \qquad \text{By definition of substitution } \\ [C]\Gamma \vdash [\tau/\alpha][\Omega]A_0 \leq [\Omega]B \qquad \text{By definition of substitution } \\ [C]\Gamma \vdash [\tau/\alpha][\Omega]A_0 \leq [\Gamma]B \qquad D \qquad By definition of substitution } \\ [C]\Gamma \vdash [\tau/\alpha][\Omega]A_0 \leq [\Gamma]B \rightarrow \Delta_0 \qquad By definition of substitution } \\ [C]\Gamma \vdash [\tau/\alpha][\Omega]A_0 \leq [\Gamma]B \rightarrow \Delta_0 \qquad By definition of substitution } \\ [C]\Gamma \vdash [\tau/\alpha][\Omega]A_0 \leq [\Gamma]B \rightarrow \Delta_0 \qquad By definition of substitution } \\ [C]\Gamma \vdash [\tau/\alpha][\Omega]A_0 \leq [\Gamma]B \rightarrow \Delta_0 \qquad By definition of substitution } \\ [C]\Gamma \vdash [\tau/\alpha][\Omega]A_0 \leq [\Gamma]B \rightarrow \Delta_0 \qquad By definition of substitution } \\ [C]\Gamma \vdash [\tau/\alpha][\Omega]A_0 \leq [\Gamma]B \rightarrow \Delta_0 \qquad By definition of substitution } \\ [C]\Gamma \vdash [\tau/\alpha][\Omega]A_0 \leq [\Gamma]B \rightarrow \Delta_0 \qquad By definition of substitution } \\ [C]\Gamma \vdash [\tau/\alpha][\Omega]A_0 \leq [\Gamma]B \rightarrow \Delta_0 \qquad By definition of substitution } \\ [C]\Gamma \vdash [\tau/\alpha][\Omega]A_0 \leq [\Gamma]B \rightarrow \Delta_0 \qquad By definition of substitution } \\ [C]\Gamma \vdash [\tau/\alpha][\Omega]A_0 \leq [\Gamma]B \rightarrow \Delta_0 \qquad By definition of substitution } \\ [C]\Gamma \vdash [\tau/\alpha][\Omega]A_0 \leq [\Gamma]B \rightarrow \Delta_0 \qquad By definition of substitution } \\ [C]\Gamma \vdash [\tau/\alpha][\Omega]A_0 \leq [\Gamma]B \rightarrow \Delta_0 \qquad By definition of substitution } \\ [C]\Gamma \vdash [\tau/\alpha][\Omega]A_0 \leq [\Gamma]B \rightarrow \Delta_0 \qquad By definition of substitution } \\ [C]\Gamma \vdash [\tau/\alpha][\Omega]A_0 \leq [\Gamma]B \rightarrow \Delta_0 \qquad By definition of substitution }$$

By above equality

- Case 2.AEx: A is an existential variable $[\Gamma]A = \hat{\alpha}$:

 $\Gamma \vdash \forall \alpha. A' <: [\Gamma]B \dashv \Delta$

We split on the form of $[\Gamma]B$.

* Case 2.AEx.SameEx: $[\Gamma]B$ is the same existential variable $[\Gamma]B = \hat{\alpha}$:

- * Case 2.AEx.OtherEx: $[\Gamma]B$ is a different existential variable $[\Gamma]B = \hat{\beta}$ where $\hat{\beta} \neq \hat{\alpha}$: Either $\hat{\alpha} \in FV([\Gamma]\hat{\beta})$, or $\hat{\alpha} \notin FV([\Gamma]\hat{\beta})$.
 - $\hat{\alpha} \in FV([\Gamma]\hat{\beta})$:

We have $\hat{\alpha} \leq [\Gamma] \hat{\beta}$.

Therefore $\hat{\alpha} = [\Gamma] \hat{\beta}$, or $\hat{\alpha} \prec [\Gamma] \hat{\beta}$.

But we are in Case 2.AEx.OtherEx, so the former is impossible.

Therefore, $\hat{\alpha} \prec [\Gamma] \hat{\beta}$.

Since Γ is predicative, $[\Gamma]\hat{\beta}$ cannot have the form $\forall \beta. \dots$, so the only way that $\hat{\alpha}$ can be a proper subterm of $[\Gamma]\hat{\beta}$ is if $[\Gamma]\hat{\beta}$ has the form $B_1 \to B_2$ such that $\hat{\alpha}$ is a subterm of B_1 or B_2 , that is: $\hat{\alpha} \neq [\Gamma]\hat{\beta}$.

Then by a property of substitution, $[\Omega] \hat{\alpha} \stackrel{?}{\supset} [\Omega] [\Gamma] \hat{\beta}$.

By Lemma 18 (Substitution Extension Invariance), $[\Omega][\Gamma]\hat{\beta} = [\Omega]\hat{\beta}$, so $[\Omega]\hat{\alpha} \not\subset [\Omega]\hat{\beta}$. We have $[\Omega]\Gamma \vdash [\Omega]\hat{\alpha} \leq [\Omega]\hat{\beta}$, and we know that $[\Omega]\hat{\alpha}$ is a monotype, so we can use Lemma 8 (Occurrence) (ii) to show that $[\Omega]\hat{\alpha} \not\subset [\Omega]\hat{\beta}$, a contradiction.

 $\cdot \hat{\alpha} \notin FV([\Gamma]\hat{\beta})$:

* Case 2.AEx.Unit: $[\Gamma]B = 1$:

$$\begin{array}{cccc} \Gamma \longrightarrow \Omega & \text{Given} \\ 1 = [\Omega]1 & \text{By definition of substitution} \\ \hat{\alpha} \notin \text{FV}(1) & \text{By definition of FV}(-) \\ [\Omega]\Gamma \vdash [\Omega]\hat{\alpha} \leq [\Omega]1 & \text{Given} \\ & \Gamma \vdash \hat{\alpha} : \stackrel{\leq}{=} 1 \ \neg \Delta & \text{By Theorem 13 (1)} \\ & \Omega \longrightarrow \Omega' & " \\ & & \Delta \longrightarrow \Omega' & " \end{array}$$

 $\begin{array}{ll} 1 = [\Gamma] 1 & \text{By definition of substitution} \\ \hat{\alpha} \notin FV(1) & \text{By definition of } FV(-) \end{array}$

 $\Gamma \vdash \hat{\alpha} <: 1 \dashv \Delta$ By <: InstantiateL

* Case 2.AEx.Uvar: $[\Gamma]B = \beta$:

Similar to Case 2.AEx.Unit, using $\beta = [\Omega]\beta = [\Gamma]\beta$ and $\hat{\alpha} \notin FV(\beta)$.

* Case 2.AEx.Arrow: $[\Gamma]B = B_1 \rightarrow B_2$:

Since $[\Gamma]B$ is an arrow, it cannot be exactly $\hat{\alpha}$.

Suppose, for a contradiction, that $\hat{\alpha} \in FV([\Gamma]B)$.

```
\hat{\alpha} \leq [\Gamma]B
                                                                    \hat{\alpha} \in FV([\Gamma]B)
           [\Omega] \hat{\alpha} \leq [\Omega] [\Gamma] B
                                                                    By a property of substitution
              \Gamma \longrightarrow \Omega
       [\Omega][\Gamma]B = [\Omega]B
                                                                    By Lemma 18 (Substitution Extension Invariance)
            [\Omega] {\hat{\alpha}} \preceq [\Omega] B
                                                                    By above equality
             [\Gamma]B \neq \hat{\alpha}
                                                                    Given (2.AEx.Arrow)
                                                                    By a property of substitution
       [\Omega][\Gamma]B \neq [\Omega]\hat{\alpha}
           [\Omega]B \neq [\Omega]\hat{\alpha}
                                                                    By Lemma 18 (Substitution Extension Invariance)
           [\Omega] \hat{\alpha} \prec [\Omega] B
                                                                    Follows from \prec and \neq
          [\Omega] \hat{\alpha} \stackrel{?}{\supset} [\Omega] B
                                                                    [\Omega]A has the form \cdots \rightarrow \cdots
             [\Omega]\Gamma \vdash [\Omega]\hat{\alpha} \leq [\Omega]B
                                                                    Given
                          [\Omega]B is a monotype
                                                                    \Omega is predicative
          [\Omega] \hat{\alpha} \not \subset [\Omega] B
                                                                    By Lemma 8 (Occurrence) (ii)
                \Rightarrow \Leftarrow
\hat{\alpha} \notin FV([\Gamma]B)
                                                                    By contradiction
              \Gamma \vdash \hat{\alpha} : \leq [\Gamma] B \dashv \Delta
                                                         By Theorem 13 (1)
         \Delta \longrightarrow \Omega'
        \Omega \longrightarrow \Omega'
B
              \Gamma \vdash \hat{\alpha} \mathrel{<:} \underbrace{[\Gamma]B}_{B_1 \to B_2} \dashv \Delta \quad By \mathrel{<:} \mathsf{InstantiateL}
```

- Case 2.BEx: $[\Gamma]A$ is not polymorphic and $[\Gamma]B$ is an existential variable: $[\Gamma]B = \hat{\beta}$ We split on the form of $[\Gamma]A$.
 - * Case 2.BEx.Unit ($[\Gamma]A = 1$), Case 2.BEx.Uvar ($[\Gamma]A = \alpha$), Case 2.BEx.Arrow ($[\Gamma]A = A_1 \rightarrow A_2$):

Similar to Cases **2.AEx.Unit**, **2.AEx.Uvar** and **2.AEx.Arrow**, but using part (2) of Theorem 13 instead of part (1), and applying <:InstantiateR instead of <:InstantiateL as the final step.

- Case 2.Units: $[\Gamma]A = [\Gamma]B = 1$:

$$\begin{array}{cccc} & & \Gamma \vdash 1 <: \ 1 \dashv \Gamma & \ By <: Unit \\ & & \Gamma \longrightarrow \Omega & \ Given \\ & & \Delta \longrightarrow \Omega & \Delta = \Gamma \end{array}$$

 $\Omega \longrightarrow \Omega'$ By Lemma 20 (Reflexivity) and $\Omega' = \Omega$

- Case 2.Uvars: $[\Gamma]A = [\Gamma]B = \alpha$:

$$\begin{array}{ccc} \alpha \in \Omega & \text{By inversion on } \leq \mathsf{Var} \\ \Gamma \longrightarrow \Omega & \text{Given} \\ \alpha \in \Gamma & \text{By Lemma 24 (Extension Order)} \\ \Gamma \vdash \alpha <: \alpha \dashv \Gamma & \text{By } <: \mathsf{Var} \\ \Delta \longrightarrow \Omega & \Delta = \Gamma \end{array}$$

By Lemma 20 (Reflexivity) and $\Omega' = \Omega$

- Case 2.Arrows: $A = A_1 \rightarrow A_2$ and $B = B_1 \rightarrow B_2$: Only rule ≤→ could have been used.

```
[\Omega]\Gamma \vdash [\Omega]B_1 \leq [\Omega]A_1
                                                               Subderivation
        \Gamma \vdash [\Gamma]B_1 <: [\Gamma]A_1 \dashv \Theta
                                                               By i.h.
    \Theta \longrightarrow \Omega_0
    \Omega \longrightarrow \Omega_0
     \Gamma \longrightarrow \Omega
                                                               Given
     \Gamma \longrightarrow \Omega_0
                                                               By Lemma 21 (Transitivity)
    \Theta \longrightarrow \Omega_0
                                                               Above
  [\Omega]\Gamma = [\Omega]\Theta
                                                               By Lemma 52 (Confluence of Completeness)
   [\Omega]\Gamma \vdash [\Omega]A_2 \leq [\Omega]B_2
                                                               Subderivation
  [\Omega]\Theta \vdash [\Omega]A_2 \leq [\Omega]B_2
                                                               By above equality
[\Omega]A_2 = [\Omega][\Gamma]A_2
                                                               By Lemma 18 (Substitution Extension Invariance)
[\Omega]B_2=[\Omega][\Gamma]B_2
                                                               By Lemma 18 (Substitution Extension Invariance)
  [\Omega]\Theta \vdash [\Omega][\Gamma]A_2 \leq [\Omega][\Gamma]B_2
                                                               By above equalities
        \Theta \vdash [\Theta][\Gamma]A_2 <: [\Theta][\Gamma]B_2 \dashv \Delta By i.h.
    \Delta \longrightarrow \Omega'
 \Omega_0 \longrightarrow \Omega'
     \Gamma \vdash ([\Gamma]A_1) \to ([\Gamma]A_2) \mathrel{<:} ([\Gamma]B_1) \to ([\Gamma]B_2) \dashv \Delta \quad By \mathrel{<:} \to
     \Gamma \vdash [\Gamma](A_1 \to A_2) \mathrel{<:} [\Gamma](B_1 \to B_2) \dashv \Delta
                                                                                           By definition of substitution
\Omega \longrightarrow \Omega'
                                                                                            By Lemma 21 (Transitivity)
```

Corollary 55 (Completeness of Subtyping). *If* $\Psi \vdash A \leq B$ *then there is a* Δ *such that* $\Psi \vdash A \leq B \dashv \Delta$.

Proof. Let $\Omega = \Psi$ and $\Gamma = \Psi$.

By Lemma 20 (Reflexivity), $\Psi \longrightarrow \Psi$, so $\Gamma \longrightarrow \Omega$.

By Lemma 4 (Well-Formedness), $\Psi \vdash A$ and $\Psi \vdash B$; since $\Gamma = \Psi$, we have $\Gamma \vdash A$ and $\Gamma \vdash B$.

By Theorem 14, there exists Δ such that $\Gamma \vdash [\Gamma]A <: [\Gamma]B \dashv \Delta$.

Since $\Gamma = \Psi$ and Ψ is a declarative context with no existentials, $[\Psi]C = C$ for all C, so we actually have $\Psi \vdash A \lt : B \dashv \Delta$, which was to be shown.

L' Completeness of Typing

Theorem 15 (Completeness of Algorithmic Typing). *Given* $\Gamma \longrightarrow \Omega$ *and* $\Gamma \vdash A$:

```
(i) If [\Omega]\Gamma \vdash e \Leftarrow [\Omega]A
then there exist \Delta and \Omega'
such that \Delta \longrightarrow \Omega' and \Omega \longrightarrow \Omega' and \Gamma \vdash e \Leftarrow [\Gamma]A \dashv \Delta.
```

(ii) If
$$[\Omega]\Gamma \vdash e \Rightarrow A$$
 then there exist Δ , Ω' , and A' such that $\Delta \longrightarrow \Omega'$ and $\Omega \longrightarrow \Omega'$ and $\Gamma \vdash e \Rightarrow A' \dashv \Delta$ and $A = [\Omega']A'$.

(iii) If
$$[\Omega]\Gamma \vdash [\Omega]A \bullet e \Rightarrow C$$

then there exist Δ , Ω' , and C'
such that $\Delta \longrightarrow \Omega'$ and $\Omega \longrightarrow \Omega'$ and $\Gamma \vdash [\Gamma]A \bullet e \Rightarrow C' \dashv \Delta$ and $C = [\Omega']C'$.

Proof. By induction on the given declarative derivation.

 $\Gamma \vdash e \Leftarrow A \dashv \Delta$

$$\begin{array}{ll} \bullet \ \, \mathbf{Case} & \frac{(x:A) \in [\Omega] \Gamma}{[\Omega] \Gamma \vdash x \Rightarrow A} \, \, \mathrm{DeclVar} \\ & (x:A) \in [\Omega] \Gamma & \mathrm{Premise} \\ & \Gamma \longrightarrow \Omega & \mathrm{Given} \\ & (x:A') \in \Gamma \, \mathrm{where} \, [\Omega] A' = [\Omega] A & \mathrm{From \ definition \ of \ context \ application} \\ & \mathrm{Let} \, \, \Delta = \Gamma. \\ & \mathrm{Let} \, \, \Omega' = \Omega. \\ & \mathbb{S} \quad \Gamma \longrightarrow \Omega & \mathrm{Given} \\ & \mathbb{S} \quad \Gamma \longrightarrow \Omega & \mathrm{Given} \\ & \mathbb{S} \quad \Omega \longrightarrow \Omega & \mathrm{By \ Lemma \ 20 \ (Reflexivity)} \\ & \mathbb{S} \quad \Gamma \vdash x \Rightarrow A' \dashv \Gamma & \mathrm{By \ Var} \\ & [\Omega] A' = [\Omega] A & \mathrm{Above} \\ & \mathbb{S} \quad = A & \mathrm{FEV}(A) = \emptyset \\ \end{array}$$

• Case
$$\frac{[\Omega]\Gamma \vdash e \Rightarrow B \qquad [\Omega]\Gamma \vdash B \leq [\Omega]A}{[\Omega]\Gamma \vdash e \Leftrightarrow [\Omega]A} \qquad \text{DeclSub}$$

$$[\Omega]\Gamma \vdash e \Rightarrow B \qquad \text{Subderivation}$$

$$\Gamma \vdash e \Rightarrow B' \dashv \Theta \qquad \text{By i.h.}$$

$$B = [\Omega]B' \qquad "$$

$$\Theta \longrightarrow \Omega_0 \qquad "$$

$$\Omega \longrightarrow \Omega_0 \qquad "$$

$$\Gamma \longrightarrow \Omega \qquad \qquad \text{Given}$$

$$\Gamma \longrightarrow \Omega_0 \qquad \qquad \text{By Lemma 21 (Transitivity)}$$

$$[\Omega]\Gamma \vdash B \leq [\Omega]A \qquad \qquad \text{Subderivation}$$

$$[\Omega]\Gamma = [\Omega]\Theta \qquad \qquad \text{By Lemma 52 (Confluence of Completeness)}$$

$$[\Omega]\Theta \vdash B \leq [\Omega]A \qquad \qquad \text{By above equalities}$$

$$\Theta \longrightarrow \Omega_0 \qquad \qquad \text{Above}$$

$$\Theta \vdash [\Theta]B' <: [\Theta]A \dashv \Delta \qquad \text{By Theorem 14}$$

$$\Delta \longrightarrow \Omega' \qquad "$$

$$\Omega_0 \longrightarrow \Omega' \qquad "$$

$$\Omega_0 \longrightarrow \Omega' \qquad \qquad "$$

$$\text{By Lemma 21 (Transitivity)}$$

$$\text{By Lemma 21 (Transitivity)}$$

By Sub

Case

$$\frac{}{[\Omega]\Gamma\vdash\ () \Leftarrow 1} \mathsf{Decl1I}$$

We have $[\Omega]A = 1$. Either $[\Gamma]A = 1$ or $[\Gamma]A = \hat{\alpha} \in \mathsf{unsolved}(\Gamma)$.

In the former case:

In the latter case:

$$\begin{array}{lll} \Gamma \vdash () \Rightarrow 1 \dashv \Gamma & \text{By 1I} \Rightarrow \\ [\Omega]\Gamma \vdash 1 \leq 1 & \text{By } \leq \text{Unit} \\ \\ 1 = [\Omega]1 & \text{By definition of substitution} \\ = [\Omega][\Gamma] \hat{\alpha} & \text{By } [\Omega]A = 1 \\ = [\Omega] \hat{\alpha} & \text{By Lemma 18 (Substitution Extension Invariance)} \\ \\ [\Omega]\Gamma \vdash [\Omega]1 \leq [\Omega] \hat{\alpha} & \text{By above equalities} \\ \Gamma \vdash 1 <: \hat{\alpha} \dashv \Delta & \text{By Theorem 13 (1)} \\ 1 = [\Gamma]1 & \text{By definition of substitution} \\ \hat{\alpha} = [\Gamma] \hat{\alpha} & \hat{\alpha} \in \text{unsolved}(\Gamma) \\ \Gamma \vdash [\Gamma]1 <: [\Gamma] \hat{\alpha} \dashv \Delta & \text{By above equalities} \\ \\ \Omega \longrightarrow \Omega' & '' \\ \Delta \longrightarrow \Omega' & '' \\ \Gamma \vdash () \Leftarrow \hat{\alpha} \dashv \Delta & \text{By Sub} \\ \Gamma \vdash () \Leftarrow [\Gamma]A \dashv \Delta & \text{By Sub} \\ \Gamma \vdash () \Leftarrow [\Gamma]A \dashv \Delta & \text{By } [\Gamma]A = \hat{\alpha} \\ \end{array}$$

$$\bullet \ \, \textbf{Case} \ \, \frac{[\Omega]\Gamma, \alpha \vdash \, e \Leftarrow A_0}{[\Omega]\Gamma \vdash \, e \Leftarrow \forall \alpha.\, A_0} \, \, \mathsf{Decl} \forall \mathsf{I}$$

3

```
[\Omega]A = \forall \alpha. A_0
                       = \forall \alpha. [\Omega] A'
                                                            By def. of subst. and predicativity of \Omega
                A_0 = [\Omega]A'
                                                            Follows from above equality
          [\Omega]\Gamma, \alpha \vdash e \Leftarrow [\Omega]A'
                                                            Subderivation and above equality
                \Gamma \longrightarrow \Omega
                                                            Given
           \Gamma, \alpha \longrightarrow \Omega, \alpha
                                                            By \longrightarrow Uvar
         [\Omega]\Gamma, \alpha = [\Omega, \alpha](\Gamma, \alpha)
                                                            By definition of context substitution
 [\Omega, \alpha](\Gamma, \alpha) \vdash e \Leftarrow [\Omega]A'
                                                            By above equality
 [\Omega, \alpha](\Gamma, \alpha) \vdash e \Leftarrow [\Omega, \alpha]A' By definition of substitution
                     \Gamma, \alpha \vdash e \Leftarrow [\Gamma, \alpha]A' \dashv \Delta'
                                                                                    By i.h.
                                                                                    "
                  \Delta' \longrightarrow \Omega'_0
              \Omega, \alpha \longrightarrow \Omega'_0
                \Gamma, \alpha \longrightarrow \Delta'
                                                                                    By Lemma 54 (Typing Extension)
                      \Delta' = \Delta, \alpha, \Theta
                                                                                    By Lemma 24 (Extension Order) (i)
          \Delta, \alpha, \Theta \longrightarrow \Omega'_0
                                                                                    By above equality
                     \Omega_0'=\Omega',\alpha,\Omega_Z
                                                                                    By Lemma 24 (Extension Order) (i)
                    \Delta \longrightarrow \Omega'
13
                   \Omega \longrightarrow \Omega'
                                                                                    By Lemma 24 (Extension Order) on \Omega, \alpha \longrightarrow \Omega'_0
13
                     \Gamma, \alpha \vdash e \Leftarrow [\Gamma, \alpha] A' \dashv \Delta, \alpha, \Theta
                                                                                    By above equality
                     \Gamma, \alpha \vdash e \Leftarrow [\Gamma]A' \dashv \Delta, \alpha, \Theta
                                                                                    By definition of substitution
                         \Gamma \vdash e \Leftarrow \forall \alpha. [\Gamma] A' \dashv \Delta
                                                                                    By \forall I
                         \Gamma \vdash e \Leftarrow [\Gamma](\forall \alpha. A') \dashv \Delta
                                                                                    By definition of substitution
13°
                                     \frac{[\Omega]\Gamma \vdash [\tau/\alpha]A_0 \bullet e \Rightarrow C}{C} \text{ Decl}\forall App}
                         [\Omega]\Gamma \vdash \forall \alpha. A_0 \bullet e \Rightarrow C
                                          [Ω]A
                                         [\Omega]\Gamma \vdash \tau
                                                                                                                      Subderivation
                                        [\Omega]A = \forall \alpha. A_0
                                                                                                                      Given
                                                   = \forall \alpha. [\Omega] A'
                                                                                                                      By def. of subst. and predicativity of \Omega
                                          [\Omega]\Gamma \vdash [\tau/\alpha][\Omega]A' \bullet e \Longrightarrow C
                                                                                                                      Subderivation and above equality
                                           \Gamma \longrightarrow \Omega
                                                                                                                      Given
                                       \Gamma, \hat{\alpha} \longrightarrow \Omega, \hat{\alpha} = \tau
                                                                                                                      By \longrightarrow Solve
                                         [\Omega]\Gamma = [\Omega, \hat{\alpha} = \tau](\Gamma, \hat{\alpha})
                                                                                                                      By definition of context application
                    [\Omega, \hat{\alpha} = \tau](\Gamma, \hat{\alpha}) \vdash [\tau/\alpha][\Omega]A' \bullet e \Longrightarrow C
                                                                                                                      By above equality
                    [\Omega, \hat{\alpha} = \tau](\Gamma, \hat{\alpha}) \vdash [\tau/\alpha][\Omega, \hat{\alpha} = \tau]A' \bullet e \Rightarrow C
                                                                                                                      By def. of subst.
 (\lceil [\Omega]\tau/\alpha \rceil \lceil \Omega, \hat{\alpha} = \tau \rceil A') = (\lceil \Omega, \hat{\alpha} = \tau \rceil \lceil \hat{\alpha}/\alpha \rceil A')
                                                                                                                      By distributivity of substitution
                                               \tau = [\Omega]\tau
                                                                                                                      \mathsf{FEV}(\tau) = \emptyset
        (\lceil \tau/\alpha \rceil \lceil \Omega, \hat{\alpha} = \tau \rceil A') = (\lceil \Omega, \hat{\alpha} = \tau \rceil \lceil \hat{\alpha}/\alpha \rceil A')
                                                                                                                      By above equality
                    [\Omega, \hat{\alpha} = \tau](\Gamma, \hat{\alpha}) \vdash [\Omega, \hat{\alpha} = \tau][\hat{\alpha}/\alpha]A' \bullet e \Rightarrow C
                                                                                                                      By above equality
           \Gamma, \hat{\alpha} \vdash [\hat{\alpha}/\alpha]A' \bullet e \Longrightarrow C' \dashv \Delta By i.h.
              C = [\Omega]C'
137
          \Delta \longrightarrow \Omega'
         \Omega \longrightarrow \Omega'
               \Gamma \vdash \forall \alpha. A' \bullet e \Rightarrow C' \dashv \Delta
                                                                            Ву ∀Арр
3
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Given

• Case
$$\frac{[\Omega]\Gamma, x : A_1' \vdash e_0 \Leftarrow A_2'}{[\Omega]\Gamma \vdash \lambda x. e_0 \Leftarrow A_1' \to A_2'} \text{ Decl} \to I$$

We have $[\Omega]A = A_1' \to A_2'$. Either $[\Gamma]A = A_1 \to A_2$ where $A_1' = [\Omega]A_1$ and $A_2' = [\Omega]A_2$ —or $[\Gamma]A = \hat{\alpha}$ and $[\Omega]\hat{\alpha} = A_1' \to A_2'$.

In the former case:

$$\Omega_0' = \Omega', x : A_1', \Theta$$
 By Lemma 24 (Extension Order) (v)
 $\Omega \longrightarrow \Omega'$
 $\Gamma, x : [\Gamma]A_1 \longrightarrow \Delta'$ By Lemma 54 (Typing Extension)
$$\Delta' = \Delta, x : \cdots, \Theta$$
 By Lemma 24 (Extension Order) (v)

 $\Delta, x: \cdots, \Theta \longrightarrow \Omega', x: A'_1, \Theta$ By above equalities $\Delta \longrightarrow \Omega'$ By Lemma 24 (Extension Order) (v)

$$\Gamma, x : [\Gamma]A_1 \vdash e_0 \Leftarrow [\Gamma]A_2 \dashv \Delta, \alpha, \Theta$$
 By above equality
$$\Gamma \vdash \lambda x. \ e_0 \Leftarrow ([\Gamma]A_1) \rightarrow ([\Gamma]A_2) \dashv \Delta \quad \text{By } \rightarrow \text{I}$$

$$\Gamma \vdash \lambda x. \ e_0 \Leftarrow [\Gamma](A_1 \rightarrow A_2) \dashv \Delta \quad \text{By definition of substitution}$$

In the latter case:

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13

3

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\Gamma, \hat{\alpha}, \hat{\beta}, x : \hat{\alpha} \longrightarrow \Delta'
                                                                                                                  By Lemma 54 (Typing Extension)
                                                                  egin{aligned} \widehat{lpha} & \longrightarrow \Delta' \ \Delta' &= \Delta, x : \widehat{lpha}, \Theta \ artheta & \longrightarrow \Omega'_0 \end{aligned}
                                                                                                                 By Lemma 24 (Extension Order) (v)
                                                \Delta, x: \hat{\alpha}, \Theta \longrightarrow \Omega_0'
                                                                                                                 By above equality
                                                                 \Omega_0' = \Omega'', x : \cdots, \Omega_Z By Lemma 54 (Typing Extension)
                                                                \Delta \longrightarrow \Omega''
       3
                                                       \Gamma, \hat{\alpha}, \hat{\beta} \longrightarrow \Delta
\Omega_0 \longrightarrow \underbrace{\Omega'', x : \cdots, \Omega_Z}_{\Omega_0'}
By above equality
\Omega_-
Bv def. of \Omega_0
                \Omega, \hat{\alpha} = A_1', \hat{\beta} = A_2', x : A_1' \longrightarrow \Omega'', x : \cdots, \Omega_Z By def. of \Omega_0
                                                                \Omega'' = \Omega', \hat{\alpha} = ..., ... By Lemma 24 (Extension Order) (iii)
                                                               \Omega \longrightarrow \Omega'
       137
                 \Gamma, \hat{\alpha}, \hat{\beta}, x : \hat{\alpha} \vdash e_0 \Leftarrow \hat{\beta} \dashv \Delta, x : \hat{\alpha}, \Theta
                                                                                           By above equality
                                    \Gamma \vdash \lambda x. e_0 \Leftarrow \hat{\alpha} \rightarrow \hat{\beta} \dashv \Delta
                                                                                                            Bv \rightarrow l \Rightarrow
                               [\Gamma] \hat{\alpha} = \hat{\alpha}
                                                                                                            By definition of substitution
                               [\Gamma]\hat{\beta} = \hat{\beta}
                                                                                                            By definition of substitution
                                    \Gamma \vdash \lambda x. e_0 \Leftarrow ([\Gamma] \hat{\alpha}) \rightarrow ([\Gamma] \hat{\beta}) \dashv \Delta
                                                                                                            By above equalities
                                    \Gamma \vdash \lambda x. e_0 \Leftarrow [\Gamma](\hat{\alpha} \rightarrow \hat{\beta}) \dashv \Delta
       13F
                                                                                                            By definition of substitution
 \bullet \  \, \textbf{Case} \  \, \underbrace{ [\Omega] \Gamma \vdash \, e_1 \Rightarrow B \qquad [\Omega] \Gamma \vdash \, B \bullet \, e_2 \Longrightarrow A }_{ [\Omega] \Gamma \vdash \, e_1 \, e_2 \Longrightarrow A} \  \, \textbf{Decl} \rightarrow \textbf{E}  
                      [\Omega]\Gamma \vdash e_1 \Rightarrow B
                                                                                     Subderivation
                       \Gamma \longrightarrow \Omega
                                                                                     Given
                            \Gamma \vdash e_1 \Rightarrow B' \dashv \Theta
                                                                                     By i.h.
                           B = [\Omega]B'
                      \Theta \longrightarrow \Omega_0'
                      \Omega \longrightarrow \Omega'_0
                      [\Omega]\Gamma \vdash B \bullet e_2 \Rightarrow A
                                                                                     Subderivation
                      [\Omega]\Gamma \vdash [\Omega]B' \bullet e_2 \Rightarrow A
                                                                                     By above equality
                       \Gamma \longrightarrow \Omega'_0
                                                                                     By Lemma 21 (Transitivity)
                     [\Omega]\Gamma = [\Omega]\Omega
                                                                                     By Lemma 49 (Stability of Complete Contexts)
                               = [\Omega'_0]\Omega'_0
                                                                                     By Lemma 51 (Finishing Completions)
                               = [\Omega'_0]\Gamma
                                                                                     By Lemma 49 (Stability of Complete Contexts)
                               = [\Omega'_0]\Theta
                                                                                     By Lemma 52 (Confluence of Completeness)
                   By above equality
                   [\Omega]B' = [\Omega'_0]B'
                                                                                     By Lemma 50 (Finishing Types)
                 [\Omega_0']B' = [\Omega_0'][\Theta]B'
                                                                                     By Lemma 18 (Substitution Extension Invariance)
                   [\Omega'_0]\Theta \vdash [\Omega][\Theta]B' \bullet e_2 \Rightarrow A
                                                                                     By above equalities
                           \Theta \vdash [\Theta]B' \bullet e_2 \Longrightarrow A' \dashv \Delta
                                                                                  By i.h. with \Omega'_0
                           A = [\Omega]A'
       137
                       \Delta \longrightarrow \Omega'
                    \Omega_0' \longrightarrow \Omega'
                      \Omega \longrightarrow \Omega'
                                                                                     By Lemma 21 (Transitivity)
                            \Gamma \vdash e_1 e_2 \Rightarrow A' \dashv \Delta
                                                                                     By \rightarrow E
```

• Case
$$\frac{[\Omega]\Gamma \vdash e \Leftarrow B}{[\Omega]\Gamma \vdash \underbrace{B \to C}_{[\Omega]A} \bullet e \Rightarrow C} \text{ Decl} \to \mathsf{App}$$

We have $[\Omega]A = B \to C$. Either $[\Gamma]A = B_0 \to C_0$ where $B = [\Omega]B_0$ and $C = [\Omega]C_0$ —or $[\Gamma]A = \hat{\alpha}$ where $\hat{\alpha} \in \mathsf{unsolved}(\Gamma)$ and $[\Omega]\hat{\alpha} = B \to C$.

In the former case:

$$\begin{split} [\Omega]\Gamma \vdash e \Leftarrow B & \text{Subderivation} \\ B = [\Omega]B_0 & \text{Known in this subcase} \end{split}$$

$$\Gamma \longrightarrow \Omega & \text{Given} \\ \Gamma \vdash e \Leftarrow [\Gamma]B_0 \dashv \Delta & \text{By i.h.} \\ \Gamma \vdash ([\Gamma]B_0) \rightarrow ([\Gamma]C_0) \bullet e \Rightarrow [\Gamma]C_0 \dashv \Delta & \text{By} \rightarrow \text{App} \\ & \Delta \longrightarrow \Omega' & " \\ \text{Let } C' = [\Gamma]C_0. & " \\ \text{Let } C' = [\Gamma]C_0. & \text{Known in this subcase} \\ & = [\Omega][\Gamma]C_0 & \text{By Lemma 18 (Substitution Extension Invariance)} \\ & = [\Omega]C' & [\Gamma]C_0 = C' \\ & \Box \Gamma \vdash [\Gamma](B_0 \rightarrow C_0) \bullet e \Rightarrow [\Gamma]C_0 \dashv \Delta & \text{By definition of substitution} \end{split}$$

In the latter case, $\hat{\alpha} \in \mathsf{unsolved}(\Gamma)$, so the context Γ must have the form $\Gamma_0[\hat{\alpha}]$.

$$\begin{array}{c} \Gamma \longrightarrow \Omega \\ \Gamma_0 [\aa] \longrightarrow \Omega \\ \Gamma_0 [\aa] \longrightarrow \Omega \\ [\Omega] A = B \to C \\ [\Omega] \aa = B \to C \\ \Omega = \Omega_0 [\aa = A_0] \text{ and } [\Omega] A_0 = B \to C \\ \text{Let } \Gamma' = \Gamma_0 [\aa_2, \aa_1, \aa = \aa_1 \to \aa_2]. \\ \text{Let } \Omega'_0 = \Omega_0 [\aa_2 = [\Omega] C, \aa_1 = [\Omega] B, \aa = \aa_1 \to \aa_2]. \\ \Gamma' \longrightarrow \Omega'_0 \\ \text{By Lemma 29 (Parallel Admissibility) (ii) twice} \\ [\Omega] \Gamma \vdash e \Leftarrow B \\ \text{Subderivation} \\ \Omega \longrightarrow \Omega'_0 \\ \text{By Lemma 27 (Solved Variable Addition for Extension)} \\ \text{then Lemma 29 (Parallel Admissibility) (iii)} \\ [\Omega] \Gamma = [\Omega] \Omega \\ \text{By Lemma 49 (Stability of Complete Contexts)} \\ = [\Omega'_0] \Omega'_0 \\ \text{By Lemma 51 (Finishing Completions)} \\ = [\Omega'_0] \Gamma' \\ \text{By Lemma 52 (Confluence of Completeness)} \\ B = [\Omega'_0] \aa_1 \\ [\Omega'_0] \Gamma' \vdash e \Leftarrow [\Omega'_0] \aa_1 \\ \text{By above equalities} \\ \Gamma' \vdash e \Leftarrow [\Gamma'] \aa_1 + \Delta \\ \Omega \longrightarrow \Omega' \\ \square \Omega \longrightarrow \Omega' \\ \text{By Lemma 21 (Transitivity)} \\ [\Gamma'] \aa_1 = \aa_1 \\ \Omega'_1 \in \text{unsolved}(\Gamma') \\ \Gamma' \vdash e \Leftarrow \aa_1 + \Delta \\ \text{By above equality} \\ \end{array}$$

Case

 $\Gamma \vdash \lambda x. e_0 \Rightarrow \hat{\alpha} \rightarrow \hat{\beta} \dashv \Delta$

Case
$$\frac{[\Omega]\Gamma \vdash \sigma \to \tau \qquad [\Omega]\Gamma, x : \sigma \vdash e_0 \Leftarrow \tau}{[\Omega]\Gamma \vdash \lambda x. e_0 \Rightarrow \sigma \to \tau} \text{ Decl} \to l \Rightarrow$$

By $\rightarrow l \Rightarrow$

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Let A' = (\hat{\alpha} \to \hat{\beta}).
               \Gamma \vdash \lambda x. e_0 \Rightarrow A' \dashv \Delta
                                                        By above equality
        \sigma \to \tau = ([\Omega_0] \hat{\alpha}) \to ([\Omega_0] \hat{\beta})
                                                        By definition of \Omega_0
        \sigma \to \tau = [\Omega_0](\hat{\alpha} \to \hat{\beta})
                                                        By definition of substitution
              A = [\Omega_0]A'
                                                        By above equalities
               A = [\Omega']A'
                                                        By Lemma 50 (Finishing Types)
₽
           \Gamma' \longrightarrow \Delta'
                                                        By Lemma 54 (Typing Extension)
           \Omega \longrightarrow \Omega'
                                                        By Lemma 21 (Transitivity)
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References

Frank Pfenning. Structural cut elimination. In LICS, 1995.