MEASURE, BAIRE CATEGORY, PERFECT SETS, AND GAMES OF INFINITE LENGTH

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May 15, 2009

Abstract. We offer a survey of some known connections between games of infinite length and Lebesgue measurability, the Baire Property, and the perfect set property. In particular, we show that the Axiom of Determinacy implies all subsets of the real numbers possess these three properties, whereas the Axiom of Choice provides a single, simultaneous counterexample to Determinacy and those properties.

Contents

1. Introduction

The purpose of this exposition is to introduce an important yet accessible and reasonably self-contained web of ideas from descriptive set theory to non-specialists: the surprising interaction between Lebesgue measure, point-set topology, and game theory. To that end, basic knowledge of measure and topology are assumed, but nothing beyond what is covered in a typical undergraduate analysis course. We focus our investigation on three regularity properties sets may have—Lebesgue measurability, the Property of Baire (Section 3), and the perfect set property (Section 4)—and the relation of these properties to certain games and to the Axiom of Choice.

In particular, we discuss the Axiom of Determinacy, which roughly says that certain infinitely long two-player games of perfect information are "games of skill" in that, like their finite counterparts, one player always has an optimal strategy which will guarantee victory. The main results presented in this paper are those in the final Section 7, that the Axiom of Determinacy implies all subsets of $\mathbb{R}^n$ have the three properties listed above. This contradicts the Axiom of Choice, and as such, we will have to limit ourselves to working with the weaker Axiom of Dependent Choice, whose use is introduced and demonstrated in Section 2.

It is our hope that the exposition here highlights the connections within our small selection of topics better than the very broad introductions to descriptive set theory such as [J02], [K95], [M02], or [Mo80], and the very narrow original research articles on the topic such as [My64] and [MS64]. For example, while a "perfect set theorem" in the vein of Proposition 6.1 is not needed to construct a counterexample to Determinacy (assuming Choice), we find the approach more satisfying than a completely direct one, and moreover it allows us to give a single sweeping counterexample to Determinacy *and* the three regularity properties AD is later proved to imply (Proposition 4.6). We acknowledge that tightly interwoven Propositions and Lemmas make a paper more difficult to read quickly or nonlinearly, which we think is an acceptable sacrifice for additional cohesion.

It should be said that the only novelty of this paper is in the exposition. With the possible exception of a few simple examples, we do not claim any of the results as our own.

This paper was written in partial fulfilment of the requirements of the degree Bachelor of Science in Mathematics with Honors at Brown University. We wish to thank our advisor Jill Pipher, who initially funded and oversaw our readings in set theory during the summer of 2008, and who provided helpful comments during the research and writing of this paper.

2. Polish Spaces and Dependent Choice

The **Axiom of Choice** (AC) is the statement that for any collection of nonempty sets $\{A_\alpha : \alpha \in I\}$, there exists a **choice function** $f : I \to \bigcup_{\alpha \in I} A_\alpha$ which "picks" one element out of each, $f(\alpha) \in A_\alpha$ for all $\alpha \in I$. Equivalently, the product $\prod_{\alpha \in I} A_\alpha$ of these nonempty sets A_α is nonempty. We take the position that the AC is perfectly acceptable for the development of "classical" mathematics, yet

(i) Most of the theory of the real numbers (and similar spaces) can be developed with a weaker version of Choice, called the Axiom of Dependent Choice (Axiom 2.1), and
(ii) Some interesting things in set theory can only be done in the presence of this weaker version of Choice.

The purpose of this section will be to elaborate on these two points. We treat (ii) first, as we have less to say about it up front.

We will be exploring basic consequences of the Axiom of Determinacy (AD), which contradicts AC, for example by implying that all subsets of the real numbers are Lebesgue measurable (Theorem 7.7). These proofs are quite interesting, but one may be left with the feeling that such results are mere "fiction" on account of being incompatible with Choice. There are two ways out, which amount to weakening the Axiom of Determinacy or weakening the Axiom of Choice. For the first way, stronger *local* versions of these consequences of Determinacy are true—for instance if AD is interpreted to hold just for a limited collection of definable sets (such that it no longer contradicts Choice), then one can often conclude that this collection of sets is Lebesgue measurable. This is done, for example, in Chapter 6 of [Mo80]. The second way out is to cite the result that there exists an important model of set theory *minus the Axiom of Choice* called $L(\mathbb{R})$ in which both AD and the weaker Axiom of Dependent Choice are thought to be the "right" axioms so that the results stated here are actually true there. This sentiment is expressed in Chapter 33 of [J02]. Rather than get lost in such details, we present consequences of AD *as they are*, leaving it to the reader to decide how they ought to be interpreted. Of course, we make sure to be clear about which of our theorems depend Determinacy, and which depend on Choice.

Philosophical and/or metamathematical obligations now met (we hope!), we devote the rest of this section to point (i). The Axiom of Choice is invaluable in dealing with large infinite sets, especially those with little known structure. For example, it is equivalent to the "trichotomy of cardinals," the statement that any two sets have comparable cardinalities in that there is an injection of one into the other.[1] Without Choice, we still may talk about two sets having the *same* cardinality (when there is a bijection between them), and we may talk about a set A having cardinality *at most* that of B (when there is an injection of A into B). The Schröder-Bernstein Theorem, which can be proved without using AC at all, says that a pair of injections $A \to B$ and $B \to A$ can be used to construct a bijection between A and B and hence that the definitions are meaningful. However, without the full Axiom of Choice, there may exist sets of incomparable size, so we will have to be somewhat careful with cardinality in what follows.

By contrast, the topology of metric spaces is completely determined by which sequences converge, so that the most important use of the Axiom of Choice in this setting is just to choose sequences, for example in the proof that a function f between metric spaces X and Y is continuous if and only if it respects all *sequential* limits. This prompts us to state the Axiom of Dependent Choice, which formalizes the use of AC to choose sequences.

Axiom 2.1 (Dependent Choice)**.** *Let A be a nonempty set and let R be a* (*binary*) *relation on A such that for any $x \in A$, there exists $y \in A$ such that $x\ R\ y$. Then there exists a sequence $(a_n)_{n=0}^{\infty}$ of points in A such that $a_n\ R\ a_{n+1}$ for all n.*

Intuitively, Dependence Choice (DC) says that if a finite sequence can always be extended by choosing one more element, then one can actually construct an infinite sequence. For comparison, DC implies another weak form of AC, the "Axiom of Countable Choice:" that for any countable collection of nonempty sets $(A_n)_{n=0}^{\infty}$, there is a choice function $f : \omega \to \bigcup_{n=0}^{\infty} A_n$ (where $\omega = \{0, 1, 2, ...\}$ is the set of nonnegative integers) such that $f(n) \in A_n$ for all n (equivalently, $\prod_{n=0}^{\infty} A_n$ is nonempty). Appealing to the intuition of choosing sequences $(a_n)_{n=0}^{\infty}$ gives the thrust of the obvious proof: simply choose $a_n \in A_n$ for each n. Formally, we make the disjoint union $A = \bigcup_{n=0}^{\infty} \big(A_n \times \{n\}\big)$, and the relation to consider on A is $(x, m)\ R\ (y, n)$ whenever $x \in A_n$, $y \in A_m$, and $n = m + 1$. However, it is known that DC is strictly stronger than Countable Choice ([J73], Theorem 8.12).

[1] Assuming Choice, Zorn's Lemma guarantees the existence of "maximal" injections between subsets of a given set into another set. A proof that the trichotomy of cardinals implies Choice (a result known as Hartog's Theorem) can be found in [SF96].

One of the most illustrative, famous, and useful examples of choosing sequences via Dependent Choice is the Baire Category Theorem for complete metric spaces. Dependent Choice is in fact the weakest version of Choice under which one can prove the Baire Category Theorem in that they are equivalent in the presence of the other axioms of set theory by [B77].

Theorem 2.2 (Baire Category Theorem)**.** *If V_n is a sequence of open, dense subsets of a complete metric space, then $\bigcap_{n=0}^{\infty} V_n$ is dense (and in particular is nonempty).*

Proof. A set is dense if and only if it meets every nonempty open set; thus for a fixed nonempty open U, we show that $\bigcap_{n=0}^{\infty} V_n$ meets U. We inductively choose a sequence of open sets B_n. Let $B_0 = U$. If B_n has been chosen for some $n \geq 0$, then because V_n is open and dense, $B_n \cap V_n$ is open and nonempty. Then we can choose an open ball B_{n+1} whose closure is contained in $B_n \cap V_n$ and whose diameter is at most $1/(n+1)$. By DC, we can construct a whole sequence of such B_n.

We use these B_n to show $U \cap \bigcap_{n=0}^{\infty} V_n$ is not empty. Choose some point x_n out of each B_n, and then the sequence $(x_n)_{n=0}^{\infty}$ is Cauchy (by the condition of shrinking diameters), hence converges to some point $x \in \bigcap_{n=1}^{\infty} \overline{B_n}$. But $\overline{B_n} \subseteq B_{n-1} \cap V_{n-1} \subseteq U \cap V_{n-1}$ for all $n \geq 1$, so in fact $x \in U \cap \bigcap_{n=0}^{\infty} V_n$. □

The conclusions of the Baire Category Theorem depend only on the topology of space rather than on the metric, so in fact the result holds in any **completely metrizable** space, that is, any space whose topology coincides with the one induced by some complete metric on its underlying set. For example, the open interval $(0, 1)$ is not complete in its usual metric, but any homeomorphism f from $(0, 1)$ to the complete metric space $\mathbb{R}$ transports the problem there: $d(x, y) = |f(x) - f(y)|$ is a complete metric on $(0, 1)$ inducing the right topology.

Though Dependent Choice is already a strong tool for metric (or metrizable) spaces, assuming even more structure will make spaces even more susceptible to our methods of choosing sequences.

Definition 2.3. *A topological space is a* **Polish space**[2] *if it is completely metrizable and is* **separable** *(has a countable dense subset).*

We note that the separability condition should imply Polish spaces have cardinality at most 2^ω (the cardinality of the continuum), because every point is the limit of some sequence of points in the countable dense subset. However, we have come to another awkward point in talking about cardinality without the full Axiom of Choice. Namely, a surjection $f : A \to B$ does not necessarily give an injection $B \to A$, as the construction of such an injection would require choosing one point out of $f^{-1}(b)$ for each $b \in B$. Still, it is true that any Polish space injects into a set of cardinality 2^ω (which is what it really means to have cardinality at most 2^ω), but we need a little more machinery to prove it (see Example 2.8).

Polish spaces will be the domain of discourse for the rest of the paper. While some of our definitions and theorems may be valid for more general kinds of spaces, we will not

[2]Apparently named after the Polish mathematicians Sierpiński, Tarski, Kuratowski, and others, who first popularized their study [W09].

emphasize such generality. Familiar examples of Polish spaces arising in analysis are $\mathbb{R}^n$, separable Banach spaces such as $L^p(\mathbb{R}^n)$, and discrete countable spaces (a metric for discrete spaces is $d(x,y) = 1$ whenever $x \neq y$, which is trivially complete). We note that separable metric spaces are always **second countable**, meaning there is a countable collection of open sets, called a **base**, which generates the topology—here we take balls of rational radii around the points of any countable dense subset. As such, many applications of Dependent Choice on Polish spaces can be reduced to applications of Countable Choice or Choice may even be avoidable altogether,[3] but we will not make a point of avoiding DC.

Lemma 2.4. *Every Polish space X has a complete metric inducing its topology which takes only values less than* 1.

Proof. Let d be a metric on X. We claim the function $\widetilde{d}(x,y) = d(x,y)/\big(1 + d(x,y)\big)$ does the job. It is a metric; for instance the triangle inequality holds because the function $G(t) = t/(1+t)$ is increasing but concave on $[0,\infty)$:

$$\begin{aligned}\frac{d(x,z)}{1+d(x,z)} &= G\big(d(x,z)\big) \leq G\big(d(x,y) + d(y,z)\big) \leq G\big(d(x,y)\big) + G\big(d(y,z)\big)\\ &= \frac{d(x,y)}{1+d(x,y)} + \frac{d(y,z)}{1+d(y,z)}.\end{aligned}$$

Furthermore $\widetilde{d}$ induces the same topology as d and preserves completeness because G is a homeomorphism of $[0,\infty)$ onto $[0,1)$, so that points are "close" under d if and only if they are "close" under $\widetilde{d}$. □

There are several important ways to construct new Polish spaces from old ones. First, there are countable products (which are nonempty under DC or Countable Choice):

Lemma 2.5. *If X_n are Polish spaces, then $\prod_{n=0}^{\infty} X_n$ is Polish.*

We recall the definition of the product topology: it is the topology on the set $\prod_{n=0}^{\infty} X_n$ generated by subsets of the form $\prod_{n=0}^{\infty} A_n$, where finitely many A_n are arbitrary open subsets of X_n and all the rest are equal to X_n. Identifying points of the product space with functions $x : \omega \to \prod_{n=0}^{\infty} X_n$ such that $x(n) \in X_n$ for every n, the product topology is also characterized as the topology of pointwise convergence: a sequence of elements $\big(x_k(n)\big)_{k=0}^{\infty}$ of the product converges to a point y if and only if $x_k(n) \to y(n)$ as $k \to \infty$ for all n. In any case, the proof of Lemma 2.5 is not hard.

Proof. For each n, use Lemma 2.4 (and DC or Countable Choice) to choose a metric $d_n < 1$ on X_n inducing its topology. For variables $x, y \in \prod_{n=0}^{\infty} X_n$, the function $d(x,y) = \sum_{n=0}^{\infty} 2^{-n} d_n\big(x(n), y(n)\big)$ is a metric. Moreover d is complete, because a sequence x_k is Cauchy if and only if $x_k(n)$ is Cauchy for all n; by the same reasoning, d induces

[3] Perhaps a surprising fact is that the Baire Category Theorem can be proved for Polish spaces with no use of Choice at all [Lé79]. The result is less surprising once you know the trick: construct an enumeration of the basic open sets specified above; then we have a rule for choosing basic open sets with desired properties (take the first appearing in this enumeration), as well as a rule for choosing a point out of each (keep track of the center of each of these balls).

the topology of pointwise convergence. By the first characterization of the product topology, it is second countable because all of the factors are. Choose one element from each of these countably many basic open sets to get a countable dense subset. □

Some of the most important examples of Polish spaces are constructed as products.

Example 2.6. Cantor space, denoted $\mathscr{C}$, is the product of countably many copies of the two-point discrete space $\{0, 1\}$. That is, it consists of all infinite sequences of 0's and 1's, and two such sequences are "close" in $\mathscr{C}$ if and only if they share "long" initial sequences. This justifies the name, as $\mathscr{C}$ is homeomorphic to Cantor's famous "middle thirds" set. The (countable) set of finite sequences of 0's and 1's is denoted $2^{<\omega}$; this includes the empty sequence $\emptyset$. For $s \in 2^{<\omega}$, we let $|s|$ be its length, and for x in either $\mathscr{C}$ or in $2^{<\omega}$, we write $s \subset x$ to mean that s is an initial segment of x. With this notation in place, the sets $N_s = \{x \in \mathscr{C} : s \subset x\}$ for $s \in 2^{<\omega}$ are open and generate the topology on $\mathscr{C}$.

We note that $\mathscr{C}$ is similar to the real interval $[0, 1]$ by considering elements of $\mathscr{C}$ as binary expansions for real numbers. Corresponding to the basic open sets N_s are the closed dyadic intervals, as in Figure 1 below. Since all non-dyadic numbers have unique binary expansions and all dyadic numbers have only two binary expansions, the indicated surjective map $\mathscr{C} \to [0, 1]$ restricts to a homeomorphism after removing countable sets from domain and range. Thus Lebesgue measure, which is unaffected by countable sets, can be easily transported from $[0, 1]$ to $\mathscr{C}$ where the standard name for it is the **coin-flipping measure** (because N_s has measure $2^{-|s|}$).

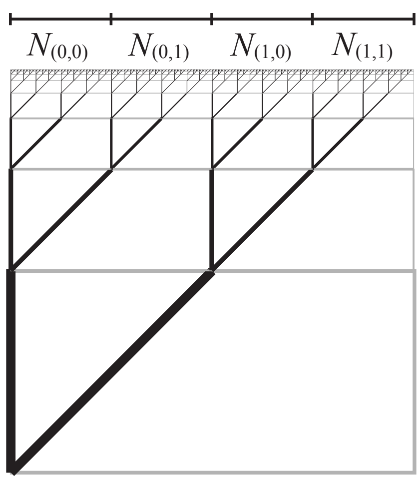

FIGURE 1. Identifying points of $\mathscr{C}$ with the branches of an infinite complete binary tree, the relationship between basic open neighborhoods N_s in $\mathscr{C}$ and closed dyadic intervals in $[0, 1]$ becomes clear. Labeled are the N_s for $|s| = 2$, positioned under the appropriate dyadic intervals.

Example 2.7. The second example we wish to introduce is called **Baire space**, which is constructed as the countable product of the discrete space ω with itself. That is, it is the space of all sequences of nonnegative integers. Common notations are $\mathscr{N}$, $\mathbb{N}^{\mathbb{N}}$, or ω^ω, of which we prefer the last. Letting $\omega^{<\omega}$ denote the finite sequences of integers, we use notation analogous to that for Cantor space in Example 2.6 above: N_s will be the basic open sets $\{x \in \omega^\omega : s \subset x\}$ for $s \in \omega^{<\omega}$, and so on. Baire space is also quite similar to the real numbers, in that it can be shown to be homeomorphic to the space of irrationals in $(0,1)$ by the continued fraction expansion

$$x \mapsto \frac{1}{1 + x(0) + \frac{1}{1+x(1)+\ldots}}.$$

The proof makes for a good exercise, but we do not wish to give it here. Transportation of Lebesgue measure from $[0,1]$ to ω^ω is thus also possible, though its interpretation is not as simple as flipping coins.

Example 2.8. The **Hilbert cube** $[0,1]^\omega$ is the product of countably many copies of $[0,1]$. We claim any Polish space X is homeomorphic to a subset of $[0,1]^\omega$, and in particular has cardinality at most 2^ω. To see this, simply let d be a metric on X taking only values less than 1, and let $(a_n)_{n=0}^\infty$ be a countable dense subset. Then the map $f : X \to [0,1]^\omega$ defined by

$$x \mapsto \big(d(a_0, x),\, d(a_1, x),\, d(a_2, x),\, \ldots\,\big)$$

is continuous and injective. Also, if points $f(x)$ and $f(y)$ in the image are close in $[0,1]^\omega$, that means for some large n that $|d(a_k, x) - d(a_k, y)|$ is small for all $k = 0, ..., n$, which means x and y are close. Thus f has a continuous inverse.

We conclude the section with the following lemma, which gives another construction; part (b) will be very useful in the following sections. The proof is only moderately enlightening, but we give the details, for the sake of completeness.

Lemma 2.9. *Let X be a Polish space with topology $\mathscr{T}$.*

(a) *All open and closed subspaces of X are Polish.*
(b) *For any Borel subset $B \subseteq X$, there is a stronger Polish topology on X, $\mathscr{T}' \supseteq \mathscr{T}$, such that B is clopen in $\mathscr{T}'$.*

Proof. (a) Since second countability passes from X down to its subspaces, so does separability. Closed subspaces of completely metrizable spaces are complete under the same metric. However, open subspaces must be remetrized.

Let $d < 1$ be a complete metric on X inducing its topology and define the following metric $\widehat{d}$ on an open subset U, which blows up towards the boundary (basically so that no $\widehat{d}$-Cauchy sequence can escape U):

$$\widehat{d}(x,y) = d(x,y) + \left| \frac{1}{d(x, U^c)} - \frac{1}{d(y, U^c)} \right|,$$

where distance to the closed set U^c, $d(x,U^c) = \inf\{d(x,a) : a \in U^c\}$, is positive and at most 1. For example the triangle inequality holds because

$$\begin{aligned}
\widehat{d}(x,z) &= d(x,z) + \left|\tfrac{1}{d(x,U^c)} - \tfrac{1}{d(z,U^c)}\right| \\
&\leq d(x,y) + d(y,z) + \left|\tfrac{1}{d(x,U^c)} - \tfrac{1}{d(y,U^c)}\right| + \left|\tfrac{1}{d(y,U^c)} - \tfrac{1}{d(z,U^c)}\right| \\
&= \widehat{d}(x,y) + \widehat{d}(y,z).
\end{aligned}$$

We'll now see that d and $\widehat{d}$ induce the same topology on U. Since $d \leq \widehat{d}$, it is clear that any open $\widehat{d}$-ball contains an open d-ball of the same radius, so that the topology induced by $\widehat{d}$ is no stronger than the one induced by d. Conversely because d-distance to U^c is continuous (with respect to d), the term $\left|\frac{1}{d(x,U^c)} - \frac{1}{d(y,U^c)}\right|$ is small when $d(x,y)$ is small. Thus any d-ball contains a $\widehat{d}$-ball, and the induced topologies are the same.

We indicated in an above parenthetical why $\widehat{d}$ is complete, and we now provide a few more details. Let $(x_n)_{n=0}^{\infty}$ be a $\widehat{d}$-Cauchy sequence in U; then it is also a d-Cauchy sequence, so converges under d to a point $x \in \overline{U}$. If we knew $x \in U$, then it would follow that x_n converges under $\widehat{d}$ to x. Since x_n is $\widehat{d}$-Cauchy, it follows that the sequence $\big(1/d(x_n,U^c)\big)_{n=0}^{\infty}$ is Cauchy in $[1,\infty)$, hence converges. By continuity with respect to d, this implies $d(x,U^c) > 0$, so that $x \in U$.

(b) There is a standard way to treat Borel sets: let $\mathscr{A}$ be the collection of all subsets of X for which there exists some Polish topology $\mathscr{T}' \supseteq \mathscr{T}$ under which A is clopen. We will show that $\mathscr{A}$ is a σ-algebra containing all the open sets, and hence that it contains all Borel sets as well. Clearly $\mathscr{A}$ is closed under complements, because A is clopen (under some topology) if and only A^c is clopen.

Part (a) almost shows that $\mathscr{A}$ contains the open (and closed) sets. An open set U of X and its complement are both Polish; what we must show is that the disjoint union of U and U^c also gives a Polish topology on (all of) X. For this, we must only define a new metric. Let d_U and d_{U^c} be complete metrics on the two subspaces agreeing with their topologies and only taking values less than 1. Then we "break apart" U and U^c with the following metric:

$$\widehat{d}(x,y) = \begin{cases} d_U(x,y) & \text{if } x,y \in U \\ d_{U^c}(x,y) & \text{if } x,y \in U^c \\ 1 & \text{otherwise.} \end{cases}$$

It now suffices to show that $\mathscr{A}$ is closed under countable intersections. If $(A_n)_{n=0}^{\infty}$ is a sequence of sets in $\mathscr{A}$, each paired with a complete metric $d_n < 1$ on X inducing a topology $\mathscr{T}_n$ under which A_n is clopen, we define a metric on X very similar to the "product metric" constructed in Lemma 2.5: $d(x,y) = \sum_{n=0}^{\infty} 2^{-n} d_n(x,y)$. A sequence converges [respectively, is Cauchy] under this metric if and only if it converges [is Cauchy] under every d_n; hence d is complete and the topology it induces is at least as strong as each $\mathscr{T}_n$. Most importantly, the set $A = \bigcap_{n=0}^{\infty} A_n$ is closed under d, because each A_n is. There is no reason that A ought to be open as well, but by the treatment of open and closed sets above, the topology can be refined to a stronger Polish topology in which A is clopen. $\square$

3. Measure and Category

The title of this section is taken from Oxtoby's book [O71] which develops in detail the ideas and analogies presented here.

With respect to Lebesgue measure on $\mathbb{R}^n$, we recall that the null sets are "small" in the following sense:

(i) any subset of a null set is null,
(ii) any countable union of null sets is null, and
(iii) $\mathbb{R}^n$ is not itself null.

Any collection $\mathscr{I}$ of subsets of a set S satisfying these three conditions (with "set in $\mathscr{I}$" replacing "null" and S replacing $\mathbb{R}^n$) is called a **σ-ideal on** S. A rather trivial example of a σ-ideal on any uncountable set S is the set of all its countable subsets.

We introduce another σ-ideal on $\mathbb{R}^n$, or more generally, on any Polish space. Say a set is **meager** or **of first category** if it is a subset of a countable union of closed sets with empty interior; equivalently, if it is a countable union of nowhere dense sets. Clearly then, the collection of meager sets satisfies conditions (i) and (ii) for being a σ-ideal. That a Polish space is a nonmeager subset of itself (condition (iii)) follows from the Baire Category Theorem, because a set is closed with empty interior if and only if its complement is open and dense. In fact, any nonempty open subset of a Polish space is nonmeager.

Since the meager sets are "negligible" in the sense of being a σ-ideal, the Baire Category Theorem is a useful tool for proving existence results. For example, one can prove [O71] there exists a continuous, nowhere differentiable function on $[0, 1]$ by showing that the nowhere differentiable functions have meager complement in the Banach space $C([0, 1])$ of continuous functions $[0, 1] \to \mathbb{R}$ under the "sup norm"

$$d(f, g) = \sup\{|f(x) - g(x)| : x \in [0, 1]\}.$$

Completeness of this space is just a restatement of the fact that a uniform limit of continuous functions is continuous. However, we note that the use of the Baire Category Theorem is not essential here, as a continuous, nowhere differentiable function can be constructed explicitly; see [GO03] Example 3.8. More serious applications of the Baire Category Theorem (to functional analysis) can be found in Chapter 5 of [F99].

Of course, measure and category do not indicate the same kind of smallness, in that $\mathbb{R}^n$ can be partitioned as the union of a null set and a meager set. For example, by covering the rational points with quickly shrinking open balls, one can construct open, dense sets of arbitrarily small measure. For any sequence of open, dense sets $V_k \subseteq \mathbb{R}^n$ with measures shrinking to 0 as $k \to \infty$, the intersection $\bigcap_{k=0}^{\infty} V_k$ is null but its complement is meager.

We define another property, which will be in many ways analogous to Lebesgue measurability.

Definition 3.1. *Say a subset of a Polish space X has the* **Baire Property** *if its symmetric difference ($A \triangle B = (A \cap B^c) \cup (A^c \cap B)$) with some open set is meager. Let $BP(X)$ or simply BP denote the collection of all subsets of X having the Baire Property.*

Proposition 3.2. *For any Polish space X, $BP(X)$ is a σ-algebra.*

Proof. First we show BP is closed under complements. If U is open, then its boundary $\overline{U}\backslash U$, is closed and has empty interior (in particular is meager), because any nonempty open subset of $\overline{U}\backslash U$ would correspond to closed set containing U but smaller than $\overline{U}$. Thus if $A \in BP$ with $A \triangle U$ meager, then $A \triangle \overline{U} = A^c \triangle \overline{U}^c$ is also meager, and of course $\overline{U}^c$ is open, so $A^c \in BP$.

For countable unions: given a sequence of sets $A_n \in BP$ and open U_n such that each $A_n \triangle U_n$ is meager, we have that

$$\left(\bigcup_{n=0}^{\infty} A_n\right) \triangle \left(\bigcup_{n=0}^{\infty} U_n\right) \subseteq \bigcup_{n=0}^{\infty} (A_n \triangle U_n)$$

is meager. (To see the containment: if x is in the left hand side, then x is in some A_n or x is in some U_n but not both; for such a choice of n, x is in $A_n \triangle U_n$.) $\square$

Thus BP contains all the Borel sets (since it clearly contains all open sets: $U \triangle U = \emptyset$). Much as the σ-algebra of measurable sets contains all Borel sets as well as all null sets, BP contains all the meager sets. Moreover, the Borel sets provide good approximations to measurable or BP sets in that any such set has null or meager symmetric difference (respectively) with some Borel set (for BP, this is just because open sets are Borel).

We can give a more precise description of this approximation by Borel sets. Define a G_δ **set** to be a countable intersection of open sets and an F_σ **set** to be a countable union of closed sets. For example, these are Borel sets, the complement of a G_δ set is an F_σ set and conversely, and the Baire Category Theorem implies that a dense G_δ set is nonmeager. Moreover, closed sets are G_δ (and hence open sets are F_σ), because distance to a closed set C is a continuous function $x \mapsto d(x, C) = \inf\{d(x, y) : y \in C\}$ which vanishes precisely on C, hence

$$C = \bigcap_{n=1}^{\infty} \{x \in X : d(x, C) < 1/n\}$$

is a countable intersection of open sets. We now recall the following characterization of measurability.

Proposition 3.3. *A subset of $A \subseteq \mathbb{R}^n$ is measurable if and only if there is a G_δ set G and an F_σ set F such that $G \supseteq A \supseteq F$ and $G\backslash F$ is null.*

Proof. To construct G and F (the nontrivial direction), one considers the outer measure (inf over open sets outside A) and inner measure (sup over closed sets inside), which are equal for measurable sets. For A of finite measure, one can choose appropriate sequences of open or closed sets whose measure converges to that of A and then take the intersection or union respectively. For A of infinite measure, just restrict to disjoint cubes of finite measure first. $\square$

We provide an analogous result for BP, but note the inclusions are reversed.

Proposition 3.4. *A set A is in BP if and only if there exist a G_δ set G and an F_σ set F such that $G \subseteq A \subseteq F$ and $F\backslash G$ is meager.*

Proof. The direction requiring proof is the construction of F and G, given $A \in BP$ with open U such that $A \triangle U$ is meager. By restriction, both $A \cap U^c$ and $A^c \cap U$ are meager, so we may take sequences C_n and D_n of closed sets with empty interior such that

$$A \cap U^c \subseteq \bigcup_{n=0}^{\infty} C_n \qquad \text{and} \qquad A^c \cap U \subseteq \bigcup_{n=0}^{\infty} D_n.$$

Set $F = U \cup \bigcup_{n=0}^{\infty} C_n$ (remember that open sets are F_σ), so that $A \subseteq F$ and $F \backslash A \subseteq (A^c \cap U) \cup \bigcup_{n=0}^{\infty} C_n$ is meager. For G, we have $A^c \cap U \subseteq U \cap \bigcup_{n=0}^{\infty} D_n$, so taking local complements (in U) gives $A \cap U \supseteq U \cap \bigcap_{n=0}^{\infty} D_n^c$. Let G be the right side of this containment, so that G is a G_δ subset of A. Moreover, $A \backslash G \subseteq (A \cap U^c) \cup \bigcup_{n=0}^{\infty} D_n$ is meager. □

We present a lemma whose content is very similar to Propositions 3.3 and 3.4, but which is more useful when we don't know whether the set in question is measurable or has the Property of Baire.

Lemma 3.5. *Let A be any subset of a Polish space X.*

(a) *There exists a set $S \supseteq A$ with the Baire Property such any subset of $S \backslash A$ with the Baire Property is meager.*
(b) *If $X = \mathbb{R}^n$, then there is a measurable set $T \supseteq A$ such that any measurable subset of $T \backslash A$ is null.*

Proof. The proof of (b) is exactly as in Proposition 3.3. The proof for (a) is a matter of deciding where A locally looks meager or nonmeager and just using definitions appropriately.

Let $(U_n)_{n=0}^{\infty}$ be a countable base for the topology, and let G be the union of all the U_n whose intersection with A is meager. Since the union is countable, $G \cap A$ is also meager. The closed set G^c is the set of all points $x \in X$ such that all neighborhoods of x have nonmeager intersection with A. Put $S = G^c \cup (G \cap A)$. As the union of a closed set and a meager one, S has the Baire Property; also S contains $A = (G^c \cap A) \cup (G \cap A)$.

It remains to show that any set $B \subseteq S \backslash A$ with the Baire Property must be meager. The part of B in $G \cap A$ is of course meager, so it suffices to show that $B \cap G^c$ is meager. Suppose on the contrary that there is a set $B \subseteq S \backslash A$ and an open set U with $B \triangle U$ meager, but $B \cap G^c$ nonmeager. Then $U \cap G^c$ is nonmeager, and in particular nonempty. Choose a point $y \in U \cap G^c$; then all neighborhoods of y have nonmeager intersection with A, so $U \cap A$ is nonmeager. This implies $B \cap A$ is nonmeager, which is impossible because B and A were assumed disjoint. □

With just a little more work, Lemma 3.5 can be combined with Propositions 3.3 and 3.4. We will not need the full strength of this result, but it is worth mentioning.

Corollary 3.6. *Let A be any subset of a Polish space X.*

(a) *There is a G_δ set G and an F_σ set F with $G \subseteq A \subseteq F$ such that any subset of $F \backslash A$ or of $A \backslash G$ with the Baire property is meager.*
(b) *If $X = \mathbb{R}^n$, there is a G_δ set $\widehat{G}$ and an F_σ set $\widehat{F}$ with $\widehat{G} \supseteq A \supseteq \widehat{F}$ such that any measurable subset of $\widehat{G} \backslash A$ or of $A \backslash \widehat{F}$ is null.*

4. Perfect Set Property

In this section, we define a third regularity property (Definition 4.3), having to do with perfect sets. A nonempty subset A of a Polish space is said to be **perfect** if it is closed and all its points are limit points. In $\mathbb{R}$, the most familiar perfect sets are the closed intervals $[a,b]$ and the Cantor set. In fact any continuous injective image of $\mathscr{C}$ is perfect. In proving this, we recall a very general fact.

Lemma 4.1.
(a) *A continuous bijection f from a compact space X to a Hausdorff space Y is a homeomorphism.*[4]
(b) *Any continuous injection f from $\mathscr{C}$ to a Polish space is a homeomorphism onto its image, and $f(\mathscr{C})$ is perfect.*

Proof. (a) We must show that f^{-1} is continuous, for which it suffices to show that $f(K)$ is closed for all closed $K \subseteq X$. Closed subsets of compact spaces are compact, continuous images of compact sets are compact, and compact subsets of Hausdorff spaces are closed.

(b) $\mathscr{C}$ is compact, being homeomorphic to the closed and bounded Cantor set in the line, and Polish spaces are Hausdorff, so f is a homeomorphism by (a). The image $f(\mathscr{C})$ is compact, hence closed; all its points are limit points because the same is true of $\mathscr{C}$ and f is a homeomorphism. $\square$

Proposition 4.2. *Any perfect set A in a Polish space contains a homeomorphic copy of $\mathscr{C}$ (and in particular has cardinality 2^ω).*

We introduce a little more notation for the finite sequences in $2^{<\omega}$. For $s \in 2^{<\omega}$ of length n and x in either $2^{<\omega}$ or $\mathscr{C}$, let $s^\frown x$ be the concatenated sequence

$$\big(s(0),\, \ldots,\, s(n-1),\, x(0),\, x(1),\, \ldots\big).$$

Also use the notation $x|_k$ for the truncated sequence $\big(x(0),\, \ldots,\, x(k-1)\big)$, and similarly $s|_k = \big(s(0),\, \ldots,\, s(k-1)\big)$ provided $k \leq n$.

Proof. We define for all $s \in 2^{<\omega}$ an open ball B_s and a point $x_s \in B_s$ which will guide us in constructing a continuous injection of $\mathscr{C}$ into A (this suffices to prove the result by Lemma 4.1(b)). The construction is illustrated in Figure 2 below. Pick $x_\emptyset \in A$ arbitrarily, and let $B_\emptyset$ be an open ball containing $x_\emptyset$. Now because all points of A are limits, we may choose $x_{(1)} \neq x_\emptyset$ in $A \cap B_\emptyset$. Put $x_{(0)} = x_\emptyset$ and pick disjoint open balls $B_{(0)}$ and $B_{(1)}$ containing $x_{(0)}$ and $x_{(1)}$ respectively which are contained in $B_\emptyset$, have diameters less than 1, and themselves are separated by a positive distance. Continue this process inductively: if x_s and B_s have been defined whenever $|s| \leq n$, then for each such s, put $x_{s^\frown(0)} = x_s$ and choose $x_{s^\frown(1)} \neq x_s$ arbitrarily in $A \cap B_s$ (possible because x_s is a limit point of A). Then separate $x_{s^\frown(0)}$ and $x_{s^\frown(1)}$ with open balls $B_{s^\frown(0)}$ and $B_{s^\frown(1)}$ in B_s with diameters less than $1/(n+1)$.

[4]Of course not all continuous bijections are homeomorphisms; for instance if $\mathscr{T}$ and $\mathscr{T}'$ are topologies on the same space X with $\mathscr{T}' \supsetneq \mathscr{T}$, then the identity map is a continuous bijection from $(X, \mathscr{T}')$ to $(X, \mathscr{T})$, but discontinuous from $(X, \mathscr{T})$ to $(X, \mathscr{T}')$. Perhaps a more natural example is the continuous bijection $[0,1) \to S^1$ by $t \mapsto e^{2\pi i t}$ from a noncompact space to a compact one.

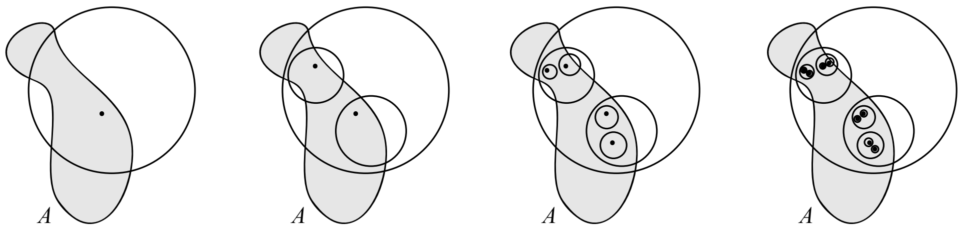

FIGURE 2. We inductively choose shrinking balls and points which they separate to construct a copy of the Cantor set inside any perfect set A.

For any $c \in \mathscr{C}$, the condition of shrinking diameters on balls $B_{c|_n}$ guarantees the sequence $(x_{c|_n})_{n=0}^{\infty}$ is Cauchy, so converges to some point of A, since A is closed. Thus we may define a function $f : \mathscr{C} \to A$ which takes $c \in \mathscr{C}$ to this associated point $\lim_{n\to\infty} x_{c|_n}$. Since balls at the same level are separated by positive distance, f is injective. By construction, if c and d in $\mathscr{C}$ agree in the first n places, then $f(c)$ and $f(d)$ are both in the ball $B_{c|_n} = B_{d|_n}$, which has diameter less than $1/n$; this proves f is continuous. $\square$

Definition 4.3. *A subset A of a Polish space has the* **perfect set property** *if it is either countable or has a perfect subset.*

In light of Proposition 4.2, a set can't both be countable and have a perfect subset, but for sets with the perfect set property, this is a dichotomy. The observation that perfect sets actually have cardinality 2^ω says that there are no counterexamples to the Continuum Hypothesis among sets with the perfect set property. The following Cantor-Bendixson Theorem shows that the definition is meaningful.

Theorem 4.4 (Cantor-Bendixson)**.** *Let X be a Polish space, and let $A \subseteq X$ be an uncountable closed subset. Then A is the disjoint union of a perfect set P and a countable set C. In particular, all closed subsets of any Polish space have the perfect set property.*

Proof. We need a way to strip away all the isolated points of A until we are left with a perfect set. Say a point $x \in X$ is a **condensation point of** A if every neighborhood of x contains uncountably many points of A. In particular these are limit points, so by closure are actually contained in A. Let A^* be the set of all condensation points of A; this is our candidate for P.

Let $(U_n)_{n=0}^{\infty}$ be a countable base for the topology on X, and let G be the union of all the U_n's whose intersection with A is at most countable, so $G \cap A$ is countable. A point is in G^c if and only if every neighborhood of it contains uncountably many points of A; thus $G^c = A^*$. This proves A^* is closed and $A \backslash A^*$ is countable.[5]

To see that any point $x \in A^*$ is a limit point, we just recall that any neighborhood U of x contains uncountably many points of A; since $A \backslash A^*$ is countable, it follows that

[5]The set up here is reminiscent of Lemma 3.5(b), with the σ-ideal of countable sets taking the place of the σ-ideal of meager sets. But the present result being more fundamental, it would be more accurate to say the proof of Lemma 3.5(b) is modeled after this one.

U contains uncountably many points of A^*. Thus every point of A^* is a condensation point of A^* and in particular is a limit point of A^*. $\square$

In fact, we have already done enough work in previous sections to extend the perfect set property from closed sets to Borel sets.

Proposition 4.5. *In any Polish space X, all Borel sets have the perfect set property.*

Proof. Let A be an uncountable Borel set. If $\mathscr{T}$ is the topology on X, we invoke Lemma 2.9(b) to obtain a stronger Polish topology $\mathscr{T}'$ such that A is clopen in $\mathscr{T}'$. By the Cantor-Bendixson Theorem, A contains a subset A_0 which is perfect under $\mathscr{T}'$, so by Proposition 4.2 contains a homeomorphic copy of $\mathscr{C}$. The identity map id from $(X, \mathscr{T}')$ to $(X, \mathscr{T})$ is a continuous injection, so the composition

$$\mathscr{C} \hookrightarrow (A, \mathscr{T}') \hookrightarrow (X, \mathscr{T}') \xrightarrow{\text{id}} (X, \mathscr{T})$$

is a continuous injection. By Lemma 4.1(b), the image in $(X, \mathscr{T})$ is perfect. $\square$

We conclude this section with a construction requiring the full Axiom of Choice. It is presently of interest because it connects perfect sets to the theory developed in Section 3. This construction will come up again in Section 6.

Proposition 4.6. *Assuming the Axiom of Choice, any uncountable Polish space X has a subset B such that B and B^c both intersect every perfect set. Such a set lacks both the Baire Property and the perfect set property, and if $X = \mathbb{R}^n$, this set is not Lebesgue measurable.*

Any set B with this property is called a **Bernstein set**.

Proof. Since we are working with AC, there is no reason to be careful about cardinality. By Proposition 4.2, every perfect subset of X has cardinality 2^ω. We claim that X has 2^ω perfect subsets. Certainly it has at most 2^ω of them, because perfect sets are closed, and the open sets of X are countably generated. On the other hand, the Cantor-Bendixson Theorem guarantees at least one perfect subset (because X is closed in itself), which in turn must contain a homeomorphic copy of the Cantor set; it now suffices to show that $\mathscr{C}$ has 2^ω perfect subsets. For this, we define a total order on $\mathscr{C}$: say $x < y$ if n is the first place where x and y differ, and $x(n) < y(n)$. Then for every x not ending with $(1, 1, 1, ...)$, the set $\{y \in \mathscr{C} : x \leq y\}$ is perfect.

Now we are set up for a "diagonalization" argument. Choose two points out of each perfect set *such that no choice is ever repeated*,[6] and then we may specify that exactly one point chosen from each is in B, so that B and B^c both intersect every perfect set.

Clearly any such set B lacks the perfect set property because it has cardinality 2^ω and no perfect subset (as does B^c). But Borel sets do have the perfect set property, so there are no nontrivial G_δ or F_σ sets contained in or containing B. In light of Propositions 3.3 and 3.4, B neither has the Property of Baire nor is Lebesgue measurable. $\square$

[6]It is intuitively obvious enough that choices can be made without repetition, though a fully rigorous proof would use induction on the ordinals less than 2^ω.

5. Analytic Sets

The construction of a Bernstein set in Proposition 4.6 relied heavily on the Axiom of Choice. This section, by contrast, treats a particularly well-behaved class of sets, which are measurable, have the Baire Property, and have the perfect set property. They are the **analytic sets**—the continuous images of Borel sets (from any Polish spaces). Clearly all Borel sets are analytic, and note that the Borel sets do indeed have all three desired properties.

While it is true that continuous *preimages* of Borel sets are Borel, there are known examples of non-Borel analytic sets.[7] In fact the discovery of analytic sets by Luzin was prompted by Lebesgue's erroneous "proof" that the continuous images of Borel sets are themselves Borel; Lebesgue briefly retells the story in [L85].

The modern definition of an analytic set is one which is the continuous image of the Baire space ω^ω. This seems to be much narrower than the definition we have given, but we show that the two definitions are equivalent, which will be a useful tool for proving regularity properties.

Proposition 5.1. *For any Polish space X, there is a continuous surjection $f : \omega^\omega \to X$.*

Proof. Much as in Proposition 4.2, we construct a branching tree of open sets which will define a function $\omega^\omega \to X$. This time we are interested in surjectivity rather than injectivity, so we cover X with a countable union of open balls, not worrying about overlap, then we cover each of those balls with countably many smaller balls (with closures contained in the larger ball). We continue the process inductively, as indicated in Figure 3 below, and leave the rest of the details to the reader. □

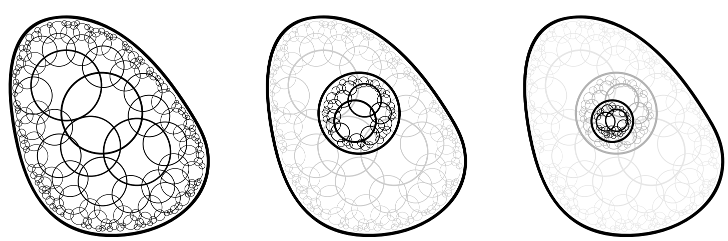

Figure 3. Create a branching tree of open sets by covering X with a sequence of small open balls, then cover each of them with smaller open balls, and so on. To avoid a very cluttered illustration, we focus on just a single branch of this tree.

[7] A concrete example is given in [Lu27]. Say a point $x \in \omega^\omega$ is **composite** if there is an increasing sequence of integers $n_{k+1} > n_k$ such that $x(n_k)$ is positive and $x(n_k)$ properly divides $x(n_{k+1})$ for all $k \in \omega$. Then the set of all composite points is a non-Borel analytic subset of ω^ω. A different construction is given in Chapter 4 of [M02], guided by the observation that an analytic set whose complement is not analytic cannot be Borel.

Proposition 5.2. *For any Borel set A in a Polish space X, there is a continuous map $f : \omega^\omega \to X$ whose image is A.*

Proof. Let $\mathscr{T}$ denote the topology on X, and use Lemma 2.9(b) to find a stronger Polish topology $\mathscr{T}'$ under which A is clopen, so that A is a Polish subspace of $(X, \mathscr{T}')$. Now by Proposition 5.1 there is a continuous surjection $f : \omega^\omega \to A$ (with respect to the relative topology on A induced by $\mathscr{T}'$). Since $\mathscr{T}' \supseteq \mathscr{T}$, the identity map id from $(X, \mathscr{T}')$ to $(X, \mathscr{T})$ is continuous, so the composition

$$\omega^\omega \xrightarrow{f} (A, \mathscr{T}') \hookrightarrow (X, \mathscr{T}') \xrightarrow{\text{id}} (X, \mathscr{T})$$

is also continuous. □

Corollary 5.3. *A subset A of a Polish space X is analytic if and only if it is the continuous image of ω^ω.*

Proof. Any continuous image of a Borel set is also the continuous image of ω^ω by Proposition 5.2. □

We are now ready to prove the claimed regularity properties of analytic sets.

Theorem 5.4. *All analytic subsets of a Polish space X have the perfect set property.*

Proof. Let $A \subseteq X$ be an uncountable analytic set and let $f : \omega^\omega \to X$ be continuous with image A. We will show that there is a copy of Cantor space inside ω^ω which f maps injectively into A, which will complete the proof by Lemma 4.1(b).

We observe for any $s \in \omega^{<\omega}$ that if $f(N_s)$ is uncountable, then for some $m \in \omega$, $f\big(N_{s^\frown(m)}\big)$ must also be uncountable. By assumption, $f(N_\emptyset) = f(\omega^\omega)$ is uncountable, so there is in fact a nested sequence $s_n \subset s_{n+1}$ of elements of $\omega^{<\omega}$ such that $f(N_{s_n})$ is uncountable for all n. Letting $x \in \omega^\omega$ be the union of the s_n's, we have that $f(N_{x|_n})$ is uncountable for all n.

But we need to prove the stronger result that for any $s \in \omega^{<\omega}$ with $f(N_s)$ uncountable, there exist nonempty p and q in $\omega^{<\omega}$ such that $f(N_{s^\frown p})$ and $f(N_{s^\frown q})$ are both uncountable and are separated by some positive distance. This will allow us to inductively create the kind of branching-tree structure that defines a Cantor set, which we illustrate in the following Figure 4.

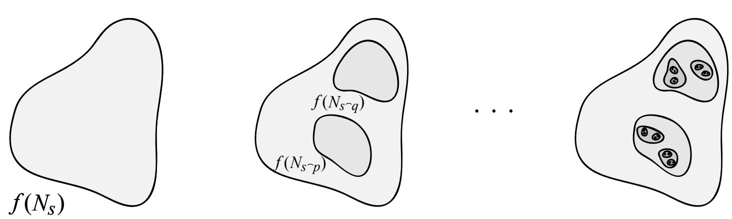

Figure 4. The inductive scheme gives a procedure for constructing a Cantor set inside any uncountable analytic set.

Fix s such that $f(N_s)$ is uncountable, and take $x \in \omega^\omega$ extending s such that $f(N_{x|_n})$ is uncountable for all n. Then there is some positive $k \in \omega$ such that there are uncountably many points of $f(N_s)$ with distance at least $1/k$ to $f(x)$. By continuity of f, we can choose $p \in \omega^{<\omega}$ such that $s^\frown p \subset x$ and $f(N_{s^\frown p})$ stays within distance $1/3k$ from $f(x)$; by definition of x, this set $f(N_{s^\frown p})$ is uncountable.

To construct q, let $B = B(f(x), 1/k)$ and recall that there are uncountably many points of $f(N_s)$ in B^c. Just as before, we may inductively extend s to a sequence $y \in \omega^\omega$ such that $f(N_t)$ contains uncountably many points of B^c for every finite initial segment $t \subset y$. By continuity of f and closure of B^c, it must be that $f(y) \in B^c$, so that the distance between $f(x)$ and $f(y)$ is at least $1/k$. Again by continuity, there is some $q \in \omega^{<\omega}$ such that $s^\frown q \subset y$ and $f(N_{s^\frown q})$ stays at within distance $1/3k$ from $f(y)$. Then $f(N_{s^\frown p})$ and $f(N_{s^\frown q})$ are separated by distance at least $1/3k$, and indeed $f(N_{s^\frown q})$ is uncountable by definition of y. $\square$

We need one more tool before proving measurability and Baire Property for analytic sets.

Definition 5.5. *For a collection of sets A_s indexed by $s \in \omega^{<\omega}$, define the* **Suslin operation** $\mathscr{A}$ *by*

$$\mathscr{A}\{A_s : s \in \omega^{<\omega}\} = \bigcup_{x \in \omega^\omega} \bigcap_{s \subset x} A_s.$$

Theorem 5.6. *All analytic subsets of a Polish space X have the Baire Property; all analytic subsets of $\mathbb{R}^n$ are Lebesgue measurable.*

Proof. The proofs of the two statements in the Theorem are practically identical, only differing in which part of Lemma 3.5 is used to approximate A. As such, we only give the proof for Baire Property.

Let $A \subseteq X$ be an analytic set, and $f : \omega^\omega \to X$ continuous with image A. For each $s \in \omega^{<\omega}$, let $A_s = f(N_s)$ so that $A_\emptyset = A$ and $A_s = \bigcup_{k=0}^\infty A_{s^\frown(k)}$. We note that for every $x \in \omega^\omega$, we have by continuity that

$$\bigcap_{s \subset x} A_s = \{f(x)\} = \bigcap_{s \subset x} \overline{A_s},$$

and so $A = \mathscr{A}\{A_s\} = \mathscr{A}\{\overline{A_s}\}$.

With Lemma 3.5(a), we now approximate each A_s by a set $B_s \in BP(X)$ containing A_s such that any subset of $B_s \backslash A_s$ with the Baire Property is meager. By intersecting B_s with the closed set $\overline{A_s}$, we may assume $B_s \subseteq \overline{A_s}$. It suffices to show that first approximation is a good one, in that $B_\emptyset \backslash A$ is meager (so that A is the union of $B_\emptyset$ and a meager set). First, we note by $A_s \subseteq B_s \subseteq \overline{A_s}$ that $A = \mathscr{A}\{B_s\}$, so that

$$B_\emptyset \backslash A = B_\emptyset \Big\backslash \Bigg(\bigcup_{x \in \omega^\omega} \bigcap_{s \subset x} B_s \Bigg).$$

We claim this is a subset of

$$\bigcup_{s \in \omega^{<\omega}} \Bigg(B_s \Big\backslash \bigcup_{k=0}^{\infty} B_{s^\frown(k)} \Bigg),$$

which is meager because each $B_s \setminus \bigcup_{k=0}^{\infty} B_{s^\frown(k)}$ is a subset of $B_s \setminus \bigcup_{k=0}^{\infty} A_{s^\frown(k)} = B_s \setminus A_s$. Now it suffices to prove the containment

$$B_\emptyset \Big\backslash \Big(\bigcup_{x \in \omega^\omega} \bigcap_{s \subset x} B_s \Big) \subseteq \bigcup_{s \in \omega^{<\omega}} \Big(B_s \Big\backslash \bigcup_{k=0}^{\infty} B_{s^\frown(k)} \Big).$$

Suppose $y \in B_\emptyset$ but is not in the right hand side. This means for every s that if $y \in B_s$, then $y \in B_{s^\frown(k)}$ for some k. By the assumption $y \in B_\emptyset$, we can inductively construct partial sequences s_n of length n such that $s_n \subset s_{n+1}$ and $y \in \bigcap_{n=0}^{\infty} B_{s_n}$. Letting $z \in \omega^\omega$ be the union of the s_n's, we have $y \in \bigcap_{s \subset z} B_s$, which completes the proof. $\square$

6. Games of Infinite Length

In what may seem at first glance to be a departure from the kind of material presented in the preceding sections, we define a certain kind of two-player game. It will be the goal of this section and the next to explore the relations between games, measure, category, and perfect sets.

For a fixed set $A \subseteq \mathscr{C}$ called the **payoff set** or sometimes the **winning set**, players P_1 and P_2 alternate choosing 0 or 1. Together they play out a sequence of bits $x \in \mathscr{C}$; if $x \in A$, we say P_1 wins, otherwise P_2 wins. Formally, a **strategy** for one of the players is a function $2^{<\omega} \to \{0,1\}$, which we think of as rule dictating whether that player will move 0 or 1 in any given game state. If σ is a strategy for P_1 and τ is a strategy for P_2, the resulting **play** according to these strategies is the sequence of bits

$$\big(\sigma(\emptyset),\ \tau(\sigma(\emptyset)),\ \sigma(\sigma(\emptyset), \tau(\sigma(\emptyset))),\ \ldots\big),$$

which we will often write more simply as $(\sigma, \tau, \sigma, \tau, ...)$. A strategy is called a **winning strategy** if it beats every one of the other player's strategies; clearly at most one of the two players can have a winning strategy. If one player does have a winning strategy, we say the game (or simply the set A) is **determined**.

The **Axiom of Determinacy** (AD) is the statement that every $A \subseteq \mathscr{C}$ is determined. We will soon show that AD contradicts the Axiom of Choice, though as mentioned in Section 2, we will continue to use Dependent Choice.[8] In fact, the main content of Section 7 is completely incompatible with the construction of the Bernstein set in Proposition 4.6, showing that AD implies all subsets of the real numbers are Lebesgue measurable, have the Property of Baire, and have the perfect set property.

For some games, it is easy to construct a winning strategy for one of the players. As the most trivial possible example, if $A = \mathscr{C}$, then any strategy for P_1 is winning, and if $A = \emptyset$, then any strategy for P_2 is winning. Extrapolating from here, we see that any game whose payoff set is a finite union of basic open neighborhoods N_s, $s \in 2^{<\omega}$ is also determined, as the game can be represented by a finite tree and solved by induction. The idea that a game could be undetermined is a little stranger: given any strategy σ for P_1, P_2 has a strategy τ which beats it, and P_1 has a strategy ξ which beats *that*, and so on. But as promised, the Axiom of Choice can produce such an anomaly, which we establish through a kind of "perfect set theorem" for determined games.

[8] For example [K84] shows that if AD is consistent with the usual axioms of set theory minus Choice, then it is consistent with those axioms plus DC.

Proposition 6.1. *For any subset $A \subseteq \mathscr{C}$, if P_1 has a winning strategy, then A contains a perfect set; if P_2 has a winning strategy, then A^c contains a perfect set.*

Proof. Suppose P_1 has a winning strategy σ. Then in particular, σ beats every "non-interactive" strategy for P_2 which simply chooses consecutive bits from some $x \in \mathscr{C}$, so that the play $(\sigma, x(0), \sigma, x(1), ...)$ is a point of A. Then the map $f : \mathscr{C} \to \mathscr{C}$ defined by $f(x) = (\sigma, x(0), \sigma, x(1), ...)$ carries $\mathscr{C}$ into A. But f is continuous—if long initial segments of x and y in $\mathscr{C}$ agree, then long initial segments of $(\sigma, x(0), \sigma, x(1), ...)$ and $(\sigma, y(0), \sigma, y(1), ...)$ also agree—and f is injective, so $f(\mathscr{C})$ is perfect by Lemma 4.1(b).

The same proof shows that a winning strategy for P_2 produces a Cantor set in A^c. $\square$

Corollary 6.2. *Assuming the Axiom of Choice, there exists an undetermined set in $\mathscr{C}$.*

Proof. The Bernstein set constructed in Proposition 4.6 contains no perfect set, and neither does its complement. $\square$

Proposition 6.1 is our first indication that games of infinite length are not a totally discrete phenomenon. We continue in this vein, with what might be considered a bit of good news compared to the above.

Theorem 6.3 (Gale-Stewart [GS53])**.** *All closed sets $A \subseteq \mathscr{C}$ are determined.*

Proof. Suppose P_2 has no winning strategy for A; we will construct a winning strategy for P_1. By assumption, there must be some choice $a_0 \in \{0, 1\}$ such that for all $b_0 \in \{0, 1\}$, P_2 has no winning strategy in the "subgame" starting with (a_0, b_0). And then there must be some choice $a_1 \in \{0, 1\}$ (depending on b_0) such that for all $b_1 \in \{0, 1\}$, P_2 has no winning strategy in the subgame starting with (a_0, b_0, a_1, b_1). This continues in the obvious way, so that if P_1 makes such choices, P_2 has no winning strategy in the subgame starting with $(a_0, b_0, ..., a_n, b_n)$ for any n. In particular, the basic open neighborhood $N_{(a_0,b_0,...,a_n,b_n)}$ meets A for every n, so *by closure*, $\bigcap_{n=0}^{\infty} N_{(a_0,b_0,...,a_n,b_n)}$ meets A (choose a sequence $x_n \in N_{(a_0,b_0,...,a_n,b_n)} \cap A$, which must be Cauchy). But $\bigcap_{n=0}^{\infty} N_{(a_0,b_0,...,a_n,b_n)}$ consists of a single point, and this point is the play of the game, so P_1 wins. Thus we have outlined a strategy for P_1 which beats any strategy for P_2. $\square$

Corollary 6.4. *All open sets $A \subseteq \mathscr{C}$ are determined.*

Proof. Consider the two subgames G_0 and G_1 where P_1 has made the first move, 0 or 1 respectively, and it is P_2's turn to move, trying to play into A^c. A winning strategy for P_1 in the current game amounts to a winning strategy in *either* G_0 or G_1, whereas a winning strategy for P_2 in the current game amounts to a winning strategy in *both* G_0 and G_1. Since A^c is closed in both $N_{(0)}$ and $N_{(1)}$, these are the only two possibilities. $\square$

An amusing way to view the argument in Theorem 6.3 is in the following context. Quite informally, we may think of the statement "P_1 has a winning strategy" (for A) as an infinitely long formula

$$\exists a_0 \, \forall b_0 \, \exists a_1 \, \forall b_1 \, \ldots \text{ such that } (a_0, b_0, a_1, b_1, \ldots) \in A.$$

Similarly, the statement "P_2 has a winning strategy" would be rendered as

$$\forall a_0 \, \exists b_0 \, \forall a_1 \, \exists b_1 \, \ldots \text{ such that } (a_0, b_0, a_1, b_1, \ldots) \notin A,$$

which appears to be the negation of "P_1 has a winning strategy" by the rule that $\neg\exists$ is equivalent to $\forall\neg$. If we take this view, then AD appears to be manifestly true, because either P_1 has a winning strategy or doesn't. Now, traditionally mathematics is built up out of only finitely long formulas, and moreover it is far from obvious that one *ought* to be able to move the negation symbol through infinitely many quantifiers, so the above is certainly no proof for AD. However, the proof of Theorem 6.3 essentially shows that for *closed* sets A, arbitrarily long finite approximations to the above infinite formulas (which actually *are* each other's negations) suffice.

At this point, we cite a much stronger result, due to Martin, though its proof is far beyond the scope of this paper, and we shall have no occasion to employ its conclusion.

Theorem 6.5 (Martin [Ma75])**.** *All Borel sets $A \subseteq \mathscr{C}$ are determined.*

But the statement does suggest an immediate false proof, which we find worth investigating. That is, once we know all the open sets are determined, we ought to be able to prove that the collection of determined sets is a σ-algebra, and hence contains all Borel sets. We show that this line of reasoning is doomed to failure.

Proposition 6.6. *Assuming the Axiom of Choice, there exist two determined sets whose union is not determined, and there exists a determined set whose complement is not determined.*

Before giving the proof, we note that the statement about complements seems truly bizarre, as complementation appears to just switch the roles of the two players (with P_1 now trying to play into A^c and P_2 trying to play into A). However, one bit of asymmetry does persist, in that P_1 still moves first; this is why the proof of Corollary 6.4, while still easy, does take more than a single line. Naturally, the constructions will be centered around Bernstein sets as constructed in Proposition 4.6, as they are the only undetermined sets we have identified.

Proof. Fix a set B such that neither B nor B^c contains a perfect subset.

For the statement about unions, we consider two "quarter-sized" versions of B which are small enough that P_1 is guaranteed a loss on either, but large enough that their union can be undetermined. That is, let $B_1 = \{(0,0)^\frown x : x \in B\}$ and $B_2 = \{(0,1)^\frown x : x \in B\}$, so that P_2 can guarantee a win on B_1 by moving 1 at the first opportunity and a win on B_2 by moving 0 at the first opportunity, regardless of P_1's move. However if P_1 makes the first move 0 on the union $B_1 \cup B_2$, then no matter what P_2 moves next, the players are playing a subgame on B; thus neither player has a winning strategy for $B_1 \cup B_2$.

For complementation, consider the "half-sized" Bernstein set $B_3 = \{(0)^\frown x : x \in B\}$. Then P_1 can guarantee a win for the set $B_3 \cup N_{(1)}$ by simply moving 1 in the first place. The complement, however, is just $\{(0)^\frown x : x \in B^c\}$. If P_1's first move is 0, then P_2 is left trying to play into B, so this game is undetermined. $\square$

In fact, one of the above constructions can be strengthened: there exists a determined set whose union *with a single point* is undetermined. For this, we need many "scaled-down" copies of a Bernstein set B. For $n \in \omega$, let $s_n = (0, 1, ..., 0, 1)$ be the sequence of length $2n$ whose digits alternate 0 and 1. If $B_n = \{s_n{}^\frown(0,0)^\frown x : x \in B\}$, then the set $A = \bigcup_{n=0}^{\infty} B_n$ is determined, as P_2 has a winning strategy in always playing 1. If, however, the point $(0, 1, 0, 1, ...)$ is adjoined to A, this strategy is no longer winning for P_2, as P_1 can beat it by always playing 0. Then to have any chance at winning, P_2 must eventually play 0; if P_1 has always played 0 up until this point, the players come to a subgame with P_1 trying to move into B. Thus $A \cup \{(0, 1, 0, 1, ...)\}$ is undetermined.

The rest of this section is dedicated to more bad news that demonstrates why we have our work cut out for us in the next section.[9] In particular, we investigate a certain non-interaction of determinacy with Lebesgue measure and Baire Property, which ought to be a bit surprising given our promise to show in the next section that AD implies all sets of reals are measurable and in BP.

To state the following propositions, we recall from Example 2.6 the transportation of Lebesgue measure from $[0, 1]$ to $\mathscr{C}$, by identification of the basic open neighborhoods N_s with closed dyadic intervals of measure $2^{-|s|}$. We start with an important pair of counterexamples, whose constructions are completely trivial.

Proposition 6.7. *Assuming the Axiom of Choice, there is a determined set which is not Lebesgue measurable, and there is a determined set which does not have the Property of Baire.*

Proof. Every subset of the basic open neighborhood $N_{(0,0)}$ is determined, as P_2 can guarantee a win by moving 1 at the first opportunity, but there are plenty of nonmeasurable and/or non-BP subsets of $N_{(0,0)}$. □

We note that Proposition 6.7 has something like a converse in Proposition 6.9 below. It relies on the following intermediate construction, which actually does not depend on the Axiom of Choice.

Proposition 6.8. *There is a closed null set for which P_1 has a winning strategy. In particular, this set is also meager.*

Proof. For every n, let A_n be the set of all points in $\mathscr{C}$ whose zeroth, second, ..., $2n$-th bits are all 0. Each $A_n \supseteq A_{n+1}$ and A_n is clopen with measure 2^{-n}. Then the set $A = \bigcap_{n=0}^{\infty} A_n$ is closed and null and is the set of all points in $\mathscr{C}$ whose every even-positioned bit is 0. As such, a winning strategy for P_1 is to always play 0.

For the second statement, any null set has empty interior, so any closed null set is meager. □

It will follow in the proof below that there is a null and meager set into which every game can be *embedded*, in some sense. However, it is not terribly important to us to make a precise definition of what it ought to mean to embed a game into some set, as we are more interested in a simple statement of the proposition.

[9]Of course, it is only with tongue planted firmly in cheek that we can call the following counterexamples “bad news.” Besides being surprising constructions in their own right, they guarantee the next section’s theorems will not have dull, straightforward proofs.

Proposition 6.9. *Assuming the Axiom of Choice, there is a null and meager set which is undetermined. In particular, this set is Lebesgue measurable and has the Property of Baire.*

Proof. We begin with a Bernstein set B. Our first task is to "spread out" the points of B so that the resulting set is null and meager. For this, we define

$$B^* = \{(x(0), x(1), 0, 0, x(2), x(3), 0, 0, ...) \in \mathscr{C} : x \in B\}$$

which is indeed both null and meager. However, this will not be an appropriate payoff set for our new game, as P_2 has a winning strategy in just playing 1 in some position where all points of B^* have 0.

It will then be our second task to adjoin a null and meager set to B^* which will discourage P_2 from choosing 1 at these times. The result will be that *both* players are essentially forced to play 0 every other time, but otherwise are free to play as before, so that this new game is undetermined exactly because B is undetermined. Let A be a null and meager set on which P_1 has a winning strategy, as constructed in Proposition 6.8. To guarantee that P_2 ought to follow the "rule" we have put forth, we adjoin countably many scaled down copies of A in such a way that if P_2 ever breaks the rule and chooses 1 inappropriately, the players come to a subgame in which it is P_1's turn, trying to play into this scaled down copy of A. We illustrate the placement of these small copies of A in Figure 5. This new payoff set is a countable union of null and meager sets, hence is again null and meager. $\square$

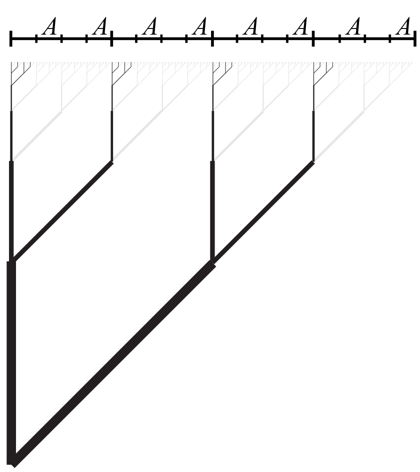

FIGURE 5. After transforming B to the null and meager set B^* (contained in the black subtree), we adjoin scaled copies of a null and meager set A which P_1 can win so as to punish P_2 for ever moving off the black subtree. Only the largest of these copies of A are labeled.

7. Games with Rules

This section delivers the promised proofs that AD implies all subsets of all Polish spaces have the perfect set property and the Baire Property, and that all subsets of $\mathbb{R}^n$ are Lebesgue measurable. As observed in Proposition 6.7, the proofs cannot be as simple as "all determined sets are Lebesgue measurable, therefore AD implies all sets are Lebesgue measurable." Rather, the approach is through defining games with additional "rules" which

(i) can be *implemented* as a game without these rules (by crafting the payoff set so that the first player to break the rules is guaranteed to lose, i.e. there is a winning strategy for the other player on the resulting subgame) and
(ii) give information about a specified set, under the assumption that one player has a winning strategy.

We have developed some familiarity with both ideas already; the construction in Proposition 6.9 involved a fair amount of rule making, and Proposition 6.1 used a winning strategy to construct a perfect set.

We move to our next example of a game with rules. Referring to the games introduced in the previous section as **Cantor space games**, we define a completely analogous class of games called **Baire space games**, where P_1 and P_2 pick nonnegative integers at each step rather than just 0's and 1's. Strategies, winning strategies, and so on can all be defined appropriately for Baire space games, and moreover, all the proofs in Section 6 can be adapted for Baire space games. An Axiom of Determinacy can all be stated for Baire space games; what we wish to establish via some kind of "game with rules" is the following.

Proposition 7.1. *The respective Axioms of Determinacy for Cantor space games and Baire space games are equivalent.*

Proof. It is easy to implement any Cantor space game as a Baire space game with rules: just prohibit either player from ever moving anything other than 0 or 1. That is, a payoff set $A \subseteq \mathscr{C}$ for a Cantor space game can be viewed as a subset of ω^ω by the obvious inclusion $\mathscr{C} \hookrightarrow \omega^\omega$. To prohibit P_2 from moving outside of $\mathscr{C} \subseteq \omega^\omega$, adjoin to $A \subseteq \omega^\omega$ all the basic open neighborhoods N_s for $s \in \omega^{<\omega}$ of even length whose last entry is 2 or greater. Then we see AD for Baire space implies AD for Cantor space.

Conversely, we can implement Baire space games as Cantor space games, by encoding integers as strings of 0's and 1's. Given a Baire space game with payoff set $A \subseteq \omega^\omega$, we define a Cantor space game with the following rules (which we enforce by altering the payoff set):

(i) A player may not play 1 if the opponent has just played 1.
(ii) A player may not play 1 infinitely many times in a row.

When the players follow these rules, the outcome of the game is as follows: P_1 plays 1 an integer number of times (possibly 0 times), while P_2 simply plays 0's, until eventually P_1 plays 0. Then P_2 plays 1 an integer number of times (again, possibly 0 times), while P_1 must play 0's, and so on. There is an obvious way to take a sequence of bits arranged in this manner to a sequence of integers; say P_1 wins exactly when this sequence of integers is a point in A. Thus AD for Cantor space implies AD for Baire space. $\square$

The terminology of Cantor space games and Baire space games is not standard, and for the most part, set theorists are only interested in the latter type. Baire space offers more immediate flexibility for defining games with rules, and the above proposition guarantees there is no intrinsic reason to prefer one over the other. However, we believe the relation between $\mathscr{C}$ and $\mathbb{R}$ is clearer than that between ω^ω and $\mathbb{R}$ in that binary expansions are more tractable than continued fractions. Thus Cantor space games were more appropriate for Section 6.

At this point, we move on to consequences of AD, by constructing games where winning strategies can be interpreted as objects outside the context of games. As a warm-up, we give the following, which we find too amusing not to include, especially given the incompatibility of AD and full Choice.

Proposition 7.2. *Assuming the Axiom of Determinacy, any countable collection of subsets ω^ω has a choice function. In fact the same is true for any Polish space X.*

This is of course a weaker version of Choice than DC, which we have been using without reservation, but the proof below does not require any Choice at all.

Proof. Given subsets $A_n \subseteq \omega^\omega$, define a Baire space game where P_2 wins the play $(a_0, b_0, a_1, b_1, ...)$ exactly when $(b_0, b_1, b_2, ...) \in A_{a_0}$. Then P_1 does not have a winning strategy, because once P_1 plays a_0, P_2 may choose any $(b_0, b_1, b_2, ...) \in A_{a_0}$ and guarantee a win. To say that P_2 has a winning strategy means these individual choices can be collected to a choice function for the sets $(A_n)_{n=0}^\infty$.

For other Polish spaces, we simply note that Proposition 5.1, which guarantees X is the continuous image of ω^ω, can be proved without any Choice at all: whenever a set must be covered by countably many basic open balls with certain properties, use all of them. Also, a nested union of basic open balls with radii shrinking to 0 and closures contained in the ones before has nonempty intersection because the specified *centers* are a Cauchy sequence. $\square$

We move to some of the real content of this section with the perfect set property, which ought to be somewhat reminiscent of Proposition 6.1. Since sets with the perfect set property are either countable or have cardinality 2^ω, we may think of this theorem as the derivation of a weak version of the Continuum Hypothesis from the Axiom of Determinacy—weak in that it only applies to subsets of Polish spaces, which would be equivalent if only we could use the Axiom of Choice (to construct maximal injective maps between arbitrary sets). In any case, we think it is beneficial to prove the theorem first for the Cantor space below, then outline a modification of the proof to arbitrary Polish spaces in Remark 7.5.

Theorem 7.3 (Perfect Set Property)**.** *Assuming the Axiom of Determinacy, every subset of $\mathscr{C}$ has the perfect set property.*

Proof. For a fixed subset $A \subseteq \mathscr{C}$, we define a variation on the Cantor space game, called the **perfect set game**. Here, P_1 can choose any finite sequence in $2^{<\omega}$ while P_2 is still limited to choosing 0 or 1. The resulting play (concatenate the moves in the obvious way) is a point x in $\mathscr{C}$, and we say P_1 wins exactly when $x \in A$. Note that by enumerating $2^{<\omega}$, the perfect set game can be implemented as a Baire space game, and as such, it follows from AD that the perfect set game is also determined.

If P_1 has a winning strategy σ, then we can construct a perfect subset of A exactly as in Proposition 6.1. That is, σ must beat all those strategies in which P_2 just reads off bits of a point $x \in \mathscr{C}$, so that the map $\mathscr{C} \to \mathscr{C}$ taking x to the point played out by these strategies, which is continuous and injective, has its image in A.

On the other hand, we will show that if P_2 has winning strategy, then A is countable. Given any strategy τ for P_2, a point $x \in \mathscr{C}$, and a finite "position" $p \in 2^{<\omega}$, we say τ **rejects** x **at** p if $p \subset x$ and for every move $s \in 2^{<\omega}$ which P_1 could make with $p^\frown s \subset x$, the very next move $\tau(p^\frown s)$ that P_2 would make ensures the play of the game is not x (meaning $p^\frown s^\frown(\tau(p^\frown s))$ is not an initial segment of x). The first thing to note is that if τ does not reject a point x at some position p, then P_1 has a strategy which guarantees the play of the game is x. Then any winning strategy for P_2 must reject every point of A at some position.

The second thing to note is that τ can only reject one point at any position. Suppose both $x, y \in \mathscr{C}$ are rejected at the same position $p \in 2^{<\omega}$ with $p = \big(p(0), \ldots, p(n-1)\big)$. In particular, x and y agree in places 0 through $n-1$. To see that they agree in the n-th place as well, we observe that if P_1 plays the empty sequence $\emptyset$ at p (where of course $p^\frown\emptyset = p$ is still an initial segment of both x and y), then P_2's next move $\tau(p)$ is *not* equal to either $x(n)$ or $y(n)$. Since there are only two possible values for $x(n)$ and $y(n)$, they must be equal. But this argument continues inductively; if x and y share initial segment $p^\frown s$, then they also share one more bit because $\tau(p^\frown s)$ is not equal to the next bit of either. Thus $x = y$ by induction. This finishes the proof that if P_2 has a winning strategy, then A is countable. □

Remark 7.4. In the perfect set game above, the Axiom of Determinacy guarantees that if A is countable, then (since P_1 has no winning strategy) P_2 must have a winning strategy, and similarly if A has a perfect subset then P_1 must have a winning strategy. Interestingly enough, we can say what those strategies are. That is, without Determinacy, we can constructively prove that P_1 has a winning strategy in the perfect set game if and only if A has a perfect subset and that P_2 has a winning strategy if and only if A is countable. The role of Determinacy is to conclude that these are the only two possibilities.

If A is countable, enumerated $(a_n)_{n=0}^{\infty}$, then one winning strategy for P_2 is to always use the n-th move to ensure that the play of the game is not a_n by just choosing the "wrong" bit.

If A has a perfect subset A_0, then the opening move for P_1 should be a finite sequence $s_0 \in 2^{<\omega}$ such that both of the basic open neighborhoods $N_{s_0{}^\frown(0)}$ and $N_{s_0{}^\frown(1)}$ meet A_0, which is possible because all points of A_0 are limits. No matter what P_2's next move b_0 is, there must be another finite sequence $s_1 \in 2^{<\omega}$ such that both basic open neighborhoods $N_{s_0{}^\frown(b_0)^\frown s_1{}^\frown(0)}$ and $N_{s_0{}^\frown(b_0)^\frown s_1{}^\frown(1)}$ meet A_0. Continue inductively, and the play of the game will be a point in A_0 by closure.

Remark 7.5. There is still more to say about the perfect set game; in particular, we have not yet delivered on our promise to prove that AD implies all subsets of *every* Polish space X have the perfect set property. If X is countable, then there is nothing to prove. For any uncountable X, there is a perfect subset X_0 which contains all but countably many points of X (Theorem 4.4); thus it suffices to prove the result for Polish spaces X which are perfect subsets of themselves. We now outline a modified version of the perfect set game on such spaces to which the proof of Theorem 7.3 generalizes. The key difference is that the two players now choose basic open sets each round instead of numbers.

Fix a payoff set $A \subseteq X$, and let $\mathscr{B}$ be the usual countable base consisting of open balls $B(a, q)$, where a ranges over a countable dense set and q ranges over the positive rational numbers. The players take turns to construct a nested sequence of sets $C_{n+1} \subseteq C_n$ by the following rules (see Figure 6):

(i) $C_{-1} = X$.
(ii) If C_n has been constructed and n is odd, then P_1 must choose C_{n+1} to be any union of two balls $B(x, q)$ and $B(y, r)$ in $\mathscr{B}$ that are separated by a positive distance, both have closures contained in C_n, and both q and r are less than $1/(n+2)$ (such a choice is always possible for P_1 because X is perfect).
(iii) Then P_2 chooses C_{n+2} to be one of the two balls making up C_{n+1}.

The conditions on shrinking radii and closures simply guarantee $\bigcap_{n=0}^{\infty} C_n$ is a single point of X; say P_1 wins just in case that point is in A. This modified perfect set game can be implemented as a Baire space game, because P_1 has countably many choices each round, and P_2 has just two choices.

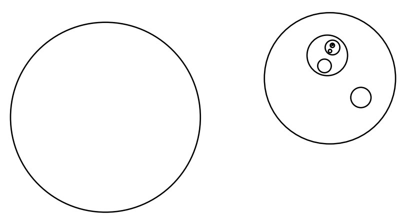

FIGURE 6. In the modified perfect set game, P_1 chooses two basic open balls, and P_2 chooses in which one of those balls the game should continue.

We claim that exactly the same reasoning going into Theorem 7.3 and Remark 7.4 yields the result that P_1 has a winning strategy if and only if A contains a perfect subset, and P_2 has a winning strategy if and only if A is countable. Thus AD implies all subsets of all Polish spaces have the perfect set property.

Theorem 7.6 (Baire Property). *Assuming the Axiom of Determinacy, every subset of any Polish space X has the Baire Property.*

We wish to make a guiding analogy before giving the proof. The perfect set game, as introduced in the proof of Theorem 7.3 is a natural variation on Cantor space games, by allowing one player to "cheat" by playing any finite sequences from $2^{<\omega}$ each round. Here is another variation: allow *both* players to choose from $2^{<\omega}$ each round (and require the choices to be nonempty to guarantee that the play of the game is actually an element of $\mathscr{C}$ rather than of $2^{<\omega}$). This will be a special case of the Banach-Mazur game which we introduce in the following proof, and as such, one could use this game to prove Theorem 7.6 in the case $X = \mathscr{C}$.

Proof. Fix a set $A \subseteq X$, and let $\mathscr{B}$ be a countable base of open balls for the topology on X. In the **Banach-Mazur game**, the players alternate choosing balls $C_0, C_1, ...$ from $\mathscr{B}$ such that the closure of each ball is contained in the ball most recently picked by the other player, and the n-th chosen ball has diameter less than $1/n$. When the game has been played, the intersection of all the chosen balls contains a single point; P_1 wins exactly when that point lies in A. Because each player has countably many options each round, the Banach-Mazur game can be implemented as a Baire space game.

We claim that if P_2 has a winning strategy for the game, then A is meager. But the game is nearly symmetric for the two players, the only difference being that P_1 moves first, so it will follow that if P_1 has a winning strategy, then there is some ball $B \in \mathscr{B}$ such that $A^c \cap B$ is meager.[10]

To prove this, we proceed as in the perfect set game; we need to say what it means for a strategy τ for P_2 to reject a point $x \in X$ after some finite number of choices $C_0 \supseteq C_1 \supseteq ... \supseteq C_n$ have been made. The right definition is: $x \in C_n$ is rejected if for all possible balls B containing x that P_1 could choose at this point in the game, the next ball $\tau(C_0, C_1, ..., C_n, B)$ that P_2 would choose in response does not contain x. If P_2 is to have a winning strategy for A, then that strategy must reject each point of A at some game state.

For any game state $C_0 \supseteq C_1 \supseteq ... \supseteq C_n$ we must show that the set R of points of X which P_2 can reject is meager. First, R is closed in the relative topology of C_n, because if $x \in C_n \backslash R$, then there is some possible choice B for P_1 containing x such that $x \in \tau(C_0, C_1, ..., C_n, B)$. That is, $\tau(C_0, C_1, ..., C_n, B)$ is an open ball around x containing only points which are not in R. Second, we show that R has empty interior. If we suppose on the contrary that the interior is nonempty, then there is a ball $\widehat{B} \subseteq R$ which P_1 could play after $C_0, C_1, ..., C_n$. But then P_2's response $\tau(C_0, C_1, ..., C_n, \widehat{B})$ is a set of points which are certainly not rejected at the specified game state, contradicting $\tau(C_0, C_1, ..., C_n, \widehat{B}) \subseteq \widehat{B} \subseteq R$. Thus R is meager in the relative topology of C_n, so it is meager in X. Since there are only countably many game states (they are indexed by $\omega^{<\omega}$), this proves that the set of points a strategy for P_2 can reject from any state is meager. So if P_2 has a winning strategy, A must be meager.

The intermediate conclusion we have come to is that AD implies every subset $S \subseteq X$ is either meager or there is some open set $B \in \mathscr{B}$ such that $S^c \cap B$ is meager. What we really wish to prove is that A has the Property of Baire. If P_2 has a winning strategy, then A is meager, so is in BP. Otherwise, there is some $B \in \mathscr{B}$ such that $A^c \cap B$ is meager; let G be the union of all such B. Then G is open and $A^c \cap G$ is meager, as it is a countable union of meager sets $A^c \cap B$. We claim that $A \cap G^c$ is also meager, or else (by the intermediate conclusion) there would be some $\widehat{B} \in \mathscr{B}$ such that $(A \cap G^c)^c \cap \widehat{B}$ is meager. This would imply both $G \cap \widehat{B}$ and $A^c \cap \widehat{B}$ are meager, so by definition of G, $\widehat{B} \subseteq G$. Thus the supposedly meager set $G \cap \widehat{B}$ is just $\widehat{B}$—but no nonempty open set

[10]These conditions are actually *if and only if*. For meager A, a winning strategy for P_2 can be constructed by going through the proof of the Baire Category Theorem.

is meager by the Baire Category Theorem. The contradiction proves $A \cap G^c$ is meager, thus $A \triangle G$ is meager, so A has the Property of Baire. $\square$

Theorem 7.7 (Lebesgue Measurability)**.** *Assuming the Axiom of Determinacy, all subsets of* $\mathbb{R}^d$ *are Lebesgue measurable.*

Proof. It suffices to prove that all subsets of the unit cube $[0,1]^d$ are measurable. In turn, it suffices (by Lemma 3.5(b)) to show that AD implies any set $A \subseteq [0,1]^d$ whose every measurable subset is null must itself be null. We give the proof for $d = 1$, then outline a simple modification for higher dimensions which would unfortunately clutter up notation if written out in full. For any $\varepsilon > 0$, the goal is to show that such a set A can be covered by an open set with measure less than ε. The idea is to construct a game in which P_1 picks bits 0 or 1 for a real number $x \in A$, while P_2 picks quickly shrinking open sets in an attempt to cover x. By the hypothesis that A is "not large," we will be able to show that P_1 has no winning strategy; by Determinacy, the only other possibility is that P_2 has a winning strategy, from which we will conclude that A is in fact "small."

So we set down rules for the **covering game**. Take the usual countable base $\mathscr{B}$ for the topology on $[0,1]$ of (relatively) open intervals with rational endpoints. For all n, let $\mathscr{B}_n$ be the set of finite unions of elements of $\mathscr{B}$ with measure less than $2^{-2n-2}\varepsilon$. Each round of the game, P_1 chooses $x_n \in \{0,1\}$, and P_2 chooses an open set $G_n \in \mathscr{B}_n$. Say P_1 wins exactly when the binary expansion of these choices $\sum_{n=0}^{\infty} 2^{-n-1} x_n$ is in A but not in the union of sets chosen by P_2, $\bigcup_{n=0}^{\infty} G_n$. Since each $\mathscr{B}_n$ is countable, the covering game can be implemented as a Baire space game, and in particular its determinacy follows from AD.

We now suppose for the sake of contradiction that P_1 has a winning strategy σ for this game. In particular, σ must beat the "noninteractive" strategies in which P_2 has already decided which sets to pick each round. That is, for all $y \in \omega^\omega$, σ beats the strategy by which P_2 plays the $y(n)$-th member of $\mathscr{B}_n$ in the n-th round for all n. Consider the map $f : \omega^\omega \to [0,1]$ which takes $y \in \omega^\omega$ to the real number that P_1 would play by σ if P_2 is playing by the strategy corresponding to y. This map is continuous, so its image is analytic and hence measurable by Theorem 5.6. Since σ is a winning strategy, f takes ω^ω into A, so $f(\omega^\omega)$ is null by the assumption on A. As such, there is some sequence of open sets $G_n \in \mathscr{B}_n$ such that $f(\omega^\omega) \subseteq \bigcup_{n=0}^{\infty} G_n$. But this means that the "noninteractive" strategy in which P_2 picks exactly these G_n's each round beats σ, which is a contradiction.

Now by the Axiom of Determinacy, P_2 has a winning strategy τ, so that P_2 can cover any $x \in A$ "as it is played." We introduce a bit of temporary notation. For any nonempty finite sequence $s = \big(s(0), \ldots, s(n)\big)$ in $2^{<\omega}$, consider the sequence of sets $G_0, \ldots, G_n$ with which P_2 would respond, playing by τ. These sets depend only on s and τ (which is fixed), so we may label the last one G_n as $G(s)$. For all s of length $n+1$, we have $G(s) \in \mathscr{B}_n$, so that the measure of $G(s)$ is less than $2^{-2n-2}\varepsilon$. Notation in place, to say that τ is a winning strategy means

$$A \subseteq \bigcup_{s \neq \emptyset} G(s).$$

This set on the right can be rewritten

$$\bigcup_{n=0}^{\infty} \bigcup_{|s|=n+1} G(s),$$

and this set has measure less than

$$\sum_{n=0}^{\infty} \sum_{|s|=n+1} 2^{-2n-2}\varepsilon = \sum_{n=0}^{\infty} 2^{n+1} \cdot 2^{-2n-2}\varepsilon = \varepsilon.$$

This proves that A has outer measure less than ε, and because ε was arbitrary, A is null. Thus the proof is complete for $d = 1$.

For $d > 1$, $\mathscr{B}$ is the countable set of (relatively) open balls of rational radii around points whose coordinates in $[0,1]^d$ are all rational, and the $\mathscr{B}_n$ are defined analogously. Much as before, P_1 picks $x_n \in \{0,1\}$ each round while P_2 picks an open set $G_n \in \mathscr{B}_n$. All we need is a way to interpret the sequence $(x_0, x_1, x_2, ...) \in \mathscr{C}$ as a point of $[0,1]^d$. This can be done by the following map:

$$(x_0, x_1, x_2, ...) \mapsto \left(\sum_{k=0}^{\infty} 2^{-k-1} x_{dk},\ \sum_{k=0}^{\infty} 2^{-k-1} x_{1+dk},\ \ldots,\ \sum_{k=0}^{\infty} 2^{-k-1} x_{(d-1)+dk} \right).$$

The rest of the proof goes through unchanged. □

References

[B77] Charles Blair. The Baire category theorem implies the principle of dependent choices. *Bull. Acad. Polon. Sci.* **25** (1977) 933-934.

[F99] Gerald Folland. *Real Analysis: Modern Techniques and Their Applications.* 2nd ed. John Wiley & Sons, Inc. 1999.

[GS53] David Gale and Frank Stewart. Infinite games with perfect information. *Contributions to the theory of games, Ann. Math. Stud.* **28** Princeton University Press 1953.

[GO03] Bernard Gelbaum and John Olmstead. *Counterexamples in Analysis.* Dover 2003.

[J73] Thomas Jech. *The Axiom of Choice.* North-Holland 1973.

[J02] Thomas Jech. *Set Theory.* 3rd ed. Springer-Verlag Berlin Heidelberg New York 2002.

[K84] Alexander Kechris. The axiom of determinacy implies dependent choices in $L(\mathbb{R})$. *J. Symbolic Logic.* **49** (1984) 161-173.

[K95] Alexander Kechris. *Classical Descriptive Set Theory.* Springer-Verlag New York 1995.

[L85] Henri Lebesgue. Preface to Luzin's book "Lektsii ob analiticheskikh mnozhestvakh i ikh prilozheniyakh." *Russ. Math. Surv.* **40** (1985) 9-14.

[Lé79] Azriel Lévy. *Basic Set Theory.* Springer-Verlag Berlin Heidelberg New York 1979.

[Lu27] Nikolai Luzin. Sur les ensembles analytiques. *Fund. Math.* **10** (1927) 1-95.

[M02] David Marker. *Descriptive Set Theory.* Lecture notes Fall 2002. Available online `http://www.math.uic.edu/~marker/math512/dst.pdf` Accessed 2008.

[Ma75] Donald Martin. Borel Determinacy. *Ann. Math.* **102** (1975) 363-371.

[Mo80] Yiannis Moschovakis. *Descriptive Set Theory.* North-Holland 1980.

[My64] Jan Mycielski. On the axiom of determinateness. *Fund. Math.* **53** (1964) 205-224.

[MS64] Jan Mycielski and Stanisław Świerczkowski. On the Lebesgue measurability and the axiom of determinateness. *Fund. Math.* **54** (1964) 67-71.

[O71] John Oxtoby. *Measure and Category.* Springer-Verlag New York 1971.

[SF96] Raymond Smullyan and Melvin Fitting. *Set Theory and the Continuum Problem.* Oxford University Press, 1996.

[W09] Wikipedia. Polish space. Available online http://en.wikipedia.org/wiki/Polish_space Accessed 2009.