STA261: Problems 3

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This assignment is not for credit. Complete the questions as preparation for the quiz in tutorial 3 on July 16th. The questions on the quiz will be very similar to the questions on the assignment.

1. Sufficiency: Suppose random variable X has density $f_X(x;\theta)$ depending on parameter θ, we have a random sample of independent and identically distributed $X_i \overset{d}{=} X$, and a sufficient statistic for θ $T_1(\mathbf{X})$. Let $r(\cdot)$ be an invertible function, and $T_2 = r(T_1)$. Prove that T_2 is sufficient for θ.

2. Sufficiency: Show the following estimators are sufficient for their respective population parameters, for the following independent random samples and corresponding distributions. If you use the factorization theorem, be sure to state the functions $g(\hat{\theta}, \theta)$ and $h(\mathbf{x})$.

   (a) $X_i \sim Gamma(\alpha, \beta)$ with density $f_{x_i}(x_i) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x_i^{\alpha-1} e^{-\frac{x_i}{\beta}}$, $(\hat{\alpha}, \hat{\beta}) = \left(\prod_{i=1}^n x_i, \sum_{i=1}^n x_i\right)$
   (b) $X_i \sim N(\mu, \sigma)$, $(\hat{\mu}, \hat{\sigma}) = \left(\sum_{i=1}^n x_i, \sum_{i=1}^n x_i^2\right)$
   (c) $X_i \sim N(\mu, \sigma)$, $(\hat{\mu}, \hat{\sigma}) = \left(\bar{x}, \bar{x^2}\right)$
   (d) $X_i \sim Beta(\alpha, \beta)$ with $f_{x_i} = \frac{\Gamma(\alpha+\beta)}{\Gamma\alpha\Gamma\beta} x_i^{\alpha-1} (1-x_i)^{\beta-1}$, $\left(\hat{\alpha}, \hat{\beta}\right) = \left(\prod_{i=1}^n x_i, \prod_{i=1}^n (1-x_i)\right)$
   (e) $X_i \sim Beta(\alpha, \beta)$ as before, $\left(\hat{\alpha}, \hat{\beta}\right) = \left(\sum_{i=1}^n \log x_i, \sum_{i=1}^n \log(1-x_i)\right)$

3. Sufficiency. For $X_i \sim Unif(a, b)$, the continuous uniform distribution on (a, b), find a sufficient statistic for (a, b). Hint: the density is only defined over a certain subset of $\mathbb{R}$, what is it? Make sure to include the corresponding indicator function of the support when you write out the density, i.e.

$$f_{x_i}(x_i) = \frac{1}{b-a} \times I(support)$$

4. Show that the following two statistics are sufficient for any parameter from any distribution:

   (a) The full dataset, $\mathbf{x} = (x_1, \ldots, x_n)$
   (b) The order statistics, which are just the ordered sample values $(x_{(1)}, \ldots, x_{(n)})$ with $x_{(1)} \leq \ldots \leq x_{(n)}$

5. State and prove the factorization theorem.