STA261: Lecture 10

Hypothesis Tests & Confidence Intervals I

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Disclaimer

The materials in these slides are intended to be a companion to the course textbook, *Mathematical Statistics and Data Analysis, Third Edition*, by John A Rice. Material in the slides may or may not be taken directly from this source. These slides were organized and typeset by Alex Stringer.

A big thanks to Jerry Brunner as well for providing inspiration for assignment questions.

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Hypothesis Tests

We built up the following procedure for hypothesis tests:

- Decide on a **null hypothesis** H_0, which we did in the form of a candidate value θ_0 for parameter θ
- Form a **test statistic** which has a known sampling distribution if H_0 is in fact true, called its **null distribution**
- Assume H_0 is true, and calculate a **p-value**: the probability of observing a dataset giving a test statistic as extreme as what we did see, the crucial part being the *assumption that* H_0 *is true*
- Reject H_0 if $p_0 < \alpha$, a pre-determined **significance level** that directly controls our chance of making a **Type I Error** (false positive)

Hypothesis Tests

This was done by comparing likelihoods, leading to a **Likelihood Ratio Test**.

This is very powerful, but its sampling distribution is approximate.

Recall the coin toss example: flip a coin 10 times, get 7 heads, want to know if the coin is fair. With such a low number of flips, we might question whether the approximate sampling distribution is that accurate.

Furthermore, since there are only 11 possible outcomes of that experiment, there are only 11 possible values for the likelihood ratio- it's distribution isn't even close to *continuous*!

Can we get an exact test of this hypothesis?

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Normal Approximation

We have previously proved another approximate result (that's exact for the normal distribution):

If $\hat{\theta}$ is the MLE for θ, θ_0 is the true value, $j(\hat{\theta})$ is the observed information at $\hat{\theta}$ and $i(\theta_0)$ is the Expected Information at θ_0, then

$$\frac{\hat{\theta} - \theta_0}{1/\sqrt{i(\theta_0)}} \dot{\sim} N(0,1)$$

Also,

$$\frac{\hat{\theta} - \theta_0}{1/\sqrt{j(\hat{\theta})}} \dot{\sim} N(0,1)$$

Normal Approximation

This can be used to compute approximate p-values just like the LRT procedure. Let's compare the results for our previous examples:

$X_i \overset{IID}{\sim} N(\mu, 1)\ldots$ (exact)

$X_i \overset{IID}{\sim} N(\mu, \sigma^2)\ldots$ (approximate... or exact?)

$X_i \overset{IID}{\sim} Exp(\theta)\ldots$

$X_i \overset{IID}{\sim} Bern(\theta)\ldots$

Confidence Intervals

A natural question to ask is: what values of $\hat{\theta}$ would cause me to reject H_0? That is, for which values of $\hat{\theta}$ would p_0 be less than α?

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A related question is: can we find an *interval* $(L(\mathbf{X}), U(\mathbf{X})) \subset \Omega$, depending on the data $\mathbf{X}$, such that the interval contains/covers θ_0 with high probability? Say, probability $1 - \alpha$?

...

Confidence Intervals

We just found an approximate $1 - \alpha$ **Confidence Interval** for θ:

$$] \left(\hat{\theta} - \frac{1}{\sqrt{j(\hat{\theta})}} z_{1-\alpha/2}, \hat{\theta} + \frac{1}{\sqrt{j(\hat{\theta})}} z_{1-\alpha/2} \right)$$

where $z_{1-\alpha/2}$ is the $1 - \alpha/2$ quantile of a $N(0, 1)$ distribution, $P(Z < z_{1-\alpha/2}) = 1 - \alpha/2$ for $Z \sim N(0, 1)$.

We replaced (estimated) $i(\theta_0)$ with $j(\hat{\theta})$ because of the difficulties in isolating θ_0 in the middle of the interval.

Interpretation

If $\alpha = 0.05 = 5\%$, say, we say that $1 - \alpha = 95\%$ of intervals calculated in this way would contain the true value θ_0.

It is mathematically equivalent but *very misleading* to say that

$$P(\theta_0 \in (L(\mathbf{X}), U(\mathbf{X}))) = 1 - \alpha$$

because θ_0 is *not random*, the interval $(L(\mathbf{X}), U(\mathbf{X}))$ is. We typically don't write probability statements led with non-random quantities, mostly for clarity.

Examples

If $X_i \overset{IID}{\sim} Exp(\theta)$ as in the bus waiting-times example from before, find a 95% confidence interval for θ_0, the average waiting time, if you wait for 30 days and observe an average waiting time of 6 minutes.

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Examples

Now let $X_i \overset{IID}{\sim} Exp(\beta)$ with the "rate" parameterization, i.e. $E(X) = 1\beta$. We could repeate the above steps to get a confidence interval for β. Or, could we use the above answer?

. . .

Examples

Suppose the number of patients arriving at an emergency room in an hour is $Possion(\lambda)$ distributed. The hospital hires you to tell them when the average number of patients in an hour can reasonably be expected to exceed 10 based on current data.

You watch for two full days and observe an average of 7 patients per hour. What do you tell the hospital?

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Examples

If $X_i \overset{IID}{\sim} N(\mu, 1)$, calculate an *exact* 95% confidence interval for μ, using the fact that

$$\frac{\bar{X} - \mu_0}{1/\sqrt{n}} \sim N(0,1)$$

...

Examples

If $X_i \overset{IID}{\sim} N(0, \sigma^2)$, calculate an *exact* 95% confidence interval for σ^2, using the fact that

$$\frac{ns_n^2}{\sigma^2} \sim \chi_n^2$$

where

$$s_n^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$$

Examples

If $X_i \overset{IID}{\sim} N(0, \sigma^2)$, calculate an *exact* 95% confidence interval for σ^2, using the fact that

$$\frac{(n-1)s_{n-1}^2}{\sigma^2} \sim \chi_{n-1}^2$$

where

$$s_{n-1}^2 = \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2$$

Examples

If $X_i \overset{IID}{\sim} N(\mu, \sigma^2)$, calculate an *exact* 95% confidence interval for μ, using the fact that

$$\frac{\bar{X} - \mu_0}{s_{n-1}/\sqrt{n}} \sim t_{n-1}$$

...

Examples

In the news you hear "this poll is accurate to plus/minus 2 percentage points, 19 times out of 20".

Suppose the poll is to estimate some unknown quantity p, and that it was based off of an IID random sample from a relevant population. Interpret the above claim statistically.

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