STA261: Problems 5

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This assignment is not for credit. Complete the questions as preparation for the quiz in tutorial 5 on July 23rd. The questions on the quiz will be very similar to the questions on the assignment.

1. t-distribution. Let $Z \sim N(0,1)$, $U \sim \chi^2_k$, and $Z \perp U$. In this question you will derive the density of

$$T = \frac{Z}{\sqrt{U/k}}$$

   (a) Use the change of variables formula (textbook, section 2.3, page 62, proposition B) to show that the density of $V = \sqrt{U/k}$ is

$$f_V(v) = \frac{k^{k/2}}{\Gamma(k/2)2^{k/2-1}} v^{k-1} e^{-kv^2/2}$$

   Note that $U > 0$ with probability 1, so the transformation here is invertible.

   (b) Recall the formula for the density of the quotient of two independent random variables (textbook section 3.6.1, page 98): if $Z = Y/X$, and $X > 0$ with probability 1, then

$$f_Z(z) = \int_0^\infty x f_X(x) f_Y(xz) dx$$

   (I added the condition that $X > 0$ with probability one, so $f_X(x) = 0$ if $x < 0$, to simplify the formula for use in this problem). Use this and the independence of Z and U to show that the density of $T = Z/V$ is given by

$$f_T(t) = \frac{\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{k\pi}\,\Gamma(k/2)} \left(1 + \frac{t^2}{k}\right)^{-\frac{k+1}{2}}$$

   You can use the fact that if $X \sim Gamma(\alpha, \beta)$,

$$f_X(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}$$

   which lets you evaluate expressions of the form

$$\int_0^\infty x^{\alpha-1} e^{-x/\beta} dx = \Gamma(\alpha)\beta^\alpha$$

2. The professor who taught me introductory statistics (see youtube channel: jbstatistics) went out and bought $n = 16$ boxes of Raisin Bran and measured their weights, with the goal of testing Kellogs' claim that the mean weight of cereal present in boxes of Raisin Bran is 755g (family size). I don't have these data but here are some synthetic datapoints representing the weights (in grams) of some randomly purchased boxes of family size raisin bran:

$$782.1, 765, 708.9, 768.6, 751.5, 732, 763.3, 766.5, 768.7, 749.2, 750.5, 748.4, 781.8, 779, 775.2, 746.9$$

   (a) Compute $\bar{X}$ and s_{n-1}. I would recommend doing it by hand (with a calculator), since you might have to do something similar on a test.

   (b) Assume that

$$T = \frac{\bar{X} - \mu}{s_{n-1}/\sqrt{n}} \sim t_{n-1}$$

and show that

$$P\left(\bar{X} - \frac{s_{n-1}}{\sqrt{n}} \times t_{n-1,1-\alpha/2} < \mu < \bar{X} + \frac{s_{n-1}}{\sqrt{n}} \times t_{n-1,1-\alpha/2}\right) = 1 - \alpha$$

where $0 < \alpha < 0.5$, and $t_{n-1,1-\alpha/2}$ is the $1 - \alpha/2$ quantile of a t_{n-1} distribution, i.e. the value such that

$$P(t_{n-1} < t_{n-1,1-\alpha/2}) = 1 - \alpha/2$$

For example if $n = 16$ and $\alpha = 0.05$, you can run the command `qt(0.975,15)` in `R` to obtain $t_{15,0.975} = 2.13$

(c) Compute the above interval for these data (i.e. plug in $\bar{X}$ and s_{n-1}, you should get a pair of numbers).

(d) Do you think it is reasonable to suggest that $\mu = 755$ from these data? Why or why not?

3. Let $X_i \sim Unif(0, \theta)$, be an IID sample from a continuous uniform distribution, having density $f_X(x) = (1/\theta), 0 < x < \theta$. Since the maximum value X can take is θ, I think a good estimator of θ would be $\hat{\theta}_1 = X_{(n)}$, the sample maximum, i.e. $X_{(n)} = \max(X_1, \ldots, X_n)$.
   (a) Show that $X/\theta \sim Unif(0, 1)$
   (b) Let $W_n = X_{(n)}/\theta$, where $X_{(n)}$ is the sample maximum defined in the question. Show that W_n has cumulative distribution function $F_{W_n}(w) = w^n$ and density function $f_{W_n}(w) = nw^{n-1}$.
   (c) Find $E(W_n)$ and $Var(W_n)$ (remember, $Var(X) = E(X^2) - E(X)^2$), and use these to find $E(X_{(n)})$ and $Var(X_{(n)})$
   (d) Suggest an estimator $\hat{\theta}_2$ of θ that satisfies $E(\hat{\theta}_2) = \theta$.
   (e) Evaluate the variance of your estimator, and compare it to the variance of $\hat{\theta}_1 = X_{(n)}$.
   (f) Which estimator would you prefer, say, for a sample size of 5? Of 50? 500? There is no "right" or "wrong" answer here.