UNIVERSITY OF TORONTO
Faculty of Arts and Science
August 2019 Examinations
STA261H1S
Probability and Statistics II
Duration: 3 hours
Aids Allowed: Non-programmable calculator

First Name:____

Last Name:____

Student Number:_____________________________________

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Question	Out of	Question	Out of
1	4	4	4
2	4	5	6
3	4		
Total	30		

1. (4) Let $X_1, \ldots, X_n$ be an IID sample from a standard Normal distribution having density $f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right)$. Let $S_n = X_1 + \cdots + X_n$.

a) (1) Show that the moment generating function of X_1 is $M_X(t) = \exp(t^2/2)$.

b) (.5) Show that $M_X(t)$ is a fixed point of the transformation $\phi(x) \to \phi(x/\sqrt{n})^n$, that is, $M_X(t/\sqrt{n})^n = M_X(t)\ \forall\ n \in \mathbb{N}$.

c) (1.5) Show that $S_n/n \xrightarrow{p} 0$. For full marks show all work including computing all necessary quantities and stating any theorems you use and what conditions must be true for them to hold.

d) (1) Show that $S_n/\sqrt{n} \xrightarrow{d} \text{Normal}(0,1)$ directly, i.e. without just invoking the Central Limit Theorem.

2. (4) Suppose $X_1, \ldots, X_n$ is an IID random sample from a distribution with density f_θ and parameter $\theta \in \Theta$. We put a prior distribution $\pi(\theta)$ on θ.

a) (1) Define (mathematically) what it means for a statistic $T = T(X)$ to be sufficient for θ.

b) (1) Give a mathematical expression for the posterior distribution of $\theta | X_1, \ldots, X_n$.

c) (2) Show that T is sufficient if and only if the posterior distribution of $\theta | X_1, \ldots, X_n$ is equal to the posterior distribution of $\theta | T(X)$ for every $\theta \in \Theta$ and for every choice of prior.

3. (4) Suppose we observe an IID random sample $X_1, \dots, X_n$ from a Binomial(n, θ) distribution, with $P(X = x) = \binom{n}{x}\theta^x(1-\theta)^{n-x}$. We wish to perform Bayesian inference on θ.

a) (.5) The beta distribution with parameters $\alpha > 0, \beta > 0$ has density $f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$ for $0 < x < 1$. Use this fact to evaluate the integral $\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx$.

b) (1) Suppose we put a Beta(α, β) prior on θ. Evaluate the normalizing constant, i.e. the denominator of the posterior distribution.

c) (.5) Define what is meant by a conjugate prior distribution.

d) (1) Show that the posterior distribution of $\theta|X_1, \ldots, X_n$ under the prior from b) is

$$\text{Beta}\left(\alpha + \sum_{i=1}^{n} X_i, \beta + n - \sum_{i=1}^{n} X_i\right)$$

e) (1) Write down an expression for a normal approximation to the posterior for this example. You don't have to compute any likelihood-related quantities expicitly, I'm just looking for a general formula.

4. (8) Let Y_i be an independent but **not identically distributed** random sample from the family of normal distributions, $Y_i \overset{IND}{\sim} \text{Normal}(\mu_i, \sigma_0^2)$ where σ_0^2 is known. Suppose we also observe fixed covariates x_i, one for each Y_i, and we wish to estimate μ_i according to the linear model $\mu_i = \beta x_i$ for some fixed, unknown parameter $\beta \in \mathbb{R}$. To repeat: x_i are fixed and known; β is fixed and unknown. We are estimating β only.

a) (2) With σ_0^2 a fixed, known constant, write down the likelihood and log-likelihood for β.

b) (2) Write down the score function for β and find the maximum likelihood estimator for β.

c) (2) Find the observed information and expected information for β and the variance of the maximum likelihood estimator for β.

d) (2) If we didn't know σ_0^2 but wished to estimate it together with β, would our estimates for β change? What about its estimated variance? You may (actually, you should) use facts about normal families to answer this question; you should not do all the calculations again.

5. (6) Suppose we have data $X_1, \ldots, X_n$ independently and identically distributed according to a parametric statistical model $\{F_\theta : \theta \in \Theta\}$. Let $U = U(X)$ be an unbiased estimator for θ.

a) (2) What is $\mathbb{E}(U)$?

b) (1) Let $T = T(X)$ be a sufficient statistic. Define a new estimator $S(X) = \mathbb{E}(U(X)|T(X))$. Show that $S(X)$ is unbiased. *Hint: recall that for any random variables* X, Y, $\mathbb{E}(X) = \mathbb{E}\left(\mathbb{E}(X|Y)\right)$.

c) (1) Show that $S(X)$ has variance which is at least as small as the variance of $U(X)$. *Hint: for any random variables* X, Y, $\mathrm{Var}(X) = \mathrm{Var}(\mathbb{E}(X|Y)) + \mathbb{E}\left(\mathrm{Var}(X|Y)\right)$. *Variance is non-negative.*

d) (2) Suppose, in addition, $T(X)$ has the property that for any function h, $\mathbb{E}\left(h(T(X))\right) = 0 \implies h \equiv 0$, called "completeness". Prove that $S(X)$ has minimum variance out of all unbiased estimators. This question is very hard, but give it a shot anyways— I believe in you. *Hint: define two unbiased estimators $U_1(X)$ and $U_2(X)$. Condition them on $T(X)$, and subtract the result from each other. Use completeness. How does what you get imply the result?*

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