STA261: Problems 2

Alex Stringer

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This assignment is not for credit. Complete the questions as preparation for the quiz in tutorial 2 on July 11th. The questions on the quiz will be very similar to the questions on the assignment.

1. Consistency. For independent random samples from the following families of distributions, show that the given estimator is consistent for the population parameter.
   (a) $X_i \sim Pois(\lambda)$, $\hat{\lambda} = \bar{X}$.
   (b) $X_i \sim Exp(\theta)$, where $f_\theta(x) = \theta e^{-\theta x}$, $\hat{\theta} = 1/\bar{X}$.
   (c) $X_i \sim Exp(\beta)$, where $f_\beta(x) = \frac{1}{\beta}e^{-\frac{x}{\beta}}$, $\hat{\beta} = \bar{X}$
   (d) $X_i \sim \chi^2_\nu$, $\hat{\nu} = \bar{X}$
   (e) $X_i \sim Gamma(\alpha, \beta)$ with $f_{\alpha,\beta}(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha-1}e^{-\frac{x}{\beta}}$, $\hat{\alpha} = \frac{(\bar{X})^2}{s^2}$ and $\hat{\beta} = \frac{s^2}{\bar{X}}$, where $s^2 = \frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})^2$. Note you can use $\frac{1}{n-1}$ in the sample variance if you want- they both give a consistent estimator of σ^2.
2. Consistency: For $X_i \sim N(\mu, \sigma)$, state and prove whether each estimator is consistent or not. Be sure to say exactly where you are assuming a function is continuous in your proofs.
   (a) $\hat{\mu} = \bar{X}$
   (b) $\hat{\sigma} = s = \sqrt{\frac{1}{n-1}\sum_{i=1}^n \left(X_i - \bar{X}\right)^2}$
   (c) $\hat{\sigma} = \sqrt{\frac{1}{n}\sum_{i=1}^n \left(X_i - \bar{X}\right)^2}$
   (d) $\hat{\mu} = \frac{1}{n+1{,}000{,}000}\sum_{i=1}^n X_i$
   (e) $\hat{\mu} = \frac{1}{\frac{1}{n}\sum_{i=1}^n \frac{1}{X_i}}$
   (f) $\hat{\sigma^2} = \frac{1}{\frac{1}{n}\sum_{i=1}^n \frac{1}{X_i^2} - \left(\frac{1}{n}\sum_{i=1} \frac{1}{X_i}\right)^2}$
3. Method of Moments. For independent random samples from the following families of distributions, find a consistent estimator for the population parameter using the method of moments. Some of these are tricky; if you need more practice, first try doing the previous questions in reverse. The method of moments is just a reverse consistency proof, as discussed in lecture.
   (a) The gamma distribution as asked in question 1. Note the support of x is $x > 0$.
   (b) $X_i \sim Beta(\alpha, \beta)$ with $f_{\alpha,\beta}(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$, $x \in (0,1)$.
   (c) $X_i \sim LogNormal(\mu, \sigma)$, with $f_{\mu,\sigma}(x) = \frac{1}{x\sigma\sqrt{2\pi}}e^{-\frac{(\log x - \mu)^2}{2\sigma^2}}$, $x > 0$
   (d) $X \sim Unif(0, \beta), x \in (0, \beta)$
4. Method of Moments. Let $X_i \sim Unif(\alpha, \beta)$. Find a method of moments estimator of (α, β). This question is messy.