STA261: Midterm Solutions

Section L0101

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1. (a) Get this from first principles:

$$\begin{aligned} E(I(Y < y)) &= (1) \times P(I(Y < y) = 1) \times (0) \times P(I(Y < y) = 0) \\ &= P(Y < y) \end{aligned}$$

   (b) By the LLN,

$$\frac{1}{n}\sum_{i=1}^{n} I(Y_i < y) \xrightarrow{p} E(I(Y_i < y))$$

   We showed in part a) that $E(I(Y_i < y)) = P(Y < y) = F(y)$. Hence $\hat{F}(y)$ is a consistent estimator of $F(y)$. Take off 2 marks if the LLN is not mentioned. If the students try to solve this question in another way, they can get full marks if their answer is correct, but this was the intended way to solve this question.

2. (a)
   - 3 marks for solving the equations

$$\beta = \frac{Var(X)}{E(X)}$$
$$\alpha = \frac{E(X)^2}{Var(X)}$$

     Don't take off marks if students did the question in terms of $E(X^2)$ instead of $Var(X)$, even if they forget to simplify $E(X^2) - E(X)^2$ into $Var(X)$.
   - 3 marks for plugging sample moments in for population moments

$$\hat{\beta} = \frac{s^2}{\bar{X}}$$
$$\hat{\alpha} = \frac{\bar{X}^2}{s^2}$$

   (b)
   - 2 marks for solving the equation

$$\log \beta = E(\log X) - \psi(1)$$

   - 2 marks for plugging in $\frac{1}{n}\sum_{i=1}^{n} \log X_i$ for $E(\log X)$.

3. (a)

$$f(x_1,\ldots,x_n|\beta) = \prod_{i=1}^{n} \frac{1}{\beta^2} x_i \exp\left(-\frac{x_i}{\beta}\right)$$
$$= \beta^{-2n}\left(\prod_{i=1}^{n} x_i\right)\exp\left(-\frac{1}{\beta}\sum_{i=1}^{n} x_i\right)$$

(b) The joint density factors as

$$f(x_1,\ldots,x_n|\beta) = \left(\prod_{i=1}^{n} x_i\right) \times \beta^{-2n}\exp\left(-\frac{1}{\beta}\sum_{i=1}^{n} x_i\right)$$

so by the factorization theorem, $T_1 = \sum_{i=1}^{n} X_i$ is sufficient for β.

(c) $T_2 = \exp(-T_1)$, which is a one-to-one function of a sufficient statistic. Hence T_2 is sufficient.

4. (a) $\ell(p) = \sum_{i=1}^{n} \log\binom{m}{x_i} + \sum_{i=1}^{n} x_i \log p + (nm - \sum_{i=1}^{n} x_i)\log 1 - p)$

(b) The score function is

$$S(p) = \frac{\sum_{i=1}^{n} x_i}{p} - \frac{nm - \sum_{i=1}^{n} x_i}{1-p}$$

Setting to zero and solving gives $\hat{p} = \frac{\sum_{i=1}^{n} X_i}{nm}$.

(c) The observed information is minus the derivative of the score,

$$J(p) = \frac{\sum_{i=1}^{n} x_i}{p^2} + \frac{nm - \sum_{i=1}^{n} x_i}{(1-p)^2}$$

(d) The fisher information is the expectation of the observed information,

$$I(p) = \frac{nm}{p} + \frac{nm}{(1-p)}$$

(e) In this single parameter problem, the asymptotic variance of $\hat{p}$ is given by

$$Var(\hat{p}) = \frac{1}{I(p_0)} = \frac{1}{nm}\left(\frac{1}{p_0} + \frac{1}{1-p_0}\right)$$

That answer is fine- students may also have simplified to

$$Var(\hat{p}) = \frac{1}{I(p_0)} = \frac{p_0(1-p_0)}{nm}$$

(f)

$$\hat{p} \sim N\left(p_0, \frac{1}{I(p_0)}\right)$$