# STA261: Midterm

Section L5101

*February 14, 2018*

First Name:______________________________________________________________

Last Name:______________________________________________________________

Student Number:______________________________________________________________

- This midterm consists of 10 multiple choice questions and 3 written answer questions, for a total of 40 marks.
- This midterm contains 10 pages.
- Your TCard must be displayed on your desk at all times.
- **Circle your final answer to each problem. Questions that do not have a circled answer will receive zero marks**.
- All final answers must be written on the **front** of the page. Nothing written on the backs of pages will be marked. You should do your rough work on the back of the page first, then write and circle your final answer on the front of the page.
- Because of this requirement, note that the amount of space given to answer a question might be *much* more than is required.
- Do not write at the top of the page, above the question number. This will mess with the scanning of the QR code.
- Non-programmable calculators may be used. No other aids are permitted. No other papers are allowed on your desk
- If you need extra paper for writing, raise your hand. You must hand in all extra sheets of paper used. Nothing written on extra sheets of paper will be marked.
- Use pen. Questions answered in pencil will not be eligible for remark requests.

Each of the following *multiple choice* questions is worth 1 mark. Circle the letter corresponding to the correct option.

1. Maximum likelihood estimators
   (a) Depend on the data only through a sufficient statistic
   (b) are always unbiased
   (c) are asymptotically unbiased
   (d) (a) and (b)
   (e) (a) and (c)
2. Let $\hat{\theta}$ be any estimator of $\theta$, let $T$ be any sufficient statistic for $\theta$, and let $\tilde{\theta} = E(\hat{\theta}|T)$. Assuming that $\hat{\theta} \neq \tilde{\theta}$, we can say that
   (a) $Var(\hat{\theta}) < Var(\tilde{\theta})$
   (b) $Var(\hat{\theta}) > Var(\tilde{\theta})$
   (c) $Var(\hat{\theta}) \leq Var(\tilde{\theta})$
   (d) $Var(\hat{\theta}) \geq Var(\tilde{\theta})$
   (e) Impossible to say
3. The defintion of a *consistent* estimator $\hat{\theta}$ of $\theta$ is
   (a) $\hat{\theta} \xrightarrow{d} \theta$
   (b) $\hat{\theta} \xrightarrow{p} \theta$
   (c) $P(\hat{\theta} = \theta) = 1$
   (d) $P(\mathbf{X}|\hat{\theta})$ does not depend on $\theta$
4. Consider the random variable $X$ drawn from some population, with $E(X) = \mu$. We take a sample $X_i = 1 \ldots n$, and get $\hat{\mu} = \bar{X} = 3$. In this example, $\mu$ refers to a
   (a) Parameter
   (b) Estimator
   (c) Sufficient statistic
   (d) Estimate
5. In the above example, the number 3 refers to a
   (a) Parameter
   (b) Estimator
   (c) Sufficient statistic
   (d) Estimate

6. In the above example, $\bar{X}$ refers to a
   (a) Parameter
   (b) Estimator
   (c) Sufficient statistic
   (d) Estimate
7. Method of Moments estimators are always
   (a) Consistent
   (b) Sufficient
   (c) Unbiased
   (d) (a) and (b)
   (e) (b) and (c)
   (f) (a) and (c)
   (g) (a) and (b) and (c)
8. The *factorization theorem* states that for a sample $\mathbf{X}$ with joint distribution $f(\mathbf{X}|\theta)$,
   (a) $f(\mathbf{X}|\theta)$ doesn't depend on $\theta$
   (b) $f(\mathbf{X}|\theta)$ can be factored into the form $g(\hat{\theta}, \theta) \times h(\mathbf{x})$ for any estimator $\hat{\theta}$ of $\theta$
   (c) $f(\mathbf{X}|\theta)$ can be factored into the form $g(\hat{\theta}, \theta) \times h(\mathbf{x})$ if and only if $\hat{\theta}$ is not a function of $\mathbf{x}$
   (d) $f(\mathbf{X}|\theta)$ can be factored into the form $g(\hat{\theta}, \theta) \times h(\mathbf{x})$ if and only if $\hat{\theta}$ is sufficient for $\theta$
9. Consider some sample $\mathbf{x} = (x_1 \ldots x_n)$ with joint density $f(x_1, \ldots, x_n|\theta)$. The *likelihood function* is
   (a) The function that is closest to the true value $\theta$
   (b) The most *efficient* estimator of $\theta$
   (c) The joint density of the sample, treated as a function of $\theta$
   (d) The density of $\theta$ treated as a function of the sample
10. In the above question, the *maximum likelihood estimator* of $\theta$ is
   (a) The value of $\theta$ that generated the observed data
   (b) The value of $\theta$ that most likely generated the observed data
   (c) The estimator of $\theta$ that is based on the simplest possible sufficient statistic
   (d) Always the same as the Method of Moments estimator for $\theta$ if the data are IID

1. (10 marks) Let $X$ be a discrete random variable with mass function $P(x) = P(X = x)$. Let $X_i$, $i = 1 \dots n$ be an independent, identically distributed random sample from this distribution. For any fixed $x$, consider the random variable

$$I(X = x) = \begin{cases} 1 \text{ if } X = x \\ 0 \text{ if } X \neq x \end{cases}$$

(a) (4 marks) Show that $E(I(X = x)) = P(x)$.

(b) (6 marks) Show that

$$\hat{P}(x) = \frac{1}{n}\sum_{i=1}^{n} I(X_i = x)$$

is a *consistent* estimator of the mass function $P(x) = P(X = x)$.

2. (10 marks) Let $X_i \sim Gamma(\alpha, 2)$, $i = 1 \ldots n$ independently with density

$$f(x|\alpha) = \frac{1}{\Gamma(\alpha)2^\alpha} x^{\alpha-1} \exp\left(-\frac{x}{2}\right)$$

(a) (2 marks) Write down the joint density for this independent, identically distributed random sample.

(b) (4 marks) Use the *factorization theorem* to show that

$$T_1 = \prod_{i=1}^{n} X_i$$

is *sufficient* for $\alpha$.

(c) (4 marks) Show that

$$T_2 = \sum_{i=1}^{n} \log X_i$$

is *sufficient* for $\alpha$.

3. (10 marks) Let $X_i \sim Exp(\theta)$, $i = 1 \ldots n$ independently, with density

$$f(x|\theta) = \frac{1}{\theta} e^{-x/\theta}$$

The support is $X_i > 0$ and the parameter space is $\theta > 0$. You may use without proof that $E(X) = \theta$. You may also assume that the regularity conditions are satisfied.

(a) (2 marks) Write down the log-likelihood function for $\theta$ for this IID sample.

(b) (2 marks) Find the maximum likelihood estimator $\hat{\theta}$ for $\theta$.

(c) (2 marks) Find the *Observed Information* as a function of $\theta$.

(d) (1 mark) Let $\theta_0$ be the true value of $\theta$. Find the *Fisher Information* at $\theta = \theta_0$.

(e) (1 mark) Evaluate the *asymptotic variance* of $\hat{\theta}$.

(f) (2 marks) State the *large-sample* distribution of $\hat{\theta}$.