STA261: Problems 8

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This assignment is not for credit. Complete the questions as preparation for the quiz in tutorial 8 on August 1st. The questions on the quiz will be very similar to the questions on the assignment.

1. Suppose $X_i \sim N(\mu, \sigma^2)$ is an IID sample from a normal distribution with mean parameter μ and known variance σ^2. We are interested in developing a likelihood ratio procedure for seeing whether the observed data supports a candidate value $\mu = \mu_0$.
   (a) State the full parameter space Ω and its dimension.
   (b) State the restricted parameter space Ω_0 and its dimension.
   (c) Show that the likelihood ratio as defined in lecture can be written
   $$-2\log\Lambda = \sum_{i=1}^{n}\left(\frac{X_i - \mu_0}{\sigma}\right)^2 - \sum_{i=1}^{n}\left(\frac{X_i - \bar{X}}{\sigma}\right)^2$$
   (d) We proved in class that Λ has an approximate χ^2_1 distribution when defined this way. Show that in this example, $-2\log\Lambda$ has an exact χ^2_1 distribution:
   $$-2\log\Lambda \sim \chi^2_1$$
   Remember the proof from a few lectures back that $\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$? This is similar.
   (e) Show that the likelihood ratio simplifies to
   $$-2\log\Lambda = \left(\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}\right)^2$$
2. Consider the previous question, but now the variance is unknown.
   (a) Show that the likelihood ratio simplifies to
   $$-2\log\Lambda = \log\hat{\sigma}^2_0 - \log\hat{\sigma}^2$$
   (b) Show that it simplifies even further to
   $$-2\log\Lambda = n\log\left(1 + \frac{t^2}{n-1}\right)$$
   where $t^2 = \frac{(\bar{X}-\mu_0)^2}{s^2/n}$.
3. Show that if $X_i \overset{IID}{\sim} N(\mu, 1)$, the distribution of $-2\log\Lambda(\mathbf{x})$ is exact, not approximate. You already showed this in question 1 but just computing it and its distribution; now we do it using Taylor series like with the CLT for the MLE.
   (a) Write down the likelihood and the log-likelihood for μ.
   (b) Write $-2\log\Lambda(\mathbf{x}) = 2(\ell(\hat{\mu}) - \ell(\mu_0)$. Take a second order Taylor expansion of $\ell(\mu_0)$ about the point $\mu = \hat{\mu}$.
   (c) What is the remainder term equal to and why?
   (d) Explain why this completes the proof.
4. Suppose now that we have K random samples from normal distributions, all mutually independent, with different means and the same known variance. That is, we have
   $$X_{ij} \sim N(\mu_i, \sigma^2)$$
   $$i = 1 \ldots K$$
   $$j = 1 \ldots n$$
   We wish to develop a likelihood ratio for seeing whether the observed data supports all the means being equal to each other, $\mu_i = \mu_0$ for $i = 1 \ldots K$. We do not specify what μ_0 is- we estimate it from the data.

(a) State the full parameter space Ω.

(b) State the restricted parameter space Ω_0. What type of geometrical object is this?

(c) Show that the numerator of the likelihood ratio

$$L_0(\mu_0|\mathbf{x}) = c \times \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{K}\sum_{j=1}^{n}\left(x_{ij}-\mu_0\right)^2\right)$$

where c is some constant that does not depend on μ (you don't need to say what it is).

(d) Show that the (restricted) MLE of μ_0 under the assumption that $\mu_i = \mu_0$ is

$$\hat{\mu_0} = \frac{1}{nK}\sum_{i=1}^{K}\sum_{j=1}^{n} x_{ij} \equiv \bar{x}_{..}$$

(e) Show that the denominator of the likelihood ratio is

$$L_1(\mu_1,\ldots,\mu_K|\mathbf{x}) = c \times \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{K}\sum_{j=1}^{n}\left(x_{ij}-\mu_i\right)^2\right)$$

where c is some constant that does not depend on μ_0 (you don't need to say what it is).

(f) Show that the (unrestricted) MLEs of $\mu_1 \ldots \mu_K$ are

$$\hat{\mu_i} = \frac{1}{n}\sum_{j=1}^{n} x_{ij} \equiv \bar{x}_i$$

(g) Show that the likelihood ratio in this problem can be written

$$-2\log\Lambda = \frac{1}{\sigma^2}\left(\sum_{i=1}^{K}\sum_{j=1}^{n}\left(x_{ij}-\bar{x}_{..}\right)^2 - \sum_{i=1}^{K}\sum_{j=1}^{n}\left(x_{ij}-\bar{x}_i\right)^2\right)$$

(h) What is the distribution of $-2\log\Lambda$, including the relevant degrees of freedom, if $\mu_i = \mu_0$?