STA261 S19: Test 1

Please write your information clearly and legibly.

First Name:____

Last Name:____

Student Number:____

U of T Email:____

- No aids permitted except a non-programmable calculator,
- 60 minutes.
- Write all answers directly beneath where the question is asked.
- Use the backs of the pages and the final page for rough work. Write all your final answers directly below the question. **Only what is written in the space beneath the question will be marked. Use your space wisely.**
- Test is out of 10 marks. 4 marks are designated "basic" and test base knowledge. 4 marks are designated "adept" and test application of base knowledge to new problems. 2 marks are dedicated "advanced" and require in-depth understanding and problem solving skills. Use your time wisely.

1. *Basic, 4 marks*

a) (2) Define what it means for a sequence of random variables X_n to converge in probability to a random variable X.

b) (2) If $X_n \overset{i.i.d.}{\sim} \text{Normal}(0, \sigma^2)$ then X_n are independent, $\mathbb{E}X_n = 0$ and $\text{Var}(X_n) = \sigma^2 < \infty$. Let $S_n = X_1 + \cdots + X_n$. Show $S_n/n \xrightarrow{p} 0$. State all conditions of any theorem(s) you use and make sure to say why they are satisfied.

2. *Adept, 4 marks.* Suppose we want to evaluate a very complicated integral of a one-dimensional function $f : \mathbb{R} \to \mathbb{R}$:

$$I = \int_0^1 f(x)dx$$

f is too complicated to evaluate I analytically. One numerical method to evaluate I is as follows: sample $U_1, \ldots, U_n \overset{iid}{\sim} \text{Unif}(0, 1)$, and compute

$$\widehat{I} = \frac{1}{n} \sum_{i=1}^{n} f(U_i) \tag{0.1}$$

a) (1) Compute $\mathbb{E}(\widehat{I})$. The Unif(a, b) density is $g(x) = \frac{1}{b-a}$ for $a \leq x \leq b$.

b) (1) Compute $\text{Var}(\widehat{I})$.

c) (2) Show that $\widehat{I} \xrightarrow{p} I$. What conditions on f must be assumed for this to be true?

3. *Advanced, 2 marks.* Let $X_n \xrightarrow{p} X$. Suppose $f : \mathbb{R} \to \mathbb{R}$ be a L-Lipschitz continuous function, which means for some $L > 0$ and every $x, y \in \mathbb{R}$ we have

$$|f(x) - f(y)| \leq L|x - y| \tag{0.2}$$

Prove that $f(X_n) \xrightarrow{p} f(X)$.

THIS PAGE IS FOR ROUGH WORK. NOTHING WRITTEN ON THIS PAGE WILL BE MARKED.