STA261: Assignment 4

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January, 2018

This assignment is not for credit. Complete the questions as preparation for quizzes and tests.

Suggested reading: Textbook sections 8.5

Do the following problems from the textbook section 8.10 (starting on page 312): 4(c),(d); 5(b),(c); 7(a),(b),(c); 16; 17(c),(d),(e); 21(b),(c)

The textbook problems are especially important in this assignment.

1. *Maximum Likelihood*: Let $X_i \sim Unif(a, b)$.
   (a) Write down the likelihood.
   (b) Find the MLE $(\hat{a}, \hat{b})$ of (a, b).
   (c) Explain why you can't use the distributional results we proved in Lecture 4 in this example. Consider the "Regularity Conditions" from the lecture slides, and reference explicitly which one(s) is/are violated.

2. *Maximum Likelihood*: Let $X_i \sim$ *Laplace*(θ), with density $f_{x_i}(x_i) = \frac{1}{2}\exp\left(-|x_i - \mu|\right)$
   (a) Write down the likelihood.
   (b) Find the MLE $\hat{\mu}$ of μ.
   (c) Can you use the distributional results we derived in Lecture 4? Check each of the "Regularity Conditions" as in question 1.

3. *Likelihood Inference*: Prove the identity used in deriving the variance of the score function,

$$\frac{\partial}{\partial\theta}\int_x \frac{\partial \log f(x_i|\theta)}{\partial\theta} f(x_i|\theta)dx = \int_x \frac{\partial^2 \log f(x_i|\theta)}{\partial\theta^2} f(x_i|\theta)dx + \int_x \left(\frac{\partial \log f(x_i|\theta)}{\partial\theta}\right)^2 f(x_i|\theta)dx$$

   It looks frightening, but it's just calculus. Use the fact that

$$\frac{\partial \log f(x)}{\partial x} = \frac{1}{f(x)} \times \frac{\partial f(x)}{\partial x}$$

4. *Maximum Likelihood*: As on assignment 2, let $\epsilon_i \sim N(0, \sigma)$ be independent and identically distributed random variables. Let $\beta \in \mathbb{R}$ be a fixed, unknown constant and $x_i \in \mathbb{R}, i = 1 \dots n$ be fixed, *known* quantities. Let $Y_i = \beta x_i + \epsilon_i, i = 1 \dots n$.
   (a) Find the MLE for β

(b) What is its *exact* sampling distribution? Don't use any limiting approximations; work it out exactly. You can answer this by remembering a question from assignment 1 that dealt with the distribution of a sum of independent normal random variables.

(c) Find the MLE for σ^2

(d) What is its *exact* sampling distribution? Don't use any limiting approximations; work it out exactly. You can answer this by remembering a question from assignment 1 that dealt with the distribution of a sum of squares of independent normal random variables.

(e) Evaluate the Observed Information and Fisher Information matrices for $\hat{\beta}, \hat{\sigma}^2$. Invert the Fisher Information matrix (using the formula for the inverse of a 2×2 matrix); do the resulting variance estimates agree with the ones you derived exactly? Why or why not?

5. *Maximum Likelihood*: As in the previous question, let $\epsilon_i \sim N(0, \sigma)$ be independent and identically distributed random variables. Now, let $\beta_0, \beta_1 \in \mathbb{R}$ be fixed, unknown constants and $x_i \in \mathbb{R}, i = 1 \ldots n$ be fixed, *known* quantities. Let $Y_i = \beta_0 + \beta_1 x_i + \epsilon_i, i = 1 \ldots n$.

   (a) Find the MLE for $\boldsymbol{\beta} = (\beta_0, \beta_1)$

   (b) Find the MLE for σ^2

   (c) Find the Observed Information and Fisher information matrices for $(\hat{\beta}_0, \hat{\beta}_1, \hat{\sigma}^2)$. These are 3×3 matrices.