STA261 Summer 2018

Quiz 5

July 23rd, 2018

First Name: SOLUTIONS.

Last Name:____

Student Number:____

This quiz is out of 10 marks. Do ALL of your work on the back of the quiz, where the questions are. You can use the front for rough work, but nothing on the front will be marked, or even seen by the TAs.

If $X_i \sim Unif(0,\theta)$ then its density is $f_X(x) = (1/\theta), 0 < x < \theta$, and $X/\theta \sim Unif(0,1)$, and $W_n = X_{(n)}/\theta$ has cumulative distribution function $F_{W_n}(w) = w^n$ and density function $f_{W_n}(w) = nw^{n-1}$, where $X_{(n)}$ is the sample maximum, i.e. $X_{(n)} = \max(X_1, \dots, X_n)$.

BELOW SPACE IS FOR ROUGH WORK. NOTHING WRITTEN HERE WILL BE READ OR MARKED.

1. For W_n, $X_{(n)}$ and θ as defined on the front page,

(a) (4 marks) Find $E(X_{(n)})$ and $Var(X_{(n)})$.

$$W_n = X_{(n)}/\theta, \quad f_W(w) = n w^{n-1}$$

1 $$EW_n = \int_0^1 n w^{n} dw = \frac{n}{n+1} w^{n+1} \Big|_0^1 = \frac{n}{n+1}$$

$$EW_n^2 = \int_0^1 n w^{n+1} dw = \frac{n}{n+2} w^{n+2} \Big|_0^1 = \frac{n}{n+2}$$

1 $$Var(W_n) = \frac{n}{n+2} - \left(\frac{n}{n+1}\right)^2 = \frac{n(n+1)^2 - n^2(n+2)}{(n+2)(n+1)} = \frac{n^3+2n^2+n-n^3-2n^2}{(n+2)(n+1)} = \frac{n}{(n+2)(n+1)}$$

1 $$\Rightarrow E(X_{(n)}) = \theta E(W_n) = \frac{n}{n+1}\theta$$

1 $$Var(W_n) = Var(X_{(n)}/\theta) \Rightarrow Var(X_n) = \theta^2 Var(W_n) = \frac{n\theta^2}{(n+1)(n+2)}$$

(b) (2 marks) Suggest an estimator $\hat{\theta}$ of θ that satisfies $E(\hat{\theta}) = \theta$.

2 $$\hat{\theta} = \frac{n+1}{n} X_{(n)} \Rightarrow E\hat{\theta} = \frac{n+1}{n} EX_{(n)} = \theta$$

(c) (4 marks) Evaluate the variance of your estimator, and compare it to the variance of $X_{(n)}$ (say whether it is smaller or larger, or if you can't tell).

2

$$Var\,\hat{\theta} = \left(\frac{n+1}{n}\right)^2 Var(X_{(n)}) > Var(X_{(n)}) \ \forall n \in \mathbb{N}, n \geq 2$$

$$= \frac{(n+1)^2 n\theta^2}{(n+1)n^2(n+2)} = \frac{(n+1)\theta^2}{n(n+2)}$$ 2