STA261 Summer 2018

Quiz 8

August 1st, 2018

First Name: SOLUTIONS.

Last Name:

Student Number:

This quiz is out of 10 marks. Do ALL of your work on the back of the quiz, where the questions are. You can use the front for rough work, but nothing on the front will be marked, or even seen by the TAs.

If $X_i \overset{IID}{\sim} N(\mu, \sigma^2)$ then the density is $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$, for $x \in \mathbb{R}$, $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}^+$. - and MLE is $\hat{\mu} = \bar{X}$

BELOW SPACE IS FOR ROUGH WORK. NOTHING WRITTEN HERE WILL BE READ OR MARKED.

1. Suppose $X_i \overset{IID}{\sim} N(\mu_x, \sigma^2)$ and $Y_i \overset{IID}{\sim} N(\mu_y, \sigma^2)$, where the common variance σ^2 is known. Derive a likelihood ratio procedure for determining whether $\mu_x = \mu_y$ is supported by the observed data.

   (a) (2 marks) State the full parameter space and the restricted parameter space, and their dimensions.

1 Full: $\Omega = \mathbb{R}^2 \quad \dim \Omega = 2$

1 Restricted: $\Omega_0 = \{\mu \binom{1}{1} : \mu \in \mathbb{R}\}$, a line in $\mathbb{R}^2$. $\dim \Omega_0 = 1$

   (b) (8 marks) Find the Likelihood Ratio Statistic $-2\log\Lambda$. Simplify as far as possible. State its asymptotic distribution under the null hypothesis. You may state MLE without proof.

$$L_x(\mu_x) = (2\pi\sigma^2)^{-n/2} \exp\left(\frac{-1}{2\sigma^2}\sum (X_i - \mu_x)^2\right)$$
$$L_y(\mu_y) = (2\pi\sigma^2)^{-n/2} \exp\left(\frac{-1}{2\sigma^2}\sum (Y_i - \mu_y)^2\right)$$

2 $X_i \perp Y_i \Rightarrow L_{x,y}(\mu_x, \mu_y) = (2\pi\sigma^2)^{-n} \exp\left(\frac{-1}{2\sigma^2}\left(\sum (X_i - \mu_x)^2 + \sum (Y_i - \mu_y)^2\right)\right)$

1 Under H_0: $\mu_x = \mu_y$, $L_{x,y}(\mu) = (2\pi\sigma^2)^{-n} \exp\left(\frac{-1}{2\sigma^2}\left(\sum (X_i - \mu)^2 + \sum (Y_i - \mu)^2\right)\right)$

MLEs: $\hat{\mu}_x = \bar{X}$, $\hat{\mu}_y = \bar{Y}$, $\hat{\mu} = \frac{\sum X_i + \sum Y_i}{2n}$.

2 $$\Lambda = \frac{L_{x,y}(\hat{\mu})}{L_{x,y}(\hat{\mu}_x, \hat{\mu}_y)} = \exp\left(\frac{-1}{2\sigma^2}\left(\sum (X_i - \hat{\mu})^2 - \sum (X_i - \bar{X})^2 + \sum (Y_i - \hat{\mu})^2 - \sum (Y_i - \bar{Y})^2\right)\right)$$

2 $$-2\log\Lambda = \frac{1}{\sigma^2}\left(\sum (X_i - \hat{\mu})^2 - \sum (X_i - \bar{X})^2 + \sum (Y_i - \hat{\mu})^2 - \sum (Y_i - \bar{Y})^2\right)$$

1 $\dot{\sim}\ \chi^2_1$ under H_0.