1. For W_n, $X_{(n)}$ and θ as defined on the front page,
   (a) (4 marks) Find $E(X_{(n)})$ and $Var(X_{(n)})$.

$f_{W_n}(w) = nw^{n-1}$, $W_n = X_{(n)}/\theta$

$E(W_n) = \int_0^1 nw^n dw = \frac{nw^{n+1}}{n+1}\Big|_0^1 = \boxed{\frac{n}{n+1}}$ — ①

$E(W_n^2) = \int_0^1 nw^{n+1} dw = \frac{nw^{n+2}}{n+2}\Big|_0^1 = \boxed{\frac{n}{n+2}}$

$\Rightarrow Var(W_n) = E(W_n^2) - E(W_n)^2 = \frac{n}{n+2} - \left(\frac{n}{n+1}\right)^2$

$\Rightarrow Var(W_n) = \frac{n(n+1)^2 - n^2(n+2)}{(n+1)^2(n+2)} = \frac{\cancel{n^3} + \cancel{2n^2} + n - \cancel{n^3} - \cancel{2n^2}}{(n+1)^2(n+2)}$

$\Rightarrow Var(W_n) = \boxed{\frac{n}{(n+1)^2(n+2)}}$ — ①

$E(X_{(n)}) = E(\theta W_n) = \boxed{\frac{\theta n}{n+1}}$ — ①

$Var(X_{(n)}) = \theta^2 Var(W_n) = \boxed{\frac{\theta^2 n}{(n+1)^2(n+2)}}$ — ①

(b) (2 marks) Suggest an estimator $\hat{\theta}$ of θ that satisfies $E(\hat{\theta}) = \theta$.

$\hat{\theta} = \frac{n+1}{n} X_{(n)} \Rightarrow E(\hat{\theta}) = \frac{n+1}{n} E(\theta W_n) = \theta$ — ②

OR $\hat{\theta} = 2\bar{X} \Rightarrow E(2\bar{X}) = 2E(\bar{X}) = \frac{2}{n}\sum_{i=1}^{n} E(X_i) = 2E(X_i) = 2\int_0^\theta \frac{x}{\theta} dx$

$= 2\left[\frac{x^2}{2\theta}\right]_0^\theta = \frac{2\theta^2}{2\theta} = \theta$ — ②

(c) (4 marks) Evaluate the variance of your estimator, and compare it to the variance of $X_{(n)}$ (say whether it is smaller or larger, or if you can't tell).

$Var\left(\frac{n+1}{n} X_{(n)}\right) = \left(\frac{n+1}{n}\right)^2 Var(X_{(n)}) > Var(X_{(n)})$ — ②

$\Rightarrow Var(\hat{\theta}) = \left(\frac{n+1}{n}\right)^2 \frac{\theta^2 n}{(n+1)^2(n+2)} = \boxed{\frac{\theta^2}{n(n+2)}}$ ②

OR $Var(2\bar{X}) = \frac{4}{n^2}\sum_{i=1}^{n} Var(X_i) = \frac{4}{n} Var(X_i)$

$Var(X_i) = \left[\int_0^\theta \frac{x^2}{\theta} dx\right] - \left(\frac{\theta}{2}\right)^2 = \frac{x^3}{3\theta}\Big|_0^\theta - \frac{\theta^2}{4}$

$\Rightarrow Var(X_i) = \frac{\theta^2}{3} - \frac{\theta^2}{4} = \frac{4\theta^2 - 3\theta^2}{12} = \frac{\theta^2}{12}$ — ②

$\therefore Var(\hat{\theta}) = \frac{4}{n}\left(\frac{\theta^2}{12}\right) = \boxed{\frac{\theta^2}{3n}} = \frac{(n+1)^2(n+2)}{3n^2} Var(X_{(n)})$

$> Var(X_{(n)})$ — ②

etc. (for any other estimator)