STA261: Problems 1

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This assignment is not for credit. Complete the questions as preparation for the quiz in tutorial 1 on July 9th. The questions on the quiz will be very similar to the questions on the assignment.

1. Probability Distributions:
   (a) Let $Z \sim N(0,1)$. Let $Y = aZ$, where $a \in \mathbb{R}$. Show that $Z \overset{d}{=} Y$ if and only if $|a| = 1$.
   (b) Let $Z \sim N(0,1)$. Find the distribution of Z^2.
   (c) Let $Z_1 \sim N(0,1)$ and $Z_2 \sim N(0,1)$ independently. Show that $Z_1 + Z_2 \sim N(0,2)$.
   (d) Use your answer to the previous question to prove that if $Z_i \sim N(0,1)$ independently for $i = 1 \ldots n$, then $\sum_{i=1}^{n} Z_i \sim N(0,n)$.
   (e) Let $Z_1 \sim N(0,1)$ and $Z_2 \sim N(0,1)$ independently. Find the distribution of $Z_1^2 + Z_2^2$. You can find hints in chapter 6 of the textbook.
   (f) Let $Z_i \sim N(0,1), i = 1 \ldots n$ independently as in the previous questions. Use your answers to the previous questions to find the distribution of $\sum_{i=1}^{n} Z_i^2$.

2. Independence: Let X and Y be random variables with distribution functions $F_X(x)$ and $F_Y(y)$, and corresponding density functions $f_X(s)$ and $f_Y(y)$. Define X and Y to be independent if $X|Y \overset{d}{=} X$ and $Y|X \overset{d}{=} Y$. This means that $F_{X|Y}(x) = F_X(x)$ for all x (and the same for y). We write $X \perp Y$.
   (a) Show that this definition is mathematically equivalent to the usual definition of independence, $f_{X,Y}(x,y) = f_X(x)f_Y(y)$.
   (b) Recall the definition of the covariance between two random variables: $Cov(X,Y) = E((X - E(X))(Y - E(Y)))$. Show that if $X \perp Y$, cov(X,Y) = 0.
   (c) The converse is not true. As a counter example, let $Z \sim N(0,1)$.
      (i) Find the distribution of $Z|Z^2 = z$.
      (ii) Find the distribution of $Z^2|Z = z$.
      (iii) Are Z and Z^2 independent?
      (iv) Show $cov(Z, Z^2) = 0$. You can use your answers from question 1 if you need to.
   (d) The converse is still not true, in general, even if X and Y are normal, because they might not be jointly normal. As a counter example, let $X \sim N(0,1)$, let $W \sim Unif\{-1,1\}$ (the discrete uniform distribution, $P(W = -1) = P(W = 1) = 1/2$) be independent of X, and let $Y = WX$ (this example is from an interesting wikipedia article here).
      (i) Show that $X \overset{d}{=} Y$. This question is straightforward if you use the definition of the CDF and the rules of probability to show $F_Y(y) = F_X(y) \forall y \in \mathbb{R}$.
      (ii) Show that $cov(X,Y) = 0$
      (iii) Show that $X \not\perp Y$
   (e) However, the converse is true if X and Y are jointly Normal. That is, if (X,Y) has a bivariate normal distribution. Let $X \sim N(0,1)$ and $Y \sim N(0,1)$, with $cov(X,Y) = 0$, and assume that (X,Y) has a bivariate normal distribution. Show that $X \perp Y$. Hint: use the bivariate normal density on page 81 of the textbook.

3. Convergence in Probability. Let $\{X_n\}$ be a sequence of random variables with $E(X_i) = \mu$ and $\lim_{n\to\infty} Var(X_n) = 0$. Show $X_n \xrightarrow{p} \mu$.

4. Convergence in Distribution. Let $\{X_n\}$ be a sequence of random variables and let $\mu \in \mathbb{R}$. Suppose $X_n \xrightarrow{d} \mu$. Show $X_n \xrightarrow{p} \mu$.

5. Law of Large Numbers. Let $\{X_i\}$ be a sequence of independent random variables with $E(X_i) = \mu$ and $Var(X_i) = \sigma^2$.

(a) Evaluate $\lim_{n \to \infty} P(|\bar{X_n} - \mu| > 0.01)$.

(b) Can you evaluate $P(|\bar{X_{100}} - \mu| > 0.01)$? Why or why not?

6. Central Limit Theorem. Suppose we measure the heights of $n = 100$ randomly selected people on the University of Toronto campus at lunchtime. Let $X_1 \dots X_n$ be the random variables that represent the heights we might measure, in cm. Suppose we know from a previous experiment that these heights have a mean of $160cm$ and a standard deviation of $20cm$.

   (a) Why can't you evaluate the probability that the $10th$ person's height is greater than 170cm?

   (b) Approximate the probability that the sample mean height of the people measured is greater than 170cm.

   (c) Approximate the probability that the sample mean height of the people measured is less than $150cm$.

   (d) Approximate the probability that the sample mean height of the people measured is between $150cm$ and $170cm$.

   (e) Suppose we observed a sample mean of $164cm$. What is the probabilty of observing this, or something farther from the true mean of $160cm$ (in either direction, lower or higher), in a sample of this size?