STA261: Lecture 6

Unbiasedness & Efficiency

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Disclaimer

The materials in these slides are intended to be a companion to the course textbook, *Mathematical Statistics and Data Analysis, Third Edition*, by John A Rice. Material in the slides may or may not be taken directly from this source. These slides were organized and typeset by Alex Stringer.

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Unbiasedness

Definition: suppose $\hat{\theta}$ is an estimator for θ. The **bias** of $\hat{\theta}$ is defined as

$$bias(\hat{\theta}) = E(\hat{\theta} - \theta) = E(\hat{\theta}) - \theta$$

The bias measures the degree by which we expect $\hat{\theta}$ to differ from θ systematically, or on average.

If we repeated our experiment many times and calculated the average of all the resulting estimates of θ, we would expect this to be $bias(\hat{\theta})$ away from θ.

Unbiasedness

Property 3 of an estimator is called **unbiasedness**.

Definition: an estimator $\hat{\theta}$ of θ is called **unbiased** if $E(\hat{\theta}) = \theta$.

This is equivalent to $bias(\hat{\theta}) = 0$.

Why Unbiasedness?

This comes from the principle that we want to pick an estimation procedure that we don't expect to give wrong answers, at least on average.

This is because we are using our estimates to make decisions about the process that generated the data.

It is a property of the sampling distribution of $\hat{\theta}$. It doesn't directly say anything about what kind of value we can expect in any given sample. Only that if we repeated our experiement many times and calculated $\hat{\theta}$ for each given dataset, on average, we would expect that $\hat{\theta}$'s we get to equal θ.

Example

Let $X_i \sim N(\mu, \sigma)$ and show $\bar{X}$ is unbiased.

We showed before that $E(X) = \mu$, and $E(\bar{X}) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \mu$. Because $E(\bar{X}) = \mu$, $\bar{X}$ is unbiased for μ.

Example

Let $X_i \sim Exp(\theta)$, with $f(x) = \frac{1}{\theta}e^{-x/\theta}$. Is $\hat{\theta} = \bar{X}$ unbiased for θ?

Compute $E(X) = \theta$, either by integrating or using the MGF (integrating is easier in this example). Then compute

$$E(\bar{X}) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \theta$$

so $\hat{\theta}$ is unbiased for θ.

Example

Let $X_i \sim Exp(\beta)$, with $f(x) = \beta e^{-\beta x}$. Is $\hat{\beta} = \frac{1}{\bar{X}}$ unbiased for θ?

Compute $E(\hat{\beta}) = E\left(\frac{1}{\bar{X}}\right) = \ldots$

But $E\left(\frac{1}{\bar{X}}\right) \neq \frac{1}{E(\bar{X})}$, so this question is harder than it looks!

Example

We can show:

$$
\begin{aligned}
X_i &\overset{IID}{\sim} Exp(\beta) \\
&\implies \sum_{i=1}^{n} X_i \sim Gamma(n, \beta) \\
&\implies \frac{1}{\sum_{i=1}^{n} X_i} \sim InvGamma(n, \beta) \\
&\implies E\left(\frac{1}{\sum_{i=1}^{n} X_i}\right) = \frac{\beta}{n-1}
\end{aligned}
$$

so

$$E\left(\frac{1}{\bar{X}}\right) = \frac{n}{n-1}\beta$$

and $bias(1/\bar{X}) = \frac{\beta}{n-1}$, which increases with β and decreases with n.

Example

We can "correct" $\hat{\beta} = 1/\bar{X}$ to be unbiased:

$$\hat{\beta}_2 = \frac{n-1}{n} \times \frac{1}{\bar{X}}$$

has $E(\hat{\beta}_2 = \beta$. It also has lower variance:

$$Var(\hat{\beta}_2) = \left(\frac{n-1}{n}\right)^2 Var(\hat{\beta})$$

Is there any other unbiased estimator of β that has even lower variance?

The Cramer-Rao Lower Bound (Textbook, page 300 - 301)

Theorem: Suppose $\hat{\theta}$ is any *unbiased* estimator of θ. Then

$$Var(\hat{\theta}) \geq \frac{1}{I(\theta)}$$

where $I(\theta)$ denotes the Fisher Information in the entire IID sample of size n (recall from a previous lecture).

There are some regularity conditions required here that we will discuss next lecture.

Efficiency

This lets us state property 4.

Definition: an estimator $\hat{\theta}$ of θ is **efficient** if it attains the Cramer-Rao Lower bound, that is if

$$Var(\hat{\theta}) = \frac{1}{I(\theta)}$$

Example

Let $X_i \sim N(\mu, 1)$. Show $\bar{X}$ is an efficient estimator of μ.

You have to show it's unbiased, which we did above: $E(\bar{X}) = \mu$.

Then compute the variance of the estimator,

$$
\begin{aligned}
Var(\bar{X}) &= \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) \\
&= \frac{n(1)}{n^2} \\
&= \frac{1}{n}
\end{aligned}
$$

Example

Now compute the Fisher Information for a single datapoint,

$$i_0(\mu) = -E\left(\frac{\partial^2 \log f(x|\mu)}{\partial \mu^2}\right)$$
$$= 1$$

so $i(\mu) = n$ and $\frac{1}{i(\mu)} = \frac{1}{n}$. Therefore $\bar{X}$ is an efficient estimator of μ, because it is unbiased and has variance attaining the Cramer-Rao lower bound.

Example

For $X_i \sim N(\mu, \sigma^2)$, are either $\hat{\sigma}_1^2 = s_n^2$ or $\hat{\sigma}^2 = s_{n-1}^2$ unbiased and efficient?

Unbiased: ...

Efficient: ...

Example

For the $Exponential(\theta)$ (with $E(X) = \theta$) example, is the estimator $\hat{\theta} = \bar{X}$ efficient?

Yes: . . .

Example

What about the other exponential example, the $Exponential(\beta)$ with $\hat{\beta} = 1/\bar{X}$?

We can't talk about the efficiency of $\hat{\beta} = 1\bar{X}$, because we showed it's not unbiased.

What about the corrected estimator $\hat{\beta}_2 = \frac{n-1}{n} \times \frac{1}{\bar{X}}$? ...