STA261: Midterm Solutions

Section L5101

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1. (a)
$$\begin{aligned} E(I(X=x)) &= 1 \times P(I(X=x)=1) + 0 \times P(I(X=x)=0) \\ &= P(X=x) \end{aligned}$$

   (b) By the LLN, $\hat{P}(x) \xrightarrow{p} E(I(X=x)) = P(X=x) = P(x)$, so $\hat{P}(x)$ is consistent for $P(x)$.

2. - 6 marks for correctly solving the equations. Either of the below representations is correct:
$$\begin{aligned} \mu &= 2\log E(X) - \frac{1}{2}\log E(X^2) = \log\left(\frac{E(X)^2}{\sqrt{E(X^2)}}\right) \\ \sigma^2 &= logE(X^2) - 2\log E(X) = \log\left(\frac{E(X^2)}{E(X)^2}\right) \end{aligned}$$

   - 4 marks for plugging in the sample moments

3. (a) The joint density is
$$f(x_1, \ldots, x_n|\alpha) = \left(\frac{1}{\Gamma(\alpha)2^\alpha}\right)^n \left(\prod_{i=1}^n x_i\right)^\alpha \exp\left(-\frac{1}{2}\sum_{i=1}^n\right)$$

   (b) The joint density factors as
$$f(x_1, \ldots, x_n|\alpha) = \left(\frac{1}{\Gamma(\alpha)2^\alpha}\right)^n \left(\prod_{i=1}^n x_i\right)^\alpha \times \exp\left(-\frac{1}{2}\sum_{i=1}^n\right)$$
   so by the factorization theorem, $T_1 = \prod_{i=1}^n X_i$ is sufficient for α.

   (c) $T_2 = \log T_1$ is a one-to-one function of a sufficient statistic and therefore is sufficient.

4. (a) The log-likelihood is $\ell(\theta) = -n\log\theta - \frac{1}{\theta}\sum_{i=1}^n x_i$

   (b) The score function is
$$S(\theta) = -\frac{n}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^n x_i$$
   Setting to zero and solving gives $\hat{\theta} = \bar{X}$.

   (c) $J(\theta)$ is minus the derivative of $S(\theta)$,
$$J(\theta) = -\frac{n}{\theta^2} + \frac{2}{\theta^3}\sum_{i=1}^n x_i$$

(d) The fisher information is the expected observed information,

$$I(\theta) = E(J(\theta)) = \frac{n}{\theta^2}$$

(e) In this single parameter problem, the asymptotic variance of $\hat{\theta}$ is the inverse of the fisher information at the true value θ_0,

$$Var(\hat{\theta}) = 1/I(\theta_0) = \frac{\theta_0^2}{n}$$

(f) The asymptotic distribution of $\hat{\theta}$ is $N(\theta_0, 1/I(\theta_0))$,

$$\hat{\theta} \sim N\left(\theta_0, \frac{\theta_0^2}{n}\right)$$