STA261: Assignment 9

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March, 2018

This assignment is not for credit. Complete the questions as preparation for quizzes and tests.

1. Suppose $X_i \sim N(\mu, \sigma^2)$ is an IID sample from a normal distribution with mean parameter μ and known variance σ^2. We are interested in testing the hypothesis $H_0 : \mu = \mu_0$ against the alternative $H_1 : \mu \neq \mu_0$ using a likelihood ratio test.
   (a) State the full parameter space Ω.
   (b) State the restricted parameter space Ω_0
   (c) Are the null and alternative hypotheses simple, or composite?
   (d) Show that the likelihood-ratio test statistic as defined in lecture can be written
   $$-2\log\Lambda = \sum_{i=1}^{n}\left(\frac{X_i - \mu_0}{\sigma}\right)^2 - \sum_{i=1}^{n}\left(\frac{X_i - \bar{X}}{\sigma}\right)^2$$
   (e) Check: show that $-2\log\Lambda > 0$ if H_0 is true, for any sample $\mathbf{x}$.
   (f) We proved in class that Λ has an approximate χ^2_1 distribution when defined this way. Show that in this example, $-2\log\Lambda$ has an *exact* χ^2_1 distribution:
   $$-2\log\Lambda \sim \chi^2_1$$
   You should use results you proved on assignment 7, and from the relevant textbook section (see assignment 7 for reference).
   (g) State formulae for the critical region and p-value for testing this hypothesis at the α significance level.
   (h) Show that this test is *exactly* the same as the original normal-theory hypothesis test we derived in lecture 7. That is, show
   $$-2\log\Lambda = \left(\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}\right)^2$$
   so that the signed square root of $-2\log\Lambda$ equals our test statistic from before.
2. Consider the previous question, but now the variance is unknown. In lecture we showed that the appropriate likelihood ratio test statistic for testing this hypothesis is
   $$-2\log\Lambda = \log\hat{\sigma}^2_0 - \log\hat{\sigma}^2$$
   Show that
   $$-2\log\Lambda = n\log\left(1 + \frac{t^2}{n-1}\right)$$

where $t^2 = \frac{(\bar{X}-\mu_0)^2}{s^2/n}$. There is a hint in the lecture slides.

3. Suppose now that we have K random samples from normal distributions, all mutually independent, with different means and the same *known* variance. That is, we have

$$X_{ij} \sim N(\mu_i, \sigma^2)$$
$$i = 1 \dots K$$
$$j = 1 \dots n$$

We wish to test

$$H_0 : \mu_1 = \dots = \mu_K = \mu_0$$

against

$$H_1 : \text{ at least one } \mu_i \neq \mu_j$$

That is, we wish to test that all the means are equal to some common value μ_0. However, we do not specify what μ_0 is- we estimate it from the data.

(a) State the full parameter space Ω.

(b) State the restricted parameter space Ω_0.

(c) Are the null and alternative hypotheses simple, or composite?

(d) Show that the likelihood under the null hypothesis is

$$L_0(\mu_0|\mathbf{x}) = c \times \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{K}\sum_{j=1}^{n}\left(x_{ij}-\mu_0\right)^2\right)$$

where c is some constant that does not depend on μ (you don't need to say what it is).

(e) Show that the MLE of μ_0 under H_0 is

$$\hat{\mu_0} = \frac{1}{nK}\sum_{i=1}^{K}\sum_{j=1}^{n} x_{ij} \equiv \bar{x}_{..}$$

(f) Show that the likelihood under the alternative hypothesis is

$$L_1(\mu_1,\dots,\mu_K|\mathbf{x}) = c \times \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{K}\sum_{j=1}^{n}\left(x_{ij}-\mu_i\right)^2\right)$$

where c is some constant that does not depend on μ_0 (you don't need to say what it is).

(g) Show that the MLE of $\mu_1 \dots \mu_K$ under H_1 is

$$\hat{\mu_i} = \frac{1}{n}\sum_{j=1}^{n} x_{ij} \equiv \bar{x}_i$$

(h) Show that the likelihood ratio statistic in this problem is

$$-2\log\Lambda = \frac{1}{\sigma^2}\left(\sum_{i=1}^{K}\sum_{j=1}^{n}\left(x_{ij}-\bar{x}_{..}\right)^2 - \sum_{i=1}^{K}\sum_{j=1}^{n}\left(x_{ij}-\bar{x}_i\right)^2\right)$$

(i) What is the distribution of $-2\log\Lambda$? You know it's χ^2- what are the degrees of freedom?

(j) State formulae for the critical region and p-value for testing this hypothesis at the α significance level.

4. For IID random samples from the following distributions, derive a likelihood ratio test for testing $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$.

   (a) Exponential, $f(x|\theta) = \frac{1}{\theta}e^{-x/\theta}$, $X > 0, \theta > 0$

   (b) Poisson, $P(X = x|\theta) = \frac{\theta^x e^{-\theta}}{x!}$, $X > 0, \theta > 0$

   (c) Bernoulli, $P(X = x) = \theta^x(1-\theta)^{1-x}$

   (d) Binomial, $P(X = x) = \binom{m}{x}\theta^x(1-\theta)^{m-x}$

5. Consider the contingency table example discussed in lecture.

   (a) Show that the likelihood ratio test statistic is

$$-2\log\Lambda = 2\sum_{i=1}^{R}\sum_{j=1}^{C} y_{ij}\log\left(\frac{Ny_{ij}}{r_i c_j}\right)$$

   (b) For the following 2×2 table, test the hypothesis of independence at the 5% significance level. Report the test statistic, critical value, p-value, and your conclusion. I got a p-value of about 0.042.

65	11
98	35