STA261 S19: Test 1 Solutions

No aids. 60 minutes. Write all answers directly beneath where the question is asked. Use the backs of the pages for rough work.

1. *Basic, 4 marks*

a) (2) Define what it means for a sequence of random variables X_n to converge in probability to a random variable X.

Solution. $\forall \epsilon > 0,\ \lim_{n\to\infty} \mathbb{P}\left(|X_n - X| > \epsilon\right) = 0.$ (2 marks)

b) (2) If $X_n \overset{i.i.d.}{\sim}$ Normal$(0, \sigma^2)$ then X_n are independent, $\mathbb{E}X_n = 0$ and $\mathrm{Var}(X_n) = \sigma^2 < \infty$. Let $S_n = X_1 + \cdots + X_n$. Show $S_n/n \xrightarrow{p} 0$. State all conditions of any theorem(s) you use and make sure to say why they are satisfied.

Solution. $\mathrm{Var}(X_n) < \infty$ (1 mark), so by independence the weak law of large numbers (1 mark) implies $S_n/n \xrightarrow{p} \mathbb{E}X_1 = 0$.

Alternative solution. Under independence, $S_n \sim \mathcal{N}(0, n\sigma^2)$, so $S_n/n \sim \mathcal{N}(0, \sigma^2/n)$.
Then, for any $\epsilon > 0$,

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{P}\left(|S_n/n| > \epsilon\right) &\leq \lim_{n\to\infty} \frac{\mathrm{Var}(S_n/n)}{\epsilon^2} && \text{(Chebyshev's inequality)} \\
&= \lim_{n\to\infty} \frac{\sigma^2}{n\epsilon^2} \\
&= 0.
\end{aligned}$$

2. *Adept, 4 marks.* Suppose we want to evaluate a very complicated integral of a function $f : \mathbb{R} \to \mathbb{R}$:

$$I = \int_0^1 f(x)dx$$

f is too complicated to evaluate I analytically. One numerical method to evaluate I is as follows: sample $U_1, \dots, U_n \overset{iid}{\sim} \text{Unif}(0,1)$, and compute

$$\widehat{I} = \frac{1}{n}\sum_{i=1}^{n} f(U_i) \tag{0.1}$$

a) (1) Compute $\mathbb{E}(\widehat{I})$. The Unif(a,b) density is $g(x) = \frac{1}{b-a}$ for $a \le x \le b$.

Solution.

$$\begin{aligned}
\mathbb{E}\widehat{I} &= \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} f(U_i)\right] \\
&= \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left[f(U_i)\right] && \text{(linearity of expectation)} \\
&= \frac{1}{n}\sum_{i=1}^{n}\int_a^b f(x)g(x)dx && \text{(definition of expectation – 0.5 marks)} \\
&= \frac{1}{n}\sum_{i=1}^{n}\int_0^1 f(x)dx && (a = 0 \text{ and } b = 1) \\
&= \frac{1}{n}n\int_0^1 f(x)dx && \text{(the integral does not depend on } i \text{ – 0.5 marks)} \\
&= I.
\end{aligned}$$

b) (1) Compute $\text{Var}(\widehat{I})$.

Solution. For any i,

$$\begin{aligned}
\text{Var}(f(U_i)) &= \mathbb{E}f^2(U_i) - \mathbb{E}^2 f(U_i) && \text{(common formula for variance)} \\
&= \mathbb{E}f^2(U_i) - I^2 && \text{(by 2.a))} \\
&= \int_0^1 f^2(x)dx - I^2. && \text{(applying Unif(0,1) density – 0.5 marks)}
\end{aligned}$$

Then,

$$\begin{aligned}
\text{Var}(\widehat{I}) &= \frac{1}{n^2}\sum_{i=1}^{n}\text{Var}(f(U_i)) && \text{(by independence of } U_i \text{ – 0.5 marks)} \\
&= \frac{1}{n}\left[\int_0^1 f^2(x)dx - I^2\right]. && \text{(by above calculation which doesn't depend on } i\text{)}
\end{aligned}$$

c) (2) Show that $\widehat{I} \xrightarrow{p} I$. What conditions on f must be assumed for this to be true?

Solution. If $\int_0^1 f^2(x)dx < \infty$ (1 mark) (which happens as long as $f(x) < \infty$ on $[0,1]$ since it is an integral over a bounded domain; as well, this is implied by $I < \infty$) then since $U_1, \ldots, U_n$ are independent, $f(U_1), \ldots, f(U_n)$ are independent, and $\widehat{I}$ is a sum of independent terms (0.5 marks) with finite variance (0.5 marks). So, by the weak law of large numbers, $\widehat{I} \xrightarrow{p} I$.

Alternative solution. Suppose that f satisfies $I < \infty$. Then, from 2.b), $\lim_{n\to\infty} \mathrm{Var}(\widehat{I}) = 0$. So, for any $\epsilon > 0$,

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{P}\left[\left|\widehat{I} - I\right| > \epsilon\right] &= \lim_{n\to\infty} \mathbb{P}\left[\left|\widehat{I} - E\widehat{I}\right| > \epsilon\right] && \text{(from 1.a))} \\
&\leq \lim_{n\to\infty} \frac{\mathrm{Var}(\widehat{I})}{\epsilon^2} && \text{(Chebyshev's inequality)} \\
&= 0.
\end{aligned}$$

Thus, since $X_n \xrightarrow{p} X \iff (X_n - X) \xrightarrow{p} 0$, $\widehat{I} \xrightarrow{p} I$.

3. *Advanced, 2 marks.* Let $X_n \xrightarrow{p} X$. Suppose $f : \mathbb{R} \to \mathbb{R}$ be a L-Lipschitz continuous function, which means for some $L > 0$ and every $x, y \in \mathbb{R}$ we have

$$|f(x) - f(y)| \leq L|x - y| \tag{0.2}$$

Prove that $f(X_n) \xrightarrow{p} f(X)$.

Solution. By the Lipschitz condition, for any $\epsilon > 0$,

$$\{\omega : |f(X_n(\omega)) - f(X(\omega))| > \epsilon\} \subseteq \{\omega : |X_n(\omega) - X(\omega)| > \epsilon/L\}\,. \text{ (1 mark)}$$

Thus, since $X_n \xrightarrow{p} X$, taking $\epsilon' = \epsilon/L$,

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{P}\left[|f(X_n) - f(X)| > \epsilon\right] &\leq \lim_{n\to\infty} \mathbb{P}\left[|X_n - X| > \epsilon/L\right] && \text{(0.5 marks)} \\
&= 0. && \text{(0.5 marks)}
\end{aligned}$$