STA261: Assignment 1

Alex Stringer

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This assignment is not for credit. Complete the questions as preparation for the quiz in tutorial. The questions on the quiz will be very similar to the questions on the assignment.

For this assignment, the questions on the quiz will be related to the material on convergence of sequences/LLN/CLT covered in lecture. The rest of the questions are provided here so that you are clear about your expected background in this course. It is okay to have to think/put in effort at doing the below questions (I certainly had to review these concepts myself!), but the concepts should all at least be familiar. It may be helpful to refer to chapters 1 - 5 of the textbook.

1. *Probability Distributions*:
   (a) Let $Z \sim N(0,1)$. Let $Y = aZ$, where $a \in \mathbb{R}$. Show that $Z \overset{d}{=} Y$ if and only if $|a| = 1$.
   (b) Let $Z \sim N(0,1)$. Find the distribution of the squared length of $Z, |Z|^2$. The "length" of a number is just its distance from 0, i.e. its absolute value.
   (c) Let $Z_1 \sim N(0,1)$ and $Z_2 \sim N(0,1)$ independently. Show that $Z_1 + Z_2 \sim N(0, \sqrt{2})$. I am using the notation $N(mean, standarddeviation)$.
   (d) Use your answer to the previous question to prove that if $Z_i \sim N(0,1)$ independently, then $\sum_{i=1}^n Z_i \sim N(0, \sqrt{n})$.
   (e) Let $Z_i \sim N(0,1)$ independently as in the previous questions. Use your answers to the previous questions to find the distribution of the squared length of the random vector $\mathbf{Z} = (Z_1, \ldots, Z_n)$, which equals $\sum_{i=1}^n Z_i^2$.

2. *Independence*. Let X and Y be random variables with distribution functions $F_X(x)$ and $F_Y(y)$, and corresponding density functions $f_X(s)$ and $f_Y(y)$. Define X and Y to be *independent* if $X|Y \overset{d}{=} X$ and $Y|X \overset{d}{=} Y$. This means that $F_{X|Y}(x) = F_X(x)$ for all x (and the same for y). We write $X \perp Y$.
   (a) Show that this definition is mathematically equivalent to the usual definition of independence, $f_{X,Y}(x,y) = f_X(x)f_Y(y)$.
   (b) Show that if $X \perp Y$, cov(X,Y) = 0.
   (c) The converse is not true. As a counter example, let $Z \sim N(0,1)$.
      (i) Find the distribution of $Z|Z^2 = z$.
      (ii) Find the distribution of $Z^2|Z = z$.
      (iii) Are Z and Z^2 independent?

(iv) Show $cov(Z, Z^2) = 0$

(d) The converse is still not true, in general, even if X and Y are normal, because they might not be *jointly* normal. As a counter example, let $X \sim N(0,1)$, let $W \sim Unif\{-1,1\}$ (the *discrete* uniform distribution, $P(W=-1) = P(W=1) = 1/2$) be independent of X, and let $Y = WX$ (this example is from an interesting wikipedia article here). Showing why X and Y are not jointly normally distributed is slightly out of scope for this course; just take my word for it.

(i) Show that $X \overset{d}{=} Y$. This question is straightforward if you use the definition of the CDF and the rules of probability to show $F_Y(y) = F_X(y) \forall y \in \mathbb{R}$.

(ii) Show that $cov(X, Y) = 0$

(iii) Show that $X \not\perp Y$

(e) However, the converse *is* true if X and Y are *jointly* Normal. That is, if (X, Y) has a bivariate normal distribution. Let $X \sim N(0,1)$ and $Y \sim N(0,1)$, with $cov(X, Y) = 0$, and assume that (X, Y) has a bivariate normal distribution. Show that $X \perp Y$. *Hint*: use the bivariate normal density on page 81 of the textbook.

3. Prove **Chebyshev's** inequality (see lecture slides).

4. Prove **Markov's** inequality (see lecture slides).

5. *Convergence in Probability.* Let $\{X_n\}$ be a sequence of random variables with $E(X_i) = \mu$ and $\lim_{n\to\infty} Var(X_n) = 0$. Show $X_n \xrightarrow{p} \mu$.

6. *Convergence in Distribution.* Let $\{X_n\}$ be a sequence of random variables and let $\mu \in \mathbb{R}$. Suppose $X_n \xrightarrow{d} \mu$. Show $X_n \xrightarrow{p} \mu$.

7. *Law of Large Numbers.* Let $\{X_i\}$ be a sequence of independent random variables with $E(X_i) = \mu$ and $Var(X_i) = \sigma^2$.

(a) Evaluate $\lim_{n\to\infty} P(|\bar{X_n} - \mu| > 0.01)$.

(b) Can you evaluate $P(|\bar{X_{100}} - \mu| > 0.01)$? Why or why not?

8. *Central Limit Theorem.* Let $\{X_i\}$ be a sequence of independent random variables having the discrete uniform distribution, $X_i \sim unif\{-1,1\}$.

(a) Evaluate $E(X_i)$ and $Var(X_i)$

(b) Use the central limit theorem to calculate

$$\lim_{n\to\infty} P\left(\left|\frac{\sum_{i=1}^n X_i}{\sqrt{n}}\right| \le 1\right)$$

(c) Derive the exact distribution of $\sum_{i=1}^n X_i$. *Hint*: find a simple transformation that makes each X_i Bernoulli. Then the distribution of the sum is known.

(d) Use your distribution to verify *computationally* (that means, use a computer to calculate some values) that $\lim_{n\to\infty} P\left(\left|\frac{\sum_{i=1}^{n} X_i}{\sqrt{n}}\right| \leq 1\right)$ equals what you calculated in step 2. Just calculate the probability for some values of n using the pbinom function in R, and make a simple plot.

(e) *Bonus*: investigate how large n needs to be for the approximation using the CLT to be good. Create a plot with n on the x-axis and two lines on the y-axis: one for the exact probability, and one for the approximate probability given by the CLT. How large does n have to be before the lines start overlapping?

Remember you won't be evaluated on your use of any statistical programming languages on any quizzes/tests in this course. It is still a good idea to do (d) and (e) above, for your own development. Knowing how to make simple graphs and calculate simple probablities is an essential skill.

9. *Central Limit Theorem.* Suppose we measure the heights of n randomly selected people on the University of Toronto campus at lunchtime. Let $X_1 \dots X_n$ be the random variables that represent the heights we might measure, in cm. Suppose we know from a previous experiment that these heights have a mean of $160cm$ and a standard deviation of $20cm$. And suppose one more time that we measure $n = 100$ individuals.

   (a) Why can't you evaluate the probability that the $10th$ person's height is greater than 170cm?

   (b) Approximate the probability that the sample mean height of the people measured is greater than 170cm.

   (c) Approximate the probability that the sample mean height of the people measured is less than $150cm$.

   (d) Approximate the probability that the sample mean height of the people measured is between $150cm$ and $170cm$.

   (e) Suppose we observed a sample mean of $190cm$. What is the probabilty of observing this, or something farther from the true mean of $160cm$ (in either direction, lower or higher)?

Textbook questions:

- Section 5.4, questions 1, 2, 5, 7 (remember, a continuous function satifies $\lim_{x\to a} g(x) = g(\lim_{x\to a} x) = g(a)$), 10, 15, 16, 17, 18, 26