STA261: Assignment 2

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This assignment is not for credit. Complete the questions as preparation for quizzes and tests.

Suggested reading: Textbook sections 8.3 and 8.4.

1. In the following problems, identify the *parameter*, the *estimator*, and the *estimate*. There might be 0, 1, or 2 of each thing in each question.

   (a) We sample the heights of U of T students, which we know to have a mean of $170cm$. We get a sample mean of $168cm$.

   (b) An auto insurance company knows that the probability of someone with your age and driving record making a claim in the next 30 days is 0.0001. They count the number of claims you make in this period, and find you made 0.

   (c) An auto insurance company wants to figure out the expected number of claims that someone with your age and driving record should make in the next 30 days. They look at similar 30 day periods, and calculate that out of 40,000 such individuals, 37 claims were made.

   (d) The news reports that for a poll in an election campaign, candidate A has a popularity of 49% and candidate B has a popularity of 51%, and that this means candidate B is going to get the most votes in the election.

2. *Consistency.* For independent random samples from the following families of distributions, show that the given estimator is consistent for the population parameter.

   (a) $X_i \sim Pois(\lambda)$, $\hat{\lambda} = \bar{X}$.

   (b) $X_i \sim Exp(\theta)$, where $f_\theta(x) = \theta e^{-\theta x}$, $\hat{\theta} = 1/\bar{X}$.

   (c) $X_i \sim Exp(\beta)$, where $f_\beta(x) = \frac{1}{\beta} e^{-\frac{x}{\beta}}$, $\hat{\beta} = \bar{X}$

   (d) $X_i \sim \chi^2_\nu$, $\hat{\nu} = \bar{X}$

   (e) $X_i \sim Gamma(\alpha, \beta)$ with $f_{\alpha,\beta}(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}}$, $\hat{\alpha} = \frac{(\bar{X})^2}{s^2}$ and $\hat{\beta} = \frac{s^2}{\bar{X}}$, where $s^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$. Note you can use $\frac{1}{n-1}$ in the sample variance if you want- they both give a consistent estimator of σ^2.

3. *Consistency.* For $X_i \sim N(\mu, \sigma)$, state and prove whether each estimator is consistent or not. Be sure to say exactly where you are assuming a function is continuous in your proofs.

   (a) $\hat{\mu} = \bar{X}$

(b) $\hat{\sigma} = s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2}$

(c) $\hat{\sigma} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2}$

(d) $\hat{\sigma} = \frac{1}{n}\sum_{i=1}^{n}\left|X_i - \bar{X}\right|$

(e) $\hat{\mu} = \frac{1}{n+1{,}000{,}000}\sum_{i=1}^{n} X_i$

(f) $\hat{\mu} = \frac{1}{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_i}}$

(g) $\hat{\sigma^2} = \frac{1}{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_i^2} - \left(\frac{1}{n}\sum_{i=1}\frac{1}{X_i}\right)^2}$

4. *Consistency.* Recall the covariance between two random variables X and Y is defined as

$$Cov(X, Y) = E((X - E(X))(Y - E(Y)))$$

   (a) Show $Cov(X, Y) = E(XY) - E(X)E(Y)$

   (b) Using what you know about convergence in probability of continuous functions of random variables, show that the *sample correlation coefficient*

$$R = \frac{\sum_{i=1}^{n}(X_i - \bar{X})(Y_i - \bar{Y})}{\sqrt{\sum_{i=1}^{n}(X_i - \bar{X})^2 \sum_{i=1}^{n}(Y_i - \bar{Y})^2}}$$

   is consistent for the population correlation,

$$\rho = \frac{Cov(X, Y)}{\sqrt{Var(X)Var(Y)}}$$

   Remember, Slutky's lemma tells us that basically everything you have learned thusfar using one random variable works in the multiple-variable case too.

5. *Method of Moments.* For independent random samples from the following families of distributions, find a consistent estimator for the population parameter using the method of moments. Some of these are tricky; if you need more practice, first try doing the previous questions in reverse. The method of moments is just a reverse consistency proof, as discussed in lecture.

   (a) The gamma distribution as asked in question 1. Note the support of x is $x > 0$.

   (b) $X_i \sim t_\nu$. This is the t-distribution- if you haven't seen it before, either look up its properties or just skip this question. We'll cover it in detail later.

   (c) $X_i \sim Beta(\alpha, \beta)$ with $f_{\alpha,\beta}(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$, $x \in (0, 1)$.

   (d) $X_i \sim LogNormal(\mu, \sigma)$, with $f_{\mu,\sigma}(x) = \frac{1}{x\sigma\sqrt{2\pi}}e^{-\frac{(\log x - \mu)^2}{2\sigma^2}}, x > 0$

   (e) $X \sim Unif(0, \beta), x \in (0, \beta)$

6. *Method of Moments.* Let $X_i \sim Unif(\alpha, \beta)$. Find a method of moments estimator of (α, β). This question is messy.

7. *Method of Moments.* Let $\epsilon_i \sim N(0, \sigma)$ be independent and identically distributed random variables. Let $\beta \in \mathbb{R}$ be a fixed, unknown constant and $x_i \in \mathbb{R}, i = 1 \dots n$ be fixed, *known* quantities. Let $Y_i = \beta x_i + \epsilon_i, i = 1 \dots n$

(a) Identify all the *parameters* in this question. There are two.

(b) Find a Method of Moments estimator for β.

(c) Even though I told you MoM estimators are always consistent- prove that your estimator for β is consistent by going through the usual motions.

(d) Find a Method of Moments estimator of σ^2, and show that it too is consistent.

Do textbook questions from Chapter 8: 4(a), 4(b), 5(a), 7(a), 17(b), 18(a), 21(a) (this one is very important), 52(a)

The reason I am telling you to do only certain parts of the questions is because these questions cover all of chapter 8, which we haven't done yet.