STA261: Assignment 10

Alex Stringer

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This assignment is not for credit. Complete the questions as preparation for quizzes and tests.

1. For our usual normal-theory hypothesis test, where we reject $H_0 : \mu = \mu_0$ at the α significance level if

$$\left| \frac{\bar{X} - \mu_0}{\sigma_0 / \sqrt{n}} \right| > z_{\alpha/2}$$

   (a) Show that the power of this test to detect an effect of size $d = \frac{\mu_1 - \mu_0}{\sigma}$ is

$$\eta(d, n, \alpha) = 1 - (\Phi(d\sqrt{n} + z_{1-\alpha/2}) - \Phi(d\sqrt{n} - z_{1-\alpha/2}))$$

   (b) Show that for any fixed $n \in \mathbb{N}, \alpha \in (0, 1)$:

$$\lim_{d \to \infty} \eta(d, n, \alpha) = 1$$

   That is, for any sample size, there always exists an effect that we can detect with as high a probability as we want.

   (c) Show that for any fixed $d \in \mathbb{R}, \alpha \in (0, 1)$:

$$\lim_{n \to \infty} \eta(d, n, \alpha) = 1$$

   That is, for any fixed effect we would like to detect, we can choose the sample size large enough to be able to do so with as high a probability as we want.

   (d) Show that η is a *symmetric* function of the effect size d: $\eta(d, n, \alpha) = \eta(-d, n, \alpha)$. Many sources define the effect size in terms of its absolute value; this question shows that in this case, we don't have to do that and can speak about positive effects without loss of generality.

   (e) Show that for any $n > 0, \alpha \in (0, 1)$,

$$\eta(0, n, \alpha) = \alpha$$

   That is, if the null hypothesis is true, the power and the significance level are equal. Explain in words (1 or 2 sentences) what this means in terms of the two types of errors we could make.

2. Consider the coin-flip example from class: $X_i \sim Bern(\theta), i = 1 \dots n$, and we wish to test $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$.

   (a) Show that the power of this test to detect $\theta = \theta_1$ is given by

$$\eta = 1 - \Phi\left(d\sqrt{n} + \sqrt{\frac{\theta_0}{\theta_1} \frac{1-\theta_0}{1-\theta_1}} z_{1-\alpha/2} \right) + \Phi\left(d\sqrt{n} - \sqrt{\frac{\theta_0}{\theta_1} \frac{1-\theta_0}{1-\theta_1}} z_{1-\alpha/2} \right)$$

(b) Evaluate the power of the test to detect that a coin has probability of heads of 0.6 using 50 flips. Your answer should be a number, but note that if asked a similar question on the test, you could leave it in terms of Φ. I got $\eta = 0.289$.

3. *Optional: won't be tested on final* Prove that there does not exist a UMP test for testing $H_0 : \mu = \mu_0$ aganst $H_1 : \mu \neq \mu_0$ for $X_i \sim N(\mu, \sigma)$ at the α significance level. Hint: consider the two separate regions $\mu_1 > \mu_0$ and $\mu_1 < \mu_0$. Are there UMP tests for these regions, and are they the same?

4. For the two sample t-test with equal sample sizes n described in lecture and in the textbook on page 433,

   (a) For the *independent samples* two-sample t-test, show that the power function is

   $$\eta(d, n, \alpha) = 1 - (\Phi(d\sqrt{n} + z_{1-\alpha/2}) - \Phi(d\sqrt{n} - z_{1-\alpha/2}))$$

   as before, except that here

   $$d = \frac{\mu_X - \mu_Y}{\sigma\sqrt{2}}$$

   This is still called "effect size" in this context. Explain why the power of the t-test involves the normal distribution and not the t-distribution.

   (b) If we have *paired* data, and we are testing $H_0 : \mu_X = \mu_Y$ against $H_1 : \mu_X \neq \mu_Y$ at the 0.05 significance level, write down the power function of the test. This should be a familiar formula.

   (c) For the following effect sizes and sample sizes, compare the power of the two tests. My answers are included. You don't have to do *all* of these; just make sure you are comfortable with the calculations. Would you say that if the data is naturally paired, a paired t-test is a good thing to do? Explain.

d	n	power_paired	power_independent
0.1	10	0.0615326	0.0557472
0.2	10	0.0969354	0.0732097
0.5	10	0.3526081	0.2009556
1.0	10	0.8853791	0.6087795
0.1	20	0.0732097	0.0615326
0.2	20	0.1454725	0.0969354
0.5	20	0.6087795	0.3526081
1.0	20	0.9940005	0.8853791
0.1	50	0.1089546	0.0790975
0.2	50	0.2929889	0.1700750
0.5	50	0.9424375	0.7054180
1.0	50	0.9999998	0.9988173
0.1	100	0.1700750	0.1089546

d	n	power_paired	power_independent
0.2	100	0.5160053	0.2929889
0.5	100	0.9988173	0.9424375
1.0	100	1.0000000	0.9999998