80818 Intuitionistic Logic - Exercise Sheet 5

September 30, 2021

1. Consider the standard Heyting valued model of **HAS** on the open sets of Cantor space. Define $A : \mathbb{N} \to \mathcal{O}_{2^{\mathbb{N}}}$ as follows.

$$A(n) := \{f : \mathbb{N} \to 2 \mid f(n) = 1\}$$

   (a) Show that $[\![\forall x\, x \in A \,\vee\, x \notin A]\!] = \top$.

   (b) Show that $[\![\forall x\, x \in A]\!] = \bot$.

   (c) Show that $[\![\exists x\, x \notin A]\!] \neq \top$.

   (d) Show that the following formulation of Markov's principle is not provable in **HAS**:

$$\forall X\, (\forall x\, x \in X \,\vee\, x \notin X) \to (\neg(\forall x\, x \in X) \;\to\; \exists x\, x \notin X)$$

2. Consider the standard Heyting valued model of **HAS** on the open sets of $I^{\mathbb{N}}$. Define $A, B : \mathbb{N} \to \mathcal{O}_{I^{\mathbb{N}}}$ as follows:

$$A(n) := \{f : \mathbb{N} \to I \mid f(n) = 0 \,\vee\, f(n) = 1\}$$
$$B(n) := \{f : \mathbb{N} \to I \mid f(n) = 2 \,\vee\, f(n) = 1\}$$

   (a) Show that $[\![\forall x\, x \in A \,\vee\, x \in B]\!] = \top$.

   (b) Let $f : \mathbb{N} \to I$ be an element of $I^{\mathbb{N}}$ and n any natural number. Show that if $f(n) = 0$, then $f \notin [\![n \in B]\!]$, and if $f(n) = 2$, then $f \notin [\![n \in A]\!]$.

   (c) Let $(a_n)_{n \in \mathbb{N}}$ be a binary sequence (i.e. $a_n \in 2$ for each $n \in \mathbb{N}$). Show that

$$\bigwedge_{n \in \mathbb{N}} [\![(a_n = 0 \to n \in A) \;\wedge\; (a_n = 1 \to n \in B)]\!] \;=\; \bot$$

3. Let $(B, \leq)$ be any poset. Define a relation $\lhd$ on B and subsets of B as follows:

$$p \lhd U \quad \text{iff} \quad \forall q \leq p\, \exists r \leq q\, r \in U^{\leq}$$

   (a) Show that if $p \leq q$ and $q \lhd U$, then $p \lhd U$.

   (b) Verify that $(B, \leq, \lhd)$ satisfies the axioms of a formal topology.

(c) Show that for any proper formal topology $(B, \leq, \triangleleft)$, any formula φ, any variable assignment σ and all $p \in B$, we have $p \Vdash_\sigma \neg\varphi$ if and only if for all $q \leq p$, $q \nVdash \varphi$.

(d) Show that for the particular formal topology above, any formula φ, any variable assignment σ and any $p \in B$ we have $p \Vdash_\sigma \varphi \vee \neg\varphi$. (You may assume the law of excluded middle in the metatheory.)

Remark: This kind of forcing is sometimes used when we want models of theories in classical logic such as Zermelo-Fraenkel set theory.