

80818 Intuitionistic Logic - Exercise Sheet 7

October 21, 2021

- Let (B, \leq) be a Heyting algebra (not necessarily complete). Let U be a downwards closed subset of B . Write \overline{U} for the set

$$\overline{U} := \{\bigvee S \mid S \subseteq U \text{ and } \bigvee S \text{ exists in } B\}$$

- Show that if U is downwards closed, then \overline{U} is the smallest c-ideal containing U .
 - For $p \in B$ and $S \subseteq B$, write $p \triangleleft S$ to mean $p \in \overline{S^{\leq}}$. Show that (B, \leq, \triangleleft) is a formal topology. (You might find it helpful to recall that complete Heyting algebras satisfy the distributive law $x \wedge \bigvee S = \bigvee_{y \in S} x \wedge y$.)
 - Show that \triangleleft is the smallest subset of $B \times \mathcal{P}(B)$ satisfying the two conditions
 - (B, \leq, \triangleleft) is a formal topology.
 - For all $S \subseteq B$ if $\bigvee S$ exists in (B, \leq) , then $\bigvee S \triangleleft S$.
 - Show that for $U \subseteq B$, U is an open set of the formal topology (B, \leq, \triangleleft) above if and only if it is a c-ideal.
- Let (X, \mathcal{O}_X) and (Y, \mathcal{O}_Y) be topological spaces. A *homeomorphism* from X to Y is a bijective function $f : X \rightarrow Y$ such that both f and its inverse are continuous. We say (X, \mathcal{O}_X) is *homogeneous* if for all points x, x' of X , there is a homeomorphism $f : X \rightarrow X$ such that $f(x) = x'$. Let $(P, \bigvee, \rightarrow)$ and $(Q, \bigvee, \rightarrow)$ be complete Heyting algebras. A isomorphism from P to Q is a bijective function θ such that both θ and its inverse preserve all meets, joins and Heyting implication. An *automorphism* of $(P, \bigvee, \rightarrow)$ is an isomorphism from $(P, \bigvee, \rightarrow)$ to itself.
 - Show that Cantor space is homogeneous.
 - Let (X, \mathcal{O}_X) be any homogeneous topological space and let $(\mathcal{O}_X, \bigvee, \rightarrow)$ be the Heyting algebra of open sets. Show that if $\pi(U) = U$ for every $U \in \mathcal{O}_X$ and every automorphism π of the Heyting algebra $(\mathcal{O}_X, \bigvee, \rightarrow)$, then either $U = \perp$ or $U = \top$. (You may assume the law of excluded middle, and you may assume that if π is a bijection between Heyting algebras such that both π and π^{-1} preserve the ordering relation, then π and π^{-1} preserve all of the Heyting algebra structure.)

- (c) Let $(P, \vee, \wedge, \rightarrow)$ be any complete Heyting algebra and π any automorphism of $(P, \vee, \wedge, \rightarrow)$. Show that in the standard Heyting valued model of **HAS** we can define bijective functions $\tilde{\pi}_S : \mathcal{M}_S \rightarrow \mathcal{M}_S$ and $\tilde{\pi}_N : \mathcal{M}_N \rightarrow \mathcal{M}_N$ such that for all $n \in \mathcal{M}_N$ and $A, B \in \mathcal{M}_S$ we have $\llbracket \tilde{\pi}_N(n) \in \tilde{\pi}_S(A) \rrbracket = \pi(\llbracket n \in A \rrbracket)$ and $\llbracket \tilde{\pi}_S(A) = \tilde{\pi}_S(B) \rrbracket = \pi(\llbracket A = B \rrbracket)$. (Hint: $\tilde{\pi}_N$ is trivial to define.)
- (d) Given an automorphism π as above, and a variable assignment σ , write $\tilde{\pi} \circ \sigma$ for the result of replacing $\sigma(x)$ with $\tilde{\pi}_N(\sigma(x))$ for each number variable x and replacing $\sigma(X)$ with $\tilde{\pi}_S(\sigma(X))$ for each set variable X . Show that for every formula φ , we have $\llbracket \varphi \rrbracket_{\tilde{\pi} \circ \sigma} = \pi(\llbracket \varphi \rrbracket_\sigma)$. (Don't worry about all the details, but include at least one set quantifier of the inductive argument).
- (e) Let φ be any closed formula of **HAS** and σ any variable assignment. Show that in the standard topological model on Cantor space we have either $\llbracket \varphi \rrbracket_\sigma = \perp$ or $\llbracket \varphi \rrbracket_\sigma = \top$.