80818 Intuitionistic Logic - Exercise Sheet 9

November 4, 2021

1. We say a closed formula φ of first order arithmetic is true if it holds in the meta theory where we are working (or equivalently if it holds in the standard model of **HA** on the 2 element Heyting algebra). We consider the standard intensional realizability model of $\mathbf{HA}_\omega$. We say a closed formula φ is *self realizing* if the following hold

   (i) If there exists $e \in \mathcal{A}$ such that $e \Vdash \varphi$, then φ is true.

   (ii) There is an $e \in \mathcal{A}$ such that if φ is true, then $e \Vdash \varphi$.

   A formula of **HA** (possibly with free variables) is *negative* if it is in the smallest class containing atomic formulas and closed under conjunction, implication and universal quantifiers.

   Show that every closed negative formula of **HA** is self realizing in the intensional realizability model of $\mathbf{HA}_\omega$.

2. Let X be any set and $\mathcal{A}$ a non trivial pca. Given $f, g : X \to \mathcal{P}(\mathcal{A})$, we write $f \leq g$ to mean that there exists $e \in \mathcal{A}$ such that for all $x \in X$ and all $a \in f(x)$, we have $ea\downarrow$ with $ea \in g(x)$. We say $f \sim g$ to mean that $f \leq g$ and $g \leq f$.

   (a) Show that $\leq$ gives a preorder structure on $\mathcal{P}(\mathcal{A})^X$ (i.e. it is reflexive and transitive).

   (b) Given any set with preorder $(X, \leq)$, and $x, y \in X$ we write $x \sim y$ to mean $x \leq y$ and $y \leq x$. Show that $\sim$ is an equivalence relation and $X/\!\sim$ has the structure of a poset (i.e. a reflexive, transitive and anti symmetric relation $\leq$) with $[x] \leq [y]$ in the poset precisely when $x \leq y$ in the preorder.

   (c) Show that $\mathcal{P}(\mathcal{A})^X/\!\sim$ with the ordering above is a Heyting algebra.

   (d) Let $\mathcal{A}$ be a non trivial pca with decidable equality. Take the X above to be $\mathcal{A} \times \mathcal{A}$. For $a, b \in \mathcal{A}$, define $f_{a,b} : \mathcal{A} \times \mathcal{A} \to \mathcal{P}(\mathcal{A})$ as follows:

$$f_{a,b}(a', b') := \begin{cases} \{\mathbf{p}(\mathbf{p}ab)\top\} & a' = a, b' = b \text{ and } ab\downarrow \\ \{\mathbf{p}(\mathbf{p}ab)\bot\} & a' = a, b' = b \text{ and } ab\uparrow \\ \{\mathbf{p}(\mathbf{p}a'b')\top, \mathbf{p}(\mathbf{p}a'b')\bot\} & a' \neq a \text{ or } b' \neq b \end{cases}$$

Define $g : \mathcal{A} \times \mathcal{A} \to \mathcal{P}(\mathcal{A})$ as follows.

$$g(a, b) := \{\mathbf{p}ab\}$$

i. Show that for all $a, b \in \mathcal{A}$, $f_{a,b} \sim g$.
ii. Show that for $a, b, c, d \in \mathcal{A}$ we have $\lambda a', b'.(f_{a,b}(a', b') \cap f_{c,d}(a', b')) \sim g$.
iii. Show $\lambda a', b'.\bigcap_{a,b} f_{a,b}(a', b')$ is *strictly* smaller than g (you might find question 2 from the previous sheet useful).