

3 Omniscience Principles

In **HAS** and **HA_ω** we now have formal languages where we can talk about mathematical statements that constructive mathematicians might be interested in. In this section we will see some important examples known as *omniscience principles*. These are used by constructive mathematicians to provide prototypical examples of nonconstructive statements. They are often used to illustrate to classical mathematicians the kind of statement that is not allowed when working constructively. They are also a useful tool for showing that something is not provable constructively: if we can use a statement φ to prove an omniscience principle, then φ must also be non constructive. However, for this argument to really work, we first need to show that the omniscience principles are not provable in the formal theories we are working in. By having a range of different principles, we can classify mathematical statements by “how non constructive” they are. This is one of the key ideas of a field known as *constructive reverse mathematics*.

Each omniscience principle has a universal quantifier ranging over all *binary sequences*, that is, functions from \mathbb{N} to 2. When we are working in **HA_ω** we implement this by viewing $\forall f \in 2^{\mathbb{N}} \varphi$ as notational shorthand for $\forall f^{N \rightarrow N} ((\forall n f(n) = 0 \vee f(n) = 1) \rightarrow \varphi)$. We first consider the strongest omniscience principle, the limited principle of omniscience.

Definition 3.1. The *limited principle of omniscience (LPO)* is the following statement:

$$\forall f \in 2^{\mathbb{N}} (\forall n f(n) = 0) \vee (\exists n f(n) = 1)$$

We first observe that **LPO** is easily provable in classical mathematics:

Proposition 3.2. *The law of excluded middle implies LPO.*

Proof. Let f be any binary sequence. By the law of excluded middle we know $\exists n f(n) = 1$ is either true or false. If it is true, then we are done. Suppose then that it is false. We need to show $\forall n f(n) = 0$. For each natural number n , we know that $f(n) = 0$ or $f(n) = 1$ (exercise!). However, if we had $f(n) = 1$, this would contradict $\neg(\exists n f(n) = 1)$. Hence we must have $f(n) = 0$. But we can now deduce $\forall n f(n) = 0$, as we needed. \square

The intuition for why **LPO** is not constructively acceptable is that in order to know whether the sequence f contains a 1, we need to look at the entire sequence at once. We cannot tell whether or not there exists n such that $f(n) = 1$ by only looking at a finite portion of the sequence, whether that is the first five values, the first one million values, or first Graham’s number values or higher. If we ever find a number n such that $f(n) = 1$, then we know for sure that $\exists n f(n) = 1$, but if $f(n) = 0$ for every value of n we have checked so far, then we have no way of knowing whether this will continue to be the case forever, or if we will find an n with $f(n) = 1$ sometime in the future.

We can also motivate the idea by thinking about physical measurements. As we improve our equipment and carry out more experiments we can know

physical quantities with greater and greater precision, but we will never reach absolute precision. For example, if we are given two platinum bars, we will eventually find out if they have different lengths, but if they have the same length we will never know for sure. Hence, there is no way in general to decide which of the two cases we are in.

Finally, we can understand this idea through *computability*. Given a computer program that outputs the numbers 0 and 1, we have no way to decide, in general whether it will output 0 forever, or whether it will eventually output 1 given enough time on an ideal computer. We will later make this last intuition precise using realizability.

The remaining omniscience principles are the weak limited principle of omniscience, the lesser limited principle of omniscience and Markov's principle.

Definition 3.3. The *weak limited principle of omniscience* (**WLPO**) is the following statement.

$$\forall f \in 2^{\mathbb{N}} (\forall n f(n) = 0) \vee \neg(\forall n f(n) = 0)$$

Definition 3.4. *Markov's principle* (**MP**) is the following statement.

$$\forall f \in 2^{\mathbb{N}} \neg(\forall n f(n) = 0) \rightarrow \exists n f(n) = 1$$

Markov's principle is not always included as an omniscience principle, and many constructive mathematicians view it as a perfectly reasonable axiom to use in constructive proofs. For example, it is viewed as acceptable according to the philosophy of recursive or "Russian" constructive mathematics. The idea is that existential quantifiers need to be justified by computable functions. In the case of Markov's principle, we imagine the binary sequence as a computer program outputting a sequence of 0's and 1's as it is given different inputs. If it is false that the program will always output 0, then we can write a program to find a number n such that the program outputs 1 given input n - we simply keep running the original program on higher and higher input values until it returns 1, and then return the input value where this happened.

Note however that Markov's principle is exactly what we need to get from **WLPO** to **LPO**. More precisely, we have the following proposition.

Proposition 3.5. **LPO** is equivalent to the conjunction of **WLPO** and **MP**.

Proof. There is a short proof in intuitionistic natural deduction that $\exists n f(n) = 1$ implies $\neg(\forall n f(n) = 0)$. From this it is clear that **LPO** implies **WLPO**. Similarly, given Markov's principle, we also have the converse statement, that $\neg(\forall n f(n) = 0)$ implies $\exists n f(n) = 1$, and so we also have that **WLPO** and **MP** together imply **LPO**.

We just need to check that **LPO** implies **MP**, but this is again straightforward: if $\exists n f(n) = 1$, then we are done, and if $\forall n f(n) = 0$, we can apply ex falso together with the assumption $\neg(\forall n f(n) = 0)$ to deduce $\exists n f(n) = 1$. \square

Definition 3.6. The *lesser limited principle of omniscience* (**LLPO**) is the following statement.

$$\forall f \in 2^{\mathbb{N}} \forall n, m ((f(n) = 1 \wedge f(m) = 1) \rightarrow n = m) \longrightarrow (\forall n f(2n) = 0) \vee (\forall n f(2n + 1) = 0)$$

Most people find lesser limited principle of omniscience to be the least intuitive of the omniscience principles. To explain it a bit more, the clause $\forall n, m ((f(n) = 1 \wedge f(m) = 1) \rightarrow n = m)$ says that f has “at most one 1.” That is, $f(n)$ is equal to 0 for almost all n . It could be equal to 0 for all n , for example. However, we also allow for the possibility that $f(n)$ is equal to 1 for some n . In this case we know that $f(m)$ is equal to 0 whenever $m \neq n$. The statement $\forall n f(2n) = 0$ is telling us that if $f(n) = 1$, then n must be odd. Similarly, $\forall n f(2n + 1) = 0$ tells us that if $f(n) = 1$, then n must be even. So **LLPO** is telling us that even if we don’t know whether there is an n such that $f(n) = 1$, we can say either “if there is such an n it is odd” or “if there is such an n it is even.”

Proposition 3.7. *The weak limited principle of omniscience implies the lesser limited principle of omniscience.*

Proof. Let f be a binary sequence with at most one 1. We apply **WLPO** to the sequence f' defined by $f'(n) := f(2n)$. If we have $\forall n f'(n) = 0$, then we are done, since $f(2n) = 0$ for all n .

Now suppose that we have $\neg(\forall n f'(n) = 0)$. We will show that for all n , we have $f(2n + 1) = 0$. First recall that we know for each n that either $f(2n + 1) = 0$ or $f(2n + 1) = 1$. If we had $f(2n + 1) = 1$, then it would imply that $f(2m) = 0$ for all numbers m , since we have $2m \neq 2n + 1$ for all m . However, this would contradict $\neg(\forall n f'(n) = 0)$. Hence we have $f(2n + 1) = 0$, and since this applies for all numbers n , we are done. \square

Although **LLPO** is usually seen as not acceptable in constructive mathematics, in settings where we do not have countable choice it can be surprisingly harmless. A kind of realizability called *Lifschitz realizability* can be used to show that **LLPO** is consistent with *Church’s thesis* (an anti classical axiom that says “all functions $\mathbb{N} \rightarrow \mathbb{N}$ are computable”).