9 Independence of choice and omiscience principles over $\mathbf{HA}_\omega$

9.1 The topological model over Cantor space

Theorem 9.1. *Markov's principle does not hold in the standard topological model of $\mathbf{HA}_\omega$ on Cantor space.*

Proof. We first construct a global element of the sort $N \to N$. It suffices to find a function $g : \mathbb{N} \to S(\mathbb{N})$. So for each n, we need $g(n)$ to be a function from $\mathbb{N}$ to the open subsets of $2^{\mathbb{N}}$. We take this to be the "generic" function

$$g(n)(m) := \{f \in 2^{\mathbb{N}} \mid f(n) = m\}$$

We can then calculate

$$\begin{aligned}
[\![\neg \forall n\, g(n) = 0]\!] &= (2^{\mathbb{N}} \setminus [\![\forall n\, g(n) = 0]\!])^{\circ} \\
&= (2^{\mathbb{N}} \setminus (\bigcap_{n \in \mathbb{N}} [\![g(n) = 0]\!])^{\circ})^{\circ} \\
&= (2^{\mathbb{N}} \setminus \{\lambda n.0\}^{\circ})^{\circ} \\
&= (2^{\mathbb{N}} \setminus \emptyset)^{\circ} \\
&= 2^{\mathbb{N}}
\end{aligned}$$

However, we also have

$$\begin{aligned}
[\![\exists n\, g(n) = 1]\!] &= \{f \in 2^{\mathbb{N}} \mid \exists n\, f(n) = 1\} \\
&= 2^{\mathbb{N}} \setminus \{\lambda n.0\}
\end{aligned}$$

Hence

$$\begin{aligned}
[\![\neg \forall n\, g(n) = 0 \ \to\ \exists n\, g(n) = 1]\!] &= (2^{\mathbb{N}} \setminus \{\lambda n.0\})^{\circ} \\
&= 2^{\mathbb{N}} \setminus \{\lambda n.0\} \\
&\neq 2^{\mathbb{N}}
\end{aligned}$$

$\square$

Corollary 9.2. LPO *does not hold in the standard topological model of $\mathbf{HA}_\omega$ on Cantor space.*

9.2 Independence of countable choice

To show that the axiom of countable choice does not hold in $\mathbf{HA}_\omega$, we first consider a weaker version of the result, that has a simpler proof. We extend the signature of $\mathbf{HA}_\omega$ in include a binary relation symbol A of sort N, N. Write $\mathbf{HA}_\omega^+$ for the theory with the same axioms as $\mathbf{HA}_\omega$ over the larger signature.

We note that the axiom scheme of countable choice now includes some extra formulas, that do not occur for countable choice over $\mathbf{HA}_\omega$, namely those formulas where A occurs somewhere. In particular $\mathbf{AC}^{N,N}$ now includes the following formula.

$$\forall x^N \, \exists y^N \, Axy \;\to\; \exists f^{N \to N} \, \forall x^N \, Axf(x)$$

Theorem 9.3. $\mathbf{AC}^{N,N}$ *is not provable in* $\mathbf{HA}_\omega^+$*.*

Proof. We work in the standard topological model of $\mathbf{HA}_\omega$ over $I^{\mathbb{N}}$.

In order to make this a Heyting valued model over the extended signature, we need to show how to interpret the binary relation symbol A. We define it as follows:

$$[\![Anm]\!] := \begin{cases} \{f : \mathbb{N} \to I \mid f(n) = 0 \text{ or } f(n) = 1\} & m = 0 \\ \{f : \mathbb{N} \to I \mid f(n) = 2 \text{ or } f(n) = 1\} & m = 1 \\ \emptyset & \text{otherwise} \end{cases}$$

We clearly have

$$[\![\forall x \, \exists y \, Axy]\!] = \top$$

Hence, if $\mathbf{AC}^{N,N}$ held, we would have a global element f of $N \to N$ such that

$$[\![\forall x \, Axf(x)]\!] = \top$$

It follows from that for all n, we have $[\![f(n) = 0 \vee f(n) = 1]\!] = \top$, and $[\![f(n) = 0 \wedge f(n) = 1]\!] = \bot$. Hence $I^{\mathbb{N}} = [\![f(n) = 0]\!] \cup [\![f(n) = 1]\!]$ and $[\![f(n) = 0]\!] \cap [\![f(n) = 1]\!] = \emptyset$. Since $I^{\mathbb{N}}$ is connected we can deduce that for each n, either $[\![f(n) = 0]\!] = \top$ or $[\![f(n) = 1]\!] = \top$. That is, f has to correspond to an actual function $\mathbb{N} \to 2$ in the metatheory where we are working.

To get a contradiction from the assumption, we need to show

$$[\![\forall x \, Axf(x)]\!] \neq \top$$

We will show in fact that

$$\lambda n.1 \notin [\![\forall x \, Axf(x)]\!]$$

Suppose that $\lambda n.1 \in [\![\forall x \, Axf(x)]\!]$. In this case it would have a basic open neighbourhood $U_\sigma \subseteq [\![\forall x \, Axf(x)]\!]$ for some finite sequence σ of elements of I. Let n be any number greater than the length of σ.

We assume that $[\![f(n) = 0]\!] = \top$, with a similar proof applying for the case $[\![f(n) = 1]\!] = \top$.

Note that we can easily define a function $g : \mathbb{N} \to I$ such that $g(i) = \sigma(i)$ for $i < |\sigma|$ and such that $g(n) = 2$. Since $[\![f(n) = 0]\!]$, we have $g \in [\![f(n) = 0]\!]$. But since $[\![f(n) = 0]\!] \leq [\![An0]\!]$, this implies $g \in [\![An0]\!]$, which contradicts the definition of $[\![A]\!]$. $\square$

We will now show that $\mathbf{AC}^{N,N}$ also fails for $\mathbf{HA}_\omega$ itself. The rough idea is to combine the above proof for $\mathbf{HA}_\omega^+$ with an idea based on the omniscience principle **LLPO**. **LLPO** says that given a binary sequence f with at most one 1, either $f(2n) = 0$ for all n or $f(2n+1) = 0$. However, if $f(n) = 0$ for all n, then both cases hold, and there is no canonical way to choose one. This leads us to consider the following instance of countable choice. Suppose we have a countable family of binary sequences $f_m : \mathbb{N} \to 2$ for $m \in \mathbb{N}$ and for each m there exists $i \in \{0, 1\}$ such that for all n, $f_m(2n + i) = 0$. Countable choice would imply there is a function $g : \mathbb{N} \to 2$ such that for all m and for all n, $f_m(n + g(m)) = 0$. We will show that this is not provable in $\mathbf{HA}_\omega$.

The proof also uses some less precise general rules of thumb:

1. If we want to find a topological model where an implication does not hold, it is often helpful to consider a topological space that "looks similar" to the antecedent of the implication.

2. If we want to combine the ideas of two constructions together, it can be useful to combine the topological spaces together in a very simple way, such as binary product.

Theorem 9.4. *The following instance of countable choice is not provable in* $\mathbf{HA}_\omega$.

$$\forall f^{N \times N \to N}\, \forall m^N\, \exists i^N\, (i = 0 \vee i = 1) \wedge \forall n^N\, f(m, 2n+i) = 0 \;\to$$
$$\exists g^{N \to N}\, \forall m^N\, (g(m) = 0 \vee g(m) = 1) \wedge \forall n^N\, f(m, 2n + g(m)) = 0$$

Proof. We take X to be the topological space defined as the following subspace of $2^{\mathbb{N}\times\mathbb{N}} \times I^{\mathbb{N}}$.

$$X := \{(h, k) \in 2^{\mathbb{N}} \times I^{\mathbb{N}} \mid \forall m\, k(m) \geq 0 \to \forall n\, h(m, 2n) = 0 \wedge$$
$$k(m) \geq 2 \to \forall n\, h(m, 2n+1) = 0\}$$

We work over the standard topological model of $\mathbf{HA}_\omega$ on X and define a global element f of $\mathcal{M}_{N \times N \to N}$ as follows.

$$[\![f(m, n) = i]\!] := \{(h, k) \in X \mid h(n, m) = i\}$$

Note that the above does give a functional relation from $\mathcal{M}_N \times \mathcal{M}_N$ to $\mathcal{M}_N$. In particular we can see that each $[\![f(m, n) = i]\!]$ is an open set, since it is the intersection of X with an open set of $2^{\mathbb{N}\times\mathbb{N}} \times I^{\mathbb{N}}$. Hence there is a global element f of $\mathcal{M}_{N \times N \to N}$ satisfying it.

Furthermore, we have the following equalities for all m

$$\begin{aligned}
[\![\forall n\, f(m, 2n) = 0]\!] &= \{(h, k) \in X \mid k(m) \geq 0\} \\
&= X \,\cap\, 2^{\mathbb{N}\times\mathbb{N}} \times \{k \in I^{\mathbb{N}} \mid k(m) \geq 0\} \\
[\![\forall n\, f(m, 2n+1) = 0]\!] &= \{(h, k) \in X \mid k(m) \geq 2\} \\
&= X \,\cap\, 2^{\mathbb{N}\times\mathbb{N}} \times \{k \in I^{\mathbb{N}} \mid k(m) \geq 2\}
\end{aligned}$$

However, we can now show there is no global element g such that $[\![\forall m^N\, (g(m) = 0 \vee g(m) = 1) \wedge \forall n^N\, f(m, 2n + g(m)) = 0]\!] = \top$ by a similar argument to theorem 9.3. In fact we can show that for any global element g we have

$$(\lambda m.\lambda n.0, \lambda m.1) \notin [\![\forall m^N\, (g(m) = 0 \vee g(m) = 1) \wedge \forall n^N\, f(m, 2n + g(m)) = 0]\!]$$

$\square$