5 Heyting Valued Models

In this section we will see our first example of models of intuitionistic logic. The essential idea is that we think of the elements of a Heyting algebra as "truth values." We then assign each sentence a truth value in such a way that logical connectives $\wedge$, $\vee$ and $\rightarrow$ are sent to the corresponding operation in the Heyting algebra. This would work directly for propositional logic, but for first order we also need to deal with quantifiers. The way we will deal with this here is by making the additional assumption that we are given a *complete* Heyting algebra. We will then use this to interpret quantifiers; namely by sending universal quantifiers to meets, and existential quantifiers to joins.

5.1 Heyting valued models

Throughout this section we fix a signature with set of sorts $\mathfrak{S}$, set of operator symbols $\mathfrak{O}$ and set of relation symbols $\mathfrak{R}$.

We furthermore assume we are given a complete Heyting algebra P.

Definition 5.1. A *Heyting valued model* consists of the following data:

1. For each sort $S \in \mathfrak{S}$ a set $\mathcal{M}_S$, together with a map $E_S : \mathcal{M}_S \rightarrow P$
2. For each operator symbol $O \in \mathfrak{O}$, of sort $S_1, \ldots, S_n \rightarrow T$, a function $[\![O]\!] : \mathcal{M}_{S_1} \times \ldots \times \mathcal{M}_{S_n} \rightarrow \mathcal{M}_T$, satisfying the following for any $a_1, \ldots, a_n$ with $a_i \in \mathcal{M}_{S_i}$:
$$E([\![O]\!](a_1, \ldots, a_n)) \;\geq\; E(a_1) \wedge \ldots \wedge E(a_n)$$
3. For each relation symbol $R \in \mathfrak{R}$ of sort $S_1, \ldots, S_n$, a function $[\![R]\!] : \mathcal{M}_{S_1} \times \ldots \times \mathcal{M}_{S_n} \rightarrow P$.

Finally, for technical reasons,[1] we will also assume the following non triviality condition. For every sort $S \in \mathfrak{S}$ there exists an element a of $\mathcal{M}_S$ with $E(a) = \top$.

Definition 5.2. A *topological model* is a Heyting valued model where the Heyting algebra is the lattice of open sets of a topological space.

For $a \in \mathcal{M}_S$, we refer to $E(a)$ as the *extent* of a. One way to think about this is that the object a does not exist absolutely, but only with truth value $E(a)$. We technically allow for the case $E(a) = \bot$ ("a does not exist"), but such objects will not affect any truth values in the Heyting valued model, so we can always just ignore them. When P is the lattice of open sets of a topological space, we can visualise $E(a)$ as the region of space where a is present.

Definition 5.3. A *variable assignment* is a function σ with domain the set of free variables such that for each free variable x^S of sort S, $\sigma(x^S) \in \mathcal{M}_S$.

Given a variable assignment σ, a free variable x^S and an element a of $\mathcal{M}_S$, we write for $\sigma[x^S \mapsto a]$ for the assigment that sends x^S to a, and sends y^T to $\sigma(y^T)$ when $y^T \neq x^S$.

[1] This relates to the formulation of first order logic we are using.

For each term t of sort $S \in \mathfrak{S}$, and each variable assignment σ, we define an element $[\![t]\!]_\sigma$ of $\mathcal{M}_S$ by induction on terms.

$$\begin{aligned} [\![x^S]\!]_\sigma &:= \sigma(x^S) \\ [\![Ot_1 \dots t_n]\!]_\sigma &:= [\![O]\!]([\![t_1]\!]_\sigma, \dots, [\![t_n]\!]_\sigma) \end{aligned}$$

Note that we can show the following lemmas by induction on terms.

Lemma 5.4. *Let s be a term of sort S and σ a variable assignment. Let $x_1, \dots, x_n$ be a list of variables including any occuring free in s. Then,*

$$E(\sigma(x_1)) \wedge \dots \wedge E(\sigma(x_n)) \leq [\![s]\!]_\sigma$$

Lemma 5.5. *Let t be a term and σ a variable assignment. Suppose x^S is a free variable, and s is a term of sort S. Then we have the following equality.*

$$[\![t[x^S/s]]\!]_\sigma = [\![t]\!]_{\sigma[x^S \mapsto [\![s]\!]_\sigma]}$$

For each formula φ and each variable assignment σ defined on all variables occuring free in φ, we define an element $[\![\varphi]\!]_\sigma$ of P, by induction on terms.

$$\begin{aligned} [\![Rt_1 \dots t_n]\!]_\sigma &:= [\![R]\!]([\![t_1]\!]_\sigma, \dots, [\![t_n]\!]_\sigma) \\ [\![\bot]\!]_\sigma &:= \bot \\ [\![\varphi \wedge \psi]\!]_\sigma &:= [\![\varphi]\!]_\sigma \wedge [\![\psi]\!]_\sigma \\ [\![\varphi \vee \psi]\!]_\sigma &:= [\![\varphi]\!]_\sigma \vee [\![\psi]\!]_\sigma \\ [\![\varphi \to \psi]\!]_\sigma &:= [\![\varphi]\!]_\sigma \to [\![\psi]\!]_\sigma \\ [\![\exists x^S\, \varphi(x)]\!]_\sigma &:= \bigvee_{a \in \mathcal{M}_S} E_S(a) \wedge [\![\varphi]\!]_{\sigma[x \mapsto a]} \\ [\![\forall x^S\, \varphi(x)]\!]_\sigma &:= \bigwedge_{a \in \mathcal{M}_S} E_S(a) \to [\![\varphi]\!]_{\sigma[x \mapsto a]} \end{aligned}$$

We again get a substitution lemma.

Lemma 5.6. *Let φ be a formula, x^S a free variable of sort $S \in \mathfrak{S}$ and s a term of sort S, such that the substitution $\varphi[x^S/s]$ avoids free variable capture. Then we have the following equality.*

$$[\![\varphi[x^S/s]]\!]_\sigma = [\![\varphi]\!]_{\sigma[x^S \mapsto [\![t]\!]_\sigma]}$$

Proof. By induction on formulas. □

We can similarly show the following lemma by induction on formulas.

Lemma 5.7. *Let φ be a formula, x^S a free variable of sort $S \in \mathfrak{S}$ that does not occur in φ and a any element of $\mathcal{M}_S$. Then we have*

$$[\![\varphi]\!]_\sigma = [\![\varphi]\!]_{\sigma[x^S \mapsto a]}$$

Given a finite list of formulas $\Gamma := \varphi_1, \ldots, \varphi_n$, we write $[\![\Gamma]\!]_\sigma$ for $[\![\varphi_1]\!]_\sigma \wedge \ldots \wedge [\![\varphi_n]\!]_\sigma$.

We can now prove the *soundness theorem* for Heyting valued models.

Theorem 5.8. *Suppose that $\Gamma \vdash \varphi$ is provable in intuitionistic first order logic, where $\Gamma = \psi_1, \ldots, \psi_n$, and $x_1, \ldots, x_m$ is a list of free variables including all those occuring in Γ or φ. Then for any free variable assignment σ, we have.*

$$[\![\Gamma]\!]_\sigma \ \wedge \ E(\sigma(x_1)) \wedge \ldots \wedge E(\sigma(x_m)) \ \leq \ [\![\varphi]\!]_\sigma$$

Proof. We show this by induction on proofs. We will just do some of the cases to illustrate the idea (the remainder are left as an exercise).

The case $\wedge I$ Suppose that we have deduced $\Gamma \vdash \varphi \wedge \psi$ by $\wedge I$, together with $\Gamma \vdash \varphi$ and $\Gamma \vdash \psi$. By the inductive hypothesis, we know that for any variable assignment σ and any list of free variables $x_1, \ldots, x_m$ including all those occuring free in Γ, φ or ψ, we have the following.

$$[\![\Gamma]\!] \wedge E(\sigma(x_1)) \wedge \ldots \wedge E(\sigma(x_n)) \leq [\![\varphi]\!]_\sigma$$
$$[\![\Gamma]\!] \wedge E(\sigma(x_1)) \wedge \ldots \wedge E(\sigma(x_n)) \leq [\![\psi]\!]_\sigma$$

It follows directly from the fact that $[\![\varphi \wedge \psi]\!]_\sigma = [\![\varphi]\!]_\sigma \wedge [\![\psi]\!]_\sigma$ by definition, and the definition of meet in a lattice that

$$[\![\Gamma]\!] \wedge E(\sigma(x_1)) \wedge \ldots \wedge E(\sigma(x_n)) \leq [\![\varphi \wedge \psi]\!]_\sigma$$

which is exactly what we needed to show.

The case $\wedge E$ Suppose we have deduce $\Gamma \vdash \varphi$ from $\Gamma \vdash \varphi \wedge \psi$. By the inductive hypothesis, we know that for any variable assignment σ and any list of free variables $x_1, \ldots, x_m$ including all those occuring free in Γ, φ or ψ, we have the following.

$$[\![\Gamma]\!]_\sigma \wedge E(\sigma(x_1)) \wedge \ldots \wedge E(\sigma(x_m)) \ \leq \ [\![\varphi \wedge \psi]\!]_\sigma$$

We can easily deduce the following from the definitions.

$$[\![\Gamma]\!]_\sigma \wedge E(\sigma(x_1)) \wedge \ldots \wedge E(\sigma(x_m)) \ \leq \ [\![\varphi]\!]_\sigma$$

It only remains to deal with the case where we are given a list of variables, say $x_1, \ldots, x_m$ including all those in Γ and φ, but missing out some free variables occuring in ψ but not Γ or φ. Let $y_1^{T_1}, \ldots, y_k^{T_k}$ be a list of all the free variables occuring in ψ, but not equal to x_i for any i. Let $a_1, \ldots, a_k$ be a list with $a_j \in \mathcal{M}_{T_j}$ and $E(a_j) = \top$ for $j = 1, \ldots, k$. Let τ be the variable assignment sending y_j to a_j and otherwise equal to σ. By lemma 5.7 we have $[\![\Gamma]\!]_\tau = [\![\Gamma]\!]_\sigma$ and $[\![\varphi]\!]_\tau = [\![\varphi]\!]_\sigma$, and by the basic properties of meet, we have

$$[\![\Gamma]\!]_\tau \wedge E(\sigma(x_1)) \wedge \ldots \wedge E(\sigma(x_m)) \wedge E(\tau(y_1)) \wedge \ldots \wedge E(\tau(y_k)) = \\ [\![\Gamma]\!]_\tau \wedge E(\sigma(x_1)) \wedge \ldots \wedge E(\sigma(x_m))$$

We can thereby deduce the following, where this time the list $x_1, \ldots, x_m$ only needs to contain variables occuring free in Γ and φ.

$$[\![\Gamma]\!]_\sigma \wedge E(\sigma(x_1)) \wedge \ldots \wedge E(\sigma(x_m)) \;\leq\; [\![\varphi]\!]_\sigma$$

The remaining logical connectives are very similar and are left as an exercise. However, for quantifiers we need to be a bit careful and make use of the substitution lemmas.

The case $\exists E$ Suppose we have deduced $\Gamma \vdash \psi$ from $\Gamma \vdash \exists y\, \varphi[x/y]$ and $\Gamma, \varphi \vdash \psi$. We want to show the following, for all variable assignments σ and lists of variables $x_1, \ldots, x_m$ containing all the free variables in Γ and ψ:

$$[\![\Gamma]\!]_\sigma \wedge E(\sigma(x_1)) \wedge \ldots \wedge E(\sigma(x_m)) \leq [\![\psi]\!]_\sigma \tag{1}$$

For any $a \in \mathcal{M}_S$, we have the following by the inductive hypothesis,

$$[\![\Gamma]\!]_\sigma \wedge [\![\varphi]\!]_{\sigma[x \mapsto a]} \wedge E(\sigma(x_1)) \wedge \ldots \wedge E(\sigma(x_m)) \wedge E(a) \;\leq\; [\![\psi]\!]_\sigma$$

We deduce that we have the following:

$$\bigvee_{a \in \mathcal{M}_S} ([\![\Gamma]\!]_\sigma \wedge [\![\varphi]\!]_{\sigma[x \mapsto a]} \wedge E(\sigma(x_1)) \wedge \ldots \wedge E(\sigma(x_m)) \wedge E(a)) \;\leq\; [\![\psi]\!]_\sigma$$

By distributivity, we have

$$[\![\Gamma]\!]_\sigma \wedge E(\sigma(x_1)) \wedge \ldots \wedge E(\sigma(x_m)) \wedge \bigvee_{a \in \mathcal{M}_S} (E(a) \wedge [\![\varphi]\!]_{\sigma[x \mapsto a]}) \;\leq\; [\![\psi]\!]_\sigma$$

which is just

$$[\![\Gamma]\!]_\sigma \wedge E(\sigma(x_1)) \wedge \ldots \wedge E(\sigma(x_m)) \wedge [\![\exists x\, \varphi]\!]_\sigma \;\leq\; [\![\psi]\!]_\sigma$$

However, again by the inductive hypothesis we have

$$[\![\Gamma]\!]_\sigma \wedge E(\sigma(x_1)) \wedge \ldots E(\sigma(x_n)) \leq [\![\exists x\, \varphi]\!]_\sigma$$

It follows that

$$[\![\Gamma]\!]_\sigma \wedge E(\sigma(x_1)) \wedge \ldots \wedge E(\sigma(x_m)) \wedge [\![\exists x\, \varphi]\!]_\sigma = [\![\Gamma]\!]_\sigma \wedge E(\sigma(x_1)) \wedge \ldots \wedge E(\sigma(x_m))$$

However, we can now deduce (1).

The case $\exists I$ Suppose we have deduced $\Gamma \vdash \exists x^S\, \varphi(x)$ from $\Gamma \vdash \varphi(s)$, where s is a term of sort S, avoiding free variable capture. We want to show the following, for all variable assignments σ:

$$[\![\Gamma]\!]_\sigma \;\wedge\; E(\sigma(x_1)) \wedge \ldots \wedge E(\sigma(x_m)) \;\leq\; [\![\exists x^S\, \varphi]\!]_\sigma$$

Let $y_1^{T_1}, \dots, y_k^{T_k}$ be a list of the free variables occuring in s but not equal to x_i for any i. Let $a_1, \dots, a_k$ be a list with $a_j \in \mathcal{M}_{T_j}$ and $E(a_j) = \top$ for $j = 1, \dots, k$. Let τ be the variable assignment obtained by setting the value at y_j to be a_j for each j, and otherwise agreeing with σ. By the inductive hypothesis, we have

$$[\![\Gamma]\!]_\tau \wedge E(\sigma(x_1)) \wedge \ldots \wedge E(\sigma(x_m)) \leq [\![\varphi[x^S/s]]\!]_\tau$$

By lemma 5.6 we have $[\![\varphi[x^S/s]]\!]_\tau = [\![\varphi]\!]_{\tau[x^S \mapsto [\![t]\!]_\tau]}$. We can therefore reason as follows,

$$\begin{aligned}
[\![\Gamma]\!]_\tau \wedge E(\sigma(x_1)) \wedge \ldots \wedge E(\sigma(x_m)) &\leq [\![\varphi[x^S/s]]\!]_\tau \\
&= [\![\varphi]\!]_{\tau[x \mapsto [\![t]\!]_\tau]} \\
&\leq \bigvee_{a \in \mathcal{M}_S} [\![\varphi]\!]_{\tau[x \mapsto a]} \\
&= [\![\exists x\, \varphi]\!]_\tau \\
&= [\![\exists x\, \varphi]\!]_\sigma
\end{aligned}$$

The case $\forall E$ Suppose that we have derived $\Gamma \vdash \varphi[x/s]$ from $\Gamma \vdash \forall x^S \varphi$. Suppose that we are given a list of variables $x_1, \dots, x_m$ including all those occuring free in Γ and $\varphi[x/s]$, and σ is a variable assignment. By induction, we may assume we already have the following.

$$\begin{aligned}
[\![\Gamma]\!] \wedge E(\sigma(x_1)) \wedge \ldots \wedge E(\sigma(x_m)) &\leq [\![\forall x\, \varphi]\!]_\sigma \\
&= \bigwedge_{a \in \mathcal{M}_S} (E(a) \to [\![\varphi]\!]_{\sigma[x^S \mapsto a]}) \\
&\leq E([\![s]\!]_\sigma) \to [\![\varphi]\!]_{\sigma[x \mapsto [\![s]\!]_\sigma]} \\
&= E([\![s]\!]_\sigma) \to [\![\varphi[x/s]]\!]_\sigma
\end{aligned}$$

By the definition of Heyting implication, we can deduce

$$[\![\Gamma]\!] \wedge E(\sigma(x_1)) \wedge \ldots \wedge E(\sigma(x_m)) \wedge E([\![s]\!]_\sigma) \leq [\![\varphi[x/s]]\!]_\sigma$$

Note that the list of variables $x_1, \dots, x_n$ includes any occuring free in s. Hence we have $E([\![s]\!]_\sigma) \geq E(x_1) \wedge \ldots E(x_m)$ by lemma 5.4. Hence, we have

$$\begin{aligned}
&[\![\Gamma]\!] \wedge E(\sigma(x_1)) \wedge \ldots \wedge E(\sigma(x_m)) \wedge E([\![s]\!]_\sigma) = \\
&\qquad\qquad [\![\Gamma]\!] \wedge E(\sigma(x_1)) \wedge \ldots \wedge E(\sigma(x_m))
\end{aligned}$$

But we are now done. □

Corollary 5.9. *Suppose that* $\vdash \varphi$*. Then for every variable assignment* σ *such that* $E(\sigma(x^S)) = \top$ *for all variables* x^S *occuring free in* φ*, we have* $[\![\varphi]\!]_\sigma = \top$*. (We also write this* $\mathcal{M} \vDash \varphi$*).*

5.2 Some simple examples of Heyting valued models

We now give some basic examples of Heyting valued models to illustrate how they work, and the use of the soundness theorem to get independence results.

5.2.1 Trivial Heyting valued models

We first consider the simplest Heyting algebra, 2, which just has two elements $\bot$ and $\top$. Assuming the law of excluded middle, this is a complete Heyting algebra.[2] We can also see this Heyting algebra as the lattice of open sets of the topological space with exactly one point.

In this case, we have a set $\mathcal{M}_S$ for each sort $S \in \mathfrak{S}$. As mentioned, the elements x with $E_S(x) = \bot$ don't play any role when assigning truth values, so by removing them, we may assume without loss of generality that $E_S(x) = \top$ for all $x \in \mathcal{M}_S$. We then see that the interpretation of an operator symbol O of sort $S_1, \ldots, S_n \to T$ is just a function $\mathcal{M}_{S_1} \times \ldots \times \mathcal{M}_{S_n} \to \mathcal{M}_T$. The interpretation of a relation symbol R of sort $S_1, \ldots, S_n$ is just a subset of $\mathcal{M}_{S_1} \times \ldots \times \mathcal{M}_{S_n}$.

We can see that this has recovered the usual notion of model for classical logic. Furthermore, note that the assigment of truth values is the same as usual for models. That is, we have the following equivalences:

$$
\begin{array}{lll}
[\![\varphi \wedge \psi]\!]_\sigma = \top & \text{iff} & [\![\varphi]\!]_\sigma = \top \text{ and } [\![\psi]\!]_\sigma = \top \\
[\![\varphi \vee \psi]\!]_\sigma = \top & \text{iff} & [\![\varphi]\!]_\sigma = \top \text{ or } [\![\psi]\!]_\sigma = \top \\
[\![\varphi \to \psi]\!]_\sigma = \top & \text{iff} & [\![\varphi]\!]_\sigma = \top \text{ implies } [\![\psi]\!]_\sigma = \top \\
[\![\exists x^S \, \varphi(x)]\!] = \top & \text{iff} & \text{there exists } a \in \mathcal{M}_S \text{ such that } [\![\varphi]\!]_{\sigma[x \mapsto a]} = \top \\
[\![\forall x^S \, \varphi(x)]\!] = \top & \text{iff} & \text{for all } a \in \mathcal{M}_S, [\![\varphi]\!]_{\sigma[x \mapsto a]} = \top
\end{array}
$$

Note that if the law of excluded middle is true in the metatheory where we are working, then it also holds in any such model.

5.2.2 Sierpiński space

In order to get results about intuitionistic logic that aren't already true for classical logic, we need to consider Heyting valued models for non trivial Heyting algebras.

We first consider the open sets of Sierpiński space. These consist of $\bot = \emptyset$, $\top = \{0, 1\}$ and an "intermediate truth value" $\{0\}$, with $\bot \leq \{0\} \leq \top$ corresponding to the open set containing just 0.

This is already enough to get some simple separation results, showing that some theorems easily provable in classical logic cannot be proved in intuitionistic logic. To illustrate the idea we will consider a very simple signature, which just one sort, no relation symbols and a single 0-ary predicate symbol Q.

Theorem 5.10. *The formula $Q \vee \neg Q$ is not provable in intuitionistic logic.*

[2]When we are working in a constructive metatheory, we can instead use the set of all subsets of a singleton set.

Proof. We consider the Heyting valued model on Sierpiński space over the above signature defined as follows. We take $\mathcal{M}$ to be the one element set $\{*\}$, and define $E(*) := \top$. It remains to define the interpretation of the 0-ary relation symbol $[\![Q]\!]$, which just needs to be an element of the Heyting algebra. We take $[\![Q]\!]$ to be the intermediate truth value $\{0\}$. By definition this gives us the following for the unique variable assignment σ.

$$[\![Q]\!]_\sigma = \{0\}$$

We now consider the truth value of $\neg Q$. Recall that this is just notation for $Q \to \bot$.

$$\begin{aligned}
[\![Q \to \bot]\!] &= [\![Q]\!] \to [\![\bot]\!] \\
&= \{0\} \to \emptyset \\
&= \{1\}^{\mathrm{o}} \\
&= \emptyset
\end{aligned}$$

Note that even though the truth value of Q was non trivial, the truth value of $\neg Q$ is still $\bot$. We can now calculate,

$$\begin{aligned}
[\![Q \vee \neg Q]\!] &= [\![Q]\!] \cup [\![\neg Q]\!] \\
&= \{0\} \cup \emptyset \\
&= \{0\} \\
&\neq \top
\end{aligned}$$

Now we observe that if $Q \vee \neg Q$ was provable in intuitionistic logic we would have by theorem 5.8 that $[\![Q \vee \neg Q]\!] = \top$. Since this is not the case, we can deduce that $Q \vee \neg Q$ is not provable. $\square$

We give one more example using Sierpiński space, this time to illustrate the use of the extent predicate. We work over a signature with one sort, no operator symbols, and two relation symbols, a nulllary relation Q, and a unary relation R.

It's useful to note that we have the following lemmas. The first is for any Heyting algebra.

Lemma 5.11. *Let p, q be elements of a Heyting algebra. Then $p \to q = \top$ if and only if $p \leq q$.*

Proof. Suppose that $p \leq q$. Then $p \wedge r \leq q$ for any element r of the Heyting algebra. This implies that $p \wedge r \leq q$ if and only if $r \leq \top$. In other words $\top$ satisfies the property that uniquely characterises $p \to q$. It follows that $\top = p \to q$. Note that we can reverse each step of the above argument to show the converse. $\square$

Lemma 5.12. *Let U be an open set of a topological space X. Then $U \to \bot = (X \setminus U)^{\mathrm{o}}$ (the interior of the complement of U).*

Proof. By expanding out the definition of implication for the opens of a topological space. ☐

Theorem 5.13. *The following statement (sometimes referred to as* constant domain*) is not provable in intuitionistic logic.*

$$\forall x\,(Q \vee Rx) \;\to\; (Q \vee \forall x\, Rx)$$

Proof. We define a topological model on Sierpiński space as follows. We define $\mathcal{M}$ to have 2 elements, say $\mathcal{M} = \{a, b\}$. In order to satisfy the non triviality condition, we need $E(x) = \top$ for some x. Say $E(a) = \top$. We take the other extent to be the intermediate truth value $E(b) := \{0\}$. We again take Q to have the intermediate truth value $[\![Q]\!] = \{0\}$. We define the interpretation of R by $[\![R]\!](a) = \top$ and $[\![R]\!](b) = \bot$.

Note that $[\![\forall x\,(Q \vee Rx) \;\to\; (Q \vee \forall x\, Rx)]\!] = \top$ if and only if $[\![\forall x\,(Q \vee Rx)]\!] \subseteq [\![Q \vee \forall x\, Rx]\!]$. We explicitly compute both values as follows.

$$\begin{aligned}
[\![\forall x\,(Q \vee Rx)]\!] &= \bigwedge_{c \in \mathcal{M}} E(c) \to [\![Q \vee Rx]\!]_{x \mapsto c} \\
&= \bigwedge_{c \in \mathcal{M}} E(c) \to ([\![Q]\!] \cup [\![Rx]\!]_{x \mapsto c}) \\
&= (E(a) \to ([\![Q]\!] \cup [\![Rx]\!]_{x \mapsto a})) \wedge (E(b) \to ([\![Q]\!] \cup [\![Rx]\!]_{x \mapsto b})) \\
&= (\top \to (\{0\} \cup \top)) \cap (\{0\} \to (\{0\} \cup \bot)) \\
&= (\top \to \top) \cap (\{0\} \to \{0\}) \\
&= \top
\end{aligned}$$

$$\begin{aligned}
[\![Q \vee \forall x\, Rx]\!] &= \{0\} \cup \bigwedge_{c \in \mathcal{M}} E(c) \to [\![Rx]\!]_{x \mapsto c} \\
&= \{0\} \cup ((E(a) \to [\![Rx]\!]_{x \mapsto a}) \cap (E(b) \to [\![Rx]\!]_{x \mapsto b})) \\
&= \{0\} \cup ((\top \to \top) \cap (\{0\} \to \bot)) \\
&= \{0\} \cup (\top \cap \{1\}^{\circ}) \\
&= \{0\} \cup (\top \cap \emptyset) \\
&= \{0\} \\
&\neq [\![\forall x\,(Q \vee Rx)]\!]
\end{aligned}$$

☐