80818 Intuitionistic Logic - Exercise Sheet 4

September 23, 2021

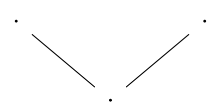
1. Work over a signature with one sort, and two 0-ary relation symbols A and B. Show the following are not provable in intuitionistic logic.

   (a) $\neg A \vee \neg\neg A$

   (b) $\neg(A \wedge B) \ \to \ \neg A \vee \neg B$

   (c) $(A \to B) \vee (B \to A)$

   Hint: In each case you might find it useful to consider a Kripke model on the following three element poset:

2. Work over a signature with one sort, a 0-ary relation A and a 1-ary relation B. The axiom **CD** states

   $$\forall x\,(A \vee Bx) \ \to \ (A \vee \forall x\, Bx)$$

   We say a Heyting valued model is *global* if $E(a) = \top$ for all $a \in \mathcal{M}$.

   (a) Let $(Q, \leq)$ be a poset. Let $(V_i)_{i \in I}$ be a family of open sets in the upset topology. Show that $\bigwedge_{i \in I} V_i = \bigcap_{i \in I} V_i$.

   (b) Show that **CD** holds in every global topological model on the upset topology of some poset. You will need to assume the law of excluded middle in the metatheory where you are working.

3. Let $(X, \mathcal{O})$ be a topological space. For any subset $Y \subseteq X$, we define the *closure* of Y, $\overline{Y}$ to be the set of $x \in X$ such that for every open neighbourhood U of x, $U \cap Y$ is non empty. You may assume the law of excluded middle throughout this question.

   (a) Show that $\overline{U} = X \setminus (X \setminus U)^{\circ}$.

   (b) Show that $U \subseteq (\overline{U})^{\circ}$ for any open set U.

(c) Suppose we are given two collections of open sets $(U_i)_{i\in I}$ and $(V_i)_{i\in I}$. Suppose that $x \in \bigwedge_{i\in I}(U_i \to V_i)$. Show x has a neighbourhood W such that for all $y \in W$ and all $i \in I$, if $y \in U_i$, then $y \in V_i$.

(d) Let V be an open set of Cantor space, and let $f \in V$. Show that $V \setminus \{f\}$ is an open set and that $(\overline{V \setminus \{f\}})^{\circ}$ contains f.

(e) Let $((\mathcal{M}_S)_{S\in\mathfrak{S}}, (\llbracket R \rrbracket)_{R\in\mathfrak{R}}, (\llbracket O \rrbracket)_{O\in\mathfrak{O}})$ be a Heyting valued model over some signature $(\mathfrak{S}, \mathfrak{R}, \mathfrak{O})$. Show that for any formula φ and any variable assignment σ, $\llbracket \neg\neg\varphi \rrbracket_\sigma = (\overline{\llbracket \varphi \rrbracket_\sigma})^{\circ}$.

(f) Consider the signature with one sort, a unary relation symbol A, and no operator symbols. Consider the Heyting valued model on Cantor space defined by taking $\mathcal{M}$ to be the set of open sets of Cantor space. Define $E(U) = \top$ for all $U \in \mathcal{M}$. Take $\llbracket A \rrbracket$ to be the identity function, i.e. given U viewed as an element of $\mathcal{M}$, $\llbracket A \rrbracket(U)$ is U, viewed as an element of the Heyting algebra of open sets.

Show that for all variable assignments σ,

$$\llbracket \forall x\,(\neg\neg A(x) \to A(x)) \rrbracket_\sigma = \bot$$