6 Kripke Models and Formal Topological Models

6.1 Forcing notation for Heyting valued models

It can be sometimes be useful to look at Heyting valued models from a different angle. Rather than looking at the truth value of a formula, which could be any element of the Heyting algebra, we fix a set of elements that are simple to describe or otherwise well behaved, and use this to give our alternative viewpoint.

For this we use the definition of basis that we have seen for topological spaces but generalises to all complete Heyting algebras (or in fact any complete poset).

Definition 6.1. A *basis* or *generating set* for a Heyting algebra P is a set $B \subseteq P$ with the following property. For any element p of P, we have

$$p = \bigvee \{q \in B \mid q \leq p\}$$

We can use this to define *forcing notation* or *Kripke-Joyal semantics* for Heyting valued models.

Definition 6.2. Let p be an element of a basis B of a Heyting algebra. We write $p \Vdash_\sigma \varphi$ for $p \leq [\![\varphi]\!]_\sigma$. If $p \Vdash_\sigma \varphi$, we say p *forces* φ.

Note that we can recover the truth value of φ with respect to a variable assignment σ from the forcing relation $\Vdash_\sigma$. Namely, we have the following equality.

$$[\![\varphi]\!]_\sigma = \bigvee \{p \in B \mid p \Vdash_\sigma \varphi\}$$

We can directly describe the forcing relation for logical formulas using the propositions below.

Proposition 6.3. *For all $p \in B$, and all formulas φ and ψ, we have the following.*

1. *$p \Vdash_\sigma \bot$ if and only if $p = \bot$.*
2. *$p \Vdash_\sigma \varphi \wedge \psi$ if and only if $p \Vdash_\sigma \varphi$ and $p \Vdash_\sigma \psi$.*
3. *$p \Vdash_\sigma \varphi \to \psi$ if and only if, for all $q \in B$ such that $q \leq p$ and $q \Vdash_\sigma \varphi$, we also have $q \Vdash_\sigma \psi$.*
4. *$p \Vdash_\sigma \forall x\, \varphi$ if and only if for all $a \in \mathcal{M}_S$ and for all $q \in B$ with $q \leq p$, if $q \Vdash E(a)$, then $q \Vdash_{\sigma[x \mapsto a]} \varphi$.*

Proof. 1 and 2 follow directly from the definitions of $\bot$ and $\wedge$ in a lattice.

We now show 3.

$$\begin{aligned} p \leq [\![\varphi]\!]_\sigma \to [\![\psi]\!]_\sigma & \qquad \text{iff} \\ p \wedge [\![\varphi]\!]_\sigma \leq [\![\psi]\!]_\sigma & \qquad \text{iff} \\ [\![\varphi]\!]_\sigma \leq p \to [\![\psi]\!]_\sigma & \qquad \text{iff} \\ \bigvee \{r \mid r \Vdash_\sigma \varphi\} \leq p \to [\![\psi]\!]_\sigma & \end{aligned}$$

However, the final inequality holds if and only if for every r such that $r \Vdash \varphi$, we have $r \wedge p \leq [\![\psi]\!]_\sigma$. However, this holds precisely when for all $q \in B$ such that $q \leq r$ and $q \leq p$ we have $q \Vdash_\sigma \psi$. Finally, observe that q is less than or equal to an element r such that $r \Vdash_\sigma \varphi$ if and only if $q \Vdash_\sigma \varphi$. Hence we now have the condition in 3.

We can show 4 by a very similar argument to 3. $\square$

We say a basis B is *proper* if it does not contain $\perp$. Note that in this case we have $p \nVdash_\sigma \perp$ for all p.

We don't have such a different looking characterisation of the other logical connectives, but we can reformulate them a little, by using covering notation.

Definition 6.4. Let $S \subseteq P$ and $p \in B$. We write $p \lhd S$ and say p is *covered by* S when $p \leq \bigvee S$.

Proposition 6.5. *For all $p \in B$, and all formulas φ and ψ, we have the following.*

1. $p \Vdash_\sigma \varphi \vee \psi$ *if and only if* $p \lhd \{q \in B \mid q \Vdash_\sigma \varphi \text{ or } q \Vdash_\sigma \psi\}$.
2. $p \Vdash_\sigma \exists x\, \varphi$ *if and only if* $p \lhd \bigcup_{a \in \mathcal{M}_S} \{q \in B \mid q \leq E(a) \text{ and } q \Vdash_{\sigma[x \mapsto a]} \varphi\}$.

6.2 Kripke models

Let $(Q, \leq)$ be a poset with a bottom element. We will define Kripke models for the poset. To keep things simple, we will only consider signatures with no operator symbols.[1] Write $\mathfrak{S}$ for the set of sorts, and $\mathfrak{R}$ for the set of relation symbols.

Definition 6.6. A *Kripke model* consists of the following data for each $q \in Q$,

1. For each sort $S \in \mathfrak{S}$, an inhabited set $\mathcal{M}_{S,q}$
2. For each relation symbol $R \in \mathfrak{R}$ of sort $S_1, \ldots, S_n$ and $q \in Q$, a set $[R]_q \subseteq \mathcal{M}_{S_1,q} \times \ldots \times \mathcal{M}_{S_n,q}$

satisfying the following conditions for all $p, q \in Q$ such that $p \leq q$:

[1] In order to deal with operator symbols it is necessary to use a construction called *sheafification* that makes the definition more complicated. We will see a version of this later for topological models of $\mathbf{HA}_\omega$. Exercise: Think about what goes wrong in the definition of Heyting valued model here when we have a binary operation symbol.

1. $\mathcal{M}_{S,p} \subseteq \mathcal{M}_{S,q}$ for all sorts S.
2. For each relation symbol $R \in \mathfrak{R}$ we have $[R]_p \subseteq [R]_q$.

Given a Kripke model, we define a topological model as follows. We first need to specify the topological space. We take it to be Q with the upset topology.

1. For each sort $S \in \mathfrak{S}$, we define $\mathcal{M}_S := \bigcup_{q \in Q} \mathcal{M}_{S,q}$.
2. For each sort $S \in \mathfrak{S}$, and $a \in \mathcal{M}_S$, we define $E_S(a) := \{q \in Q \mid a \in \mathcal{M}_{S,q}\}$.
3. For each relation symbol $R \in \mathfrak{R}$ of sort $S_1, \ldots, S_n$ and for elements $a_1, \ldots, a_n$ with each a_i belonging to $\mathcal{M}_{S_i}$, we define $[\![R]\!](a_1, \ldots, a_n)$ to be $\{q \in Q \mid (a_1, \ldots, a_n) \in [R]_q\}$.

Note that by the definition of Kripke model, we can see that for each a, $E_S(a)$ is an open set in the topology, i.e. an upwards closed subset of Q. Similarly, the condition on relations in the definition of Kripke model tells us that $[\![R]\!](a_1, \ldots, a_n)$ is always an open set. Hence we do indeed have a well defined topological model on the upset topology.

We can now explicitly describe the forcing relation for topological models derived from Kripke models in this way. First recall that we can view the elements of Q itself as a basis for the upset topology, where $q \in Q$ corresponds to the upset $\{r \in Q \mid q \leq r\}$. Under this correspondence, we have for any upwards closed set U that $p \lhd U$ if and only if $p \in U$. Also recall that the correspondence between elements of Q ordered by $\leq$ and upwards closed sets with least element ordered by $\subseteq$, the order is reversed. Hence if we want to describe the forcing relation in terms of the order on Q, we need to reverse all the inequalities in proposition 6.3. This gives us the following definition:

$p \nVdash_\sigma \bot$		always
$p \Vdash_\sigma \varphi \wedge \psi$	iff	$p \Vdash_\sigma \varphi$ and $p \Vdash_\sigma \psi$
$p \Vdash_\sigma \varphi \vee \psi$	iff	$p \Vdash_\sigma \varphi$ or $p \Vdash_\sigma \psi$
$p \Vdash_\sigma \varphi \to \psi$	iff	for all $q \geq p$ if $q \Vdash_\sigma \varphi$ then $q \Vdash_\sigma \psi$
$p \Vdash_\sigma \exists x\, \varphi$	iff	there exists $a \in \mathcal{M}_p$ such that $p \Vdash_{\sigma[x \mapsto a]} \varphi$
$p \Vdash_\sigma \forall x\, \varphi$	iff	for all $q \geq p$ and $a \in \mathcal{M}_q$, $q \Vdash_{\sigma[x \mapsto a]} \varphi$

6.3 Formal topological models

Kripke models turn out to be useful in some situations, but are limited in some ways. For this reason, we consider a generalisation that includes some more examples of Heyting valued models, even some that are not topological models, but behaves very in a similar way to Kripke models, with a similar explicit description of their forcing relation. Instead of a poset, we consider the more general definition of formal topology. Fix a formal topology B with ordering relation $\leq$ and covering relation $\lhd$. We again restrict to signatures without operator symbols for simplicity.

Definition 6.7. A *formal topological model* consists of the following data for each $p \in B$,

1. For each sort $S \in \mathfrak{S}$, a set $\mathcal{M}_{S,p}$
2. For each relation symbol $R \in \mathfrak{R}$ of sort $S_1, \ldots, S_n$ and $p \in B$, a set $[R]_p \subseteq \mathcal{M}_{S_1,p} \times \ldots \times \mathcal{M}_{S_n,p}$

satisfying the following conditions for all $q, p \in B$:

1. If $q \leq p$, then $\mathcal{M}_{S,p} \subseteq \mathcal{M}_{S,q}$. If $q \triangleleft U$ and a is an element of $\bigcap_{p \in U} \mathcal{M}_{S,p}$ then $a \in \mathcal{M}_{S,q}$.
2. For each relation symbol $R \in \mathfrak{R}$, if $q \leq p$, then $[R]_p \subseteq [R]_q$. If $q \triangleleft U$ and $a \in \bigcap_{p \in U} [R]_p$ then $a \in [R]_q$.

Similarly to Kripke models, we can define a Heyting valued model over the open sets of the formal topology as follows.

1. For each sort $S \in \mathfrak{S}$, we define $\mathcal{M}_S := \bigcup_{q \in B} \mathcal{M}_{S,q}$.
2. For each sort $S \in \mathfrak{S}$, and $a \in \mathcal{M}_S$, we define $E_S(a) := \{q \in B \mid a \in \mathcal{M}_{S,q}\}$.
3. For each relation symbol $R \in \mathfrak{R}$ of sort $S_1, \ldots, S_n$ and for elements $a_1, \ldots, a_n$ with each a_i belonging to $\mathcal{M}_{S_i}$, we define $[\![R]\!](a_1, \ldots, a_n)$ to be $\{q \in B \mid (a_1, \ldots, a_n) \in [R]_q\}$.

Note that we have chosen the definition of formal topological model so that extent and relation symbols are interpreted as open sets with respect to the formal topology, and so we do get a Heyting valued model over the complete Heyting algebra of open sets.

Note furthermore that for each element p of B, we can define U_p to be the least open set containing p to get a basis for the Heyting algebra of open sets. We also have $U_p \triangleleft V$ if and only if $p \triangleleft V$. We can then use propositions 6.3 and 6.5 to explicitly describe the forcing relation with respect to elements p of B as follows.

$p \Vdash_\sigma \bot$	iff	$p \triangleleft \emptyset$
$p \Vdash_\sigma \varphi \wedge \psi$	iff	$p \Vdash_\sigma \varphi$ and $p \Vdash_\sigma \psi$
$p \Vdash_\sigma \varphi \vee \psi$	iff	$p \triangleleft \{q \in B \mid q \Vdash_\sigma \varphi$ or $q \Vdash_\sigma \psi\}$
$p \Vdash_\sigma \varphi \to \psi$	iff	for all $q \leq p$ if $q \Vdash_\sigma \varphi$ then $q \Vdash_\sigma \psi$
$p \Vdash_\sigma \exists x \, \varphi$	iff	$p \triangleleft \{q \in B \mid \exists a \in \mathcal{M}_q \, q \Vdash_{\sigma[x \mapsto a]} \varphi\}$
$p \Vdash_\sigma \forall x \, \varphi$	iff	for all $q \leq p$ and $a \in \mathcal{M}_q$, $q \Vdash_{\sigma[x \mapsto a]} \varphi$

We say a formal topology is *proper* if it is never the case that $p \triangleleft \emptyset$. In this case we can see that $p \nVdash_\sigma \bot$ for all $p \in B$.