80818 Intuitionistic Logic - Solutions to Exercise Sheet 2

1. (a) There was some ambiguity in this question. We will construct a closed term whose sort is the single finite type $\sigma \to (\rho \to \tau)$.

   It is suprisingly tricky to show this directly, so we will use the λ-abstraction lemma (Lemma 2.5). Namely, we define

   $$t' := \lambda x^\sigma.\lambda y^\rho.tx$$

   This is fine as an answer, but if we wanted to do so, we can explicitly write out a term using the proof of the λ-abstract lemma. We first show what we would get if just expand out the proof in the most naïve way.

   $$\begin{aligned} t' &:= \lambda x.\lambda y.(tx) \\ &= \lambda x.(\mathbf{s}(\mathbf{k}t))(\mathbf{k}x) \\ &= \mathbf{s}(\lambda x.\mathbf{s}(\mathbf{k}t))(\lambda x.\mathbf{k}x) \\ &= \mathbf{s}(\mathbf{s}(\mathbf{ks})(\mathbf{s}(\mathbf{kk})(\mathbf{k}t)))(\mathbf{s}(\mathbf{kk})\mathbf{i}) \end{aligned}$$

   However, we can make some simplifications to get a simpler term. Note that instead of $\lambda y.tx$ we can use $\mathbf{k}(tx)$, and when we see $\lambda x.tx$ we can just use t instead and it will work just as well. This gives the following alternative definition t'':

   $$\begin{aligned} t'' &:= \lambda x.\mathbf{k}(tx) \\ &= \mathbf{s}(\mathbf{kk})t \end{aligned}$$

   Let's verify that t'' actually works:

   $$\begin{aligned} t''xy &= \mathbf{s}(\mathbf{kk})txy \\ &= ((\mathbf{kk}x)(tx))y \\ &= \mathbf{k}(tx)y \\ &= tx \end{aligned}$$

   From now on we won't explicitly calculate the λ-terms - they can get quite complicated in general.

(b) We use the recursor $\mathbf{r}$ for this. We first define the open term with variables n and m:

$$n + m := \mathbf{r}n(\lambda x.\lambda y.Sx)m$$

We verify that this works by inducttion:

$$\begin{aligned} n + 0 &= \mathbf{r}n(\lambda x.\lambda y.Sx)0 \\ &= n \\ n + (Sm) &= \mathbf{r}n(\lambda x.\lambda y.Sx)(Sm) \\ &= (\lambda x.\lambda y.Sx)(\mathbf{r}n(\lambda x.\lambda y.Sx)m) \\ &= S(\mathbf{r}n(\lambda x.\lambda y.Sx)m) \\ &= S(n + m) \end{aligned}$$

Finally, to get a closed term we use λ-abstraction again to get $\lambda n.\lambda n.\mathbf{r}n(\lambda x.\lambda y.Sx)m$.

(c) There are multiple correct answers, but e.g. we can take $\mathrm{prd} := \mathbf{r}0(\mathbf{ki})$. We verify that this works:

$$\begin{aligned} \mathrm{prd}\, 0 &:= \mathbf{r}0(\mathbf{ki})0 \\ &= 0 \\ \mathrm{prd}(Sn) &:= \mathbf{r}0(\mathbf{ki})(Sn) \\ &= \mathbf{ki}(\mathbf{r}0(\mathbf{ki})n)n \\ &= \mathbf{i}n \\ &= n \end{aligned}$$

Once again, we can get a closed term by λ-abstraction.

(d) This is exactly the same as $+$, but with prd in place of S. Explicitly this gives us the term

$$\lambda n.\lambda m.\mathbf{r}n(\lambda x.\lambda y.\, \mathrm{prd}\, x)m$$

(e) E.g. we can define $\mathbf{d}_0 := \mathbf{r}0(\mathbf{k}(\mathbf{k}(S0)))$. Verification is similar to before.

2. (a) We define this for each $f \in 2^{\mathbb{N}}$ by induction on n.

For $n = 0$, we have nothing to do by $\neg(n < 0)$.

For $n = (Sm)$, we have by the inductive hypothesis that either for all $k < m$, $f(k) = 0$, or there exists $k < m$ such that $f(k) = 1$. We deal with the latter case first. If $k < m$, then also $k < Sm$, and so there exists $k < Sm$ with $f(k) = 1$. In the former case, we have by the definition of $2^{\mathbb{N}}$ that either $f(m) = 0$ or $f(m) = 1$. First suppose $f(m) = 0$. For every $k < m$, we have either $k < m$ or $k = m$. However in either case we have $f(k) = 0$. Now suppose $f(m) = 1$. In this case we are again done, since $m < Sm$.

(b) Suppose first that $n = 0$. In that case it is impossible that there is $m < n$ with $f(m) = 1$, since we have $\neg(m < 0)$.

Now suppose that $n = Sk$. Suppose that $f(m) = 1$ for some m with $m < Sk$. We have either $m < k$ or $m = k$. In the former case, we can apply the inductive hypothesis and we are done. In the latter case, we know by part (a) that either $f(i) = 0$ for all $i < k$ or $f(i) = 1$ for some $i < k$. In the former case, note that we can take $m' = k$ to get a witness there exists $m' < Sk$ satisfying the required condition, and we are done. In the latter case, we can again imply the inductive hypothesis.

3. (a) Given $f : \mathbb{N} \to \mathbb{N}$, we compose with $\mathbf{d}_0$ to get $g := \lambda x.\mathbf{d}_0(fx)$. By **WLPO** we can assume that either $g(n) = 0$ for all n or it is false that $g(n) = 0$ for all n. Suppose first that $g(n) = 0$ for all n. We will show $f(n) = 0$ for all n. From last time, we know that either $f(n) = 0$ or $f(n) = Sk$ for some k. The latter implies $g(n) = S0 \neq 0$, giving a contradiction, and so $f(n) = 0$.

Now suppose that it is false that $g(n) = 0$ for all n. We will show that it is false that $f(n) = 0$ for all n. Suppose that this was the case. If so, we would have for each n, $g(n) = \mathbf{d}_0 0 = 0$, giving a contradiction.

(b) By part (a) we know that for every $f : \mathbb{N} \to \mathbb{N}$ there exists k such that either $k = 0$ and we have $\forall n\, f(n) = 0$ or $k = 1$ and we have $\neg(\forall n\, f(n) = 0)$. We need to show k is unique, so suppose k' is another number satisfying the same condition. We want to show $k = k'$. We know that $k = 0$ or $k = 1$ and similarly $k' = 0$ or $k' = 1$. This gives us four cases to consider. If $k = 0$ and $k' = 0$ then we are done, and similarly if $k = 1$ and $k' = 1$. If $k = 0$ and $k' = 1$ then we have both $\forall n\, f(n) = 0$ and $\neg(\forall n\, f(n) = 0)$, so we can apply ex falso. The same argument applies when $k = 1$ and $k' = 0$, and so we are done.

Applying unique choice, we get $F : \mathbb{N}^{\mathbb{N}} \to 2$ such that if $F(f) = 0$ then $\forall n\, f(n) = 0$ and if $F(f) = 1$, then $\neg(\forall n\, f(n) = 0)$. In particular if we define $f(m)$ to be $\lambda m.0$, then we have $F(f) = 0$. There are several ways to define g_n. E.g. we can take $g_n(m) = 1$ if $m = n$ and $g_n(m) = 0$ if $m \neq n$. If $m < n$, then $m \neq n$ and so $g_n(m) = 0 = f(m)$. But $g_n(n) = 1$, and so $F(g_n) = 1$.

(c) Let h be any function $\mathbb{N} \to 2$. We first check that for every n there is a unique m such either $m = f(n)$ and $h(i) = 0$ for all $i < n$, or there is a least $i < n$ such that $h(i) = 1$ and $m = g_i(n)$. We first show that there exists such a number m. By 2.(b) we have either that $h(i) = 0$ for all $i < n$, in which case we can take $m = f(n)$ or there exists $i < n$ which is least such that $h(i) = 1$, in which case we can take $m = g_i(n)$. We now check uniqueness. Suppose m, m' are two values both satisfying the condition. We again split into the two cases that either $h(i) = 0$ for all $i < n$, or there exists $i < n$ which is least such that $h(i) = 1$. In the former case we have $m = f(n) = m'$.

In the latter case the least i such that $h(i) = 1$ is unique (because if there were two i, i' we would have $i < i'$ and $i' < i$). Hence we have $m = g_i(n)$ and $m' = g_i(n)$, giving $m = m'$ as required. We can hence apply unique choice to show there is a function k such that for each n, $k(n)$ is the number m satisfying the condition above.

We will now deduce **WLPO**. Let h be any binary sequence. If k is as above, we have either $F(k) = F(f)$ or $F(k) \neq F(f)$. We will use this to show the instance of **WLPO** for h.

Note that if we had $\forall n\, h(n) = 0$, it would follow (by function extensionality) that $k = f$. It would follow that $F(k) = F(f)$. Hence if $F(k) \neq F(f)$ we can deduce $\neg(\forall n\, h(n) = 0)$. We will now show that if $F(k) = F(f)$ then we have $\forall n\, h(n) = 0$. For each n, will show $h(n) = 0$. It suffices to derive a contradiction from the assumption $h(n) = 1$. In this case, there is a least i such that $h(i) = 1$ by 2.(b). In this case we have that for all m, $k(m) = g_i(m)$, since if $m \leq i$, then $k(m) = f(m) = g_i(m)$, and if $m > i$, then $k(m) = g_i(m)$ by definition. Hence by function extensionality we have $k = g_i$. However, this gives a contradiction, since then $F(k) = F(g_i) \neq F(f)$.