4 Heyting Algebras and Topology

4.1 Posets and lattices

We first recall some basic theory about posets.

Definition 4.1. A *poset* is a set P, together with a binary relation $\leq$ satisfying the following axioms.

1. $x \leq x$ (reflexivity)
2. $x \leq y \wedge y \leq z \;\rightarrow\; x \leq z$ (transitivity)
3. $x \leq y \wedge y \leq x \;\rightarrow\; x = y$ (anti-symmetry)

We will sometimes write $x \leq y$ as $y \geq x$.

Definition 4.2. The *top* or *greatest* element of a poset P is $\top \in P$ such that for all $x \in P$, $x \leq \top$.

The *bottom* or *least* element of a poset P is $\bot \in P$ such that for all $x \in P$, $\bot \leq x$.

Definition 4.3. Let S be a set of elements of a poset P. We say z is the *least upper bound* or *join* of S if

1. for all $x \in S$, $x \leq z$ (z is an upper bound)
2. if $y \in P$ is such that for all x in S $x \leq y$, then $z \leq y$ (any other upper bound is greater)

Note that any set has at most one join. If it exists, we write it as $\bigvee S$. Given a two element set $\{x, y\}$, we write $\bigvee\{x, y\}$ as $x \vee y$.

We similarly define the *greatest lower bound* or *meet* of S as an element z such that

1. for all $x \in S$, $z \leq x$
2. if $y \in P$ is such that for all x in S $z \leq x$, then $y \leq z$.

The meet of a set is also unique, and when it exists we write it as $\bigwedge S$, and for two element sets $\{x, y\}$, we write $\bigwedge\{x, y\}$ as $x \wedge y$.

Definition 4.4. We say a poset P is *complete* if for every set $S \subseteq P$, $\bigvee S$ and $\bigwedge S$ both exist.

Proposition 4.5. *A poset P is complete if and only if it has all joins (or all meets).*

Proof. Let S be a set. We want to construct the greatest lower bound of S. Define $L := \{x \in P \mid \forall y \in S\, x \leq y\}$ the set of all lower bounds of S. We claim $\bigvee L$ is the greatest lower bound of S. For each $y \in S$, we know that for every x in L, $x \leq y$. It follows that $\bigvee L \leq y$. By applying this to each $y \in S$ we see $\bigvee L$ is a lower bound for S and it is clear it is the greatest one (since L contains every other lower bound). $\square$

Definition 4.6. A *lattice* is a poset P with least and greatest elements $\perp$ and $\top$, and any two elements $x, y \in P$ have a meet $x \wedge y$ and a join $x \vee y$.

Example 4.7. If P is any collection of sets, then it has a canonical ordering given by $x \leq y$ whenever $x \subseteq y$. If P is closed under binary intersection $x \cap y$ and union $x \cup y$, then $(P, \subseteq)$ is a lattice.

It is possible to recover the poset relation on a lattice from the operations $\wedge$ and $\vee$. Because of this, we can also think of lattices as *algebraic* structures (i.e. sets with operations satisfying equations).

4.2 Heyting algebras

To motivate Heyting algebras, we first consider an important example, the *Lindenbaum-Tarski algebra* of an intuitionistic theory.

Let T be a theory over some signature. We define an equivalence relation on formulas by $\varphi \sim \psi$ when $T \vdash \varphi \leftrightarrow \psi$. Let P be the quotient of the set of formulas by $\sim$. Note that we have a canonical ordering on P: we say $[\varphi] \leq [\psi]$ if $T \vdash \varphi \to \psi$, and note that this is preserved by the equivalence relation.

We see that P is a lattice, and moreover each part of the lattice structure corresponds naturally to logical connectives. For example $[\perp] \leq [\varphi]$ for all formulas φ, by ex falso. Similarly $[\top]$ is the greatest element of the lattice. We can show that $[\varphi \vee \psi]$ is greater than $[\varphi]$ and $[\psi]$ by the $\vee$ introduction rule, and the $\vee$ elimination rule precisely tells us that any other upper bound is greater than $[\varphi \vee \psi]$. Similarly we can use conjunction to construct meets in the Lindenbaum-Tarski algebra.

However, there is one logical connective that does not appear as part of the lattice structure, namely *implication*, $\to$. As before, we can translate the introduction and elimination rules into properties of the order structure. For all formulas φ, ψ and χ, we can use implication introduction to show that if $[\chi \wedge \varphi] \leq [\psi]$, then $[\chi] \leq [\varphi \to \psi]$. Using introduction elimination, we can also show the converse: if $[\chi] \leq [\varphi \to \psi]$, then $[\chi \wedge \varphi] \leq [\psi]$.

We can see Heyting algebras as posets that behave similar to Lindenbaum-Tarski algebras. We can think of the elements of a Heyting algebra as "truth values." We will use them to define certain models (*Heyting valued models*) of theories in intuitionistic logic. Formally, we define them as follows.

Definition 4.8. A *Heyting algebra* is a lattice $(P, \top, \perp, \wedge, \vee)$ together with a binary operation $\to$, *implication*, satisfying the condition below.

$$z \leq x \to y \qquad \text{if and only if} \qquad z \wedge x \leq y$$

4.3 Topological spaces

For this course, our main source of examples of Heyting algebras are going to be *topological spaces*. From the point of view of Heyting algebras, we can think of topological spaces as concrete examples of Heyting algebras, whose elements are all subsets of a fixed set.

The original motivation for topological spaces is to model "spaces" that appear in mathematics, such a spheres or 3-dimensional space.

Another idea that will be important in this course is the notion of morphism between topological spaces, *continuous* functions.

4.3.1 Some basic definitions and examples

Definition 4.9. Let X be a set. A *topology* on X is a collection $\mathcal{O}$ of subsets of X, satisfying the following conditions.

1. X and $\emptyset$ are elements of $\mathcal{O}$.
2. If U and V are elements of $\mathcal{O}$, their intersection $U \cap V$ also belongs to $\mathcal{O}$.
3. If $S \subseteq \mathcal{O}$ is a set of elements of $\mathcal{O}$, then their union $\bigcup S$ also belongs to $\mathcal{O}$.

We refer to the elements of $\mathcal{O}$ as *open sets*, and the elements of X as *points*. We say a set X together with a topology is a *topological space*.

Example 4.10. If X is any set, then we can define a topology by taking $\mathcal{O}$ to be the collection of *all* subsets of X. This is referred to as the *discrete* topology on X.

We can also define a topology by taking $\mathcal{O}$ to have just two elements, $\emptyset$ and X. This is referred to as the *indiscrete topology*.

Example 4.11. We define a topology on the set with two elements $2 := \{0, 1\}$ as the set of subsets $\{\emptyset, \{0\}, \{0, 1\}\}$. This topological space is referred to as *Sierpiński space*, S.

Example 4.12. We define a topology on the set with three elements $I = \{0, 1, 2\}$ as the set of subsets $\{\emptyset, \{0, 1\}, \{1\}, \{1, 2\}, \{0, 1, 2\}\}$. We will refer to this space as the *abstract interval*, or just the *interval*.

Example 4.13. If $(Q, \leq)$ is any poset, we can define a topology on Q as follows. A set $U \subseteq Q$ is *upwards closed* or an *upset* if whenever $x \in U$ and $x \leq y$, we have $y \in U$. The set of all upsets defines a topology on Q referred to as the *upset* or *Alexandrov* topology on the poset.

Example 4.14. If $(X, \mathcal{O})$ is a topological space, and Y is a subset of X, then we can also define a topology on Y as follows. We say a set $U \subseteq Y$ is open if for some open set V of X, we have $U = V \cap Y$. We refer to this as the *subspace topology* on Y.

Definition 4.15. Let $(X, \mathcal{O})$ be a topological space, and $Y \subset X$ any subset. We define the *interior* of Y, Y° to be the union of all open sets U such that $U \subseteq Y$.

Proposition 4.16. *We observe the following properties of interior.*

1. *For all Y, $Y^\circ \subseteq Y$.*

2. *For all Y, Y° is an open set.*

3. *If Y is already an open set, then $Y^{\circ} = Y$.*

Example 4.17. In Sierpiński space, the interior of $\{1\}$ is the empty set.

Proposition 4.18. *For any topological space $(X, \mathcal{O})$, the poset $\mathcal{O}$ of open sets ordered by inclusion is a complete Heyting algebra.*

Proof. We define joins using union and meets using (binary) intersection. Top and bottom element are given by X and $\emptyset$ respectively. It only remains to define the implication operator. We might try to define this using the canonical implication on subsets, i.e. for sets U and V, the set

$$\{x \in X \mid x \in U \to x \in V\}.$$

However, the above set is not necessarily open, even when U and V are. To get an open set, we use the interior operator. Namely, we define

$$U \to V := \{x \in X \mid x \in U \to x \in V\}^{\circ}.$$

It only remains to check that this satisfies the necessary condition to be the implication of a Heyting algebra (exercise!). □

NB: Using the law of excluded middle, we could alternatively define implication by

$$U \to V := (V \cup (X \setminus U))^{\circ}$$

Remark 4.19. *Since lattices of open sets have all joins, they must also have meets, by proposition 4.5. Whereas the join of a set $S \subseteq \mathcal{O}$ is just the union, $\bigvee S = \bigcup S$, the intersection $\bigcap S$ is not necessarily open. However, we can still explicitly describe the meet of S using the interior operator, $\bigwedge S = (\bigcap S)^{\circ}$.*

Definition 4.20. Let $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$ be topological spaces. We say a function $f : X \to Y$ is *continuous* if for every open set U of Y, the preimage, $f^{-1}(U)$ is an open set of X.

4.3.2 Neighbourhoods in a topological space

Definition 4.21. Let $(X, \mathcal{O})$ be a topological space. A *neighbourhood* of a point $x \in X$ is an open set $U \in \mathcal{O}$ such that $x \in U$.

We think of a neighbourhood of a point x as a set of points that are "nearby" x. For example, in a discrete topology, every point x has a neighbourhood $\{x\}$ that does not contain any other points. We can visualise this as a clear space around x that does not contain any elements. On the other hand in an indiscrete space, the only neighbourhood of a point x is the entire space. The points are so "close together" that you can't look at one without looking at the whole space. In Sierpiński space, every neighbourhood of 1 also contains 0, so 0 is "infinitely close" to 1, but this is not a symmetric relation: 0 has a neighbourhood that does not contain 1.

Definition 4.22. Let $(X, \mathcal{O})$ be a topological space. A *basis* of the topology, is a set of open sets $B \subseteq \mathcal{O}$ with the following property. For any open neighbourhood U of a point x, there exists an open neighbourhood V of x with $V \subseteq U$ and $V \in B$.

Equivalently, a basis is a set B such that for every open set U, there is a set $S \subseteq B$ such that $U = \bigcup S$.

Remark 4.23. *Note that the set of all open sets is a basis. This satisfies the definition, but is typically not useful.*

Note that if B is a basis of a topology, then a set is open if and only if it is equal to the union $\bigcup S$ for some $S \subseteq B$.

Example 4.24. If $(Q, \leq)$ is any poset, then we can define a basis of the upset topology as the set of upwards closed sets with least element. Note that there is a precise correspondence between the elements of the basis and elements of Q.

4.3.3 Product topologies

Definition 4.25. We define a topology on $\mathbb{N}^\mathbb{N}$ as follows. Given a finite sequence of numbers σ of length k, we define the set U_σ as follows.

$$U_\sigma := \{f : \mathbb{N} \to \mathbb{N} \mid \forall i < k\, f(i) = \sigma(i)\}$$

We define a set U to be open if it can be written as a union of sets of the form U_σ for finite sequences σ. We refer to the topological space with these open sets as *Baire space.*

We refer to the subspace topology on the set of binary sequences $2^\mathbb{N} \subseteq \mathbb{N}^\mathbb{N}$ as *Cantor space.*

We refer to the set of binary sequences with at most one 1, with the subspace topology as $\mathbb{N}_\infty$.

Proposition 4.26. *Assume* **LPO**. *Then* $\mathbb{N}_\infty$ *is isomorphic to the set* $\mathbb{N} \amalg \{\infty\}$*, where a set* $U \subseteq \mathbb{N} \amalg \{\infty\}$ *is open when it satisfies the following. If* $\infty \in U$*, then for some* $n \in \mathbb{N}$*,* U *contains every* $m \in \mathbb{N}$ *with* $m \geq n$*.*

Proposition 4.27. *Given* $\mathbb{N}^\mathbb{N}$ *with the Baire space topology, and* $\mathbb{N}$ *with the discrete topology, a function* $F : \mathbb{N}^\mathbb{N} \to \mathbb{N}$ *is continuous if and only if it satisfies the following condition. For every* $f \in \mathbb{N}$*, there is a natural number* n *such that for any* $g \in \mathbb{N}$ *satisfying the condition that* $g(i) = f(i)$ *for* $i < n$*, we have* $F(g) = F(f)$*.*

Proof. Since $\mathbb{N}$ has the discrete topology, every singleton set $\{m\}$ is open, and in any case we can write any set as a union of singletons. It follows that F is continuous if and only if $F^{-1}(\{m\})$ is open for every m. This says exactly that any element f of $F^{-1}(\{m\})$ has a basic open neighbourhood contained in $F^{-1}(\{m\})$. This precisely says that the condition described in the proposition holds whenever $F(f) = m$. However, F is continuous precisely when this condition holds for arbitrary m, so given a function f, we can just apply it with $m := F(f)$. $\square$

The Baire space topology on $\mathbb{N}^{\mathbb{N}}$ is an instance of a more general construction known as the product topology.

Definition 4.28. Suppose we are given a set I, and a family of topological spaces $(X_i, \mathcal{O}_i)$ for each $i \in I$. We define a topology on the product $\prod_{i \in I} X_i$ as follows. Given a pair $\sigma = (F, (U_j)_{j \in F})$ consisting of a finite subset F of I, say $i_1, \ldots, i_k$, together with open sets $U_j \in \mathcal{O}_{i_j}$ for $j = 1, \ldots, k$, we define the set U_σ by the equation

$$U_\sigma := \{f \in \prod_{i \in I} X_i \mid \forall j \in F \, f(j) \in U_j\}$$

We refer to sets of the form U_σ as basic opens, and define a set to be open if it is a union of basic opens. We refer to the resulting topological space as the *product topology* of the family.

Proposition 4.29. *Baire space is the product of the (constant) family consisting of countably many copies of* $\mathbb{N}$ *with the discrete topology.*

4.3.4 Connectedness

An important concept when looking at the behaviour of topological models is that of connectedness, which informally is the idea that a space is "indecomposable" - it is impossible to break the space up cleanly into separate pieces.

Definition 4.30. A topological space X is *connected* if given open sets U and V such that $U \cup V = X$ and $U \cap V = \emptyset$ we have either $X = U$ or $X = V$.

Proposition 4.31. *A topological space* X *is connected if and only if for every discrete topological space* Y*, every map* $X \to Y$ *is constant.*

Example 4.32. Sierpiński space and the abstract interval are connected.

Example 4.33. Any discrete space with at least two distinct points is not connected.

Example 4.34. Baire space, Cantor space and $\mathbb{N}_\infty$ are not connected.

Definition 4.35. We say a topological space X is *locally connected* if every point x of X has an open neighbourhood U such that U is connected as a topological space with the subspace topology.

4.4 Formal topologies

It is sometimes convenient to take an alternative approach to topology, where we ignore the points of the topological space, and instead focus on the elements of a basis for the topology.

Definition 4.36. A *formal topology* is a poset $(B, \leq)$ together with a relation $\lhd$ of sort $B, \mathcal{P}(B)$ (i.e. a relation on B and sets of elements of B), satisfying the following axioms, for all $a, b \in B$ and $U, V \subseteq B$. We write $U^{\leq}$ to mean the *downwards closure of* U, i.e. $\{a \in B \mid \exists b\, b \in U \wedge a \leq b\}$.

1. $a \in U$ implies $a \lhd U$.
2. $a \lhd U$ and $a \lhd V$ implies $a \lhd U^{\leq} \cap V^{\leq}$.
3. If $a \lhd U$ and for all $b \in U$, $b \lhd V$, then $a \lhd V$.
4. $a \leq b$ implies $a \lhd \{b\}$

We refer to the relation $\lhd$ as the *covering relation* of the formal topology. When $b \lhd U$, we say b is *covered by* U.

Definition 4.37. We say an *open set* is a subset U of B satisfying the following conditions:

1. If $a \leq b$ and $b \in U$ then $a \leq b$. (downwards closure)
2. If $a \lhd U$, then $a \in U$.

Proposition 4.38. *The open sets of a formal topology ordered by inclusion form a complete Heyting algebra.*

Example 4.39. If $(B, \leq)$ is any poset, we can define a minimal covering relation by $b \lhd U$, whenever $b \leq a$ for some $a \in U$.

Example 4.40. Let $(X, \mathcal{O})$ be a topological space with a basis B. We define a formal topology as follows. We take the underlying poset to be B ordered by subset inclusion. If U is a set of basic open sets, We say $b \lhd U$ if $b \subseteq \bigcup U$. Then open sets of the formal topology correspond precisely to the open sets of the topological space. If $V \subset X$ is an open set of the topological space, we define an open set U of the formal topology, by taking U to be the set of basic opens b such that $b \subseteq V$. Given an open set U of the formal topology, we define an open set $V \subset X$ of the topological space by $V := \bigcup U$.

Example 4.41. As a special case of the previous example, we can define *formal Baire space* as follows. We take B to be the set of finite sequences of natural numbers, ordered by *reverse extension*. That is, for finite sequences σ and τ, we say $\sigma \leq \tau$ if the length of σ is greater than than of τ, and whenever i is less than the length of τ, we have $\tau(i) = \sigma(i)$. We define the covering relation $\lhd$ to be the smallest relation satisfying the axioms of a formal topology, and such that if $\sigma * \langle n \rangle \lhd U$ for all n, then $\sigma \lhd U$.