80818 Intuitionistic Logic - Solutions to Exercise Sheet 3

1. First note that by $\exists I$ and $\rightarrow I$, we have that $T \vdash \varphi[x/t] \;\rightarrow\; \exists x\, \varphi$ for all terms t free for x in φ. Hence $[\varphi[x/t]] \leq [\exists x\, \varphi]$ for all t, and so $[\exists x\, \varphi]$ is an upper bound.

   Now suppose that $[\psi]$ is also an upper bound for S. In particular, if we choose y to be a variable not occuring free in ψ, then we have $[\varphi[x/y]] \leq [\psi]$. Hence we have $T \vdash \varphi[x/y] \rightarrow \psi$, and so $T, \exists x\, \varphi, \varphi[x/y] \vdash \psi$. However, we can now apply $\exists E$, to show $T, \exists x\, \varphi \vdash \psi$, and hence $T \vdash \exists x\, \varphi \;\rightarrow\; \psi$. It follows that $[\exists x\, \varphi] \leq \psi$, and so $[\exists x\, \varphi]$ is the least upper bound.

2. We will make the additional assumption that P is inhabited (the statement is vacuously true when P is empty). Say that $p \in P$.

   Suppose we have open sets U and V such that $P = U \cup V$ and $U \cap V = \emptyset$. We then have either $p \in U$ or $p \in V$. We just consider the case $p \in U$, since we can apply exactly the same argument when $p \in V$. We will show that $V = \emptyset$. Suppose that $q \in V$. In this case p and q have an upper bound r. Since $p \leq r$ and P is upwards closed $r \in U$. But since $q \leq r$ we also have $r \in V$, and so $r \in U \cap V$, contradicting $U \cap V = \emptyset$.

3. We define $\mathcal{O}$ to be the collection of all sets U such that for all $x \in V$ there exists $a \in B$ such that $x \in a$ and $a \subseteq U$.

   Note that this vacuously holds for $\emptyset$ and by condition (i) also holds for X.

   Suppose that S is a subset of $\mathcal{O}$. We need to check that $\bigcup S$ is open. Suppose that $x \in \bigcup S$. Then $x \in U$ for some $U \in S$. Hence there exists $a \in B$ such that $x \in a$ and $a \subseteq U$. But then we also have $a \subseteq \bigcup S$. Hence $\bigcup S$ is open.

   Finally, we need to check that $\mathcal{O}$ is closed under binary intersection. Suppose that $U, V \in \mathcal{O}$. Suppose that $x \in U \cap V$. Hence there exists $a \in B$ such that $x \in a$ and $a \subseteq U$ and there exists $b \in B$ such that $x \in b$ and $b \subseteq V$. By (ii) there exists c such that $x \in c$ and $c \subseteq a \cap b$. The latter implies that $c \subseteq U \cap V$. Hence $U \cap V$ is open.

4. (a) Note that the inclusion $2^{\mathbb{N}} \hookrightarrow \mathbb{N}^{\mathbb{N}}$ is continuous, from the fact that $2^{\mathbb{N}}$ has the subspace topology as a subset of $\mathbb{N}^{\mathbb{N}}$.
   We define the map $F : \mathbb{N}^{\mathbb{N}} \rightarrow 2^{\mathbb{N}}$ by taking $F(f)(n)$ to be $\max(n, 1)$. To show that F is continuous, it suffices to show that the preimage

of any basic open set in $2^{\mathbb{N}}$ is a union of basic open sets in $\mathbb{N}^{\mathbb{N}}$. If σ is any finite binary sequence, note that $f \in F^{-1}(U_\sigma)$ if and only if $\max(F(f)(n), 1) = \sigma(n)$ for all $n < |\sigma|$. However, this is true if and only if $F(f)(n) = \tau(n)$ for some finite sequence of numbers τ of the same length as σ, such that $\max(\tau(n), 1) = \sigma(n)$. Hence $F^{-1}(U_\sigma)$ is precisely the union of all $U_\tau \subseteq \mathbb{N}^{\mathbb{N}}$ where τ has the above property, as required.

(b) We will show that in fact every basic open U_σ is connected. It follows that $I^{\mathbb{N}}$ is locally connected.

Suppose that $U_\sigma \subseteq V \cup W$ where $V \cap W \cap U_\sigma = \emptyset$. We need to show that either $U_\sigma \subseteq V$ or $U_\sigma \subseteq W$. We define a map $f : \mathbb{N} \to I$ as follows

$$f(n) := \begin{cases} \sigma(n) & n < |\sigma| \\ 1 & n \geq |\sigma| \end{cases}$$

We have ensured by definition that $f \in U_\sigma$. Hence either $f \in V$ or $f \in W$. We just consider the case $f \in V$, the other case being similar. Let $g \in U_\sigma$. In that case we have $g(n) \leq f(n)$ for all n, by considering separately the cases $n \leq |\sigma|$ and $n > |\sigma|$. Note however, that every open set is upwards closed with respect to the pointwise ordering on $I^{\mathbb{N}}$, so if $g \in W$, then also we would have $f \in W$. However, this contradicts $V \cap W \cap U_\sigma = \emptyset$. Therefore $g \in V$. We have now shown $U_\sigma \subseteq V$, as required.