

## 4 Heyting Algebras and Topology

### 4.1 Posets and lattices

We first recall some basic theory about posets.

**Definition 4.1.** A *poset* is a set  $P$ , together with a binary relation  $\leq$  satisfying the following axioms.

1.  $x \leq x$  (reflexivity)
2.  $x \leq y \wedge y \leq z \rightarrow x \leq z$  (transitivity)
3.  $x \leq y \wedge y \leq x \rightarrow x = y$  (anti-symmetry)

We will sometimes write  $x \leq y$  as  $y \geq x$ .

**Definition 4.2.** The *top* or *greatest* element of a poset  $P$  is  $\top \in P$  such that for all  $x \in P$ ,  $x \leq \top$ .

The *bottom* or *least* element of a poset  $P$  is  $\perp \in P$  such that for all  $x \in P$ ,  $\perp \leq x$ .

**Definition 4.3.** Let  $S$  be a set of elements of a poset  $P$ . We say  $z$  is the *least upper bound* or *join* of  $S$  if

1. for all  $x \in S$ ,  $x \leq z$  ( $z$  is an upper bound)
2. if  $y \in P$  is such that for all  $x$  in  $S$   $x \leq y$ , then  $z \leq y$  (any other upper bound is greater)

Note that any set has at most one join. If it exists, we write it as  $\bigvee S$ . Given a two element set  $\{x, y\}$ , we write  $\bigvee \{x, y\}$  as  $x \vee y$ .

We similarly define the *greatest lower bound* or *meet* of  $S$  as an element  $z$  such that

1. for all  $x \in S$ ,  $z \leq x$
2. if  $y \in P$  is such that for all  $x$  in  $S$   $z \leq x$ , then  $y \leq z$ .

The meet of a set is also unique, and when it exists we write it as  $\bigwedge S$ , and for two element sets  $\{x, y\}$ , we write  $\bigwedge \{x, y\}$  as  $x \wedge y$ .

**Definition 4.4.** We say a poset  $P$  is *complete* if for every set  $S \subseteq P$ ,  $\bigvee S$  and  $\bigwedge S$  both exist.

**Proposition 4.5.** A poset  $P$  is complete if and only if it has all joins (or all meets).

*Proof.* Let  $S$  be a set. We want to construct the greatest lower bound of  $S$ . Define  $L := \{x \in P \mid \forall y \in S, x \leq y\}$  the set of all lower bounds of  $S$ . We claim  $\bigvee L$  is the greatest lower bound of  $S$ . For each  $y \in S$ , we know that for every  $x$  in  $L$ ,  $x \leq y$ . It follows that  $\bigvee L \leq y$ . By applying this to each  $y \in S$  we see  $\bigvee L$  is a lower bound for  $S$  and it is clear it is the greatest one (since  $L$  contains every other lower bound).  $\square$

**Definition 4.6.** A *lattice* is a poset  $P$  with least and greatest elements  $\perp$  and  $\top$ , and any two elements  $x, y \in P$  have a meet  $x \wedge y$  and a join  $x \vee y$ .

**Example 4.7.** If  $P$  is any collection of sets, then it has a canonical ordering given by  $x \leq y$  whenever  $x \subseteq y$ . If  $P$  is closed under binary intersection  $x \cap y$  and union  $x \cup y$ , then  $(P, \subseteq)$  is a lattice.

It is possible to recover the poset relation on a lattice from the operations  $\wedge$  and  $\vee$ . Because of this, we can also think of lattices as *algebraic* structures (i.e. sets with operations satisfying equations).

## 4.2 Heyting algebras

To motivate Heyting algebras, we first consider an important example, the *Lindenbaum-Tarski algebra* of an intuitionistic theory.

Let  $T$  be a theory over some signature. We define an equivalence relation on formulas by  $\varphi \sim \psi$  when  $T \vdash \varphi \leftrightarrow \psi$ . Let  $P$  be the quotient of the set of formulas by  $\sim$ . Note that we have a canonical ordering on  $P$ : we say  $[\varphi] \leq [\psi]$  if  $T \vdash \varphi \rightarrow \psi$ , and note that this is preserved by the equivalence relation.

We see that  $P$  is a lattice, and moreover each part of the lattice structure corresponds naturally to logical connectives. For example  $[\perp] \leq [\varphi]$  for all formulas  $\varphi$ , by ex falso. Similarly  $[\top]$  is the greatest element of the lattice. We can show that  $[\varphi \vee \psi]$  is greater than  $[\varphi]$  and  $[\psi]$  by the  $\vee$  introduction rule, and the  $\vee$  elimination rule precisely tells us that any other upper bound is greater than  $[\varphi \vee \psi]$ . Similarly we can use conjunction to construct meets in the Lindenbaum-Tarski algebra.

However, there is one logical connective that does not appear as part of the lattice structure, namely *implication*,  $\rightarrow$ . As before, we can translate the introduction and elimination rules into properties of the order structure. For all formulas  $\varphi$ ,  $\psi$  and  $\chi$ , we can use implication introduction to show that if  $[\chi \wedge \varphi] \leq [\psi]$ , then  $[\chi] \leq [\varphi \rightarrow \psi]$ . Using introduction elimination, we can also show the converse: if  $[\chi] \leq [\varphi \rightarrow \psi]$ , then  $[\chi \wedge \varphi] \leq [\psi]$ .

We can see Heyting algebras as posets that behave similar to Lindenbaum-Tarski algebras. We can think of the elements of a Heyting algebra as “truth values.” We will use them to define certain models (*Heyting valued models*) of theories in intuitionistic logic. Formally, we define them as follows.

**Definition 4.8.** A *Heyting algebra* is a lattice  $(P, \top, \perp, \wedge, \vee)$  together with a binary operation  $\rightarrow$ , *implication*, satisfying the condition below.

$$z \leq x \rightarrow y \quad \text{if and only if} \quad z \wedge x \leq y$$

## 4.3 Topological spaces

For this course, our main source of examples of Heyting algebras are going to be *topological spaces*. From the point of view of Heyting algebras, we can think of topological spaces as concrete examples of Heyting algebras, whose elements are all subsets of a fixed set.

The original motivation for topological spaces is to model “spaces” that appear in mathematics, such as spheres or 3-dimensional space.

Another idea that will be important in this course is the notion of morphism between topological spaces, *continuous* functions.

#### 4.3.1 Some basic definitions and examples

**Definition 4.9.** Let  $X$  be a set. A *topology* on  $X$  is a collection  $\mathcal{O}$  of subsets of  $X$ , satisfying the following conditions.

1.  $X$  and  $\emptyset$  are elements of  $\mathcal{O}$ .
2. If  $U$  and  $V$  are elements of  $\mathcal{O}$ , their intersection  $U \cap V$  also belongs to  $\mathcal{O}$ .
3. If  $S \subseteq \mathcal{O}$  is a set of elements of  $\mathcal{O}$ , then their union  $\bigcup S$  also belongs to  $\mathcal{O}$ .

We refer to the elements of  $\mathcal{O}$  as *open sets*, and the elements of  $X$  as *points*.

We say a set  $X$  together with a topology is a *topological space*.

**Example 4.10.** If  $X$  is any set, then we can define a topology by taking  $\mathcal{O}$  to be the collection of *all* subsets of  $X$ . This is referred to as the *discrete* topology on  $X$ .

We can also define a topology by taking  $\mathcal{O}$  to have just two elements,  $\emptyset$  and  $X$ . This is referred to as the *indiscrete topology*.

**Example 4.11.** We define a topology on the set with two elements  $2 := \{0, 1\}$  as the set of subsets  $\{\emptyset, \{0\}, \{0, 1\}\}$ . This topological space is referred to as *Sierpiński space*,  $S$ .

**Example 4.12.** We define a topology on the set with three elements  $I = \{0, 1, 2\}$  as the set of subsets  $\{\emptyset, \{0, 1\}, \{1\}, \{1, 2\}, \{0, 1, 2\}\}$ . We will refer to this space as the *abstract interval*, or just the *interval*.

**Example 4.13.** If  $(Q, \leq)$  is any poset, we can define a topology on  $Q$  as follows. A set  $U \subseteq Q$  is *upwards closed* or an *upset* if whenever  $x \in U$  and  $x \leq y$ , we have  $y \in U$ . The set of all upsets defines a topology on  $Q$  referred to as the *upset* or *Alexandrov* topology on the poset.

**Example 4.14.** If  $(X, \mathcal{O})$  is a topological space, and  $Y$  is a subset of  $X$ , then we can also define a topology on  $Y$  as follows. We say a set  $U \subseteq Y$  is open if for some open set  $V$  of  $X$ , we have  $U = V \cap Y$ . We refer to this as the *subspace topology* on  $Y$ .

**Definition 4.15.** Let  $(X, \mathcal{O})$  be a topological space, and  $Y \subset X$  any subset. We define the *interior* of  $Y$ ,  $Y^\circ$  to be the union of all open sets  $U$  such that  $U \subseteq Y$ .

**Proposition 4.16.** We observe the following properties of interior.

1. For all  $Y$ ,  $Y^\circ \subseteq Y$ .

2. For all  $Y$ ,  $Y^\circ$  is an open set.
3. If  $Y$  is already an open set, then  $Y^\circ = Y$ .

**Example 4.17.** In Sierpiński space, the interior of  $\{1\}$  is the empty set.

**Proposition 4.18.** For any topological space  $(X, \mathcal{O})$ , the poset  $\mathcal{O}$  of open sets ordered by inclusion is a complete Heyting algebra.

*Proof.* We define joins using union and meets using (binary) intersection. Top and bottom element are given by  $X$  and  $\emptyset$  respectively. It only remains to define the implication operator. We might try to define this using the canonical implication on subsets, i.e. for sets  $U$  and  $V$ , the set

$$\{x \in X \mid x \in U \rightarrow x \in V\}.$$

However, the above set is not necessarily open, even when  $U$  and  $V$  are. To get an open set, we use the interior operator. Namely, we define

$$U \rightarrow V := \{x \in X \mid x \in U \rightarrow x \in V\}^\circ.$$

It only remains to check that this satisfies the necessary condition to be the implication of a Heyting algebra (exercise!).  $\square$

NB: Using the law of excluded middle, we could alternatively define implication by

$$U \rightarrow V := (V \cup (X \setminus U))^\circ$$

**Remark 4.19.** Since lattices of open sets have all joins, they must also have meets, by proposition 4.5. Whereas the join of a set  $S \subseteq \mathcal{O}$  is just the union,  $\bigvee S = \bigcup S$ , the intersection  $\bigcap S$  is not necessarily open. However, we can still explicitly describe the meet of  $S$  using the interior operator,  $\bigwedge S = (\bigcap S)^\circ$ .

**Definition 4.20.** Let  $(X, \mathcal{O}_X)$ ,  $(Y, \mathcal{O}_Y)$  be topological spaces. We say a function  $f : X \rightarrow Y$  is *continuous* if for every open set  $U$  of  $Y$ , the preimage,  $f^{-1}(U)$  is an open set of  $X$ .

### 4.3.2 Neighbourhoods in a topological space

**Definition 4.21.** Let  $(X, \mathcal{O})$  be a topological space. A *neighbourhood* of a point  $x \in X$  is an open set  $U \in \mathcal{O}$  such that  $x \in U$ .

We think of a neighbourhood of a point  $x$  as a set of points that are “nearby”  $x$ . For example, in a discrete topology, every point  $x$  has a neighbourhood  $\{x\}$  that does not contain any other points. We can visualise this as a clear space around  $x$  that does not contain any elements. On the other hand in an indiscrete space, the only neighbourhood of a point  $x$  is the entire space. The points are so “close together” that you can’t look at one without looking at the whole space. In Sierpiński space, every neighbourhood of 1 also contains 0, so 0 is “infinitely close” to 1, but this is not a symmetric relation: 0 has a neighbourhood that does not contain 1.

**Definition 4.22.** Let  $(X, \mathcal{O})$  be a topological space. A *basis* of the topology, is a set of open sets  $B \subseteq \mathcal{O}$  with the following property. For any open neighbourhood  $U$  of a point  $x$ , there exists an open neighbourhood  $V$  of  $x$  with  $V \subseteq U$  and  $V \in B$ .

Equivalently, a basis is a set  $B$  such that for every open set  $U$ , there is a set  $S \subseteq B$  such that  $U = \bigcup S$ .

Note that if  $B$  is a basis of a topology, then a set is open if and only if it is equal to the union  $\bigcup S$  for some  $S \subseteq B$ .

**Example 4.23.** If  $(Q, \leq)$  is any poset, then we can define a basis of the upset topology as the set of upwards closed sets with least element. Note that there is a precise correspondence between the elements of the basis and elements of  $Q$ .

### 4.3.3 Product topologies

**Definition 4.24.** We define a topology on  $\mathbb{N}^{\mathbb{N}}$  as follows. Given a finite sequence of numbers  $\sigma$  of length  $k$ , we define the set  $U_\sigma$  as follows.

$$U_\sigma := \{f : \mathbb{N} \rightarrow \mathbb{N} \mid \forall i < k \ f(i) = \sigma(i)\}$$

We define a set  $U$  to be open if it can be written as a union of sets of the form  $U_\sigma$  for finite sequences  $\sigma$ . We refer to the topological space with these open sets as *Baire space*.

We refer to the subspace topology on the set of binary sequences  $2^{\mathbb{N}} \subseteq \mathbb{N}^{\mathbb{N}}$  as *Cantor space*.

We refer to the set of binary sequences with at most one 1, with the subspace topology as  $\mathbb{N}_\infty$ .

**Proposition 4.25.** Assume **LPO**. Then  $\mathbb{N}_\infty$  is isomorphic to the set  $\mathbb{N} \amalg \{\infty\}$ , where a set  $U \subseteq \mathbb{N} \amalg \{\infty\}$  is open when it satisfies the following. If  $\infty \in U$ , then for some  $n \in \mathbb{N}$ ,  $U$  contains every  $m \in \mathbb{N}$  with  $m \geq n$ .

**Proposition 4.26.** Given  $\mathbb{N}^{\mathbb{N}}$  with the Baire space topology, and  $\mathbb{N}$  with the discrete topology, a function  $F : \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}$  is continuous if and only if it satisfies the following condition. For every  $f \in \mathbb{N}^{\mathbb{N}}$ , there is a natural number  $n$  such that for any  $g \in \mathbb{N}^{\mathbb{N}}$  satisfying the condition that  $g(i) = f(i)$  for  $i < n$ , we have  $F(g) = F(f)$ .

*Proof.* Since  $\mathbb{N}$  has the discrete topology, every singleton set  $\{m\}$  is open, and in any case we can write any set as a union of singletons. It follows that  $F$  is continuous if and only if  $F^{-1}(\{m\})$  is open for every  $m$ . This says exactly that any element  $f$  of  $F^{-1}(\{m\})$  has a basic open neighbourhood contained in  $F^{-1}(\{m\})$ . This precisely says that the condition described in the proposition holds whenever  $F(f) = m$ . However,  $F$  is continuous precisely when this condition holds for arbitrary  $m$ , so given a function  $f$ , we can just apply it with  $m := F(f)$ .  $\square$

The Baire space topology on  $\mathbb{N}^{\mathbb{N}}$  is an instance of a more general construction known as the product topology.

**Definition 4.27.** Suppose we are given a set  $I$ , and a family of topological spaces  $(X_i, \mathcal{O}_i)$  for each  $i \in I$ . We define a topology on the product  $\prod_{i \in I} X_i$  as follows. Given a pair  $\sigma = (F, (U_j)_{j \in F})$  consisting of a finite subset  $F$  of  $I$ , say  $i_1, \dots, i_k$ , together with open sets  $U_j \in \mathcal{O}_{i_j}$  for  $j = 1, \dots, k$ , we define the set  $U_\sigma$  by the equation

$$U_\sigma := \{f \in \prod_{i \in I} X_i \mid \forall j \in F f(j) \in U_j\}$$

We refer to sets of the form  $U_\sigma$  as basic opens, and define a set to be open if it is a union of basic opens. We refer to the resulting topological space as the *product topology* of the family.

**Proposition 4.28.** *Baire space is the product of the (constant) family consisting of countably many copies of  $\mathbb{N}$  with the discrete topology.*

#### 4.3.4 Connectedness

An important concept when looking at the behaviour of topological models is that of connectedness, which informally is the idea that a space is “indecomposable” - it is impossible to break the space up cleanly into separate pieces.

**Definition 4.29.** A topological space  $X$  is *connected* if given open sets  $U$  and  $V$  such that  $U \cup V = X$  and  $U \cap V = \emptyset$  we have either  $X = U$  or  $X = V$ .

**Proposition 4.30.** *A topological space  $X$  is connected if and only if for every discrete topological space  $Y$ , every map  $X \rightarrow Y$  is constant.*

**Example 4.31.** Sierpiński space and the abstract interval are connected.

**Example 4.32.** Any discrete space with at least two distinct points is not connected.

**Example 4.33.** Baire space, Cantor space and  $\mathbb{N}_\infty$  are not connected.

### 4.4 Formal topologies

It is sometimes convenient to take an alternative approach to topology, where we ignore the points of the topological space, and instead focus on the elements of a basis for the topology.

**Definition 4.34.** A *formal topology* is a poset  $(B, \leq)$  together with a relation  $\triangleleft$  of sort  $B, \mathcal{P}(B)$  (i.e. a relation on  $B$  and sets of elements of  $B$ ), satisfying the following axioms, for all  $a, b \in B$  and  $U, V \subseteq B$ . We write  $U^\leq$  to mean the *downwards closure* of  $U$ , i.e.  $\{a \in B \mid \exists b \in U \wedge a \leq b\}$ .

1.  $a \in U$  implies  $a \triangleleft U$ .

2.  $a \triangleleft U$  and  $a \triangleleft V$  implies  $a \triangleleft U^\leq \cap V^\leq$ .
3. If  $a \triangleleft U$  and for all  $b \in U$ ,  $b \triangleleft V$ , then  $a \triangleleft V$ .
4.  $a \leq b$  implies  $a \triangleleft \{b\}$

We refer to the relation  $\triangleleft$  as the *covering relation* of the formal topology. When  $b \triangleleft U$ , we say  $b$  is *covered by*  $U$ .

**Definition 4.35.** We say an *open set* is a subset  $U$  of  $B$  satisfying the following conditions:

1. If  $a \leq b$  and  $b \in U$  then  $a \leq b$ . (downwards closure)
2. If  $a \triangleleft U$ , then  $a \in U$ .

**Proposition 4.36.** *The open sets of a formal topology ordered by inclusion form a complete Heyting algebra.*

**Example 4.37.** If  $(B, \leq)$  is any poset, we can define a minimal covering relation by  $b \triangleleft U$ , whenever  $b \leq a$  for some  $a \in U$ .

**Example 4.38.** Let  $(X, \mathcal{O})$  be a topological space with a basis  $B$ . We define a formal topology as follows. We take the underlying poset to be  $B$  ordered by subset inclusion. If  $U$  is a set of basic open sets, We say  $b \triangleleft U$  if  $b \subseteq \bigcup U$ . Then open sets of the formal topology correspond precisely to the open sets of the topological space. If  $V \subset X$  is an open set of the topological space, we define an open set  $U$  of the formal topology, by taking  $U$  to be the set of basic opens  $b$  such that  $b \subseteq V$ . Given an open set  $U$  of the formal topology, we define an open set  $V \subset X$  of the topological space by  $V := \bigcup U$ .

**Example 4.39.** As a special case of the previous example, we can define *formal Baire space* as follows. We take  $B$  to be the set of finite sequences of natural numbers, ordered by *reverse extension*. That is, for finite sequences  $\sigma$  and  $\tau$ , we say  $\sigma \leq \tau$  if the length of  $\sigma$  is greater than than of  $\tau$ , and whenever  $i$  is less than the length of  $\tau$ , we have  $\tau(i) = \sigma(i)$ . We define the covering relation  $\triangleleft$  to be the smallest relation satisfying the axioms of a formal topology, and such that if  $\sigma * \langle n \rangle \triangleleft U$  for all  $n$ , then  $\sigma \triangleleft U$ .