10 Completeness and existence properties

10.1 Completeness theorem for Heyting valued models

In logic, it is common to consider not just soundness theorems (as we have seen so far) but converse versions, known as *completeness* theorems, where we show that if something holds in every model, then it is provable. For the kinds of model we commonly study in intuitionistic logic, we can often do something stronger: for a given theory T, we can find a single *canonical* model such that a closed formula φ is provable in T if and only if it holds in the model. In this section we will see how this works for Heyting valued models on a complete Heyting algebra.

The essential idea is to take the Heyting algebra to be the Lindenbaum-Tarski algebra of a theory T, take the domains $\mathcal{M}_S$ to be terms of sort S and then show $[\![\varphi]\!] = [\varphi]$. However, there are a few issues to deal with to make this precise.

The first problem is that we have only defined Heyting valued model for complete Heyting algebras and the Lindenbaum-Tarski algebra is typically not complete. To deal with this we use a construction called *Dedekind-MacNeille completion.*

Lemma 10.1. *Let $(P, \wedge, \vee, \top, \bot, \to)$ be a Heyting algebra (not necessarily complete). Then we can construct a complete Heyting algebra $(\overline{P}, \bigwedge, \to)$ and a function $\iota : P \to \overline{P}$ such that ι preserves Heyting implication, and any meets and joins that already exist in P.*

Proof. We define an *c-ideal* to be a set $U \subseteq P$ satisfying the following conditions:

1. $\bot \in U$
2. If $p \in U$ and $q \leq p$, then $q \in U$ (i.e. U is downwards closed)
3. If $S \subseteq P$ and $\bigvee S$ already exists in P, then $\bigvee S \in U$

We take $\overline{P}$ to be the set of all c-ideals, ordered by inclusion. We define $\iota(p)$ to be the downwards closure of $\{p\}$.

Note that $\overline{P}$ is a complete Heyting algebra. For example, given c-ideals U and V, we define $U \to V$ to be

$$U \to V := \{p \in P \mid \forall q \leq p\, q \in U \to q \in V\}$$

We can see in particular that $\{p\}^{\leq} \to \{q\}^{\leq} = \{p \to q\}^{\leq}$.

Given $S \subseteq \overline{P}$, we can define the join of S by

$$\bigvee S := \left\{\bigvee X \mid X \subseteq \bigcup S \text{ and } \bigvee X \text{ exists}\right\}$$

One again note that the definition of c-ideal was chosen precisely to ensure that given $S \subseteq P$ such that $\bigvee S$ exists, we have

$$\bigvee \iota(S) = \left\{\bigvee S\right\}^{\leq}$$

This precisely ensures that any joins that exist in P are preserved by ι. $\square$

Now fix a theory T over a signature $(\mathfrak{S}, \mathfrak{O}, \mathfrak{R})$. If L is the Lindenbaum-Tarski algebra on T, we will define a Heyting valued model on $\overline{L}$.

Given a sort $S \in \mathfrak{S}$, we define $\mathcal{M}_S$ to be the set of all terms of sort S. It is an important point that this really means all terms, not just closed terms. In particular, for the completeness proof to work correctly at quantifiers we will need to use the fact that free variables of sort S are included in $\mathcal{M}_S$. We define $E(t) := \top$, i.e. we construct a global model.

From this definition, it is clear that for each operator symbol $O \in \mathfrak{O}$, we can define $[\![O]\!](t_1, \ldots, t_n)$ simply to be $Ot_1 \ldots t_n$. Similarly, for each relation symbol $R \in \mathfrak{R}$, we can define $[\![R]\!](t_1, \ldots, t_n)$ to be $\iota([Rt_1 \ldots t_n])$ (that is, we send the formula $Rt_1 \ldots t_n$ to the correspond equivalence class $[Rt_1 \ldots t_n]$ in the Lindenbaum-Tarski algebra, and then include it into the Dedekind-MacNeille completion with ι.

Theorem 10.2. *For the Heyting valued model defined above, we have for all formulas φ and all variable assigments σ,*

$$[\![\varphi]\!]_\sigma = \iota([\varphi[\sigma]])$$

Proof. This is proved by induction on formulas. $\square$

10.2 Existence properties in logic

It is a key characteristic of constructive mathematics that proving statements of the form $\exists x\, \varphi(x)$ should require something more than the same statement in classical logic. Namely, we should only be able to prove this if we can explicitly find a witness. We formalise this idea through *existence properties*.

Definition 10.3. A theory T satisfies the *disjunction property* if whenever $T \vdash \varphi \vee \psi$, either $T \vdash \varphi$ or $T \vdash \psi$.

Definition 10.4. A theory T satisfies the *term existence property* if whenever $T \vdash \exists x\, \varphi(x)$, there is a closed term t such that $T \vdash \varphi(t)$.

Definition 10.5. A theory T satisfies the *definable existence property* if whenever φ is a formula whose only free variable is x and $T \vdash \exists x\, \varphi(x)$ there is a formula ψ, whose only free variable is x such that $T \vdash \exists! x\, (\varphi(x) \wedge \psi(x))$.

Note that the term existence property implies the definable existence property, and if a theory T satisfies the definable existence property, we can extend it to a theory with the term existence property by adding a constant symbol c_φ and axiom $\varphi(c_\varphi)$ whenever φ is a formula whose only free variable is x such that $T \vdash \exists! x\, \varphi(x)$.

Definition 10.6. Suppose T has a sort N together with a constant symbol 0 of sort N and a unary operation symbol S of sort $N \to N$. For each natural number $n \in \mathbb{N}$, we define a term $\underline{n}$ of sort N by induction:

$$\underline{0} := 0$$
$$\underline{n+1} := S\underline{n}$$

We say T satisfies the *numerical existence property* if whenever $T \vdash \exists x^N \varphi(x)$, there exists $n \in \mathbb{N}$ such that $T \vdash \varphi(\underline{n})$.

Although we only need 0 and S in the signature for the above definition to make sense, in practice we only consider theories that are extensions of **HA**, i.e. every theorem of **HA** is also provable in T.

The disjunction and numerical existence property hold for all theories widely viewed as foundations for constructive mathematics. On the other hand, they can never hold for theories used as foundations of mathematics based on classical logic. We can see this using an argument based on Gödel's incompleteness theorem. If T is a theory in classical logic where we can interpret Peano arithmetic, then we can formalise the statement "T is either consistent or not consistent" inside T, and prove it as a direct instance of the law of excluded middle. However, by Gödel's result we can show that T does not prove either of the disjuncts (as long as T is consistent and recursively axiomatisable).

The status of the definable existence property is less clear. There are examples of theories in classical logic that satisfy it, including Peano arithmetic and any set theory extending $\mathbf{ZF} + V = \mathbf{OD}$. There are also examples of theories used in constructive mathematics that do not satisfy the definable existence property, such as the set theories **IZF** and **CZF**

10.3 Numerical existence property for Heyting arithmetic

We now show a simple technique for proving that a theory has the numerical existence property. To illustrate the idea, we will just show the result for **HA**. However, this is a robust argument that can be generalised and adapted to diverse theories, including for example **HAS** and $\mathbf{HA}_\omega$.

Definition 10.7. Given a complete Heyting algebra $(P, \bigvee, \wedge, \to)$, we define the *connectification* P^* to have underlying set $P \amalg \{\top_*\}$ with the ordering $\leq$ defined so that $\top_*$ is the top element of the poset and otherwise the ordering agrees with that of P.

Note that P^* is itself a complete Heyting algebra, and it is connected in the following very strong sense.

Lemma 10.8. *Suppose that $S \subseteq P^*$ is such that $\bigvee S = \top_*$. Then for some $p \in S$ we have $p = \top_*$.*

We write $\top_P$ for the "old" top element, which is now strictly below the new top. We make use of the following key lemma, whose proof is left as an exercise.

Lemma 10.9. *The map $\pi : P^* \to P$ sending p to $p \wedge \top_P$ preserves Heyting implication and all meets and joins.*

Theorem 10.10. **HA** *satisfies the disjunction property and numerical existence property.*

Proof. We take P to be the Dedekind MacNeille completion of the Lindenbaum-Tarski algebra for **HA**. We then define a Heyting valued model over the connectification P^* as follows.

We define $\mathcal{M}$ to be the set of all terms, including terms with free variables, as we did for the completeness theorem. However, we define a non trivial existence predicate as follows:

$$E(t) := \begin{cases} \top_* & \text{if } t = \underline{n} \text{ for some } n \in \mathbb{N} \\ \top_P & \text{otherwise} \end{cases}$$

If we write $[\![\varphi]\!]_\sigma^P$ for the interpretation of φ in the canontical model and $[\![\varphi]\!]_\sigma^*$ in the new model on P^* defined above, then by lemma 10.9 and induction on formulas, we have

$$[\![\varphi]\!]_\sigma^P = [\![\varphi]\!]_\sigma^* \wedge \top_P$$

Now assume that **HA** $\vdash \exists n\, \varphi(n)$. One can show that the axioms of **HA** hold in the model (exercise). Hence by the soundness theorem for intuitionistic logic, we have (for any variable assignment σ)

$$[\![\exists n\, \varphi]\!]_\sigma^* = \top_*$$

By lemma 10.8 and the interpretation of existential quantifiers it follows that there exists a term t such that

$$E(t) \wedge [\![\varphi(t)]\!]_\sigma^* = \top_*$$

However, it follows that $E(t) = \top_*$, and therefore that $t = \underline{n}$ for some $n \in \mathbb{N}$.

Now by the observation above, we can deduce

$$[\![\varphi(\underline{n})]\!]_\sigma^P = \top_P$$

However, finally we can apply the completeness theorem to show **HA** $\vdash \varphi(\underline{n})$, as required. □