15 Realizability with Truth

One the key features of realizability is that it gives us models that satisfy axioms that do not necessarily hold externally in the metatheory where we are working (such as Church's thesis). However, it is sometimes preferable to consider a variation with the same idea that realizers capture the computational information implicit in intuitionisitic proofs, but where only true sentences can be realized. We will refer to this variation as *realizability with truth*.

This allows us to show properties such as *Church's rule*: Given a constructive proof (e.g. in $\mathbf{HA}_\omega$) that $\forall x^N \, \exists y^N \, \varphi(x, y)$, there is an algorithm e such that $\forall x^N \, \exists y^N \, \varphi(x, e \cdot y)$, provably in $\mathbf{HA}_\omega$. Moreover, in contrast to the anticlassical axioms of Kleene realizability, this algorithm remains valid when we move to stronger theories, even with classical logic, and with careful phrasing, we can even deduce the statement is "true" in the metatheory where we are working.

15.1 An external description

We will first give an "external" description of realizability with truth along similar lines to the other models with have considered in the course so far.

We fix a signature $(\mathfrak{S}, \mathfrak{O}, \mathfrak{R})$ of sorts, operator symbols and relation symbols, and a pca$^+$, $\mathcal{A}$.

Definition 15.1. A *realizability model with truth* consists of the following:

1. For each sort $S \in \mathfrak{S}$ a set $\mathcal{M}_S$
2. For each sort S a function $E_S : \mathcal{M}_S \to \mathcal{P}(\mathcal{A})$
3. For each relation symbol $R \in \mathfrak{R}$ of sort $S_1, \ldots, S_n$, a set $[\![R]\!]^\top \subseteq \mathcal{M}_{S_1} \times \ldots \times \mathcal{M}_{S_n}$ and a function $[\![R]\!] : \mathcal{M}_{S_1} \times \ldots \times \mathcal{M}_{S_n} \to \mathcal{P}(\mathcal{A})$. We require that if $[\![R]\!](a_1, \ldots, a_n)$ is a non empty subset of $\mathcal{A}$, then $(a_1, \ldots, a_n) \in [\![R]\!]^\top$.
4. For each operator symbol $O \in \mathfrak{O}$ of sort $S_1, \ldots, S_n \to T$ a function $[\![O]\!] : \mathcal{M}_{S_1} \times \ldots \times \mathcal{M}_{S_n} \to \mathcal{M}_T$ such that there exists $e \in \mathcal{A}$ with the following property. For all $x_1, \ldots, x_n$ with $x_i \in \mathcal{M}_{S_i}$ and all $a_1, \ldots, a_n$ with $a_i \in E_{S_i}(x_i)$ we have $ea_1 \ldots a_n \downarrow$ and $ea_1 \ldots a_n \in E_T([\![O]\!](x_1, \ldots, x_n))$.

We think of elements of $[\![R]\!]^\top$ as being those tuples $(a_1, \ldots, a_n)$ for which R is *true*. Hence the condition states that if R is realized, then it is true. However, we do not require the converse statement, so there can be true statements that are not realized. We think of the realizers as providing evidence that a formula is true, so if we have evidence that something is true, then it is true, but there are also true statements that we don't have enough evidence to know for certain.

For each variable assigment α, we define the interpretation of each term $[\![t]\!]_\alpha$ the same as for realizability models.

For formulas, we first define what it means for a formula to be *true* in the model.

$R(t_1, \ldots, t_n)$	true wrt α iff	$([\![t_1]\!]_\alpha, \ldots, [\![t_n]\!]_\alpha) \in [\![R]\!]$
$\bot$	true wrt α	never
$\varphi \wedge \psi$	true wrt α iff	φ is true and ψ is true
$\varphi \vee \psi$	true wrt α iff	φ is true or ψ is true
$\varphi \to \psi$	true wrt α iff	φ is true implies ψ is true
$\exists x^S \, \varphi$	true wrt α iff	there exists $a \in \mathcal{M}_S$ such that φ is true wrt $\alpha[x \mapsto a]$
$\forall x^S \, \varphi$	true wrt α iff	for all $a \in \mathcal{M}_S, \varphi$ is true wrt $\alpha[x \mapsto a]$

We extend realizability with truth from atomic formulas to all formulas by induction, as follows. Note that this is almost the same as the realizability interpretation we saw before, except that we adjust the definition for implication and universal quantifiers.

$e \nVdash_\alpha \bot$		always
$e \Vdash_\alpha \varphi \wedge \psi$	iff	$\mathbf{p}_0 e \Vdash_\alpha \varphi$ and $\mathbf{p}_1 e \Vdash_\alpha \psi$
$e \Vdash_\alpha \varphi \vee \psi$	iff	either $\mathbf{p}_0 e = \top$ and $\mathbf{p}_1 e \Vdash_\alpha \varphi$, or $\mathbf{p}_0 e = \bot$ and $\mathbf{p}_1 e \Vdash_\alpha \psi$
$e \Vdash_\alpha \varphi \to \psi$	iff	if $f \Vdash_\alpha \varphi$, then $ef \downarrow$ and $ef \Vdash_\alpha \psi$, and $\varphi \to \psi$ is true
$e \Vdash_\alpha \exists x^S \, \varphi$	iff	there exists $a \in \mathcal{M}_S$ such that $\mathbf{p_0} e \in E_S(a)$ and $\mathbf{p}_1 e \Vdash_{\alpha[x \mapsto a]} \varphi$
$e \Vdash_\alpha \forall x^S \, \varphi$	iff	for all $a \in \mathcal{M}_S$ and for all $f \in E_S(a), ef \downarrow$ and $ef \Vdash_{\alpha[x \mapsto a]} \varphi$ and $\forall x^S \, \varphi$ is true

As usual, we have a soundness theorem for realizability with truth:

Theorem 15.2. *If $\Gamma \vdash \varphi$ is provable in intuitionistic logic, and $x_1^{S_1}, \ldots, x_n^{S_n}$ is a list of variables including all of those that occur free in Γ and φ, then we can find $e \in \mathcal{A}$ such that for all variable assignments α, all $f_1, \ldots, f_n$ with $f_i \in E_{S_i}(\alpha(x_i))$, and all g such that $g \Vdash_\alpha \Gamma$, we have $ef_1 \ldots f_n g \downarrow$ and*

$$ef_1 \ldots f_n g \Vdash_\alpha \varphi$$

The key new result for realizability with truth is that if a formula is realized, then it is true, in the following sense.

Theorem 15.3. *Let α be a variable assignment. Suppose there is $e \in \mathcal{A}$ such that $e \Vdash_\alpha \varphi$. Then φ is true wrt α.*

Proof. We prove this by induction on formulas. Note that the definition of realizability with truth was adjusted precisely to give us the cases of implication and universal quantifiers in the inductive argument.

We still need to check the other cases, but these are straightforward.

We show the case of disjunction as an example. Suppose that $e \Vdash_\alpha \varphi \vee \psi$. Then either $\mathbf{p}_0 e = \top$ and $\mathbf{p}_1 e \Vdash_\alpha \varphi$ or $\mathbf{p}_0 e = \bot$ and $\mathbf{p}_1 e \Vdash_\alpha \psi$. In the former case we have φ is true by the inductive hypothesis, and so $\varphi \vee \psi$ is true, and similarly in the latter case, ψ and so also $\varphi \vee \psi$ is true. $\square$

We can define realizability with truth models of $\mathbf{HA}_\omega$ by the following definition:

We take $\mathcal{M}_N$ to be $\mathbb{N}$, and for sorts σ and τ, we take $\mathcal{M}_{\sigma\times\tau}$ to be $\mathcal{M}_\sigma \times \mathcal{M}_\tau$ just as in the extensional model of $\mathbf{HA}_\omega$. However, we adjust the definition of $\mathcal{M}_{\sigma\to\tau}$ by taking it to be the set of *all* functions from $\mathcal{M}_\sigma$ to $\mathcal{M}_\tau$. As usual we take $E_{\sigma\to\tau}(f)$ to be the set of all $e \in \mathcal{A}$ that track f. We again define equality to be absolute.

We can use the realizability with truth model to extract algorithms from proofs in $\mathbf{HA}_\omega$ that give us information about true formuals of arithmetic. For example, we can show the following:

Theorem 15.4. *Suppose that $\varphi(x, y)$ is a formula of $\mathbf{HA}_\omega$ whose only free variables are x^N and y^N, and that $\mathbf{HA}_\omega \vdash \forall x\, \exists y\, \varphi(x, y)$. Then we can find a computable function, say $e \in \mathcal{T}_0$ such that for all $n \in \mathbb{N}$, we have $e\underline{n} \downarrow$ and $\varphi(\underline{n}, \underline{e \cdot n})$ is true.*

Proof. Suppose that $\mathbf{HA}_\omega \vdash \forall x\, \exists y\, \varphi(x, y)$. Then using the soundness theorem for the realizability with truth model of $\mathbf{HA}_\omega$ for the pca $\mathcal{T}_0^+$, we get $a \in \mathcal{T}_0^+$ such that $a \Vdash \forall x\, \exists y\, \varphi(x, y)$. Hence we can take $e := \lambda x.\mathbf{p}_0(ax)$. For each n, we have $\mathbf{p}_1(a\underline{n}) \Vdash \varphi(\underline{n}, \underline{en})$, and so $\varphi(\underline{n}, \underline{en})$ is true. □

15.2 The internal version

In theorem 15.4 we saw that given a proof in $\mathbf{HA}_\omega$, we could extract an algorithm witnessing $\forall x\, \exists y\, \varphi(x, y)$. However, we were only able to show that for each n, $\varphi(\underline{n}, \underline{e \cdot n})$ is true, not that the statement is provable in $\mathbf{HA}_\omega$. It is also possible to give this stronger result using a similar technique. However, to do this, it is necessary to consider a variation where we carry out parts of the definition of realizability with truth inside $\mathbf{HA}_\omega$, and use this to prove parts of the soundness theorem, again inside $\mathbf{HA}_\omega$.

We won't cover this in complete detail, but to give a rough idea, one can follow this outline:

1. Instead of working with the general theory of pcas, we only look at specific pcas that we can formalise in $\mathbf{HA}_\omega$, with the application operator in the pca appearing as a formula with 3 free variables, which is provably functional. It is possible to do this for $\mathcal{T}_0^+$ by working with terms via their Gödelnumbering. However, it turns out to be more useful to consider $\mathcal{K}_1$.

2. Instead of defining realizability models in general, we work directly with ones that we can define in $\mathbf{HA}_\omega$. We only consider the realizability with truth model of $\mathbf{HA}_\omega$ above. We define $\mathcal{M}_\sigma$ and E_σ by external induction on the sorts σ. We take $\mathcal{M}_\sigma$ to be the sort σ itself. We define E_σ to be a formula with at most two free variables, x^σ and e^N. E.g. we define $E_{\sigma\to\tau}(x, e)$ to be the internal statement that e tracks (as an element of $\mathcal{K}_1$) the function x.

3. We define the realizability interpretation by external induction on formulas. Namely, for each (external) formula φ, we define a formula of $\mathbf{HA}_\omega$ with an additional free variable, e, which we denote $e \Vdash \varphi$.

4. We also prove by induction on formulas, that it is provable in $\mathbf{HA}_\omega$ that $\forall e^N \, e \Vdash \varphi \;\to\; \varphi$.

5. We prove the soundness theorem again by an external inductive argument, this time on proofs. Namely, given a proof $\Gamma \vdash \varphi$, we can find $e \in \mathbb{N}$ such that $\mathbf{HA}_\omega \vdash \; e \Vdash \varphi$.

By following this outline, we can sketch a proof of a stronger version of theorem 15.4.

Theorem 15.5. *Suppose that $\varphi(x, y)$ is a formula of $\mathbf{HA}_\omega$ whose only free variables are x^N and y^N, and that $\mathbf{HA}_\omega \vdash \forall x \, \exists y \, \varphi(x, y)$. Then we can find $n \in \mathbb{N}$ such that $\mathbf{HA}_\omega \vdash \forall x^N \, \varphi(x, \underline{n} \cdot x)$. We say $\mathbf{HA}_\omega$ satisfies* Church's rule.

Proof. Suppose that $\mathbf{HA}_\omega \vdash \forall x^N \, \varphi(x, \underline{n} \cdot x)$. Then we can find $e \in \mathbb{N}$ such that $\mathbf{HA}_\omega \vdash \underline{e} \Vdash \forall x^N \, \varphi(x, y)$. Hence $\mathbf{HA}_\omega \vdash \forall x^N \, (\mathbf{p}_1 \underline{e} x) \Vdash \varphi(x, \mathbf{p}_0 \underline{e})$, and so we have $\mathbf{HA}_\omega \vdash \forall x^N \, \varphi(x, \mathbf{p}_0 \underline{e})$. It turns out that one can show $\mathbf{HA}_\omega \vdash \underline{\lambda x. \mathbf{p}_0(ex)} = \lambda x. \mathbf{p}_0(\underline{e} x)$, and so taking $e' := \lambda x. \mathbf{p}_0(ex)$, we have $\mathbf{HA}_\omega \vdash \forall x^N \, \varphi(x, \underline{e'} x)$. ☐