13 Kleene Realizability

13.1 Encoding $\mathcal{T}_0$ in arithmetic

So far the main non trivial examples of pca that we've seen are $\mathcal{T}_0$ and $\mathcal{T}_0^+$, the pca of normal forms and inside first reduction and the extended version. As it stands, this is not something we can formalise in **HA**, or even in $\mathbf{HA}_\omega$; we cannot even define the set of normal forms in this setting. Hence it is useful to have way to view normal terms as numbers. That is, we need a Gödelnumbering of terms. To define this, first note that we can define an bijection from $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$. There are various ways to do this. For example, note that the function $\langle -, - \rangle : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ given by the following definition is definable and provably a bijection already in **HA**.

$$\langle n, m \rangle := \frac{1}{2}(n+m)(n+m+1) + m$$

We can then define an injective function $\ulcorner - \urcorner : \mathcal{T}^+ \to \mathbb{N}$ as follows:

$$\begin{array}{cccc}
\ulcorner 0 \urcorner := 0 = \langle 0, 0 \rangle & \ulcorner S \urcorner := \langle 0, 1 \rangle & \ulcorner P \urcorner := \langle 0, 2 \rangle & \ulcorner \mathbf{d} \urcorner := \langle 0, 3 \rangle \\
\ulcorner \mathbf{k} \urcorner := \langle 0, 4 \rangle & \ulcorner \mathbf{s} \urcorner := \langle 0, 5 \rangle & & \\
\ulcorner \mathbf{p} \urcorner := \langle 0, 6 \rangle & \ulcorner \mathbf{p}_0 \urcorner := \langle 0, 7 \rangle & \ulcorner \mathbf{p}_1 \urcorner := \langle 0, 8 \rangle & \\
& s \cdot t := \langle 1, \langle \ulcorner s \urcorner, \ulcorner t \urcorner \rangle \rangle & &
\end{array}$$

Theorem 13.1. *There is a total computable function $f : \mathbb{N} \to \mathbb{N}$ such that*

$$f(\langle \langle n, m \rangle, k \rangle) = \begin{cases} \langle 1, \ulcorner r \urcorner \rangle & n = \ulcorner t \urcorner \text{ for } t \in \mathcal{T}_0^+ \text{ and } t\underline{m} \to_k r \\ \langle 0, 0 \rangle & \text{otherwise} \end{cases}$$

As it turns out, we can now define application in **HA**, in the following sense.

Theorem 13.2. *There is a formula $\varphi(l, m, n)$ in the language of arithmetic such that $\varphi(l, m, n)$ is true if and only if $l = \ulcorner s \urcorner$ for a term s, $m = \ulcorner t \urcorner$ for a normal term t, and s reduces to t at stage n. Furthermore, we may assume φ is a* negative *formula, i.e. it does not contain disjunction or existential quantifiers and that* **HA** *proves $\forall l, m, n\, \varphi(l, m, n) \vee \neg\varphi(l, m, n)$.*

Definition 13.3. The axiom of Church's thesis $\mathbf{CT}_0!$ is the following sentence of $\mathbf{HA}_\omega$.

$$\forall f^{N \to N}\, \exists e^N\, \forall n^N\; e \cdot n \downarrow \wedge\, e \cdot n = fn$$

Lemma 13.4. *There is an element $\mathbf{e}$ of $\mathcal{T}_0^+$ with the property that for all $t \in \mathcal{T}^+$, we have $\mathbf{e}\ulcorner t \urcorner = t'$ if t evaluates to t' at some stage n, and otherwise is undefined. In particular, if t is normal, then $\mathbf{e}\ulcorner t \urcorner = t$.*

Proof. First note that using the decidability combinator for numbers, $\mathbf{d}$, and the fact that projection is computable, we can ensure that $\mathbf{e}\langle 0, 0 \rangle = 0$, that $\mathbf{e}\langle 0, 1 \rangle = S$, that $\mathbf{e}\langle 0, 2 \rangle = P$, and similarly for all the other constants. Using the $\mathbf{y}$ combinator we can also ensure $\mathbf{e}\langle 1, \langle n, m \rangle \rangle \simeq \mathbf{e}n(\mathbf{e}m)$. In particular, we can see that $\mathbf{e}n$ is defined whenever n is the Gödelnumber for a normal term. $\square$

13.2 The first Kleene algebra

The first Kleene algebra, $\mathcal{K}_1$ is often seen as the key example of pca$^+$ and is the one that was originally used for realizability, the general theory of pca's being a later generalisation of this example. $\mathcal{K}_1$ is the first example we will see of an ω-pca, so we can assume that the underlying set is equal to $\mathbb{N}$, and that in the pca$^+$ structure 0 is the actual zero of $\mathbb{N}$ and that S represents the actual successor function. Furthermore, it is defined so that the representable partial functions are exactly the computable partial functions. In fact these features characterise $\mathcal{K}_1$ uniquely up to isomorphism, using a non trivial argument due to Blum. However, for this course we will just show how to define an extended pca with these properties.

Definition 13.5. We define $\mathcal{K}_1$ to be the pca with underlying set $\mathbb{N}$, and application defined as follows:

$$n \cdot m := \begin{cases} l & \text{if } \mathbf{e}\underline{n}\underline{m} = \underline{l} \\ \text{undefined} & \text{otherwise} \end{cases}$$

Note that we cannot use $\mathbf{k}$ and $\mathbf{s}$ from $\mathcal{T}_0^+$ directly, but we can still define them as follows. For $\mathbf{k}$ we use the fact that pairing is computable.

$$\mathbf{k} := \ulcorner \lambda x.\langle 1, \langle \ulcorner \mathbf{k} \urcorner, x \rangle \rangle \urcorner$$

We define $\mathbf{s}$ using $\mathbf{e}$ from lemma 13.4. Our first attempt would be $\mathbf{s}_0$, as defined below.

$$\mathbf{s}_0 := \ulcorner \lambda x.\lambda y.\lambda z.\mathbf{e}(\mathbf{e}xz)(\mathbf{e}yz) \urcorner$$

As an element of $\mathcal{T}_0^+$, this takes the Gödelnumbers of two normal terms as input, and then evaluates them. However, we need to ensure that $\mathbf{s}x$ and $\mathbf{s}xy$ are Gödelnumbers for terms, rather than the terms themselves. Hence, we again use the computability of the pairing operator, and define

$$\mathbf{s}_1 := \ulcorner \lambda x.\lambda y.\langle 1, \langle \langle 1, \langle \mathbf{s}_0, x \rangle \rangle, y \rangle \rangle \urcorner$$
$$\mathbf{s} := \ulcorner \lambda x.\langle 1, \langle \mathbf{s}_1, x \rangle \rangle \urcorner$$

Theorem 13.6. *Church's thesis holds in both standard realizability models of* $\mathbf{HA}_\omega$ *on* $\mathcal{K}_1$.

Proof. For the intensional model, $\mathcal{M}_{N \to N}$ is precisely the set of $n \in \mathbb{N}$ representing a total function $\mathbb{N} \to \mathbb{N}$. So given such an n, it's clear that if we want a realizer for $\exists e^N, \forall m^N\, e \cdot m\downarrow \wedge\, e \cdot m = fm$, then the first component should just be n. We still need to show how to find the second component, which should be a realizer for $\forall m^N\, n \cdot m\downarrow \wedge\, n \cdot m = fm$. The key point is that from theorem 13.2 the statement $\forall m^N\, n \cdot m\downarrow \wedge\, e \cdot m = fm$ is equivalent to one of the form $\forall m\, \exists k\, \varphi(n, m, k)$ where φ is negative. Furthermore, in the presence of Markov's principle (which always holds in the realizability models we consider in this course), this is equivalent to $\forall m\, \neg \forall k\, \neg \varphi(n, m, k)$, which is entirely negative. However, negative formulas ψ are always *self-realizing*. That is, we can find f such that f realizes ψ whenever ψ is true. We can apply this here to get a realizer for $\forall m^N \, n \cdot m\downarrow \wedge \, n \cdot m = fm$. $\square$

Definition 13.7. We say an extended pca satisfies the *computability axiom* if there exists $\mathbf{c} \in \mathcal{K}_1$ with the following property. For all $a, b, c \in \mathcal{K}_1$, $\mathbf{c}abc \downarrow$ and $\mathbf{c}abc = 0$ or $\mathbf{c}abc = 1$, and for all a, b, $ab \downarrow$ if and only if there exists c such that $\mathbf{c}abc = 1$.

Lemma 13.8. *$\mathcal{K}_1$ satisfies the computability axiom.*

Proof. By theorem 13.1. $\square$

Theorem 13.9. *There is no $e \in \mathcal{K}_1$ with the following property: For all a such that a is total, $ea \downarrow$ and $ea = 0$ if $a\underline{n} = 0$ for all n, and $ea = 1$ if $a\underline{n} \neq 0$ for some n.*

Proof. If there was such a term e, then by theorem 13.1 we could use it to construct a term e' such that $e'ab = \underline{1}$ if $ab \downarrow$ and $e'ab = \underline{0}$ if $ab \uparrow$. However, this is not possible for any strictly partial pca (exercise). $\square$

Corollary 13.10. **WLPO** *does not hold in either of the standard realizability models of $\mathbf{HA}_\omega$ on $\mathcal{K}_1$.*

Proof. Suppose there was a realizer e of

$$e \Vdash \forall f^{N \to N} \, (\forall n \, fn = 0) \vee \neg(\forall n \, fn = 0)$$

We will use e to contradict theorem 13.9. Suppose a is total. Then this gives us an element of $\mathcal{M}_{N \to N}$, directly for the intensional model, and as the function that a represents for the extensional model. In either case we have $ea \Vdash (\forall n \, an = 0) \vee \neg(\forall n \, an = 0)$. Hence either $\mathbf{p}_0(ea) = \top$ and $\mathbf{p}_1(ea) \Vdash \forall n \, an = 0$, or $\mathbf{p}_0(ea) = \bot$ and $\mathbf{p}_1(ea) \Vdash \neg\forall n \, an = 0$. In the former case, we have that for all n, $\mathbf{p}_1(ea)n \Vdash an = 0$, and so $an = 0$, and in the latter case it is false that $an = 0$ for all n, because otherwise $\lambda n.\top$ would be a realizer of $\forall n \, an = 0$. Hence $e' := \lambda a.\mathbf{p}_0(ea)01$ has the required property to contradict theorem 13.9. $\square$

Theorem 13.11. *There is no $e \in \mathcal{K}_1$ with the following property: For all a such that a is total and $an \neq 0$ at most once, $ea \downarrow$ with $ea \in \{0, 1\}$ and for all n, $a(2n + (ea)) = 0$.*

Proof. Assume there is such an e. Using the $\mathbf{y}$ combinator, we can define a as follows. For each n, we will define $a(2n)$ and $a(2n+1)$. We first check if n is least such that $\mathbf{c}ean = 1$. If not, we take both $a(2n)$ and $a(2n+1)$ to be 0. If it is, then we know that $ea \downarrow$. We then check the value of ea. If it is 0, we take $a(2n) = 1$ and $a(2n+1) = 0$. If it is 1, we take $a(2n) = 0$ and $a(2n+1) = 1$. If it is anything else, then we take $a(2n) = a(2n+1) = 0$. Note that we have ensured that whatever happens a is a total binary sequence with at most one 1. Hence $ea \downarrow$, and so $\mathbf{c}ean = 1$ for some n. By considering the least such n, we get a contradiction. $\square$

Corollary 13.12. **LLPO** *does not hold in either of the standard realizability models of* $\mathbf{HA}_\omega$ *on* $\mathcal{K}_1$.

13.3 The Kleene Tree

We now work in **HAS**.

Definition 13.13. A set T of numbers is a *tree* if it is inhabited, every element of T is the encoding of a finite sequence $[a_0, \ldots, a_{n-1}]$ where $a_i \in \{0, 1\}$, and whenever $[a_0, \ldots, a_{n-1}, a_n] \in T$ we have also $[a_1, \ldots, a_n] \in T$ (we say T is *prefix closed*).

An *infinite path* through the tree is a binary sequence $f : \mathbb{N} \to 2$ such that for all n we have $[f(0), f(1), \ldots, f(n)] \in T$.

We say T is *infinite* if for all n, T contains a binary sequence of length greater than n.

We say T is *decidable* if we have $n \in T \vee n \notin T$ for all n.

Theorem 13.14. *In the standard realizability model of* **HAS** *on* $\mathcal{K}_1$ *there is an infinite tree with no infinite branch (the* Kleene tree*).*

Proof. We again use the fact that $\mathcal{K}_1$ satisfies the computability axiom. We first define a set $T \subseteq \mathbb{N}$ externally to the model. We take T to be the set of codes for sequences $[a_0, \ldots, a_{n-1}]$ such that for all $e < n$ with $\mathbf{c}eem = 1$ for some $m < n$, we have $a_i \neq ei$. Note that T cannot contain any computable infinite branches. Suppose $e \in \mathcal{K}_1$ is a total binary sequence. Then $ee\downarrow$, and so $\mathbf{c}ee = 1$ for some m. However, we can see that for all $n > \max(e, m)$ and all finite sequences $[a_0, \ldots, a_{n-1}] \in T$, we have $a_e \neq ee$, and so e cannot be a branch of the tree. However, T is infinite, and moreover for each n we can find $[a_0, \ldots, a_{n-1}] \in T$ computably: we evaluate $\mathbf{c}eem$ for each $m, e < n$, and whenever $\mathbf{c}eem = 1$, we can evaluate ee, and then ensure $a_e \neq ee$. If $\mathbf{c}eem = 0$ for all $m < n$, we can take a_e to be either 0 or 1.

We can then view T as an element $\bar{T}$ of $\mathcal{M}_S$ by defining

$$\bar{T}(n) = \begin{cases} \{\top\} & n \in T \\ \emptyset & n \notin T \end{cases}$$

By the way that we defined T, we can computably decide whether or not a binary sequence belongs to T, and so we have a realizer that $\bar{T}$ is decidable. Furthermore, using the fact that T has no computable infinite branch, we can find a realizer witnessing that $\bar{T}$ has no infinite branch, and similarly, we can find a realizer witnessing that $\bar{T}$ is infinite using the fact that T is "computably infinite." ☐

Corollary 13.15. *In the standard realizability model of* **HAS** *on* $\mathcal{K}_1$ *there is an infinite open cover of Cantor space with no finite subcover.*

Proof. We take the open cover to consist of U_σ for each finite binary sequence σ such that $\sigma \notin \bar{T}$. Now for every infinite binary sequence $f \in 2^{\mathbb{N}}$, there is n such that $[f(0), \ldots, f(n)] \notin \bar{T}$. Hence we can take $\sigma := [f(0), \ldots, f(n)]$ to get $f \in U_\sigma$ and $\sigma \notin \bar{T}$. However, given any finite set $\sigma_1, \ldots, \sigma_k$ of finite binary sequences in the complement of $\bar{T}$, we can take n to be the maximum length, and then find $\tau \in \bar{T}$ of length n. However, we can now define f to be the infinite binary sequence such that $f(j) = \tau(j)$ for $j \leq n$, and $f(j) = 0$ for $j > n$. We then have $f \notin U_{\sigma_i}$ for $1 \leq i \leq k$. $\square$