80818 Intuitionistic Logic - Solutions to Exercise Sheet 1

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1. (a)

$$\dfrac{\dfrac{\dfrac{\varphi, \psi \vdash \varphi \qquad \varphi, \psi \vdash \psi}{\varphi, \psi \vdash \varphi \wedge \psi} \wedge I}{\varphi \vdash \psi \to (\varphi \wedge \psi)} \to I}{\vdash \varphi \to (\psi \to (\varphi \wedge \psi))} \to I$$

   (b)

$$\dfrac{\dfrac{\varphi, \psi \vdash \varphi}{\varphi \vdash \psi \to \varphi} \to I}{\vdash \varphi \to (\psi \to \varphi)} \to I$$

   (c) In the below, we write Γ for the context $\varphi \to (\psi \to \chi), \varphi \to \psi, \varphi$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \vdash \varphi \to (\psi \to \chi) \qquad \Gamma \vdash \varphi}{\Gamma \vdash \psi \to \chi} \qquad \dfrac{\Gamma \vdash \varphi \to \psi \qquad \Gamma \vdash \varphi}{\Gamma \vdash \psi} \to E}{\Gamma \vdash \chi} \to E}{\varphi \to (\psi \to \chi), \varphi \to \psi \vdash \varphi \to \chi} \to I}{\varphi \to (\psi \to \chi) \vdash (\varphi \to \psi) \to (\varphi \to \chi)} \to I}{\vdash (\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))} \to I$$

2. This is an instance of a standard result in first order logic. For example, you can see Lemma 3.10.2 in Van Dalen, *Logic and Structure* for a textbook proof.

   Throughout this question, it is useful to note that given a proof of a formula φ, we can derive a proof of $\varphi[z/s]$ for any free variable z and term s, as follows:

$$\dfrac{\dfrac{\Gamma \vdash \varphi}{\Gamma \vdash \forall z\, \varphi} \forall I}{\Gamma \vdash \varphi[z/s]} \forall E$$

   We first show the implication (a) $\Rightarrow$ (b):

Since the only relation symbol is E, every atomic formula is of the form Est for terms s and t.

We first show, by induction on terms, that for all terms s, we have $Exy \to E\,s[z/x]\,s[z/y]$.

First suppose that s is a free variable. This divides into two subcases: either $s = z$, or $s = v$ for some free variable v such that $v \neq z$.

If $s = z$, then $E\,s[z/x]\,s[z/y] = Exy$. We can show $Exy \to Exy$ directly by $I \to$.

If $s = v \neq z$, then $E\,s[z/x]\,s[z/y] = Evv$. We can prove Evv by reflexivity Exx.

Finally, consider the case $s = Ot$. By substituting $t[z/x]$ for x and $t[z/y]$ for y in the axiom $Exy \to E(Ox)(Oy)$ we can derive $E\,t[z/x]\,t[z/y] \to E(Ot[z/x])(Ot[z/y])$. From this together with the induction hypothesis, $Exy \to E\,t[z/x]\,t[z/y]$, and $\to E$, we can derive $Exy \to E(Ot[z/x])(Ot[z/y])$. However, $Es[z/x]s[z/y]$ is equal to $E(Ot[z/x])(Ot[z/y])$ and so we are done.

We now use this to show $Exy \to (E\,s[z/x]\,t[z/x] \;\to\; E\,s[z/y]\,t[z/y])$. By applying the above claim to the terms s and t, we can show $Exy \to E\,s[z/x]\,s[z/y]$ and $Exy \to E\,t[z/x]\,t[z/y]$. By substitution, we can derive $Eyx \to E\,s[z/y]\,s[z/x]$.[1] Combining this with the symmetry axiom $Exy \to Eyx$ and $\to E$, we can show $Exy \to E\,s[z/y]\,s[z/x]$. We can now see that in context $Exy, E\,s[z/x]\,t[z/x]$ we can derive $E\,s[z/y]\,s[z/x]$, $E\,s[z/x]\,t[z/x]$ and $E\,t[z/x]\,t[z/y]$. By applying these with suitable substitutions of the transitivity axiom and $\to E$ we can derive $Exy, E\,s[z/x]\,t[z/x] \vdash E\,s[z/y]\,t[z/y]$ and so $\vdash Exy \to E\,s[z/x]\,t[z/x] \to E\,s[z/y]\,t[z/y]$ by two instances of $\to I$.

Next we check (b) $\Rightarrow$ (a)

Reflexivity, Exx is listed as an axiom in (b).

For symmetry $Exy \to Eyx$, we take φ to be Ezx, to prove the formula $Exy \to (Exx \to Eyx)$. By reflexivity, we can deduce $Exy \to Eyx$.

For transitivity, first note that by substitution, it suffices to show $Exy \to (Eyw \to Exw)$, where $w \neq z$. We apply the assumption in (b) with $\varphi = Ezw$, noting that again by substitution we can swap round x and y, to obtain $Eyx \to (Eyw \to Exw)$. Combining with symmetry (which we have already shown above) gives us $Exy \to (Eyw \to Exw)$ as we needed.

Finally we show (b) $\Rightarrow$ (c) (it is already clear that (c) $\Rightarrow$ (b)). We already know (b) $\Rightarrow$ (a) from above, so we can assume both (a) and (b).

We show by induction on formulas φ that for all free variables x and y, we have $Exy \to (\varphi[z/x] \to \varphi[z/y])$.

[1] Technically this would be done as a series of three substitutions - first replace x with a new variable v not occuring in s, then replace y with x, then replace v with y.

If φ is atomic, then we apply the assumption (b).

If $\varphi = \bot$, then it does not have any free variables, so $\bot[z/x] = \bot[z/y] = \bot$. We just apply $\to I$ to show $Exy \to (\bot \to \bot)$.

If $\varphi = \psi \wedge \chi$, then by the inductive hypothesis we can assume $Exy \to (\psi[z/x] \to \psi[z/y])$ and $Exy \to (\chi[z/x] \to \chi[z/y])$. By $\wedge E$ and repeatedly applying $\to E$, $Exy, \psi[z/x] \wedge \chi[z/x] \vdash \psi[z/y]$ and $Exy, \psi[z/x] \wedge \chi[z/x] \vdash \chi[z/y]$. It follows by $\wedge I$ and repeatedly applying $\to I$ that $\vdash Exy \to ((\psi[z/x] \wedge \chi[z/x]) \to (\psi[z/x] \wedge \chi[z/x]))$, as required.

The case of disjunction is very similar to conjunction, so we omit it.

If $\varphi = \psi \to \chi$, we note that we choose the statement we are proving by induction so that x and y can be any free variables. In particular, by the induction hypothesis, we may assume $Eyx \to (\psi[z/y] \to \psi[z/x])$. Together with symmetry and $\to E$, we deduce $Exy \to (\psi[z/y] \to \psi[z/x])$. We also have by the induction hypothesis that $Exy \to (\chi[z/x] \to \chi[z/y])$. We can now deduce by $\to E$ that $Exy \to (\psi[z/x] \to \chi[z/x]) \to (\psi[z/y] \to \chi[z/y])$.

For existential quantifiers $\exists x\, \varphi$ we use the elimination followed by the introduction rule in a similar way to disjunction, and the case of universal quantifiers is again similar.

3. (a) We wish to prove the formula $\varphi(x) := x = 0 \ \vee \ \exists y\, x = Sy$ for all x. By induction, it suffices to prove $\varphi(0)$ and $\varphi(x) \to \varphi(Sx)$. For $\varphi(0)$ we use $\vee I_l$ and reflexivity. For $\varphi(x) \to \varphi(Sx)$, we note that we do not need to use the assumption $\varphi(x)$ and just deduce this from $\varphi(Sx)$. Namely, by reflexivity, we have $Sx = Sx$ and so can deduce $\exists y\, x = Sy$ by $\exists I$, and applying $\vee I_r$ we get $\varphi(Sx)$.

   (b) We show this by induction on x. We first show $0 = y \vee (0 \neq y)$. By part (a) and $\vee E$, it suffices to derive this from $y = 0$ and $\exists z\, y = Sz$. In the former case, we apply $\vee I_l$ and reflexivity. In the latter case we apply $\vee I_r$, noting that $0 = y$ and $\exists z\, y = Sz$ contradict the axiom of **HA**, $\neg(Sx = 0)$.

   We now show that for all y, $Sx = y \ \vee \ Sx \neq y$, assuming that we already have $x = y \ \vee \ x \neq y$ for all y (the inductive hypothesis). As before, by part (a) and $\vee E$, it suffices to show this from $y = 0$, and from $\exists z\, y = Sz$. In the former case, we do not need to use the inductive hypothesis, and derive $Sx \neq 0$, from the axiom $\neg(Sx = 0)$ of **HA**. In the latter case, we may assume (by $\exists E$) that $y = Sz$. Applying the inductive hypothesis to z, we get $x = z \vee (x \neq z)$. By $\vee E$, we just need to derive $Sx = Sz \ \vee \ Sx \neq Sz$ from $x = z$ and from $x \neq z$. For the former case we apply the axioms of equality to show $Sx = Sz$ and apply $\vee I_l$. For the latter case, we use the axiom of **HA** $Sx = Sy \to x = y$ to show $x \neq z \to Sx \neq Sz$, and then deduce $Sx \neq Sz$ by $\to E$ and then $Sx = Sz \ \vee \ Sx \neq Sz$ from $\vee I_r$.

   (c) We first show $(\varphi \vee \neg\varphi) \to \exists n\, ((n = 0 \to \varphi) \wedge (n \neq 0 \to \neg\varphi))$.

By $\vee E$ it suffices to derive the conclusion form φ and from $\neg\varphi$. We first assume φ. We clearly have $0 = 0 \to \varphi$, and by reflexivity and $\bot E$, we have $(0 \neq 0 \to \neg\varphi)$, and so we can take $n = 0$. Now assume $\neg\varphi$. By the axiom $\neg(Sx = 0)$ of **HA** and $\bot E$, we have $1 = 0 \to \varphi$, and by assumption we have $1 \neq 0 \to \neg\varphi$, and so we can take $n = 1$.

Finally, we show the converse $\exists n\, ((n = 0 \to \varphi) \wedge (n \neq 0 \to \neg\varphi))$. By $\exists E$, we may assume $(n = 0 \to \varphi) \wedge (n \neq 0 \to \neg\varphi)$. By part (b) (for instance), we have $n = 0 \vee \neg(n = 0)$. By the elimination and introduction rules for disjunction, it suffices to show that $n = 0$ implies φ and that $n \neq 0$ implies $\neg\varphi$. In both cases we just apply $\to E$ together with $n = 0 \to \varphi$ for the former case, and $n \neq 0 \to \neg\varphi$ in the latter case.

4. (a) There are a few different ways to show this. Here is one:

By $\exists E$, we may assume $\varphi(X)$ and $\neg\varphi(Y)$ for free set variables X and Y. We take Z to be the union of X and Y. Namely, by comprehension, we can assume there is a Z satisfying the following property.

$$\forall n\, n \in Z \;\leftrightarrow\; (n \in X) \;\vee\; (n \in Y)$$

Note that we can show both $X \subseteq Z$ and $Y \subseteq Z$.

Now we apply the axiom $\forall X\, (\varphi \vee \neg\varphi)$ to Z (via $\forall E$). We consider the two cases $\varphi(Z)$ and $\neg\varphi(Z)$. In the former case, we take $X' := Z$ and $Y' := Y$. We then have $\varphi(X')$, and $\neg\varphi(Y')$ and $Y' \subseteq X'$. Similarly, if we have $\neg\varphi(Z)$, we can take $X' := X$ and $Y' := Z$.

(b) Again there are a few ways to do this (including some that don't use part (a)), but here is one way:

By part (a), we may assume we have X and Y such that $\varphi(X)$, $\neg\varphi(Y)$, and either $X \subseteq Y$ or $Y \subseteq X$. We just assume $X \subseteq Y$, since a very similar argument applies in the other case.

We define by comprehension a set Z satisfying the following.

$$\forall n\, n \in Z \;\leftrightarrow\; n \in X \vee (n \in Y \wedge \psi)$$

Note that if ψ is false, then we have $Z = X$ by extensionality, and it follows by the axioms of equality that $\varphi(Z)$ is true. Similarly, if ψ is true, then $Z = Y$ by extensionality, and so we have $\neg\varphi(Z)$.

By the axioms, we have either $\varphi(Z)$ or $\neg\varphi(Z)$. In the former case, $\varphi(Z)$ implies $Z \neq X$, which then implies $\neg\psi$. In the latter case $\neg\varphi(Z)$ implies $Z \neq Y$, which then implies $\neg\neg\psi$. In either case we then have $\neg\psi \;\vee\; \neg\neg\psi$, as required.