80818 Seminar on topics in logic: Intuitionistic logic

1 Introduction

1.1 Why intuitionistic logic?

In intuitionistic logic we remove an axiom that many would view as fundamental in logic: the law of excluded middle, the statement $\varphi \vee \neg\varphi$ for all propositions φ. The original motivation for doing this was philosophical considerations, such as Brouwer's philosophy of intuitionism. He believed that mathematics is inherently a mental construction. Mathematical statements are true if they are known to be true and false if they are known to be false. If there is neither a proof of a statement, nor a proof of its negation (for example, famous open problems in mathematics such as the Riemann hypothesis and $P = NP$), then the statement is neither true nor false, according to intuitionism. Today intuitionism in its strictest form is not commonly believed by working mathematicians, but many still work without the law of excluded middle, in an approach to mathematics referred to as *constructivism.* Reasons for working constructively in mathematics include:

1. Constructive proofs are more appealing since they are more explicit, giving us a greater understanding of why a result is true.

2. Although many at first (including Brouwer himself) believed constructive mathematics to be impractical, large parts of mathematics are now known to work fine in constructive mathematics, as long definitions and statements of theorems are chosen correctly. This started with the work of Bishop, with later developments by many others.

3. Constructive proofs can be combined with powerful techniques in logic to get stronger versions of a theorem automatically. For example, given a constructive proof that a function exists, we can show there is a function with additional properties such as *continuity* and *computability.*

It is useful to know that large parts of mathematics can be done constructively, since otherwise studying constructive proofs would have little point. However, for this course we will mostly encounter the third point, as we study constructive proofs from the outside using formal logic.

Although Brouwer himself did not formalise his ideas using logic, intuitionism and constructive mathematics have been widely studied from a logical point of view, to further understand them and make them more precise. This started with Brouwer's student Heyting, who developed what we would now call intuitionistic logic, as well as some basic theories such as Heyting arithmetic (an intuitionistic version of Peano arithmetic) that we will see in this course. Today intuitionistic logic is a rich subject with many aspects. This course is mainly going to focus on the *semantics*, i.e. models, of intuitionistic logic. In classical logic, the term *model* usually only refers to one thing - the definition appearing in model theory. This definition would not get us very far in intuitionistic logic, and even for some quite basic results we need to consider other notions of model. In this course the notions of model will include

1. Kripke models
2. Heyting valued models
3. realizability models.

As mentioned above, some of these can be used to strengthen proofs, for example turning a proof of the existence of a function into a construction of a computable function. Another major theme is going to be independence and consistency proofs. Consider the axiom of countable choice. This states that if $\forall n \in \mathbb{N}\ \exists x\, \varphi(n, x)$ then there is a function $f : \mathbb{N} \to \mathbb{N}$ such that $\forall n \in \mathbb{N}\ \varphi(n, f(n))$. Cohen famously showed countable choice is independent of classical set theory **ZF**. However, there are weaker versions of countable choice that follow simply from the law of excluded middle. If $\forall n \in \mathbb{N}\ \exists m \in \mathbb{N}\, \varphi(n, m)$ is true, then we can non constructively prove the existence of a choice function f, simply by taking $f(n)$ to be the least natural number m such that $\varphi(n, m)$ is true. However, this is not provable in intuitionistic logic, and moreover we can show this using a natural example of a Heyting valued model. We can also show intuitionistic logic is consistent with many *anti-classical* axioms that contradict the law of excluded middle, such as Church's thesis, which states all functions $\mathbb{N} \to \mathbb{N}$ are computable.

1.2 Review of intuitionistic first order logic

This course will mostly be about the semantics (i.e. models) of intuitionistic logic and some simple theories based on intuitionistic logic, mostly variants of Heyting arithmetic. Because of this we won't need to worry so much about the technicalities in the definition of proof that would arise in a course in proof theory. However, it is still useful to fix the definition of what we mean by formal proof, to refer back to later.

The system we will work with is multisorted intuitionistic natural deduction with sequent notation. To unpack this a bit more, we will consider a formal system with the following features,

1. It is intuitionistic - we will not assume the law of excluded middle (the axiom $\varphi \vee \neg\varphi$).

2. It is multisorted - we allow for theories where variables can range over different sorts. For example, we might have one sort for numbers and another sort for sets of numbers.

3. We will define proofs using natural deduction - most people find this the easiest and most intuitive form of formal logic.

4. We will use sequent notation to make natural deduction proofs easier to deal with formally.

Now for the formal definitions.

Definition 1.1. A *signature* consists of the following data:

1. A set $\mathcal{S}$ of *sorts*

2. A set $\mathcal{R}$ of *relation symbols*

3. For each relation symbol $R \in \mathcal{R}$, an *arity*, which is a finite list of sorts $S_1, \ldots, S_n \in \mathcal{S}$

4. A set $\mathcal{O}$ of *operator symbols*

5. For each operator symbol $O \in \mathcal{O}$, an *arity* of the operator, which is a finite list of sorts $S_1, \ldots, S_n \in \mathcal{S}$ together with 1 more sort, $T \in \mathcal{S}$. We will write the arity as $S_1, \ldots, S_n \to T$.

Next we define terms. We start with a countable supply of free variables of each sort. For now we will write the free variables of sort $S \in \mathcal{S}$ as $x_1^S, x_2^S, x_3^S, \ldots$. Later we will mostly drop the superscript, and indicate the sort of the variable other ways, for instance by the choice of letter or font.

Definition 1.2. Given a signature and the free variables, we inductively define terms, and simultaneously assign a sort to every term, as follows.

1. If x_i^S is a free variable of sort S, then it is also a term of sort S.

2. If $O \in \mathcal{O}$ is an operator symbol of arity $S_1, \ldots S_n \to T$ and $s_1, \ldots, s_n$ are terms of sort S_i for $i = 1, \ldots, n$, then $Os_1 s_2 \ldots s_n$ is a term of sort T.

Next, we can define formulas.

Definition 1.3. We inductively define the set of *formulas* as follows.

1. If $R \in \mathcal{R}$ is a relation symbol, with arity $S_1, \ldots, S_n \in \mathcal{S}$, and $s_1, \ldots, s_n$ are terms where s_i has sort S_i for $i = 1, \ldots, n$, then $Rs_1 s_2 \ldots s_n$ is a formula

2. $\perp$ is a formula

3. If φ and ψ are formulas, then the following are also formulas,

   (a) $\varphi \wedge \psi$

(b) $\varphi \vee \psi$

(c) $\varphi \to \psi$

4. If φ is a formula and x_i^S is a free variable, then the following are also formulas,

   (a) $\exists x_i^S \varphi$

   (b) $\forall x_i^S \varphi$

We will write substitution as $\varphi[x/t]$ where x is a free variable of sort $S \in \mathcal{S}$, and t is a term with the same sort S. We can read this as "x is replaced by t in φ." Formally, we define substitution as follows.

Definition 1.4. Let x be free variable of sort S and t a term of sort S. We first define substitution into terms $s[x/t]$ by induction on the definition of term. Namely,

1. If y is a free variable, we define $y[x/t]$ to be t if $x = y$, and otherwise $y[x/t]$ is defined to be y

2. For operator symbols $O \in \mathcal{O}$, we define $Os_1 \dots s_n[x/t]$ to be $O(s_0[x/t]) \dots (s_n[x/t])$.

We define $\varphi[x/t]$ for each formula φ again by induction

1. For a relation symbol $R \in \mathcal{R}$, we define $(Rs_1 \dots s_n)[x/t]$ to be $R(s_1[x/t]) \dots (s_n[x/t])$.

2. We define $(\varphi \Box \psi)[x/t]$ to be $\varphi[x/t] \Box \psi[x/t]$ where $\Box \in \{\wedge, \to, \vee\}$.

3. We define $(\forall y\, \varphi)[x/t]$ to be $\forall y\, (\varphi[x/t])$ when $y \neq x$ and to be $\forall y\, \varphi$ otherwise

4. We define $(\exists y\, \varphi)[x/t]$ to be $\exists y\, (\varphi[x/t])$ when $y \neq x$ and to be $\exists y\, \varphi$ otherwise

To help formulate proofs, we will first define sequents.

Definition 1.5. A *sequent* is a finite set of formulas $\varphi_1, \dots, \varphi_n$, and one more formula ψ. We will write this data as $\varphi_1, \dots, \varphi_n \vdash \psi$.

Finally, we can define proofs.

Definition 1.6. We define the set of *proofs* inductively by the following rules. Every proof proves a sequent, which is given simultaneously in the definition. We will refer to this as the *conclusion* of the proof.

To get off the ground, we first need the assumption rule. That is, whenever φ is an element of the set Γ, we have a proof

$$\frac{}{\Gamma \vdash \varphi}$$

We follow a general pattern that each logical connective has both introduction and elimination rules.

First the rules for conjunction:

$$\frac{\Gamma \vdash \varphi \qquad \Gamma \vdash \psi}{\Gamma \vdash \varphi \wedge \psi} \wedge I \qquad \frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \varphi} \wedge E_l \qquad \frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \psi} \wedge E_r$$

Disjunction:

$$\frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \vee \psi} \vee I_l \quad \frac{\Gamma \vdash \psi}{\Gamma \vdash \varphi \vee \psi} \vee I_r \quad \frac{\Gamma \vdash \varphi \vee \psi \qquad \Gamma, \varphi \vdash \chi \qquad \Gamma, \psi \vdash \chi}{\Gamma \vdash \chi} \vee E$$

Implication:

$$\frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \to \psi} \to I \qquad \frac{\Gamma \vdash \varphi \to \psi \qquad \Gamma \vdash \varphi}{\Gamma \vdash \psi} \to E$$

Now the rules for quantifiers, for all and exists. In both cases we assume x and y are free variables of the same sort, $S \in \mathcal{S}$, that t is a term of sort S. We also need to assume several technical conditions regarding free variables. Firstly, we need that the substitution $\varphi[x/t]$ "avoids free variable capture." That is, any occurrences of free variables in t do not become bound variables in $\varphi[x/t]$. For $\forall I$ and $\exists E$ we need that x does not occur free in any formula of Γ. Finally, for $\forall I$, we need that y is not free in φ unless $y = x$ and for $\exists E$, y is not free in ψ unless $y = x$.

$$\frac{\Gamma \vdash \varphi}{\Gamma \vdash \forall y\, \varphi[x/y]} \forall I \qquad \frac{\Gamma \vdash \forall x\, \varphi}{\Gamma \vdash \varphi[x/t]} \forall E$$

$$\frac{\Gamma \vdash \varphi[x/t]}{\Gamma \vdash \exists x\, \varphi} \exists I \qquad \frac{\Gamma \vdash \exists y\, \varphi[x/y] \qquad \Gamma, \varphi \vdash \psi}{\Gamma \vdash \psi} \exists E$$

Finally, we consider $\perp$, which only has an elimination rule, often referred to as *ex falso sequitur quodlibet* or just *ex falso*.

$$\frac{\Gamma \vdash \perp}{\Gamma \vdash \varphi} \perp E$$

We say a sequent $\Gamma \vdash \varphi$ is *provable* if it is the conclusion of some proof. We say a formula φ is provable if the sequent $\vdash \varphi$ is provable.

Definition 1.7. A *theory* over a given signature is a set T of formulas for that signature. We will refer to the elements of T as the *axioms* of the theory.

We write $T \vdash \varphi$ to mean that there is a finite set $\Gamma \subseteq T$ such that $\Gamma \vdash \varphi$, and say φ is a *theorem* of T.