7 Heyting Valued Models of Second Order Arithmetic

7.1 Standard models

The definition of Heyting valued model is a very general one. This has advantages - we will see later the completeness theorem for complete Heyting algebras, which says that if a formula holds in every Heyting valued model, then it is provable in intuitionistic logic. However, it can also be too flexible. If we are interested in a specific theory, then each time we want a model with a particular property, we need to specify the model completely, carefully ensuring each part is chosen to ensure the axioms of our theory of interest hold. For this reason, it is often useful to have some notion of *standard model* of a theory we are interested in. This is a set recipe for generating Heyting valued models of the theory, from any complete Heyting algebra. Once the complete Heyting algebra has been chosen, the remaining details of the model are already specified in the general definition, ensuring that the axioms of our theory of interest hold. For this course the theory of interest will be either **HAS** or $\mathbf{HA}_\omega$, but the same idea of standard model also appears for many other theories in logic, such as subtheories of classical second order arithmetic, set theory (both constructive and classical), higher order logic and type theory.

7.2 Some new notation regarding variable assignments

It is sometimes notationally inconvenient to always keep track of variable assignments. Hence we introduce some notation to avoid having to write them out every time.

Suppose we are given a signature $(\mathfrak{S}, \mathfrak{O}, \mathfrak{R})$ and a model over the signature $((\mathcal{M}_S)_{S\in\mathfrak{S}}, (\llbracket O \rrbracket)_{O\in\mathfrak{O}}, (\llbracket R \rrbracket)_{R\in\mathfrak{R}})$ and complete Heyting algebra $(P, \bigvee, \wedge, \to)$. We define a new signature by extending the set of operator symbols $\mathfrak{O}$ with a new constant of sort S for each $a \in \mathcal{M}_S$ for each sort $S \in \mathfrak{S}$. Given a variable assignment σ and a formula φ whose free variables are $x_1, \ldots, x_n$, note that $\varphi[x_1, \ldots, x_n/\sigma(x_1), \ldots, \sigma(x_n)]$ is a closed formula over the extended signature.

We use the notational convention

$$\llbracket \varphi[x_1, \ldots, x_n/\sigma(x_1), \ldots, \sigma(x_n)] \rrbracket := \llbracket \varphi \rrbracket_\sigma$$

Note that this defines a unique element $\llbracket \psi \rrbracket$ of P for any closed formula ψ of the extended language.

7.3 Standard models of second order Heyting arithmetic

We first see a standard way to construct Heyting valued models of **HAS** for any complete Heyting algebra $(P, \bigvee, \wedge, \to)$. We have two sorts to deal with: numbers and sets. We will define our Heyting valued models to be *global*. That is the extent for each sort will be defined to be constantly equal to $\top$. Because of this, the interpretation of quantifiers the models can be simplified as follows.

$$\llbracket \exists x^S \, \varphi(x) \rrbracket_\sigma := \bigvee_{a \in \mathcal{M}_S} \llbracket \varphi \rrbracket_{\sigma[x \mapsto a]}$$
$$\llbracket \forall x^S \, \varphi(x) \rrbracket_\sigma := \bigwedge_{a \in \mathcal{M}_S} \llbracket \varphi \rrbracket_{\sigma[x \mapsto a]}$$

Since we have two sorts we just need to define two sets: $\mathcal{M}_N$ for numbers and $\mathcal{M}_S$ for sets. We simply define $\mathcal{M}_N$ to be the set of (external) natural numbers $\mathbb{N}$. We define zero and successor to simply be the same as the external ones.

This leaves the question of what to use for sets $\mathcal{M}_S$ and how to interpret the relations $\in$ and $=$. In classical logic, we think of sets of numbers as either containing a number or not. However, in models of intuitionistic logic it is useful to allow for more possibilities. When we define a set X we can take the truth value of $n \in X$ to be anything, i.e. any element of the Heyting algebra P of truth values. Thinking topologically, we allow X to contain n within some regions of space, while not containing it within others. To formalise this we take $\mathcal{M}_S$ to be the set of functions assigning a truth value to each natural number:

$$\mathcal{M}_S := P^{\mathbb{N}}$$

This idea also suggests to us an interpretation of the membership relation. For $n \in \mathbb{N}$ and $A : \mathbb{N} \to P$, we define

$$\llbracket n \in A \rrbracket := A(n)$$

Finally, we need to define the equality relations on each sort. Since we can prove decidable equality for numbers in **HA**, we are forced to just take the simplest definition:

$$\llbracket n = m \rrbracket = \begin{cases} \top & n = m \\ \bot & n \neq m \end{cases}$$

The equality on sets is also now fixed. Firstly by extensionality we need to have $\bigwedge_{n \in N} \llbracket n \in X \leftrightarrow n \in Y \rrbracket \leq \llbracket X = Y \rrbracket$. However, in order to satisfy the axioms for equality we also need to have $\llbracket X = Y \rrbracket \leq \bigwedge_{n \in N} \llbracket n \in X \leftrightarrow n \in Y \rrbracket$. Hence we are forced to define equality of sets this way:

$$\llbracket X = Y \rrbracket \; := \; \bigwedge_{n \in N} \llbracket n \in X \leftrightarrow n \in Y \rrbracket$$

Theorem 7.1. *For any complete Heyting algebra $(P, \bigvee, \wedge, \to)$, the Heyting valued model above satisfies all of the theorems of* **HAS** *(i.e. $\llbracket \varphi \rrbracket_\sigma = \top$ for any formula φ provable in* **HAS** *and any variable assignment σ).*

Proof. By the soundness theorem for intuitionistic logic, it suffices to show that the model satisfies the axioms of **HAS**. In fact we can use any axiomatisation that ends up giving the same theorems. We do not need to prove first order induction, for example, because we can derive it from second order induction and comprehension.

We can also derive many axioms from the following statement.

$$X = Y \;\leftrightarrow\; (\forall n\, n \in X \leftrightarrow n \in Y) \tag{1}$$

We can easily show that the above axiom holds in our models, since we chose to define equality of sets in a way that forces this to be the case.

Clearly (1) implies extensionality, since it is the right to left direction of the bi-implication. However, it also implies axioms of equality, including all of the equality axioms for sets:

$$\begin{array}{ll} X = X & X = Y \;\rightarrow\; Y = X \\ X = Y \rightarrow (Y = Z \;\rightarrow\; X = Z) & X = Y \rightarrow (n \in X \;\rightarrow\; n \in Y) \end{array}$$

It still remains to check the equality axioms for the sort of numbers, namely, we need to show

$$\begin{array}{ll} n = n & n = m \;\rightarrow\; m = n \\ n = m \;\rightarrow\; (m = k \;\rightarrow\; n = k) & n = m \rightarrow (n \in X \;\rightarrow\; m \in X) \end{array}$$

However, all are straightforward to check for our chosen definition of equality of numbers, which matches up with the external "true" equality of numbers.

We now check comprehension. We are given a formula $\varphi(x)$ and need to verify that for all variable assignments σ we have

$$[\![\exists X\, \forall n\, (n \in X \;\leftrightarrow\; \varphi(n))]\!] = \top$$

Note that it is sufficient to find a specific element $A \in \mathcal{M}_S$ such that $[\![n \in A]\!] = [\![\varphi(n)]\!]$ for all n. However, we can do this by simply defining $A(n) := [\![\varphi(n)]\!]$.

We finally check second order induction. We need to show that for each $A \in \mathcal{M}_S$ we have

$$[\![(0 \in A \;\wedge\; \forall n\, (n \in A \rightarrow Sn \in A)) \;\rightarrow\; \forall n\, n \in A]\!] = \top$$

By unfolding the definitions and using the basic properties of implication and meet in a complete Heyting algebra, this amounts to showing the following inequality for each n:

$$A(0) \;\wedge\; \bigwedge_{m \in \mathbb{N}} (A(m) \rightarrow A(S(m))) \;\leq\; A(n) \tag{2}$$

We show this by induction on n in the metatheory where we are working. The case $n = 0$ is easy from the definition of meet. Suppose we have already shown (2) for n; we will prove it for Sn. It follows from the inductive hypothesis and the basic properties of meet that we have

$$A(0) \;\wedge\; \bigwedge_{m \in \mathbb{N}} (A(m) \rightarrow A(S(m))) = A(0) \;\wedge\; \bigwedge_{m \in \mathbb{N}} (A(m) \rightarrow A(S(m))) \;\wedge\; A(n)$$

We can then reason as follows

$$A(0) \wedge \bigwedge_{m\in\mathbb{N}} (A(m) \to A(S(m))) \wedge A(n) \leq A(n) \wedge (A(n) \to A(S(n)))$$
$$\leq A(S(n))$$

□

It is a characteristic of the standard model approach that we do not have a general completeness theorem. There are examples of formulas that hold in every standard model that are not provable within the theory we are interested in, in this case **HAS**. In fact, for standard models of **HAS**, this is the case for all true first order formulas. In other words, if a first order formula is true externally, in the metatheory where we are working, then it is also true in the model. Results of this kind are known as *absoluteness* theorems.

Theorem 7.2. *Assume the law of excluded middle and that in the Heyting algebra $(P, \bigvee, \wedge, \to)$ we have $\bot \neq \top$. Let φ be a formula over the signature of* **HA** *(i.e. no second order variables free or bound, in particular no second order quantifiers, no $\in$ and no set equality relations). Given a function σ from free variables to $\mathbb{N}$ write $\varphi[\sigma]$ for the result of replacing each free variable x by $\sigma(x)$. Then for any function σ from variables to $\mathbb{N}$, $[\![\varphi]\!]_\sigma = \top$ if and only if $\varphi[\sigma]$ is true, and otherwise $[\![\varphi]\!]_\sigma = \bot$.*

Proof. This is proved by induction on the set of formulas. We give some of the cases of the induction, with the remainder left as an exercise.

Disjunction Suppose that $\varphi \vee \psi[\sigma]$ is true. If $\varphi[\sigma]$ is true, then $[\![\varphi]\!]_\sigma = \top$ by the inductive hypothesis, and so also $[\![\varphi \vee \psi]\!]_\sigma = \top$. Similarly if $\psi[\sigma]$ is true. Hence in either case we have $[\![\varphi \vee \psi]\!]_\sigma = \top$.

Suppose on the other hand that $\varphi \vee \psi[\sigma]$ is false. Then $\varphi[\sigma]$ and $\psi[\sigma]$ are both false. Hence $[\![\varphi]\!]_\sigma = [\![\psi]\!]_\sigma = \bot$. Hence $[\![\varphi \vee \psi]\!]_\sigma = \bot$. By the law of excluded middle, it follows that if $[\![\varphi \vee \psi]\!]_\sigma = \top$ then $\varphi \vee \psi[\sigma]$ is true.

Implication Suppose that $\varphi \to \psi[\sigma]$ is true. By the law of excluded middle, either $\psi[\sigma]$ is true or $\varphi[\sigma]$ is false. In the former case we have $[\![\varphi]\!]_\sigma = \top$ and in the latter case $[\![\psi]\!]_\sigma = \bot$. In either case we can deduce $[\![\varphi \to \psi]\!]_\sigma = \top$. On the other hand, suppose that $\varphi \to \psi[\sigma]$ is false. Then $\psi[\sigma]$ is false, and by the law of excluded middle, $\varphi[\sigma]$ is true. Hence $[\![\varphi]\!]_\sigma = \bot$ and $[\![\psi]\!]_\sigma = \top$. It follows that $[\![\varphi \to \psi]\!]_\sigma = \bot$. Again using the law of excluded middle, we can also deduce that if $[\![\varphi \to \psi]\!]_\sigma = \top$, then $\varphi[\sigma]$ is true.

Existential quantification Suppose that $(\exists x\, \varphi(x))[\sigma]$ is true. Then there exists a number n such that $\varphi[\sigma[x \mapsto n]]$ is true. Hence $[\![\varphi]\!]_{\sigma[x\mapsto n]} = \top$. It follows that $[\![\exists x\, \varphi]\!]_\sigma = \top$.

Suppose that $(\exists x\, \varphi)[\sigma]$ is false. In this case, for every n in $\mathbb{N}$, $\varphi[\sigma[x \mapsto n]]$ is false. Hence for each n, $[\![\varphi]\!]_{\sigma[x\mapsto n]} = \bot$. Hence $[\![\exists x\, \varphi]\!]_\sigma = \bot$. By the law of excluded middle, it follows that if $[\![\exists x\, \varphi]\!]_\sigma = \top$, then $(\exists x\, \varphi)[\sigma]$ is true. □

7.4 Some examples of standard models of HAS

7.4.1 The trivial Heyting algebra

We now have a way to construct a model of **HAS** given any complete Heyting algebra. We first consider what this looks like in the simplest case: the trivial Heyting algebra $2 = \{\bot, \top\}$.[1]

In this case the sort of sets is modelled by functions $\mathbb{N} \to 2$. However, these correspond precisely to actual subsets of $\mathbb{N}$. As always the sort of numbers is just the external set of numbers $\mathbb{N}$. Combining this with our previous general discussion of trivial Heyting valued models we get the following theorem.

Theorem 7.3. *Let φ be a formula over the signature of* **HAS**. *Then φ holds in the trivial standard model of* **HAS** *if and only if it is true (in the metatheory where we are working).*

7.4.2 Sierpiński space

We now turn to our simplest non trivial example of a complete Heyting algebra: Sierpiński space. We can use this to show that **HAS** is different from second order Peano arithmetic. That is, we show that the law of excluded middle is not provable in **HAS**. We specifically show that the following instance of excluded middle is not provable.

$$\forall X \, 0 \in X \ \vee \ 0 \notin X$$

In order to do this, it suffices to find an element A of $\mathcal{M}_S$ in the standard model on Sierpiński space such that

$$[\![0 \in A \ \vee \ 0 \notin A]\!] \neq \top$$

We can define such an A as follows. The value of $A(n)$ for $n \neq 0$ does not matter, so we can take it to be $\emptyset$, for example. We take $A(0)$ to be the intermediate truth value $\{0\}$.

We can then calculate similarly to before,

$$\begin{aligned} [\![0 \in A \ \vee \ 0 \notin A]\!] &= A(0) \cup (\{0,1\} \setminus A(0))^{\circ} \\ &= \{0\} \cup \{1\}^{\circ} \\ &= \{0\} \cup \emptyset \\ &= \{0\} \\ &\neq \{0,1\} \end{aligned}$$

7.5 Cantor space

In the previous example, we only showed that an instance of excluded middle is not provable in **HAS**. It is a more difficult problem to give an example of

[1] Once again we assume the law of excluded middle in order to show this is complete. For a constructive version, we can instead use the power set of a singleton.

a formula that is consistent with **HAS** that contradicts the law of excluded middle. For example, for any formula ψ, we can prove $\neg\neg(\psi \vee \neg\psi)$ in intuitionistic logic. Hence $\neg(\psi \vee \neg\psi)$ is never consistent with **HAS**. However, we have previously seen for example that $A \vee \neg A$ is not provable in intuitionistic logic with a constant relation symbol A.

To see our first non trivial consistency statement with **HAS**, we will use the standard topological model on Cantor space.

Theorem 7.4. *The following formula is consistent with* **HAS***, i.e. if we add it as an axiom it is still impossible to derive* $\perp$.

$$\neg(\forall X\, 0 \in X \;\vee\; 0 \notin X)$$

Proof. We will show the formula holds in the standard topological model on Cantor space. It will follow from the soundness theorem that it is consistent. That is, we show

$$[\![\neg(\forall X\, 0 \in X \;\vee\; 0 \notin X)]\!] = \top$$

It suffices to show

$$[\![\forall X\, 0 \in X \;\vee\; 0 \notin X]\!] = \perp$$

Since we are working in a topological model, we can explicitly describe $\perp$ as $\emptyset$. Hence we will derive a contradiction from the assumption that the open set $[\![\forall X\, 0 \in X \;\vee\; 0 \notin X]\!]$ contains an element f.

By expanding out the interpretation of universal quantification and using the explicit description of meet in a topological space, we have

$$f \in \left(\bigcap_{A \in \mathcal{M}_S} [\![0 \in A \;\vee\; 0 \notin A]\!] \right)^{\circ} \subseteq \bigcap_{A \in \mathcal{M}_S} [\![0 \in A \;\vee\; 0 \notin A]\!]$$

We will show this leads to a contradiction by finding A such that f does not belong to $[\![0 \in A \;\vee\; \neg 0 \in A]\!]$. We define this A as follows.

$$A(n) = \begin{cases} 2^{\mathbb{N}} \setminus \{f\} & n = 0 \\ \emptyset & \text{otherwise} \end{cases}$$

We clearly have $f \notin [\![0 \in A]\!]$ by construction.

However, we can calculate

$$\begin{aligned} [\![0 \notin A]\!] &= (2^{\mathbb{N}} \setminus A(n))^{\circ} \\ &= \{f\}^{\circ} \\ &= \emptyset \end{aligned}$$

Hence we have $f \notin [\![0 \in A \;\vee\; 0 \notin A]\!]$, as we needed. $\square$