13 Kleene Realizability

13.1 Encoding $\mathcal{T}_0$ in arithmetic

So far the main non trivial examples of pca that we've seen are $\mathcal{T}_0$ and $\mathcal{T}_0^+$, the pca of normal forms and inside first reduction and the extended version. As it stands, this is not something we can formalise in **HA**, or even in $\mathbf{HA}_\omega$; we cannot even define the set of normal forms in this setting. Hence it is useful to have way to view normal terms as numbers. That is, we need a Gödelnumbering of terms. To define this, first note that we can define an bijection from $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$. There are various ways to do this. For example, note that the function $\langle -, - \rangle : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ given by the following definition is definable and provably a bijection already in **HA**.

$$\langle n, m \rangle := \frac{1}{2}(n+m)(n+m+1) + m$$

We can then define an injective function $\ulcorner - \urcorner : \mathcal{T}^+ \to \mathbb{N}$ as follows:

$$\begin{array}{cccc} \ulcorner 0 \urcorner := 0 = \langle 0, 0 \rangle & \ulcorner S \urcorner := \langle 0, 1 \rangle & \ulcorner P \urcorner := \langle 0, 2 \rangle & \ulcorner \mathbf{d} \urcorner := \langle 0, 3 \rangle \\ \ulcorner \mathbf{k} \urcorner := \langle 0, 4 \rangle & \ulcorner \mathbf{s} \urcorner := \langle 0, 5 \rangle & & \\ \ulcorner \mathbf{p} \urcorner := \langle 0, 6 \rangle & \ulcorner \mathbf{p}_0 \urcorner := \langle 0, 7 \rangle & \ulcorner \mathbf{p}_1 \urcorner := \langle 0, 8 \rangle & \\ & s \cdot t := \langle 1, \langle \ulcorner s \urcorner, \ulcorner t \urcorner \rangle \rangle & & \end{array}$$

Theorem 13.1. *There is a total computable function $f : \mathbb{N} \to \mathbb{N}$ such that*

$$f(\langle \langle n, m \rangle, k \rangle) = \begin{cases} \langle 1, \ulcorner r \urcorner \rangle & n = \ulcorner t \urcorner \text{ for } t \in \mathcal{T}_0^+ \text{ and } t\underline{m} \to_k r \\ \langle 0, 0 \rangle & \text{otherwise} \end{cases}$$

As it turns out, we can now define application in **HA**, in the following sense.

Theorem 13.2. *There is a formula $\varphi(l, m, n)$ in the language of arithmetic such that $\varphi(l, m, n)$ is true if and only if $l = \ulcorner s \urcorner$, $m = \ulcorner t \urcorner$ for normal terms s and t, and s reduces to t at stage n. Furthermore, we may assume φ is a* negative *formula, i.e. it does not contain disjunction or existential quantifiers and that* **HA** *proves $\forall l, m, n\, \varphi(l, m, n) \vee \neg\varphi(l, m, n)$.*

Definition 13.3. The axiom of Church's thesis $\mathbf{CT}_0!$ is the following sentence of $\mathbf{HA}_\omega$.

$$\forall f^{N \to N}\, \exists e^N\, \forall n^N\; e \cdot n \downarrow \wedge\, e \cdot n = fn$$

Lemma 13.4. *There is an element $\mathbf{e}$ of $\mathcal{T}_0^+$ with the property that for all $t \in \mathcal{T}^+$, we have $\mathbf{e} \ulcorner t \urcorner = t'$ if t evaluates to t' at some stage n, and otherwise is undefined. In particular, if t is normal, then $\mathbf{e} \ulcorner t \urcorner = t$.*

Proof. First note that using the decidability combinator for numbers, $\mathbf{d}$, and the fact that projection is computable, we can ensure that $\mathbf{e}\langle 0, 0 \rangle = 0$, that $\mathbf{e}\langle 0, 1 \rangle = S$, that $\mathbf{e}\langle 0, 2 \rangle = P$, and similarly for all the other constants. Using the $\mathbf{y}$ combinator we can also ensure $\mathbf{e}\langle 1, \langle n, m \rangle \rangle \simeq \mathbf{e}n(\mathbf{e}m)$. In particular, we can see that $\mathbf{e}n$ is defined whenever n is the Gödelnumber for a normal term. □

13.2 The first Kleene algebra

The first Kleene algebra, $\mathcal{K}_1$ is often seen as the key example of pca$^+$ and is the one that was originally used for realizability, the general theory of pca's being a later generalisation of this example. $\mathcal{K}_1$ is the first example we will see of an ω-pca, so we can assume that the underlying set is equal to $\mathbb{N}$, and that in the pca$^+$ structure 0 is the actual zero of $\mathbb{N}$ and that S represents the actual successor function. Furthermore, it is defined so that the representable partial functions are exactly the computable partial functions. In fact these features characterise $\mathcal{K}_1$ uniquely up to isomorphism, using a non trivial argument due to Blum. However, for this course we will just show how to define an extended pca with these properties.

Definition 13.5. We define $\mathcal{K}_1$ to be the pca with underlying set $\mathbb{N}$, and application defined as follows:

$$n \cdot m := \begin{cases} l & \text{if } \mathbf{e}\underline{nm} = \underline{l} \\ \text{undefined} & \text{otherwise} \end{cases}$$

Note that we cannot use $\mathbf{k}$ and $\mathbf{s}$ from $\mathcal{T}_0^+$ directly, but we can still define them as follows. For $\mathbf{k}$ we use the fact that pairing is computable.

$$\mathbf{k} := \ulcorner \lambda x.\langle 1, \langle \ulcorner \mathbf{k} \urcorner, x\rangle\rangle \urcorner$$

We define $\mathbf{s}$ using $\mathbf{e}$ from lemma 13.4. Our first attempt would be $\mathbf{s}_0$, as defined below.

$$\mathbf{s}_0 := \ulcorner \lambda x.\lambda y.\lambda z.\mathbf{e}(\mathbf{e}xz)(\mathbf{e}yz) \urcorner$$

As an element of $\mathcal{T}_0^+$, this takes the Gödelnumbers of two normal terms as input, and then evaluates them. However, we need to ensure that $\mathbf{s}x$ and $\mathbf{s}xy$ are Gödelnumbers for terms, rather than the terms themselves. Hence, we again use the computability of the pairing operator, and define

$$\mathbf{s}_1 := \ulcorner \lambda x.\lambda y.\langle 1, \langle\langle 1, \langle \mathbf{s}_0, x\rangle\rangle, y\rangle\rangle \urcorner$$
$$\mathbf{s} := \ulcorner \lambda x.\langle 1, \langle \mathbf{s}_1, x\rangle\rangle \urcorner$$

Theorem 13.6. *Church's thesis holds in both standard realizability models of* $\mathbf{HA}_\omega$ *on* $\mathcal{K}_1$.

Proof. For the intensional model, $\mathcal{M}_{N\to N}$ is precisely the set of $n \in \mathbb{N}$ representing a total function $\mathbb{N} \to \mathbb{N}$. So given such an n, it's clear that if we want a realizer for $\exists e^N, \forall m^N \, e \cdot m \downarrow \wedge e \cdot m = fm$, then the first component should just be n. We still need to show how to find the second component, which should be a realizer for . The key point is that from theorem 13.2 the statement $\forall m^N \, n \cdot m \downarrow \wedge e \cdot m = fm$ is equivalent to one of the form $\forall m \, \exists k \, \varphi(n, m, k)$ where φ is negative. Furthermore, in the presence of Markov's principle (which always holds in the realizability models we consider in this course), this is equivalent to $\forall m \, \neg \forall k \, \neg \varphi(n, m, k)$, which is entirely negative. However, negative formulas ψ are always *self-realizing*. That is, we can find f such that f realizes ψ whenever ψ is true. We can apply this here to get a realizer for $\forall m^N \, n \cdot m \downarrow \wedge e \cdot m = fm$. $\square$

Definition 13.7. We say an extended pca satisfies the *computability axiom* if there exists $\mathbf{c} \in \mathcal{K}_1$ with the following property. For all $a, b, c \in \mathcal{K}_1$, $\mathbf{c}abc\downarrow$ and $\mathbf{c}abc = 0$ or $\mathbf{c}abc = 1$, and for all a, b, $ab\downarrow$ if and only if there exists c such that $\mathbf{c}abc = 1$.

Lemma 13.8. *$\mathcal{K}_1$ satisfies the computability axiom.*

Proof. By theorem 13.1. □

Theorem 13.9. *There is no $e \in \mathcal{K}_1$ with the following property: For all a such that a is total, $ea = 0$ if $a\underline{n} = 0$ for all n, and $ea = 1$ if $a\underline{n} \neq 0$ for some n.*

Proof. If there was such a term e, then by theorem 13.1 we could use it to construct a term e' such that $e'ab = \underline{1}$ if $ab\downarrow$ and $e'ab = \underline{0}$ if $ab\uparrow$. However, this is not possible for any strictly partial pca (exercise). □

Corollary 13.10. **WLPO** *does not hold in either of the standard realizability models of $\mathbf{HA}_\omega$ on $\mathcal{K}_1$.*

Proof. Suppose there was a realizer e of

$$e \Vdash \forall f^{N\to N}\,(\forall n\, fn = 0) \vee \neg(\forall n\, fn = 0)$$

We will use e to contradict theorem 13.9. Suppose a is total. Then this gives us an element of $\mathcal{M}_{N\to N}$, directly for the intensional model, and as the function that a represents for the extensional model. In either case we have $ea \Vdash (\forall n\, an = 0) \vee \neg(\forall n\, an = 0)$. Hence either $\mathbf{p}_0(ea) = \top$ and $\mathbf{p}_1(ea) \Vdash \forall n\, an = 0$, or $\mathbf{p}_0(ea) = \bot$ and $\mathbf{p}_1(ea) \Vdash \neg\forall n\, an = 0$. In the former case, we have that for all n, $\mathbf{p}_1(ea)n \Vdash an = 0$, and so $an = 0$, and in the latter case it is false that $an = 0$ for all n, because otherwise $\lambda n.\top$ would be a realizer of $\forall n\, an = 0$. Hence $e' := \lambda a.\mathbf{p}_0(ea)01$ has the required property to contradict theorem 13.9. □