80818 Intuitionistic Logic - Exercise Sheet 1

September 6, 2021

1. Given proofs in natural deduction of the following,
   (a) $\varphi \to (\psi \to (\varphi \wedge \psi))$
   (b) $\varphi \to (\psi \to \varphi)$
   (c) $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$

2. Work over the following signature: a single sort, a unary operator symbol O and a binary relation symbol E. Show that the three statements below are equivalent to each other.
   (a) The conjunction of the four axioms below:
   $$Exx, \qquad Exy \to Eyx,$$
   $$Exy \to (Eyz \to Exz), \qquad Exy \ \to \ E(Ox)(Oy)$$
   (b) The conjunction of Exx and for each *atomic* formula φ, the formula below
   $$Exy \to (\varphi[z/x] \to \varphi[z/y])$$
   (c) The conjunction of Exx and for *every* formula φ, the formula below
   $$Exy \to (\varphi[z/x] \to \varphi[z/y])$$

3. Show that the following are provable in **HA**.
   (a) $x = 0 \ \vee \ \exists y \, x = Sy$
   (b) $x = y \ \vee \ \neg(x = y)$
   (c) $(\varphi \ \vee \ \neg\varphi) \ \leftrightarrow \ \exists n \, ((n = 0 \to \varphi) \wedge (n \neq 0 \to \neg\varphi))$ for any formula φ

4. You are given a formula φ of **HAS** with a free variable X. For this fixed φ work over **HAS** with the following three additional axioms.
   $$\forall X \, (\varphi \vee \neg\varphi), \qquad \exists X \, \varphi, \qquad \exists X \, \neg\varphi$$
   (a) Prove $\exists X \, \exists Y \, (\varphi \wedge \neg\varphi[X/Y] \wedge (X \subseteq Y \vee Y \subseteq X))$, where $X \subseteq Y$ is notational shorthand for $\forall x \, (x \in X \ \to \ x \in Y)$.
   (b) For *any* formula ψ, prove $\neg\psi \vee \neg\neg\psi$. (This is referred to as the *weak law of excluded middle.*)