8 Heyting Valued Models of Arithmetic with Finite Types

8.1 Partial equivalence relations and H-sets

The theories that we are interested in typically have equality relations for each sort. For such theories it is often conventient to merge the equality relation and extent predicate together into a single binary relation by the following method.

Suppose that we are given signature $(\mathfrak{S}, \mathfrak{O}, \mathfrak{R})$ such that $\mathfrak{R}$ includes a binary relation symbol $=$ on a sort S. Suppose further we have a Heyting valued model for that signature over a complete Heyting algebra $(H, \bigvee, \wedge, \rightarrow)$ and that $=$ satisfies the axioms for equality.

We extend the signature with a new binary relation symbol $='$ and interpret it in the Heyting valued model as follows

$$[\![a =' b]\!] := E(a) \wedge E(b) \wedge [\![a = b]\!]$$

Note that the extended Heyting valued satisfies $\forall x \, \forall y \, x =' y \leftrightarrow x = y$, which follows directly from the way that universal quantifiers are interpreted in the model. In particular $='$ satisfies all the axioms for equality, since $=$ does.

Futhermore, we can recover the extent predicate from $='$ by the following equation.

$$E(a) = [\![a =' a]\!]$$

We can also see that $[\![='\!]\!]$ has the following properties:

$$[\![a =' b]\!] = [\![b =' a]\!]$$
$$[\![a =' b]\!] \wedge [\![b =' c]\!] \leq [\![a =' c]\!]$$

We can think of this as a "Heyting valued" version of the following definition.

Definition 8.1. Let X be any set. A *partial equivalence relation* on X is a binary relation $E \subseteq X \times X$ satisfying the following conditions:

1. For all $x, y \in X$, $E(x, y)$ if and only if $E(y, x)$ (E is symmetric).
2. For all $x, y, z \in X$, if $E(x, y)$ and $E(y, z)$, then $E(x, z)$ (E is transitive).

We refer sets with H-valued relations satisfying the above as H-sets:

Definition 8.2. Let $(H, \bigvee, \wedge, \rightarrow)$ be a complete Heyting algebra. An H-set is a set X, together with a function $\approx : X \times X \rightarrow H$ satisfying the following for all $x, y, z \in X$:

$$x \approx y = y \approx x$$
$$x \approx y \wedge y \approx z \leq x \approx z$$

Given an H-set $(X, \approx)$ we will write $E(x)$ for $x \in X$ as notation for $x \approx x$.

8.2 Singletons in Heyting valued models of HAS, and H-sets

In order to motivate an important aspect of the standard model on $\mathbf{HA}_\omega$, we will first take a closer look at how singleton sets work in **HAS**.

Recall that a singleton set is one with exactly one element. We formalise this in **HAS** as follows:

Definition 8.3. We say that X is a *singleton* if there exists a number x such that $x \in X$ and for all numbers x, y such that $x \in X$ and $y \in X$ we have $x = y$.

If we apply this to the standard Heyting valued model of **HAS** on a complete Heyting algebra $(H, \bigvee, \wedge, \rightarrow)$, we obtain the following definition.

Definition 8.4. We say $A : \mathbb{N} \rightarrow H$ is a *singleton* if

1. $\bigvee_{n \in \mathbb{N}} A(n) = \top$
2. for all n, m such that $m \neq n$ we have $A(n) \wedge A(m) = \bot$

Note that given any $n \in \mathbb{N}$, we can define a singleton $\underline{n}$ as follows:

$$\underline{n}(m) := \begin{cases} \top & m = n \\ \bot & m \neq n \end{cases}$$

However, it is important to note that in general these are not the only examples of singletons. Suppose that we have $p, q \in H$ such that $p \wedge q = \bot$ and $p \vee q = \top$. In that case we can define a singleton that can take one of two different values:

$$A(k) := \begin{cases} p & k = n \\ q & k = m \\ \bot & k \neq n \text{ and } k \neq m \end{cases}$$

To give a more concrete example of this we can define the following on Cantor space:

$$A(k) := \begin{cases} \{f : \mathbb{N} \rightarrow 2 \mid f(0) = 0\} & k = 0 \\ \{f : \mathbb{N} \rightarrow 2 \mid f(0) = 1\} & k = 1 \\ \emptyset & k > 1 \end{cases}$$

We note however, that these examples only work because Cantor space is not connected. We can use connectedness to get some more control over what singletons can look like.

Definition 8.5. We say a complete Heyting algebra $(H, \bigvee, \wedge\ \rightarrow)$ (or more generally any poset with finite meets and least element) is *connected* if for all $p, q \in H$ such that $p \vee q = \top$ and $p \wedge q = \bot$ we have $p = \top$ or $q = \top$.

In particular the lattice of open sets of a topological space is connected as a Heyting algebra if and only if the topological space is connected.

Proposition 8.6. *Suppose that $(H, \bigvee, \wedge \to)$ is a connected complete Heyting algebra. Then for every singleton $A : \mathbb{N} \to H$ in the standard model of* **HAS** *such that $A(n) = \bot$ for $n > 1$, there exists $n \in \{0, 1\}$ such that $[\![A = \underline{n}]\!] = \top$.*

Proof. Define $p := A(0)$ and $q := A(1)$. We then have $p \vee q = \top$ and $p \wedge q = \bot$. By connectedness, we have either $p = \top$ or $q = \top$. In the former case we have $[\![A = \underline{0}]\!] = \top$ and in the latter case we have $[\![A = \underline{1}]\!] = \top$. $\square$

More generally, we define singletons for H-sets as follows:

Definition 8.7. Suppose $(X, \approx)$ is an H-set for a complete Heyting algebra $(H, \bigvee, \wedge, \to)$. We define a new H-set $S(X)$ of *singletons* as follows. The underlying set of $S(X)$ consists of functions $A : X \to H$. We define $\approx$ as follows.

$$A \approx B := \bigvee_{x \in X} A(x) \;\wedge\; \bigvee_{x \in X} B(x) \;\wedge\; \bigwedge_{x,y \in X} (A(x) \wedge B(y) \;\to\; x \approx y)$$

Note that for an H-set X, we can define a canonical inclusion $i : X \to S(X)$ such that $x \approx y \leq i(x) \approx i(y)$ for all $x, y \in X$ by

$$i(x)(y) := x \approx y$$

Definition 8.8. We say an H-set $(X, \approx)$ is *weakly complete* if there is a function $s : S(X) \to X$ such that for all $A, B : X \to H$ we have

$$A \approx B \leq s(A) \approx s(B)$$

and such that for all $x \in X$ and $A \in S(X)$ we have

$$x \approx x \leq s(i(x)) \approx x$$
$$A \approx A \leq i(s(A)) \approx A$$