80818 Intuitionistic Logic - Exercise Sheet 10

November 11, 2021

1. Let σ and τ be finite types. Write $\mathbf{Inj}^{\sigma,\tau}$ to mean there exists an injective function from σ to τ. Write $\mathbf{Ext}^{\sigma,\tau}$ to mean function extensionality for functions from σ to τ, and $\mathbf{Ext}$ to mean function extensionality for all sorts σ and τ.
   (a) Show that $\mathbf{Inj}^{N \to N,N}$ together with $\mathbf{Ext}^{N,N}$ implies $\mathbf{WLPO}$.
   (b) Show that $\mathbf{HA}_\omega + \mathbf{CT}_0! + \mathbf{AC}^{N \to N,N} + \mathbf{Ext}^{N,N}$ implies $\mathbf{Inj}^{N \to N,N}$.
   (c) Hence or otherwise, show that $\mathbf{HA}_\omega + \mathbf{CT}_0! + \mathbf{AC}^{N \to N,N} + \mathbf{Ext}^{N,N}$ is not consistent, i.e. it proves $\perp$.
   (d) Show that the following theories are consistent (you may assume the law of excluded middle and the axiom of choice in the metatheory).
      i. $\mathbf{HA}_\omega + \mathbf{CT}_0! + \mathbf{AC} + \mathbf{Inj}^{N \to N,N}$
      ii. $\mathbf{HA}_\omega + \mathbf{CT}_0! + \mathbf{Ext}$
      iii. $\mathbf{HA}_\omega + \mathbf{AC} + \mathbf{Ext}$

2. Suppose we are given any function $\delta : \mathbb{N} \to \mathbb{N}$. Write $\mathcal{T}^{\delta+}$ for the smallest set containing constants $\mathbf{k}, \mathbf{s}, 0, S, P, \mathbf{d}$ and δ, and containing an element $s \cdot t$ whenever $t, s \in \mathcal{T}^{\delta+}$. We define the relation $t \to_k r$ the same as for $\mathcal{T}^+$, except with the additional rule $\delta \underline{n} \to_k \underline{\delta(n)}$ for all $n, k \in \mathbb{N}$.
   (a) Verify that the subset of normal terms gives an extended pca, and that δ is $\mathcal{T}_0^{\delta+}$-computable.
   (b) Outline a proof that there is an ω-pca where δ is representable.
   (c) Show the following theory is consistent $\mathbf{HA}_\omega + \mathbf{AC} + \mathbf{Inj}^{N \to N,N} + \neg\mathbf{CT}_0!$.

3. (a) Let $\mathcal{A}$ be any ω-pca. Write T for the set $\{a \in \mathcal{A} \mid \forall b \in \mathcal{A}\, ab \downarrow\}$. Show there is no representable surjection from $\mathcal{A}$ to T.
   (b) Work in $\mathbf{HAS}$. A binary relation from a set X to a set Y is a set Z such that for all $z \in Z$ there exists $x \in X$ and $y \in Y$ such that $z = \langle x, y \rangle$. A function from X to Y is a binary relation that is functional. Show that it is consistent with $\mathbf{HAS}$ that there is no surjective function from $\mathbb{N}$ to $\{n \mid \forall m\, n \cdot m \downarrow\}$, where application is as in $\mathcal{K}_1$.