80818 Intuitionistic Logic - Exercise Sheet 6

October 7, 2021

1. Let X be a locally connected topological space, and consider the standard topological model of **HAS** on the open sets of X.

   (a) Suppose that $A \in \mathcal{M}_S$ and that $x \in [\![\forall n\, n \in A \vee n \notin A]\!]$. Show that there is an open neighbourhood U of x and a function $g : \mathbb{N} \to 2$ such that for all n, if $g(n) = 1$ then $U \subseteq [\![n \in A]\!]$ and if $g(n) = 0$ then $U \subseteq [\![n \notin A]\!]$.

   (b) Suppose that A, x and U are as above. Show that $U \subseteq [\![\exists n\, n \in A \vee \forall n\, n \notin A]\!]$. (You may assume **LPO** in the metatheory.)

   (c) Show that the following formulation of **LPO** holds in the standard topological model of **HAS** on any locally connected topological space.

$$\forall X\, (\forall x\, x \in X \vee x \notin X) \to (\exists x\, x \in X) \vee (\forall x\, x \notin X)$$

2. Let $(X, \approx_X)$, $(Y, \approx_Y)$ and $(Z, \approx_Z)$ be H-sets on a complete Heyting algebra H.

   (a) Show that $\approx_X : X \times X \to H$ is a functional relation from X to X.

   (b) Suppose that F is a functional relation from X to Y and G is a functional relation from Y to Z. Show that we can define a functional relation H from X to Z by $H(x, z) := \bigvee_{y \in Y} F(x, y) \wedge G(y, z)$.

3. Let $(X, \mathcal{O}_X)$ be any topological space. Define an H-set $(\mathbb{N}, \approx)$ on the open sets of X by setting $n \approx n = \top$ and $n \approx m = \bot$ for $m \neq n$.

   (a) Let F be a functional relation from $\mathbb{N}$ to $\mathbb{N}$. Show that for every $x \in X$ and every $n \in \mathbb{N}$ there exists a unique m such that $x \in F(n, m)$.

   (b) Show that the function $f : X \to \mathbb{N}^{\mathbb{N}}$ resulting from the construction in (a) is continuous for any functional relation F (where $\mathbb{N}^{\mathbb{N}}$ has the Baire space topology).

   (c) Given a continuous function $g : X \to \mathbb{N}^{\mathbb{N}}$, show that there is a functional relation G from $\mathbb{N}$ to $\mathbb{N}$ such that applying the construction in (a) to G yields g.