8 Heyting Valued Models of Arithmetic with Finite Types

8.1 Partial equivalence relations and H-sets

The theories that we are interested in typically have equality relations for each sort. For such theories it is often conventient to merge the equality relation and extent predicate together into a single binary relation by the following method.

Suppose that we are given signature $(\mathfrak{S}, \mathfrak{O}, \mathfrak{R})$ such that $\mathfrak{R}$ includes a binary relation symbol $=$ on a sort S. Suppose further we have a Heyting valued model for that signature over a complete Heyting algebra $(H, \bigvee, \wedge, \rightarrow)$ and that $=$ satisfies the axioms for equality.

We extend the signature with a new binary relation symbol $\approx$ and interpret it in the Heyting valued model as follows

$$[\![a \approx b]\!] := E(a) \wedge E(b) \wedge [\![a = b]\!]$$

Note that the extended Heyting valued model satisfies $\forall x \, \forall y \, x \approx y \leftrightarrow x = y$, which follows directly from the way that universal quantifiers are interpreted in the model. In particular $\approx$ satisfies all the axioms for equality, since $=$ does.

Futhermore, we can recover the extent predicate from $\approx$ by the following equation.

$$E(a) = [\![a \approx a]\!]$$

We can also see that $[\![\approx]\!]$ has the following properties:

$$[\![a \approx b]\!] = [\![b \approx a]\!]$$
$$[\![a \approx b]\!] \wedge [\![b \approx c]\!] \leq [\![a \approx c]\!]$$

We can think of this as a “Heyting valued” version of the following definition.

Definition 8.1. Let X be any set. A *partial equivalence relation* on X is a binary relation $E \subseteq X \times X$ satisfying the following conditions:

1. For all $x, y \in X$, $E(x, y)$ if and only if $E(y, x)$ (E is symmetric).
2. For all $x, y, z \in X$, if $E(x, y)$ and $E(y, z)$, then $E(x, z)$ (E is transitive).

We refer sets with H-valued relations satisfying the above as H-sets:

Definition 8.2. Let $(H, \bigvee, \wedge, \rightarrow)$ be a complete Heyting algebra. An H-set is a set X, together with a function $\approx: X \times X \rightarrow H$ satisfying the following for all $x, y, z \in X$:

$$x \approx y = y \approx x$$
$$x \approx y \wedge y \approx z \leq x \approx z$$

Given an H-set $(X, \approx)$ we will write $E(x)$ for $x \in X$ as notation for $x \approx x$.

8.2 Singletons in Heyting valued models of HAS and H-sets

In order to motivate an important aspect of the standard model on $\mathbf{HA}_\omega$, we will first take a closer look at how singleton sets work in **HAS**.

Recall that a singleton set is one with exactly one element. We formalise this in **HAS** as follows:

Definition 8.3. We say that X is a *singleton* if there exists a number x such that $x \in X$ and for all numbers x, y such that $x \in X$ and $y \in X$ we have $x = y$.

If we apply this to the standard Heyting valued model of **HAS** on a complete Heyting algebra $(H, \bigvee, \wedge, \rightarrow)$, we obtain the following definition.

Definition 8.4. We say $A : \mathbb{N} \rightarrow H$ is a *singleton* if

1. $\bigvee_{n \in \mathbb{N}} A(n) = \top$
2. for all n, m such that $m \neq n$ we have $A(n) \wedge A(m) = \bot$

Note that given any $n \in \mathbb{N}$, we can define a singleton $\underline{n}$ as follows:

$$\underline{n}(m) := \begin{cases} \top & m = n \\ \bot & m \neq n \end{cases}$$

However, it is important to note that in general these are not the only examples of singletons. Suppose that we have $p, q \in H$ such that $p \wedge q = \bot$ and $p \vee q = \top$. In that case we can define a singleton that can take one of two different values:

$$A(k) := \begin{cases} p & k = n \\ q & k = m \\ \bot & k \neq n \text{ and } k \neq m \end{cases}$$

To give a more concrete example of this we can define the following on Cantor space:

$$A(k) := \begin{cases} \{f : \mathbb{N} \rightarrow 2 \mid f(0) = 0\} & k = 0 \\ \{f : \mathbb{N} \rightarrow 2 \mid f(0) = 1\} & k = 1 \\ \emptyset & k > 1 \end{cases}$$

We note however, that these examples only work because Cantor space is not connected. We can use connectedness to get some more control over what singletons can look like.

Definition 8.5. We say a complete Heyting algebra $(H, \bigvee, \wedge \ \rightarrow)$ (or more generally any poset with finite meets and least element) is *connected* if for all $p, q \in H$ such that $p \vee q = \top$ and $p \wedge q = \bot$ we have $p = \top$ or $q = \top$.

In particular the lattice of open sets of a topological space is connected as a Heyting algebra if and only if the topological space is connected.

Proposition 8.6. *Suppose that $(H, \bigvee, \wedge \to)$ is a connected complete Heyting algebra. Then for every singleton $A : \mathbb{N} \to H$ in the standard model of* **HAS** *such that $A(n) = \bot$ for $n > 1$, there exists $n \in \{0, 1\}$ such that $[\![A = \underline{n}]\!] = \top$.*

Proof. Define $p := A(0)$ and $q := A(1)$. We then have $p \vee q = \top$ and $p \wedge q = \bot$. By connectedness, we have either $p = \top$ or $q = \top$. In the former case we have $[\![A = \underline{0}]\!] = \top$ and in the latter case we have $[\![A = \underline{1}]\!] = \top$. $\square$

More generally, we define singletons for H-sets as follows:

Definition 8.7. Suppose $(X, \approx)$ is an H-set for a complete Heyting algebra $(H, \bigvee, \wedge, \to)$. We define a new H-set $S(X)$ of *singletons* as follows. The underlying set of $S(X)$ consists of functions $A : X \to H$. We define a function $E : S(X) \to H$ by

$$E(A) := \bigvee_{x \in X} (x \approx x \wedge \bigwedge_{y \in X} A(y) \ \leftrightarrow \ x \approx y)$$

We then define the H-valued partial equivalence relation on $S(X)$ by

$$A \approx B \ := \ E(A) \wedge E(B) \wedge \bigwedge_{x \in X} A(x) \leftrightarrow B(x)$$

Note that given an H-set $(X, \approx)$ and a map $A : X \to H$, we can define a Heyting valued model for the signature with one sort, and a unary relation symbol A, and a binary relation $=$ satisfying the axioms of reflexivity, symmetry and transitivity. In this case, $E(A)$ is the interpretation of the formula

$$\exists x \, \forall y \, A(y) \leftrightarrow x = y$$

We can show, e.g. by giving a proof in intuitionisitic logic, that this is equivalent to the conjunction of the formulas

$$\exists x \, A(x) \qquad \forall x, y \, A(x) \wedge A(y) \ \to \ x = y \qquad \forall x, y \; x = y \wedge A(x) \ \to \ A(y)$$

The first two state that there is exactly one x such that $A(x)$, while the last is one of the axioms for equality.

For an H-set X, we can define a canonical inclusion $i : X \to S(X)$ such that $x \approx y \leq i(x) \approx i(y)$ for all $x, y \in X$ by

$$i(x)(y) := x \approx y$$

Note that we can define a Heyting valued model for a signature with two sorts, an equality relation on each sort and a unary operation symbol, where we interpret the two sorts as X and $S(X)$, and i as the the unary operation symbol. In this case we can show that the following formulas all evaluate to $\top$ in the interpretation of intuitionistic logic. Firstly, the axiom of equality

$$\forall x^X \, \forall y^X \; x = y \to ix = iy$$

Secondly the following two axioms, stating that i is a bijection

$$\forall z^{S(X)}\, \exists x^X\, z = i(x) \qquad \forall x^X\, \forall y^X\, ix = iy \to x = y$$

Also note that the above Heyting valued model also admits an interpretation for a relation $\in$ of arity $Y, S(Y)$, and we can show that the following axioms are satisfied.

$$\forall z\, \forall z'\, (\forall y\, y \in z \leftrightarrow y \in z') \leftrightarrow z = z'$$

$$\forall z\, \exists! y\, y \in z$$

Definition 8.8. Let $(X, \approx_X)$ and $(Y, \approx_Y)$ be H-sets. A *functional relation from X to Y* is a function $F : X \times Y \to H$ such that for all $x, x' \in X$ and $y, y' \in Y$ we have

$$x \approx x' \,\wedge\, y \approx y' \,\wedge\, F(x, y) \leq F(x', y')$$

and for all $x \in X$, we have

$$x \approx x \leq \bigvee_{y \in Y} y \approx y \wedge F(x, y)$$

and for all $x \in X$ and $y, y' \in Y$, we have

$$x \approx x \wedge y \approx y \wedge y' \approx y' \wedge F(x, y) \wedge F(x, y') \leq y \approx y'$$

We can again view these conditions as certain logical formulas evaluating to $\top$ in a Heyting valued model. Namely, we have two axioms of equality

$$\forall x\, \forall x', x = x' \wedge F(x, y) \to F(x', y) \qquad \forall x\, \forall y\, \forall y'\, y = y' \wedge F(x, y) \to F(x, y')$$

and the following two formulas stating that F is total and single valued.

$$\forall x\, \exists y\, F(x, y) \qquad \forall x\, \forall y\, \forall y'\, F(x, y) \wedge F(x, y') \to y = y'$$

Definition 8.9. We say an H-set $(Y, \approx_Y)$ is *weakly complete* or *Higgs complete* if for every H-set $(X, \approx_X)$ and every functional relation F from X to Y, there is a function $f : X \to Y$ such that for all $x \in X$ we have

$$x \approx x \leq F(x, f(x))$$

Lemma 8.10. *For any H-set $(Y, \approx_Y)$, $S(Y)$ is weakly complete.*

Proof. Suppose we have a functional relation F from $(X, \approx_X)$ to $S(Y)$. We define the function $f : X \to S(Y)$ by taking $f(x) := \lambda y.F(x, i(y))$. We need to check that $x \approx x \leq f(x) \approx f(x)$ and that $x \approx x \leq F(x, \lambda y.F(x, y))$ for all x. In both cases we will use the description of the definitions in terms of formulas evaluating to $\top$ in a Heyting valued model. First, note that to show $x \approx x \leq f(x) \approx f(x)$, it suffices to prove in intuitionistic logic that for all x, $f(x)$ is a singleton. However, we may assume that for every x there exists a unique z such that z is a singleton and we have $F(x,z)$. Since z is a singleton, there is a unique y such that $z = i(y)$. But we have now shown $f(x)$ must be the singleton $i(y)$.

The same argument also shows that $x \approx x \leq F(x, f(x))$. We have shown in intuitionistic logic that for all x, there is a unique z such that $F(x,z)$ and a unique y such that $z = i(y) = f(x)$. However, it follows by the axioms of equality that we have $F(x, f(x))$. $\square$

Definition 8.11. Suppose we are given H-sets $(X, \approx_X)$ and $(Y, \approx_Y)$, we define an H-valued partial equivalence relation $\approx_{X\times Y}$ on $X \times Y$ as follows:

$$(x,y) \approx_{X\times Y} (x',y') := x \approx_X x' \wedge y \approx_Y y'$$

Definition 8.12. Suppose we are given H-sets $(X, \approx_X)$ and $(Y, \approx_Y)$, we define an H-valued partial equivalence relation $\approx_{Y^X}$ on Y^X as follows:

$$f \approx g := \bigwedge_{x\in X} x \approx x \, \to \, f(x) \approx g(x)$$

Note that we can define a Heyting valued model on a signature with two three sorts X, Y, Y^X and a term Ap of sort $Y^X, X \to Y$, where we interpret each sort as the corresponding set of the same name, and Ap as function application. In this case the following formula evaluates to $\top$ in the Heyting valued model:

$$\forall f, g\, (\forall x\, fx = gx) \leftrightarrow f = g$$

Lemma 8.13. *If $(Y, \approx_Y)$ is a weakly complete H-set and $(X, \approx_X)$ is any H-set, then $(Y^X, \approx)$ is weakly complete.*

Proof. We need to check that given a functional relation F from an H-set $(Z, \approx_Z$ to $(Y^X, \approx)$ we can find $f : Z \to Y^X$ such that for all z, $z \approx z \leq F(z, f(z))$.

We define a new functional relation G from $Z \times X$ to Y by $G((z,x),y) := \bigvee_{h:X\to Y} F(z,h) \wedge h(x) \approx_Y y$. We will first check that G is a functional relation. We can then deduce that there exists $g : Z \times X \to Y$ such that for all $x \in X$ and $z \in Z$ $(z,x) \approx (z,x) \leq G((z,x), g(z,x))$. We then define $f : Z \to Y^X$ defined by $f(z)(x) := g(z,x)$ and check that for all $x \in X$ we have $x \approx x \leq F(x, f(x))$.

We will check both of the statements above again using the descriptions in terms of formulas evaluating to $\top$ in a Heyting valued model. We first need to prove that $\forall (z,x)\, \exists! y\, G((x,z),y)$. However, we can make the assumption $\forall z\, \exists! f\, F(x,f)$ by the definition of functional relation, and we can make the assumption $\forall f\, \forall x\, \exists! y\, f(x) = y$ by the definition of the equality relation for Y^X. From these assumptions it follows that $\forall (z,x)\, \exists! y\, G((x,z),y)$.

For the second statement we need to check, it suffices to prove in intuitionisitic logic that for all z we have $F(z, f(z))$. We may assume that there exists h of sort Y^X such that $F(z,h)$. We can then show that for all x of sort X we have $g(z,x) = h(x)$. Then since the equality relation on Y^X was chosen to satisfy extensionality, we can deduce $f(z) = h$, and thereby deduce $F(z, f(z))$, as required. $\square$

8.3 Standard Heyting valued models of $\mathbf{HA}_\omega$

Fix a complete Heyting algebra $(H, \bigvee, \wedge, \to)$. We will define the standard Heyting valued model of $\mathbf{HA}_\omega$ on H in such a way that each function sort $\sigma \to \tau$ behaves similarly to functional relations in the standard model of **HAS**. This will ensure that the models are non trivial. It will also have the side effect that the models satisfy both function extensionality and the axiom of unique choice.

We will ensure that each sort of $\mathbf{HA}_\omega$ is interpreted as a weakly complete H-set. We will then use the H-valued partial equivalence relation on the H-set to define both the extent and the equality relation in the Heyting valued model.

8.3.1 The domains of the model, extent and equality

We define $\mathcal{M}_N$ to be $S(\mathbb{N})$, with partial equivalence relation defined as in definition 8.7.

Given finite types σ and τ, we define $\mathcal{M}_{\sigma\times\tau}$ to be $\mathcal{M}_\sigma \times \mathcal{M}_\tau$, with partial equivalence relation defined as in definition 8.11.

We define $\mathcal{M}_{\sigma\to\tau}$ to be the set of functions from $\mathcal{M}_\sigma$ to $\mathcal{M}_\tau$, with partial equivalence relation defined as in definition 8.12. Note that from the definition of the partial equivalence relation, we can see that function extensionality has to hold in the model.

We can also see that unique choice holds, as follows. Let φ be any formula. We want to show $\forall x^\sigma\, \exists! y^\tau\, \varphi(x, y)$ holds in the model. Suppose that $z_1^{\rho_1}, \ldots, z_n^{\rho_n}$ is a list of the free variables occuring in φ not equal to x and y.

$$(c_1, \ldots, c_n) \approx (c'_1, \ldots, c'_n) := c_1 \approx c'_1 \wedge \ldots \wedge c_n \approx c'_n \wedge \\ [\![\forall x^\sigma\, \exists! y^\tau\, \varphi(x, y, c_1, \ldots, c_n)]\!]$$

Finally, note that we have a functional relation from $Z \times \mathcal{M}_\sigma$ to $\mathcal{M}_\tau$, which is just $[\![\varphi]\!]$, viewed as a map $Z \times \mathcal{M}_\sigma \times \mathcal{M}_\tau \to H$. Since we ensured that $\mathcal{M}_\tau$ is weakly complete, this gives us a function from $Z \times \mathcal{M}_\sigma$ to $\mathcal{M}_\tau$. Given any element $(c_1, \ldots, c_n)$ of Z, we can apply the function above to them, to get a function $\mathcal{M}_\sigma \to \mathcal{M}_\tau$, which is what we needed to find an element f of $\mathcal{M}_{\sigma\to\tau}$. By the way that we defined this function, we do indeed have the inequality below, confirming that f does witness unique choice holding in the model.

$$E(c_1) \wedge \ldots \wedge E(c_n) \wedge [\![\forall x\, \exists! y \varphi]\!] \leq E(f) \wedge [\![\forall x\, \varphi(x, f(x))]\!]$$

8.3.2 The application operation

We now need to show now to interpret the operator symbols. We define $[\![\mathrm{Ap}]\!] : \mathcal{M}_{\sigma\to\tau} \times \mathcal{M}_\sigma \to \mathcal{M}_\tau$ to be the external function application map. It is important to note that we know this is a well defined function, since we implemented $\mathcal{M}_{\sigma\to\tau}$ so that its underlying set is just the set of functions from $\mathcal{M}_\sigma$ to $\mathcal{M}_\tau$. We also need to check that for all $f : \mathcal{M}_\sigma \to \mathcal{M}_\tau$ and all $x \in \mathcal{M}_\sigma$ we have $f \approx f \wedge x \approx x \leq f(x) \approx f(x)$ to satisfy the definition of Heyting valued model. However, this is ensured by definition 8.12. Moreover, given $f, f' : \mathcal{M}_\sigma \to \mathcal{M}_\tau$

and $x, x' \in \mathcal{M}_\sigma$ we have $f \approx f' \wedge x \approx x' \leq f(x) \approx f'(x')$, which implies the above, but also shows that the Heyting valued model satisfies the axioms of equality $f = f' \to \mathrm{Ap}(f, x) = \mathrm{Ap}(f', x)$ and $x = x' \to \mathrm{Ap}(f, x) = \mathrm{Ap}(f, x')$.

8.3.3 Constant symbols

We interpret each constant symbol $\mathbf{p}, \mathbf{p}_i, \mathbf{k}, \mathbf{s}$ as the corresponding "external" version. For example, given sorts σ and τ, we need to take $\mathbf{k}$ to be a global element of $\mathcal{M}_{\sigma \to (\tau \to \sigma)}$. This means it needs to be a function $\mathcal{M}_\sigma \to \mathcal{M}_{\tau \to \sigma}$. But $\mathcal{M}_{\tau \to \sigma}$ is itself the set of functions from $\mathcal{M}_\tau$ to $\mathcal{M}_\sigma$. Hence we take $[\![\mathbf{k}]\!]$ to be the function that takes an element a of $\mathcal{M}_\sigma$ and returns the function constantly equal to x. This needs to be a global element, i.e. such that $E([\![\mathbf{k}]\!]) = \top$. This amounts to showing that $a \approx a' \leq [\![\mathbf{k}]\!](a) \approx [\![\mathbf{k}]\!](a')$. However, this is the same as showing $a \approx a' \leq \bigwedge_{b \in \mathcal{M}_\sigma} [\![\mathbf{k}]\!](a)(b) \approx [\![\mathbf{k}]\!](a')(b)$. Since $[\![\mathbf{k}]\!](a)(b) = a$ for all a and b, this is easy to show. We also need to check that this does satisfy the axiom for $\mathbf{k}$, namely $\forall x \forall y\, \mathbf{k}xy = x$. This amounts to showing $a \approx a \wedge b \approx b \leq [\![\mathbf{k}]\!]ab \approx a$, which again follows directly from the definition of $[\![\mathbf{k}]\!]$.

We can argue similarly for the other constants $\mathbf{p}, \mathbf{p}_i, \mathbf{s}$.

This only leaves the constant symbols relating directly to numbers, namely $0, S$ and the recursor $\mathbf{r}$. We take $[\![0]\!]$ to be $i(0)$. We need to take $[\![S]\!]$ to be a function $S(\mathbb{N}) \to S(\mathbb{N})$. One way to do this is to note that it suffices to find a functional relation from $S(\mathbb{N}) \to S(\mathbb{N})$. However, by composing with the function $i : \mathbb{N} \to S(\mathbb{N})$ and its inverse, which is a functional relation from $S(\mathbb{N})$ to $\mathbb{N}$, it suffices to find a function from $\mathbb{N}$ to $\mathbb{N}$, which we can take to just be the external successor function. However, we can also explicitly describe $[\![S]\!](A)$ for $A \in S(\mathbb{N})$. It is simply the function $\mathbb{N} \to H$ defined by $\lambda n.A(S(n))$. The induction axioms are proved using external induction in a similar way to the standard Heyting valued model for **HAS**.

Defining $[\![\mathbf{r}]\!]$ amounts to finding a function $S(\mathbb{N}) \to \mathcal{M}_\sigma$, given an element a of $\mathcal{M}_\sigma$ and a function $f : \mathcal{M}_\sigma \times S(\mathbb{N}) \to \mathcal{M}_\sigma$. As before, we observe that it suffices to define a function g from $\mathbb{N} \to \mathcal{M}_\sigma$. We define g by (external) recursion as $g(0) = a$, and $g(Sn) = f(g(n), i(n))$. It is clear that this satisfies the relavant equations by definition.

By the above reasoning we can deduce the soundness theorem for standard Heyting valued models of $\mathbf{HA}_\omega$:

Theorem 8.14. *The above model satisfies the axioms of* $\mathbf{HA}_\omega$*, as well as function extensionality and unique choice.*