80818 Intuitionistic Logic - Exercise Sheet 2

September 9, 2021

1. Working in $\mathbf{HA}_\omega$,

   (a) Given a term t of sort $\sigma \to \tau$ construct a term t' of sort $\sigma \to (\rho \to \tau)$ satisfying the following equation.

   $$\forall x^\sigma \, \forall y^\rho \, (t'xy = tx)$$

   (b) Construct a closed term $+$ of sort $N \to (N \to N)$ satisfying the equations below.

   $$n + 0 = n, \qquad n + (Sm) = S(n + m)$$

   (c) Construct a closed term prd of sort $N \to N$ satisfying the following equations:

   $$\mathrm{prd}\, 0 = 0, \qquad \mathrm{prd}(Sx) = x$$

   (d) Construct a closed term $\dot{-}$ of sort $N \to (N \to N)$ satisfying the following equations:

   $$x \dot{-} 0 = x, \qquad x \dot{-} Sy = \mathrm{prd}(x \dot{-} y)$$

   (e) Construct a closed term $\mathbf{d}_0$ satisfying the following equations:

   $$\mathbf{d}_0 0 = 0, \qquad \mathbf{d}_0(Sn) = S0$$

2. You may assume there is a binary relation $<$ on numbers with the following properties:

   $$\begin{array}{ll} \neg(x < x) & (x < y \land y < z) \to x < z \\ \neg(x < 0) & x < Sy \leftrightarrow (x = y \lor x < y) \end{array}$$

   (a) Prove the following "finite" version of **LPO**:

   $$\forall f \in 2^{\mathbb{N}} \, \forall n \, (\forall m \; m < n \to f(m) = 0) \lor (\exists m \; m < n \land f(m) = 1)$$

   (b) Show the following statement for all n by induction on n. If $f(m) = 1$ for some m with $m < n$ then there is a number m' such that $f(m') = 1$ and for all i such that $i < m'$ we have $f(i) = 0$. (That is, m' is the *least* number such that $f(m') = 1$)

3. Suppose we are given a function $F : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$. A *discontinuity* is a function $f \in \mathbb{N}^{\mathbb{N}}$, together with a countable sequence of functions $g_n \in \mathbb{N}^{\mathbb{N}}$ for $n \in \mathbb{N}$, with the following properties:

   (i) for all n we have $g_n(m) = f(m)$ for all $m < n$

   (ii) for all n we have $F(g_n) \neq F(f)$

   (a) Assuming **WLPO**, show that for every function $f : \mathbb{N} \to \mathbb{N}$ either $f(n) = 0$ for all n, or $f(n) \neq 0$ for some n. Hint: You might find the term $\mathbf{d}_0$ from 1(e) useful.

   (b) Assuming **WLPO**, give an example of a function $F : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ that has a discontinuity. (If working in $\mathbf{HA}_\omega$ you may also assume the axiom of unique choice.) Hint: Use unique choice to define a function using part (a), and show that it has a discontinuity.

   (c) Given a function $F : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ with a discontinuity $(f, (g_n)_{n \in \mathbb{N}})$, prove **WLPO**. (If working in $\mathbf{HA}_\omega$ you may also assume the axiom of unique choice and function extensionality.) Hint: Show for any function $h : \mathbb{N} \to 2$ there is a function $k : \mathbb{N} \to \mathbb{N}$ such that if $h(n) = 0$ for all n, then $k = f$ and if there is a number n which is least number such that $h(n) = 1$ then $k = g_n$. Question 2 will be useful for this. The value of $F(k)$ is either equal to $F(f)$ or not equal to $F(f)$.