

# 80818 Intuitionistic Logic - Exercise Sheet 9

November 4, 2021

1. We say a closed formula  $\varphi$  of first order arithmetic is true if it holds in the meta theory where we are working (or equivalently if it holds in the standard model of **HA** on the 2 element Heyting algebra). We consider the standard intensional realizability model of **HA**<sub>ω</sub>. We say a formula  $\varphi(x_1, \dots, x_k)$  is *self realizing* if the following hold
  - (i) For any numbers  $n_1, \dots, n_k$ , if there exists  $e \in \mathcal{A}$  such that  $e \Vdash \varphi(n_1, \dots, n_k)$ , then  $\varphi(n_1, \dots, n_k)$  is true.
  - (ii) There is an  $e \in \mathcal{A}$  such that for any numbers  $n_1, \dots, n_k$ , if  $\varphi(n_1, \dots, n_k)$  is true, then  $e \Vdash \varphi(n_1, \dots, n_k)$ .

A formula of **HA** (possibly with free variables) is *negative* if it is in the smallest class containing atomic formulas and closed under conjunction, implication and universal quantifiers.

Show that every closed negative formula of **HA** is self realizing in the intensional realizability model of **HA**<sub>ω</sub>.

2. Let  $X$  be any set and  $\mathcal{A}$  a non trivial pca. Given  $f, g : X \rightarrow \mathcal{P}(\mathcal{A})$ , we write  $f \leq g$  to mean that there exists  $e \in \mathcal{A}$  such that for all  $x \in X$  and all  $a \in f(x)$ , we have  $ea \downarrow$  with  $ea \in g(x)$ . We say  $f \sim g$  to mean that  $f \leq g$  and  $g \leq f$ .
  - (a) Show that  $\leq$  gives a preorder structure on  $\mathcal{P}(\mathcal{A})^X$  (i.e. it is reflexive and transitive).
  - (b) Given any set with preorder  $(X, \leq)$ , and  $x, y \in X$  we write  $x \sim y$  to mean  $x \leq y$  and  $y \leq x$ . Show that  $\sim$  is an equivalence relation and  $X/\sim$  has the structure of a poset (i.e. a reflexive, transitive and anti symmetric relation  $\leq$ ) with  $[x] \leq [y]$  in the poset precisely when  $x \leq y$  in the preorder.
  - (c) Show that  $\mathcal{P}(\mathcal{A})^X/\sim$  with the ordering above is a Heyting algebra.
  - (d) Let  $\mathcal{A}$  be a non trivial pca with decidable equality. Take the  $X$  above to be  $\mathcal{A} \times \mathcal{A}$ . For  $a, b \in \mathcal{A}$ , define  $f_{a,b} : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{P}(\mathcal{A})$  as follows:

$$f_{a,b}(a', b') := \begin{cases} \{\mathbf{p}(\mathbf{p}ab)\top\} & a' = a, b' = b \text{ and } ab \downarrow \\ \{\mathbf{p}(\mathbf{p}ab)\perp\} & a' = a, b' = b \text{ and } ab \uparrow \\ \{\mathbf{p}(\mathbf{p}a'b')\top, \mathbf{p}(\mathbf{p}a'b')\perp\} & a' \neq a \text{ or } b' \neq b \end{cases}$$

Define  $g : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{P}(\mathcal{A})$  as follows.

$$g(a, b) := \{\mathbf{p}ab\}$$

- i. Show that for all  $a, b \in \mathcal{A}$ ,  $f_{a,b} \sim g$ .
- ii. Show that for  $a, b, c, d \in \mathcal{A}$  we have  $\lambda a', b'. (f_{a,b}(a', b') \cap f_{c,d}(a', b')) \sim g$ .
- iii. Show  $\lambda a', b'. \bigcap_{a,b} f_{a,b}(a', b')$  is *strictly* smaller than  $g$  (you might find question 2 from the previous sheet useful).