12 Realizability

12.1 Realizability models for intuitionistic logic

A classic way to motivate realizability is formalise an intuitive idea known as the Brouwer-Heyting-Kolmogorov, or BHK, interpretation of intuitionistic logic. The BHK interpretation asserts for each logical connective what it means to constructively justify the truth of formulas built from that connective. The most important cases to consider are disjunction, existential quantifiers and implication. To assert a disjunction $\varphi \vee \psi$ it is essential, according to BHK, to either assert φ or to assert ψ. In particular, we need to have a choice of φ or ψ, that tells us which we going to show. Similarly, to assert $\exists x\, \varphi(x)$ we need to have both a witness t and then we must also assert $\varphi(t)$ for that particular t. To assert an implication $\varphi \to \psi$, we must have an effective rule that tells us how to produce witnesses of ψ from witnesses of φ. In realizability we make the vague term "rule" precise by defining to be a representable function in a pca.

Another viewpoint of realizability is that the powerset of a pca $\mathcal{A}$, $\mathcal{P}(\mathcal{A})$ can play a similar role to the complete Heyting algebra in Heyting valued models. In fact the definition of realizability model for intuitionistic logic with only relation symbols is identical to that of Heyting valued model.

Fix a signature $(\mathfrak{S}, \mathfrak{O}, \mathfrak{R})$ and a partial combinatory algebra $\mathcal{A}$. Will will also assume we have pairing and projection constants $\mathbf{p}$, $\mathbf{p}_0$ and $\mathbf{p}_1$. These could be either constructed in a general pca, or part of the structure of an extended pca. We will write $\top$ for $\lambda x.\lambda y.x$ and $\perp$ for $\lambda x.\lambda y.y$.

Definition 12.1. A *realizability model* for $(\mathfrak{S}, \mathfrak{O}, \mathfrak{R})$ with pca $\mathcal{A}$ consists of the following data:

1. For each sort $S \in \mathfrak{S}$ a set $\mathcal{M}_S$
2. For each sort S a function $E_S : \mathcal{M}_S \to \mathcal{P}(\mathcal{A})$
3. For each relation symbol $R \in \mathfrak{R}$ of sort $S_1, \ldots, S_n$ a function $[\![R]\!] : \mathcal{M}_{S_1} \times \ldots \times \mathcal{M}_{S_n} \to \mathcal{P}(\mathcal{A})$
4. For each operator symbol $O \in \mathfrak{O}$ of sort $S_1, \ldots, S_n \to T$ a function $[\![O]\!] : \mathcal{M}_{S_1} \times \ldots \times \mathcal{M}_{S_n} \to \mathcal{M}_T$ such that there exists $e \in \mathcal{A}$ with the following property. For all $x_1, \ldots, x_n$ with $x_i \in \mathcal{M}_{S_i}$ and all $a_1, \ldots, a_n$ with $a_i \in E_{S_i}(x_i)$ we have $ea_1 \ldots a_n \downarrow$ and $ea_1 \ldots a_n \in E_T([\![O]\!](x_1, \ldots, x_n))$. We say that e *realizes* or *tracks* $[\![O]\!]$.

Finally, we have a non triviality condition, that for every sort $S \in \mathfrak{S}$ there exists an element a of $\mathcal{M}_S$ where $E_S(a)$ is non empty.

We now describe the interpretation of intuitionistic logic in realizability models.

As for Heyting valued models we assign for each term t of sort T and each variable assignment α, an element $[\![t]\!]_\alpha$ of $\mathcal{M}_T$. For realizability models, we further ensure that these assignments are realized, in the following sense. Suppose that $x_1^{S_1}, \ldots, x_n^{S_n}$ is a list of variables including all those that occur free in t. Then there is $e \in \mathcal{A}$ with the following property. For every free variable assignment α and for $f_1, \ldots, f_n$ with each $f_i \in E_{S_i}(\alpha(x_i))$, we will ensure that $ef_1 \ldots f_n \downarrow$, with $ef_1 \ldots f_n \in [\![t]\!]_\alpha$. We define the value of $[\![t]\!]_\alpha$ exactly the same as for Heyting valued models, namely, by induction with

$$[\![x]\!]_\alpha := \alpha(x_i)$$
$$[\![Ot_1 \ldots t_n]\!]_\alpha := [\![O]\!]([\![t_1]\!]_\alpha, \ldots, [\![t_n]\!]_\alpha)$$

Lemma 12.2. *There exists a realizer $e \in \mathcal{A}$ for each term t with free variables amongst $x_1, \ldots, x_n$, as described above.*

Proof. Fix a list of variables $x_1, \ldots, x_n$. We will show how to construct realizers for terms containing only free variables in the list $x_1, \ldots, x_n$. $[\![x_i]\!]_\alpha$ is realized by $\lambda x_1, \ldots, x_n.x_i$. Suppose we are already given realizers $f_1, \ldots, f_m$ for $t_1, \ldots, t_m$, and that $[\![O]\!]$ is realized by $e \in \mathcal{A}$. Then $[\![Ot_1 \ldots t_m]\!]_\alpha$ is realized by

$$\lambda x_1, \ldots, x_n.e(f_1 x_1 \ldots x_n) \ldots (f_m x_1 \ldots x_n)$$

$\square$

We now show how to define truth values of formulas. For each formula φ and variable assignment, α, we will define $[\![\varphi]\!]_\alpha \subseteq \mathcal{A}$.

$$
\begin{aligned}
[\![Rt_1 \ldots t_n]\!]_\alpha &:= [\![R]\!]([\![t_1]\!]_\alpha, \ldots, [\![t_n]\!]_\alpha) \\
[\![\bot]\!]_\alpha &:= \emptyset \\
[\![\varphi \wedge \psi]\!]_\alpha &:= \{e \mid \mathbf{p}_0 e \in [\![\varphi]\!]_\alpha \text{ and } \mathbf{p}_1 e \in [\![\psi]\!]_\alpha\} \\
[\![\varphi \vee \psi]\!]_\alpha &:= \{e \mid (\mathbf{p}_0 e = \top \text{ and } \mathbf{p}_1 e \in [\![\varphi]\!]_\alpha) \text{ or } (\mathbf{p}_0 e = \bot \text{ and } \mathbf{p}_1 e \in [\![\psi]\!]_\alpha)\} \\
[\![\varphi \to \psi]\!]_\alpha &:= \{e \mid \text{for all } f \in [\![\varphi]\!]_\alpha,\ ef\downarrow \text{ and } ef \in [\![\psi]\!]_\alpha\} \\
[\![\exists x^S\, \varphi]\!]_\alpha &:= \bigcup_{a \in \mathcal{M}_S} \{e \mid \mathbf{p}_0 e \in E_S(a) \text{ and } \mathbf{p}_1 e \in [\![\varphi]\!]_{\alpha[x \mapsto a]}\} \\
[\![\forall x^S\, \varphi]\!]_\alpha &:= \bigcap_{a \in \mathcal{M}_S} \{e \mid \text{for all } f \in E_S(a),\ ef\downarrow \text{ and } ef \in [\![\varphi]\!]_{\alpha[x \mapsto a]}\}
\end{aligned}
$$

Even more so than for Heyting valued models, it can be useful to instead phrase this definition using forcing notation. For a given variable assignment α, we write $e \Vdash_\alpha \varphi$ to mean $e \in [\![\varphi]\!]_\alpha$. We can then describe $\Vdash_\alpha$ explicitly as follows.

$e \nVdash_\alpha \bot$		always
$e \Vdash_\alpha \varphi \wedge \psi$	iff	$\mathbf{p}_0 e \Vdash_\alpha \varphi$ and $\mathbf{p}_1 e \Vdash_\alpha \psi$
$e \Vdash_\alpha \varphi \vee \psi$	iff	either $\mathbf{p}_0 e = \top$ and $\mathbf{p}_1 e \Vdash_\alpha \varphi$, or $\mathbf{p}_0 e = \bot$ and $\mathbf{p}_1 e \Vdash_\alpha \psi$
$e \Vdash_\alpha \varphi \to \psi$	iff	if $f \Vdash_\alpha \varphi$, then $ef\downarrow$ and $ef \Vdash_\alpha \psi$
$e \Vdash_\alpha \exists x^S\, \varphi$	iff	there exists $a \in \mathcal{M}_S$ such that $\mathbf{p_0} e \in E_S(a)$ and $\mathbf{p}_1 e \Vdash_{\alpha[x \mapsto a]} \varphi$
$e \Vdash_\alpha \forall x^S\, \varphi$	iff	for all $a \in \mathcal{M}_S$ and for all $f \in E_S(a), ef\downarrow$ and $ef \Vdash_{\alpha[x \mapsto a]} \varphi$

Before showing the soundness theorem, we first introduce some notation. For any pca $\mathcal{A}$, we can encode finite lists of elements of $\mathcal{A}$ as single elements of $\mathcal{A}$, in a similar manner to how natural numbers are implemented in arbitrary pcas. We write the encoding of lists as $[e_1, \dots, e_n] \in \mathcal{A}$, which is defined by induction on the length of the list by

$$[] := \mathbf{p}\bot\bot$$
$$[e_1, \dots, e_{n+1}] := \mathbf{p}\top(\mathbf{p}e_{n+1}[e_1, \dots, e_n])$$

If Γ is a finite set of formulas that can be written as $\{\varphi_1, \dots, \varphi_n\}$ and α a variable assignment, then we write $e \Vdash_\alpha \Gamma$ to mean $e = [e_1, \dots, e_n]$ and for each i, $e \Vdash_\alpha \varphi_i$.

Theorem 12.3. *If $\Gamma \vdash \varphi$ is provable in intuitionistic logic, and $x_1^{S_1}, \dots, x_n^{S_n}$ is a list of variables including all of those that occur free in Γ and φ, then we can find $e \in \mathcal{A}$ such that for all variable assignments α, all $f_1, \dots, f_n$ with $f_i \in E_{S_i}(\alpha(x_i))$, and all g such that $g \Vdash_\alpha \Gamma$, we have $ef_1 \dots f_n g \downarrow$ and*

$$ef_1 \dots f_n g \Vdash_\alpha \varphi$$

Proof. To a large extent, this is very similar to the soundness theorem for Heyting valued models. We again work by induction on the definition of provability, proving some cases as an example, and leaving the rest as exercises.

The case $\vee E$ Suppose that we have deduced $\Gamma \vdash \chi$ from the hypotheses $\Gamma \vdash \varphi \vee \psi$, $\Gamma, \varphi \vdash \chi$ and $\Gamma, \psi \vdash \chi$. By the inductive hypothesis, we may assume we already have realizers for $\Gamma \vdash \varphi \vee \psi$, $\Gamma, \varphi \vdash \chi$ and $\Gamma, \psi \vdash \chi$. Suppose that $x_1, \dots, x_n$ is a list of free variables including any occurring in Γ or ψ. We will assume that the list $x_1, \dots, x_n$ also includes any free variables occurring in φ and ψ: we can do this using the non triviality condition in a very similar manner to Heyting valued models. Hence we have $e, f, g \in \mathcal{A}$ such that for all variable assignments α and all $h_1, \dots, h_n$ such that $h_i \in E_{S_i}(\alpha(x_i))$ and all k, l, m such that $k \Vdash_\alpha \Gamma$, $m \Vdash_\alpha \varphi$ and $l \Vdash_\alpha \psi$, we have

$$eh_1 \dots h_n k \Vdash_\alpha \varphi \vee \psi \tag{1}$$
$$fh_1 \dots h_n km \Vdash_\alpha \chi \tag{2}$$
$$gh_1 \dots h_n kl \Vdash_\alpha \chi \tag{3}$$

It follows from (1) that either $\mathbf{p}_0(eh_1 \dots h_n k) = \top$ and $\mathbf{p}_1(eh_1 \dots h_n k) \Vdash_\alpha \varphi$ or $\mathbf{p}_0 eh_1 \dots h_n k = \bot$ and $\mathbf{p}_1(eh_1 \dots h_n k) \Vdash_\alpha \psi$. In the former case we have $\mathbf{p}_0(eh_1 \dots h_n k)fg = f$ and in the latter $\mathbf{p}_0(eh_1 \dots h_n k)fg = g$. In the former case, we have by (2) that $fh_1 \dots h_n k\mathbf{p}_1(eh_1 \dots h_n k) \Vdash_\alpha \chi$, and in the latter case, by (3) we have $gh_1 \dots h_n k\mathbf{p}_1(eh_1 \dots h_n k) \Vdash_\alpha \chi$. Hence, in either case we have

$$\mathbf{p}_0(eh_1 \dots h_n k)fgh_1 \dots h_n k(\mathbf{p}_1(eh_1 \dots h_n k)) \Vdash_\alpha \chi$$

So we can take our required realizer to be

$$\lambda x_1, \dots, x_n.\lambda y.\mathbf{p}_0(ex_1 \dots x_n y)fgx_1 \dots x_n y(\mathbf{p}_1(ex_1 \dots x_n y))$$

The case $\exists I$ Suppose we have deduced $\Gamma \vdash \exists x\, \varphi$ from $\Gamma \vdash \varphi[x/t]$. Let $x, y_1, \ldots, y_n$ be a list of free variables including all those occurring in Γ and φ. By the inductive hypothesis, we have e such that for all variable assignments α, and all $f_1, \ldots, f_n$ with $f_i \in E(\alpha(x_i))$, and all $k\mathcal{A}$ such that $k \Vdash_\alpha \Gamma$, we have

$$ef_1 \ldots f_n k \Vdash_\alpha \varphi[x/t]$$

In the interpretation of terms we ensured that there is $g \in \mathcal{A}$ such that for all such α and f_i we have $gf_1 \ldots f_n \in [\![t]\!]_\alpha$. Hence we can deduce

$$\mathbf{p}(gf_1 \ldots f_n k)(ef_1 \ldots f_n k) \Vdash_\alpha \exists x\, \varphi$$

We can therefore take our realizer to be

$$\lambda x_1, \ldots, x_n.\lambda y.\mathbf{p}(gx_1 \ldots x_n y)(ex_1 \ldots x_n y)$$

$\square$

12.2 Realizability models for $\mathbf{HA}_\omega$

We now show how to construct standard realizability models for $\mathbf{HA}_\omega$. Instead of giving one definition, we will give two different models for each pca$^+$, $\mathcal{A}$, that we will refer to as the *intensional* model and the *extensional* model.

12.2.1 The intensional model

We define $\mathcal{M}_S$ and E_S for each sort S of $\mathbf{HA}_\omega$ by induction on finite types. We will ensure that $\mathcal{M}_S \subseteq \mathcal{A}$, and then define $E_S(e) := \{e\}$.

We define $\mathcal{M}_N$ to be the copy of the standard natural numbers in $\mathcal{A}$, i.e. $\{\underline{n} \mid n \in \mathbb{N}\}$.

Suppose we have already defined $\mathcal{M}_\sigma$ and $\mathcal{M}_\tau$. We define $\mathcal{M}_{\sigma\times\tau}$ and $\mathcal{M}_{\sigma\to\tau}$ as follows.

$$\mathcal{M}_{\sigma\times\tau} := \{\mathbf{p}ef \in \mathcal{A} \mid e \in \mathcal{M}_\sigma \text{ and } f \in \mathcal{M}_\tau\}$$

$$\mathcal{M}_{\sigma\to\tau} := \{e \in \mathcal{A} \mid \text{for all } f \in \mathcal{M}_\sigma,\ ef\downarrow \text{ and } ef \in \mathcal{M}_\tau\}$$

We interpret equality to be the standard one, namely

$$[\![x = y]\!] := \begin{cases} \mathcal{A} & \text{if } x = y \\ \emptyset & \text{otherwise} \end{cases}$$

We interpret application by

$$[\![\mathrm{Ap}]\!](e, f) := e \cdot f$$

We interpret each constant symbol as the corresponding pca$^+$ constant in $\mathcal{A}$. Namely,

$$\begin{array}{lll} [\![\mathbf{s}^{\sigma,\tau,\rho}]\!] := \mathbf{s} & [\![\mathbf{k}^{\sigma,\tau}]\!] := \mathbf{k} & \\ [\![\mathbf{p}^{\sigma,\tau}]\!] := \mathbf{p} & [\![\mathbf{p}_0^{\sigma,\tau}]\!] := \mathbf{p}_0 & [\![\mathbf{p}_1^{\sigma,\tau}]\!] := \mathbf{p}_1 \\ [\![0]\!] := 0 & [\![S]\!] := S & \end{array}$$

This only leaves the recursor combinator, $\mathbf{r}$, which we define by combining the fixpoint combinator $\mathbf{y}$ with $\mathbf{d}$ from the pca$^+$ structure. Our first (incorrect) attempt would be to define $[\![\mathbf{r}^{\sigma,\tau}]\!]$ as,

$$\mathbf{r} := \lambda x.\lambda y.\mathbf{y}(\lambda u.\lambda z.\mathbf{d}0zx(y(u(Pz))(Pz)))$$

This would certainly give us that for all x, y and z we have

$$\mathbf{r}xyz \simeq \mathbf{d}0zx(y(\mathbf{r}xy(Pz))(Pz))$$

and in particular

$$\begin{aligned}\mathbf{r}xy(Sz) &\simeq \mathbf{d}0(Sz)x(y(\mathbf{r}xy(P(Sz)))(Pz))\\ &\simeq y(\mathbf{r}xyz)z\end{aligned}$$

Note, however, that there is no way to show that for the above definition $\mathbf{r}xy0\downarrow$, since in order to show this, we would need that $\mathbf{d}00x(y(\mathbf{r}xy(P0))(P0))\downarrow$, which can only be defined once the subterm $\mathbf{r}xy(P0)$ is defined. Hence, we adjust the above definition to get the one below, exploiting the fact that λ-terms are always defined.

$$[\![\mathbf{r}^{\sigma}]\!] := \lambda x.\lambda y.\mathbf{y}(\lambda u.\lambda z.\mathbf{d}0z(\mathbf{k}x)(\lambda w.y(u(Pz))(Pz))\top)$$

Theorem 12.4. *The above model satisfies all the axioms (and hence all the theorems) of* $\mathbf{HA}_\omega$.

Proof. Note that we have interpreted equality in way that makes it straightforward to check the axioms of identity. By the above reasoning, each of the equations associated with the constants also holds in the model. It only remains to check that induction holds. Similarly to the recursor, we can do this using the $\mathbf{y}$ combinator. This time it is slightly simpler. Suppose we are given $e \in \mathcal{A}$ such that

$$e \Vdash_\alpha \varphi(0) \wedge \forall n\, \varphi(n) \to \varphi(Sn)$$

We can then show by induction, that for all $n \in \mathbb{N}$ we have

$$\mathbf{y}(\lambda u.\lambda z.(\mathbf{d}0z(\mathbf{k}(\mathbf{p}_0 e))(\lambda w.\mathbf{p}_1 e(u(Pz)))\top))\underline{n} \Vdash_\alpha \varphi(\underline{n})$$

It follows that

$$\lambda x.\mathbf{y}(\lambda u.\lambda z.(\mathbf{d}0z(\mathbf{k}(\mathbf{p}_0 x))(\lambda w.\mathbf{p}_1 x(u(Pz)))\top)) \Vdash \\ \varphi(0) \wedge \forall n\, \varphi(n) \to \varphi(Sn) \;\to\; \forall n\, \varphi(n)$$

$\square$

Theorem 12.5. *Markov's principle holds in the intensional models of* $\mathbf{HA}_\omega$.

Proof. First, note that we can show the following in general for a pca$^+$, $\mathcal{A}$. Suppose that e is such that for all $n \in \mathbb{N}$ $e\underline{n}$ and either $e\underline{n} = \top$ or $e\underline{n} = \bot$. If there is an n such that $e\underline{=}\top$, then we can find the first such instance computably and uniformly in e. We deduce this as a special case of the stronger statement that we can compute f such that for all n, if there is $k \in \mathbb{N}$ such that $e\underline{n+k} = \top$, then $f\underline{n}\downarrow$ and $f\underline{n} = \underline{k}$ where k is the least such value.

$$f := \mathbf{y}(\lambda u.\lambda z.((ez)(\mathbf{k}\underline{0})\lambda w.(S(u(Sz))))\top)$$

However, we can now see that Markov's principle is realized by

$$\lambda x.\mathbf{y}(\lambda u.\lambda z.((xz)(\mathbf{k}\underline{0})\lambda w.(S(u(Sz))))\top)\underline{0}$$

□

Theorem 12.6. *The axiom of choice* $\mathbf{AC}^{\sigma,\tau}$ *holds in intensional models of* $\mathbf{HA}_\omega$ *for all sorts* σ *and* τ.

Proof. Suppose that we have

$$e \Vdash_\alpha \forall x^\sigma\, \exists y^\tau\, \varphi(x, y)$$

Note that this gives us an element e' of $\mathcal{M}_{\sigma\to\tau}$ defined by $\lambda z.\mathbf{p}_0(ez)$. Meanwhile, we also have

$$\lambda z.\mathbf{p}_1(ez) \Vdash_\alpha \forall x^\sigma\, \varphi(x, e'x)$$

We can hence see that $\mathbf{AC}^{\sigma,\tau}$ is realized by $\lambda x.\mathbf{p}(\lambda z.\mathbf{p}_0(ez))(\lambda z.\mathbf{p}_1(ez))$. □

12.2.2 The extensional model

We now give a second way to construct realizability models of $\mathbf{HA}_\omega$, this time ensuring that we also get extensionality and the axiom of unique choice.

We again define $\mathcal{M}_S$ and E_S for each sort S by induction on finite types. At each stage we will ensure that for each $x \in \mathcal{M}_S$, $E_S(x)$ is inhabited. It is possible to construct these as quotients of the ones in the intensional model, but to get a more concrete description of the model will will define them directly.

As before, we define $\mathcal{M}_N := \mathbb{N}$ and $E_N(n) := \{n\}$.

For products, we define

$$\mathcal{M}_{\sigma\times\tau} := \mathcal{M}_\sigma \times \mathcal{M}_\tau$$
$$E_{\sigma\times\tau}((x, y)) := \{e \mid \mathbf{p}_0 e \in E_\sigma(x) \text{ and } \mathbf{p}_1 e \in E_\tau(y)\}$$

Finally, for function types, we recall that $e \in \mathcal{A}$ *tracks* $F : \mathcal{M}_\sigma \to \mathcal{M}_\tau$ if for all $x \in \mathcal{M}_\sigma$ and all $a \in E_\sigma(x)$ we have $ea\downarrow$ with $ea \in E_\tau(F(x))$. We then define

$$\mathcal{M}_{\sigma\to\tau} := \{F \in \mathcal{M}_\tau^{\mathcal{M}_\sigma} \mid \exists e \in \mathcal{A}\ e \text{ tracks } F\}$$
$$E_{\sigma\to\tau}(F) := \{e \in \mathcal{A} \mid e \text{ tracks } F\}$$

Note that every element of $\mathcal{M}_{\sigma\to\tau}$ is in particular a function from $\mathcal{M}_\sigma$ to $\mathcal{M}_\tau$, so we can define application simply by

$$[\![\mathrm{Ap}]\!](F, x) := F(x)$$

We again define equality to be the standard one, namely,

$$[\![x = y]\!] := \begin{cases} \mathcal{A} & \text{if } x = y \\ \emptyset & \text{otherwise} \end{cases}$$

Theorem 12.7. *The above model satisfies all the axioms of* $\mathbf{HA}_\omega$*, as well as the axioms of unique choice, function extensionality and Markov's principle.*

Proof. The proof that the axioms of $\mathbf{HA}_\omega$ and Markov's principle hold is essentially the same as for the intensional model.

For function extensionality, suppose that t and s are both terms of sort $\sigma \to \tau$, that α is any variable assignment and that for some $e \in \mathcal{A}$ we have $e \Vdash_\alpha \forall x^\sigma\, tx = sx$. Then, by the definition of the model, $[\![s]\!]_\alpha$ and $[\![t]\!]_\alpha$ are both functions from $\mathcal{M}_\sigma$ to $\mathcal{M}_\tau$, and for any $a \in \mathcal{M}_\sigma$, and $f \in E_\sigma(a)$, we have $ef \Vdash_{\alpha[x\mapsto a]} tx = sx$, which by the interpretation of equality and application implies $[\![t]\!]_\alpha(a) = [\![s]\!]_\alpha(a)$. Since this applies for arbitrary a we have, by function extensionality in our meta theory, that $[\![t]\!]_\alpha = [\![s]\!]_\alpha$. However, by the interpretation of equality, this implies that for any $g \in \mathcal{A}$, we have $g \Vdash_\alpha s = t$. Hence, we have a realizer for function extensionality given simply by $\mathbf{i}$.

We can show unique choice holds by similar arguments. □

12.3 Standard realizability models for HAS

We now show how to construct realizability models of **HAS**. As before, we take the natural number sort $\mathcal{M}_N$ to simply be the usual one. This leaves the problem of how to define the sort of sets. One way to motivate the definition, is to view definition as similar to the one for Heyting valued models of **HAS**. Instead of defining a set to be a function from $\mathbb{N}$ to the Heyting algebra, we take sets to be functions $A : \mathbb{N} \to \mathcal{P}(\mathcal{A})$. We recall that the standard Heyting valued models were defined to be global. It turns out that the way to do this in realizability models is to take E_S to be constantly equal to the same, non empty set. We will take this to be $E_S(A) := \{\top\}$.

We interpret membership and equality, again similarly to Heyting valued models.

$$[\![n \in A]\!] := A(n)$$
$$[\![A = B]\!] := \bigcap_{n\in\mathbb{N}} \{e \in \mathcal{A} \mid \forall f \in A(n)\, \mathbf{p}_0 ef \in B(n) \ \wedge\ \forall f \in B(n)\, \mathbf{p}_1 f \in A(n)\}$$

Theorem 12.8. *The above model satisfies all the axioms (and hence all the theorems) of* **HAS**.

Proof. Just as for Heyting valued models of **HAS**, extensionality follows directly from the interpretation of equality in the model.

We check comprehension. Let φ be any formula. We define $A : \mathbb{N} \to \mathcal{P}(\mathcal{A})$ by $A(n) := [\![\varphi(\underline{n})]\!]_\alpha$. We then have the following, immediately from our interpretation of $\in$.

$$\mathbf{k}(\mathbf{pii}) \Vdash_\alpha \forall n\, n \in A \leftrightarrow \varphi(n)$$

It only remains to check second order induction. As before, we will use the $\mathbf{y}$-combinator. Suppose that

$$e \Vdash_\alpha 0 \in A \ \wedge\ \forall n\, n \in A \to Sn \in A$$

Similarly to before, we have

$$\lambda x.\mathbf{d}0x(\mathbf{k}(\mathbf{p}_0 e))(\lambda y.(\mathbf{p}_1 e)x)\top \Vdash_\alpha \forall x\, x \in A$$

□

Theorem 12.9. *Every standard realizabilty of* **HAS** *satisfies the uniformity principle.*

Proof. Suppose that $e \Vdash \forall X\, \exists n\, \varphi(X, n)$. Let $A \in \mathcal{M}_S$. We define the model so that $E_S(A) = \{\top\}$. It follows that $e\top\downarrow$ and $e\top \Vdash \exists n\, \varphi(X, n)$. Hence there exists $n \in \mathcal{M}_N = \mathbb{N}$ such that $\mathbf{p}_0(e\top) \in E_N(n)$ and $\mathbf{p}_1(e\top) \Vdash \varphi(A, n)$. We hence have $\mathbf{p}_0(e\top) = \underline{n}$. However, $\mathbf{p}_0(e\top)$ can only be equal to at most one numeral, and this value of independent of A, i.e. will get the same numeral $\mathbf{p}_0(e\top)$ for any $A \in \mathcal{M}_S$. It follows that we have

$$\lambda x.\mathbf{p}(\mathbf{p}_0(x\top))(\lambda y.\mathbf{p}_1(x\top)) \Vdash \forall X\, \exists n\, \varphi(X, n) \ \to\ \exists n\, \forall X\, \varphi(X, n)$$

□