80818 Intuitionistic Logic - Exercise Sheet 7

October 21, 2021

1. Let $(B, \leq)$ be a Heyting algebra (not necessarily complete). Let U be a downwards closed subset of B. Write $\overline{U}$ for the set

$$\overline{U} := \{\bigvee S \mid S \subseteq U \text{ and } \bigvee S \text{ exists in B}\}$$

   (a) Show that if U is downwards closed, then $\overline{U}$ is the smallest c-ideal containing U.

   (b) For $p \in B$ and $S \subseteq B$, write $p \lhd S$ to mean $p \in \overline{S^{\leq}}$. Show that $(B, \leq, \lhd)$ is a formal topology. (You might find it helpful to recall that complete Heyting algebras satisfy the distributive law $x \wedge \bigvee S = \bigvee_{y \in S} x \wedge y$ and that the same proof applies for Heyting algebras in general as soon as we know $\bigvee S$ exists.)

   (c) Show that $\lhd$ is the smallest subset of $B \times \mathcal{P}(B)$ satisfying the two conditions

      i. $(B, \leq, \lhd)$ is a formal topology.

      ii. For all $S \subseteq B$ if $\bigvee S$ exists in $(B, \leq)$, then $\bigvee S \lhd S$.

   (d) Show that for $U \subseteq B$, U is an open set of the formal topology $(B, \leq, \lhd)$ above if and only if it is a c-ideal.

2. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. A *homeomorphism* from X to Y is a bijective function $f : X \to Y$ such that both f and its inverse are continuous. We say $(X, \mathcal{O}_X)$ is *homogeneous* if for all points x, x' of X, there is a homeomorphism $f : X \to X$ such that $f(x) = x'$. Let $(P, \bigvee, \to)$ and $(Q, \bigvee, \to)$ be complete Heyting algebras. A isomorphism from P to Q is a bijective function θ such that both θ and its inverse preserve all meets, joins and Heyting implication. An *automorphism* of $(P, \bigvee, \to)$ is an isomorphism from $(P, \bigvee, \to)$ to itself.

   (a) Show that Cantor space is homogeneous.

   (b) Let $(X, \mathcal{O}_X)$ be any homogeneous topological space and let $(\mathcal{O}_X, \bigvee, \to)$ be the Heyting algebra of open sets. Show that if $\pi(U) = U$ for every $U \in \mathcal{O}_X$ and every automorphism π of the Heyting algebra $(\mathcal{O}_X, \bigvee, \to)$, then either $U = \bot$ or $U = \top$. (You may assume the law of excluded middle, and you may assume that if π is a bijection between Heyting algebras such that both π and π^{-1} preserve the ordering relation, then π and π^{-1} preserve all of the Heyting alegbra structure.)

(c) Let $(P, \bigvee, \wedge, \rightarrow)$ be any complete Heyting algebra and π any automorphism of $(P, \bigvee, \wedge, \rightarrow)$. Show that in the standard Heyting valued model of **HAS** we can define bijective functions $\tilde{\pi}_S : \mathcal{M}_S \rightarrow \mathcal{M}_S$ and $\tilde{\pi}_N : \mathcal{M}_N \rightarrow \mathcal{M}_N$ such that for all $n \in \mathcal{M}_N$ and $A, B \in \mathcal{M}_S$ we have $[\![\tilde{\pi}_N(n) \in \tilde{\pi}_S(A)]\!] = \pi([\![n \in A]\!])$ and $[\![\tilde{\pi}_S(A) = \tilde{\pi}_S(B)]\!] = \pi([\![A = B]\!])$. (Hint: $\tilde{\pi}_N$ is trivial to define.)

(d) Given an automorphism π as above, and a variable assignment σ, write $\tilde{\pi} \circ \sigma$ for the result of replacing $\sigma(x)$ with $\tilde{\pi}_N(\sigma(x))$ for each number variable x and replacing $\sigma(X)$ with $\tilde{\pi}_S(\sigma(X))$ for each set variable X. Show that for every formula φ, we have $[\![\varphi]\!]_{\tilde{\pi} \circ \sigma} = \pi([\![\varphi]\!]_\sigma)$. (Don't worry about all the details, but include at least one set quantifier of the inductive argument).

(e) Let φ be any closed formula of **HAS** and σ any variable assignment. Show that in the standard topological model on Cantor space we have either $[\![\varphi]\!]_\sigma = \bot$ or $[\![\varphi]\!]_\sigma = \top$.