14 The Second Kleene Algebra and Function Realizability

14.1 The second Kleene algebra

Definition 14.1. A partial function $F : \mathbb{N}^{\mathbb{N}} \rightharpoonup \mathbb{N}$ is *continuous* if for all $f \in \mathbb{N}^{\mathbb{N}}$ such that $F(f)\downarrow$ there is $n \in \mathbb{N}$ such that for all $g \in \mathbb{N}^{\mathbb{N}}$, if $g(i) = f(i)$ for $i < n$, then $F(g)\downarrow$ and $F(g) = F(f)$.

A partial function $F : \mathbb{N}^{\mathbb{N}} \rightharpoonup \mathbb{N}^{\mathbb{N}}$ is *continuous* if for all n the partial function sending f to $F(f)(n)$ is continuous.

Note that every continuous function $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is in particular a continuous function.

The key idea behind the second Kleene algebra is that we can encode partial continuous functions $\mathbb{N}^{\mathbb{N}} \rightharpoonup \mathbb{N}^{\mathbb{N}}$ as elements of $\mathbb{N}^{\mathbb{N}}$. We first show how to encode partial continuous functions $\mathbb{N}^{\mathbb{N}} \rightharpoonup \mathbb{N}$. Write $\mathbb{N}^{<\omega}$ for the set of finite sequences of natural numbers. Note that we can view any function $f : \mathbb{N}^{<\omega} \to \mathbb{N} + \{\bot\}$ as a continuous partial function function $F : \mathbb{N}^{\mathbb{N}} \rightharpoonup \mathbb{N}$ defined by

$$F(g) := \begin{cases} f(g(0), \ldots, g(n-1)) & f(g(0), \ldots, g(n-1)) \in \mathbb{N} \text{ and } n \text{ is least such} \\ \text{undefined} & \text{otherwise} \end{cases}$$

This in fact defines a surjective function from $(\mathbb{N} + \{\bot\})^{\mathbb{N}^{<\omega}}$ to continuous partial functions $\mathbb{N}^{\mathbb{N}} \rightharpoonup \mathbb{N}$. Given continuous $F : \mathbb{N}^{\mathbb{N}} \rightharpoonup \mathbb{N}$, we can define $f : \mathbb{N}^{<\omega} \to \mathbb{N} + \{\bot\}$ on $(a_0, \ldots, a_{n-1})$ as follows. If $F(g) = F(h)$ whenever $g(i) = h(i) = a_i$ for $i < n$, then we take $f(a_0, \ldots, a_{n-1}) := F(g)$, and otherwise we take $f(a_0, \ldots, a_{n-1})$ to be $\bot$. We say f is an *associate* of the function F.

However, we have a canonical bijection between $(\mathbb{N} + \{\bot\})^{\mathbb{N}^{<\omega}}$ and $\mathbb{N}^{\mathbb{N}}$ by composing with bijections $\mathbb{N}^{<\omega} \cong \mathbb{N}$ and $\mathbb{N} + \{\bot\} \cong \mathbb{N}$. This is one way of understanding the explicit definition below.

Definition 14.2. We define a function $|$ from $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ to partial functions $\mathbb{N} \rightharpoonup \mathbb{N}$. We define $f|g(n)$ to be $f(\langle n, [g(0), \ldots, g(m-1)]\rangle) - 1$ if m is the least such number with $f(\langle n, [g(0), \ldots, g(m-1)]\rangle) > 0$. If there is no such m, then $f|g(n)$ is undefined. We then convert this into a partial binary operator giving a partial applicative structure on $\mathbb{N}^{\mathbb{N}}$ by

$$f \cdot g(n) := \begin{cases} f|g & f|g \text{ is total} \\ \text{undefined} & \text{otherwise} \end{cases}$$

The partial applicative structure has elements $\mathbf{s}$ and $\mathbf{k}$ making it a partial combinatory algebra, that we call the *second Kleene algebra*, $\mathcal{K}_2$.

We have a canonical way to make $\mathcal{K}_2$ into an extended pca.

We define 0 to be the function constantly equal to 0. Note that the function sending $f : \mathbb{N} \to \mathbb{N}$ to the function $\lambda n.f(n) + 1$ is evidently continuous, and so has an associate, S that we use for the successor combinator. Note that for each n, the numeral $\underline{n}$ is precisely the constant function $\lambda x.n$.

14.2 Function realizability

We refer to realizability over $\mathcal{K}_2$ as *function realizability*. We will show two key properties of function realizability: that every function $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ is continuous, and that we have the axiom of choice $\mathbf{AC}^{N\to N,N}$. These two axioms are sometimes combined together into a single axiom called *continuous choice*, which states that whenever $\forall f^{N\to N}\, \exists x^N\, \varphi(f,x)$ there exists a continous function $F : (N \to N) \to N$ such that for all $f \in \mathbb{N}^{\mathbb{N}}$ we have $\varphi(f, F(f))$. However, we will consider them separately. We first look at the axiom of choice.

Note that we have a continuous way to take a function $f : \mathbb{N} \to \mathbb{N}$ and evaluate it: i.e. return the numeral $\underline{f(n)}$ given f and $\underline{n}$ as input. We can also go the other way, and given an associate f for a continuous function F such that $F(\underline{n})$ is a numeral for all n, we can find, continuously in f, a function $g : \mathbb{N} \to \mathbb{N}$ such that $F(\underline{N}) = \underline{g(n)}$.

Using this, we can show the following for realizability models on $\mathcal{K}_2$.

Theorem 14.3. $\mathbf{AC}^{N\to N,N\to N}$ *holds in the standard extensional realizability model of* $\mathbf{HA}_\omega$ *on* $\mathcal{K}_2$.

Definition 14.4. Let $F : \mathbb{N}^{\mathbb{N}} \rightharpoonup \mathbb{N}^{\mathbb{N}}$. A *modulus of convergence function* is a function $M : \mathbb{N}^{\mathbb{N}} \rightharpoonup \mathbb{N}$ such that if $F(g)\downarrow$, then also $M(g)\downarrow$ and for all $h : \mathbb{N} \to \mathbb{N}$ if $h(i) = g(i)$ for $i < M(g)$ then $F(h)\downarrow$ and $F(h) = F(g)$.

Note that assuming $\mathbf{AC}^{N\to N,N}$, F is continuous if and only if it admits a modulus of convergence function. In fact we have the following theorem.

Theorem 14.5. *We can find* $m \in \mathcal{K}_2$ *with the following property. For all* f, *if* $F : \mathbb{N}^{\mathbb{N}} \rightharpoonup \mathbb{N}^{\mathbb{N}}$ *is the partial continuous function that* f *represents, then* mf *represents the modulus of convergence function for* F.

Using this result we can show the following for realizability models.

Theorem 14.6. *In both standard realizabliity models of* $\mathbf{HA}_\omega$ *on* $\mathcal{K}_2$, *every function* $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ *is continuous.*

Moreover, in the intensional model, there is a function $m : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ *such that for all* f, $m(f)$ *is a modulus of convergence function for* f.