80818 Intuitionistic Logic - Exercise Sheet 3

September 16, 2021

1. Fix a signature $(\mathcal{O}, \mathcal{R})$ and a theory T over $(\mathcal{O}, \mathcal{R})$. Recall that the *Lindenbaum-Tarski algebra* is defined by quotienting the set of formulas over $(\mathcal{O}, \mathcal{R})$ by the equivalence relation $\sim$ defined by $\varphi \sim \psi$ when $T \vdash \varphi \leftrightarrow \psi$, and has an order relation $\leq$ defined by $[\varphi] \leq [\psi]$ when $T \vdash \varphi \to \psi$. Fix a formula φ and a free variable x. Let S be the set of all equivalence classes of the form $[\varphi[x/t]]$ for each term t (avoiding free variable capture).

   Show that $[\exists x\, \varphi]$ is the least upper bound of S.

2. Let $(P, \leq)$ be a poset such that any two elements $p, q \in P$ have an upper bound, i.e. r such that $p \leq r$ and $q \leq r$. Show that the upset topology on $(P, \leq)$ is connected.

3. Let X be any set. Let B be a collection of subsets of X with the following properties:

   (i) For every element x of X, there exists $b \in B$ such that $x \in b$.

   (ii) For all $a, b \in B$ and every element x of $a \cap b$, there exists $c \in B$ such that $x \in c$ and $c \subseteq a \cap b$.

   Show that there is a topology $\mathcal{O}$ on X such that B is a basis for $\mathcal{O}$.

4. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. We say X is a *retract* of Y if there is a pair of continuous maps $s : X \to Y$ and $r : Y \to X$ such that $r \circ s$ is the identity function on X.

   (a) Show that Cantor space is a retract of Baire space.

   (b) Show that $\mathbb{N}_\infty$ is a retract of Cantor space.

   (c) Let X be a retract of Y and let $(Z, \mathcal{O}_Z)$ be another topological space. Suppose that every function $Y \to Z$ is continuous. Show that every function $X \to Z$ is continuous. (Avoid using the law of excluded middle, but don't worry about whether this can be formalised in $\mathbf{HA}_\omega$ or $\mathbf{HAS}$.)

5. Recall that I is the poset with three elements $\{0, 1, 2\}$ with $0 \leq 1$ and $2 \leq 1$. Given a finite sequence $\sigma := (\sigma(0), \sigma(1), \ldots, \sigma(k-1))$ with $\sigma(i) \in I$ define $U_\sigma := \{f : \mathbb{N} \to I \mid \forall i < k\, f(i) \geq \sigma(i)\}$.

(a) Show there is a topology on $I^{\mathbb{N}}$ with a basis consisting of sets of the form U_σ for finite sequences σ.

(b) Show that $I^{\mathbb{N}}$ is locally connected. (Try to give a constructive proof, but don't worry about if it can be formalised in $\mathbf{HA}_\omega$ or $\mathbf{HAS}$.)