Towards computable homotopy theory

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What is an oracle?

Definition (Turing '36)

A partial function $\mathbb{N} \rightharpoonup \mathbb{N}$ is *computable* if it can be computed by a Turing machine (a computer program).

Key idea: We can encode computer programs as natural numbers. We write the partial function encoded by e as φ_e.

Theorem (Turing '36)

There is at least one non computable function.

Proof.

$$\kappa(n) := \begin{cases} 1 & \varphi_n(n) \downarrow = 0 \\ 0 & \text{otherwise} \end{cases}$$

$\square$

Definition (Turing '39)

An *oracle Turing machine* is a computer program that can query information from an outside source (an *oracle*). We say a partial function $f : \mathbb{N} \rightharpoonup 2$ is *computable relative to* $\chi : \mathbb{N} \to 2$ if we can compute f using χ as an oracle.
We also say that f is *Turing reducible to* χ and write $f \leq_T \chi$.
Note that this defines a preorder on functions $\mathbb{N} \to 2$. We refer to the poset reflection of this preorder as the *Turing degrees*.

Example

A web browser can send queries (http requests) to a server and receive back information (webpages).

Queries can depend on the result of previous queries. E.g. a webbrowser can request all the images mentioned on a webpage that it just received.

Theorem (Hyland '82)

The Turing degrees embed into the lattice of subtoposes of the effective topos, $\mathcal{E}ff$.

We can generalise Hyland's result to HoTT using cubical assemblies and higher modalities.

Cubical Assemblies

Theorem (Uemura)

The category of cubical assemblies consists of cubical sets constructed internally in the lcc of assemblies. Cubical assemblies form a model of cubical type theory and thereby HoTT.

Theorem (S, Uemura)

Cubical assemblies have a reflective subuniverse that satisfies Church's thesis "all functions are computable."

Theorem (S)

If $\Omega_{\neg\neg}$ is a classifier for $\neg\neg$-stable propositions in the metatheory, then the discrete cubical set $\Delta(\Omega_{\neg\neg})$ is a classifier for for $\neg\neg$-stable propositions in cubical sets.

Modalities

Definition (Rijke, Shulman, Spitters)

A *uniquely eliminating modality* is an operation on types $\bigcirc : \mathcal{U} \to \mathcal{U}$ together with unit $\eta_X : X \to \bigcirc X$ for each $X : \mathcal{U}$ such that the canonical map $\prod_{z:\bigcirc X} \bigcirc(P(z)) \to \prod_{x:X} \bigcirc P(\eta_X(x))$ is an equivalence for $X : \mathcal{U}$ and $P : \bigcirc X \to \mathcal{U}$:

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "\eta_X"] & \sum_{z:\bigcirc X} Pz \arrow[d, two heads] \\
\bigcirc X \arrow[r] \arrow[ur, dotted] & \bigcirc X
\end{tikzcd}$$

A type X is

- $\bigcirc$-*modal* if $\eta_X : X \to \bigcirc X$ is an equivalence.
- $\bigcirc$-*separated* if for all $x, y : X$, $x = y$ is $\bigcirc$-modal.
- $\bigcirc$-*connected* if $\bigcirc X$ is contractible.

Definition

Given two modalities $\bigcirc$ and $\bigcirc'$, we write $\bigcirc \leq_T \bigcirc'$ if every $\bigcirc$-connected type is $\bigcirc'$-connected, or equivalently if every $\bigcirc'$-modal type is $\bigcirc$-modal.

Definition (Rijke-Shulman-Spitters)

The *nullification* of a family of types $i : I \vdash A(i)\,\mathsf{Type}$ is the smallest modality $\bigcirc$ such that $A(i)$ is $\bigcirc$-connected for all $i : I$.

Theorem (Rijke-Shulman-Spitters)

Nullification exists, and can be described explicitly as a higher inductive type.

The definition below works for any modality ∇, but we will only apply it where ∇ is the modality of $\neg\neg$-sheafification in cubical assemblies (nullify all $\neg\neg$-dense propositions).

Definition

An *oracle function* from A to B is a function $\chi : A \to \nabla B$.

We can think of the elements of ∇B as partial elements of A and write $b\downarrow$ for the type $\mathsf{hFibre}_{\eta_B^\nabla}(b)$.

Definition

The *oracle modality*, $\bigcirc_\chi$ on an oracle $\chi : A \to \nabla B$ is the nullification of the family of types $a : A \vdash \chi(a)\downarrow$.
We write $\mathcal{U}[\chi]$ for the corresponding reflective subuniverse of $\mathcal{U}$, i.e. the set of all $\bigcirc_\chi$-modal types.

We think of $\bigcirc_\chi$ as the smallest modality that forces χ to be a total function, and $\mathcal{U}[\chi]$ as the largest subuniverse of $\mathcal{U}$ that contains the map χ.

Proposition

Assuming $\neg\neg$-resizing we can show for sets B that ∇B has the same universe level as B.

Proposition

For all $\chi : A \to \nabla(B)$, $\bigcirc_\chi \leq_T \nabla$.

Proposition

If B is a $\neg\neg$-separated set, then η^∇_B is an embedding. Hence for all $a : A$, $\chi(a)\downarrow$ is a proposition.

Rijke, Shulman, Spitters: modalites generated by propositions are called *topological* modalities. They are in particular lex, i.e. $\mathcal{U}[\chi]$ is $\bigcirc_\chi$-modal itself.

When working with oracle modalities it's useful to use three additional axioms that hold in cubical assemblies:

1. A classifier for $\neg\neg$-stable propositions, $\Omega_{\neg\neg}$.
2. Computable choice: a generalisation of Church's thesis to all $\neg\neg$-stable relations (possibly partial and multivalued) with $\neg\neg$-stable domain.
3. Markov induction: $\nabla\mathbb{N}$ is well founded.

Overall aim: to use this as a setting to look for interactions between homotopy theory and computability theory.

Given an oracle χ, we can consider the group $\Omega(\mathcal{U}, \bigcirc_\chi \mathbb{N})$ of permutations of $\mathbb{N}$ computable using χ. By default this includes information telling us the oracle Turing machine that computes the permumation. We can erase this information using ∇, leaving only a group in sets.

Theorem (S)

For oracles $\chi, \chi' : \mathbb{N} \to \nabla 2$. If $\nabla(\Omega(\mathcal{U}, \bigcirc_\chi \mathbb{N})) \cong \nabla(\Omega(\mathcal{U}, \bigcirc_{\chi'} \mathbb{N}))$, then $\neg\neg(\chi \equiv_T \chi')$.

This can be proved directly,[1] but we will give a new proof combining homotopy theory with synthetic computability theory using Markov induction.

[1] Question for the audience: do you know anywhere in the literature this is done?

Key ideas from computability theory.

- "Finite sets are always computable:" For any finite set $F \subseteq_{\mathsf{fin}} \nabla A$ we have $\neg\neg \prod_{\alpha : F} \mathsf{hFibre}_{\eta_A^{\nabla}}(\alpha)$.
- Given $e, f, g : \bigcirc_{\chi}\mathbb{N} \simeq \bigcirc_{\chi}\mathbb{N}$, such that $f \neq g$ we have $\bigcirc_{\chi}(e \neq f + e \neq g)$. This is, we can find out, computably in χ whether $e \neq f$ or $e \neq g$. We can prove this synthetically using Markov induction.

Key ideas from homotopy theory, following Buchholtz, Van Doorn, Rijke:

- We can encode any oracle $\chi : \mathbb{N} \to \nabla 2$ as an element of $\Omega(\mathcal{U}^{\mathbb{N}}, \lambda x.\, \bigcirc_\chi 2)$.
- The map $\mathcal{U}^{\mathbb{N}} \to \mathcal{U}$ sending A to $\sum_{\mathbb{N}} A$ induces an inclusion of groups $\Omega(\mathcal{U}^{\mathbb{N}}, \lambda x.\, \bigcirc_\chi 2) \hookrightarrow \Omega(\mathcal{U}, \Omega(\bigcirc_\chi \mathbb{N}))$.
- It is useful to know this inclusion factors through wreath product: The map $\mathcal{U}^{\mathbb{N}} \to \mathcal{U}$ factors through $\sum_{X:\mathcal{U}} \mathcal{U}^X$:

$$
\begin{array}{ccccc}
\mathcal{U}^A & \xrightarrow{F_A} & \sum_{X:\mathcal{U}} \mathcal{U}^X & \xrightarrow{D} & \mathcal{U} \\
Y : A \to \mathcal{U} & \xmapsto{F_A} & (A, Y) & & \\
 & & (X, Y) & \xmapsto{D} & \sum_{x:X} Yx
\end{array}
$$

$\Omega(\sum_{X:\mathcal{U}} \mathcal{U}^X, (\mathbb{N}, \lambda x.\, \bigcirc_\chi 2))$ is the wreath product $S_2 \wr S_{\mathbb{N}}$

WIP: Studying other modalities based on oracle modalities.

Theorem (Christensen-Opie-Rijke-Scoccola)

For every modality $\bigcirc$, there is a modality $\bigcirc^{=}$ such that a type is $\bigcirc^{=}$-modal iff it is $\bigcirc$-separated.

We refer to $\bigcirc^{=}$ as the *suspension* of $\bigcirc$, and write the k-fold suspension as $\bigcirc^{(k)}$.[2]

For a type A we have,

- A only contains "computable" points
- ∇A includes the additional (non-computable) points of A that can be proved to exist using classical logic
- $\bigcirc_{\chi} A$ includes new points that can be computed using χ as oracle
- $\bigcirc_{\chi}^{=} A$ has the same points as A, but we can use the oracle to construct new paths

[2] Question for the audience: what is good notation/terminology for this?

We can compute some homotopy groups, but so far only have general results that don't require any computability theory:

Proposition

If $n \geq k + 2$ then $\pi_k(\bigcirc^{(n)} A) = A$ for all A.

Proposition

If $\pi_k(A)$ is $\neg\neg$-separated, then $\pi_k(\bigcirc_\chi^{(k+1)} A) = \pi_k(A)$.

To show the assumption of $\neg\neg$-separation is necessary:

Example

If Y_n is the nth generator of $\bigcirc_\chi^{(k+1)}$ we have $\bigcirc_\chi^{(k+1)} Y_n = 1$ by construction. However, one can show that $\Omega^k(\bigcirc_\chi^{(k+1)} Y_n)$ is $\Omega^k(\mathbb{S}^k) * \chi(n)\downarrow$, which is trivial precisely when $\chi(n)\downarrow$.

Conjecture

If A is a modest cubical assembly, $\pi_k(A)$ is $\neg\neg$-separated and $n \leq k$ then $\pi_k(\bigcirc^{(n)}A) = \bigcirc A$

NB: Spheres are modest. One can check several cases directly: $\pi_1(\bigcirc^{(1)}\mathbb{S}^1)$, $\pi_2(\bigcirc^{(2)}(\mathbb{S}^2))$, $\pi_3(\bigcirc^{(3)}(\mathbb{S}^2))$ and $\pi_3(\bigcirc^{(2)}(\mathbb{S}^2))$ are all $\bigcirc\mathbb{Z}$.

More open problems:

1. More examples of modalities in cubical assemblies.
2. Is the category of cubical assemblies hypercomplete?
3. "HoTT-style" synthetic proofs of classic results in computable group theory e.g. Higman embedding theorem.
4. Computable structures is a subtopic in computability theory studying countable algebraic structures in the effective topos. What about computable higher structures?
5. Higher domain theory?

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Thanks for your attention!