80818 Intuitionistic Logic - Exercise Sheet 8

October 28, 2021

1. We write $t \to_n r$ to mean t reduces to r at stage n, as in definition 11.17 of the notes. Verify the following.
   (a) $\mathbf{k}(\mathbf{ks})\mathbf{ss} \to_0 \mathbf{s}$
   (b) $\mathbf{s}(\mathbf{ss})\mathbf{ks} \to_2 \mathbf{s}(\mathbf{ks})(\mathbf{s}(\mathbf{ks}))$
   (c) $\mathbf{kk}(\mathbf{skks}) \to_1 \mathbf{k}$
   (d) It is false that $\mathbf{kk}(\mathbf{skks}) \to_0 \mathbf{k}$?

2. Define $\top := \lambda x.\lambda y.x$ and $\bot := \lambda x.\lambda y.y$.
   (a) Let $(\mathcal{A}, \cdot)$ be a pca. Let d be any element of $\mathcal{A}$ with the following properties:
      i. For all $a \in \mathcal{A}$, $da\downarrow$ and either $da = \bot$ or $da = \top$.
      ii. If $a, a' \in \mathcal{A}$ are such that for all $b \in \mathcal{A}$, $ab \simeq a'b$, then $da = da'$.

      Show that d is constant, i.e. $da = da'$ for all $a, a' \in \mathcal{A}$. (Hint: You might find it useful to consider the term $\mathbf{y}(\lambda x.dxa'a)$ when $a = \top$ and $a' = \bot$.)
   (b) Suppose there is a term d such that $dab = \top$ if $ab\downarrow$ and $dab = \bot$ if $ab\uparrow$. Show, by part (a) or otherwise that $ab\downarrow$ for all $a, b \in \mathcal{A}$ (we say $\mathcal{A}$ is a *total pca* or just *ca*).
   (c) Let $\mathcal{A}$ be a pca with *decidable equality*. That is, there is $e \in \mathcal{A}$ such that for all $a, b \in \mathcal{A}$, $eab\downarrow$ and $eab = \top$ when $a = b$ and $eab = \bot$ when $a \neq b$. Define $c := \lambda x.e\mathbf{k}(xx)\mathbf{sk}$. Show that cc is not defined.
   (d) An *ω-pca* is a pca$^+$ $\mathcal{A}$ such that for all $a \in \mathcal{A}$ there exists $n \in \mathbb{N}$ such that $a = \underline{n}$. Show there is no total ω-pca.

   Remark: The argument in part (a) is known in computability theory as Rice's theorem.

3. Suppose we are given two extended pcas with the same underlying partial applicative structure, say $(\mathcal{A}, \cdot)$. Write the respective pca$^+$ structures as $\mathbf{s}, \mathbf{k}, \mathbf{p}, \mathbf{p}_0, \mathbf{p}_1, 0, S, \mathbf{d}$ and $\mathbf{s}', \mathbf{k}', \mathbf{p}', \mathbf{p}_0', \mathbf{p}_1', 0', S', \mathbf{d}'$. Write N for the smallest subset of $\mathcal{A}$ such that $0 \in N$ and $Sx \in N$ whenever $x \in N$ and N' for the smallest subset of $\mathcal{A}$ such that $0' \in N'$ and $S'x \in N'$ whenever $x \in N'$.

(a) Show there is a *representable bijection* between N and N'. That is, $a, b \in \mathcal{A}$ such that $ac \in N'$, for all $c \in N$, $bc' \in N$ for all $c' \in N'$, and $a(bc') = c'$ and $b(ac) = c$ for all such c, c'.

(b) Suppose we are given two extended pcas $\mathcal{A}$ and $\mathcal{A}'$ with the same underlying partial applicative structure. Show a partial function $f : \mathbb{N} \rightharpoonup \mathbb{N}$ is $\mathcal{A}$-computable if and only if it is $\mathcal{A}'$-computable.