1. Hard direction

Let $g : V \to \mathbb{R}$ be an eigenvector achieving λ_G. Let $s : \{1, ..., n\} \to V$ be a function such that $f := g \circ s$ is monotonically decreasing. Let $S_i = \{s(1), ..., s(i)\}$. Define $\alpha_G = \min_i h_{S_i}$. Let r denote the largest integer such that $\operatorname{vol}(S_r) \leq \frac{\operatorname{vol}(G)}{2}$.

Theorem 1.1: $\frac{\alpha_G^2}{2} \leq R(g_+)$

Proof:

$$\begin{aligned}
\frac{\alpha_G^2}{2} &= \frac{\alpha_G^2}{2} \frac{\left(\sum_{i=1}^n f_+(i)^2 d_{s(i)}\right)^2}{\left(\sum_{i=1}^n f_+(i)^2 d_{s(i)}\right)^2} \\
&\leq \frac{\alpha_G^2}{2} \frac{\left(\sum_{i=1}^n f_+(i)^2 \left|\widetilde{\operatorname{vol}}(S_i) - \widetilde{\operatorname{vol}}(S_{i-1})\right|\right)^2}{\left(\sum_{i=1}^n f_+(i)^2 d_{s(i)}\right)^2} && \text{Theorem 1.2} \\
&= \frac{\alpha_G^2}{2} \frac{\left(\sum_{i=1}^{n-1} \left(f_+(i)^2 - f_+(i+1)^2\right) \widetilde{\operatorname{vol}}(S_i)\right)^2}{\left(\sum_{i=1}^n f_+(i)^2 d_{s(i)}\right)^2} && \text{Theorem 1.3} \\
&\leq \frac{\left(\sum_{i=1}^{n-1} \left(f_+(i)^2 - f_+(i+1)^2\right) |\partial(S_i)|\right)^2}{2\left(\sum_{i=1}^n f_+(i)^2 d_{s(i)}\right)^2} && \text{Theorem 1.4} \\
&\leq \frac{\left(\sum_{s(i)\sim s(j)} f_+(i)^2 - f_+(j)^2\right)^2}{\left(2\sum_{i=1}^n f_+(i)^2 d_{s(i)}\right)^2} \\
&\leq \frac{\left(\sum_{s(i)\sim s(j)} (f_+(i) + f_+(j))^2\right)\left(\sum_{s(i)\sim s(j)} (f_+(i) - f_+(j))^2\right)}{\left(\sum_{i=1}^n f_+(i)^2 d_{s(i)}\right)^2} \\
&\leq \frac{\left(\sum_{s(i)\sim s(j)} (f_+(i) + f_+(j))^2\right)\left(\sum_{s(i)\sim s(j)} (f_+(i) - f_+(j))^2\right)}{\left(\sum_{i=1}^n f_+(i)^2 d_{s(i)}\right)\left(\sum_{s(i)\sim s(j)} (f_+(i) + f_+(j))^2\right)} \\
&= \frac{\sum_{s(i)\sim s(j)} (f_+(i) - f_+(j))^2}{\sum_{i=1}^n f_+(i)^2 d_{s(i)}} \\
&= \frac{\sum_{u\sim v} (g_+(u) - g_+(v))^2}{\sum_v g_+(v)^2 d_v} \\
&= R(g_+).
\end{aligned}$$

□

Theorem 1.2:

$$d_{s(i)} \leq |\min\{\operatorname{vol}(S_i), \operatorname{vol}(S_i^c)\} - \min\{\operatorname{vol}(S_{i-1}), \operatorname{vol}(S_{i-1}^c)\}|$$

Proof:

- Case $\operatorname{vol}(S_i) \leq \operatorname{vol}(S_i^c)$ and $\operatorname{vol}(S_{i-1}) \leq \operatorname{vol}(S_{i-1}^c)$: TODO
- Case $\operatorname{vol}(S_i) \geq \operatorname{vol}(S_i^c)$ and $\operatorname{vol}(S_{i-1}) \geq \operatorname{vol}(S_{i-1}^c)$: TODO
- Case $\operatorname{vol}(S_i) \leq \operatorname{vol}(S_i^c)$ and $\operatorname{vol}(S_{i-1}) \geq \operatorname{vol}(S_{i-1}^c)$: This is a contradiction, indeed $(S_i) \leq (S_i^c) \leq (S_{i-1}^c) \leq (S_{i-1})$
- Case $\operatorname{vol}(S_i) \geq \operatorname{vol}(S_i^c)$ and $\operatorname{vol}(S_{i-1}) \leq \operatorname{vol}(S_{i-1}^c)$: In this case we have $(S_{i-1}) \leq \frac{G}{2}$ and therefore $(G) - 2 \cdot (S_{i-1}) - d_i \geq (G) - 2 \cdot \frac{G}{2} - d_i$

□

Theorem 1.3: $\sum_{i=1}^{n} f_+(i)^2 \left|\widetilde{\operatorname{vol}}(S_i) - \widetilde{\operatorname{vol}}(S_{i-1})\right| = \sum_{i=1}^{n-1} \left(f_+(i)^2 - f_+(i+1)^2\right) \widetilde{\operatorname{vol}}(S_i)$.

Proof:

$$\sum_{i=1}^{n} f_+(i)^2 \left|\widetilde{\operatorname{vol}}(S_i) - \widetilde{\operatorname{vol}}(S_{i-1})\right| = \left|\sum_{i=1}^{n} f_+(i)^2 \widetilde{\operatorname{vol}}(S_i) - \sum_{i=1}^{n} f_+(i)^2 \widetilde{\operatorname{vol}}(S_{i-1})\right|$$

$$= \left|\sum_{i=1}^{n-1} f_+(i)^2 \widetilde{\operatorname{vol}}(S_i) + \cancel{f_+(n)^2 \widetilde{\operatorname{vol}}(S_n)} - \sum_{i=1}^{n-1} f_+(i+1)^2 \widetilde{\operatorname{vol}}(S_i) - \cancel{f_+(1)^2 \widetilde{\operatorname{vol}}(S_0)}\right|$$

$$= \left|\sum_{i=1}^{n-1} \left(f_+(i)^2 - f_+(i+1)^2\right) \widetilde{\operatorname{vol}}(S_i)\right|.$$

The theorem follows from the fact that every summand is positive. □

Theorem 1.4:

$$\alpha_G \, \widetilde{\operatorname{vol}}(S_i) \leq |\partial(S_i)|$$

Proof:

$$\min_j \frac{|\partial(S_j)|}{\min\left\{\operatorname{vol}(S_j), \operatorname{vol}\left(S_j^c\right)\right\}} \cdot \min\{\operatorname{vol}(S_i), \operatorname{vol}(S_i^c)\}$$

$$\leq \frac{|\partial(S_i)|}{\min\{\operatorname{vol}(S_i), \operatorname{vol}(S_i^c)\}} \cdot \min\{\operatorname{vol}(S_i), \operatorname{vol}(S_i^c)\} = |\partial(S_i)|.$$

□

Theorem 1.5: $g_+(v_i)^2 \geq g_+(v_{i+1})^2$

Proof: The only troublesome cases is when $g_+(v_i) = 0$ and $g_+(v_{i+1}) = g(v_{i+1})$. In this case we have $g(v_i) < g(v_r)$ and $g(v_{i+1}) \geq g(v_r)$ which implies $r < i$ and $i + 1 \leq r$. This is a contradiction. □

Theorem 1.6:

$$\left(\sum_{u\sim v} g_+(u)^2 - g_+(v)^2\right)^2 \leq 2\left(\sum_{u\sim v} (g_+(u)+g_+(v))^2\right)\left(\sum_{u\sim v} (g_+(u)-g_+(v))^2\right)$$

Proof: Consider the matrices $X(i,j) = 1_{s(i)\sim s(j)}(f_+(i)+f_+(j))$ and $Y(i,j) = 1_{s(i)\sim s(j), i<j}(f_+(i) - f+(j))$. Their Frobenius inner product is

$$\langle X, Y\rangle = \sum_{i,j} X^T(i,j)Y(i,j)$$

$$= \sum_{i,j} 1_{s(i)\sim s(j), i<j}(f_+(i)+f_+(v))(f_+(i)-f_+(j))$$

$$= \sum_{s(i)\sim s(j)} f_+(i)^2 - f_+(j)^2.$$

The norms of X and Y are

$$\|X\|_F^2 = \sum_{u,v} X(u,v)^2 = \sum_{u,v} 1_{u\sim v}(g_+(u)+g_+(v))^2 = 2\sum_{u\sim v}(g_+(u)+g_+(v))^2,$$

$$\|Y\|_F^2 = \sum_{u,v} Y(u,v)^2 = \sum_{u,v} 1_{u\sim v, u<v}(g_+(u)-g_+(v))^2 = \sum_{u\sim v}(g_+(u)-g_+(v))^2.$$

By the Cauchy-Schwarz inequality we have

$$\langle X, Y\rangle^2 \leq \|X\|_F^2 \|Y\|_F^2,$$

and the theorem follows. □

Theorem 1.7:

$$\sum_{u\sim v}(g_+(u)+g_+(v))^2 \leq \sum_u g_+(u)^2 d_u$$

Proof:

$$\sum_{u\sim v}(g_+(u)+g_+(v))^2$$

$$\leq \sum_{u\sim v} g_+(u)^2 + g_+(v)^2$$

$$= \sum_{u\sim v} g_+(u)^2 + \sum_{u\sim v} g_+(v)^2$$

$$= 2\sum_{u\sim v} g_+(u)^2$$

$$= 2\cdot\frac{1}{2}\sum_u g_+(u)^2 \sum_v 1_{u\sim v}$$

$$= \sum_u g_+(u)^2 d_u.$$

$\square$

Theorem 1.8:

$$\sum_{i=1}^{n}\left(f_+(i)^2 - f_+(i+1)^2\right)\partial(S_i) \leq \sum_{u\sim v} g_+(u)^2 - g_+(v)^2$$

Proof:

$$\begin{aligned}\sum_{i=1}^{n}\left(f_+(i)^2 - f_+(i+1)^2\right)\partial(S_i) &= \sum_{i=1}^{n}\left(f_+(i)^2 - f_+(i+1)^2\right)\left(\sum_{j=(i+1)}^{n} 1_{s(i)\sim s(j)}\right) \\ &= \sum_{i=1}^{n-1}\sum_{j=(i+1)}^{n}\left(f_+(i)^2 - f_+(i+1)^2\right)1_{s(i)\sim s(j)}.\end{aligned}$$

By monotonicity we have $f_+(i)^2 - f_+(i+1)^2 \geq 0$ for $i < j$. Therefore the above is less or equal

$$\sum_{i=1}^{n-1}\sum_{j=(i+1)}^{n}\left(f_+(i)^2 - f_+(j)^2\right)1_{s(i)\sim s(j)} = \sum_{s(i)\sim s(j)}\left(f_+(i)^2 - f_+(j)^2\right).$$

$\square$

Theorem 1.9:

$$\sum_{v} g(v)^2 d_v \leq \sum_{v} (g(v) - g(v_r))^2 d_v$$

Proof:

$$\begin{aligned}\sum_{v} g(v)^2 d_v &= \min_c \sum_{v}\left(g(v)^2 + c^2\right)d_v - 2c\underbrace{\sum_{v} g(v)d_v}_{0} \\ &= \min_c \sum_{v} (g(v) - c)^2 d_v \\ &\leq \sum_{v} (g(v) - g(v_r))^2 d_v\end{aligned}$$

$\square$

Theorem 1.10: $R(g_+) \leq \lambda_G$ or $R(g_-) \leq \lambda_G$

Proof:

$$\lambda_G = \frac{\sum_{u \sim v} \left(g(u) - g(v)\right)^2}{\sum_v g(v)^2 d_v}$$
$$\geq \frac{\sum_{u \sim v} \left(g(u) - g(v)\right)^2}{\sum_v g(v)^2 d_v}$$

- Case $g(v_r) \leq g(u)$ and $g(u) \leq g(v_r)$
  $g_+(u) - g_+(v) = g(u) - g(v_r)$ and $g_-(u) - g_-(v) = 0$
  $\left(g_+(u) - g_+(v)\right)^2 + \left(g_+(u) - g_+(v)\right) = \left(g(v) - g(v)\right)^2$
- Case $g(v_r) > g(u)$ and $g(u) > g(v_r)$
  $g_+(u) - g_+(v) = g(v) - g(v)$ and $g_-(u) - g_-(v) = 0$
  $\left(g_+(u) - g_+(v)\right)^2 + \left(g_+(u) - g_+(v)\right) = \left(g(v) - g(v)\right)^2$
- Case $g(v_r) > g(u)$ and $g(u) \leq g(v_r)$
  TODO
- Case $g(v_r) \leq g(u)$ and $g(u) > g(v_r)$
  TODO

$\square$