1. Hard direction

Let $g : V \to \mathbb{R}$ be an eigenvector achieving λ_G.

- Let $s : \{0, ..., n-1\} \to V$ be a function such that $g \circ s$ is monotonically decreasing. Write v_i for $s(i)$ and $g(i)$ for $g(s(i))$.
- Let $S_i = \{v_j \mid j < i\}$.
- Define $\alpha_G = \min_i h_{S_i}$.
- Let r denote the largest integer such that $\operatorname{vol}(S_r) < \frac{\operatorname{vol}(G)}{2}$, or in other words $\operatorname{vol}(S_r) < \operatorname{vol}(S_r^c)$.
- Define $f(v) := g(v) - g(v_r)$. Denote the positive and negative part of f with f_+ and f_- respectively.

Theorem 1.1: $i < j \Rightarrow f_+(i)^2 \geq f_+(j)^2$

Proof: The only possible troublesome case is $f_+(i) = 0$ and $f_+(j) = f(j)$. In this case we have $g(i) < g(r) \Rightarrow i > r$ and $g(j) \geq g(r) \Rightarrow j \leq r$, a contradiction. □

Theorem 1.2: $R(g_+) \leq \lambda_G$ or $R(g_-) \leq \lambda_G$

Proof:

$$\begin{aligned}
\lambda_G &= \frac{\sum_{u \sim v} (g(u) - g(v))^2}{\sum_v g(v)^2 d_v} \\
&\geq \frac{\sum_{u \sim v} (g(u) - g(v_r) + g(v_r) - g(v))^2}{\sum_v (g(v) - g(v_r))^2 d_v} \\
&= \frac{\sum_{u \sim v} (f(u) - f(v))^2}{\sum_v f(v)^2 d_v} \\
&\geq \frac{\sum_{u \sim v} (f_+(u) - f_+(v))^2 + (f_-(u) - f_-(v))^2}{\sum_v (f_+(v) - f_-(v))^2 d_v} \\
&\geq \frac{\sum_{u \sim v} (f_+(u) - f_+(v))^2 + \sum_{u \sim v} (f_-(u) - f_-(v))^2}{\sum_v f_+(v)^2 d_v + \sum_v f_-(v)^2 d_v}.
\end{aligned}$$

We close using the fact that

$$\frac{a+b}{c+d} \geq \min\left\{\frac{a}{c}, \frac{b}{d}\right\}.$$

□

Theorem 1.3:

$$\sum_v g(v)^2 d_v \leq \sum_v (g(v) - g(v_r))^2 d_v$$

Proof:

$$\begin{aligned}
\sum_v g(v)^2 d_v &= \min_c \sum_v \left(g(v)^2 + c^2\right) d_v - 2c \underbrace{\sum_v g(v) d_v}_{0} \\
&= \min_c \sum_v (g(v) - c)^2 d_v \\
&\leq \sum_v (g(v) - g(v_r))^2 d_v
\end{aligned}$$

□

Theorem 1.4:

$$(f_+(x) - f_+(y))^2 + (f_-(x) - f_-(y))^2 \leq (f(x) - f(y))^2$$

Proof: Done. □

Theorem 1.5: $\frac{\alpha_G^2}{2} \leq R(g_+)$

Proof:

$$
\begin{aligned}
\frac{\alpha_G^2}{2} &= \frac{\alpha_G^2}{2} \frac{\left(\sum_{0\leq i<n} f_+(i)^2 d_i\right)^2}{\left(\sum_{0\leq i<n} f_+(i)^2 d_i\right)^2} \\
&= \frac{\alpha_G^2}{2} \frac{\left(\sum_{0\leq i<n-1} \left(f_+(i)^2 - f_+(i+1)^2\right) \operatorname{vol}(S_i)\right)^2}{\left(\sum_{0\leq i<n} f_+(i)^2 d_i\right)^2} && \text{Theorem 1.7} \\
&= \frac{\alpha_G^2}{2} \frac{\left(\sum_{0\leq i<n-1} \left(f_+(i)^2 - f_+(i+1)^2\right) \min\{\operatorname{vol} S_i, \operatorname{vol} S_i^c\}\right)^2}{\left(\sum_{0\leq i<n} f_+(i)^2 d_i\right)^2} && \text{since} \operatorname{vol} S_i^c \leq \operatorname{vol} S_i \Rightarrow f_+(i) = 0. \\
&\leq \frac{\left(\sum_{0\leq i<n-1} \left(f_+(i)^2 - f_+(i+1)^2\right) |\partial(S_i)|\right)^2}{2\left(\sum_{0\leq i<n} f_+(i)^2 d_i\right)^2} && \text{Theorem 1.8} \\
&\leq \frac{\left(\sum_{i\sim j} f_+(i)^2 - f_+(j)^2\right)^2}{\left(2\sum_{0\leq i<n} f_+(i)^2 d_i\right)^2} && \text{Theorem 1.9} \\
&\leq \frac{\left(\sum_{i\sim j} \left(f_+(i) + f_+(j)\right)^2\right)\left(\sum_{i\sim j} \left(f_+(i) - f_+(j)\right)^2\right)}{\left(\sum_{0\leq i<n} f_+(i)^2 d_i\right)^2} && \text{Theorem 1.10} \\
&\leq \frac{\left(\sum_{i\sim j} \left(f_+(i) + f_+(j)\right)^2\right)\left(\sum_{i\sim j} \left(f_+(i) - f_+(j)\right)^2\right)}{\left(\sum_{0\leq i<n} f_+(i)^2 d_i\right)\left(\sum_{i\sim j} \left(f_+(i) + f_+(j)\right)^2\right)} && \text{Theorem 1.11} \\
&= \frac{\sum_{i\sim j} \left(f_+(i) - f_+(j)\right)^2}{\sum_{i=1}^n f_+(i)^2 d_i} \\
&= \frac{\sum_{u\sim v} \left(g_+(u) - g_+(v)\right)^2}{\sum_v g_+(v)^2 d_v} \\
&= R(g_+).
\end{aligned}
$$

□

Theorem 1.6:

$$\sum_{0 \le i < n} f_+(i)^2 d_i = \sum_{0 \le i < n-1} \left(f_+(i)^2 - f_+(i+1)^2\right) \operatorname{vol}(S_i)$$

Proof: Using https://leanprover-community.github.io/mathlib4_docs/Mathlib/Algebra/BigOperators/Module.html#Finset.sum_range_by_parts

$$\begin{aligned} \sum_{0 \le i < n} f_+(i)^2 d_i &= \cancel{f_+(n-1)^2} \sum_{0 \le i < n} d_i - \sum_{0 \le i < n-1} \left(f_+(i+1)^2 - f_+(i)^2\right) \sum_{0 \le j < i} d_j \\ &= \sum_{0 \le i < n-1} \left(f_+(i)^2 - f_+(i+1)^2\right) \operatorname{vol}(S_i) \end{aligned}$$

□

Theorem 1.7: $\sum_{i=1}^{n} f_+(i)^2 (\operatorname{vol}(S_i) - \operatorname{vol}(S_{i-1})) = \sum_{i=1}^{n-1} \left(f_+(i)^2 - f_+(i+1)^2\right) \operatorname{vol}(S_i)$.

Proof:

$$\begin{aligned} \sum_{i=1}^{n} f_+(i)^2 (\operatorname{vol}(S_i) - \operatorname{vol}(S_{i-1})) &= \sum_{i=1}^{n} f_+(i)^2 \operatorname{vol}(S_i) - \sum_{i=1}^{n} f_+(i)^2 \operatorname{vol}(S_{i-1}) \\ = \sum_{i=1}^{n-1} f_+(i)^2 \operatorname{vol}(S_i) + \cancel{f_+(n)^2 \operatorname{vol}(S_n)} &- \sum_{i=1}^{n-1} f_+(i+1)^2 \operatorname{vol}(S_i) - \cancel{f_+(1)^2 \operatorname{vol}(S_0)} \\ &= \sum_{i=1}^{n-1} \left(f_+(i)^2 - f_+(i+1)^2\right) \operatorname{vol}(S_i). \end{aligned}$$

□

Theorem 1.8:

$$\alpha_G \min\{\operatorname{vol}(S_i), \operatorname{vol}(S_i^c)\} \le |\partial(S_i)|$$

Proof:

$$\begin{aligned} &\min_j \frac{|\partial(S_j)|}{\min\left\{\operatorname{vol}(S_j), \operatorname{vol}\left(S_j^c\right)\right\}} \cdot \min\{\operatorname{vol}(S_i), \operatorname{vol}(S_i^c)\} \\ \le &\frac{|\partial(S_i)|}{\min\{\operatorname{vol}(S_i), \operatorname{vol}(S_i^c)\}} \cdot \min\{\operatorname{vol}(S_i), \operatorname{vol}(S_i^c)\} = |\partial(S_i)|. \end{aligned}$$

□

Theorem 1.9:

$$\sum_{i=1}^{n}\left(f_+(i)^2 - f_+(i+1)^2\right)\partial(S_i) \le \sum_{i\sim j} f_+(i)^2 - f_+(j)^2$$

Proof:

$$\begin{aligned}
\sum_{i=1}^{n}\left(f_+(i)^2 - f_+(i+1)^2\right)\partial(S_i) &= \sum_{i=1}^{n}\left(f_+(i)^2 - f_+(i+1)^2\right)\left(\sum_{j=(i+1)}^{n} 1_{i\sim j}\right) \\
&= \sum_{i=1}^{n-1}\sum_{j=(i+1)}^{n}\left(f_+(i)^2 - f_+(i+1)^2\right)1_{i\sim j} \\
&\le \sum_{i=1}^{n-1}\sum_{j=(i+1)}^{n}\left(f_+(i)^2 - f_+(j)^2\right)1_{i\sim j} && j > i \Rightarrow f_+(j)^2 \le f_+(i+1)^2 \\
&= \sum_{i\sim j}\left(f_+(i)^2 - f_+(j)^2\right).
\end{aligned}$$

□

Theorem 1.10:

$$\left(\sum_{i\sim j} f_+(i)^2 - f_+(j)^2\right)^2 \le 2\left(\sum_{i\sim j}\left(f_+(i) + f_+(j)\right)^2\right)\left(\sum_{i\sim j}\left(f_+(i) - f_+(j)\right)^2\right)$$

Proof: Consider the matrices $X(i,j) = 1_{i\sim j}(f_+(i) + f_+(j))$ and $Y(i,j) = 1_{i\sim j, i<j}(f_+(i) - f(j))$. Their Frobenius inner product is

$$\langle X, Y\rangle = \sum_{i,j} X^T(i,j)Y(i,j)$$

$$= \sum_{i,j} 1_{i\sim j, i<j}(f_+(i) + f_+(v))(f_+(i) - f_+(j))$$

$$= \sum_{i\sim j} f_+(i)^2 - f_+(j)^2.$$

The norms of X and Y are

$$\begin{aligned}
\|X\|_F^2 &= \sum_{i,j} X(i,j)^2 = \sum_{i,j} 1_{i\sim j}(f_+(i) + f_+(j))^2 = 2\sum_{i\sim j}(f_+(i) + f_+(j))^2, \\
\|Y\|_F^2 &= \sum_{i,j} Y(i,j)^2 = \sum_{i,j} 1_{i\sim j, i<j}(f_+(i) - f_+(j))^2 = \sum_{i\sim j}(f_+(i) - f_+(j))^2.
\end{aligned}$$

By the Cauchy-Schwarz inequality we have

$$\langle X, Y\rangle^2 \le \|X\|_F^2\|Y\|_F^2,$$

and the theorem follows. □

Theorem 1.11:

$$\sum_{i \sim j} \left(f_+(i) + f_+(j)\right)^2 \leq \sum_i f_+(i)^2 d_i$$

Proof:

$$\begin{aligned}
\sum_{i \sim j} \left(g_+(i) + g_+(j)\right)^2 &\leq \sum_{i \sim j} g_+(i)^2 + g_+(j)^2 \\
&= \sum_{i \sim j} g_+(i)^2 + \sum_{i \sim j} g_+(j)^2 \\
&= 2 \sum_{i \sim j} g_+(i)^2 \\
&= 2 \cdot \frac{1}{2} \sum_i g_+(i)^2 \sum_j 1_{i \sim j} = \sum_i g_+(i)^2 d_i.
\end{aligned}$$

□