

# 1. Hard direction

Let  $g : V \rightarrow \mathbb{R}$  be an eigenvector achieving  $\lambda_G$ .

- Let  $s : \{1, \dots, n\} \rightarrow V$  be a function such that  $g \circ s$  is monotonically decreasing. Write  $v_i$  for  $s(i)$  and  $g(i)$  for  $g(s(i))$ .
- Let  $S_i = \{v_1, \dots, v_i\}$ .
- Define  $\alpha_G = \min_i h_{S_i}$ .
- Let  $r$  denote the largest integer such that  $\text{vol}(S_r) < \frac{\text{vol}(G)}{2}$ , or in other words  $\text{vol}(S_r) < \text{vol}(S_r^c)$ .
- Define  $f(v) := g(v) - g(v_r)$ . Denote the positive and negative part of  $f$  with  $f_+$  and  $f_-$  respectively.

**Theorem 1.1:**  $i < j \Rightarrow f_+(i)^2 \geq f_+(j)^2$

*Proof:* The only possible troublesome case is  $f_+(i) = 0$  and  $f_+(j) = f(j)$ . In this case we have  $g(i) < g(r) \Rightarrow i > r$  and  $g(j) \geq g(r) \Rightarrow j \leq r$ , a contradiction.  $\square$

**Theorem 1.2:**  $R(g_+) \leq \lambda_G$  or  $R(g_-) \leq \lambda_G$

*Proof:*

$$\begin{aligned}
 \lambda_G &= \frac{\sum_{u \sim v} (g(u) - g(v))^2}{\sum_v g(v)^2 d_v} \\
 &\geq \frac{\sum_{u \sim v} (g(u) - g(v_r) + g(v_r) - g(v))^2}{\sum_v (g(v) - g(v_r))^2 d_v} \\
 &= \frac{\sum_{u \sim v} (f(u) - f(v))^2}{\sum_v f(v)^2 d_v} \\
 &\geq \frac{\sum_{u \sim v} (f_+(u) - f_+(v))^2 + (f_-(u) - f_-(v))^2}{\sum_v (f_+(v) - f_-(v))^2 d_v} \\
 &\geq \frac{\sum_{u \sim v} (f_+(u) - f_+(v))^2 + \sum_{u \sim v} (f_-(u) - f_-(v))^2}{\sum_v f_+(v)^2 d_v + \sum_v f_-(v)^2 d_v}.
 \end{aligned}$$

We close using the fact that

$$\frac{a+b}{c+d} \geq \min \left\{ \frac{a}{c}, \frac{b}{d} \right\}.$$

$\square$

**Theorem 1.3:**

$$\sum_v g(v)^2 d_v \leq \sum_v (g(v) - g(v_r))^2 d_v$$

*Proof:*

$$\begin{aligned}
 \sum_v g(v)^2 d_v &= \min_c \sum_v (g(v)^2 + c^2) d_v - \underbrace{2c \sum_v g(v) d_v}_0 \\
 &= \min_c \sum_v (g(v) - c)^2 d_v \\
 &\leq \sum_v (g(v) - g(v_r))^2 d_v
 \end{aligned}$$

□

**Theorem 1.4:**

$$(f_+(x) - f_+(y))^2 + (f_-(x) - f_-(y))^2 \leq (f(x) - f(y))^2$$

*Proof:* Done.

□

**Theorem 1.5:**  $\frac{\alpha_G^2}{2} \leq R(g_+)$

*Proof:*

$$\begin{aligned}
\frac{\alpha_G^2}{2} &= \frac{\alpha_G^2}{2} \frac{\left(\sum_{i=1}^n f_+(i)^2 d_i\right)^2}{\left(\sum_{i=1}^n f_+(i)^2 d_i\right)^2} \\
&\leq \frac{\alpha_G^2}{2} \frac{\left(\sum_{i=1}^n f_+(i)^2 (\text{vol}(S_i) - \text{vol}(S_{i-1}))\right)^2}{\left(\sum_{i=1}^n f_+(i)^2 d_i\right)^2} \\
&= \frac{\alpha_G^2}{2} \frac{\left(\sum_{i=1}^{n-1} (f_+(i)^2 - f_+(i+1)^2) \text{vol}(S_i)\right)^2}{\left(\sum_{i=1}^n f_+(i)^2 d_i\right)^2} && \text{Theorem 1.6} \\
&= \frac{\alpha_G^2}{2} \frac{\left(\sum_{i=1}^{n-1} (f_+(i)^2 - f_+(i+1)^2) \min\{\text{vol } S_i, \text{vol } S_i^c\}\right)^2}{\left(\sum_{i=1}^n f_+(i)^2 d_i\right)^2} \text{ since } \text{vol } S_i^c \leq \text{vol } S_i \Rightarrow f_+(i) = 0. \\
&\leq \frac{\left(\sum_{i=1}^{n-1} (f_+(i)^2 - f_+(i+1)^2) |\partial(S_i)|\right)^2}{2 \left(\sum_{i=1}^n f_+(i)^2 d_i\right)^2} && \text{Theorem 1.7} \\
&\leq \frac{\left(\sum_{i \sim j} f_+(i)^2 - f_+(j)^2\right)^2}{\left(2 \sum_{i=1}^n f_+(i)^2 d_i\right)^2} && \text{Theorem 1.8} \\
&\leq \frac{\left(\sum_{i \sim j} (f_+(i) + f_+(j))^2\right) \left(\sum_{i \sim j} (f_+(i) - f_+(j))^2\right)}{\left(\sum_{i=1}^n f_+(i)^2 d_i\right)^2} && \text{Theorem 1.9} \\
&\leq \frac{\left(\sum_{i \sim j} (f_+(i) + f_+(j))^2\right) \left(\sum_{i \sim j} (f_+(i) - f_+(j))^2\right)}{\left(\sum_{i=1}^n f_+(i)^2 d_i\right) \left(\sum_{i \sim j} (f_+(i) + f_+(j))^2\right)} && \text{Theorem 1.10} \\
&= \frac{\sum_{i \sim j} (f_+(i) - f_+(j))^2}{\sum_{i=1}^n f_+(i)^2 d_i} \\
&= \frac{\sum_{u \sim v} (g_+(u) - g_+(v))^2}{\sum_v g_+(v)^2 d_v} \\
&= R(g_+).
\end{aligned}$$

□

**Theorem 1.6:**  $\sum_{i=1}^n f_+(i)^2 (\text{vol}(S_i) - \text{vol}(S_{i-1})) = \sum_{i=1}^{n-1} (f_+(i)^2 - f_+(i+1)^2) \text{vol}(S_i).$

*Proof:*

$$\begin{aligned}
& \sum_{i=1}^n f_+(i)^2 (\text{vol}(S_i) - \text{vol}(S_{i-1})) = \sum_{i=1}^n f_+(i)^2 \text{vol}(S_i) - \sum_{i=1}^n f_+(i)^2 \text{vol}(S_{i-1}) \\
& = \sum_{i=1}^{n-1} f_+(i)^2 \text{vol}(S_i) + \cancel{f_+(n)^2 \text{vol}(S_n)} - \sum_{i=1}^{n-1} f_+(i+1)^2 \text{vol}(S_i) - \cancel{f_+(1)^2 \text{vol}(S_0)} \\
& = \sum_{i=1}^{n-1} (f_+(i)^2 - f_+(i+1)^2) \text{vol}(S_i).
\end{aligned}$$

□

**Theorem 1.7:**

$$\alpha_G \min\{\text{vol}(S_i), \text{vol}(S_i^c)\} \leq |\partial(S_i)|$$

*Proof:*

$$\begin{aligned}
& \min_j \frac{|\partial(S_j)|}{\min\{\text{vol}(S_j), \text{vol}(S_j^c)\}} \cdot \min\{\text{vol}(S_i), \text{vol}(S_i^c)\} \\
& \leq \frac{|\partial(S_i)|}{\min\{\text{vol}(S_i), \text{vol}(S_i^c)\}} \cdot \min\{\text{vol}(S_i), \text{vol}(S_i^c)\} = |\partial(S_i)|.
\end{aligned}$$

□

**Theorem 1.8:**

$$\sum_{i=1}^n (f_+(i)^2 - f_+(i+1)^2) \partial(S_i) \leq \sum_{i \sim j} f_+(i)^2 - f_+(j)^2$$

*Proof:*

$$\begin{aligned}
\sum_{i=1}^n (f_+(i)^2 - f_+(i+1)^2) \partial(S_i) & = \sum_{i=1}^n (f_+(i)^2 - f_+(i+1)^2) \left( \sum_{j=(i+1)}^n 1_{i \sim j} \right) \\
& = \sum_{i=1}^{n-1} \sum_{j=(i+1)}^n (f_+(i)^2 - f_+(i+1)^2) 1_{i \sim j} \\
& \leq \sum_{i=1}^{n-1} \sum_{j=(i+1)}^n (f_+(i)^2 - f_+(j)^2) 1_{i \sim j} \quad j > i \Rightarrow f_+(i)^2 - f_+(j)^2 \geq 0 \\
& = \sum_{i \sim j} (f_+(i)^2 - f_+(j)^2).
\end{aligned}$$

□

**Theorem 1.9:**

$$\left( \sum_{i \sim j} f_+(i)^2 - f_+(j)^2 \right)^2 \leq 2 \left( \sum_{i \sim j} (f_+(i) + f_+(j))^2 \right) \left( \sum_{i \sim j} (f_+(i) - f_+(j))^2 \right)$$

*Proof:* Consider the matrices  $X(i, j) = 1_{i \sim j}(f_+(i) + f_+(j))$  and  $Y(i, j) = 1_{i \sim j, i < j}(f_+(i) - f_+(j))$ . Their Frobenius inner product is

$$\begin{aligned} \langle X, Y \rangle &= \sum_{i, j} X^T(i, j) Y(i, j) \\ &= \sum_{i, j} 1_{i \sim j, i < j} (f_+(i) + f_+(j)) (f_+(i) - f_+(j)) \\ &= \sum_{i \sim j} f_+(i)^2 - f_+(j)^2. \end{aligned}$$

The norms of  $X$  and  $Y$  are

$$\begin{aligned} \|X\|_F^2 &= \sum_{i, j} X(i, j)^2 = \sum_{i, j} 1_{i \sim j} (f_+(i) + f_+(j))^2 = 2 \sum_{i \sim j} (f_+(i) + f_+(j))^2, \\ \|Y\|_F^2 &= \sum_{i, j} Y(i, j)^2 = \sum_{i, j} 1_{i \sim j, i < j} (f_+(i) - f_+(j))^2 = \sum_{i \sim j} (f_+(i) - f_+(j))^2. \end{aligned}$$

By the Cauchy-Schwarz inequality we have

$$\langle X, Y \rangle^2 \leq \|X\|_F^2 \|Y\|_F^2,$$

and the theorem follows. □

**Theorem 1.10:**

$$\sum_{i \sim j} (f_+(i) + f_+(j))^2 \leq \sum_i f_+(i)^2 d_i$$

*Proof:*

$$\begin{aligned}
& \sum_{i \sim j} (g_+(i) + g_+(j))^2 \\
& \leq \sum_{i \sim j} g_+(i)^2 + g_+(j)^2 \\
& = \sum_{i \sim j} g_+(i)^2 + \sum_{i \sim j} g_+(j)^2 \\
& \quad = 2 \sum_{i \sim j} g_+(i)^2 \\
& = 2 \cdot \frac{1}{2} \sum_i g_+(i)^2 \sum_j 1_{i \sim j} \\
& \quad = \sum_i g_+(i)^2 d_i.
\end{aligned}$$

□