The (Small) Extended Handbook for Undergraduate Math

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1 Disclaimer

1. This handbook contains a wide range of formulas spanning from college math. It's meant to serve as an aid when you practice problem-solving, as well as when you try to speed up the pace when doing so. Memorizing these formulas is **not enough** to improve your performance in a math test; rather, becoming adept at employing them through extensive practice such as doing past papers or going over textbook exercises would. As an example of such, consider the following formula which denotes the solution to a quadratic equation: Given ax^2+bx+c, then $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$. It's not guaranteed that by simply memorizing it would help you develop the creativity necessary to solve the quadratic equation $\log^2 x - 5\log x + 8 = 0$

2. A bit contrary to advice 1, some formulas, or definitions, are worth memorizing in order to help you speed up the pace in which you solve math problems under testing conditions. Examples of such include trigonometric identities, definitions of sets and derivatives. It's hoped that through the aid of this handbook, you can identify which concepts you need most reviewing and further polishing; for instance, if you're adroit in the topic of algebra and functions, but seem to struggle on

the topic of abstract topology, then mostly focus on finding problems to solve in the latter topic rather than partition your studying schedule evenly for all topics appearing in this handbook.

2 Pre-calculus

2.1 Functions

- A **composition** of function: $(g \circ f)(x) = g(f(x))$. This is not necessarily the same as $(f \circ g)(x)$
- An **inverse** function: $f^{-1}(y) = x \iff f(x) = y$. Therefore $f^-1(f(x)) = x$. Not to confuse with reciprocal $\frac{1}{f(x)}$.

2.2 Analytical Geometry

- A ***line*** is of the form: $ax+by+c = 0 \vee y - y_1 = m(x - x_1) \vee m = \frac{\Delta y}{\Delta x}$
- A ***parabola*** is of the form: $y = \pm\frac{1}{4p}x^2$ (for cup shaped parabola) or $x = \pm\frac{1}{4p}y^2$ for horizontal parabola
  - $(0, \pm p) \vee (\pm p, 0)$ are the focuses of the parabola
  - $y = \pm p \vee x = \pm p$ are the directrices content...
- A ***circle*** is of the form: $(x - h)^2 + (y - k)^2 = r^2$
  - (h, k) is the center of the circle
  - r is the radius
- An ***ellipse*** is of the form: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
  - the *foci* are two fixed points in the ellipse which the sum of the distances from every point on the ellipse to them is a constant.
  - the *eccentricity* measures the flatness of the ellipse

    - If $a > b$:
        - foci:$(\pm c, 0), c = \sqrt{a^2 - b^2}$
        - vertices:$(\pm a, 0)$
        - major axis length: $2a$
        - minor axis length: $2b$
        - eccentricity: $e = \frac{c}{a}$
    - If $a < b$:
        - foci:$(0, \pm c), c = \sqrt{b^2 - a^2}$
        - vertices:$(0, \pm b)$
        - major axis length: $2b$
        - minor axis length: $2a$
        - eccentricity: $e = \frac{c}{b}$

- A ***hyperbola*** is of the form: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ or $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$.
  - the *foci* are two fixed points in which the difference of the distances from every point on the hyperbola to them is a constant.
  - a point that the hyperbola will never touch is an *asymptote*
    - When $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$:
        - foci: $(\pm c, 0), c = \sqrt{a^2 + b^2}$
        - vertices: $(\pm a, 0)$
        - asymptotes: $y = \pm \frac{b}{a} x$
    - When $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$:
        - foci: $(0, \pm c), c = \sqrt{a^2 + b^2}$
        - vertices: $(0, \pm b)$
        - asymptotes: $y = \pm \frac{b}{a} x$

2.3 Polynomial Equations

- Given $ax^2 + bx + c$, then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, where:

- $\Delta > 0$ means two solutions
- $\Delta = 0$ means one solution
- $\Delta < 0$ means two complex roots

- The ***division algorithm*** guarantees that dividing polynomial function p(x) by d(x) yield unique polynomials, a quotient q(x) and a remainder r(x) following the form: $p(x) = q(x) * d(x) + r(x)$.
  - The degree of r(x) is either zero or is less than the degree of d(x).

- The ***Remainder Theorem*** states that if a polynomial *p(x)* is divided by a linear function $d(x) = x - k$, the remainder $r(x) = r = p(k)$, therefore $p(x) = q(x) * (x - k) + p(k)$.
  - d(x) = (x - k) becomes a divisor of p(x) when p(k) = 0

- The ***Factor Theorem*** follows that $x = k$ is a root of the polynomial *p(x) = 0* iff *x - k* is a factor of *p(x)*

- The ***Fundamental Theorem of Algebra*** states that any real polynomial of degree n is as follows:
  $p(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_o$ where $n \geq 1$
  Where $p(x) = 0$ has exactly n roots; $p(x)$ can also be rewritten as a product of unique, linear factors:
  $p(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_o = a_n(x - k_1)(x - k_2)...(x - k_n))$, where $x = k_i$ is a root.

- The ***Rational Roots Theorem*** states that for a real polynomial
  $p(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_o$
  If all coefficients are integers, then if $p(x) = 0$ has any rational roots r, they're among $r = \frac{s}{t}$, where s is a factor of a_0 and t is a factor of a_n

- The ***Conjugate Radical Roots Theorem*** states that for a real polynomial

$p(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_o$
If all coefficients are rational, then if $p(x) = 0$ has a rational root of the form $r = s + t\sqrt{u}$, where $\sqrt{u}$ is an irrational number, then there must be another root of the form $r' = s - t\sqrt{u}$, called the radical conjugate of r.

- The ***Complex Conjugate Roots Theorem*** states that for a real polynomial
$p(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_o$
If all coefficients are real, then if $p(x) = 0$ has a rational root of the form $r = s + ti$, where i is the imaginary number, then there must be another root of the form $r' = s - ti$, called the complex conjugate of r.

- The ***Sum and Product of the Roots*** states that for a real polynomial:
$p(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_o (a_n \neq 0)$
The sum of its n roots is: $\sum_1^n r = -\frac{a_{n-1}}{a_n}$
The product of the roots is: $\Pi_1^n r = (-1)^n \frac{a_0}{a_n}$

2.4 Logarithms

- $\log_a x = y \leftrightarrow a^y = x$

- $a^{log_a x} = x$

- If $f(x) = \log_a x$, then $f^-1(x) = a^x$

- $\log_a(x_1 x_2) = \log_a x_1 + \log_a x_2$

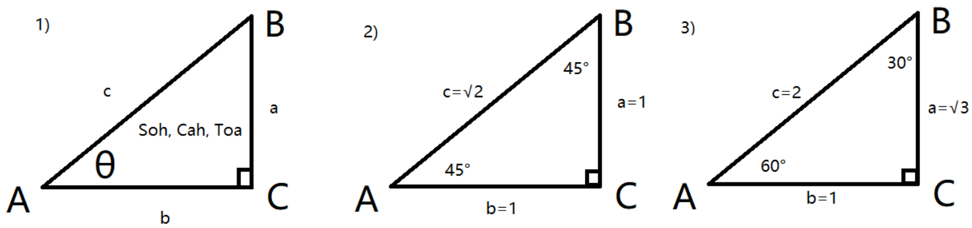
- $\log_a(\frac{x_1}{x_2}) = \log_a x_1 - \log_a x_2$

- $\log_a x^b = b \log_a x$

- $\log_a b = \frac{\log_a x}{\log_b x}(x \neq 0 \leftrightarrow b \neq 1) = \frac{log_x b}{log_x a}$

2.5 Trigonometry

- Given a 1) triangle $\triangle ABC$, let's denote angle $\angle BAC$ as θ, then:



  - $\sin\theta = \frac{a}{c}$
  - $\cos\theta = \frac{b}{c}$
  - $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{a}{b}$
  - $\csc\theta = \frac{1}{\sin\theta} = \frac{c}{a}$
  - $\sec\theta = \frac{1}{\cos\theta} = \frac{c}{b}$
  - $\cot\theta = \frac{1}{\tan\theta} = \frac{b}{a}$

  Where 2) and 3) are special cases where the exact ratios are known.

- Unit circle helps determine the value for angles $90°, 180°, 270°$ *and* $360°$.

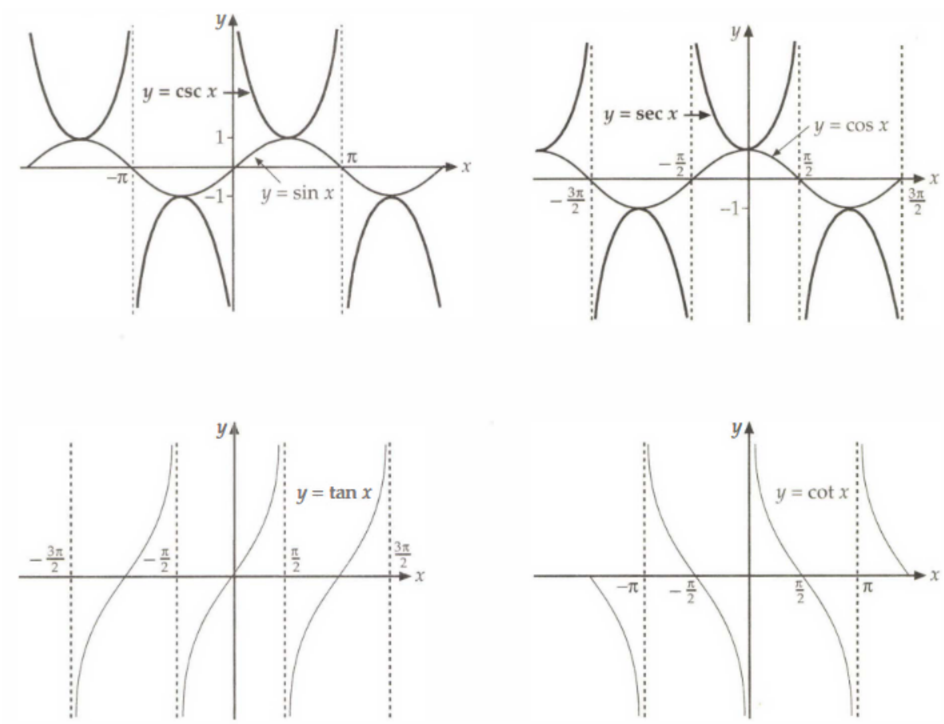
- $\frac{\pi}{180°}$

- Opposite-Angle identities:
  - $\sin(-\theta) = -\sin\theta$
  - $\cos(-\theta) = \cos\theta$

- Pythagorean Identities:
  - $\sin^2\theta + \cos^2\theta = 1$
  - $1 + \tan^2\theta = \sec^2\theta$
  - $1 + \cot^2\theta = \csc^2\theta$

- Addition and Subtraction of angles:
  - $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$ (if left is +, then right is +)

- $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \pm \sin\alpha\sin\beta$ (if left is +, then right is -)
- $\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \pm \tan\alpha\tan\beta}$ (if left is +, then right is $\pm$, else if left is -, then right is $\mp$)
- $\sin 2\theta = 2\sin\theta\cos\theta$
- $\cos 2\theta = \cos^2\theta - \sin^2\theta$
- $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

* $\sin(\frac{\pi}{2} - \theta) = \cos\theta \vee \cos(\frac{\pi}{2} - \theta) = \sin\theta$
* $\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$
* $\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$
* $\tan\frac{\theta}{2} = \pm\frac{\sin\theta}{1+\cos\theta}$
* 

  Source: Review (2016)

Function	Domain	Range
$\arcsin\theta$	$\lvert\theta\rvert \leq 1$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
$\arccos\theta$	$\lvert\theta\rvert \leq 1$	$[0, \pi$
$\arctan\theta$	$\forall\theta$	$(-\frac{\pi}{2}, \frac{\pi}{2})$
$arccsc\ \theta$	$\lvert\theta\rvert \geq 1$	$[-\frac{\pi}{2}, \frac{\pi}{2}]\ except\ 0$
$arcsec\ \theta$	$\lvert\theta\rvert \geq 1$	$[0, \pi]\ except\ \frac{\pi}{2}$
$arccot\ \theta$	$\forall\theta$	$(0, \pi)$

2.6 Limits

- $\lim_{x\to+\infty} a_x = A \coloneqq \forall\varepsilon > 0\ \exists N \ \epsilon\ \mathbb{N}.\ \forall n > N,\ \ |a_x - A| < \varepsilon$
  - Given a sequence of numbers (a_x), as the term at the x^{th} position approaches to infinity, it converges to a value A.
  - This is read as "the limit of x approaching to ∞ for the x^{th} value of a series being equal to A is defined as for all epsilon greater than zero, there exists some natural number such that for all n greater than N, the absolute difference between $a_x - A$ is less than epsilon.
- $(a_x) : \lim_{x\to+\infty}$
- $\lim_{x\to a} a = a$
- $\lim_{x\to a} x = a$
- $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
- $\lim_{x\to a}[f(x) * g(x)] = \lim_{x\to a} f(x) * \lim_{x\to a} g(x)$
- $\lim_{x\to a} \frac{f(x))}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, where $\lim_{x\to a} g(x) \neq 0$
- $\lim_{x\to a} kf(x) = k * \lim_{x\to a} f(x)$, where k is a constant
- $\lim_{x\to a}(f(x)^{p/q}) = (\lim_{x\to a} f(x))^{p/q}, where\ \lim_{x\to a} f(x) > 0\ if\ p/q\ is\ even$

- $\lim_{x\to a} f(g(x)) = f(L),\ where\ L = \lim_{x\to a} g(x)$. Here, I have to know $\lim_{x\to L} f(x)$

- $\lim_{x\to+\infty}(1+\frac{1}{x})^x = \lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$

- $\lim_{x\to 0}[\ln(1+x)^{\frac{1}{x}}] = \lim_{x\to 0}\frac{\ln(1+x)}{x} = \lim_{x\to 0}\frac{e^x-1}{x} = 1$

- $\lim_{x\to 0}\frac{a^x-1}{x} = \ln a$

- $\lim_{x\to+\infty}\frac{1}{x^n} = 0$

- If $\sum_1^\infty f(x) = L \wedge g(x) < f(x),\ then\ \sum_1^\infty g(x) = L'(converges)$

- If $\sum_1^\infty f(x) = \infty \wedge g(x) > f(x),\ then\ \sum_1^\infty g(x) = \infty(diverges)$

- If a sequence is both monotonic and bounded, then it's **convergent**. In a *monotonic* function every subsequent term is greater or lower than the previous one. A *bounded* function is one in which there exists some a_x such that all values are not greater than $a_x \wedge a_x \neq \infty$. A **divergent** function stretches to infinity. $x_n = (-1)^n$ is neither convergent nor divergent.

- Some rules of convergence:

  - Let (a_x) be a series of numbers $(a_1, a_2, ...)$. If (a_x) *and* (b_x) converge to A and B respectively, then:

    * $a_x \pm b_x \to A \pm B$
    * $a_x * b_x \to AB$
    * $\frac{a_x}{b_x} \to \frac{A}{B}, B \neq 0$
    * If (a_x) converges to L, then $k(a_n) = (k * a_n) \to kL$
    * If $k > 0$, $(\frac{1}{x^k}) \to 0$
    * If $|k| > 1, (\frac{1}{k^x}) \to 0$

* **Sandwich theorem**: If $(a_n) \wedge (c_n)$ converge to the same limit L and $\forall x, a_x \leq b_x \leq c_x$ for every $x > N$, then b_x *converges* to L
* **Sandwich theorem for limit function**: Let $\lim_{x\to a} f(x) = L \wedge \lim_{x\to a} h(x) = L \wedge \exists \delta . \forall x \ f(x) \leq g(x) \leq h(x)$. $0 < |x-a| < \delta$, then $\lim_{x\to a} g(x) = L$
* $\int_{-\infty}^{\infty} f(x)\, dx = \int_{c}^{\infty} f(x)\, dx + \int_{-\infty}^{c} f(x)\, dx$, if both $\int_{c}^{\infty} f(x)\, dx \wedge \int_{-\infty}^{c} f(x)\, dx$ converge.
* If $f(n) = a_n$, then (a_n) converges to L if $\lim_{x\to+\infty} f(x) = L$ (may use L'Hopital's rule)

- Some tests for convergence. Assume (a_x) is made of non-negative terms:
  - Ratio test: $\lim_{x\to+\infty} \frac{a_{x+1}}{a_x} = L$. If
    $L < 1 \Rightarrow \sum a \ converges$
    $L > 1 \Rightarrow \sum a \ diverges$
    $L = 1$ don't know
  - Root test: $\lim_{x\to+\infty} \sqrt[x]{a^x} = L$. If
    $L < 1 \Rightarrow \sum a \ converges$
    $L > 1 \Rightarrow \sum a \ diverges$
    $L = 1$ don't know
  - $\sum_1^{\infty} a_x \ converges \ \leftrightarrow \int_1^{\infty} f(x) dx \ converges$
  - Assume $\sum_1^{\infty} a_x$ has infinite negative terms. If $\sum_1^{\infty} |a_n|$ converges, then $\sum_1^{\infty} a_x$ is *absolutely convergent*. Otherwise we don't know much else.

2.7 Continuous functions

- A functions is *continuous at a* if $\lim_{x\to a} f(x) = f(a)$:
  - f(a) exists

- $\lim_{x \to a} f(x)$ exists
- The above holds $\forall x \; in \; f(x)$

- If f and g are continuous at a, so is $f \pm g, f * g, \frac{f}{g}[g(a) \neq 0)]$.

- If f is continuous at a, and g is continuous at f(a), then f(g(x)) is continuous at x = a.

- **Extreme value theorem:**
  - If f(x) is a function and is continuous in $[a, b]$.
  - Then $\exists c \exists d. \; a \leq c, b \geq d$ such that f(c) is an absolute minimum and f(d) is an absolute maximum **of this interval only**

- **Bolzano's theorem**:
  - If f(x) is a function and is continuous in $[a, b]$.
  - If $f(a) \wedge f(b)$ have opposite signs.
  - Then $\exists c f(c) = 0$, c is a critical point.

- **Brower's fixed point theorem**:
  - If f(x) is a function and is continuous in $[0, 1]$.
  - $\forall x \; \epsilon \; [0, 1]. \; 0 \leq f(x) \leq 1$
  - Then $\exists c \; \epsilon \; [0, 1]. \;\; f(c) = c$

- **Intermediate value theorem:**
  - If f(x) is a function and is continuous in $[a, b]$.
  - $\exists c \int_a^b f(x) \; dx = f(c) f(b - a)$

3 Calculus

3.1 Derivation formulas

- $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \Rightarrow h = \Delta x$
- $\frac{d}{dx}\frac{g(x)}{h(x)} = \frac{\frac{d}{dx}[g(x)]h(x) - g(x)\frac{d}{dx}[h(x)]}{h(x)^2}$
- $\frac{d}{dx} f(x) * g(x) = f'(x) * g(x) + f(x) * g'(x)$
- $\frac{d}{dx} f(g(x)) = f'(g(x)) * g'(x)$
- $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$
- $\frac{d}{dx} a = 1$
- $\frac{d}{dx} ax = a$
- $\frac{d}{dx} cf(x) = c * f'(x)$
- $\frac{d}{dx}|x| = \begin{cases} 1 & \text{if} \quad x \geq 0 \\ -1 & \text{if} \quad x < 0 \end{cases}$
- $\frac{d}{dx}\log_a x = \frac{1}{x \log_a e} \Rightarrow \; if \; a = e, \frac{d}{dx}\ln x = \frac{1}{x}$
- $\frac{d}{dx} a^x = a^x \ln(a)$
- $\frac{d}{dx}\frac{1}{x} = -\frac{1}{x^2}$
- $\frac{d}{dx} e^x = e^x$
- $\frac{d}{dx}\sin x = \cos x$
- $\frac{d}{dx}\cos x = -\sin x$
- $\frac{d}{dx}\tan x = \sec^2 x = \frac{1}{\cos^2 x}$
- $\frac{d}{dx}\sec x = \sec x \tan x = \sin x \sec^2 x$
- $\frac{d}{dx}\cot x = -\csc^2 x$

- $\frac{d}{dx} \csc x = -\csc x \cot x$

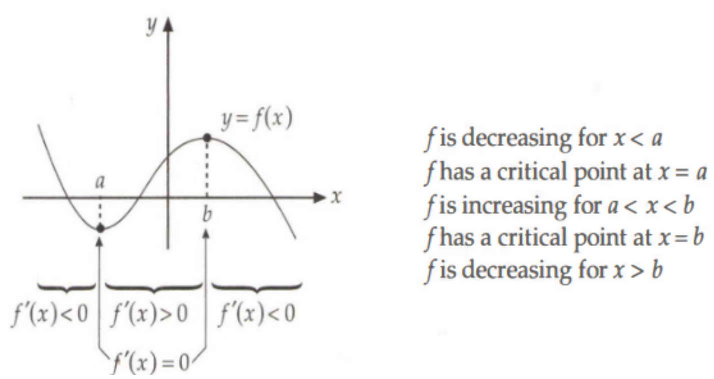
- $\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$

- $\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}$

- $\frac{d}{dx} \arccos x = \frac{-1}{\sqrt{1-x^2}}$

- Approximate a value: $y = f(a) + f'(a)(x - a)$

- The first derivative tells us:



- For the second derivative:
  - If $f''(x) > 0$, then the curve is convex, lies above the tangent line.
  - If $f''(x) < 0$, then the curve is concave and lies below the tangent line.
  - If $f''(x) = c$ and c is an inflection point, then it's convex at one side, and concave at the other.

- If $f'(x) = 0 \wedge f''(x) > 0, then\ f(x)$ is local minima.

- If $f'(x) = 0 \wedge f''(x) < 0, then\ f(x)$ is a local maxima

- **Rolle's Theorem**: If f is continuous and differentiable at every point on a closed interval [a, b] with $f(a) = f(b)$, then $\exists c, c \epsilon (a, b), f'(c) = 0$

- **Mean-Value theorem for derivatives**: Assume f is continuous and differentiable at every point on a closed interval [a, b], then $\exists c, c\,\epsilon\,(a, b), f'(c) = \frac{f(b)-f(a)}{b-a}$

- **Partial derivatives:** For a function $f(w, x, y, z, ..) = c$, its partial derivative w.r.t a variable, say x, is calculated by holding all other variables as constants.

- **Higher order partial derivatives:** Given a function $f(x, y)$, its derivative with respect to x in Leibniz's notation is written as $\frac{df}{dx}$. If we want its second derivative with respect to y, then it is $\frac{d^2f}{dydx}$ (order is from right to left). For almost all encountered functions $\frac{d^2f}{dydx} = \frac{d^2f}{dxdy}$

- Given a surface $z = f(x, y)$, the equation of the tangent plane to the surface at $P = (x_0, y_0, z_0)$ is $z - z_0 = \frac{df}{dx}|_P * (x - x_0) + \frac{df}{dy}|_P * (y - y_0)$

- If the surface is $f(x, y, z) = c$ where z can't be expressed in terms of x and y, then equation of tangent plane to it is at $P = (x_0, y_0, z_0)$ is $z - z_0 = \frac{dz}{dx}|_P * (x - x_0) + \frac{dz}{dy}|_P * (y - y_0)$ and find partial derivatives through implicit differentiation (or find implicit derivatives using partial differentiation).

- $f(x, y, z) = c \neq z = f(x, y)$

- Linear Approximations $z_1 \approx z_0 + f_x|_P \cdot (x - x_0) + f_y|_P \cdot (y - y_0)$

- The directional derivative or gradient of $z = f(x, y)$ at point $P = (x_0, y_0)$ in the direction of $\vec{u}$, a vector whose initial point is P is: $D_{\vec{u}}f|_P = \nabla f|_P * \hat{\vec{u}}$, where $\nabla f = \frac{df}{dx}\vec{i} + \frac{df}{dy}\vec{j} + \frac{df}{dz}\vec{k}$ and $\hat{\vec{u}}$ is the unit vector

- Given P_0, it is a critical point of $f(x, y)$ if: $(\frac{df}{dx})_{P_0} = 0 \wedge (\frac{df}{dy})_{P_0} = 0$.

  - The **Hessian** of $f(x, y)$ is $\Delta = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}_{P_0} = f_{xx}(P_0)f_{yy}(P_0) - [f_{xy}(P_0)]^2$

    * If $\Delta > 0$ and $f_{xx}(P_0) < 0$, then f attains a local maximum at P_0.
    * If $\Delta > 0$ and $f_{xx}(P_0) > 0$, then f attains a local minimum at P_0.
    * If $\Delta < 0$, then f attains a saddle point at P_0.
    * If $\Delta = 0$, can't draw a definite conclusion.
  - Assume we want to find the extreme values of $f(x, y)$ subject to the constraints $g(x, y) = c$, then $\nabla f = \lambda \nabla g$ where λ is the **Lagrange multiplier** and $\nabla g \neq 0$

3.2 Integration formulas

- $\int_a^b f(x)\ dx = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x * f(x_i), where\ \Delta x = \frac{b-a}{n}, x_i = a + \Delta x i$
- $\int x^n\ dx = \begin{cases} \frac{x^n+1}{n+1} + C & \text{if} \quad n \neq -1 \\ \ln|x| + C & \text{if} \quad n = -1 \end{cases}$
- $\int u\ dv = uv - \int v\ du$
- If $\forall x f(x) \leq g(x), then\ \int_a^b f(x) \leq \int_a^b g(x)$
- $\int_a^b f(x) dx = -\int_b^a f(x)\ dx$
- $\int_a^a f(x)\ dx = 0$
- $\int_a^b c\ dx = cx + C \Rightarrow c * (b - a)$
- $\int_a^b f(x) \pm g(x) = \int_a^b f(x)\ dx \pm \int_a^b g(x)\ dx$
- $\int_a^c = \int_b^c f(x)\ dx + \int_a^b f(x)\ dx$
- If I have $\int_a^b \frac{1}{x^n}\ dx \Rightarrow \int_a^b x^{-n} dx\ instead\ of\ \frac{1}{u}$
- $\int \sin x = -\cos x + C$
- $\int \cos x = \sin x + C$

- $\int_a^b \frac{1}{1+x^2} dx = \arctan x + C$
- $\int_a^b \frac{1}{\sqrt{1-x^2}} = \arcsin x + C$
- $\int_a^b \frac{-1}{\sqrt{1-x^2}} = \arccos x + C$
- $\int n^x \ dx = \frac{1}{\ln n} n^x + C, n > 0$
- $\int \ln x \ dx = x(\ln x - 1) + C$
- $\int \sec^2 x \ dx = \tan x + C$
- $\int \csc^2 x \ dx = -\cot x + C$
- $\int \sec x \tan x \ dx = \sec x + C$
- $\csc x \cot x \ dx = -\csc x + C$
- Trigonometric substitutions:
  - $\sqrt{a^2 - u^2} \rightarrow u = a \sin\theta \Rightarrow \frac{du}{d\theta} = a\cos\theta$
  - $\sqrt{a^2 + u^2} \rightarrow u = a \tan\theta \Rightarrow \frac{du}{d\theta} = a\sec^2\theta$
  - $\sqrt{u^2 - a^2} \rightarrow u = a \sec\theta \Rightarrow \frac{du}{d\theta} = a\sec\theta\tan\theta$
- **Fundamental theorem of calculus:** $\frac{d}{dx}\int_{a(x)}^{b(x)} f(t) \ dt = f(b(x)) * b'(x) - f(a(x)) * a'(x)$
- The **line integral with respect to arc length** is $\int_C f ds = \int_a^b f(x(t), y(t)) \frac{ds}{dt} dt$, where $\frac{ds}{dt} = \pm\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} = \pm\sqrt{[x'(t)]^2 + [y'(t)]^2}$
- The **line integral of the vector field F** along C is $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M \ dx + N \ dy = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt$, where $\mathbf{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}$
- The **fundamental theorem of calculus for line integrals**: $\oint \mathbf{F} \cdot d\mathbf{r} = \oint_C \nabla f \cdot dr = f(B) - f(A)$, where $\mathbf{F}$ is a gradient field

- To get volume of revolving surface:
$\int_c^d \int_a^b f(x,y)\ dxdy, a \leq x \leq b; c \leq y \leq d$
  - May need to change to polar coordinates:
    $x = r\cos\theta$
    $y = r\sin\theta$
    $r = \sqrt{x^2 + y^2}$
    $dA \rightarrow rdrd\theta$
  - $\int\int_D (x^2 + y^2)\ dA = \int_0^{2\pi} \int_0^6 r^2 r\ drd\theta = \int_0^{2\pi} \int_0^6 r^3\ drd\theta$
- Revolution around x axis with y = f(x): $V = \int_a^b \pi * [f(x)]^2 dx$
- Revolution around y axis with x = f(y): $V = \int_c^d \pi * [f(y)]^2 dy$
- Revolution of region encircled by a solid and the axis (washer method):

  $V = \int_a^b \pi\{[f(x) - g(x)]^2\} dx$
- To get arc length s:
$s = \int_a^b ds\ dx = \int_c^d ds\ dy$
$= \int_a^b \sqrt{1 + \frac{dy}{dx}^2}\ dx = \int_c^d \sqrt{\frac{dy}{dx}^2 + 1}\ dy$
- **Green's theorem** states that: $\oint_C M\ dx + N\ dy = \int\int_R (\frac{dN}{dx} - \frac{dM}{dy}) dA$
- **L' Hopital's Rule:** $lim_{x \rightarrow a} \frac{f(x)}{g(x)}$
$g'(x) \neq 0$
$f(a) = g(a) = 0 \vee \pm\infty$(are indefinite simultaneously)
$f'(x) \wedge g'(x)\ exist$
  - Use $\ln[f(x)]^{g(x)} = g(x)\ln(f(x)$ if the indefinite form is similar to $1^\infty, \infty^0, 0^0$
- **Improper integral:** $\int_a^\infty f(x)\ dx = \lim_{b\rightarrow\infty} \int_a^b f(x)\ dx$

- **Taylor series and Taylor polynomials**:
  - Let $f(x) = \sum_0^\infty a_n * x^n$, then $a_n = \frac{f^{(n)}(0)}{n!}$
  - **Taylor series:** $f(x) = \sum_0^\infty \frac{f^{(n)}(0)}{n!} x^n$
  - $\frac{1}{1-x} == x^0 + x^1 ... = \sum_0^\infty x^n \quad (-1 < x < 1)$
  - $\frac{1}{1+x} == x^0 - x^1 ... = \sum_0^\infty (-1)^n x^n \quad -1 < x < 1$
  - $\ln(1+x) = x - \frac{x^2}{2} ... = \sum_1^\infty (-1)^{n+1} \frac{x^n}{n} \quad -1 < x \leq 1$
  - $e^x = \frac{x^0}{0!} + \frac{x^1}{1!} = \sum_0^\infty \frac{x^n}{n!} \quad \forall x$
  - $\sin x = x - \frac{x^3}{3!} = \sum_0^\infty \frac{(-1)^n}{(2n+1)!} x^{2n+1} \quad \forall x$
  - $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} = \sum_0^\infty \frac{(-1)^n}{(2n)!} x^{2n} \quad \forall x$

3.3 Analytic Geometry of R^3

- Given $\hat{i} = (1,0,0), \hat{j} = (0,1,0), \hat{k} = (0,0,1)$, and vector **v**=$x\hat{i}+y\hat{j}+z\hat{k}$, then its magnitude, or norm, , is: $||v|| = \sqrt{x^2 + y^2 + z^2}$

- The distance between endpoints of two vectors v_1 and v_2 is $||(x_2 - x_1, y_2 - y_1, z_2 - z_1) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

- Given two vectors $\vec{A}$ and $\vec{B}$, which magnitudes are A and B and coordinates are (x_1, y_1, z_1) and (x_2, y_2, z_2):
  - $proj_{\vec{A}} \vec{B} = B \cos\theta = \frac{\vec{A} \cdot \vec{B}}{\vec{A} \cdot \vec{A}} \vec{A}$
  - Their dot product is $\vec{A} \cdot \vec{B} = AB \cos\theta = x_1 * x_2 + y_1 * y_2 + z_1 * z_2$
  - Dot product is commutative ($\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$) and associative with respect to vector addition $[\vec{A} \cdot (\vec{B} + \vec{C}) = (\vec{A} \cdot \vec{B}) + (\vec{A} \cdot \vec{C})]$
  - Their cross product is $\vec{A} \times \vec{B} = AB \sin\theta = (y_1 * z_2 - z_1 * y_2)\vec{i} + (z_2 * x_1 - x_2 * z_1)\vec{j} + (x_1 * y_2 - x_2 * y_1)\vec{k}$

- The cross product is anticommutative $[(B \times A = -(A \times B)) \wedge A \neq cB\ ]$. It's also associative with respect to vector addition $[(A \times (B + C) = (A \times B) + (A \times C))]$
- The area of the parallelogram formed by **A** and **B** is $||A \times B||$
- $A \perp B \iff A \cdot B = 0$
- If given another vector $\vec{C}$ which is not a multiple of either $\vec{A} \vee \vec{B}$, then $|(A \times B) \cdot C|$ is the volume of the parallelepiped formed by $\vec{A} \wedge \vec{B} \wedge \vec{C}$

- Given a point $P_0 = (p_0, p_1, p_2)$, a vector $\vec{v} = (v_1, v_2, v_3)$, and a constant t, a line L in 3D-space is of the form $L : P_0 + t(\vec{v})$
  - A point $P = (x, y, z)$ lies on L if $(x-p_0, y-p_1, z-p_2) = t(v_1, v_2, v_3)$
  - $t = \frac{x-p_0}{v_1}$
- A plane P in 3D space is defined given a point $P_0 = (p_0, p_1, p_2)$, and a vector $\vec{n} = (n_1, n_2, n_3)$ that's **normal** to it.
  - A point $P = (x, y, z)$ lies on P if $(x-p_0, y-p_1, z-p_2) \cdot (n_1, n_2, n_3) = 0$. It can be rewritten as $n_1x + n_2y + n_3z = d$, for a constant $d = n_1p_0 + n_2p_1 + n_3p_2$
-

If the curve	is revolved around the	then replace	by	to get surface of revolution
$f(x,y)=0$	x-axis	y	$\pm\sqrt{y^2+z^2}$	$f(x,\pm\sqrt{y^2+z^2})=0$
	y-axis	x	$\pm\sqrt{x^2+z^2}$	$f(\pm\sqrt{x^2+z^2},y)=0$
$f(x,z)=0$	x-axis	z	$\pm\sqrt{y^2+z^2}$	$f(x,\pm\sqrt{y^2+z^2})=0$
	z-axis	x	$\pm\sqrt{x^2+y^2}$	$f(\pm\sqrt{x^2+y^2},z)=0$
$f(y,z)=0$	y-axis	z	$\pm\sqrt{x^2+z^2}$	$f(y,\pm\sqrt{x^2+z^2})=0$
	z-axis	y	$\pm\sqrt{x^2+y^2}$	$f(\pm\sqrt{x^2+y^2},z)=0$

4 Trigonometry

- $\tan x \Rightarrow Domain : n(\frac{\pi}{2}, -\frac{\pi}{2}).Range : (-\infty, \infty)$
- $\tan \text{inverse}\, x \Rightarrow Domain : (-\infty, \infty).Range : n(\frac{\pi}{2}, -\frac{\pi}{2})$

5 Linear algebra

- $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

6 Additional topics

6.1 Probability

Bernoulli trials

- $P(\text{exactly k successes in n trials}) = \binom{n}{k} * p^k q^{n-k}$

Random variable

- $F_X(t) = P(\{w : X(w) \leqslant t\})$
- $P(t_1 < X \leq t_2) = F_X(t_2) - F_X(t_1)$. Note, F_X is a distribution (not density function), and P(t) is a probability.
- $E(X) = \mu(X) = \int_{-\infty}^{\infty} t f_X(t) dt$
- $Var(X) = \sigma^2(X) = \int\limits_{-\infty}^{\infty} [t - \mu(X)]^2 f_X(t) dt$
- $std(X) = \sigma(X) = \sqrt{\sigma^2(x)}$
- For normal distribution:
  - $f_X(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(t-\mu)^2/2\mu^2} dt$
  - $P(t_1 < X \leqslant t_2) = \int_{t_1}^{t_2} \frac{1}{\sigma\sqrt{2\pi}} e^{-(t-\mu)^2/2\mu^2} dt$
  - $f_Z(u) = \frac{1}{\sqrt{2\pi}} e^{\mu^2/2}$, where $u = \frac{t-\mu}{\sigma}$
  - $\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{(\mu)^2/2} du$
  - $\Phi(z_1 < Z \leqslant z_2) = \Phi(z_2) - \Phi(z_1)$
  - $P(a_1 \leqslant X \leqslant a_2) = \sum_{k=a_1}^{a_2} \binom{n}{k} p^k q^{n-k}$
  - $P(a_1 \leqslant X \leqslant a_2) \approx \Phi(\frac{a_2-\mu+1/2}{\sigma}) - \Phi(\frac{a_1-\mu+1/2}{\sigma})$

z	0	0.5	1	1.5	2	2.5	3 or more
$\Phi(z)$	0.5	0.69	0.84	0.93	0.97	0.99	≈ 1

6.2 Point-set topology

- A topology T on non-empty set X is a family of subsets of X where:
  1. $\emptyset$ and X are in T
  2. If O_1 and O_2 are in T, so is their intersection $O_1 \cap O_2$, O is an open set.

3. If $\{O_i\}_{i\epsilon I}$ is any collection of sets from T, then their union, $\cup_{i\epsilon I} O_i$, is also in T

- (X, T) is a topological space
  - (X, T) is a Hausdorff space if $\forall x \; \forall y \; \epsilon \; X, \exists O_x \; \exists O_y \; . \; x \; \epsilon \; O_x \vee y \epsilon O_y$
- $\{\emptyset, X\}$ is an indiscrete or trivial topology on non-empty set X. It's the coarsest topology X can have.
- If $P(X)$ is the power set of X, *all* subsets of X, then it's a discrete topology on X. It's the finest topology non-empty set X can have.
- If T_1 and T_2 are topologies on X, and if $T_1 \supseteq T_2$, then T_1 is **finer (more detailed, maybe larger set)** than T_2 or T_2 is **coarser (simpler, maybe smaller set)** than T_1
- If O is an open set, then $O^c = X - O$ is a closed set.

For example, let $X = \{\, a, b, c, d \,\}$, $T_1 = \{\, \emptyset, \{\, a, b \,\}, \{\, c, d \,\}, X \,\}$, and $A = \{\, \emptyset, \{\, a \,\}, \{\, a, b \,\}, \{\, c, d \,\}, X \,\}$ then T_1 is a topology on X, but A is not because it lacks the union $\{\, a, c, d \,\}$

- Let (X, T) be a topological space on set X, O is open in X, and let S be a subset of X.
  - Let U be a subset of S, then U of S is **open** in S if U is equal to $O \cap S$.
  - S is a subspace of X.
  - T_s is a topology on S, and is called the subspace (or relative) topology.

- Let (X, T) be a topological space
  let A be a (not necessarily open) subset of X.
  - The **interior of A**, int(A), is the union of all open sets contained within A; it's the largest open set contained in A.
  - The **exterior of A**, ext(A), is the union of all open sets that don't *intersect* A (set U *intersects* A if $U \cap A \neq \emptyset$).
    * The ext(A) is also the int(A^c).
  - The **boundary of A**, bd(A), is set of all x in X such that every open set containing x intersects both A and A^c
  - A point x in X is the **limit or accumulation or cluster point of A** if every open set that contains x also contains at elast one point of A, other than x.
    * The set of all limit points of A, A', is the **derived set** of A.
  - The **closure of A**, cl(A) , is equal to $int(A) \cup bd(A)$, or also the union $A \cup A'$. The set A is closed in and only if it contains all of its boundary points and all of its limit points.

1. Let A be the subset $(1, 2]$ in $\mathbf{R}$:

Then we have:

$$\text{int}(A) = (1, 2)$$

$$\text{ext}(A) = (-\infty, 1) \cup (2, \infty)$$

$$\text{bd}(A) = \{1, 2\}$$

$$A' = [1, 2]$$

$$\text{cl}(A) = [1, 2]$$

- Let X be a non-empty set, and **B** be a collection of subsets of X. **B** is a **basis**, and the sets in **B** are **basis elements** if:

  1. For every x in X, there is at least one set B in **B** such that $x \epsilon B$
  2. If B_1 and B_2 are sets in **B** and $x \epsilon B_1 \cap B_2$, then there exists a set B_3 in **B** such that $x \epsilon B_3 \subseteq B_1 \cap B_2$

- A basis generates (X, T). A subset O of X is open (O belongs to the topology generated by B), if for every x in O there exists a basis element B such that $x \epsilon B \subseteq O$. Equivalently, a topology generate by basis **B** consists of all possible union of basis elements.

- If (X, T_X) and (Y, T_Y) are topological spaces, we can define the topology on the cartesian product $X \times Y$ as follows: $\mathbf{B} = \{ O_X \times O_Y : O_X \epsilon T_X \wedge O_Y \epsilon T_Y \}$. Here, basis **B** is the product topology on $X \times Y$

- Let (X, T) be a topological space. If there exists disjoint, non-empty open sets $O_1 \wedge O_2 . O_1 \cup O_2 = X$, then X is said to be **disconnected**.

- If T contains no pair of subsets of X that are disjoint and no pair of subsets whose union is X, then X is a **connected** space.

  1. If A and B are connected and they intersect (intersection is non-empty), then their union is also connected.
  2. Let A be a connected set, and let B be any set such tat $A \subseteq B subseteq cl(A)$, then B is connected.
  3. The cartesian product of connected spaces is connected.
  4. Let X be a topological space such any two points $x_1\ and\ x_2$ in X can be joined by a continuous path. This means that there exists a continuous function $p : [0, 1]X$ such that $p(0) = x_1\ and\ p(1) = x_2$. Then X is **path-connected**, and any path-connected space is connected. The converse isn't true.

- If S is a subspace of X, then:
  - S is disconnected if there are two open subsets of X, $O_1 \wedge O_2$ such that $S \subseteq O_1 \cup O_2$ and both $O_1 \cap S \ \wedge \ O_2 \cap S$ are disjoint and non-empty.
  - Otherwise S is connected.
- Let (X, T) be a topological space. A **covering** of X is a collection of subsets of X whose union is X.
- An **open covering** is a covering that consists entirely of open sets.
- If every open covering of X contains a finite subcollection that also covers X, then X is a **compact** space.
  1. Let X be a compact topological space, and let S be a subset of X. If S is closed, then it's compact. The converse is true if X is Hausdorff.
  2. The cartesian product of compact spaces is compact.
  3. **Heiner-Borel theorem:** A subset of R_n (in the standard topology) is compact if and only if it's both closed and bounded.
  4. Here, a subset A of R_n is said to be **bounded** if there exists some positive number M such that the *norm* of every point in A is less than M, where *norm* of a point $\mathbf{x} = (x_1, x_2, ..., x_n$ is $\|x\| = \sqrt{(x_1)^2 + (x_2)^2 + ...(x_n)^2}$
- Let X be a non-empty set, and d:$X \times X \to R$. The function d is a **metric on X** if the following hold true for all *x, y, z* in X:
  1. *d(x, y)* ≥ 0
     d(x, y) $= 0$ if and only if $x = y$
  2. *d(x, y)* $=$ *d(y, x)*

3. *d(x, z)* $\leq$ *d(x, y)* + *d(y, z)*
4. Where *d(x, y)* is the **distance** between x and y. A set X, together with a metric on X, is a **metric space**.If ϵ is a positive number, then the set: $B_d(x, \epsilon) = \{ x' \epsilon X :$ *d(x, x')* $< \epsilon \}$ is an **ϵ-ball, (open) ball of radius ϵ centered on x**

- The collection of all ϵ-balls, $\mathbf{B} = \{ B_d(x, \epsilon) : x \epsilon X, \epsilon > 0 \}$ is a basis for a topology on X, called a **metric topology (induced by d)**. Here, O is open if for every x in O, there exists some positive number ϵ_x such that $B_d(x, \epsilon_x) \subseteq O$.

- The standard topology on R^n is given by the square metrics $\sigma(x, x') = max \{ \mid x_1 - x_1'|, |x_2 - x_2'|, ..., |x_n - x_n'| \}$ and/or $d(x, x') = \|x - x'\| = \sqrt{x_1 - x_1'}$

- Let (X_1, d_1) *and* (X_2, d_2) be metric spaces. A function $f \colon X_1 \to X_2$ is **continuous at point** x_0 if, for every $\epsilon > 0$, there exists a $\delta > 0$ such that $d_1(x, x_0) < \delta \Rightarrow d_2(f(x), f(x_0)) < \epsilon$

- We can say that *f*:$X_1 \to X_2$ is **contiuous at the point** x_0 if for every open set O containing $f(x_0)$, the inverse image $f^{-1}(O)$ is an open set containing x_0

- Let $(X_1, T_1)$and (X_2, T_2) be topological spaces, we say the **map** (or function) $f : X_1 \to X_2$ is
  if, for every $O \epsilon T_2$, the inverse image $f^{-1}(O)$ is open in X_1, that is, $f^{-1}(O) \epsilon T_!$. Particularly, f is **continuous at the point** x_0 if for every open set O in X_2 containing *f(*x_0*)*, there's an O_1 containing x_0 such that $f(O_1) \subseteq O$.If the topology of the range space, T_2 is generate by basis **B** then, to check whether $f : (X_1, T_1) \to (X_2, T_2)$ is continuous, we check that every $f^1(B \epsilon \mathbf{B})$ is open in X_1.

- A map $f : X_1 \to X_2$ is an **open** map if the image of every open set in $X_1$is open inX_2
- If $f : X_1 \to X_2$ is a bijection, and both $f\ and\ f^{-1}$ are continuous, then f is a **homeomorphism**.
- If two topological spaces, $X_1\ and\ X_2$ are **homeomorphic**, then a homeomorphism $f : X_1 \to X_2$ establishes a one-to-one correspondence between points $X_1\ and\ X_2$ as well as between their open sets, preserving the topological structure.

  All formulas and text are extracted from (Review, 2016)

References

Review, P. (2016). *Cracking the GRE Math Subject test*, volume 74. UCLA, Dept. of Statistics.