LOCAL REGULARITY FOR THE HARMONIC MAP AND YANG-MILLS HEAT FLOWS

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Abstract. We establish new local regularity theorems for the Yang-Mills and harmonic map heat flows on complete manifolds of dimension greater than 4 and 2 respectively. Moreover, we show that smooth blow-ups at rapidly forming singularities of these flows are necessarily non-trivial. Finally, we use this theorem to obtain a new characterisation of singularity formation and use this to estimate the size of the singular set at the first singular time.

1. Introduction

The main result of this paper is the following local regularity theorem for the harmonic map and Yang-Mills heat flows on complete manifolds (see §2 for the setup):

Main Theorem. *Let (M, g) be a complete Riemannian manifold of dimension $n > 2k > 0$ and $\Omega \subset M$ open and bounded. Then there exist geometric constants $\varepsilon, C > 0$ such that if $u : \mathcal{D}_R(X,T) \subset \Omega \times]0,T[\to E \otimes \Lambda^{k-1}T^*M$ is a harmonic map ($k = 1$) or Yang-Mills ($k = 2$) heat flow, then the implication*

$$\sup_{(x,t)\in\mathcal{D}_{R/2}(X,T)} \frac{1}{\left(R/(2c_{n,k})\right)^{n-2k}} \iint_{E^{n-2k}_{R/(2c_{n,k})}(x,t)} e^{\xi_k(t-s)}[u,g]_{(x,t)}(y,s)\mathrm{dvol}_g(y)\,\mathrm{d}s < \varepsilon$$
$$\Rightarrow \left(\frac{7}{32}R\right)^{2k} \sup_{\mathcal{D}_{R/4}(X,T)} \frac{1}{2}|\psi|^2 \leq C$$

holds, where ψ is the differential u or the curvature two-form respectively.

The harmonic map heat flow was first introduced by Eells and Sampson [7] to deform smooth maps of Riemannian manifolds into harmonic ones. Key in their work was the fact that the target manifold had nonpositive sectional curvatures. Without this condition, the harmonic map heat flow does not necessarily exist for all time, which was shown by Coron and Ghidaglia [6] and Chang, Ding and Ye [4]. Given that singularities are inevitable, one might ask how big the set of singularities— i.e. the *singular set*— is at the maximal time of a harmonic map heat flow. This question was answered by Grayson and Hamilton [8] in the case of a compact domain manifold, where they showed that the singular set is of codimension 2; moreover, they studied blow-ups of rapidly-forming singularities. A key in their analysis is a weighted energy

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of the form

$$\int_M \frac{1}{2}|\psi|^2(\cdot,t)\cdot\Phi(\cdot,t), \tag{1.1}$$

where Φ is an appropriately weighted (backward) heat kernel.

Likewise, the Yang-Mills heat flow, introduced by Atiyah and Bott [3] to study the Morse theory of Yang-Mills connections, also tends to develop singularities in finite-time on manifolds of dimension at least 5, which was shown by Naito [10] and Grotowski [9]. In that case, the singular set has been shown to be of codimension 4 in the case of a compact domain manifold by Chen, Shen and Zhou [5], and an analysis of rapidly-forming singularities was carried out by Weinkove [14] in this case. The key ingredient here is also a weighted energy of the form (1.1). In this paper, we show that these results continue to hold in the non-compact setting and make use of a heat ball formula instead. We note that a similar approach has been taken by Ni [11] for the Ricci flow.

The structure of this paper is as follows. In §2 we describe the underlying geometric setup and introduce the harmonic map and Yang-Mills heat flows, as well as some important properties of theirs. In §3, we prove the main theorem of this paper. In §4 the local regularity theorem is used to analyse the singular set at the maximal times of these flows, which culminates in an estimate for the (lower-dimensional) Hausdorff measure of this set. In §5, we turn our attention to rapidly-forming singularities of these flows and show that they admit smooth, nontrivial blow-ups.

2. Setup

2.1. **Geometry.** Throughout this paper we will be dealing with a complete Riemannian manifold (M^n, g) of dimension $n > 2k > 0$ with k to be fixed shortly. We shall adopt the notation of [12] for all Riemannian geometric quantities and operators. Moreover, Ω will always denote an open bounded subset of M. Thus, we may find constants $i_0 > 0$, $\kappa_{-\infty}, \kappa_\infty > 0$ and $\{\Lambda_i : i \in \mathbb{N}\} \subset [0, \infty[$ such that $\mathrm{inj}_\Omega > i_0$ and the geometry bounds

$$\begin{gathered} \kappa_{-\infty} \leq \sec \leq \kappa_\infty \\ \frac{1}{2}\left|\nabla^i \mathrm{Riem}_g\right|^2 \leq \Lambda_i \end{gathered} \tag{2.1}$$

hold on Ω for all $i \in \mathbb{N}$. For simplicity, we will assume that $i_0 < \frac{\pi}{2\sqrt{\kappa_\infty}}$ if $\kappa_\infty > 0$.

For later purposes, we introduce for each $\lambda > 0$ and $X \in M$ the (crudely) *rescaled metric* $g_\lambda^X : B_{\mathrm{inj}_X/\lambda}(0) \subset \mathbb{R}^n \to T^*\mathbb{R}^n \otimes T^*\mathbb{R}^n$ defined by

$$g_\lambda^X(y) = \sum_{i,j=1}^n g_{ij}(\lambda y)\ \mathrm{d}y^i \otimes \mathrm{d}y^j,$$

where $\{g_{ij}\}$ are the components of g in geodesic normal coördinates about X and $\{y^i\}$ denote Euclidean coördinates.

2.2. **Flows.** We shall now proceed to describe the flows we are interested in. In both cases, we have a one-parameter family of (vector bundle-valued) $(k-1)$-forms $\{u(\cdot,t) : M \to E \otimes \Lambda^{k-1} T^*M\}_{t\in[0,T[}$ with a distinguished *fundamental k-form* $\{\psi(\cdot,t) : M \to E \otimes \Lambda^k T^*M\}_{t\in[0,T[}$ satisfying an equation of the form

$$\partial_t \psi + \Delta^\nabla \psi = B(u,\psi,\nabla\psi), \tag{2.2}$$

where $B(u,\psi,\nabla\psi)$ is a polynomial expression in ψ and $\nabla\psi$.

2.2.1. *Harmonic map heat flow.* The *harmonic map heat flow* is given by a one-parameter family of smooth maps $\{u(\cdot,t) : M \to N \subset \mathbb{R}^K\}_{t\in[0,T[}$ to a compact Riemannian submanifold N of $\mathbb{R}^K$ such that the equation

$$(\partial_t - \Delta_g)u = -\sum_{i,j} g^{ij}(b\circ u, \partial_i u \otimes \partial_j u) \tag{2.3}$$

holds, where b denotes the second fundamental form of N, here considered a map $N \to T^*\mathbb{R}^K \otimes T^*\mathbb{R}^K$. For this flow, $k=1$, $\psi = \mathrm{d}u$ and $E = \mathbb{R}^K$, the trivial vector bundle with standard fibre $\mathbb{R}^K$.

Given a harmonic map heat flow u, we may *rescale* it as follows: Let $\vartheta_x : B_{\mathrm{inj}_x}(0) \subset \mathbb{R}^n \to B_{\mathrm{inj}_x}(x) \subset M$ denote the the geodesic normal coördinate parametrisation centred at X. Defining

$$u_\lambda^{(x,t)}(y,s) := (\vartheta_x^* u)(\lambda y, t+\lambda^2 s) = u(\vartheta_x(\lambda y), t+\lambda^2 s),$$

we obtain a one-parameter family of maps $\{u_\lambda^{(x,t)}(\cdot,s) : B_{\mathrm{inj}_X/\lambda}(0) \to N\}_{s\in]-\frac{t}{\lambda^2},\frac{T-t}{\lambda^2}[}$ that solves the equation (2.3) *with respect to the metric g_λ^x.*

2.2.2. *Yang-Mills heat flow.* Let $G \to P \to M$ be a principal bundle with compact, connected semi-simple Lie group G as its structure group, and write $E_\mathfrak{g}$ for the vector bundle associated with P and the adjoint representation of G on its Lie algebra $\mathfrak{g}$. The (negative) Killing form on $\mathfrak{g}$ endows $E_\mathfrak{g}$ with the structure of a Riemannian vector bundle; moreover, any connection ω on P induces a unique covariant derivative operator ∇ on $E_\mathfrak{g}$ compatible with this Riemannian structure. The *Yang-Mills heat flow* is given by a one-parameter family of connections $\{\omega(t) : P \to \mathfrak{g} \otimes T^*P\}_{t\in[0,T[}$ such that the equation

$$\underline{\partial_t \omega} = \operatorname{div} \underline{\Omega}^\omega, \tag{2.4}$$

where $\underline{\partial_t\omega}(\cdot,t)$ is the unique section of $E_\mathfrak{g} \otimes T^*M$ having $\partial_t\omega : P \times]0,T[\to \mathfrak{g} \otimes T^*P$ as its lift, $\underline{\Omega}^\omega : M \times [0,T[\to E_\mathfrak{g} \otimes \Lambda^2 T^*M$ is the curvature two-form of ω, and the divergence operator div is induced by the Levi-Civita connection on TM and $\omega(t)$. Thus, for this flow, $k=2$, $E = E_\mathfrak{g}$ and $\psi = \underline{\Omega}^\omega$.

In local considerations, it is customary to pull connections on principal bundles back to the base manifold by means of local sections; to this end, we shall often treat $\omega(t)$ as a local section of the bundle $\mathfrak{g} \otimes T^*M$ by pulling it back as $u(\cdot,t) = \sigma^*\omega(\cdot,t) = \sum_i u_i(\cdot,t) \otimes \mathrm{d}x^i : U \to \mathfrak{g} \otimes T^*M$, where $\sigma : U \to P$ is a local section, in which case the curvature two-form is locally given as $F(\cdot,t) = \sum_{i<j} F_{ij}(\cdot,t) \otimes \mathrm{d}x^i \wedge \mathrm{d}x^j : U \to \mathfrak{g} \otimes \Lambda^2 T^*M$ with

$$F_{ij} = \partial_i u_j - \partial_j u_i + [u_i, u_j].$$

We may therefore write (2.4) more explicitly as

$$\partial_t u_i = \sum_{p,q} g^{pq} \left(\partial_p F_{qi} + [u_p, F_{qi}] - \sum_{r=1}^{n} \left(\Gamma^r_{pq} F_{ri} + \Gamma^r_{pi} F_{qr} \right) \right) \tag{2.5}$$

Given such a local representation, we may *rescale* a Yang-Mills heat flow as follows: Fix $x \in U \subset M$, $t \in \,]0,T[$ and a local section $\sigma : B_{\mathrm{inj}_x}(x) \to P$. As before, we let $\vartheta_x : B_{\mathrm{inj}_x}(0) \to B_{\mathrm{inj}_x}(x)$ denote the geodesic normal coördinate parametrisation. Defining

$$u_\lambda^{(x,t)}(y,s) := (\vartheta_x^* u)(\lambda y, t + \lambda^2 s) = \sum_{i=1}^{n} \lambda u_i(\theta_x(\lambda y), t + \lambda^2 s)\,\mathrm{d}y^i,$$

we obtain a smooth one-parameter family $\{u_\lambda^{(x,t)}(\cdot, s) : B_{\mathrm{inj}_X/\lambda}(0) \to \mathfrak{g} \otimes T^*\mathbb{R}^n\}_{s \in \left]-\frac{t}{\lambda^2}, \frac{T-t}{\lambda^2}\right[}$ solving the equation (2.5) *with respect to the metric* g_λ^x. This then gives rise to a locally defined connection on P.

2.2.3. *Common properties.* In both cases, local control on the fundamental form ψ ensures local control on the derivatives of u of all orders in the following sense:

Lemma 2.1. *If the estimate* $r^{2k} \sup_{\mathcal{D}_r(X,T)} \frac{1}{2}|\psi|^2 \leq c_0$ *holds for some* $0 < r < \mathrm{inj}_X$ *and* $X \in M$*, then for each* $i \in \mathbb{N}$ *there exists a constant* $c_i > 0$ *(polynomially) depending only on* c_0 *and the covariant derivatives of the curvature tensor up to order* i *such that*

$$r^{2(k+i)} \sup_{\mathcal{D}_{r/2}(X,T)} \frac{1}{2}|\nabla^i \psi|^2 \leq c_i. \tag{2.6}$$

Proof. See [8] for the case of the harmonic map heat flow and [14] for the case of the Yang-Mills heat flow. □

For later purposes, we shall need to make use of a very special local section of P, which is given in the following lemma and follows from the higher order estimates above and an application of a theorem due to Uhlenbeck [13].

Lemma 2.2 (Nice gauge)**.** *If* $\sup_{\mathcal{D}_r(X,T)} \frac{1}{2}|\underline{\Omega}^\omega|^2 < \infty$ *for some* $r > 0$*, then there exist constants* $\theta \in \left]0, \frac{1}{2}\right]$ *and* $c > 0$ *depending only on the geometry of* $\mathcal{D}_r(X,T)$ *and a local section* σ *of* $E_\mathfrak{g}$ *such that*

$$\sup_{\mathcal{D}_{\theta r}(X,T)} \frac{1}{2}|\sigma^* \omega|^2 \leq cr \sup_{\mathcal{D}_r(X,T)} \frac{1}{2}|\underline{\Omega}^\omega|^2.$$

2.3. **Heat balls and local monotonicity.** We now turn our attention to the monotonicity properties of the harmonic map and Yang-Mills heat flows.

For fixed $(x,t) \in \Omega \times \,]0,T[$ and $k < \frac{n}{2}$, the *generalised Euclidean* $(n-2k)$*-heat ball* at (x,t) associated to (Ω, g) is defined by

$$\begin{aligned} E_r^{n-2k}(x,t) &= \left\{ (y,s) \in \Omega \times \,]0,T] : \frac{1}{(4\pi(t-s))^{\frac{n-2k}{2}}} \exp\left(\frac{\mathrm{dist}_g(y,x)}{4(s-t)} \right) > \frac{1}{r^{n-2k}} \right\} \\ &= \bigcup_{s \in \left]t - \frac{r^2}{4\pi}, t\right[\cap]0,T]} \left(B_{R_r^{n-2k}(s-t)}(x) \cap \Omega \right) \times \{s\}, \end{aligned}$$

where $R_r^{n-2k}(\sigma) = \sqrt{2(n-2k)\sigma\log\left(-\frac{4\pi\sigma}{r^2}\right)}$ for $\sigma \in \left]-\frac{r^2}{4\pi}, 0\right[$. It may be seen from the definition of $E_r^{n-2k}(x,t)$ that it is relatively compact in $M \times [0,T[$. Moreover, we have the inequality $R_r^{n-2k}(\sigma) \leq \sqrt{\frac{n-2k}{2\pi e}}r =: c_{n,k}r$ so that

$$E_r^{n-2k}(x,t) \subset \mathcal{D}_{c_{n,k}r}(x,t).$$

For technical reasons, we shall restrict our attention to small r so that $E_r^{n-2k}(x,t) \subset \Omega \times]0,T[$.

We now state the local monotonicity principle for these flows. For simplicity, we introduce the shorthand notation

$$[u,g]_{(x,t)} = \frac{1}{2}|\psi|^2 \cdot \left(\partial_t\phi_{(x,t)} + |\nabla\phi_{(x,t)}|^2\right) - \left\langle \nabla\phi_{(x,t)}\llcorner\psi, \partial_t u + \nabla\phi_{(x,t)}\llcorner\psi\right\rangle,$$

where $\phi_{(x,t)}(y,s) = \frac{\mathrm{dist}(x,y)^2}{4(s-t)} - \frac{n-2k}{2}\log\left(4\pi(t-s)\right)$. Note that the right-hand side implicitly depends on the choice of metric g.

Theorem 2.3. *Fix $(x,t) \in \Omega \times]0,T]$ and suppose u is a harmonic map or Yang-Mills heat flow. Then there exist $r_0 > 0$ and $\xi_k \in C^\infty(]0,\infty[)$ with $\lim_{s\searrow 0}\xi_k(s) = 0$ depending on (x,t) and the geometry of Ω such that the quantity*

$$r \mapsto \frac{1}{r^{n-2k}}\iint_{E_r^{n-2k}(x,t)} e^{\xi_k(t-s)}[u]_{(x,t)}(y,s)\ \mathrm{dvol}_g(y)\,\mathrm{d}s \tag{2.7}$$

is monotone nondecreasing for $r < r_0$ whenever the integrand is summable over $E_{r_0}^{n-2k}(x,t)$. If $M = \mathbb{R}^n$, then $\xi_k \equiv 0$, the quantity is monotone for all $r > 0$ is an integral of the flow iff u is self-similar about (x,t).

Remark 2.4. The geometric quantities r_0 and ξ_k are explicitly given by the following expressions:

$$r_0 = \frac{1}{2}\min\left\{\frac{i_0}{c_{n,k}}, \frac{\mathrm{dist}(x,\partial\Omega)}{c_{n,k}}, \sqrt{4\pi t}, 1\right\};$$

$$\xi_k(t) = (T-t)\left[((n-1)\Lambda_\infty^+ - 2k\Lambda_{-\infty}^-)\log\left(\frac{e}{4\pi(T-t)}\right)^{\frac{n-2k}{2}}\right],$$

where $\Lambda_\pm$ arise from the geometry bounds

$$\Lambda_{-\infty}\mathfrak{r}(\cdot,t)^2 g^{\mathfrak{r}}(\cdot,t) \leq g(\cdot,t) - \nabla^2\left(\frac{1}{2}\mathfrak{r}(\cdot,t)^2\right) \leq \Lambda_\infty\mathfrak{r}(\cdot,t)^2 g^{\mathfrak{r}}(\cdot,t)$$

and may be determined explicitly in terms of $\kappa_\pm$ by means of a Hessian comparison theorem (cf. [12, Theorem 27, p.175]).

Remark 2.5 (Scale invariance)**.** We may more explicitly write $[u,g]_{(x,t)}$ in geodesic normal coördinates about (x,t) as

$$\frac{1}{2}|\psi|^2 \cdot \frac{n-2k}{2(t-s)} - \left\langle \frac{y}{2(s-t)}\llcorner\psi, \partial_t u + \frac{y}{2(s-t)}\llcorner\psi\right\rangle.$$

In particular, using this local expression, we have that

$$\frac{1}{r^{n-2k}}\iint_{E_r^{n-2k}(x,t)} e^{\xi_k(t-s)}[u,g]_{(x,t)}(y,s)\ \mathrm{dvol}_g(y)\,\mathrm{d}s$$

$$= \frac{1}{(r/\lambda)^{n-2k}} \iint_{E^{n-2k}_{r/\lambda}(0,0)} e^{\xi_k(-\lambda^2 s)}[u_\lambda^{(x,t)}, g_\lambda^{(x,t)}]_{(0,0)}(y,s) \cdot \sqrt{g_\lambda^{(x,t)}}(y)\ \mathrm{d}y\mathrm{d}s$$

for any $\lambda > 0$. We shall use this (approximate) *scale-invariance* property in the sequel.

It was shown in [2] that the quantity (2.7) is finite for $r = r_0$ whenever $\frac{1}{2}|\psi|^2 \in L^1(\mathcal{D}_{2c_{n,k}r_0}(x,t))$. In fact, we have the following estimate.

Lemma 2.6 (L^2 estimate)**.** *Suppose $\{u(\cdot,t)\}_{t\in[0,T[}$ is a harmonic map or Yang-Mills heat flow on Ω such that $|\psi|^2 \in L^1(\mathcal{D}_{2c_{n,k}r_0}(x,t))$. Then the estimate*

$$\frac{1}{r^{n-2k}} \iint_{E_r^{n-2k}(x,t)} |[u,g]_{(x,t)}|(y,s)\mathrm{dvol}_g(y)\mathrm{d}s \le \frac{C}{r^{n-2k+2}} \iint_{\mathcal{D}_{4c_{n,k}r}(x,t)} \frac{1}{2}|\psi|^2 \tag{2.8}$$

for $r < \frac{r_0}{2}$, where C depends only on the geometry of Ω.

Proof. In [2, Remark 5.8], it was shown that

$$\begin{aligned} &\frac{1}{r^{n-2k}} \iint_{E_r^{n-2k}(x,t)} |[u,g]_{x,t}|(y,s)\mathrm{dvol}_g(y)\mathrm{d}s \\ &\le C\left(\frac{1}{r^{n-2k+2}} \iint_{\mathcal{D}_{2c_{n,k}r}(x,t)} \frac{1}{2}|\psi|^2 + \frac{1}{r^{n-2k}}\int_{B_{2c_{n,k}r_0}(x)} \frac{1}{2}|\psi|^2(\cdot, t-\frac{r^2}{4\pi})\right). \end{aligned} \tag{2.9}$$

Now, it may be shown using the methods of [1] that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_M \frac{1}{2}|\psi|^2\varphi(\cdot,t) \\ &= -\int_M |\partial_t u|^2\varphi + \int_M \frac{1}{2}|\psi|^2(\partial_t - \Delta)\varphi + \left\langle \nabla^2\varphi, \sum_{i,j} \langle \partial_i \llcorner \psi, \partial_j \llcorner \psi\rangle\ \mathrm{d}x^i \otimes \mathrm{d}x^j \right\rangle \end{aligned} \tag{2.10}$$

for a smooth function $\varphi : M \times [0,T[\to [0,\infty[$ such that $\varphi(\cdot,t)$ is compactly supported in M for each t. We fix a smooth function $\chi : \mathbb{R}\times\mathbb{R} \to [0,\infty[$ such that

$$\operatorname{supp}\,\chi \subset \{(t_1,t_2) \in \mathbb{R}\times\mathbb{R} : |t_1| < 1, -1 < t_2 \le 0\}$$

and $\chi(t_1,t_2) = 1$ for $|t_1| < \frac{1}{4}$, $-\frac{1}{4} < t_2 < 0$. Setting $\varphi(y,s) = \chi\left(\left(\frac{\mathrm{dist}(y,x)}{4c_{n,k}r}\right)^2, \frac{s-t}{(4c_{n,k}r)^2}\right)$, it follows that supp $\varphi(\cdot,s) \subset B_{4c_{n,k}r}(x)$, $\varphi(\cdot,s) = 0$ for $s \le t-(4c_{n,k}r)^2$, and $\varphi \equiv 1$ on $\mathcal{D}_{2c_{n,k}r}(x,t)$. Moreover, by the Hessian comparison theorem, (2.10) implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_M \frac{1}{2}|\psi|^2\varphi(\cdot,t) \le \frac{C}{r^2}\int_{B_{4c_{n,k}r}(x)} \frac{1}{2}|\psi|^2(\cdot,t),$$

where C depends on n, k and the geometry of Ω. Integrating from $t-(4c_{n,k}r)^2$ to $t-\frac{r^2}{4\pi}$, we obtain

$$\int_{B_{2c_{n,k}r}(x)} \frac{1}{2}|\psi|^2(\cdot,t-\frac{r^2}{4\pi}) \le \int_M \frac{1}{2}|\psi|^2\varphi(\cdot,t-(2c_{n,k}r)^2) \le \frac{C}{r^2}\iint_{\mathcal{D}_{4c_{n,k}r}(x,t)} \frac{1}{2}|\psi|^2,$$

where we have used the fact that $t-\frac{r^2}{4\pi} > t-(2c_{n,k}r)^2$. Substituting this into (2.9) then yields (2.8). ∎

3. The local regularity theorem

We now turn our attention to the promised local regularity theorem.

Theorem 3.1. *There exist constants $\varepsilon, C > 0$ depending only on the geometries of Ω and $E \to \Omega$ and the structure of the underlying flow equation such that for any $R \leq \min\{2c_{n,k}r_0, i_0\}$ and Yang-Mills or harmonic map heat flow $u : \mathcal{D}_R(X,T) \subset \Omega \times]0,T[\to E \otimes \Lambda^{k-1}T^*M$, the implication*

$$\sup_{(x,t)\in\mathcal{D}_{R/2}(X,T)} \frac{1}{(R/(2c_{n,k}))^{n-2k}} \iint_{E^{n-2k}_{R/(2c_{n,k})}(x,t)} e^{\xi_k(t-s)}[u,g]_{(x,t)}(y,s)\mathrm{dvol}_g(y)\mathrm{d}s < \varepsilon$$

$$\Rightarrow \left(\frac{7}{32}R\right)^{2k} \sup_{\mathcal{D}_{R/4}(X,T)} \frac{1}{2}|\psi|^2 \leq C$$

holds.

We first prove the following lemma.

Lemma 3.2. *There exist constants $\varepsilon, C > 0$ depending only on the geometry of Ω and $E \to \Omega$ and the structure of the underlying flow equation such that for any $R \leq \min\{2c_{n,k}r_0, i_0\}$ and Yang-Mills or harmonic map heat flow $u : \overline{\mathcal{D}_R(X,T)} \subset \Omega \times]0,T[\to E \otimes \Lambda^{k-1}T^*M$, the implication*

$$\sup_{(x,t)\in\mathcal{D}_{R/2}(X,T)} \frac{1}{(R/(2c_{n,k}))^{n-2k}} \iint_{E^{n-2k}_{R/(2c_{n,k})}(x,t)} e^{\xi_k(t-s)}[u,g]_{(x,t)}(y,s)\mathrm{dvol}_g(y)\mathrm{d}s < \varepsilon$$

$$\Rightarrow \sup_{\alpha\in]0,1[} \left(\frac{\alpha R}{2}\right)^{2k} \sup_{\mathcal{D}_{(1-\alpha)R/2}(X,T)} \frac{1}{2}|\psi|^2 \leq C$$

holds.

Proof. We proceed by contradiction. Suppose this result is false. Then there exists a sequence $\{R_j\}_{j\in\mathbb{N}} \subset]0, R_0]$ and a sequence of Yang-Mills or harmonic map heat flow pairs $\{(u_j,\psi_j)\}_{j\in\mathbb{N}}$ defined on $\overline{\mathcal{D}_{R_j}(X,T)}$ for each $j \in \mathbb{N}$ such that the ε-regularity condition holds with $\varepsilon = \frac{1}{j}$, $R = R_j$, $u = u_j$ and $\psi = \psi_j$, and

$$(\beta_j)^{2k} := \sup_{\alpha\in]0,1[} \left(\frac{\alpha R_j}{2}\right)^{2k} \sup_{\mathcal{D}_{(1-\alpha)\frac{R_j}{2}}(X,T)} \frac{1}{2}|\psi_j|^2 \xrightarrow{j\to\infty} \infty.$$

By smoothness, there exist $\alpha_j \in]0,1]$ and $(x_j,t_j) \in \overline{\mathcal{D}_{(1-\alpha_j)\frac{R_j}{2}}(X,T)}$ such that

$$(\beta_j)^{2k} = \left(\frac{\alpha_j R_j}{2}\right)^{2k} \cdot \frac{1}{2}|\psi_j|^2(x_j,t_j).$$

By passing to a subsequence if necessary, we may assume that (x_j,t_j) is a convergent sequence. Now, note that:

(1) There holds

$$\sup_{\mathcal{D}_{(1-\frac{\alpha_j}{2})\frac{R_j}{2}}(X,T)} \frac{1}{2}|\psi|^2 = \left(\frac{4}{\alpha_j R_j}\right)^{2k} \cdot \left(\left(\frac{\alpha_j R_j}{4}\right)^{2k} \sup_{\mathcal{D}_{(1-\frac{\alpha_j}{2})\frac{R_j}{2}}(X,T)} \frac{1}{2}|\psi|^2\right) \leq \left(\frac{4\beta_j}{\alpha_j R_j}\right)^{2k}$$

(2) $\mathcal{D}_{\frac{\alpha_j}{2}\cdot\frac{R_j}{2}}(x_j,t_j) \subset \mathcal{D}_{(1-\frac{\alpha_j}{2})\frac{R_j}{2}}(X,T)$ since if $(y,s) \in \mathcal{D}_{\frac{\alpha_j R_j}{4}}(x_j,t_j)$, we have

$$d(y,X) \leq d(y,x_j) + d(x_j,X) < \frac{\alpha_j R_j}{4} + (1-\alpha_j)\frac{R_j}{2} = (1-\frac{\alpha_j}{2})\frac{R_j}{2},$$

whereas $s - T < t_j - T < 0$ and

$$\begin{aligned} s - T > t_j - T - \left(\frac{\alpha_j R_j}{4}\right)^2 &> -\left[\left((1-\alpha_j)\frac{R_j}{2}\right)^2 + \left(\frac{\alpha_j R_j}{4}\right)^2\right] \\ &= -\left(\left(1-\frac{\alpha_j}{2}\right)\frac{R_j}{2}\right)^2 + \underbrace{\left(\frac{R_j}{2}\right)^2 \alpha_j(1-\alpha_j)}_{>0}. \end{aligned}$$

Altogether, we have

$$\sup_{\mathcal{D}_{\frac{\alpha_j R_j}{4}}(x_j,t_j)} \frac{1}{2}|\psi|^2 \leq \left(\frac{4\beta_j}{\alpha_j R_j}\right)^{2k} =: \lambda_j^{-2k}. \tag{3.1}$$

We now rescale u_j appropriately. Set

$$u_j'(x,t) = (u_j)_{\lambda_j}^{(x_j,t_j)}.$$

This defines a smooth harmonic map (resp. Yang-Mills) heat flow $u_j' : \overline{\mathcal{D}_{\beta_j}(0,0)} \to V \otimes \Lambda^{k-1}T^*\mathbb{R}^n$. Moreover, $\frac{1}{2}|\psi_j'|^2(0,0) = \lambda_j^{2k} \cdot \frac{1}{2}|\psi_j'|^2(x_j,t_j) = \left(\frac{1}{2}\right)^{2k}$ and, in light of (3.1), we have $\sup_{\overline{\mathcal{D}_{\beta_j}(0,0)}} \frac{1}{2}|\psi_j'|^2 \leq 1$ and thus, by the Shi trick,

$$\sup_{\mathcal{D}_{\frac{\beta_j}{2}}(0,0)} \frac{1}{2}|\nabla^N \psi_j'|^2 \leq \mathrm{const}(N,B,E,\Omega)$$

for all $N \in \mathbb{N}$. Now, for $(x,t) = (x_j,t_j)$, the ε-regularity condition reads

$$\frac{1}{\left(\frac{R_j}{2c_{n,k}}\right)^{n-2k}} \iint_{E^{n-2k}_{\frac{R_j}{2c_{n,k}}}((x_j,t_j))} [u_j,g]_{(x_j,t_j)}(x,t) \cdot e^{\xi(t-t_j)} \mathrm{dvol}_g(x)\,\mathrm{d}t < \frac{1}{j}.$$

For fixed $\overline{R} > 0$, we have that $\frac{\overline{R}\alpha_j c_{n,k}}{2\beta_j} < 1$ for large j so that by monotonicity, we also have

$$\frac{1}{(\lambda_j \overline{R})^{n-2k}} \iint_{E^{n-2k}_{\lambda_j\overline{R}}((x_j,t_j))} [u_j,g]_{(x_j,t_j)}(x,t) \cdot e^{\xi(t-t_j)} \mathrm{dvol}_g(x)\,\mathrm{d}t < \frac{1}{j}.$$

Using scale invariance, we therefore have

$$\frac{1}{\overline{R}^{n-2k}} \iint_{E^{n-2k}_{\overline{R}}((0,0))} [u_j', g_{\lambda_j}^{(x_j,t_j)}]_{(0,0)}(x,t) \cdot e^{\xi(-\lambda_j^2 t)} \sqrt{g_{\lambda_j}^{(x_j,t_j)}(x)}\,\mathrm{d}x\,\mathrm{d}t < \frac{1}{j}. \tag{3.2}$$

Since we may choose a nice *gauge*, we also have a supremum bound for u_j'. Thus, by the Arzela-Ascoli theorem, there exists a subsequence of $\{u_j'\}$ and a smooth Dirichlet-type flow $u_\infty : \mathbb{R}^n \times]-\infty, 0] \to V \otimes \Lambda^{k-1}T^*\mathbb{R}^n$ such that $u_j' \xrightarrow{j\to\infty} u_\infty$ locally uniformly in

C^∞. From the above, we see that we must also have $\frac{1}{2}|\psi_\infty|^2(0,0) = \left(\frac{1}{2}\right)^{2k}$. However, (3.2) implies that for all $\overline{R} > 0$, we have

$$\frac{1}{\overline{R}^{n-2k}} \iint_{E_{\overline{R}}(0,0)} [u_\infty, \delta]_{(0,0)} \equiv 0,$$

i.e. u_∞ is a homothetic solution; but then u_∞ is constant so that $\psi_\infty \equiv 0$, which is a contradiction.

□

Proof of Theorem 3.1. Let $u : \mathcal{D}_R(X,T) \to E \otimes \Lambda^{k-1}T^*M$ be a harmonic map or Yang-Mills heat flow and consider $u_\delta : \overline{\mathcal{D}_{\frac{15}{16}R}(X,T)} \to E \otimes \Lambda^{k-1}T^*M$ defined by $u_\delta(y,s) = u(y, s-\delta)$ for $\delta \in \left]0, (1-(\frac{15}{16})^2)\frac{R^2}{4}\right[$. u_δ solves the same flow equation as u so that Lemma 3.2 applies. Choosing $\alpha = \frac{7}{15}$ then yields the desired estimate. □

4. The singular set

We now make use of the local regularity theorem to draw conclusions about the singular set at the maximal time. First of all, it follows from the local regularity theorem that if a singularity forms at $(X,T) \in \Omega \times \mathbb{R}$, then $X \in \mathcal{S}$, where

$$\mathcal{S} = \left\{ y \in \Omega : \forall R < R_0(y)\ \exists (x,t) \in \mathcal{D}_{\frac{R}{2}}(y,T) \text{ s.t. } \frac{1}{\left(\frac{R}{2c_{n,k}}\right)^{n-2k}} \iint_{E^{n-2k}_{\frac{R}{2c_{n,k}}}((x,t))} [u,g]\, \mathrm{d}\widetilde{\mu}_k \geq \varepsilon \right\}.$$

The following lemma tells us that we can actually measure $\mathcal{S}$ and gives us another necessary condition for singularity formation.

Lemma 4.1. *Suppose the pair (u,ψ) arises from a Yang-Mills or harmonic map heat flow on $\Omega \times [0,T[$. Then $\mathcal{S}$ is closed in Ω. Moreover, we have the inclusion*

$$\mathcal{S} \subset \{ y \in \Omega :\ \forall R < R_0(y)\ \ \frac{C_{n,k}}{R^{n-2k+2}} \iint_{\mathcal{D}_R(y,T)} \frac{1}{2}|\psi|^2 \geq \varepsilon \} \tag{4.1}$$

with $C_{n,k} > 0$ depending only on n, k, the geometries of Ω and $E \to \Omega$ and the structure of the corresponding flow equation.

Proof. If $X \in \Omega \backslash \mathcal{S}$, then there exists an $R < R_0(X)$ such that for all $(x,t) \in \mathcal{D}_{\frac{R}{2}}(X,T)$

$$\frac{1}{(R/(2c_{n,k}))^{n-2k}} \iint_{E^{n-2k}_{R/(2c_{n,k})}(x,t)} e^{\xi_k(t-s)} [u,g]_{(x,t)}(y,s) \mathrm{dvol}_g(y)\text{ş} < \varepsilon.$$

If $\widetilde{X} \in B_{\widetilde{R}/2}(X)$ with $\widetilde{R} = \frac{R}{2}$, then $\mathcal{D}_{\widetilde{R}/2}(\widetilde{X},T) \subset \mathcal{D}_{R/2}(X,T)$ so that by monotonicity,

$$\frac{1}{\left(\widetilde{R}/(2c_{n,k})\right)^{n-2k}} \iint_{E^{n-2k}_{\widetilde{R}/(2c_{n,k})}(x,t)} e^{\xi_k(t-s)} [u,g]_{(x,t)}(y,s) \mathrm{dvol}_g(y)\text{ş} < \varepsilon$$

for all $(x,t) \in \mathcal{D}_{\widetilde{R}/2}(\widetilde{X},T)$. Since $\widetilde{R} < R_0(\widetilde{X})$, we therefore have that $B_{\widetilde{R}/2}(X) \subset \Omega \backslash \mathcal{S}$, i.e. $\mathcal{S}$ is closed in Ω.

As for the inclusion (4.1), suppose $y \in \mathcal{S}$ and fix a sequence $R_i \searrow 0$. Then for sufficiently large i, there is a sequence $(x_i, t_i) \in \mathcal{D}_{\frac{R_i}{4}}(X,T)$ such that

$$\frac{1}{\left(\frac{R_i}{4c_{n,k}}\right)^{n-2k}} \iint_{E^{n-2k}_{\frac{R_i}{4c_{n,k}}}((x,t))} [u,g]\,\mathrm{d}\widetilde{\mu}_k \geq \varepsilon.$$

For fixed $R < R_0(y)$, monotonicity and the L^2-estimate (Lemma 2.6) give us

$$\frac{C_{n,k}}{R^{n-2k+2}} \iint_{\mathcal{D}_{R/2}(x_i,t_i)} \frac{1}{2}|\psi|^2 \geq \varepsilon.$$

For sufficiently large i, $\mathcal{D}_{R/2}(x_i,t_i) \subset \mathcal{D}_R(X,T)$, which yields the desired inclusion. □

Now that we know that $\mathcal{S}$ is measurable, we estimate its $(n-2k)$-dimensional Hausdorff measure.

Corollary 4.2. *Suppose (u,ψ) arises from a Yang-Mills or harmonic map heat flow on $\Omega\times[0,T[$ and $|\psi|^2 \in L^1(\Omega\times[0,T[)$. Then for any $K \Subset \Omega$ and $\delta_0 \in \left]0, \min\{R_0, \mathfrak{r}(K,\partial\Omega), \sqrt{T}\}\right[$, the estimate*

$$\mathcal{H}^{n-2k}(\mathcal{S}\cap K) \leq C_{n,k} \sup_{t\in\left]T-\delta_0^2,T\right[} \int_{B_{\delta_0}(\mathcal{S}\cap K)} \frac{1}{2}|\psi|^2(\cdot,t)\mathrm{dvol}_g < \infty$$

holds, where $C_{n,k,\Omega,E}$ is a constant depending only on n, k the geometries of Ω and $E \to \Omega$, and the structure of the PDE satisfied by u.

Proof. Fix $\delta \leq \delta_0$. We cover $\mathcal{S}\cap K$ by the family of balls $\{B_\delta(x)\}_{x\in\mathcal{S}\cap K}$. By compactness and the Vitali covering lemma, we may pass to a finite pairwise disjoint subfamily $\{B_\delta(x_i)\}_{i\in I_0}$ such that $\mathcal{S}\cap K \subset \bigcup_{i\in I_0} B_{5\delta}(x_i)$. Therefore, we have from Lemma 4.1 that

$$\begin{aligned}
&\sum_{i\in I_0}(5\delta)^{n-2k}\\
&\leq \frac{5^{n-2k}C_{n,k}}{\varepsilon}\sum_{i\in I_0}\delta^{-2}\int_{T-\delta^2}^{T}\int_{B_\delta(x_i)}\frac{1}{2}|\psi|^2\\
&\leq \frac{5^{n-2k}C_{n,k}}{\varepsilon}\sup_{t\in\left]T-\delta_0^2,T\right[}\int_{B_{\delta_0}(\mathcal{S}\cap K)}\frac{1}{2}|\psi|^2(\cdot,t).
\end{aligned}$$

Using the definition of $\mathcal{H}^{n-2k}$ then yields the desired estimate. □

5. Rapidly-forming singularities

We now turn our attention to rapidly-forming (or type-I) singularities of the harmonic map and Yang-Mills heat flows. The pair (u,ψ) is said to have a *rapidly-forming singularity* at (X,T) if the scale-invariant estimate

$$\sup_{(x,t)\in\mathcal{D}_R(X,T)} (T-t)^k\frac{1}{2}|\psi|^2(x,t) \leq \widetilde{C}_0 \tag{5.1}$$

holds for some $\widetilde{C} > 0$ and $R > 0$. We first show that rapidly-forming singularities admit smooth blow-ups. All of the results of this section have been established for compact base manifolds M by Grayson and Hamilton [8] for the harmonic map heat flow and Weinkove [14] for the Yang-Mills heat flow.

Lemma 5.1. *Suppose (u,ψ) arises from a Yang-Mills or harmonic map heat flow defined on $\Omega\times[0,T[$ and has a rapidly-forming singularity at $(X,T)\in\Omega\times\mathbb{R}$. Then there exists a Yang-Mills (resp. harmonic map) heat flow $\{u_\infty(\cdot,t)\}_{t<0}$ on $(\mathbb{R}^n,\delta)$ such that $u_r^{(X,T)}\xrightarrow{r\searrow 0}u_\infty$ subsequentially and locally uniformly in $C^\infty(\mathbb{R}^n)$.*

Proof. We shall assume without loss of generality that $R<i_0$. Fix $(x_0,t_0)\in\mathcal{D}_{\frac{R}{2}}(X,T)$. We have that $\mathcal{D}_{\widetilde{R}}(x_0,t_0)\subset\mathcal{D}_R(X,T)$ with $\widetilde{R}:=\sqrt{T-t_0}$. Moreover, the estimate (5.1) implies that

$$\sup_{\mathcal{D}_{\widetilde{R}}(x_0,t_0)}\frac{1}{2}|\psi|^2\le\frac{\widetilde{C}}{\widetilde{R}^{2k}}.$$

Lemma 2.1 then implies that for each $i\in\mathbb{N}$,

$$\sup_{\mathcal{D}_{\frac{\widetilde{R}}{2}}(x_0,t_0)}\frac{1}{2}|\nabla^i\psi|^2\le\frac{\widetilde{C}_i}{\widetilde{R}^{2(k+i)}},$$

where $\widetilde{C}_i$ depends on i, the local geometries of Ω and $E\to\Omega$, and $\widetilde{C_0}$. Therefore, we have the estimate

$$\sup_{(x,t)\in\mathcal{D}_{\frac{R}{2}}(X,T)}(T-t)^{k+i}\frac{1}{2}|\nabla^i\psi|^2(x,t)\le\widetilde{C}_i.$$

Using either a nice gauge for the Yang-Mills heat flow or the compactness of the target manifold of the harmonic map heat flow, we also obtain a bound $\sup_{\mathcal{D}_{\theta R}(X,T)}\frac{1}{2}|u|^2\le\frac{\widetilde{C}_{-1}}{(T-t)^{k-1}}$ with $\theta\in\left]0,\frac{1}{2}\right]$ and $\widetilde{C}_{-1}$ depending on $\widetilde{C}_1$ and the local geometries of Ω and E. Altogether, these estimates imply that

$$\sup_{(x,t)\in\mathcal{D}_{\frac{\theta R}{r}}(0,0)}(-t)^{k+i}\frac{1}{2}|\nabla^i\psi_r^{(X,T)}|^2(x,t)\le\widetilde{C}_i$$

and $\sup_{(x,t)\in\mathcal{D}_{\frac{\theta R}{r}}(0,0)}(-t)^{k-1}\frac{1}{2}|u_r^{(X,T)}|^2(x,t)\le\widetilde{C}_{-1}$. Thus, considering a sequence $\{u_i=u_{r_i}^{(X,T)}\}_{i\in\mathbb{N}}$ of rescalings of u with $r_i\searrow 0$, we obtain, for large enough i, bounds on u_i and all of its derivatives on compact subsets of $\mathbb{R}^n\times]-\infty,0[$. Therefore, by the Arzel-Ascoli theorem, we may pass to a subsequence that converges locally uniformly in C^∞ to a smooth solution $u_\infty:\mathbb{R}^n\times]-\infty,0[\to V\otimes\Lambda^{k-1}T^*\mathbb{R}^n$ to the corresponding flow equation on $(\mathbb{R}^n,\delta)$. □

The following corollary establishes that smooth blow-ups of rapidly forming singularities are in fact self-similar and gives us a lower bound on its *heat ball energy*; the latter guarantees that such blow-ups are non-trivial.

Corollary 5.2. *Let $\{u_{r_i}^{(X,T)}\}_{i\in\mathbb{N}}$ be a sequence of rescalings ($r_i\searrow 0$) of a Yang-Mills (or harmonic map) heat flow with rapidly forming singularity at (X,T). Moreover,* *suppose* $u^{(X,T)}_{r_i} \xrightarrow{i\to\infty} u_\infty$ *locally uniformly in* $C^\infty(\mathbb{R}^n)$. *Then* u_∞ *is self similar. Moreover,* u_∞ *satisfies the estimate*

$$\frac{1}{R^{n-2k}} \iint_{E_R^{n-2k}((0,0))} \frac{1}{2}|\psi_\infty|^2(\partial_t\phi_{(x,t)} + |\nabla\phi_{(x,t)}|^2)\,\mathrm{d}x\,\mathrm{d}t \geq \varepsilon \tag{5.2}$$

for all $R > 0$.

Proof. We first establish self-similarity. By virtue of the higher-order estimates derived in Lemma 5.1, the monotone quantity (?) is finite on u for sufficiently small R. Therefore, we may take its limit as $R \searrow 0$ and freely apply scale-invariance so that for any $r > 0$,

$$\begin{aligned}
&\lim_{R\searrow 0} \frac{1}{R^{n-2k}} \iint_{E_R^{n-2k}((X,T))} e^{\xi_k(T-s)}[u,g]_{(X,T)}(y,s)\ \mathrm{dvol}_g(y)\,\mathrm{d}s\\
&= \lim_{i\to\infty} \frac{1}{(r_i r)^{n-2k}} \iint_{E_{r_i r}^{n-2k}((X,T))} e^{\xi_k(T-s)}[u,g]_{(X,T)}(y,s)\ \mathrm{dvol}_g(y)\,\mathrm{d}s\\
&= \lim_{i\to\infty} \frac{1}{r^{n-2k}} \iint_{E_r^{n-2k}((0,0))} e^{\xi_k(-\lambda_i^2 s)}[u_i, g_{r_i}^{(X,T)}]_{(0,0)}(y,s)\ \sqrt{g_{r_i}^{(X,T)}}(y,s)\,\mathrm{d}y\,\mathrm{d}s\\
&= \frac{1}{r^{n-2k}} \iint_{E_r^{n-2k}((0,0))} [u_\infty,\delta]_{(0,0)}(y,s)\,\mathrm{d}y\,\mathrm{d}s.
\end{aligned}$$

Thus, this last quantity is independent of r and so, by the monotonicity formula, u_∞ must be self similar.

We now turn our attention to the estimate (5.2). We first fix a sequence $\lambda_i \searrow 0$. Since $X \in \mathcal{S}$, there exists a sequence $(x_i,t_i) \in \mathcal{D}_{c_{n,k} r_i r}(X,T)$ such that for sufficiently large i and small fixed $r > 0$, we have

$$\frac{1}{(\lambda_i r)^{n-2k}} \iint_{E_{\lambda_i r}^{n-2k}((x_i,t_i))} [u,g]_{(x_i,t_i)}(y,s)e^{\xi(t_i-s)}\mathrm{dvol}_g(y)\,\mathrm{d}s \geq \varepsilon.$$

Using monotonicity and translation invariance, we arrive at the inequality

$$\frac{1}{r^{n-2k}} \iint_{E_r^{n-2k}((0,0))} [u_1^{(x_i,t_i)}, g_1^{(x_i,t_i)}]_{(0,0)}(y,s)e^{\xi(-s)}\sqrt{g_1^{(x_i,t_i)}}(y)\,\mathrm{d}y\,\mathrm{d}s \geq \varepsilon.$$

Taking the limit $i \to \infty$, noting that $(x_i,t_i) \xrightarrow{i\to\infty} (X,T)$ and using the higher order estimates of Lemma **??**, we obtain the inequality

$$\frac{1}{r^{n-2k}} \iint_{E_r^{n-2k}((X,T))} [u,g]_{(X,T)}(y,s)e^{\xi(T-s)}\mathrm{dvol}_g(y)\,\mathrm{d}s \geq \varepsilon.$$

We now let $r = \lambda_i$, use scale-invariance and pass to the limit $i \to \infty$ to obtain (5.2). □

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