Local monotonicity for some geometric flows on evolving manifolds

Ahmad Afuni

Abstract

We establish a local monotonicity identity for vector bundle-valued differential forms on superlevel sets of appropriate heat kernel-like functions. As a consequence, we obtain local monotonicity formulæ for the harmonic map and Yang-Mills heat equations on evolving manifolds. We also show how these methods yield corresponding local monotonicity formulæ for the Yang-Mills-Higgs heat equation and mean curvature flow in an evolving background.

Keywords local monotonicity • geometric heat flows • harmonic map heat flow • Yang-Mills heat flow • mean curvature flow • evolving manifolds

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1 Introduction

It is well known that the fundamental solution representation formula, a consequence of the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n} u\cdot\Gamma_{(X,T)}(\cdot,t)=0 \tag{1.1}$$

for suitable $u:\mathbb{R}^n\times\,]0,\infty[\,\to\mathbb{R}$ with u a solution to the heat equation and $\Gamma_{(X,T)}(x,t)=\frac{1}{(4\pi(T-t))^{\frac{n}{2}}}\exp\left(\frac{|x-X|^2}{4(t-T)}\right)$ the canonical backward heat kernel concentrated at $(X,T)\in\mathbb{R}^n\times\,]0,\infty[$, is ubiquitous in the study of the behaviour of solutions to the heat equation; less well known however is the so-called *heat ball formula* [12, 26]

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r^n}\iint\limits_{\{\Gamma_{(X,T)}>r^{-n}\}} u(x,t)\cdot\frac{|x-X|^2}{4(t-T)^2}\,\mathrm{d}x\,\mathrm{d}t\right)=0 \tag{1.2}$$

for sufficiently small $r>0$ which, in contrast to (1.1), has as region of integration a relatively compact set. Analogues of (1.1) have been used with much success in the study of solutions to nonlinear heat-type equations such as the harmonic map heat equation in higher dimensions:

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Struwe [25] showed that if $u : \mathbb{R}^n \times]0,\infty[\to N \subset \mathbb{R}^n$ with N a Riemannian submanifold of $\mathbb{R}^n$ solves the *harmonic map heat equation*, i.e. for all $(x,t) \in \mathbb{R}^n \times]0,\infty[$ $(\partial_t - \Delta)u(x,t) \perp T_{u(x,t)}N$, *and* is in a suitable growth class at infinity, then

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left((T-t)\int_{\mathbb{R}^n} \frac{1}{2}|\mathrm{d}u|^2 \cdot \Gamma_{(X,T)}(x,t)\,\mathrm{d}x\right) \\
&= -(T-t)\int_{\mathbb{R}^n} \left|\partial_t u + \sum_{i=1}^n \frac{(x-X)^i}{2(t-T)}\partial_i u\right|^2 \cdot \Gamma_{(X,T)}(x,t)\,\mathrm{d}x;
\end{aligned} \tag{1.3}$$

in this case, the right-hand side is nonpositive and zero iff u is parabolically scale-invariant, a fact that has deep implications in the study of the formation of singularities. Similar formulæ have been used by Huisken to study the mean curvature flow [18] as well as Chen and Shen and Hong and Tian to study the Yang-Mills flow [5, 17], though formulæ akin to (1.2) have only recently been exploited in the nonlinear setting, most notably by Ecker for the mean curvature flow with Euclidean background, a certain reaction-diffusion equation and the harmonic map heat equation on Euclidean space [7, 9]; for the last, the formula takes the form

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r^{n-2}}\iint\limits_{E_r(X,T)} \frac{1}{2}|\mathrm{d}u|^2(x,t)\cdot\frac{n-2}{2(T-t)} - \left\langle \sum_{i=1}^n \frac{(x-X)^i}{2(t-T)}\partial_i u, \partial_t u + \sum_{i=1}^n \frac{(x-X)^i}{2(t-T)}\right\rangle \mathrm{d}x\,\mathrm{d}t\right) \\
&\qquad = \frac{n-2}{r^{n-1}}\iint\limits_{E_r(X,T)} \left|\partial_t u + \sum_{i=1}^n \frac{(x-X)^i}{2(t-T)}\partial_i u\right|^2 \mathrm{d}x\,\mathrm{d}t,
\end{aligned} \tag{1.4}$$

where $E_r(X,T) = \{(x,t) \in \mathbb{R}^n \times]0,T[\,:\, 4\pi(T-t)\Gamma_{(X,T)}(x,t) > r^{-n}\}$. In a sense, this formula is a more natural analogue of the well-known monotonicity formula for *harmonic maps* due to Schoen and Uhlenbeck and Price [24, 23], viz. solutions u to the harmonic map heat equation with $\partial_t u \equiv 0$, which reads

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r^{n-2}}\int_{B_r(X)} \frac{1}{2}|\mathrm{d}u|^2(x)\,\mathrm{d}x\right) \geq 0.$$

It was recently shown by Ecker, Knopf, Ni and Topping [11] that an analogue of (1.2) holds on *evolving* Riemannian manifolds: If $(M^n, \{g_t\}_{t\in[0,T_\infty[})$ is a smooth oriented manifold equipped with a one-parameter family of Riemannian metrics, then the identity

$$\begin{aligned}
&\left[\frac{1}{r^n}\iint\limits_{\{\Phi>r^{-n}\}} u(x,t)\cdot|\nabla\log\Phi|^2(x,t)\mathrm{dvol}_{g_t}(x)\,\mathrm{d}t\right]_{r=r_1}^{r=r_2} \\
&= \int_{r_1}^{r_2}\left(\frac{n}{r^{n+1}}\iint\limits_{\{\Phi>r^{-n}\}} -\log(r^n\Phi(x,t))\cdot(\partial_t - \Delta_{g_t})u(x,t)\right. \\
&\qquad\qquad \left. - u(x,t)\cdot\frac{(\partial_t + \Delta_{g_t} + \frac{1}{2}\mathrm{tr}_g\partial_t g)\Phi}{\Phi}\mathrm{dvol}_g(x)\,\mathrm{d}t\right)\mathrm{d}r,
\end{aligned} \tag{1.5}$$

holds for sufficiently small positive $r_1 < r_2$ whenever these integrals make sense, where now Δ_{g_t} is the Laplace-Beltrami operator of g_t, $u : M \times]0,\infty[\to \mathbb{R}$, and $v : M \times]0,T[\to \mathbb{R}^+$ is in some sense 'backward heat kernel-like'; thus, it may be seen that the right-hand side vanishes if u solves the heat equation and Φ solves the so-called backward heat equation $(\partial_t + \Delta_{g_t} + \frac{1}{2}\mathrm{tr}_g\partial_t g)\Phi = 0$, where this equation arises naturally from the identity

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_M u(x,t)\Phi(x,t)\mathrm{dvol}_{g_t}(x)\\ &\qquad= \int_M (\partial_t - \Delta_{g_t})\, u\cdot\Phi(x,t) + u\cdot\left(\partial_t + \Delta_{g_t} + \frac{1}{2}\mathrm{tr}_g\partial_t g\right)\Phi(x,t)\mathrm{dvol}_{g_t}(x).\end{aligned}$$

The main purpose of this paper is to establish the following analogue of (1.5) for vector bundle-valued differential k-forms (Theorem 5.1): If $E \to M^n$ is a finite-dimensional Riemannian vector bundle equipped with a smooth one-parameter family of Riemannian connections $\{\nabla^t\}_{t\in]0,T[}$ and $\{\psi_t = \psi(\cdot,t) : M \to E\otimes\Lambda^k T^*M\}_{t\in[0,T[}$ is a smooth one-parameter family of sections, then the identity

$$\begin{aligned}&\left[\frac{1}{r^{n-2k}}\iint\limits_{E_r^{n-2k}(\Phi)} \frac{1}{2}|\psi|^2\cdot\left(\partial_t\phi + |\nabla\phi|^2\right) - \left\langle \iota_{\nabla\phi}\psi, \iota_{\nabla\phi}\psi - \delta^\nabla\psi\right\rangle - \left\langle \iota_{\nabla\phi}\mathrm{d}^\nabla\psi, \psi\right\rangle \mathrm{dvol}_{g_t}\ \mathrm{d}t\right]_{r=r_1}^{r=r_2}\\
&= \int_{r_1}^{r_2}\left(\frac{n-2k}{r^{n-2k+1}}\iint\limits_{E_r^{n-2k}(\Phi)} -\frac{1}{2}|\psi|^2\cdot\frac{1}{\Phi}\left(\partial_t + \Delta + \frac{1}{2}\mathrm{tr}_g\partial_t g + \frac{k}{T-t}\right)\Phi(\cdot,t) - \left\langle(\partial_t + \Delta^\nabla)\psi,\psi\right\rangle\right.\\
&\qquad\qquad + |\iota_{\nabla\phi}\psi - \delta^\nabla\psi|^2 + |\mathrm{d}^\nabla\psi|^2\\
&\qquad\left. + \left\langle \nabla^2\phi + \frac{1}{2}\partial_t g + \frac{1}{2(T-s)}g, \sum_{i,j}\left\langle \iota_{\varepsilon_i}\psi, \iota_{\varepsilon_j}\psi\right\rangle \omega^i\otimes\omega^j\right\rangle \mathrm{dvol}_{g_t}\ \mathrm{d}t\right)\mathrm{d}r \qquad (1.6)\end{aligned}$$

holds for sufficiently small $r_1 < r_2$ whenever $(x,t)\mapsto \frac{|\psi(x,t)|^2}{T-t}$ is summable over $E_{r_2}^{n-2k}(\Phi)$, $n > 2k$ and these integrals make sense, where $E_r^{n-2k}(\Phi) = \{\Phi > r^{2k-n}\}$, v is 'weighted backward heat kernel-like' in an appropriate sense, $\phi = \log\Phi$, d^∇, δ^∇ and Δ^∇ are the exterior covariant differential, codifferential and Laplacian, ι is the interior product, and $\{\varepsilon_i\}_{i=1}^n$ and $\{\omega^i\}_{i=1}^n$ form local frames for TM and T^*M respectively (cf. §2 for details).

This formula shares many of the features of (1.5): The first term in the integrand of the right-hand side vanishes if $\Phi(x,t) = (4\pi(T-t))^k\Gamma(x,t)$ and Γ solves the backward heat equation, and the second term vanishes if $(\partial_t + \Delta^\nabla)\psi = 0$, i.e. ψ solves the heat equation; however, though the following two terms are nonnegative, the last term, which in the case $\partial_t g \equiv 0$ reduces to Hamilton's *matrix Harnack form* and in the case of Ricci flow ($\partial_t g = -2\mathrm{Ric}$) is an expression that characterises certain solitons, does not in general have a sign. Nevertheless, in certain cases (cf. Corollary 5.5), it may be controlled in such a way as to allow us to derive monotonicity formulæ from (1.6).

Besides being an interesting formula in its own right, (1.6) may be used to prove local monotonicity formulæ for solutions to nonlinear heat-type equations possessing a canonical differential form satisfying a heat-type equation, such as is the case for the Yang-Mills heat equation, where ψ is taken to be the curvature of the connection, and the harmonic map heat equation, where ψ is taken to be the differential of the map (cf. Examples 2.5 and 2.6); for the latter choice of ψ, we obtain a generalisation of (1.4), and for the former, a new local monotonicity formula (cf. §5).

The structure of this paper is as follows. In §2, we fix notation, describe our geometric setup, give a brief exposition of the backward heat kernel-like functions and heat-type equations to be considered in the sequel. A generalised notion of heat ball adapted from that in [11] is then introduced in §3, where examples and integration-by-parts formulæ à la [7] are given. In §4 we derive a monotonicity identity along the lines of (1.3) which shall serve to provide an adaptation of that formula to this setting, as well as an estimate that provides us with natural conditions under which the integrals of (1.6) are finite for suitable ψ and Φ. In §5 we derive (1.6) and apply it to ψ solving the heat-type equations introduced in §2 and suitable Φ. Finally, in §6 we generalise Ecker's heat ball formula for the mean curvature flow to the case where the embeddings map into an evolving manifold.

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2 Setup

2.1 Notation. We denote by $\mathbb{R}^+$ the positive reals and $\mathbb{N}$ the positive integers. For $x \in \mathbb{R}$, we set $x^+ = \max\{0, x\}$ and $x^- = \min\{0, x\}$. Moreover, given smooth manifolds M, N and P, we write $C^{k,l}(M \times N, P)$ for the set of all maps $M \times N \to P$ k-times differentiable in the first entry and l-times differentiable in the second, writing $C^k(M \times N, P)$ whenever $k = l$. By $\mathrm{pr}_1 : M \times N \to M$ and $\mathrm{pr}_2 : M \times N \to N$ we denote the respective (smooth) projections. Furthermore, if $E \to M$ is any vector bundle, $\Gamma(E)$ shall denote the $C^\infty(M)$-module of (smooth) sections of E. Finally, we fix a function $\chi \in C^2(\mathbb{R}, [0, 1])$ such that $\chi|_{]-\infty,\frac{1}{2}]} \equiv 0$, $\chi' \geq 0$ and $\chi|_{[1,\infty[} \equiv 1$ and define for each $q \in \mathbb{N}$ the function $\chi_q(x) = \chi(2^q x)$. Writing χ_A for the characteristic function of a set A, we may verify that $\chi_q \xrightarrow{q\to\infty} \chi_{]0,\infty[}$ and $x\chi_q'(x) \xrightarrow{q\to\infty} 0$ pointwise.

2.2 Geometry. Throughout this paper, $(M, \{g_t\}_{t\in[0,T_\infty[})$ shall be an *evolving* oriented manifold, i.e. M is to be assumed oriented and equipped with a smooth one-parameter family $\{g_t\}_{t\in[0,T_\infty[}$ of Riemannian metrics; if $\partial_t g \equiv 0$, we call M *static* and set $T_\infty = \infty$. We denote by TM its tangent bundle, T^*M its cotangent bundle, $\Lambda^k T^*M$ the kth exterior product bundle of T^*M and $(\Lambda T^*M, \wedge)$ the exterior algebra bundle of T^*M. We shall append g_t (or t for brevity) to all of the usual quantities and operators of Riemannian geometry and $x \in M$ as a subscript whenever appropriate, whence we shall write tr_{g_t} for the trace with respect to g_t, ∇_{g_t} for the gradient of a function, div_{g_t} for the divergence, $\nabla^2_{g_t}$ for the Hessian, $\Delta_{g_t} := \mathrm{tr}_{g_t} \circ \nabla^2_{g_t}$ for the Laplace-Beltrami operator on functions, $d^{g_t}(x, \cdot)$ for the geodesic distance from x, $\mathrm{inj}^{g_t}_x$ for the injectivity radius at x, dvol_{g_t} for the volume form and Vol_{g_t} for the volume of a set with respect to the Borel measure induced by dvol_{g_t}. We furthermore introduce a Borel measure μ on $M \times [0, T_\infty[$ induced by $\mathrm{d}\mu := \mathrm{dvol}_{g_t} \wedge \mathrm{d}t$ and introduce the shorthand notation

$$\iint_{\mathcal{D}} f \,\mathrm{d}\mu(x,t) := \iint_{\mathcal{D}} f(x,t) \,\mathrm{d}\mu(x,t)$$

and

$$\int_\Omega f \mathrm{dvol}_{g_t} := \int_\Omega f(x,t) \mathrm{dvol}_{g_t}(x),$$

where $\mathcal{D} \subset M \times [0, T_\infty[$ and $\Omega \subset M$ are Borel-measurable sets; thus, the latter notation indicates that the integrand is to be restricted to the time slice $\Omega \times \{t\}$. If

$$\iint_{\mathcal{D}} |f|^p \, \mathrm{d}\mu(x,t) < \infty$$

for a μ-measurable function $f : \mathcal{D} \to \mathbb{R}$ and $p \geq 1$, we write $f \in L^p(\mathcal{D})$ (or simply $f \in L^p$) and furthermore write $f \in L^\infty$ if f is essentially bounded.

We denote the g_t-geodesic ball of radius r centered at $X \in M$ by $B_r^t(X)$ and define for $(X,T) \in M \times]0, T_\infty]$ and $r_1, r_2 > 0$ the *spacetime cylinder*

$$\mathcal{D}_{r_1,r_2}(X,T) = \bigcup_{t \in](T-r_2)^+, T[} B_{r_1}^t(X) \times \{t\}.$$

In applications, we shall be interested in spacetime points (X,T) about which there is a spacetime cylinder where the injectivity radius at X may be suitably controlled; to this end, whenever $T < T_\infty$, we may fix $\delta \in]0, T[$ such that for all $t \in]T-\delta, T[$

$$\mathrm{inj}_X^{g_t} > \frac{\mathrm{inj}_X^{g_T}}{2} =: j_0 \tag{2.1}$$

which furnishes us with local geometry bounds of the form

$$\begin{aligned} \Lambda_{-\infty}\mathfrak{r}(\cdot,t)^2 g_t^{\mathfrak{r}} \leq g_t - \nabla^2\left(\frac{1}{2}\mathfrak{r}(\cdot,t)^2\right) &\leq \Lambda_\infty \mathfrak{r}(\cdot,t)^2 g_t^{\mathfrak{r}} \\ \lambda_{-\infty}\mathfrak{r}(\cdot,t)^2 \leq \partial_t \mathfrak{r}(\cdot,t)^2 &\leq \lambda_\infty \mathfrak{r}(\cdot,t)^2 \\ \vartheta_{(X,t)}^* \mathrm{dvol}_{g_t} &\leq \sigma \, \mathrm{d}x \end{aligned} \tag{2.2}$$

on $B_{j_0}^t(X)$ for all $t \in]T-\delta, T[$, where $\mathfrak{r}(x,t) := d^{g_t}(x,X)$, $\Lambda_{\pm\infty}, \sigma \in \mathbb{R}$ are constants depending on the geometry of M in $\mathcal{D}_{j_0,\delta}(X,T)$; $\lambda_{\pm\infty} \in \mathbb{R}$ are such that

$$\lambda_{-\infty} g_t \leq \partial_t g_t \leq \lambda_\infty g_t \tag{2.3}$$

on $B_{j_0}^t(X)$ for all $t \in]T-\delta, T[$; $g_t^{\mathfrak{r}} = g_t - \mathrm{d}\mathfrak{r}(\cdot,t) \otimes \mathrm{d}\mathfrak{r}(\cdot,t)$; $\vartheta_{(X,t)} : B_{j_0}(0) \to B_{j_0}^t(X)$ defines g_t-exponential coördinates at X; and $\mathrm{d}x = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$ is the canonical volume form on $\mathbb{R}^n$.

We shall also *always* suppose given a finite-dimensional Riemannian vector bundle $E \to M$ equipped with a smooth one-parameter family of Riemannian connections $\{\nabla^t\}_{t \in]0,T[}$, $T \leq T_\infty$. The Riemannian metrics on TM and E induce a metric on $E \otimes \Lambda^k T^*M$ for each $k \in \mathbb{N}$ and $t \in [0, T_\infty[$ and thus on $E \otimes \Lambda T^*M$ which we denote simply by $\langle \cdot, \cdot \rangle_t$ (with norm $|\cdot|_t$) and make the identification $E \otimes \Lambda^0 T^*M \cong E$; moreover, we write ι for the (fibrewise) interior product $TM \times (E \otimes \Lambda T^*M) \to E \otimes \Lambda T^*M$ and $(\cdot,\cdot)$ for the (fibrewise) canonical nondegenerate bilinear pairing $(E \otimes \Lambda T^*M) \times \Lambda TM \to E$.

We denote by d^{∇^t} and δ^{∇^t} the exterior covariant differential and codifferential respectively; these are induced by ∇^t and the Levi-Civita connection on TM induced by g_t, and given in any local g_t-orthonormal frame $\{\varepsilon_i(\cdot,t)\}$ for TM with dual coframe $\{\omega^i(\cdot,t)\}$ by

$$\mathrm{d}^{\nabla^t}\psi = \sum_{i=1}^n \omega^i(\cdot,t) \wedge \nabla^t_{\varepsilon_i(\cdot,t)}\psi$$

and

$$\delta^{\nabla^t}\psi = -\sum_{i=1}^{n} \iota_{\varepsilon_i(\cdot,t)} \nabla_{\varepsilon_i(\cdot,t)}\psi,$$

where $\psi : M \to E\otimes\Lambda T^*M$ is a smooth section of $E\otimes\Lambda T^*M$ and ∇^t denotes the connection on $E\otimes\Lambda T^*M$ induced by the connection ∇^t on E and the Levi-Civita connection on TM induced by g_t. These operators satisfy the relation

$$\left\langle \mathrm{d}^{\nabla^t}\psi_1, \psi_2 \right\rangle = \left\langle \psi_1, \delta^{\nabla^t}\psi_2 \right\rangle + \mathrm{div}\left(\sum_{i=1}^{n} \langle \psi_1, \iota_{\varepsilon_i}\psi_2\rangle\, \omega^i\right) \tag{2.4}$$

for all $\psi_1, \psi_2 \in \Gamma(E\otimes\Lambda T^*M)$, which motivates the so-called *Hodge Laplacian* $\Delta^{\nabla^t} := \mathrm{d}^{\nabla^t}\delta^{\nabla^t} + \delta^{\nabla^t}\mathrm{d}^{\nabla^t}$.

In the sequel, we shall omit t from all of the symbols introduced above for brevity and write e.g. $\mathrm{d}^{\nabla}\psi$ for $(x,t) \mapsto \left(\mathrm{d}^{\nabla^t}\psi_t\right)(x)$ and $|\psi|$ for $(x,t) \mapsto |\psi_t|_t(x)$.

2.3 Kernels. We proceed to introduce the heat kernel-type objects to be used in our considerations.

If the metrics $\{g_t\}_{t\in[0,T_\infty[}$ are complete, it is well known that the system

$$\left(\partial_t + \Delta + \frac{1}{2}\mathrm{tr}_g\partial_t g\right) v = 0 \text{ on } M\times\,]0,T[$$
$$\lim_{t\nearrow T} v(\cdot,t) = \delta_X,$$

where δ_X is the delta distribution and the limit is to be interpreted in the distributional sense, admits a unique minimal positive solution $\mathrm{P}_{(X,T)} : M\times[0,T[\,\to \mathbb{R}^+$ for each $(X,T) \in M\times\,]0,T_\infty]$ which we refer to as the *canonical backward heat kernel* concentrated at (X,T). A related function defined independently of the completeness of $\{g_t\}$ that shall be of use to us in the sequel is the *Euclidean backward heat kernel* concentrated at (X,T); this function is defined as the (locally Lipschitz) map $\Gamma_{(X,T)} : M\times[0,T[\,\to\mathbb{R}^+$ such that

$$\Gamma_{(X,T)}(x,t) = \frac{1}{(4\pi(T-t))^{n/2}} \exp\left(\frac{d^{g_t}(x,X)}{4(t-T)}\right).$$

If $(M,\{g_t\}) \equiv (\mathbb{R}^n,\delta)$, then $\mathrm{P}_{(X,T)} \equiv \Gamma_{(X,T)}$. More generally, we have the following asymptotics.

Theorem 2.1. *Suppose M is complete. For all $\varepsilon > 0$ there exist a relatively compact neighbourhood Ω of X; $\tau_0 \in]0,T[$; and $\xi \in C^\infty(\Omega\times[\tau_0,T],\mathbb{R}^+)$ with $\xi(X,T) = 1$ such that on $\Omega\times[\tau_0,T[$,*

$$\left|\mathrm{P}_{(X,T)} - \xi\cdot\Gamma_{(X,T)}\right| \le \varepsilon.$$

Proof. See [11, Lemma 21] or [28, Proposition 5.1] in the case where sec, $\partial_t g$ and $\nabla\partial_t g$ are bounded. ☐

Though $\Gamma_{(X,T)}$ is not a kernel per se, we shall make heavy use of it in the sequel as its definition, together with the local geometry bounds (2.2), enables us to explicity compute and estimate its derivatives rather easily; we summarize the relevant consequences in the following proposition. In order to tame the unwieldy inequalities that are to follow, we adopt the notation $a \sim (a_1, a_2)$ to express the inequality $a_1 \leq a \leq a_2$ whenever $a \in \mathbb{R}$ and $(a_1, a_2) \in \mathbb{R}^2$, here considered an $\mathbb{R}$-vector space; moreover, if $b_{\pm\infty} \in \mathbb{R}$, we shall write $b_{\pm\infty}$ for $(b_{-\infty}, b_\infty) \in \mathbb{R}^2$ and $b_{\mp\infty}$ for $(b_\infty, b_{-\infty}) \in \mathbb{R}^2$ in such relations.

Proposition 2.2. *If $(X,T) \in M \times]0, T_\infty[$, then $\Gamma_{(X,T)}$ is smooth on $\mathcal{D}_{j_0,\delta}(X,T)$ with j_0 and δ as before and, setting $\gamma_{(X,T)}(x,t) = \log \Gamma_{(X,T)}(x,t)$, the identities*

$$\partial_t \gamma_{(X,T)}(x,t) \sim \frac{n}{2(T-t)} - \frac{\mathfrak{r}(x,t)^2}{4(t-T)^2} + \frac{\lambda_{\mp\infty}\mathfrak{r}(x,t)^2}{4(t-T)}; \tag{2.5}$$

$$\nabla \gamma_{(X,T)}(x,t) = \frac{\mathfrak{r}(x,t)}{2(t-T)} \nabla \mathfrak{r}(x,t); \tag{2.6}$$

$$\left(\nabla^2 \gamma_{(X,T)}(x,t) + \frac{g}{2(T-t)}\right)(x,t) \sim \Lambda_{\pm\infty}\left[\log\left(\frac{1}{(4\pi(T-t))^{\frac{n}{2}}\Gamma_{(X,T)}}\right) g^{\mathfrak{r}}\right](x,t); \text{ and} \tag{2.7}$$

$$\begin{aligned}&(\partial_t + \Delta + \frac{1}{2}\mathrm{tr}_g \partial_t g)\Gamma(x,t) \\ &\sim \left[\left([(n-1)\Lambda_{\pm\infty} + \lambda_{\mp\infty}] \log\left(\frac{1}{(4\pi(T-t))^{\frac{n}{2}}\Gamma_{(X,T)}(x,t)}\right) + \frac{n}{2}\lambda_{\pm\infty}\right)\Gamma_{(X,T)}\right](x,t)\end{aligned} \tag{2.8}$$

hold for $(x,t) \in \mathcal{D}_{j_0,\delta}(X,T)$, where $\lambda_\pm$ and $\Lambda_\pm$ are as in (2.2).

Proof. We directly compute that

$$\partial_t \gamma_{(X,T)}(x,t) = \frac{n}{2(T-t)} - \frac{\mathfrak{r}(x,t)^2}{4(t-T)^2} + \frac{\partial_t\left(\mathfrak{r}(x,t)^2\right)}{4(t-T)}$$

which together with the local geometry bounds (2.2) yields (2.5). Similarly, we compute that

$$\nabla \gamma_{(X,T)}(x,t) = \frac{1}{2(t-T)} \nabla\left(\frac{1}{2}\mathfrak{r}(\cdot,t)^2\right)(x),$$

which is clearly equal to (2.6) and, differentiating once more, we obtain

$$\nabla^2 \gamma_{(X,T)}(x,t) = -\frac{1}{2(T-t)} \nabla^2\left(\frac{1}{2}\mathfrak{r}(\cdot,t)^2\right)(x),$$

whence, utilising (2.2) again, we obtain (2.7). Finally, since

$$(\partial_t + \Delta + \frac{1}{2}\mathrm{tr}_g \partial_t g)\Gamma_{(X,T)} = (\partial_t \gamma_{(X,T)} + |\nabla \gamma_{(X,T)}|^2 + \mathrm{tr}_g \nabla^2 \gamma_{(X,T)})\Gamma_{(X,T)},$$

we obtain using the preceding computations and the identity $\mathrm{tr}_g g = n$ that

$$\begin{aligned}&(\partial_t + \Delta + \frac{1}{2}\mathrm{tr}_g \partial_t g)\Gamma_{(X,T)}(x,t) \\ &\qquad = \left[\left(\frac{\partial_t \mathfrak{r}^2}{4(t-T)} + \mathrm{tr}_g\left(\nabla^2\gamma_{(X,T)} + \frac{g}{2(T-t)}\right) + \frac{1}{2}\mathrm{tr}_g \partial_t g\right)\Gamma_{(X,T)}\right](x,t)\end{aligned}$$

which, together with (2.2), (2.2) and (2.3), implies (2.8). $\square$

Analogous estimates are also at our disposal for the canonical backward heat kernel in the case where M is compact and static; we shall be content with those in the following theorem.

Theorem 2.3. [20, 13] *Set $\rho_{(X,T)} = \log \mathrm{P}_{(X,T)}$. If M is closed and static, then there exist $B, C, F \in \mathbb{R}^+$ depending on the geometry of M such that the inequalities*

$$(T-t)\left|\nabla\rho_{(X,T)}\right|^2(x,t) \leq C\log\left(\frac{B}{(4\pi(T-t))^{n/2}\mathrm{P}_{(X,T)}}\right)(x,t);$$
$$\partial_t\rho_{(X,T)}(x,t) + e^{-2K(T-t)}\left|\nabla\rho_{(X,T)}\right|^2(x,t) - e^{2K(T-t)}\frac{n}{2(T-t)} \leq 0;$$
$$(T-t)\partial_t\rho_{(X,T)}(x,t) \geq -F\left(1+\log\left(\frac{B}{(4\pi(T-t))^{n/2}\mathrm{P}_{(X,T)}}\right)\right)(x,t);\ \text{and}$$
$$\left(\nabla^2\rho_{(X,T)} + \frac{g}{2(T-t)}\right)(x,t) \geq -F\left[\left(1+\log\left(\frac{B}{(4\pi(T-t))^{n/2}\mathrm{P}_{(X,T)}}\right)\right)g\right](x,t)$$

hold for $(x,t) \in M \times [T-1, T[$, where $K \geq 0$ is such that $\mathrm{Ric} \geq -Kg$, *where* Ric *denotes the Ricci curvature tensor of g. If M is of nonnegative sectional curvature and* $\nabla\mathrm{Ric} \equiv 0$, *then*

$$\left(\nabla^2\rho_{(X,T)} + \frac{g}{2(T-t)}\right)(x,t) \geq 0.$$

2.4 Flows of interest. We now turn our attention to evolution equations that may be cast as certain heat equations involving differential forms, deferring mean curvature flow to §6.

Let $\{\psi_t = \psi(\cdot,t) \in \Gamma(E\otimes\Lambda^k T^*M)\}_{t\in[0,T[}$ be a smooth one-parameter family of E-valued k-forms with $T \leq T_\infty$ and $k \in \mathbb{N}\cup\{0\}$. In applications, we shall typically assume that $\{\psi_t\}$ solves a problem of one of the following two types:

(I) ψ_t solves the *heat equation*, viz. $(\partial_t + \Delta^\nabla)\psi = 0$ on $M \times]0,T[$.

(II) ψ_t is d^{∇^t}-closed ($\mathrm{d}^{\nabla^t}\psi_t = 0$) and there exist a smooth one-parameter family of sections $\{u_t = u(\cdot,t) \in \Gamma(E\otimes\Lambda^{k-1}T^*M)\}_{t\in[0,T[}$ and a one-parameter family of vector subbundles $\{E_0^t\}_{t\in[0,T[}$ of E such that

  (II.1) $\partial_t\psi = \mathrm{d}^\nabla\partial_t u$ on $M\times[0,T[$;

  (II.2) for all $t \in]0,T[$, $\partial_t u(\cdot,t) \in \Gamma(E_0^t\otimes\Lambda^{k-1}T^*M)$ and for all (local) vector fields X on M, $\iota_X\psi_t$ is a (local) section of $E_0^t\otimes\Lambda^{k-1}T^*M$; and

  (II.3) $(\partial_t u + \delta^\nabla\psi)(x,t) \perp (E_0^t\otimes\Lambda^{k-1}T^*M)_x$ for all $(x,t) \in M\times]0,T[$.

Though a family $\{\psi_t\}$ solving a problem of type (II) does not in general solve a problem of type (I), the following Pythagoras-type identity suffices for our purposes.

Lemma 2.4. *If $\{\psi_t \in \Gamma(E\otimes\Lambda^k T^*M)\}_{t\in]0,T[}$ solves a problem of type* (II), *then the identity*

$$\left\langle\left(\partial_t + \Delta^\nabla\right)\psi,\psi\right\rangle = \left|\iota_X\psi - \delta^\nabla\psi\right|^2 - \left|\partial_t u + \iota_X\psi\right|^2$$

holds for all $X \in \Gamma(TM)$.

Proof. Note that $\Delta^\nabla \psi = \mathrm{d}^\nabla \delta^\nabla \psi$ so that, by (II.1) and (2.4), we have

$$\begin{aligned}\left\langle \left(\partial_t + \Delta^\nabla\right)\psi, \psi\right\rangle &= \left\langle \mathrm{d}^\nabla \left(\partial_t u + \delta^\nabla \psi\right), \psi\right\rangle \\ &= \left\langle \partial_t u + \delta^\nabla \psi, \delta^\nabla \psi\right\rangle + \operatorname{div} \underbrace{\left(\sum_{i=1}^n \left\langle \partial_t u + \delta^\nabla \psi, \iota_{\varepsilon_i}\psi\right\rangle \omega^i\right)}_{=0 \text{ by (II.2) and (II.3)}},\end{aligned}$$

and since $\left\langle \partial_t u, \delta^\nabla \psi\right\rangle = -|\partial_t u|^2$ by (II.2) and (II.3), we obtain the result by adding and subtracting $|\iota_X \psi|^2 - 2\left\langle \iota_X \psi, \delta^\nabla \psi\right\rangle = |\iota_X \psi|^2 + 2\left\langle \iota_X \psi, \partial_t u\right\rangle$ to the right-hand side. □

We now turn our attention to some examples which shall serve to illustrate these problems' worthiness of our consideration.

Example 2.5 (Yang-Mills heat equation)**.** Suppose $G \to P \to M$ is a principal fibre bundle with compact connected semi-simple structure group G with Lie algebra $\mathfrak{g}$ and write E for the vector bundle associated to P and the adjoint representation of G on $\mathfrak{g}$; together with the Riemannian metric induced by minus the Killing form, E is a Riemannian vector bundle. We say that a smooth one-parameter family of connections $\{\omega_t = \omega_0 + \overline{a(t)} \in \Gamma(\mathfrak{g} \otimes T^*P)\}_{t \in [0,T[}$ on P, ω_0 some fixed connection, solves the *Yang-Mills heat equation* if

$$\partial_t a(\cdot, t) + \delta^{\nabla^t} \underline{\Omega}^{\omega_t} = 0$$

on M for all $t \in \,]0,T[$, where for each $t \in [0,T[$ $a(\cdot,t) \in \Gamma(E \otimes T^*M)$ is the unique section with lift $\overline{a(t)} \in \Gamma(\mathfrak{g} \otimes T^*P)$, $\underline{\Omega}^{\omega_t} \in \Gamma(E \otimes \Lambda^2 T^*M)$ is the *curvature* of ω^t and ∇^t is the (Riemannian) connection on E induced by ω_t and the Levi-Civita connection of g_t; this equation naturally arises as the negative gradient flow of the energy density $\frac{1}{2}|\underline{\Omega}^\omega|^2$ when M is static. Taking $\psi_t = \underline{\Omega}^{\omega_t}$, it follows from the Bianchi identity $\mathrm{d}^{\nabla^t}\underline{\Omega}^{\omega_t} = 0$ and $\partial_t \underline{\Omega}^{\omega_t} = \mathrm{d}^{\nabla^t}\partial_t a(\cdot,t)$ that $\{\psi_t\}$ solves a problem of type (II) with $u = a$ and $E_0^t = E$. On the other hand, it readily follows from the aforementioned identities that $(\partial_t + \Delta^{\nabla^t})\underline{\Omega}^{\omega_t} = 0$ so that $\{\psi_t\}$ solves a problem of type (I). □

Example 2.6 (Harmonic map heat equation)**.** Suppose $(N, g_N) \subset (\mathbb{R}^K, \delta)$ is a Riemannian submanifold. A smooth one-parameter family of maps $\{u_t := u(\cdot,t) : (M, g_t) \to (N, g_N)\}_{t \in [0,T[}$ solves the *harmonic map heat equation* if for each $(x,t) \in M \times \,]0,T[$,

$$\left(\partial_t u - \Delta_g u\right)(x,t) \perp T_{u(x,t)}N \subset \mathbb{R}^K, \tag{HM}$$

where u is viewed as a map into $\mathbb{R}^K$ so that the Laplacian is applied componentwise. By taking E to be the trivial $\mathbb{R}^K$-bundle $M \times \mathbb{R}^K \to M$ equipped with the canonical Euclidean metric and flat connection, and setting $v = (M \times [0,T[\ni (x,t) \mapsto (x, u(x,t)) \in E)$ so that $v(\cdot,t)$ is u_t viewed as a section of E, we may write (HM) in the form

$$\left(\partial_t v + \delta^\nabla \mathrm{d}u\right)(x,t) \perp \left(u_t^{-1}TN\right)_x,$$

where ∇^t is induced by the connection on E and the Levi-Civita connection of g_t, $\mathrm{d}u$ denotes the differential of u as a section of $E \otimes T^*M$ and $u_t^{-1}TN$ is the pullback bundle of TN by u_t, here viewed as a subbundle of E; this equation arises as the negative gradient flow of the energy density $\frac{1}{2}|\mathrm{d}u|^2$ when M is static. A direct computation shows that $\mathrm{d}^\nabla \mathrm{d}v = 0$ whence, setting $E_0^t = u_t^{-1}TN$, $\psi_t := \mathrm{d}u$ solves a problem of type (II). □

3 Generalised heat balls

We define a notion of heat ball in an attempt to unify those of Ecker [8, 7] and Ecker, Knopf, Ni and Topping [11], in particular allowing for different powers of r, whilst also accommodating for kernels not globally defined in spacetime.

3.1 Definition and examples. Suppose we are given $\Phi \in C^{1,1}(\mathcal{D}, \mathbb{R}^+)$, where $\mathcal{D} \subset M \times]0, T[$ ($T \in]0, T_\infty]$) is open. Set

$$E_r^m(\Phi) = \left\{ \Phi > \frac{1}{r^m} \right\} = \{\log(r^m \Phi) > 0\} \subset \mathcal{D}$$

for $r > 0$ and $0 < m \leq \dim M$ and write $\phi = \log(\Phi)$ and $\phi_r^m := \log(r^m \Phi)$.

We assume that there exists an $r_0 \in]0, 1[$ such that

(HB1) $E_{r_0}^m(\Phi) \cap (M \times]0, \tau[) \Subset \mathcal{D}$ for every $\tau \in]0, T[$;

(HB2) $|\nabla \phi|^2, \partial_t \phi \in L^1(E_{r_0}^m(\Phi))$; and

(HB3) $\displaystyle\lim_{\tau \nearrow T} \int_{\{\Phi(\cdot,\tau) > \frac{1}{r_0^m}\}} |\phi| \ \mathrm{dvol}_{g_\tau} = 0.$

Definition 3.1. With the above definition and assumptions, $E_r^m(\Phi)$ is said to be an (m, Φ)-*heat ball*.

Remark 3.2. Since $r_1 < r_2 \Rightarrow E_{r_1}^m(\Phi) \subset E_{r_2}^m(\Phi)$, if r_0 satisfies the above properties then so does $r \in]0, r_0[$.

Remark 3.3. In view of (HB1) and (HB3), $\phi \in L^1(E_{r_0}^m(\Phi))$.

Remark 3.4. Note that by Remark 3.2 and (HB3), if $r < r_0 < 1$, then $\phi > -m \log r > 0$ on $E_r^m(\Phi)$ and

$$\begin{aligned} 0 = \lim_{\tau \nearrow T} \int_{\{\Phi(\cdot,\tau) > \frac{1}{r^m}\}} |\phi| \mathrm{dvol}_{g_\tau} &\geq \lim_{\tau \nearrow T} (-m \log r) \int_{\{\Phi(\cdot,\tau) > \frac{1}{r^m}\}} \mathrm{dvol}_{g_\tau} \\ &= (-m \log r) \lim_{\tau \nearrow T} \mathrm{Vol}_{g_\tau} \left(\{\Phi(\cdot, \tau) > \frac{1}{r^m}\} \right) \end{aligned}$$

so that

$$\begin{aligned} &\int_{\{\Phi(\cdot,\tau) > \frac{1}{r^m}\}} |\phi_r^m| \mathrm{dvol}_{g_\tau} \\ &\qquad \leq \int_{\{\Phi(\cdot,\tau) > \frac{1}{r^m}\}} |\phi| \mathrm{dvol}_{g_\tau} + m|\log r| \cdot \mathrm{Vol}_{g_\tau} \left(\{\Phi(\cdot, \tau) > \frac{1}{r^m}\} \right) \xrightarrow{\tau \nearrow T} 0. \end{aligned}$$

We now proceed to give examples of heat balls. For all of the following considerations, we fix $X \in M$ and suppose the geometry bounds (2.1) and (2.2) of §2.2 hold on $\mathcal{D}_{j_0,\delta}(X, T)$ with j_0 and δ as defined there.

The first example is an analogue of the Euclidean heat balls of Watson [26] and Ecker [9], the idea being to mimic their constructions with the Euclidean heat kernel.

Example 3.5 (Euclidean m-heat balls)**.** Set

$$\begin{aligned}\Phi = {}^m\Gamma_{(X,T)} &:= \left(\mathcal{D}_{j_0,\delta}(X,T) \ni (x,t) \mapsto [4\pi(T-t)]^{\frac{n-m}{2}}\right) \cdot \Gamma_{(X,T)} \\ &= \left(\mathcal{D}_{j_0,\delta}(X,T) \ni (x,t) \mapsto \frac{1}{4\pi(T-t)^{m/2}} \exp\left(\frac{d^t(X,x)^2}{4(t-T)}\right)\right)\end{aligned}$$

for fixed $m > 0$.

Note that

$$\phi_r^m(x,t) > 0 \Leftrightarrow d^t(x,X)^2 < 2m(t-T)\log\left(\frac{4\pi(T-t)}{r^2}\right) =: R_r^m(t-T)^2,$$

where we suppose $R_r^m(t-T) \geq 0$ whenever it is defined. On the other hand, since $t-T<0$ in $\mathcal{D}_{j_0,\delta}(X,T)$, we see that

$$\begin{aligned}R_r^m(t-T)^2 > 0 &\Leftrightarrow \log\left(\frac{4\pi(T-t)}{r^2}\right) < 0 \\ &\Leftrightarrow t > T - \frac{r^2}{4\pi},\end{aligned}$$

whence it is clear that

$$E_r^m(\Phi) = \left(\bigcup_{t\in]T-\frac{r^2}{4\pi},T[} B^t_{R_r^m(t)}(X) \times \{t\}\right) \bigcap \mathcal{D}_{j_0,\delta}(X,T).$$

Let

$$r_0 = \frac{1}{2}\min\left\{j_0 \cdot \sqrt{\frac{2\pi e}{m}}, \sqrt{4\pi\delta}, 1\right\}.$$

We claim that $E_r^m(\Phi)$ is an (m,Φ)-heat ball for $r < r_0$ and now proceed to verify the conditions (HB1)-(HB3).

(HB1) A quick computation shows that

$$R_r^m \leq \sqrt{\frac{m}{2\pi e}}\, r \tag{3.1}$$

wherever R_r^m is defined. Thus, we have that $R_{r_0}^m < \frac{j_0}{2}$ and, from the definition of r_0, $T - \frac{r_0^2}{4\pi} > T - \delta$, whence

$$E_r^m(\Phi) = \bigcup_{t\in]T-\frac{r^2}{4\pi},T[} B^t_{R_r^m(t)}(X) \times \{t\} \tag{3.2}$$

and

$$\overline{E_r^m(\Phi) \cap \mathrm{pr}_2^{-1}(]T-\delta,\tau[)} = \bigcup_{t\in[T-\frac{r_0^2}{4\pi},\tau]} \overline{B^t_{R_{r_0}^m(t)}(X)} \times \{t\} \Subset \mathcal{D}_{j_0,\delta}(X,T)$$

for every $\tau \in]T-\delta, T[$.

(HB2) Using exponential coördinates about X with respect to g_t ($t \in \,]T-\delta, T[$ fixed), we see by virtue of Proposition 2.2 that

$$|\nabla\phi|^2 \circ \vartheta_{(X,t)} = \frac{|x|^2}{4(T-t)^2}$$

and

$$|\partial_t\phi| \circ \vartheta_{(X,t)} \leq \frac{n}{2(T-t)} + \frac{|x|^2}{4(T-t)^2} + \frac{\lambda|x|^2}{4(T-t)}$$

hold with $\lambda \in \mathbb{R}^+$ depending on $\lambda_\pm$ and, by (3.1), $R^m_{r_0} < j_0$ so that the volume bound of (2.2) implies that

$$\begin{aligned}\int_{B^t_{R^m_{r_0}(t-T)}(X)} |\nabla\phi|^2 \mathrm{dvol}_{g_t} &\leq \sigma \int_{B_{R^m_{r_0}(t-T)}(0)} \frac{|x|^2}{4(T-t)^2}\,\mathrm{d}x \\ &= \frac{n\omega_n\sigma}{4(n+2)} \cdot \frac{R^m_{r_0}(t-T)^{n+2}}{(T-t)^2}\end{aligned}$$

and

$$\begin{aligned}&\int_{B^t_{R^m_{r_0}(t-T)}(X)} |\partial_t\phi| \mathrm{dvol}_{g_t} \\ &\quad\leq \sigma\left\{\int_{B_{R^m_{r_0}(t-T)}(0)} \frac{n}{2(T-t)} + \frac{|x|^2}{4(T-t)^2} + \frac{\lambda|x|^2}{4(T-t)}\,\mathrm{d}x\right\} \\ &\quad= \frac{n\omega_n\sigma}{2}\left\{\frac{R^m_{r_0}(t-T)^n}{T-t} + \frac{R^m_{r_0}(t-T)^{n+2}}{2(n+2)(T-t)^2} + \frac{\lambda R^m_{r_0}(t-T)^{n+2}}{2(n+2)(T-t)}\right\},\end{aligned}$$

where ω_n is the volume of $B_1(0) \subset \mathbb{R}^n$. A straightforward but lengthy computation then shows that the bounding functions of t are summable over $\left]T - \frac{r_0^2}{4\pi}, T\right[$.

(HB3) Passing to exponential coördinates as in the verification of (HB2) and using the fact that $\{\Phi(\cdot,\tau) > \frac{1}{r^m}\} = B_{R^m_{r_0}(\tau-T)}(X)$, we see that it suffices to show that

$$\lim_{\tau \nearrow T} \int_{B_{R^m_{r_0}(\tau-T)}(0)} \left(|\phi| \circ \vartheta_{(X,\tau)}\right)(x)\,\mathrm{d}x = 0. \tag{3.3}$$

However, for $\tau \in \left]T - \frac{1}{4\pi}, T\right[$, we may bound this integral as

$$\begin{aligned}&\int_{B_{R^m_{r_0}(\tau-T)}(0)} \left(|\phi| \circ \vartheta_{(X,\tau)}\right)(x)\,\mathrm{d}x \\ &\quad\leq \int_{B_{R^m_{r_0}(\tau-T)}(0)} \frac{|x|^2}{4(T-\tau)} - \frac{m}{2}\log\left(4\pi(T-\tau)\right)\mathrm{d}x \\ &\quad= n\omega_n\left(\frac{R^m_{r_0}(\tau-T)^{n+2}}{4(n+2)(T-\tau)} - \frac{m}{2}\log\left(4\pi(T-\tau)\right) R^m_{r_0}(\tau-T)^n\right) \xrightarrow{\tau\nearrow T} 0.\end{aligned}$$

Remark 3.6. If $(M, g_t) \equiv (\mathbb{R}^n, \delta)$, the preceding example reduces to the heat balls of Watson [26] for $m = n$ and to those of Ecker [9] for $m = n - \gamma$ with $\gamma \in \,]0, n[$ fixed.

Following [11], we turn our attention to heat balls constructed from the canonical backward heat kernel on M. However, for later purposes, we shall require that the backward heat kernel satisfy certain differential inequalities in heat balls which are only at our disposal on compact static manifolds. For this reason, we now consider heat balls on static compact manifolds. Strictly speaking, these do not generalize the Euclidean heat balls of Watson and Ecker, but they provide an adaptation different from that of Example 3.5 in this setting.

Example 3.7 (m-heat balls on static compact manifolds)**.** We suppose that (M, g) is compact and static and let $\mathrm{P}_{(X,T)}$ denote the canonical backward heat kernel on M centred at $(X, T) \in M \times \mathbb{R}$. By Theorem 2.1, there exists a neighbourhood $\Omega \subset M$ of $X \in M$ and $\tau_0 \in]T - 1, T[\cap]0, T[$ such that

$$\frac{1}{2}\Phi_{(X,T)} - 1 \leq \mathrm{P}_{(X,T)} \leq 2\Phi_{(X,T)} + 1 \tag{3.4}$$

on $\Omega \times [\tau_0, T[$. Hence, we set $\mathcal{D} = \Omega \times [\tau_0, T[$ and define the map

$$\begin{aligned} {}^m\mathrm{P}_{(X,T)} &: \mathcal{D} \to \mathbb{R}^+ \\ (x, t) &\mapsto (4\pi(T - t))^{\frac{n-m}{2}}\, \mathrm{P}_{(X,T)}(x, t). \end{aligned}$$

We claim that $E^m_r({}^m\mathrm{P}_{(X,T)})$ is a heat ball for

$$r < r_0 := \frac{1}{2}\min\left\{9^{-1/m}, \left(1 + 2\varrho^{-m}\right)^{-1/m}\right\},$$

where $\varrho = \min\left\{\sqrt{4\pi(T - \tau_0)}, \sqrt{\frac{2\pi e}{m}}\sup\{y \in \mathbb{R}^+ : B_y(X) \Subset \Omega\},\ r_0 \text{ of Example 3.5}\right\}$.

To simplify notation, we write P for ${}^m\mathrm{P}_{(X,T)}$ and ρ for $\log \mathrm{P}$. Now, (3.4) by $(T - t)^{\frac{n-m}{2}}$ and noting that $T - t < 1$ for $t \in [\tau_0, T[$, it is clear that

$$\frac{1}{2}\Phi - 1 \leq \mathrm{P} \leq 2\Phi + 1, \tag{3.5}$$

on $\mathcal{D}$, where Φ is as in Example 3.5.

(HB1) The inequality (3.5) immediately implies that

$$E^m_{r_0}(\mathrm{P}) \subset \mathcal{D} \cap E^m_{\widetilde{r}_0}(\Phi)$$

with $\widetilde{r}_0 = \left(\frac{2}{\frac{1}{(r_0)^m} - 1}\right)^{1/m}$ and, by (3.2),

$$E^m_{\widetilde{r}_0}(\Phi) = \bigcup_{t \in\,]T - \frac{\widetilde{r}_0^2}{4\pi}, T[} B^t_{R^m_{\widetilde{r}_0}(t-T)}(X) \times \{t\} \subset B_{\varrho_0}(X) \times]T - \frac{\widetilde{r}_0^2}{4\pi}, T[$$

with $\varrho_0 = \sqrt{\frac{m}{2\pi e}}\widetilde{r}_0$ (cf. Example 3.5). In view of the choice of r_0 above, it is easily verified that $B_{\varrho_0}(X) \Subset \Omega$ and $]T - \frac{\widetilde{r}_0^2}{4\pi}, T[\subset]\tau_0, T[$, whence

$$E^m_{r_0}(\mathrm{P}) \subset E^m_{\widetilde{r}_0}(\Phi) \tag{3.6}$$

and

$$\begin{aligned} \overline{E^m_{\widetilde{r}_0}(\mathrm{P}) \cap \mathrm{pr}_2^{-1}(]\tau_0, \tau[)} &\subset \overline{B_{\varrho_0}(X)} \times [T - \frac{\widetilde{r}_0^2}{4\pi}, \tau] \\ &\subset \Omega \times]\tau_0, T[= \mathcal{D}. \end{aligned}$$

(HB2) By Theorem 2.3,

$$|\nabla\rho(x,t)|^2 \le \frac{C}{T-t}\Big(\log B - \log\Big((4\pi(T-t))^{m/2}\cdot \mathrm{P}(x,t)\Big)\Big)$$
$$\le \frac{C}{T-t}\Big(\log B - \frac{m}{2}\log(4\pi(T-t))\Big)$$

on $E^m_{r_0}(\mathrm{P})$ with B and C geometric constants, where we have used the fact that $\log \mathrm{P}(x,t) \ge m\log r_0$ on $E^m_{r_0}(\mathrm{P})$. Likewise, we also have that

$$\partial_t\rho \ge -\frac{F}{T-t}\left(1+\log\left(\frac{B}{(4\pi(T-t))^{m/2}\cdot\mathrm{P}}\right)\right)+\frac{m-n}{2(T-t)}$$
$$\ge -\frac{F}{T-t}\Big(1+\log B - \frac{m}{2}\log(4\pi(T-t))\Big)+\frac{m-n}{2(T-t)}$$

on $E^m_{r_0}(\mathrm{P})$. Finally, again by Theorem 2.3, the upper bound

$$\partial_t\rho \le \frac{n\left(e^{2K(T-t)}-1\right)+m}{2(T-t)} - e^{-2K(T-t)}|\nabla\rho|^2,$$

holds with K as in Theorem 2.3, but then the summability of $|\partial_t\rho|$ and $|\nabla\rho|^2$ on $E^m_{r_0}(\mathrm{P})$ immediately follows from the summability of $(x,t)\mapsto \frac{1}{T-t}$ and $(x,t)\mapsto\frac{\log(4\pi(T-t))}{T-t}$ on $E^m_{\widetilde{r}_0}(\Phi)$ by the inclusion (3.6).

(HB3) Since $r_0 < 9^{-1/m}$ and hence $\widetilde{r}_0 < 4^{-1/m}$, it is clear that $\Phi > 4$ on $E^m_{\widetilde{r}_0}(\Phi)$, whence

$$1 \le \mathrm{P} \le \frac{5}{4}\Phi \Rightarrow 0 \le \rho \le \log\frac{5}{4}+\log\Phi$$

on $E^m_{r_0}(\mathrm{P})$. Thus, to establish (HB3) it suffices to show that

$$\lim_{\tau\nearrow T}\int_{\mathrm{pr}_1(E^m_{\widetilde{r}_0}(\Phi)\cap(M\times\{\tau\}))}|\log\Phi|\;\mathrm{dvol}_{g_\tau}=0,$$

but this was established in Example 3.5 (HB3). □

The following example shows that if $E^m_r(\Phi)$ is a heat ball for sufficiently small r, then so is $E^m_r(\Phi\cdot\eta)$ provided η is a sufficiently regular function, a fact that is especially useful in certain applications (cf. Remark 6.10).

Example 3.8 (Modified Heat Balls)**.** Let $E^m_r(\Phi)$ be any (m,Φ)-heat ball and let $\eta \in L^\infty(E^m_{r_0}(\Phi))\cap C^1(E^m_{r_0}(\Phi))$ such that $|\nabla\eta|^2$ and $\partial_t\eta \in L^1(E^m_{r_0}(\Phi))$. Set $\widetilde{\Phi} := e^\eta\cdot\Phi|_{E^m_{r_0}(\Phi)}$. If we write

$$\eta_{-\infty}\le\eta\le\eta_\infty$$

for $\eta_{\pm\infty}\in\mathbb{R}^\pm$, then

$$e^{\eta_{-\infty}}\Phi \le \widetilde{\Phi} \le e^{\eta_\infty}\Phi,$$

whence

$$\Big(\Phi > \min\{r,r_0\}^{-m}\Big)\subset\Big(\widetilde{\Phi} > (re^{-\eta_{-\infty}/m})^{-m}\Big)$$
$$\text{and}\ \Big(\widetilde{\Phi} > r^{-m}\Big)\subset\Big(\Phi > (\min\{r_0, re^{\eta_\infty/m}\})^{-m}\Big),$$

so that

$$E^m_{\min\{r_0, r\exp(\eta_{-\infty}/m)\}}(\Phi) \subset E^m_r(\widetilde{\Phi}) \subset E^m_{\min\{r_0, r\exp(\eta_{\infty}/m)\}}(\Phi), \tag{3.7}$$

which in turn implies that $L^1(E^m_{r\exp(\eta_\infty/m)}(\Phi)) \hookrightarrow L^1(E^m_r(\widetilde{\Phi}))$.
Set $\widetilde{r}_0 := r_0 \exp(-\eta_\infty/m)$ so that $\tilde{r}_0 \in]0, r_0[\subset]0,1[$. We now verify (HB1)-(HB3).

(HB1) (3.7) immediately implies that

$$E^m_{\widetilde{r}_0}(\widetilde{\Phi}) \cap \mathrm{pr}_2^{-1}(]0,\tau[) \subset E^m_{r_0}(\Phi) \cap \mathrm{pr}_2^{-1}(]0,\tau[) \Subset \mathcal{D}$$

for every $\tau \in]0, T[$.

(HB2) If $\widetilde{\phi} := \log \widetilde{\Phi}$, then $\widetilde{\phi} = \phi + \eta$, whence, in view of (3.7) and the following remark, the assumptions on ϕ and η imply that $\partial_t \widetilde{\phi} = \partial_t \phi + \partial_t \eta \in L^1(E^m_{\widetilde{r}_0}(\widetilde{\Phi}))$ and, since $|\nabla(\phi+\eta)|^2 \leq 2\left(|\nabla\phi|^2 + |\nabla\eta|^2\right)$, we also have that $|\nabla(\widetilde{\phi})|^2 \in L^1(E^m_{\widetilde{r}_0}(\widetilde{\Phi}))$.

(HB3) By (3.7), it suffices to show that

$$\lim_{\tau \nearrow T} \int_{\mathrm{pr}_1\left(E^m_{\widetilde{r}_0}(\Phi) \cap \mathrm{pr}_2^{-1}(\{\tau\})\right)} |\widetilde{\phi}| \mathrm{dvol}_{g_\tau} = 0,$$

but $|\widetilde{\phi}| \leq |\phi| + |\eta| \leq |\phi| + \underbrace{\max\{\eta_\infty, -\eta_{-\infty}\}}_{=:G}$, whence by Remark 3.4,

$$\begin{aligned} &\lim_{\tau \nearrow T} \int_{\mathrm{pr}_1\left(E^m_{\widetilde{r}_0}(\Phi) \cap \mathrm{pr}_2^{-1}(\{\tau\})\right)} |\widetilde{\phi}| \mathrm{dvol}_{g_\tau} \\ &\leq \lim_{\tau \nearrow T} \int_{\mathrm{pr}_1\left(E^m_{\widetilde{r}_0}(\Phi) \cap \mathrm{pr}_2^{-1}(\{\tau\})\right)} |\phi| \mathrm{dvol}_{g_\tau} + G \lim_{\tau \nearrow T} \mathrm{Vol}_{g_\tau}\left(\mathrm{pr}_1\left(E^m_{r_0}(\Phi) \cap (M \times \{\tau\})\right)\right) = 0. \end{aligned}$$

We thus call such $E^m_r(\widetilde{\Phi})$ *η-modified (m, Φ)-heat balls*, or simply *modified heat balls*. $\square$

3.2 Integration formulæ. We now derive integration formulæ for integrals over heat balls in the spirit of [7] and [9]. These shall be used repeatedly in the sequel. To this end, we shall consider the *approximate* integrals

$$J^r_q(f) := \iint f \cdot (\chi_q \circ \phi^m_r) \cdot (\chi_{]0, T-q^{-1}[} \circ \mathrm{pr}_2) \mathrm{d}\mu,$$

where χ_q is as in §2, and analyze them, as well as their derivatives with respect to r, in the limit $q \to \infty$. The idea here is that these approximate the heat ball integrals

$$I^r(f) := \iint\limits_{E^m_r(\Phi)} f \, \mathrm{d}\mu$$

which would, with the right conditions on Φ, yield an integral over $\partial E^m_r(\Phi)$ upon differentiation with respect to r (cf. [7]). However, without additional information about Φ, we wouldn't be able to utilize this technique, which is why we follow the approach of [11].

To streamline the proofs of the integration formulæ to follow, we summarize the relevant properties of these approximate integrals in the

Lemma 3.9. *Let $f \in L^1\left(E^m_{r_0}(\Phi)\right)$ and suppose J^r_q and I^r are as above. Then*

1. *Whenever $0 < r \leq r_0$, we have $|J^r_q(f)| \leq I^{r_0}(|f|)$ and $r \mapsto J^r_q(f)$ is smooth.*
2. *For every $r \in]0, r_0]$, $J^r_q(f) \xrightarrow{q\to\infty} I^r(f)$.*
3. *Whenever $0 < r_1 < r_2 < r_0$ and $\int_{r_1}^{r_2} \frac{\mathrm{d}}{\mathrm{d}r} J^r_q(f) \xrightarrow{q\to\infty} \int_{r_1}^{r_2} J$ with $J \in L^1(]r_1, r_2[)$, the identity*

$$I^{r_2}(f) - I^{r_1}(f) = \int_{r_1}^{r_2} J$$

holds. In particular, $\frac{\mathrm{d}}{\mathrm{d}r} J^r_q(f) = J$ almost everywhere on $]0, r_0[$.

Proof. 1. It is clear from the definition of χ_q that

$$|(\chi_q \circ \phi^m_r)(\chi_{]0,T-q^{-1}[} \circ \mathrm{pr}_2)| \leq \chi_{E^m_r(\Phi)} \leq \chi_{E^m_{r_0}(\Phi)}, \tag{3.8}$$

which establishes the inequality.

As for smoothness, we note that $|\frac{\mathrm{d}}{\mathrm{d}r}(\chi_q \circ \phi^m_r)| = \frac{m}{r}|\chi'_q \circ \phi^m_r| \leq \frac{\mathrm{const}(m,q)}{r}\chi_q \circ \phi^m_r$, which is summable over $[r_1, r_2]$ as a function of r, thus allowing us to differentiate once under the integral sign by standard theorems from integration theory. Smoothness then follows by taking successive derivatives iterating the same argument.

2. The inequality (3.8) immediately implies that

$$\left|f \cdot (\chi_q \circ \phi^m_r) \cdot (\chi_{]0,T-q^{-1}[} \circ \mathrm{pr}_2)\right| \leq |f|\chi_{E^m_{r_0}(\Phi)}$$

and the convergence properties of χ_q (cf. §2.1) that

$$\lim_{q\to\infty} f \cdot (\chi_q \circ \phi^m_r) \cdot (\chi_{]0,T-q^{-1}[} \circ \mathrm{pr}_2) = f\chi_{E^m_r(\Phi)}.$$

By the dominated convergence theorem, the integral and limit may be interchanged, thus implying the claim.

3. We note that, on the one hand,

$$J^{r_2}_q(f) - J^{r_1}_q(f) = \int_{r_1}^{r_2} \frac{\mathrm{d}}{\mathrm{d}r} J^r_q(f).$$

On the other, the left-hand side tends to $I^{r_2}(f) - I^{r_1}(f)$ by the preceding part and the right-hand side tends to $\int_{r_1}^{r_2} J$ by assumption. By the Lebesgue differentiation theorem,

$$\lim_{r_1 \nearrow r} \frac{1}{r - r_1} \int_{r_1}^{r} J = \lim_{r_2 \searrow r} \frac{1}{r_2 - r} \int_{r}^{r_2} J = J(r)$$

for almost every $r \in]0, r_0[$. The equality above then implies the latter claim.

□

Proposition 3.10. *Suppose $X \in C^1(E^m_{r_0}(\Phi), TM)$ is such that $X(\cdot, t)$ is a local section of TM for each $t \in \mathrm{pr}_2(E^m_{r_0}(\Phi))$ and $|X|^2 \in L^1$, $f \in C^1(E^m_{r_0}(\Phi))$ with $\partial_t f \in L^1$ and $\mathrm{tr}_g \partial_t g \in L^1$. The following implications hold for almost every $r \in]0, r_0[$:*

(i) $\mathrm{div} X \in L^1 \Rightarrow \dfrac{\mathrm{d}}{\mathrm{d}r} \iint\limits_{E_r^m(\Phi)} \langle X, \nabla\phi\rangle \,\mathrm{d}\mu = -\dfrac{m}{r} \iint\limits_{E_r^m(\Phi)} \mathrm{div}\ X \,\mathrm{d}\mu;$

(ii) $f \in L^\infty \Rightarrow \dfrac{\mathrm{d}}{\mathrm{d}r} \iint\limits_{E_r^m(\Phi)} f \cdot \partial_t \phi \ \mathrm{d}\mu = -\dfrac{m}{r} \iint\limits_{E_r^m(\Phi)} \partial_t f + \dfrac{1}{2} f\, tr_g \partial_t g \ \mathrm{d}\mu;$

(iii) $f \phi_r^m \in L^1 \Rightarrow \dfrac{\mathrm{d}}{\mathrm{d}r} \iint\limits_{E_r^m(\Phi)} f\, \phi_r^m \,\mathrm{d}\mu = \dfrac{m}{r} \iint\limits_{E_r^m(\Phi)} f \ \mathrm{d}\mu;$

(iv) $f \in L^\infty,\ \phi_{r_0}^m \cdot \mathrm{tr}_g \partial_t g \in L^1 \Rightarrow \iint\limits_{E_r^m(\Phi)} \partial_t \left(f \cdot \phi_r^m\right) \mathrm{d}\mu = - \iint\limits_{E_r^m(\Phi)} f \cdot \phi_r^m \cdot \dfrac{1}{2} tr_g \partial_t g \,\mathrm{d}\mu;$ *and*

(v) $\mathrm{div}\ X \in L^\infty \Rightarrow \iint\limits_{E_r^m(\Phi)} \mathrm{div}(X \cdot \phi_r^m) \,\mathrm{d}\mu = 0.$

Proof. (i) Note that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}r} J_q^r(\langle X, \nabla\phi\rangle) &= \frac{m}{r} \iint \underbrace{\langle X, \nabla\phi\rangle \cdot (\chi_q' \circ \phi_r^m)}_{=\langle X, \nabla(\chi_q \circ \phi_r^m)\rangle} \cdot (\chi_{]0,T-q^{-1}[} \circ \mathrm{pr}_2) \,\mathrm{d}\mu \\ &= -\frac{m}{r} J_q^r(\mathrm{div}\ X), \end{aligned}$$

where the second line follows from the fact that

$$\mathrm{div}(X \cdot (\chi_q \circ \phi_r^m)) = \langle X, \nabla(\chi_q \circ \phi_r^m)\rangle + (\mathrm{div}\ X) \cdot (\chi_q \circ \phi_r^m)$$

and an application of Gauß' theorem. Now, since div X and $\langle X, \nabla\phi\rangle \in L^1(E_{r_0}^m(\Phi))$, Lemma 3.9 implies that

$$\frac{\mathrm{d}}{\mathrm{d}r} J_q^r(\langle X, \nabla\phi\rangle) = -\frac{m}{r} J_q^r(\mathrm{div}\ X) \xrightarrow{q \to \infty} -\frac{m}{r} I^r(\mathrm{div}\ X)$$

whence it follows from the fact that

$$|\frac{-m}{r} J_q^r(\mathrm{div}\ X)| \le \frac{m}{r_1} I^{r_0}(|\mathrm{div}\ X|) \in L^1(]r_1, r_2[),$$

for $0 < r_1 < r_2 < r_0$ that

$$\int_{r_1}^{r_2} \frac{\mathrm{d}}{\mathrm{d}r} J_q^r(\mathrm{div}\ X) \mathrm{d}r \xrightarrow{q \to \infty} \int_{r_1}^{r_2} \left(-\frac{m}{r} I^r(\mathrm{div}\ X)\right) \mathrm{d}r.$$

An application of Lemma 3.9 then yields the result.

(ii) We compute again:

$$\frac{\mathrm{d}}{\mathrm{d}r}J_q^r(f\cdot\partial_t\phi) = \frac{m}{r}\iint f\cdot\underbrace{\partial_t\phi\cdot(\chi_q'\circ\phi_r^m)}_{\partial_t(\chi_q\circ\phi_r^m)}\cdot(\chi_{]0,T-q^{-1}[})\,\mathrm{d}\mu$$

$$= \frac{m}{r}\left(\left[\int f\cdot(\chi_q\circ\phi_r^m)\,\mathrm{dvol}_{g_t}\right]_{t=0}^{t=T-q^{-1}}\right.$$

$$\left. - \iint(\partial_t f + \frac{1}{2}\mathrm{tr}_g\partial_t g\cdot f)\cdot(\chi_q\circ\phi_r^m)\cdot(\chi_{]0,T-q^{-1}[}\circ\mathrm{pr}_2)\,\mathrm{d}\mu\right)$$

$$= \frac{m}{r}\left(\int f\cdot(\chi_q\circ\phi_r^m)\mathrm{dvol}_{g_{T-q^{-1}}} - J_q^r(\partial_t f + \frac{1}{2}\mathrm{tr}_g\partial_t g\cdot f)\right)$$

where in the second line we integrated by parts with respect to t and in the third made use of the fact that

$$|\chi_q\circ\phi_r^m|(\cdot,0)\le\chi_{E_r^m(\Phi)}(\cdot,0)=0.$$

The last equality follows from (HB1), viz. the fact that $E_r^m(\Phi)\cap\mathrm{pr}_2^{-1}(]0,\tau[)\Subset\mathcal{D}\subset M\times]0,T[$ for $r\le r_0$. Now, on the one hand,

$$|f\cdot(\chi_q\circ\phi_r^m)(\cdot,T-q^{-1})|\le\sup_{E_{r_0}^m(\Phi)}|f|\cdot\chi_{\{\Phi(\cdot,T-q^{-1})>r^{-m}\}},$$

whence

$$\left|\int f\cdot(\chi_q\circ\phi_r^m)\mathrm{dvol}_{g_{T-q^{-1}}}\right|\le\sup_{E_{r_0}^m(\Phi)}|f|\cdot\mathrm{Vol}_{g_{T-q^{-1}}}(\{\Phi(\cdot,T-q^{-1})>\frac{1}{r^m}\})\xrightarrow{q\to\infty}0.$$

On the other hand, $\partial_t f+\frac{1}{2}\mathrm{tr}_g\partial_t g\cdot f\in L^1(E_r^m(\Phi))$ in light of the assumptions on f and $\mathrm{tr}_g\partial_t g$. Finally,

$$|J_q^r(\partial_t f+\frac{1}{2}\mathrm{tr}_g\partial_t g\cdot f)|\le I^{r_0}(|\partial_t f|+\frac{1}{2}|\mathrm{tr}_g\partial_t g|\cdot\sup_{E_{r_0}^m(\Phi)}|f|).$$

Thus, utilizing these bounds in the same manner as in (i), it follows from an application of Lemma 3.9 that

$$\int_{r_1}^{r_2}\frac{\mathrm{d}}{\mathrm{d}r}J_q^r(f\cdot\partial_t\phi)\mathrm{d}r=\int_{r_1}^{r_2}\left(-\frac{m}{r}I_r(\partial_t f+\frac{1}{2}\mathrm{tr}_g\partial_t g\cdot f)\right)\mathrm{d}r$$

whenever $0<r_1<r_2<r_0$.

(iii) We first compute that

$$\frac{\mathrm{d}}{\mathrm{d}r}J_q^r(f\cdot\phi_r^m)=\frac{m}{r}J_q^r(f)+\frac{m}{r}\iint f\cdot(\phi_r^m\cdot\chi_q'\circ\phi_r^m)\cdot(\chi_{]0,T-q^{-1}[}\circ\mathrm{pr}_2)\,\mathrm{d}\mu. \quad (3.9)$$

Now, $f\in L^1(E_r^m(\Phi))$ so that just as before the first term in (3.9) tends to $\frac{m}{r}I_r(f)$ as $q\to\infty$. On the other hand, by definition of χ_q,

$$|\phi_r^m\cdot\chi_q'\circ\phi_r^m|\le\sup|\chi'|\cdot\chi_{(2^{-(q+1)}<\phi_r^m<2^{-q})}\xrightarrow{q\to\infty}0$$

and

$$|f\cdot(\phi_r^m\cdot\chi_q'\circ\phi_r^m)\cdot(\chi_{]0,T-q^{-1}[})|\le\sup|\chi'|\cdot f\chi_{(\phi_{r_0}^m>0)}\in L^1(\mathcal{D})$$

so that we may proceed as in the preceding proofs.

(iv) We consider

$$J_q^r(\partial_t(f\cdot\phi_r^m)) = -\iint f\cdot\phi_r^m\cdot\chi_q'\circ\phi_r^m\cdot\chi_{]0,T-q^{-1}[}\circ\mathrm{pr}_2\cdot\partial_t\phi\,\mathrm{d}\mu \tag{3.10}$$
$$-\iint f\cdot\phi_r^m\cdot\chi_q\circ\phi_r^m\cdot\chi_{]0,T-q^{-1}[}\circ\mathrm{pr}_2\cdot\frac{1}{2}\mathrm{tr}_g\partial_t g\;\mathrm{d}\mu$$
$$+\int f\cdot\phi_r^m\cdot(\chi_q\circ\phi_r^m)\mathrm{dvol}_{g_{T-q^{-1}}}$$
$$-\int f\cdot\phi_r^m\cdot(\chi_q\circ\phi_r^m)\mathrm{dvol}_{g_0}$$

where we have integrated by parts with respect to t in the second line. Now, as in the proof of (iii), $\phi_r^m\cdot\chi_q'\circ\phi_r^m\xrightarrow{q\to\infty}0$ and

$$|f\cdot\phi_r^m\cdot\chi_q'\circ\phi_r^m\cdot\partial_t\phi|\le C\sup_{E_{r_0}^m(\Phi)}|f|\cdot|\partial_t\phi|\cdot\chi_{E_{r_0}^m(\Phi)}\in L^1(\Omega\times]0,T[),$$

whence the first integral in (3.10) tends to 0 as $q\to\infty$. Furthermore, the second term in (3.10) is equal to $-J_q^r(f\cdot\phi_r^m\cdot\frac{1}{2}\mathrm{tr}_g\partial_t g)$ which tends to $-I_r(f\cdot\phi_r^m\cdot\frac{1}{2}\mathrm{tr}_g\partial_t g)$ as $q\to\infty$ in light of the inequality

$$|f\cdot\phi_r^m\cdot\frac{1}{2}\mathrm{tr}_g\partial_t g|\le\frac{1}{2}\sup_{E_{r_0}^m(\Phi)}|f|\cdot|\mathrm{tr}_g\partial_t g|\cdot|\phi_r^m|\in L^1(E_r^m(\Phi)).$$

Moreover, the third integral may be handled by estimating as follows:

$$\left|\int f\cdot\phi_r^m\cdot(\chi_q\circ\phi_r^m)\mathrm{dvol}_{g_{T-q^{-1}}}\right|\le\sup_{E_{r_0}^m(\Phi)}|f|\cdot\int|\phi_r^m|\cdot\chi_{E_r^m(\Phi)}\mathrm{dvol}_{g_{T-q^{-1}}}\xrightarrow{q\to\infty}0.$$

Finally, the fourth integral is equal to 0 since $\chi_q(\phi_r^m(x,0))\le\chi_{E_{r_0}^m(\Phi)}(x,0)=0$ by (HB1) (cf. proof of (ii)).

(v) As in the preceding proof,

$$\begin{aligned}J_q^r(\mathrm{div}(X\cdot\phi_r^m)) &= \iint\mathrm{div}(X\cdot\phi_r^m)\cdot(\chi_q\circ\phi_r^m)\cdot(\chi_{]0,T-q^{-1}[}\circ\mathrm{pr}_2)\mathrm{d}\mu\\ &= -\iint\left\langle X,\underbrace{\nabla(\chi_q\circ\phi_r^m)}_{=(\chi_q'\circ\phi_r^m)\cdot\nabla\phi}\right\rangle\cdot\phi_r^m\cdot(\chi_{]0,T-q^{-1}[}\circ\mathrm{pr}_2)\mathrm{d}\mu\\ &= -\iint\langle X,\nabla\phi\rangle\cdot\underbrace{(\phi_r^m\cdot\chi_q'\circ\phi_r^m)}_{\xrightarrow{q\to\infty}0}\cdot\chi_{]0,T-q^{-1}[}\circ\mathrm{pr}_2\;\mathrm{d}\mu,\end{aligned}$$

where the second line is a consequence of Gauß' theorem. Since

$$\left|\langle X,\nabla\phi\rangle\cdot(\phi_r^m\cdot\chi_q'\circ\phi_r^m)\cdot\chi_{]0,T-q^{-1}[}\circ\mathrm{pr}_2\right|\le C|\langle X,\nabla\phi\rangle\,\chi_{E_r^m(\Phi)}\in L^1(\Omega\times]0,T[),$$

it follows from the dominated convergence theorem that

$$J_q^r(\mathrm{div}(X\cdot\phi_r^m))\xrightarrow{q\to\infty}0.$$ □

4 Nonlocal monotonicity formulæ

In this section and the next we shall make use of the *energy-momentum tensor* $\Psi_t \in \Gamma(T^*M \otimes T^*M)$ of $\psi_t \in \Gamma(E \otimes \Lambda^k T^*M)$ given in any local frame $\{\varepsilon_i\} \leftrightarrow \{\omega^i\}$ for TM and T^*M by

$$\Psi_t = \sum_{i,j=1}^{n} \left\langle \iota_{\varepsilon_i}\psi_t, \iota_{\varepsilon_j}\psi_t \right\rangle \omega^i \otimes \omega^j - \frac{1}{2}|\psi_t|^2 g_t; \tag{4.1}$$

a straightforward albeit lengthy computation (cf. [1]) shows that it satisfies the identity

$$\mathrm{div}\ \Psi_t = -\sum_{i=1}^{n} \left(\left\langle \delta^{\nabla^t}\psi_t, \iota_{\varepsilon_i}\psi_t \right\rangle + \left\langle \iota_{\varepsilon_i} \mathrm{d}^{\nabla^t}\psi_t, \psi_t \right\rangle \right) \omega^i. \tag{4.2}$$

Theorem 4.1. *If $\varphi \in C^{2,1}(M \times [0,T[\,, \mathbb{R})$ with $\varphi(\cdot,t) \in C^2_0(M)$ for each $t \in [0,T[$, $\Phi \in C^2(\mathcal{D}, \mathbb{R}^+)$ with $\mathcal{D}$ open such that* supp $\varphi(\cdot,t) \subset \mathrm{pr}_1\,(\mathcal{D} \cap (M \times \{t\}))$ *for each $t \in \,]0,T[$, and $\{\psi_t \in \Gamma(E \otimes \Lambda^k T^*M)\}_{t\in]0,T[}$ a smooth one-parameter family of sections, then*

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_M \frac{1}{2}|\psi|^2 \Phi \varphi^2 \mathrm{dvol}_{g_t}\right) \\
&= \int_M \Bigg\{ \left[\left\langle \psi, (\partial_t + \Delta^\nabla)\psi \right\rangle \Phi + \frac{1}{2}|\psi|^2 \left(\partial_t + \Delta + \frac{1}{2}\mathrm{tr}_g \partial_t g + \frac{k}{s-t} \right) \Phi \right] \\
&\qquad - \Phi \cdot \left[|\mathrm{d}^\nabla \psi|^2 + |\iota_{\frac{\nabla\Phi}{\Phi}}\psi - \delta^\nabla\psi|^2 \right] \\
&\qquad -\Phi \left\langle \nabla^2 \log\Phi + \frac{1}{2}\partial_t g + \frac{1}{2(s-t)} g, \sum_{i,j} \left\langle \iota_{\varepsilon_i}\psi, \iota_{\varepsilon_j}\psi \right\rangle \omega^i \otimes \omega^j \right\rangle \Bigg\} \varphi^2 \\
&\qquad + |\psi|^2 \Phi\varphi \left(\partial_t \varphi + \left\langle \nabla\varphi, \frac{\nabla\Phi}{\Phi} \right\rangle \right) \\
&\qquad + 2\varphi\Phi \left[\left\langle \iota_{\nabla\varphi}\psi, \delta^\nabla\psi - \iota_{\frac{\nabla\Phi}{\Phi}}\psi \right\rangle - \left\langle \iota_{\nabla\varphi} \mathrm{d}^\nabla\psi, \psi \right\rangle \right] \mathrm{dvol}_{g_t}.
\end{aligned} \tag{4.3}$$

on $]0,T[$ for $s \geq T$.

Remark 4.2. Note that if M is compact, $\varphi \equiv 1$, $s = T$ and $\Phi(x,t) = (4\pi(T-t))^k \Gamma(x,t)$ is defined for $(x,t) \in M \times [0,T[$, we obtain the more tractable formula

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_M \frac{1}{2}|\psi|^2 (4\pi(T-t))^k \Gamma \mathrm{dvol}_{g_t}\right) \\
&= \int_M (4\pi(T-t))^k \Gamma \cdot \left(\left\langle \psi, (\partial_t + \Delta^\nabla)\psi \right\rangle - \left[|\mathrm{d}^\nabla\psi|^2 + |\iota_{\frac{\nabla\Gamma}{\Gamma}}\psi - \delta^\nabla\psi|^2 \right] \right) \\
&\qquad + \frac{1}{2}|\psi|^2 \left(\partial_t + \Delta + \frac{1}{2}\mathrm{tr}_g \partial_t g \right) \Gamma \\
&\qquad - (4\pi(T-t))^k \Gamma \cdot \left\langle \nabla^2 \log\Gamma + \frac{1}{2}\partial_t g + \frac{1}{2(T-t)} g, \sum_{i,j} \left\langle \iota_{\varepsilon_i}\psi, \iota_{\varepsilon_j}\psi \right\rangle \omega^i \otimes \omega^j \right\rangle \mathrm{dvol}_{g_t};
\end{aligned} \tag{4.4}$$

this immediately implies that the quantity $\int_M \frac{1}{2}|\psi|^2 (4\pi(T-t))^k \Gamma \mathrm{dvol}_{g_t}$ is monotone nonincreasing provided:

1. $\{\psi_t\}_{t\in[0,T[}$ solves a problem of type (I) or (II), in which case $\left\langle\left(\partial_t + \Delta^\nabla\right)\psi, \psi\right\rangle - \left[|\mathrm{d}^\nabla\psi|^2 + |\iota_{\frac{\nabla\Gamma}{\Gamma}}\psi - \delta^\nabla\psi|^2\right] = -|\mathrm{d}^\nabla\psi|^2 - |\mathcal{S}_{\psi,\Gamma}|^2$, where

$$\mathcal{S}_{\psi,\Gamma} = \begin{cases} \iota_{\frac{\nabla\Gamma}{\Gamma}}\psi - \delta^\nabla\psi, & \{\psi_t\} \text{ solves a problem of type (I)} \\ \partial_t u + \iota_{\frac{\nabla\Gamma}{\Gamma}}\psi, & \{\psi_t\} \text{ solves a problem of type (II);} \end{cases} \tag{4.5}$$

2. Γ is a (nonnegative) subsolution to the backward heat equation, i.e. $(\partial_t + \Delta + \frac{1}{2}\mathrm{tr}_g\partial_t g)\Gamma \leq 0$; and

3. The *matrix Harnack expression*

$$\nabla^2 \log\Gamma + \frac{1}{2}\partial_t g + \frac{g}{2(T-t)}$$

   is nonnegative-definite. This holds e.g. if M is static, of nonnegative sectional curvature and Ricci parallel, and Γ is a nonnegative solution to the backward heat equation on M. Another notable example is when g evolves by the Ricci flow, i.e. $\partial_t g = -2\mathrm{Ric}$, and Φ is a nonnegative solution to the backward heat equation, in which case this expression *vanishes* whenever g is a *gradient shrinking soliton* (cf. [19, Appendix C]). Note that if $k = 0$, i.e. $\psi_t \in \Gamma(E)$ for all $t \in [0, T[$, this term is absent from (4.4).

In fact, if we assume that Φ solves the backward heat equation and the matrix Harnack expression vanishes, then (4.4) yields a conservation law whenever $\mathrm{d}^\nabla\psi = 0$ and $\mathcal{S}_{\psi,f} = 0$, where the latter condition may be recast as one characterising symmetric solutions when ψ is associated to the Yang-Mills heat equation as in Example 2.5 or the the harmonic map heat equation as in Example 2.6; in the latter case, this condition is precisely $\left(\partial_t + \partial_{\frac{\nabla f}{f}}\right)u = 0$, whereas in the former case, after lifting $\frac{\nabla f}{f}(\cdot, t)$ to an ω_t-horizontal vector field X_t^ω on the total space of the bundle, it may be written as $\partial_t\omega + \mathcal{L}_{X^\omega}\omega = 0$, where $\mathcal{L}$ is the Lie derivative acting component-wise on the $\mathfrak{g}$-valued one-form ω.

Remark 4.3. More generally, if (M, g_t) is complete for each $t \in [0, T_\infty[$, the proof of Theorem 4.1 may be carried out with $\varphi \equiv 1$ to yield the formula (4.4) provided ψ does not grow too rapidly at infinity, e.g. if both integrals of (4.4) and the integral

$$\int_M \left\{ \left(\left|\frac{\nabla\Gamma}{\Gamma}\right| + \left|\frac{\nabla\Gamma}{\Gamma}\right|^2 + \left|\frac{\nabla^2\Gamma}{\Gamma}\right| \right) |\psi|^2 + \left(1 + \left|\frac{\nabla\Gamma}{\Gamma}\right|^2\right) |\mathrm{d}^\nabla\psi|^2 + |\delta^\nabla\psi|^2 + |\Delta^\nabla\psi|^2 \right\} \Gamma \mathrm{dvol}_{g_t} \tag{4.6}$$

are finite for each $t \in [0, T[$. If in particular $(M, g_t) = (\mathbb{R}^n, \delta)$ for every t, the matrix Harnack expression vanishes if we choose Γ to be the canonical backward heat kernel concentrated at (X, T) for some $X \in \mathbb{R}^n$; in this case, (4.4) yields a conservation law for self-similar solutions to the harmonic map and Yang-Mills heat equations, where in the latter case an appropriate 'gauge' must be chosen (cf. [27, Lemma 3.2]).

Proof of Theorem 4.1. We first note that

$$
\begin{aligned}
&\partial_t \left(\frac{1}{2}|\psi|^2\varphi^2\Phi \mathrm{dvol}_g\right)(\cdot,t) \\
&= \Bigg\{ \left[\left\langle \partial_t\psi + \Delta^\nabla\psi, \psi\right\rangle - \left\langle \frac{1}{2}\partial_t g, \sum_{i,j=1}^n \left\langle \iota_{\varepsilon_i}\psi, \iota_{\varepsilon_j}\psi\right\rangle \omega^i\otimes\omega^j\right\rangle\right]\varphi^2\Phi + |\psi|^2\varphi\cdot\partial_t\varphi\cdot\Phi \\
&\qquad + \frac{1}{2}|\psi|^2\varphi^2\left(\partial_t\Phi + \Delta\Phi + \frac{1}{2}\mathrm{tr}_g\partial_t g\cdot\Phi\right) \\
&\qquad - \left\langle \Delta^\nabla\psi,\psi\right\rangle\varphi^2\Phi - \frac{1}{2}|\psi|^2\varphi^2\Delta\Phi \Bigg\}(\cdot,t)\mathrm{dvol}_{g_t}. \qquad (4.7)
\end{aligned}
$$

Using (2.4), we may easily compute that

$$
\begin{aligned}
\left\langle \Delta^\nabla\psi,\psi\right\rangle\varphi^2\Phi &= \left(|\mathrm{d}^\nabla\psi|^2 + \left\langle -\delta^\nabla\psi, \iota_{\frac{\nabla\Phi}{\Phi}}\psi - \delta^\nabla\psi\right\rangle + \left\langle \iota_{\frac{\nabla\Phi}{\Phi}}\mathrm{d}^\nabla\psi,\psi\right\rangle\right)\varphi^2\Phi \\
&\quad - 2\varphi\Phi\left(\left\langle \delta^\nabla\psi, \iota_{\nabla\varphi}\psi\right\rangle - \left\langle \iota_{\nabla\varphi}\mathrm{d}^\nabla\psi,\psi\right\rangle\right) \\
&\quad + \mathrm{div}\left(\varphi^2\Phi\sum_{i=1}^n\left(\left\langle\delta^\nabla\psi,\iota_{\varepsilon_i}\psi\right\rangle - \left\langle \iota_{\varepsilon_i}\mathrm{d}^\nabla\psi,\psi\right\rangle\right)\omega^i\right).
\end{aligned}
$$

Furthermore, using (4.1) and (4.2), we may write

$$
\begin{aligned}
\frac{1}{2}|\psi|^2\Delta\Phi &= -\left\langle \nabla^2\Phi,\Psi\right\rangle + \left\langle \nabla^2\Phi, \sum_{i,j=1}^n\left\langle \iota_{\varepsilon_i}\psi,\iota_{\varepsilon_j}\psi\right\rangle\omega^i\otimes\omega^j\right\rangle \\
&= -\mathrm{div}\ \iota_{\nabla\Phi}\Psi + \iota_{\nabla\Phi}\mathrm{div}\ \Psi + \left\langle \nabla^2\Phi, \sum_{i,j=1}^n\left\langle \iota_{\varepsilon_i}\psi,\iota_{\varepsilon_j}\psi\right\rangle\omega^i\otimes\omega^j\right\rangle
\end{aligned}
$$

so that we obtain

$$
\begin{aligned}
&\frac{1}{2}|\psi|^2\Delta\Phi\cdot\varphi^2 \\
&= \left(\left\langle \nabla^2\log\Phi, \sum_{i,j=1}^n\left\langle \iota_{\varepsilon_i}\psi,\iota_{\varepsilon_j}\psi\right\rangle\omega^i\otimes\omega^j\right\rangle + \left\langle \iota_{\frac{\nabla\Phi}{\Phi}}\psi - \delta^\nabla\psi, \iota_{\frac{\nabla\Phi}{\Phi}}\psi\right\rangle - \left\langle \iota_{\frac{\nabla\Phi}{\Phi}}\mathrm{d}^\nabla\psi,\psi\right\rangle\right)\varphi^2\Phi \\
&\qquad + 2\varphi\Phi\left(\left\langle \iota_{\frac{\nabla\Phi}{\Phi}}\psi, \iota_{\nabla\varphi}\psi\right\rangle - \frac{1}{2}|\psi|^2\left\langle\frac{\nabla\Phi}{\Phi},\nabla\varphi\right\rangle\right) - \mathrm{div}\left(\iota_{\nabla\Phi}\Psi\cdot\varphi^2\right).
\end{aligned}
$$

Hence, incorporating these into (4.7) and integrating over M, noting that the t-derivative and integral may be interchanged in light of the fact that the integrand is compactly supported in M for fixed t, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_M \frac{1}{2}|\psi|^2\Phi\varphi^2 \mathrm{dvol}_{g_t}\right) = \int_M \Bigg\{ &\left[\left\langle \psi, (\partial_t + \Delta^\nabla)\psi\right\rangle \Phi + \frac{1}{2}|\psi|^2 \left(\partial_t + \Delta + \frac{1}{2}\mathrm{tr}_g \partial_t g\right)\Phi\right] \\ &- \Phi \cdot \left[|\mathrm{d}^\nabla \psi|^2 + |\iota_{\frac{\nabla\Phi}{\Phi}}\psi - \delta^\nabla\psi|^2\right] \\ &-\Phi \left\langle \nabla^2 \log \Phi + \frac{1}{2}\partial_t g, \sum_{i,j} \left\langle \iota_{\varepsilon_i}\psi, \iota_{\varepsilon_j}\psi\right\rangle \omega^i \otimes \omega^j \right\rangle \Bigg\} \varphi^2 \\ &+ |\psi|^2 \Phi\varphi \left(\partial_t\varphi + \left\langle \nabla\varphi, \frac{\nabla\Phi}{\Phi}\right\rangle\right) \\ &+ 2\varphi\Phi\left[\left\langle \iota_{\nabla\varphi}\psi, \delta^\nabla\psi - \iota_{\frac{\nabla\Phi}{\Phi}}\psi\right\rangle - \left\langle \iota_{\nabla\varphi}\mathrm{d}^\nabla\psi, \psi\right\rangle\right] \mathrm{dvol}_{g_t}.\end{aligned}$$

Adding and subtracting

$$\Phi \left\langle \frac{g}{2(t-s)}, \sum_{i,j=1}^n \left\langle \iota_{\varepsilon_i}\psi, \iota_{\varepsilon_j}\psi\right\rangle \omega^i \otimes \omega^j \right\rangle \varphi^2 = \frac{k}{t-s} \cdot \frac{1}{2}|\psi|^2\Phi\varphi^2$$

in the integrand of the right-hand side then implies the result. □

We now restrict our attention to problems of type (I) and (II). Note that (4.3) reduces to

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_M \frac{1}{2}|\psi|^2\Phi\varphi^2 \mathrm{dvol}_{g}\right) = \int_M \Bigg\{ &\left[\frac{1}{2}|\psi|^2 \left(\partial_t + \Delta + \frac{1}{2}\mathrm{tr}_g \partial_t g + \frac{k}{s-t}\right)\Phi\right] \\ &- \Phi \cdot \left[|\mathrm{d}^\nabla \psi|^2 + |\mathcal{S}_{\psi,\Phi}|^2\right] \\ &-\Phi \left\langle \nabla^2 \log \Phi + \frac{1}{2}\partial_t g + \frac{1}{2(s-t)}g, \sum_{i,j} \left\langle \iota_{\varepsilon_i}\psi, \iota_{\varepsilon_j}\psi\right\rangle \omega^i \otimes \omega^j \right\rangle \Bigg\} \varphi^2 \\ &+ |\psi|^2 \Phi\varphi \left(\partial_t\varphi + \left\langle \nabla\varphi, \frac{\nabla\Phi}{\Phi}\right\rangle\right) \\ &- 2\varphi\Phi\left[\left\langle \iota_{\nabla\varphi}\psi, \mathcal{S}_{\psi,\Phi}\right\rangle + \left\langle \iota_{\nabla\varphi}\mathrm{d}^\nabla\psi, \psi\right\rangle\right] \mathrm{dvol}_{g}. \qquad (4.8)\end{aligned}$$

with $\mathcal{S}_{\psi,\Phi}$ defined by (4.5). Note that the last two terms in the integrand of the right-hand integral in (4.8) may, by the Cauchy-Schwarz and Young inequalities, be estimated as

$$\begin{aligned}&|\psi|^2\Phi\varphi\left(\partial_t\varphi + \left\langle \nabla\varphi, \frac{\nabla\Phi}{\Phi}\right\rangle\right) - 2\varphi\Phi\left(\left\langle \iota_{\nabla\varphi}\psi, \mathcal{S}_{\psi,\Phi}\right\rangle + \left\langle \iota_{\nabla\varphi}\mathrm{d}^\nabla\psi, \psi\right\rangle\right) \\ &\le |\psi|^2\left(\varphi\left(|\partial_t\varphi|\Phi + |\nabla\varphi|\cdot|\nabla\Phi|\right) + 2|\nabla\varphi|^2\Phi\right) + \frac{1}{2}\left(|\mathcal{S}_{\psi,\Phi}|^2 + |\mathrm{d}^\nabla\psi|^2\right)\Phi\varphi^2; \qquad (4.9)\end{aligned}$$

this observation immediately leads to the following monotonicity-type identity:

Lemma 4.4. *Let* $\varphi \in C^{2,1}(M \times]T-\delta_0, T[\,, [0,1])$ *be such that*

$$\varphi|_{\mathcal{D}_{r_1,\delta_0}(X,T)} \equiv 1 \text{ and } \varphi|_{(M\times]T-\delta_0,T[)\setminus\mathcal{D}_{r_2,\delta_0}(X,T)} \equiv 0$$

for $0 < r_1 < r_2 < R$ *with* $R > 0$ *fixed,* $\delta_0 \in\,]0, T[$ *and* $\Phi \in C^{2,1}(\mathcal{D}_{R,\delta_0}(X,T), \mathbb{R}^+)$*, and suppose* $\{\psi_t \in \Gamma(E \otimes \Lambda^k T^*M)\}_{t \in]0,T[}$ *solves a problem of type (I) or (II).*

If Φ, $|\nabla\Phi|$ *are bounded on* $\mathcal{D}_{r_2,\delta_0}(X,T) \backslash \mathcal{D}_{r_1,\delta_0}(X,T)$ *and the inequalities*

$$\begin{aligned} \left(\partial_t + \Delta_g + \frac{1}{2}\mathrm{tr}_g \partial_t g + \frac{k}{s-t}\right) \Phi(x,t) &\leq a_0 + a_1(t)\Phi(x,t) \ \text{and} \\ \Phi \cdot \left(\nabla^2 \log \Phi + \frac{1}{2}\partial_t g + \frac{g}{2(s-t)}\right)(x,t) &\geq (b_0 + b_1(t)\Phi(x,t))\, g_t(x) \end{aligned} \tag{4.10}$$

hold for $(x,t) \in \mathcal{D}_{r_2,\delta_0}(X,T)$ *with* $a_1, b_1 \in C(]T-\delta_0, T[) \cap L^1(]T-\delta_0, T[)$, $a_0, b_0 \in \mathbb{R}$ *and some* $s \geq T$*, then*

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(\exp\left(\int_t^T l\right)\int_M \frac{1}{2}|\psi_t|^2 \Phi\varphi^2(\cdot,t)\mathrm{dvol}_{g_t}\right) \\ &\qquad \leq \exp\left(\int_t^T l\right)\int_M -\frac{1}{2}\Phi\varphi^2(\cdot,t)\cdot\left(|\mathrm{d}^{\nabla^t}\psi_t|^2 + |\mathcal{S}_{\psi,\Phi}|^2(\cdot,t)\right)\mathrm{dvol}_{g_t} \\ &\qquad\qquad + C_0 \int_{B^t_{r_2}(X)} \frac{1}{2}|\psi_t|^2 \mathrm{dvol}_{g_t}, \end{aligned}$$

where $l_k(t) = a_1(t) - 2kb_1(t)$ *and* $C_0 = C_0(l_k, a_0, b_0, \Phi, \varphi, r_1, r_2) > 0$.

Proof. In light of the boundedness of Φ and $|\nabla\Phi|$ on $\mathcal{D}_{r_2,\delta_0}(X,T)\backslash\mathcal{D}_{r_1,\delta_0}(X,T)$ as well as the definition of φ, the right-hand side of (4.9) may be bounded from above by

$$C_1 \chi_{\mathcal{D}_{r_2,\delta}(X,T)}|\psi|^2 + \frac{1}{2}\left(|\mathcal{S}_{\psi,\Phi}|^2 + |\mathrm{d}^\nabla\psi|^2\right)\Phi\varphi^2$$

with

$$C_1 = \sup_{\mathcal{D}_{r_2,\delta}(X,T)\backslash\mathcal{D}_{r_1,\delta}(X,T)} \left[(|\partial_t\varphi| + 2|\nabla\varphi|^2)\Phi + |\nabla\varphi|\cdot|\nabla\Phi|\right].$$

Incorporating this and (4.10) into (4.8) then yields

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_M \frac{1}{2}|\psi|^2\Phi\varphi^2\mathrm{dvol}_{g_t}\right) \\ &\qquad \leq -\frac{1}{2}\int_M \left(|\mathrm{d}^\nabla\psi|^2 + |\mathcal{S}_{\psi,\Phi}|^2\right)\varphi^2\Phi\mathrm{dvol}_{g_t} \\ &\qquad\quad + l_k(t)\int_M \frac{1}{2}|\psi|^2\cdot|\varphi|^2\Phi\mathrm{dvol}_{g_t} + \frac{a_0 - 2kb_0 + 2C_1}{2}\int_M |\psi|^2\mathrm{dvol}_{g_t}. \end{aligned}$$

Multiplying through by the integrating factor $\exp(\int_t^T l_k)$ then implies the claim with

$$C_0 = \frac{\exp(\int_{T-\delta_0}^T |l|)}{2}(a_0 - 2kb_0 + 2C_1). \tag{4.11}$$

□

Remark 4.5. Note that if

$$\int_M \frac{1}{2}|\psi|^2 \mathrm{dvol}_{g_t} \leq E_0.$$

for every $t \in]0, T[$, then Lemma 4.4 yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\exp\left(\int_t^T l_k\right)\int_M \frac{1}{2}|\psi|^2\Phi\varphi^2 \mathrm{dvol}_g + C_0 E_0(T-t)\right)$$
$$\leq \exp\left(\int_t^T l_k\right)\int_M -\frac{1}{2}\Phi\varphi^2 \cdot \left(|\mathrm{d}^\nabla \psi|^2 + |\mathcal{S}_{\psi,\Phi}|^2\right)\mathrm{dvol}_g.$$

This is exactly the observation made by Chen and Struwe [4] and Chen and Shen [5] which, when $s = T < T_\infty$ and $\Phi = {}^{n-2k}\Gamma_{(X,T)}$, then implies monotonicity formulæ for problems of type (I) and (II); to see this, note that it is clear from the definition of Φ that Φ and $|\nabla\Phi|$ are bounded on $\mathcal{D}_{r_2,\delta}(X,T)\backslash\mathcal{D}_{r_1,\delta}(X,T)$ with δ fixed as in §2.2 so that the local geometry bounds (2.1)-(2.3) hold whence, by Proposition 2.2 and the inequality $x(1+\log(y/x)) \leq 1 + x\log y$ (cf. [14]), the inequalities (4.10) hold with

$$\begin{aligned}
a_0 &= \max\{\frac{n}{2}\lambda^+_{+\infty}, [(n-1)\Lambda_{+\infty} + \lambda_{-\infty}]^+\}\\
a_1(t) &= -\frac{(n-2k)a_0}{2}\log\left(4\pi(T-t)\right)\\
b_0 &= \min\{\frac{\lambda^-_{-\infty}}{2}, \Lambda^-_{-\infty}\}\\
b_1(t) &= -\frac{(n-2k)b_0}{2}\log\left(4\pi(T-t)\right).
\end{aligned} \tag{4.12}$$

Similarly, in the case where M is compact and static and $\{\psi_t\}_{t\in[0,T[}$ solves a problem of type (I) or (II), the choices $s = T$ and $\Phi = {}^{n-2k}\mathrm{P}_{(X,T)}$ lead to cut-off analogues of monotonicity formulæ due to Hamilton [14], where, using and retaining the notation of Theorem 2.3, the inequalities (4.10) hold with

$$\begin{aligned}
a_0 &= a_1(t) = 0\\
b_0 &= -F\\
b_1(t) &= -F\log\left(\frac{B}{(4\pi(T-t))^{\frac{n-2k}{2}}}\right).
\end{aligned}$$

Note, however, that in either case despite the introduction of the cut-off function φ, this formula is still nonlocal due to the presence of E_0.

We now turn our attention to an estimate guaranteeing the finiteness of the singular integrals that occur in the local monotonicity formulæ to be derived in the next section

Lemma 4.6. *Suppose $\{\psi_t \in \Gamma\left(E\otimes\Lambda^k T^*M\right)\}_{t\in]0,T[}$, $T < T_\infty$, solves a problem of type (I) or (II), and R^m_r is as in Example 3.5. For every $0 < r < \min\left\{1, \frac{j_0}{2c_{n,k}}, \sqrt{4\pi\delta}\right\}$ with*

$c_{n,k} := \sqrt{\frac{n-2k}{2\pi e}}$ *and* $t \in \left]T - r^2 \exp\left(-\frac{1}{2(n-2k)}\right)/4\pi, T\right[$*, the estimates*

$$\frac{1}{R_r^{n-2k}(t-T)^{n-2k}} \int_{B^t_{R_r^{n-2k}(t-T)}(X)} \frac{1}{2}|\psi|^2 \mathrm{dvol}_{g_t} \leq \frac{\widetilde{C}_0}{r^{n-2k}} \left(r^{-2} \iint\limits_{\mathcal{D}_{2c_{n,k}r, \frac{r^2}{4\pi}}(X,T)} \frac{1}{2}|\psi|^2 \mathrm{d}\mu + \int_{B^{T-\frac{r^2}{4\pi}}_{2c_{n,k}r}(X)} \frac{1}{2}|\psi|^2 \mathrm{dvol}_{g_{T-\frac{r^2}{4\pi}}} \right) \tag{4.13}$$

and

$$\iint\limits_{E_r^{n-2k}\left({}^{n-2k}\Gamma_{(X,T)}\right)} |\mathcal{S}_{\psi, {}^{n-2k}\Gamma_{(X,T)}}|^2 + |\mathrm{d}^\nabla \psi|^2 \,\mathrm{d}\mu \leq \widetilde{C}_0 \left(r^{-2} \iint\limits_{\mathcal{D}_{2c_{n,k}r, \frac{r^2}{4\pi}}(X,T)} \frac{1}{2}|\psi|^2 \mathrm{d}\mu + \int_{B^{T-\frac{r^2}{4\pi}}_{2c_{n,k}r}(X)} \frac{1}{2}|\psi|^2 \mathrm{dvol}_{g_{T-\frac{r^2}{4\pi}}} \right) \tag{4.14}$$

hold, where $\widetilde{C}_0$ *is a constant depending only on* n*,* k*,* χ *and local geometry bounds (2.2).*

Proof. (cf. [9, Appendix]) We first apply Lemma 4.4 by taking

$$\varphi(x,t) = \begin{cases} \chi\left(\frac{c_{n,k}r}{\mathfrak{r}(x,t)}\right), & x \neq X \\ 1, & x = X \end{cases}$$

with χ and $\mathfrak{r}$ as in §2.1 and §2.2, and $r < \frac{j_0}{2c_{n,k}}$, whence it is easily verified that φ satisfies the hypotheses of Lemma 4.4 with $r_1 = c_{n,k}r$ and $r_2 = 2r_1$. We moreover take $\Phi = {}^{n-2k}\Gamma_{(X,s)}$, here considered a function $\mathcal{D}_{j_0,\delta}(X,T) \to \mathbb{R}$, with $s \geq T$ to be fixed. It may be shown exactly as in Proposition 2.2 that the inequalities (4.10) hold with a_0, $a_1(t)$, b_0 and $b_1(t)$ given by (4.12); moreover, a straightforward computation shows that

$$\max\{\Phi, r|\nabla\Phi|\} \leq \frac{d_{n,k}}{r^{n-2k}}$$

on $\mathcal{D}_{2c_{n,k}r,\delta}(X,T)\backslash\mathcal{D}_{c_{n,k}r,\delta}(X,T)$, where $d_{n,k}$ is a positive constant depending only on n and k. Since

$$|\nabla\varphi| \leq \frac{\sup|\chi'|}{c_{n,k}r} \cdot \chi_{\mathcal{D}_{2c_{n,k}r,\delta}(X,T)\backslash\mathcal{D}_{c_{n,k}r,\delta}(X,T)}$$

and

$$|\partial_t\varphi| \leq \frac{\sup|\chi'|}{2} \cdot \lambda \cdot \chi_{\mathcal{D}_{2c_{n,k}r,\delta}(X,T)\backslash\mathcal{D}_{c_{n,k}r,\delta}(X,T)}$$

with $\lambda = \max\{|\lambda^-_{-\infty}|, |\lambda^+_\infty\}$ and $r < 1$, the constant C_0 of Lemma 4.4 defined by (4.11) may be written

$$\begin{aligned} C_0 &= \frac{\exp\left(\int_0^T |l_k|\right)}{2} \left(a_0 - 2kb_0 + d_{n,k} \sup|\chi'| \cdot \left[\frac{\frac{4\sup|\chi'|}{c_{n,k}^2} + \frac{2}{c_{n,k}} + r^2\lambda}{r^{n-2k+2}} \right] \right) \\ &\leq \exp\left(\int_0^T |l_k|\right) \cdot \frac{\varsigma_0}{r^{n-2k+2}} \end{aligned}$$

where ς_0 is a positive constant depending only on n, k, χ and the local geometry of M about (X,T). Hence, after an integration, Lemma 4.4 implies that

$$\begin{aligned}\int_M \frac{1}{2}|\psi|^2\Phi\varphi^2 \mathrm{dvol}_{g_{t_0}} &+ \int_{T-\frac{r^2}{4\pi}}^{t_0}\int_M \frac{1}{2}\Phi\varphi^2\cdot\left(|\mathrm{d}^\nabla\psi|^2+|\mathcal{S}_{\psi,\Phi}|^2\right)\mathrm{dvol}_{g_t}\,\mathrm{d}t\\ &\le \exp\left(2\int_0^T|l_k|\right)\cdot\left[\frac{\varsigma_0}{r^{n-2k+2}}\iint\limits_{\mathcal{D}_{2c_{n,k}r,\frac{r^2}{4\pi}}(X,T)}\frac{1}{2}|\psi|^2\,\mathrm{d}\mu+\int_M\frac{1}{2}|\psi|^2\Phi\varphi^2\mathrm{dvol}_{g_{T-\frac{r^2}{4\pi}}}\right]\end{aligned}$$

for $t_0\in\left]T-\frac{r^2}{4\pi},T\right[$. Now, using $\chi_{B^t_{c_{n,k}r}(X)}\le\varphi(\cdot,t)\le\chi_{B^t_{2c_{n,k}r}(X)}$ and noting that

$$\Phi(\cdot,T-\frac{r^2}{4\pi})\le\frac{1}{(4\pi(s-T+\frac{r^2}{4\pi})^{\frac{n-2k}{2}}}\le\frac{1}{r^{n-2k}},$$

we obtain

$$\begin{aligned}\int_{B^{t_0}_{c_{n,k}r}(X)}\frac{1}{2}|\psi|^2\Phi\mathrm{dvol}_{g_{t_0}} &+\int_{T-\frac{r^2}{4\pi}}^{t_0}\int_{B^t_{c_{n,k}r}(X)}\frac{1}{2}\Phi\cdot\left(|\mathrm{d}^\nabla\psi|^2+|\mathcal{S}_{\psi,\Phi}|^2\right)\mathrm{dvol}_{g_t}\,\mathrm{d}t\\ &\le\frac{\exp\left(2\int_0^T|l_k|\right)}{r^{n-2k}}\cdot\left[\frac{\varsigma_0}{r^2}\iint\limits_{\mathcal{D}_{2c_{n,k}r,\frac{r^2}{4\pi}}(X,T)}\frac{1}{2}|\psi|^2\,\mathrm{d}\mu+\int_{B^{T-\frac{r^2}{4\pi}}_{c_{n,k}r}(X)}\frac{1}{2}|\psi|^2\mathrm{dvol}_{g_{T-\frac{r^2}{4\pi}}}\right].\end{aligned}\tag{4.15}$$

We first take $s=T$, discard the first term on the left-hand side of (4.15), use the inclusion $B^t_{R_r^{n-2k}(t-T)}(X)\subset B^t_{c_{n,k}r}(X)$ as well as the identity

$$\Phi(\cdot,t)|_{B^t_{R_r^{n-2k}(t-T)}(X)}\ge\frac{1}{r^{n-2k}},$$

and take the limit $t_0\nearrow T$ on the left hand side to obtain

$$\begin{aligned}\frac{1}{r^{n-2k}}\iint\limits_{E_r^{n-2k}\left({}^{n-2k}\Gamma_{(X,T)}\right)}&\frac{1}{2}\cdot\left(|\mathrm{d}^\nabla\psi|^2+|\mathcal{S}_{\psi,\Phi}|^2\right)\mathrm{dvol}_{g_t}\,\mathrm{d}t\\ &\le\frac{\varsigma_0\cdot\exp\left(2\int_0^T|l_k|\right)}{r^{n-2k+2}}\int_{T-\frac{r^2}{4\pi}}^T\int_{B^t_{2c_{n,k}r}(X)}\frac{1}{2}|\psi|^2\mathrm{dvol}_{g_t}\,\mathrm{d}t\\ &\quad+\frac{\exp\left(2\int_0^T|l|\right)}{r^{n-2k}}\int_{B^{T-\frac{r^2}{4\pi}}_{c_{n,k}r}(X)}\frac{1}{2}|\psi|^2\mathrm{dvol}_{g_{T-\frac{r^2}{4\pi}}},\end{aligned}$$

thus establishing (4.14). As for (4.13), we discard the second term on the left-hand side of (4.15) and take $s=t_0+R_r(t_0-T)^2$, which, for $t_0\in\left]T-e^{-\frac{1}{2(n-2k)}}\frac{r^2}{4\pi},T\right[$, is greater than T, so that

$$\Phi(\cdot,t_0)|_{B^{t_0}_{R_r^{n-2k}(t_0-T)}(X,T)}\ge\frac{\exp(-\frac{1}{4})}{(4\pi)^{\frac{n-2k}{2}}R_r^{n-2k}(t_0-T)^{n-2k}}.$$

Therefore, using that $R_r^{n-2k}(t_0-t) \le c_{n,k} r$, (4.15) implies that

$$\frac{\exp(-\frac{1}{4})}{(4\pi)^{\frac{n-2k}{2}} R_r^{n-2k}(t_0-T)^{n-2k}} \int_{B_{R_r^{n-2k}(t_0-T)}(X)} \frac{1}{2}|\psi|^2 \mathrm{dvol}_{g_{t_0}}$$
$$\le \frac{\exp\left(2\int_0^T |l_k|\right)}{r^{n-2k}} \cdot \left[\frac{\varsigma_0}{r^2} \iint\limits_{\mathcal{D}_{2c_{n,k}r, \frac{r^2}{4\pi}}(X,T)} \frac{1}{2}|\psi|^2 \mathrm{d}\mu + \int_{B^{T-\frac{r^2}{4\pi}}_{c_{n,k}r}(X)} \frac{1}{2}|\psi|^2 \mathrm{dvol}_{g_{T-\frac{r^2}{4\pi}}} \right] \tag{4.16}$$

for $t_0 \in \left]T - e^{-\frac{1}{2(n-2k)}} \frac{r^2}{4\pi}, T\right[$ which establishes (4.13). $\square$

5 Local monotonicity formulæ

We now proceed to establish local monotonicity formulæ for evolving k-forms. Throughout this section, we suppose $\dim M = n > 2k$, that $E_r^{n-2k}(\Phi)$ is an $(n-2k, \Phi)$ heat ball for $r < r_0$ and set $\phi = \log \Phi$.

Theorem 5.1. *If $\left\{\psi_t \in \Gamma(E \otimes \Lambda^k T^*M)\right\}_{t\in[0,T[}$ is a smooth one-parameter family of sections, then*

$$\left[\frac{1}{r^{n-2k}} \iint\limits_{E_r^{n-2k}(\Phi)} \frac{1}{2}|\psi|^2 \cdot \left(\partial_t\phi + |\nabla\phi|^2\right) - \left\langle \iota_{\nabla\phi}\psi, \iota_{\nabla\phi}\psi - \delta^\nabla\psi\right\rangle - \left\langle \iota_{\nabla\phi}\mathrm{d}^\nabla\psi, \psi\right\rangle \mathrm{d}\mu\right]_{r=r_1}^{r=r_2}$$
$$\ge \int_{r_1}^{r_2} \left(\frac{n-2k}{r^{n-2k+1}} \iint\limits_{E_r^{n-2k}(\Phi)} -\frac{1}{2}|\psi|^2 \cdot \left(\partial_t\phi + \Delta\phi + |\nabla\phi|^2 + \frac{1}{2}tr_g\partial_t g + \frac{k}{T-t}\right) \right.$$
$$- \left\langle(\partial_t + \Delta^\nabla)\psi, \psi\right\rangle + |\iota_{\nabla\phi}\psi - \delta^\nabla\psi|^2 + |\mathrm{d}^\nabla\psi|^2$$
$$\left. + \left\langle \nabla^2\phi + \frac{1}{2}\partial_t g + \frac{g}{2(T-t)}, \sum_{i,j}\left\langle\iota_{\varepsilon_i}\psi, \iota_{\varepsilon_j}\psi\right\rangle \omega^i\otimes\omega^j \right\rangle \mathrm{d}\mu(x,t) \right) \mathrm{d}r \tag{5.1}$$

holds whenever $0 < r_1 < r_2 < r_0$ provided both spacetime integrands are in $L^1\left(E_{r_0}^{n-2k}(\Phi)\right)$. If

$$\left((x,t) \mapsto \frac{|\psi_t(x)|^2}{T-t}\right) \in L^1\left(E_{r_0}^{n-2k}(\Phi)\right), \tag{5.2}$$

then (5.1) holds with equality.

Remark 5.2. Note that the right-hand integrand of (5.1) is equal to $-\frac{1}{\Phi}$ times that of the nonlocal monotonicity formula (4.4) so that the conditions outlined in Remark 4.2 imply that the quantity

$$\frac{1}{r^{n-2k}} \iint\limits_{E_r^{n-2k}(\Phi)} \frac{1}{2}|\psi|^2 \cdot \left(\partial_t\phi + |\nabla\phi|^2\right) - \left\langle \iota_{\nabla\phi}\psi, \iota_{\nabla\phi}\psi - \delta^\nabla\psi\right\rangle - \left\langle \iota_{\nabla\phi}\mathrm{d}^\nabla\psi, \psi\right\rangle \mathrm{d}\mu \tag{5.3}$$

is monotone nondecreasing in r; moreover, if ψ solves a problem of type (I) or (II) and the summability condition (5.2) holds, Lemma 2.4 implies that

$$|\iota_{\nabla\phi}\psi - \delta^{\nabla}\psi|^2 - \big\langle(\partial_t + \Delta^{\nabla})\psi, \psi\big\rangle = |\mathcal{S}_{\psi,\Phi}|^2$$

with $\mathcal{S}_{\psi,\Phi}$ defined by (4.5) so that (5.1) reads

$$\begin{aligned}
&\left[\frac{1}{r^{n-2k}}\iint\limits_{E_r^{n-2k}(\Phi)} \frac{1}{2}|\psi|^2\cdot\left(\partial_t\phi + |\nabla\phi|^2\right) - \langle\iota_{\nabla\phi}\psi, \mathcal{S}_{\psi,\Phi}\rangle - \Big\langle \iota_{\nabla\phi}\mathrm{d}^{\nabla}\psi, \psi\Big\rangle\,\mathrm{d}\mu\right]_{r=r_1}^{r=r_2} \\
&= \int_{r_1}^{r_2}\Bigg(\frac{n-2k}{r^{n-2k+1}}\iint\limits_{E_r^{n-2k}(\Phi)} -\frac{1}{2}|\psi|^2\cdot\left(\partial_t\phi + \Delta\phi + |\nabla\phi|^2 + \frac{1}{2}\mathrm{tr}_g\partial_t g + \frac{k}{T-t}\right) + |\mathcal{S}_{\psi,\Phi}|^2 \\
&\qquad + |\mathrm{d}^{\nabla}\psi|^2 + \left\langle \nabla^2\phi + \frac{1}{2}\partial_t g + \frac{g}{2(T-t)}, \sum_{i,j}\big\langle\iota_{\varepsilon_i}\psi, \iota_{\varepsilon_j}\psi\big\rangle\,\omega^i\otimes\omega^j\right\rangle\mathrm{d}\mu(x,t)\Bigg)\,\mathrm{d}r.
\end{aligned} \tag{5.4}$$

Hence, if $\Phi(x,t) = (T-t)^k\Gamma(x,t)$ with Γ a solution to the backward heat equation, the matrix Harnack form vanishes and ψ is symmetric in the sense that $\mathrm{d}^{\nabla}\psi$ and $\mathcal{S}_{\psi,\Phi}$ vanish, (5.3) does not depend on r. In the special case where ψ arises from a solution to the Yang-Mills or harmonic map heat equation (Examples 2.5 and 2.6), the symmetry condition may be restated in terms of one on the solution as in Remark 4.2 which, as pointed out in Remark 4.3, is equivalent to scale invariance of the solution when $(M, g_t) = (\mathbb{R}^n, \delta)$, in which case (5.4) is structurally the same as (1.4) if Γ is chosen to be the canonical backward heat kernel on $\mathbb{R}^n$.

Proof of Theorem 5.1. We first assume that $\psi_t \equiv 0$ for $\tau < t < T$, whence, since ϕ and ψ are smooth on $E_r^{n-2k}(\Phi)\cap\mathrm{pr}_2^{-1}(]0,\tau[) \Subset M\times]0,T[$, all terms occurring in the integrands of (5.1) are summable over $E_{r_0}^{n-2k}(\Phi)$. Now, note that the left-hand integrand may be written as

$$\begin{aligned}
i_1(\psi) &:= \frac{1}{2}|\psi|^2(\partial_t\phi + |\nabla\phi|^2) - \big\langle\iota_{\nabla\phi}\psi, \iota_{\nabla\phi}\psi - \delta^{\nabla}\psi\big\rangle - \Big\langle\iota_{\nabla\phi}\mathrm{d}^{\nabla}\psi, \psi\Big\rangle \\
&= -\left\langle(\iota_{\nabla\phi}\Psi)^\sharp + \sum_{i=1}^n\left(\Big\langle\iota_{\varepsilon_i}\mathrm{d}^{\nabla}\psi, \psi\Big\rangle - \big\langle\iota_{\varepsilon_i}\psi, \delta^{\nabla}\psi\big\rangle\right)\varepsilon_i, \nabla\phi\right\rangle + \frac{1}{2}|\psi|^2\partial_t\phi,
\end{aligned}$$

where Ψ is defined by (4.1). Hence, adopting the approximate integral notation of §3.2 and using the proofs of Propositions 3.10(i) and 3.10(ii), we have that

$$\begin{aligned}
\left[\frac{J_q^r(i_1(\psi))}{r^{n-2k}}\right]_{r=r_1}^{r=r_2} &= \int_{r_1}^{r_2}\frac{2k-n}{r^{n-2k+1}}J_q^r(i_1(\psi)) + \frac{1}{r^{n-2k}}\frac{\mathrm{d}}{\mathrm{d}r}J_q^r(i_1(\psi))\,\mathrm{d}r \\
&= \int_{r_1}^{r_2}\frac{n-2k}{r^{n-2k+1}}J_q^r\left(i_2(\psi) - i_1(\psi)\right)\mathrm{d}r + o(1) \text{ as } q\to\infty,
\end{aligned} \tag{5.5}$$

where

$$i_2(\psi) = \mathrm{div}\left((\iota_{\nabla\phi}\Psi)^\sharp + \sum_{i=1}^n\left(\Big\langle\iota_{\varepsilon_i}\mathrm{d}^{\nabla}\psi, \psi\Big\rangle - \big\langle\iota_{\varepsilon_i}\psi, \delta^{\nabla}\psi\big\rangle\right)\varepsilon_i\right) - \partial_t\left(\frac{1}{2}|\psi|^2\right) - \frac{1}{2}|\psi|^2\cdot\frac{1}{2}\mathrm{tr}_g\partial_t g.$$

Taking the limit $q \to \infty$ in (5.5), we obtain

$$\left[\frac{1}{r^{n-2k}} \iint\limits_{E_r^m(\Phi)} i_1(\psi)\,\mathrm{d}\mu \right]_{r=r_1}^{r=r_2} = \int_{r_1}^{r_2} \left[\iint\limits_{E_r^m(\Phi)} (i_2(\psi) - i_1(\psi))\,\mathrm{d}\mu \right] \mathrm{d}r. \tag{5.6}$$

Thus, it suffices to show that $i_2(\psi) - i_1(\psi)$ is equal to the innermost integrand of the right-hand side of (5.1). Using (4.2) and (2.4), it is easily computed that

$$\begin{aligned} i_2(\psi) - i_1(\psi) = &-\frac{1}{2}|\psi|^2 \left(\partial_t\phi + \Delta\phi + |\nabla\phi|^2 + \frac{1}{2}\mathrm{tr}_g\partial_t g \right) - \left\langle (\partial_t + \Delta^\nabla)\psi, \psi \right\rangle \\ &+ |\iota_{\nabla\phi}\psi - \delta^\nabla\psi|^2 + |\mathrm{d}^\nabla\psi|^2 + \left\langle \nabla^2\phi + \frac{1}{2}\partial_t g, \sum_{i,j=1}^{n} \left\langle \iota_{\varepsilon_i}\psi, \iota_{\varepsilon_j}\psi \right\rangle \omega^i \otimes \omega^j \right\rangle, \end{aligned}$$

whence adding and subtracting

$$(x,t) \mapsto \left\langle \frac{g}{2(T-t)}, \sum_{i,j} \left\langle \iota_{\varepsilon_i}\psi, \iota_{\varepsilon_j}\psi \right\rangle \omega^i \otimes \omega^j \right\rangle (x,t) = \frac{k}{T-t} \cdot \frac{1}{2}|\psi|^2(x,t)$$

on the right-hand side then implies the claim.

Now let $\{\psi_t\}$ be arbitrary. We apply (5.6) to the time dependent section $x \mapsto \psi_t(x) \cdot \chi_q(T-t)$ with $q \in \mathbb{N}$, which clearly vanishes near T, to obtain

$$\begin{aligned} &\left[\frac{1}{r^{n-2k}} \iint\limits_{E_r^{n-2k}(\Phi)} \chi_q(T-t)^2 \cdot i_1(\psi)\,\mathrm{d}\mu(x,t) \right]_{r=r_1}^{r=r_2} \\ &\quad = \int_{r_1}^{r_2} \left(\frac{n-2k}{r^{n-2k+1}} \iint\limits_{E_r^{n-2k}(\Phi)} \chi_q(T-t)^2 \cdot [i_2(\psi) - i_1(\psi)]\,\mathrm{d}\mu(x,t) \right) \mathrm{d}r \\ &\qquad + \int_{r_1}^{r_2} \left(\frac{n-2k}{r^{n-2k+1}} \iint\limits_{E_r^{n-2k}(\Phi)} \chi_q(T-t)\chi_m'(T-t)|\psi|^2\,\mathrm{d}\mu(x,t) \right) \mathrm{d}r. \end{aligned}$$

If the condition (5.2) holds, we may pass to the limit $q \to \infty$, where the latter integral on the right-hand side tends to 0 due to (5.2), the observation that

$$\chi_q(T-t)\chi_q'(T-t)|\psi|^2 = \chi_q(T-t) \cdot (T-t)\chi_q'(T-t) \cdot \frac{|\psi|^2}{T-t}$$

and the fact that $(T-t)\chi_q'(T-t) \xrightarrow{q\to\infty} 0$; moreover, the other interchangings of limit and integral are justified by the summability over $E_{r_0}^{n-2k}(\Phi)$ of both (innermost) integrands of (5.1). If however (5.2) doesn't hold, we may simply discard the latter integral on the right-hand side and then take limits since $\chi_q' \geq 0$. $\square$

Analogously to Lemma 4.4, even if the matrix Harnack expression and backward heat operator applied to Φ do not have the right signs, we may nevertheless obtain a local monotonicity formula provided they satisfy certain inequalities.

Corollary 5.3. *If* $\left\{\psi_t \in \Gamma(E \otimes \Lambda^k T^*M)\right\}_{t\in[0,T[}$ *is a smooth one-parameter family of sections and the inequalities*

$$\partial_t\phi + \Delta\phi + |\nabla\phi|^2 + \frac{1}{2}\mathrm{tr}_g\partial_t g + \frac{k}{T-t} \leq a(t) \tag{5.7}$$

and

$$\nabla^2\phi + \frac{1}{2}\partial_t g + \frac{g}{2(T-t)} \geq b(t)g \tag{5.8}$$

hold on $E^{n-2k}_{r_0}(\Phi)$ *with* $a, b \in C(\mathrm{pr}_2(E^{n-2k}_{r_0}(\Phi))) \cap L^1\left(\mathrm{pr}_2(E^{n-2k}_{r_0}(\Phi))\right)$*, then*

$$\left[\frac{1}{r^{n-2k}} \iint\limits_{E^{n-2k}_r(\Phi)} \frac{1}{2}|\psi|^2(\partial_t\phi + |\nabla\phi|^2) - \left\langle \iota_{\nabla\phi}\psi, \iota_{\nabla\phi}\psi - \delta^\nabla\psi\right\rangle - \left\langle \iota_{\nabla\phi}\mathrm{d}^\nabla\psi, \psi\right\rangle \mathrm{d}\widetilde{\mu}_k\right]_{r=r_1}^{r=r_2}$$
$$\geq \int_{r_1}^{r_2} \left(\frac{n-2k}{r^{n-2k+1}} \iint\limits_{E^{n-2k}_r(\Phi)} \left|\mathrm{d}^\nabla\psi\right|^2 + \left|\iota_{\nabla\phi}\psi - \delta^\nabla\psi\right|^2 - \left\langle(\partial_t + \Delta^\nabla)\psi, \psi\right\rangle \mathrm{d}\widetilde{\mu}_k(x,t)\right) \mathrm{d}r$$

for $0 < r_1 < r_2 < r_0$ *whenever the spacetime integrands are in* $L^1(E^{n-2k}_{r_0}(\Phi), \mu)$*, where*

$$\xi_k(t) = \int_t^T a - 2kb$$

and $\widetilde{\mu}_k$ *is the Borel measure on the domain of* Φ *induced by* $\mathrm{d}\widetilde{\mu}_k := \mathrm{dvol}_{g_t} \wedge e^{\xi_k(t)}\,\mathrm{d}t$*. If only the left-hand spacetime integrand is known to be summable over* $E^{n-2k}_{r_0}(\Phi)$ *and*

$$|\mathrm{d}^\nabla\psi|^2 + |\iota_{\nabla\phi}\psi - \delta^\nabla\psi|^2 - \left\langle\left(\partial_t + \Delta^\nabla\right)\psi, \psi\right\rangle \geq 0 \tag{5.9}$$

on $E^{n-2k}_{r_0}(\Phi)$*, then*

$$\left[\frac{1}{r^{n-2k}} \iint\limits_{E^{n-2k}_r(\Phi)} \frac{1}{2}|\psi|^2(\partial_t\phi + |\nabla\phi|^2) - \left\langle \iota_{\nabla\phi}\psi, \iota_{\nabla\phi}\psi - \delta^\nabla\psi\right\rangle - \left\langle \iota_{\nabla\phi}\mathrm{d}^\nabla\psi, \psi\right\rangle \mathrm{d}\widetilde{\mu}_k\right]_{r=r_1}^{r=r_2} \geq 0$$

for $0 < r_1 < r_2 < r_0$*, i.e. the parenthetical quantity is monotone nondecreasing.*

Proof. We apply Theorem 5.1 to $\psi_m(x,t) := e^{\xi_k(t)/2}\chi_m(T-t)\psi_t(x)$ to obtain

$$\left[\frac{1}{r^{n-2k}}\iint\limits_{E_r^{n-2k}(\Phi)} \chi_m(T-t)^2 e^{\xi_k(t)} i_1(\psi)\,\mathrm{d}\mu(x,t)\right]_{r=r_1}^{r=r_2}$$
$$= \int_{r_1}^{r_2}\left(\frac{n-2k}{r^{n-2k+1}}\iint\limits_{E_r^{n-2k}(\Phi)}\left(-\frac{1}{2}|\psi|^2\cdot\left(\partial_t\phi+\Delta\phi+|\nabla\phi|^2+\frac{1}{2}\mathrm{tr}_g\partial_t g+\frac{k}{T-t}\right)\right.\right.$$
$$+|\iota_{\nabla\phi}\psi-\delta^\nabla\psi|^2+|\mathrm{d}^\nabla\psi|^2-\left\langle(\partial_t+\Delta^\nabla)\psi,\psi\right\rangle$$
$$\left.+\left\langle\nabla^2\phi+\frac{1}{2}\partial_t g+\frac{g}{2(T-t)},\sum_{i,j=1}^n\left\langle\iota_{\varepsilon_i}\psi,\iota_{\varepsilon_j}\psi\right\rangle\omega^i\otimes\omega^j\right\rangle\right)e^{\xi_k(t)}\chi_m(T-t)^2$$
$$\left.+\chi_m'(T-t)\chi_m(T-t)e^{\xi_k(t)}|\psi|^2-\frac{\partial_t\xi_k}{2}\cdot e^{\xi_k(t)}\chi_m(T-t)^2|\psi|^2\,\mathrm{d}\mu(x,t)\right)\mathrm{d}r, \tag{5.10}$$

where $i_1(\psi)$ is as in the proof of Theorem 5.1. Making use of inequalities (5.7) and (5.8) and noting that $\chi_m'(T-t)\geq 0$, $\partial_t\xi = 2kb-a$ and

$$\left\langle g,\sum_{i,j=1}^n\left\langle\iota_{\varepsilon_i}\psi,\iota_{\varepsilon_j}\psi\right\rangle\omega^i\otimes\omega^j\right\rangle = k|\psi|^2,$$

we may estimate the r-integrand of the right-hand integral of equation (5.10) from below by

$$\frac{n-2k}{r^{n-2k+1}}\iint\limits_{E_r^{n-2k}(\Phi)}\chi_m(T-t)^2e^{\xi_k(t)}\left(-\frac{1}{2}|\psi|^2a+|\iota_{\nabla\phi}\psi-\delta^\nabla\psi|^2+|\mathrm{d}^\nabla\psi|^2\right.$$
$$\left.-\left\langle(\partial_t+\Delta^\nabla)\psi,\psi\right\rangle+kb|\psi|^2-(2kb-a)\cdot\frac{1}{2}|\psi|^2\right)\mathrm{d}\mu(x,t)$$
$$=\frac{n-2k}{r^{n-2k+1}}\iint\limits_{E_r^{n-2k}(\Phi)}\left(\left|\iota_{\nabla\phi}\psi-\delta^\nabla\psi\right|^2+\left|\mathrm{d}^\nabla\psi\right|^2-\left\langle(\partial_t+\Delta^\nabla)\psi,\psi\right\rangle\right)\chi_m(T-t)^2e^{\xi_k(t)}\,\mathrm{d}\mu(x,t). \tag{5.11}$$

Since $t\mapsto e^{\xi_k(t)}$ is bounded on $\mathrm{pr}_2(E_{r_0}^{n-2k}(\Phi))$, we may take limits exactly as in the preceding theorem, thus establishing the first claim. For the second, we bound the right-hand side of (5.11) from below by 0 and then take limits. □

Remark 5.4. It is clear from Remark 5.2 that if $\{\psi\}_{t\in[0,T[}$ solves a problem of type (I) or (II), the positivity condition (5.9) holds.

We now turn our attention to concrete $n-2k$-heat balls, viz. Examples 3.5 and 3.7 for $m=n-2k$, fix $X\in M$ and suppose $T<T_\infty$, assuming the local geometry bounds (2.2) and (2.3); in both cases, the corresponding kernel satisfies inequalities of the form (5.7)-(5.8) on the respecive heat ball as may be seen either by appealing to Remark 4.5 or by computing directly from Proposition 2.2 and Theorem 2.3. For $\Phi = {}^{n-2k}\Gamma_{(X,T)}$, the latter approach yields the inequalities (5.7)-(5.8) with

$$a(t) = [(n-1)\Lambda_\infty+\lambda_{-\infty}]^+\log\left(\frac{1}{(4\pi(T-t))^{\frac{n-2k}{2}}}\right)+\frac{n}{2}\lambda_\infty \tag{5.12}$$

and

$$b(t) = \Lambda_{-\infty}^{-} \log\left(\frac{1}{(4\pi(T-t))^{\frac{n-2k}{2}}}\right) + \frac{\lambda_{-\infty}}{2}, \tag{5.13}$$

and similarly for $\Phi = {}^{n-2k}\mathrm{P}_{(X,T)}$ when M is compact and static, in which case we obtain $a(t) = 0$ and

$$b(t) = -F\left[1 + \log\left(\frac{B}{(4\pi(T-t))^{\frac{n-2k}{2}}}\right)\right] \tag{5.14}$$

with F and B as in Theorem 2.3.

Corollary 5.5. *Fix $X \in M$, suppose $T < T_\infty$ and assume the local geometry bounds (2.2) and (2.3). If $\{\psi_t \in \Gamma(E \otimes \Lambda^k T^* M)\}_{t\in[0,T[}$ solves a problem of type (I) or (II), then the monotonicity formula*

$$\begin{aligned}\left[\frac{1}{r^{n-2k}} \iint_{E_r^{n-2k}(\Phi)} \frac{1}{2}|\psi|^2(\partial_t\phi + |\nabla\phi|^2) - \langle \iota_{\nabla\phi}\psi, \mathcal{S}_{\psi,\Phi}\rangle - \left\langle \iota_{\nabla\phi}\mathrm{d}^\nabla\psi, \psi\right\rangle \mathrm{d}\widetilde{\mu}_k\right]_{r=r_1}^{r=r_2} \\ \geq \int_{r_1}^{r_2}\left(\frac{n-2k}{r^{n-2k+1}} \iint_{E_r^{n-2k}(\Phi)} \left|\mathrm{d}^\nabla\psi\right|^2 + |\mathcal{S}_{\psi,\Phi}|^2\, \mathrm{d}\widetilde{\mu}_k\right)\mathrm{d}r \geq 0\end{aligned} \tag{5.15}$$

holds for $0 < r_1 < r_2 < r_0$ whenever $|\psi|^2 \in L^1\left(\mathcal{D}_{2c_{n,k}r_0, \frac{r_0^2}{4\pi}}(X,T), \mu\right)$, where $\Phi = {}^{n-2k}\Gamma_{(X,T)}$, $c_{n,k} = \sqrt{\frac{n-2k}{2\pi e}}$, r_0 is as in Example 3.5 with $m = n-2k$, $\widetilde{\mu}_k$ is the Borel measure on the domain of Φ induced by $\mathrm{d}\widetilde{\mu}_k = \mathrm{dvol}_{g_t} \wedge e^{\xi_k(t)}\,\mathrm{d}t$ with

$$\xi_k(t) = (T-t)\left[([(n-1)\Lambda_\infty + \lambda_{-\infty}]^+ - 2k\Lambda_{-\infty}^-)\log\left(\frac{e}{4\pi(T-t)}\right)^{\frac{n-2k}{2}} + \frac{n\lambda_\infty - 2k\lambda_{-\infty}}{2}\right]$$

and $\mathcal{S}_{\psi,\Phi}$ is defined by (4.5).

If M is static and compact, then the monotonicity formula (5.15) holds for $0 < r_1 < r_2 < \widetilde{r_0}$ whenever $|\psi|^2 \in L^1(E_{\widetilde{r_0}}^{n-2k}(\Phi), \mu)$, where instead $\Phi = {}^{n-2k}\mathrm{P}_{(X,T)}$, r_0 is as before, $\widetilde{r_0}$ is chosen such that $E_{\widetilde{r_0}}^{n-2k}({}^{n-2k}\mathrm{P}_{(X,T)}) \subset E_{r_0}^{n-2k}({}^{n-2k}\Gamma_{(X,T)})$ (cf. r_0 of Example 3.7) and $\widetilde{\mu}_k$ is the Borel measure on the domain of Φ induced by $\mathrm{d}\widetilde{\mu}_k = \mathrm{dvol}_{g_t} \wedge e^{\xi_k(t)}\,\mathrm{d}t$ with

$$\xi_k(t) = 2kF(T-t)\left[\log\left(\frac{e}{4\pi(T-t)}\right)^{\frac{n-2k}{2}} + 1 + \log B\right]$$

and F and B as in Theorem 2.3.

Remark 5.6. Note that this corollary immediately yields local monotonicity formulæ for solutions to the Yang-Mills and harmonic map heat equations, viz. taking $k = 2$ and assuming the setup of Example 2.5 and that ω solves the Yang-Mills heat equation, there holds

$$\left[\frac{1}{r^{n-4}}\iint\limits_{E_r^{n-4}(\Phi)}\frac{1}{2}|\underline{\Omega}^\omega|^2(\partial_t\phi+|\nabla\phi|^2)-\langle\iota_{\nabla\phi}\underline{\Omega}^\omega,\partial_t a+\iota_{\nabla\phi}\underline{\Omega}^\omega\rangle\,\mathrm{d}\widetilde{\mu}_2\right]_{r=r_1}^{r=r_2} \geq \int_{r_1}^{r_2}\left(\frac{n-4}{r^{n-3}}\iint\limits_{E_r^{n-4}(\Phi)}|\partial_t a+\iota_{\nabla\phi}\underline{\Omega}^\omega|^2\,\mathrm{d}\widetilde{\mu}_2\right)\mathrm{d}r, \tag{5.16}$$

and likewise taking $k=1$ and assuming the setup of Example 2.6 and that u solves the harmonic map heat equation, there holds

$$\left[\frac{1}{r^{n-2}}\iint\limits_{E_r^{n-2}(\Phi)}\frac{1}{2}|\mathrm{d}u|^2(\partial_t\phi+|\nabla\phi|^2)-\langle\partial_{\nabla\phi}u,\partial_t u+\partial_{\nabla\phi}u\rangle\,\mathrm{d}\widetilde{\mu}_1\right]_{r=r_1}^{r=r_2} \geq \int_{r_1}^{r_2}\left(\frac{n-2}{r^{n-1}}\iint\limits_{E_r^{n-2}(\Phi)}|\partial_t u+\partial_{\nabla\phi}u|^2\,\mathrm{d}\widetilde{\mu}_1\right)\mathrm{d}r. \tag{5.17}$$

Proof of Corollary 5.5. We wish to apply Corollary 5.3. To this end, take $\Phi={}^{n-2k}\mathrm{P}_{(X,T)}$. In light of Remark 5.4 and the inequalities (5.12)-(5.13), it suffices to check that both spacetime integrands of (5.15) are summable over $E_{r_0}^{n-2k}(\Phi)$ with respect to $\widetilde{\mu}_k$ or equivalently μ, since ξ_k is bounded. Now, it is clear from Lemma 4.6 that the right-hand spacetime integrand is summable over $E_{r_0}^{n-2k}(\Phi)$, whereas the left-hand integrand may by the Cauchy-Schwarz inequality and Young's inequality be bounded from above in modulus by

$$\frac{1}{2}|\psi|^2\left(|\partial_t\phi|+3|\nabla\phi|^2\right)+\frac{1}{2}\left(|\mathcal{S}_{\psi,\Phi}|^2+|\mathrm{d}^\nabla\psi|^2\right)$$

and since by Proposition 2.2

$$\left(|\partial_t\phi|+3|\nabla\phi|^2\right)(x,t)\leq\mathrm{const}\cdot\left(\frac{1}{T-t}+\frac{R_r^{n-2k}(t-T)^2}{(t-T)^2}+\frac{R_r^{n-2k}(t-T)^2}{T-t}\right)$$

for $(x,t)\in\mathcal{D}_{j_0,\delta}(X,T)$ with R_r^{n-2k} as defined in Example 3.5 and 'const' some constant depending only on n, k, the local geometry of M about (X,T) and the auxiliary function χ, it suffices to show that

$$\iint\limits_{E_{r_0}^{n-2k}(\Phi)}\frac{1}{2}|\psi|^2\cdot\left(\frac{1}{T-t}+\frac{R_r^{n-2k}(t-T)^2}{(t-T)^2}+\frac{R_r^{n-2k}(t-T)^2}{T-t}\right)\mathrm{d}\mu(x,t)<\infty.$$

Note that since the integrand is bounded away from $t=T$ from above by a constant times $|\psi|^2$, it suffices to establish summability over $E_{r_0}^{n-2k}(\Phi)\cap\mathrm{pr}_2^{-1}(]\tau,T[)$ for τ sufficiently close to T. By Lemma 4.6, we have that

$$\int_{B^t_{R_{r_0}^{n-2k}(t-T)}(X)}\frac{1}{2}|\psi|^2\mathrm{dvol}_{g_t} \leq\frac{\widetilde{C}_0R_{r_0}^{n-2k}(t-T)^{n-2k}}{r_0^{n-2k}}\left(r_0^{-2}\iint\limits_{\mathcal{D}_{2c_{n,k}r_0,\frac{r_0^2}{4\pi}}(X,T)}\frac{1}{2}|\psi|^2\,\mathrm{d}\mu+\int_{B^{T-\frac{r_0^2}{4\pi}}_{2c_{n,k}r_0}(X)}\frac{1}{2}|\psi|^2\mathrm{dvol}_{g_{T-\frac{r_0^2}{4\pi}}}\right)$$

for $t \in \left]T - r_0^2 \exp\left(-\frac{1}{2(n-2k)}\right)/4\pi, T\right[$ so that, by Tonelli's theorem, it suffices to show that

$$\int_{T-\frac{r_0^2}{4\pi}}^{T} \frac{R_{r_0}^{n-2k}(t-T)^{n-2k}}{T-t} + R_{r_0}^{n-2k}(t-T)^{n-2k+2}\left(\frac{1}{T-t} + \frac{1}{(T-t)^2}\right) \mathrm{d}t < \infty,$$

but this follows from a straightforward computation as in Example 3.5. Hence, Corollary 5.3 applies, implying the monotonicity formula for $\Phi = {}^{n-2k}\Gamma_{(X,T)}$. The static compact case with $\Phi = {}^{n-2k}\mathrm{P}_{(X,T)}$ follows similarly. $\square$

Remark 5.7. A careful computation of the t-integrals in the preceding proof shows that we in fact have an estimate of the form

$$\frac{1}{r^{n-2k}} \iint\limits_{E_r^{n-2k}(\Phi)} \frac{1}{2}|\psi|^2(\partial_t\phi + |\nabla\phi|^2) - \langle \iota_{\nabla\phi}\psi, \mathcal{S}_{\psi,\Phi}\rangle - \left\langle \iota_{\nabla\phi}\mathrm{d}^{\nabla}\psi, \psi\right\rangle \mathrm{d}\widetilde{\mu}_k$$

$$\leq \mathrm{const}\cdot\left(\frac{1}{r_0^{n-2k+2}} \iint\limits_{\mathcal{D}_{2c_{n,k}r_0, \frac{r_0^2}{4\pi}}(X,T)} \frac{1}{2}|\psi|^2\mathrm{d}\mu + \frac{1}{r_0^{n-2k}}\int_{B^{T-\frac{r_0^2}{4\pi}}_{2c_{n,k}r_0}(X)} \frac{1}{2}|\psi|^2\mathrm{dvol}_{g_{T-\frac{r_0^2}{4\pi}}}\right)$$

under the hypotheses of the theorem, where 'const' depends only on n, k, the local geometry of M about (X,T) and the auxiliary function χ.

Remark 5.8. The rather abstract nature of the techniques used here suggests that similar results should be obtainable by similar means for related geometric heat equations. In particular, assuming the setup of Example 2.5, suppose $(V, \langle\cdot,\cdot\rangle)$ is an inner product space and $\rho : G \to \mathrm{O}(V)$ is a unitary representation. The vector bundle E_0 associated to ρ and P may thus be equipped with a Riemannian metric, again denoted $\langle\cdot,\cdot\rangle$ and, as before, a one-parameter family of connections on P gives rise to a one-parameter family of connections on E_0 which we simply denote by ∇^t. A one-parameter family of pairs $\{(\omega_t = \omega(\cdot,t) = \omega_0 + \underline{a(\cdot,t)}, u_t = u(\cdot,t)\}_{t\in[0,T[}$ consisting of a smooth family of connections ω_t on P (ω_0 a fixed connection) and sections u of E_0 is said to solve the *Yang-Mills-Higgs heat equation* with (symmetric) potential $W \in C^\infty(\mathbb{R}, [0,\infty[)$ if the equations

$$\begin{aligned} \partial_t a + \delta^{\nabla^t}\underline{\Omega}^{\omega_t} + u_t \odot \mathrm{d}^{\nabla^t}u_t &= 0 \\ \partial_t u + \delta^{\nabla^t}\mathrm{d}^{\nabla^t}u_t + 2(W' \circ |u_t|^2)u_t &= 0 \end{aligned} \tag{5.18}$$

hold, where $\odot : E_0 \times E_0 \otimes \Lambda T^*M \to E_0 \otimes \Lambda T^*M$ is a fibrewise bilinear map defined such that

$$\langle X, e_1 \odot e_2\rangle = \langle X \cdot e_1, e_2\rangle\,,$$

for all $X \in \Gamma(E)$ and $e_1, e_2 \in \Gamma(E_0)$, where $\cdot$ is the natural action of E on E_0 induced by the derivative of ρ at the identity of G and extended to the rest of $E_0 \times E_0 \otimes \Lambda T^*M$ such that for all $\eta \in \Gamma(\Lambda T^*M)$ and $e_1, e_2 \in \Gamma(E_0)$, $e_1 \odot (e_2 \otimes \eta) = (e_1 \odot e_2) \otimes \eta$. Naturally associated to such families of pairs is the energy density $e(\omega, u) = \frac{1}{2}\left(|\underline{\Omega}^\omega|^2 + |\mathrm{d}^\nabla u|^2\right) + W \circ |u|^2$ from which the equations (5.18) arise in the case where M is static. Setting $\mathcal{S}_{\omega,\Phi} = \partial_t a + \iota_{\nabla \log\Phi}\underline{\Omega}^\omega$

and $\mathcal{S}_{u,\Phi} = \partial_t u + \nabla_{\nabla \log \Phi} u$, the local monotonicity identity

$$\left[\frac{1}{r^{n-4}} \iint\limits_{E_r^{n-4}(\Phi)} e^{\xi(t)} \left(e(\omega,u)\left(\partial_t \phi + |\nabla\phi|^2\right) - \langle \iota_{\nabla\phi}\underline{\Omega}^{\omega}, \mathcal{S}_{\omega,\Phi}\rangle - \langle \nabla_{\nabla\phi} u, \mathcal{S}_{u,\Phi}\rangle\right) \mathrm{d}\mu(x,t)\right]_{r=r_1}^{r=r_2}$$
$$\geq \int_{r_1}^{r_2} \frac{n-4}{r^{n-3}} \iint\limits_{E_r^{n-4}(\Phi)} e^{\xi(t)} \left[-e(\omega,u)\left(\partial_t\phi + \Delta\phi + |\nabla\phi|^2 + \frac{1}{2}\mathrm{tr}_g \partial_t g + \frac{2}{T-t} + \xi'(t)\right)\right.$$
$$+ \frac{|\mathrm{d}^{\nabla} u|^2 + 4W \circ u}{2(T-t)} + |\mathcal{S}_{\omega,\Phi}|^2 + |\mathcal{S}_{u,\Phi}|^2$$
$$\left. + \left\langle \nabla^2\phi + \frac{1}{2}\partial_t g + \frac{g}{2(T-t)}, \sum_{i,j}\left(\langle \iota_{\varepsilon_i}\underline{\Omega}^{\omega}, \iota_{\varepsilon_j}\underline{\Omega}^{\omega}\rangle + \langle \nabla_{\varepsilon_i} u, \nabla_{\varepsilon_j} u\rangle\right)\omega^i \otimes \omega^j\right\rangle\right] \mathrm{d}\mu(x,t)\,\mathrm{d}r \tag{5.19}$$

may be established for $0 < r_1 < r_2 < r_0$ whenever $E_r^{n-4}(\Phi)$ is a heat ball for $r < r_0$ and both spacetime integrands are in $L^1(E_{r_0}^{n-4}(\Phi)$ with equality holding whenever $\left((x,t) \mapsto \frac{e(\omega,u)(x,t)}{T-t}\right) \in L^1(E_{r_0}^{n-4}(\Phi))$; the proof is identical to that of Theorem 5.1, where instead the energy-momentum tensor associated to e, viz.

$$\widetilde{\Psi}_t = \sum_{i,j=1}^{n} \left(\langle \iota_{\varepsilon_i}\underline{\Omega}^{\omega_t}, \iota_{\varepsilon_j}\underline{\Omega}^{\omega_t}\rangle + \left\langle \nabla^t_{\varepsilon_i} u, \nabla^t_{\varepsilon_j} u\right\rangle\right)\omega^i \otimes \omega^j - e(\omega,u)(\cdot,t),$$

together with the divergence identity

$$\mathrm{div}\ \widetilde{\Psi}_t = \sum_{j=1}^{n} \left(\langle \partial_t a, \iota_{\varepsilon_j}\underline{\Omega}^{\omega_t}\rangle + \langle \partial_t u, \nabla^t_j u_t\rangle\right)\omega^j$$

is used. Furthermore, if the inequalities (5.7) and (5.8) hold as in Corollary 5.3, then by setting $\xi(t) = \int_t^T a - 4b^-$, we obtain

$$\left[\frac{1}{r^{n-4}} \iint\limits_{E_r^{n-4}(\Phi)} e(\omega,u)\left(\partial_t\phi + |\nabla\phi|^2\right) - \langle \iota_{\nabla\phi}\underline{\Omega}^{\omega}, \mathcal{S}_{\omega,\Phi}\rangle - \langle \nabla_{\nabla\phi} u, \mathcal{S}_{u,\Phi}\rangle\, \mathrm{d}\widetilde{\mu}\right]_{r=r_1}^{r=r_2}$$
$$\geq \int_{r_1}^{r_2} \frac{n-4}{r^{n-3}} \iint\limits_{E_r^{n-4}(\Phi)} |\mathcal{S}_{\omega,\Phi}|^2 + |\mathcal{S}_{u,\Phi}|^2\, \mathrm{d}\widetilde{\mu}\, \mathrm{d}r$$

with $\widetilde{\mu}$ the Borel measure on the domain of Φ induced by $\mathrm{d}\widetilde{\mu} = \mathrm{dvol}_{g_t} \wedge e^{\xi(t)}\,\mathrm{d}t$ whenever both spacetime integrands are summable. Thus, for $T < T_\infty$, we may take $\Phi = {}^{n-4}\Gamma_{(X,T)}$ (or $\Phi = {}^{n-4}\mathrm{P}_{(X,T)}$ if M is static) as in Corollary 5.5 to obtain a local monotonicity formula on Euclidean (resp. static compact) heat balls whenever $0 < r_1 < r_2 < r_0$ (resp. $\widetilde{r_0}$) and $e(\omega,u) \in L^1\left(\mathcal{D}_{2c_{n,k}r_0, \frac{r_0^2}{4\pi}}(X,T), \mu\right)$, this summability condition arising from estimates obtained similarly to (4.13) and (4.14) by appealing to a nonlocal monotonicity formula due to Hong [16]; the details in the case $(M, g_t) = (\mathbb{R}^n, \delta)$ may be found in [2].

6 Mean curvature flow

We review the definition and some properties of the mean curvature flow, show under suitable conditions that heat balls may be 'pulled back' by the flow, and similarly derive a local monotonicity formula. As a rule, we follow the sign conventions of [15].

6.1 Setup. Let N^m be a smooth oriented manifold and $\{F_t = F(\cdot, t) : N \to (M, g_t)\}_{t\in[0,T[}$ a smooth one-parameter family of embeddings; furthermore suppose that F is *proper* in the sense that $(F, \mathrm{pr}_2)^{-1}(K)$ is compact for all $K \subset M \times]0, T[$.

Let

$$F^{-1}TM = \bigcup_{(x,t)\in N\times[0,T[} \{(x, t)\} \times T_{F_t(x)}M$$

denote the pullback of TM by F. We shall assume that for each $t \in [0, T[$ the pullback bundle $F_t^{-1}TM$ is realised as the point set

$$F_t^{-1}TM = \bigcup_{x\in N} \{(x, t)\} \times T_{F_t(x)}M \hookrightarrow F^{-1}TM.$$

We denote by $\mathrm{I}_t = \mathrm{I}(\cdot, t) \in \Gamma(T^*N \otimes T^*N)$ the first fundamental form of F_t, i.e. $\mathrm{I}_t = F_t^* g_t$, $\mathrm{II}_t \in \Gamma(F_t^{-1}TM \otimes T^*N \otimes T^*N)$ the second fundamental form and $\underline{\underline{\mathrm{H}}}_t = \underline{\underline{\mathrm{H}}}(\cdot, t) = \mathrm{tr}_{\mathrm{I}_t}\mathrm{II}_t \in \Gamma(F_t^{-1}TM)$ the mean curvature vector; the corresponding spacetime maps $\mathrm{I} : N \times [0, T[\to T^*M \otimes T^*M$, $\mathrm{II} : N \times [0, T[\to F^{-1}TM \otimes T^*N \otimes T^*N$ and $\underline{\underline{\mathrm{H}}} : N \times [0, T[\to F^{-1}TM$ are smooth and, in particular, $(N, \{\mathrm{I}_t\}_{t\in[0,T[})$ is an evolving manifold. If $\mathcal{D} \subset M \times]0, T[$ is open, $f : \mathcal{D} \to \mathbb{R}$ and $X : \mathcal{D} \to TM$ is such that $X_t := X(\cdot, t)$ is a local section of TM for each $t \in \mathrm{pr}_2(\mathcal{D})$, we may define 'pullbacks' $\underline{f} := f \circ (F, \mathrm{pr}_2) : F^{-1}(\mathcal{D}) \to \mathbb{R}$ and $\underline{\underline{X}} := (\mathrm{id}_{N\times]0,T[}, X \circ (F, \mathrm{pr}_2)) : F^{-1}(\mathcal{D}) \to F^{-1}TM$, where $\underline{\underline{X}}(\cdot, t)$ is a local section of $F_t^{-1}TM$ for each $t \in \mathrm{pr}_2(\mathcal{D})$. Note also that for each $t \in]0, T[$ the canonical metric induced on $F_t^{-1}TM$ by g_t together with the inclusion $TN \hookrightarrow F_t^{-1}TM$ defines an orthogonal projection $(\cdot)^T$ of $F_t^{-1}(TM)$ onto the image of this inclusion, and likewise $(\cdot)^\perp$ onto the orthogonal complement; these projections thus define smooth maps on $F^{-1}TM$, whence we may define the *tangential pullback* of X by F as the unique map $\underline{X} : (F, \mathrm{pr}_2)^{-1}(\mathcal{D}) \to TN$ such that

$$\underline{\underline{X}}(x, t)^T = (x, t, \mathrm{d}_x F_t(\underline{X}(x, t)))$$

for all $(x, t) \in N \times]0, T[$. Given a map $h : \mathcal{D} \to T^*M \otimes T^*M$ with $h_t := h(\cdot, t)$ a local section of $T^*M \otimes T^*M$ for all $t \in \mathrm{pr}_2(\mathcal{D})$, we define its *tangential trace* as the function $F^{-1}(\mathcal{D}) \ni (x, t) \mapsto \mathrm{tr}^T_{g_t} h_t(x)$ such that

$$\mathrm{tr}^T_{g_t} h_t(x) = \mathrm{tr}_{\mathrm{I}} F_t^* h_t(x)$$

and the *normal trace* as $\mathrm{tr}^\perp_g h := \underline{\mathrm{tr}_g h} - \mathrm{tr}^T_g h$; it may be checked that the tangential (resp. normal) trace may be pointwise computed as the trace with respect to the subspace $\big((F_t^{-1}TM)_x\big)^T$ (resp. $\big((F_t^{-1}TM)_x\big)^\perp$).

With an eye toward applications, we fix $(X, T) \in M \times]0, T_\infty[$ and assume the notation and local geometry bounds (2.1) and (2.2) of §2.2 throughout this section. For $R > 0$, we introduce the notation $\underline{B}^t_R(X) := \{\underline{\mathfrak{r}}(\cdot, t) = d^{g_t}(X, \cdot) < R\} = F_t^{-1}B^t_R(X)$.

6.2 Definition and properties. The family $\{F_t\}_{t\in[0,T[}$ of proper embeddings is said to evolve by *mean curvature flow* if the equation

$$\partial_t F = \underline{\underline{\mathrm{H}}} \tag{MCF}$$

holds on $N \times]0,T[$, where $\partial_t F$ is here considered a section of $F^{-1}TM$.

As remarked earlier, $(N, \{\mathrm{I}_t\})$ is an evolving manifold. It may be shown [22] [8, (B.2)] that it satisfies the evolution equation

$$(\partial_t \mathrm{I}, v \otimes w) = (F_t^* \partial_t g, v \otimes w) - 2\left\langle (\mathrm{II}_t, v \otimes w), \underline{\underline{\mathrm{H}}}_t \right\rangle$$

for all $v, w \in T_x N$ so that in particular

$$\mathrm{tr}_\mathrm{I} \partial_t \mathrm{I} = \mathrm{tr}_g^T \partial_t g - 2|\underline{\underline{\mathrm{H}}}|^2. \tag{6.1}$$

Thus, if N is compact and M is static, this identity implies the area-minimizing property

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_N \mathrm{dvol}_{\mathrm{I}_t} = -\int_N |\underline{\underline{\mathrm{H}}}|^2 \mathrm{dvol}_{\mathrm{I}_t}. \tag{6.2}$$

As it turns out, the mean curvature flow possesses a local analogue of this property in a more general setting. To this end, we recall the identity [14, §2]

$$\mathrm{div}_\mathrm{I} \underline{X} = \underline{\mathrm{div}_g X} - \mathrm{tr}_g^\perp \nabla X^\flat + \langle \underline{\underline{X}}, \underline{\underline{\mathrm{H}}} \rangle, \tag{6.3}$$

where $X : \mathcal{D} \to TM$ is such that $\mathcal{D}$ is open and $X(\cdot, t)$ is a local section of TM for $t \in \mathrm{pr}_2(\mathcal{D})$, ∇ is the (t-dependent) Levi-Civita connection on T^*M associated to $\{g_t\}$ and $X^\flat : \mathcal{D} \to T^*M$ is such that for each $t \in \mathrm{pr}_2(\mathcal{D})$ $X^\flat(\cdot, t)$ is the local one-form obtained from $X(\cdot, t)$ using the isomorphism $TM \to T^*M$ induced by g_t. The following may be found in [3, §3.6] or [6, §1] for the case $(M, g_t) \equiv (\mathbb{R}^n, \delta)$.

Lemma 6.1. *There exists a constant $\gamma \geq 0$ depending only on m, n and the constants appearing in these bounds such that for $R \in \left]0, \min\{j_0, \sqrt{4\gamma\delta}\right[$ and $t \in \left[T - \frac{R^2}{4\gamma}, T\right[$ the inequality*

$$\int_{\underline{B}^t_{R/2}(X)} \mathrm{dvol}_{\mathrm{I}_t} \leq 16 e^{\frac{m\lambda_\infty R^2}{8\gamma}} \int_{\underline{B}_R^{T-\frac{R^2}{4\gamma}}(X)} \mathrm{dvol}_{\mathrm{I}_{T-\frac{R^2}{4\gamma}}} - \int_{T-\frac{R^2}{4\gamma}}^t \int_{\underline{B}^s_{R/2}(X)} |\underline{\underline{\mathrm{H}}}|^2 \mathrm{dvol}_{\mathrm{I}_s} \,\mathrm{d}s$$

holds.

Proof. Let $\eta : \mathbb{R} \to \mathbb{R}^+$ be defined by $\eta(x) = ((1-x)^+)^4$ and define $\psi_R : N \times \left[T - \frac{R^2}{4\gamma}, T\right[\to \mathbb{R}^+$ by

$$\psi_R(x,t) := \eta\left(\frac{\underline{\mathfrak{r}}(x,t)^2 + \gamma\left(t - \left(T - \frac{R^2}{4\gamma}\right)\right)}{R^2} \right),$$

where $\gamma \geq 0$ is soon to be fixed. It is clear that ψ_R is twice differentiable, and supp $\psi_R(\cdot, t) \subset \underline{B}^t_R(X) \subset \underline{B}^t_{j_0}(X)$, since

$$1 - \frac{\underline{\mathfrak{r}}(x,t)^2 + \gamma\left(t - \left(T - \frac{R^2}{4\gamma}\right)\right)}{R^2} \geq 0 \Leftrightarrow R^2 \geq \underline{\mathfrak{r}}^2 + \gamma\left(t - \left(T - \frac{R^2}{4\gamma}\right)\right) \geq \underline{\mathfrak{r}}^2.$$

Also, $0 \leq \psi_R \leq 1$. Now, note that by (6.3) and the local geometry bounds, we have that

$$(\partial_t - \Delta_{\mathrm{I}_t})\underline{\mathfrak{r}}^2 = \underline{\partial_t \mathfrak{r}^2} - \mathrm{tr}_g^T \nabla_g^2 \mathfrak{r}^2 \geq (2m\Lambda_{-\infty}^- + \lambda_{-\infty}^-) j_0^2 - 2m \tag{6.4}$$

on $\mathcal{D}_{j_0,\delta}(X,T)$. Hence, setting $\gamma = 2m - \left(2m\Lambda_{-\infty}^- + \lambda_{-\infty}^-\right) j_0^2$, it follows from the chain rule that $(\partial_t - \Delta_{\mathrm{I}_t})\psi_R \leq 0$.

Now, since F_t is a proper embedding for each t and both $\psi_R(\cdot,t)$ and $\partial_t\psi_R(\cdot,t)$ are compactly supported in N, we may compute using (6.1) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_N \psi_R \mathrm{dvol}_{\mathrm{I}_t} &= \int_N \left(\partial_t\psi_R - |\underline{\underline{\mathrm{H}}}_t|^2\psi_R + \frac{1}{2}\mathrm{tr}_{g_t}^T \partial_t g \cdot \psi_R\right)\mathrm{dvol}_{\mathrm{I}_t} \\ &\leq -\int_N |\underline{\underline{\mathrm{H}}}_t|^2\psi_R \mathrm{dvol}_{\mathrm{I}_t} + \frac{m\lambda_\infty}{2}\int_N \psi_R \mathrm{dvol}_{\mathrm{I}_t},\end{aligned}$$

whence

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-\frac{m\lambda_\infty}{2}\left(t-\left(T-\frac{R^2}{4\gamma}\right)\right)}\int_N \psi_R \mathrm{dvol}_{\mathrm{I}_t}\right) \\ &\qquad\leq -\exp\left(-\frac{m\lambda_\infty R^2}{8\gamma}\right)\int_N |\underline{\underline{\mathrm{H}}}_t|^2\psi_R \mathrm{dvol}_{\mathrm{I}_t}\end{aligned}$$

so that, integrating from $T - \frac{R^2}{4\gamma}$ to t and estimating $\exp\left(-\frac{m\lambda_\infty}{2}\left(t-\left(T-\frac{R^2}{4\gamma}\right)\right)\right)$ from below, we obtain

$$\begin{aligned}&e^{-\frac{m\lambda_\infty R^2}{8\gamma}}\cdot\left(\int_N \psi_R(\cdot,t)\mathrm{dvol}_{\mathrm{I}_t} + \int_{T-\frac{R^2}{4\gamma}}^t \int_N |\underline{\underline{\mathrm{H}}}|^2\psi_R(\cdot,s)\mathrm{dvol}_{\mathrm{I}_s}\,\mathrm{d}s\right) \\ &\qquad\qquad\leq \int_N \psi_R(\cdot, T-\frac{R^2}{4\gamma})\mathrm{dvol}_{\mathrm{I}_{T-\frac{R^2}{4\gamma}}}. \end{aligned}\tag{6.5}$$

Now, since $\psi_R(\cdot,t) \leq \chi_{\underline{B}_R^t(X)}$, the right-hand side of (6.5) may be bounded from above by

$$\int_{\underline{B}_R^{T-\frac{R^2}{4\gamma}}(X)} \mathrm{dvol}_{\mathrm{I}_{T-\frac{R^2}{4\gamma}}}.$$

On the other hand, since

$$\underline{\mathfrak{r}} < \frac{R}{2} \Rightarrow 1 - \frac{\underline{\mathfrak{r}}^2 + \gamma\left(t-\left(T-\frac{R^2}{4\gamma}\right)\right)}{R^2} \geq 1 - \frac{\frac{R^2}{4} + \gamma\cdot\frac{R^2}{4\gamma}}{R^2} = \frac{1}{2}$$

so that

$$\psi_R(\cdot,t)|_{\underline{B}_{R/2}^t(X)} \geq \left(\frac{1}{2}\right)^4 = \frac{1}{16}. \tag{6.6}$$

Since the left-hand integrands of (6.5) are nonnegative, we may estimate their (spatial) integrals from below by the respective integrals on $\underline{B}_{R/2}^t(X)$, whence the result follows from (6.6). $\square$

We shall need a variant of this lemma involving pulled-back cross-sections of $E_r^m({}^m\Gamma_{(X,T)})$, for which purpose we recall the following well-known monotonicity formula which was first established by Ecker and Huisken [10], adapted to compact manifolds by Hamilton [14] and Ricci solitons by Lott [21], and finally generalised to this setting by Magni, Mantegazza and Tsatis [22].

Theorem 6.2 (Monotonicity Formula)**.** *If* $u \in C^2(N \times [0,T[\,,\mathbb{R})$ *is such that* supp $u(\cdot,t) \Subset N$ *for each* $t \in [0,T[$ *and* $\Phi \in C^2(\mathcal{D},\mathbb{R}^+)$ *with* $\mathcal{D} \subset M \times]0,T[$ *open with* supp $u(\cdot,t) \times \{t\} \subset (F,\mathrm{pr}_2)^{-1}\left(\mathcal{D} \cap \mathrm{pr}_2^{-1}(\{t\})\right)$*, then*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}&\left(\int_N u \cdot \underline{\Phi}\mathrm{dvol}_{\mathrm{I}_t}\right)\\ &= \int_N \underline{\Phi}(\partial_t - \Delta_{\mathrm{I}})u + u \cdot \underline{\left(\partial_t\Phi + \Delta_g\Phi + \frac{1}{2}\mathrm{tr}_g\partial_t g \cdot \Phi + \frac{n-m}{2(s-t)}\Phi\right)}\\ &\qquad - u\underline{\Phi}\mathrm{tr}_g^{\perp}\underline{\left(\nabla_g^2 \log\Phi + \frac{1}{2}\partial_t g + \frac{1}{2(s-t)}g\right)}\\ &\qquad\qquad - u\underline{\Phi}\left|\underline{\underline{\mathrm{H}}} - \underline{\nabla_g^{\perp}\log\Phi}\right|^2 \mathrm{dvol}_{\mathrm{I}_t}\end{aligned}$$

on $]0,T[$ *for every* $s \geq T$.

Proof sketch. It may be shown using (6.1) and (6.3) that the identity

$$\begin{aligned}\partial_t\left(u \cdot \Phi\mathrm{dvol}_{\mathrm{I}_t}\right) = \Big[&\mathrm{div}_{\mathrm{I}_t}\left(\underline{\Phi}\nabla u - u\nabla\underline{\Phi}\right) + \underline{\Phi}\cdot\left(\partial_t - \Delta_{\mathrm{I}_t}\right)u\\ &+ u \cdot \underline{\left(\partial_t\Phi + \Delta_g\Phi + \frac{1}{2}\mathrm{tr}_g\partial_t g \cdot \Phi + \frac{n-m}{2(s-t)}\Phi\right)}\\ &- u\Phi\mathrm{tr}_g^{\perp}\underline{\left(\nabla_g^2\log\Phi + \frac{1}{2}\partial_t g + \frac{1}{2(s-t)}g\right)}\\ &- u\underline{\Phi}\left|\underline{\underline{\mathrm{H}}} - \underline{\nabla_g^{\perp}\log\Phi}\right|^2\Big](\cdot,t)\mathrm{dvol}_{\mathrm{I}_t}\end{aligned}$$

holds; an integration and an application of Gauß' theorem and standard integration theorems to justify interchanging the derivative and integral then imply the result. ☐

If $R \in \left]0, \min\{\frac{j_0}{2}, \sqrt{\gamma\delta}\right[$ with γ as in Lemma 6.1 and Φ satisfies the inequalities (4.10) with $k = \frac{n-m}{2}$ for $(x,t) \in \mathcal{D}_{R,\delta}(X,T)$, Theorem 6.2 immediately implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{\xi_m(t)}\int_N \psi_R \cdot \underline{\Phi}\mathrm{dvol}_{\mathrm{I}_t}\right] \leq (a_0 - (n-m)b_0)e^{\sup|\xi_m|} \cdot \int_{\underline{B}_R^t(X)} \mathrm{dvol}_{\mathrm{I}_t} \tag{6.7}$$

on $]T-\delta,T[$, where ψ_R is as in the proof of Lemma 6.1 and $\xi_m(t) = \int_t^T a_1 - (n-m)b_1$. Using this, we may establish the desired analogue of Lemma 4.6 which was established by Ecker [7, Lemma 1.2] in the case $(M,g_t) \equiv (\mathbb{R}^n,\delta)$.

Lemma 6.3. *Fix* $\kappa \in \left]0, \min\{j_0, \sqrt{4\gamma\delta}\}\right[$ *with* γ *as in Lemma 6.1 and set* $\alpha = \sqrt{\frac{2\gamma}{\pi}}$*. For all* $r < \min\{1, \frac{\kappa}{2\alpha}\}$*, the estimate*

$$\int_{\underline{B}^t_{R_r^m(t-T)}(X)} \mathrm{dvol}_{\mathrm{I}_t} \leq \widetilde{C}_1 \frac{R_r^m(t-T)^m}{r^m}\int_{\underline{B}_\kappa^{T-\frac{\kappa^2}{4\gamma}}(X)} \mathrm{dvol}_{\mathrm{I}_{T-\frac{\kappa^2}{4\gamma}}} \tag{6.8}$$

holds for $t \in \left]T - \frac{e^{-\frac{1}{2m}}}{4\pi}r^2, T\right[$*, where* $\widetilde{C}_1 \geq 0$ *is a constant depending only on* m *and the local geometry of* M *about* (X,T) *and* R_r^m *is as in Example 3.5.*

Proof. Let $\Phi = {}^{n-2k}\Gamma_{(X,s)}$ considered as a function $\mathcal{D}_{j_0,\delta}(X,T) \to \mathbb{R}$ with $s \geq T$ to be fixed. Exactly as in Lemma 4.6, it may be shown that the inequalities (4.10) hold with

$$a_0 = \max\{\frac{n}{2}\lambda^+_{+\infty}, [(n-1)\Lambda_{+\infty} + \lambda_{-\infty}]^+\}, \tag{6.9}$$

$$a_1(t) = -\frac{ma_0}{2}\log\left(4\pi(T-t)\right), \tag{6.10}$$

$$b_0 = \min\{\frac{\lambda^-_{-\infty}}{2}, \Lambda^-_{-\infty}\}, \tag{6.11}$$

and

$$b_1(t) = -\frac{mb_0}{2}\log\left(4\pi(T-t)\right), \tag{6.12}$$

whence (6.7) with $R = \alpha r$ implies that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{\xi_m(t)}\int_N \psi_{\alpha r}\cdot\underline{\Phi}\mathrm{dvol}_{\mathrm{I}_t}\right] &\leq (a_0-(n-m)b_0)e^{\sup|\xi_m|}\cdot\int_{\underline{B}^t_{\alpha r}(X)}\mathrm{dvol}_{\mathrm{I}_t}\\ &\leq 16(a_0-(n-m)b_0)e^{\sup|\xi_m|+\frac{m\lambda_\infty\kappa^2}{8\gamma}}\int_{\underline{B}_\kappa^{T-\frac{\kappa^2}{4\gamma}}(X)}\mathrm{dvol}_{\mathrm{I}_{T-\frac{\kappa^2}{4\gamma}}}\end{aligned}$$

on $]T-\delta, T[$, where the last line follows from an application of Lemma 6.1, noting that $\alpha r < \frac{\kappa}{2}$. Integrating on $\left]T-\frac{R^2}{4\pi}, t_0\right[$ for $t_0 \in \left]T-\frac{R^2}{4\pi}, T\right[$ fixed and crudely estimating e^{ξ_m} from above and below then yields the inequality

$$\int_N \psi_{\alpha r}\cdot\underline{\Phi}\mathrm{dvol}_{\mathrm{I}_{t_0}} \leq e^{2\sup|\xi_m|}\left(\int_N \psi_{\alpha r}\cdot\underline{\Phi}\mathrm{dvol}_{\mathrm{I}_{T-\frac{R^2}{4\pi}}} + \widetilde{c}_1\int_{\underline{B}_\kappa^{T-\frac{\kappa^2}{4\gamma}}(X)}\mathrm{dvol}_{\mathrm{I}_{T-\frac{\kappa^2}{4\gamma}}}\right) \tag{6.13}$$

with $\widetilde{c}_1 = \frac{4}{\pi}(a_0-(n-m)b_0)e^{\frac{m\lambda^+_\infty j_0^2}{8\gamma}}$. On the one hand, we note that $\Phi(\cdot, T-\frac{r^2}{4\pi}) \leq \frac{1}{r^m}$, whence, since $\psi_{\alpha r}(\cdot,t) \leq \chi_{\underline{B}^t_{\frac{\kappa}{2}}(X)}$, Lemma 6.1 implies that the right-hand side of (6.13) may be bounded from above by

$$\frac{e^{2\sup|\xi_m|}}{r^m}\cdot(\max\{16e^{\frac{m\lambda^+_\infty j_0^2}{8\gamma}}, \widetilde{c}_1\})\int_{\underline{B}_\kappa^{T-\frac{\kappa^2}{4\gamma}}(X)}\mathrm{dvol}_{\mathrm{I}_{T-\frac{\kappa^2}{4\gamma}}}.$$

On the other hand, since $\alpha r \geq 2\sqrt{\frac{m}{\pi}}r \geq 2R^m_r$ and $\psi_{\alpha R}|_{\underline{B}^t_{\frac{\alpha r}{2}}(X)} \geq \frac{1}{16}$, the left-hand side of (6.13) may be bounded from below by

$$\frac{1}{16}\int_{\underline{B}^{t_0}_{R^m_r(t_0-T)}(X)}\underline{\Phi}\mathrm{dvol}_{\mathrm{I}_{t_0}}. \tag{6.14}$$

Finally, we restrict our attention to $t_0 \in \left]T - e^{-\frac{1}{2m}}\cdot\frac{r^2}{4\pi}, T\right[$ and take $s = t_0 + R^m_r(t_0-T)^2 \geq T$, in which case

$$\underline{\Phi}(\cdot,t_0)|_{\underline{B}^{t_0}_{R^m_r(t_0-T)}(X)} \geq \frac{e^{-\frac{1}{4}}}{(4\pi)^{\frac{m}{2}}R^m_r(t_0-T)^m};$$

using this in (6.14) then implies the result. □

6.3 Pulled-back heat balls. We now turn our attention to heat balls obtained by pulling back those of Examples 3.5 and 3.7 by mean curvature flow in an appropriate sense. Such heat balls were first considered in the case $(M, g_t) \equiv (\mathbb{R}^n, \delta)$ by Ecker [7] in a slightly different light. The following example is— in the class of maps considered— a generalisation of the heat balls introduced there.

Example 6.4 (Euclidean heat balls pulled back by MCF)**.** Fix $\kappa \in \left]0, \min\{j_0, \sqrt{4\gamma\delta}\}\right[$ with γ as in Lemma 6.1. Consider $\Phi := {}^m\Gamma_{(X,T)} : \mathcal{D}_{j_0,\delta}(X,T) \to \mathbb{R}^+$ as in Example 3.5. We claim that $E^m_r\left(\underline{{}^m\Gamma_{(X,T)}}\right) = (F, \mathrm{pr}_2)^{-1}\left(E^m_r\left({}^m\Gamma_{(X,T)}\right)\right)$ is a heat ball for $r < r_0 := \min\{\frac{\kappa}{2\alpha}, 1\}$. We verify the conditions.

(HB1) By Example 3.5 (HB1),

$$\overline{E^m_{r_0}(\Phi) \cap \mathrm{pr}_2^{-1}(]T-\delta, \tau[)} \Subset \mathcal{D}_{j_0,\delta}(X,T),$$

for $\tau \in]0, T[$ which thus implies that

$$\overline{E^m_{r_0}(\underline{\Phi}) \cap \mathrm{pr}_2^{-1}(]T-\delta, \tau[)} \subset (F, \mathrm{pr}_2)^{-1}\left(\overline{E^m_{r_0}(\Phi) \cap \mathrm{pr}_2^{-1}(]T-\delta, \tau[)}\right) \Subset N \times]0, T[$$

by properness.

(HB2) It is clear that

$$\begin{aligned} E^m_{r_0}(\underline{\Phi}) &= (F, \mathrm{pr}_2)^{-1}\left(\bigcup_{t \in \left]T - \frac{r_0^2}{4\pi}, T\right[} B^t_{R^m_{r_0}(t-T)}(X) \times \{t\}\right) \\ &= \bigcup_{t \in \left]T - \frac{r_0^2}{4\pi}, T\right[} \underline{B}^t_{R^m_{r_0}(t-T)}(X) \times \{t\}. \end{aligned} \tag{6.15}$$

Now, we note that, by the chain rule, the Cauchy-Schwarz inequality and Young's inequality, the inequality

$$|\partial_t \underline{\phi}| = |\underline{\partial_t \phi} + \left\langle H, \underline{\underline{\nabla_g \phi}}\right\rangle| \leq |\underline{\partial_t \phi}| + \frac{1}{2}\left(|\underline{\underline{\mathrm{H}}}|^2 + \underline{|\nabla_g \phi|}^2\right)$$

holds. Moreover, it is clear that $|\nabla_{\mathrm{I}} \underline{\phi}| \leq \underline{|\nabla_g \phi|}$; hence, in view of these two inequalities, (6.15) and the gradient and time-derivative bounds in Proposition 2.2 (cf. Example 3.5 (HB2)), it suffices to show that

$$\int_{T - \frac{r_0^2}{4\pi}}^T \int_{\underline{B}^t_{R^m_{r_0}(t-T)}(X)} \frac{\underline{\mathfrak{r}}^2}{(T-t)^2} \mathrm{dvol}_{\mathrm{I}_t}\, dt < \infty \tag{6.16}$$

with $\mathfrak{r}(x,t) := d^t(X, x)$ and

$$\int_{T - \frac{r_0^2}{4\pi}}^T \int_{\underline{B}^t_{R^m_{r_0}(t-T)}(X)} \frac{1}{T-t} \mathrm{dvol}_{\mathrm{I}_t}\, dt < \infty \tag{6.17}$$

since, by Lemma 6.1,

$$\int_{T-\frac{R_0^2}{4\gamma}}^{T}\int_{\underline{B}^t_{\sqrt{\frac{m}{2\pi e}}r_0}(X)} |\underline{\underline{\mathrm{H}}}|^2 \mathrm{dvol}_{\mathrm{I}_t}\,\mathrm{d}t \le 16 e^{\frac{m\lambda_\infty \kappa^2}{8\gamma}} \left(\int_{\underline{B}^{\cdot}_{R_0}(X)} \mathrm{dvol}_{\mathrm{I}_\cdot}\right)(T-\frac{R_0^2}{4\gamma}) < \infty,$$

which establishes that $|\underline{\underline{\mathrm{H}}}|^2 \in L^1\left(E^m_{r_0}(\underline{\Phi})\right)$ due to the inclusion

$$E^m_{r_0}(\underline{\Phi}) \cap \mathrm{pr}_2^{-1}(]T-\frac{R_0^2}{4\gamma}, T[) \subset \bigcup_{t\in\ ]T-\frac{R_0^2}{4\gamma},T[} \underline{B}^t_{\sqrt{\frac{m}{2\pi e}}r_0}(X)\times\{t\}$$

and the fact that $E^m_{r_0}(\underline{\Phi})\cap \mathrm{pr}_2^{-1}(]T-\frac{r_0^2}{4\pi}, T-\frac{R_0^2}{4\gamma}])$ is relatively compact in the domain of F by (HB1).

Now, in light of Lemma 6.3, the estimate

$$\int_{\underline{B}^t_{R^m_{r_0}(t-T)}(X)} \frac{\underline{\mathfrak{r}}^2}{(T-t)^2}\mathrm{dvol}_{\mathrm{I}_t} \le \widetilde{C}_1 \int_{\underline{B}_\kappa^{T-\frac{\kappa^2}{4\gamma}}(X)} \mathrm{dvol}_{\mathrm{I}_{T-\frac{\kappa^2}{4\gamma}}} \cdot \frac{R^m_{r_0}(t-T)^{m+2}}{(T-t)^2},$$

holds for $t\in\ ]\tau,T[$ for $\tau = T-\frac{\exp(-\frac{1}{2m})}{4\pi}r_0^2$ with $\widetilde{C}_1$ a geometric constant, and likewise the estimate

$$\int_{\underline{B}^t_{R^m_{r_0}(t-T)}(X)} \frac{1}{T-t}\mathrm{dvol}_{\mathrm{I}_t} \le \widetilde{C}_1 \int_{\underline{B}_\kappa^{T-\frac{\kappa^2}{4\gamma}}(X)} \mathrm{dvol}_{\mathrm{I}_{T-\frac{\kappa^2}{4\gamma}}} \cdot \frac{R^m_{r_0}(t-T)^{m}}{T-t}$$

for $t\in\ ]\tau,T[$, but both right-hand sides are summable on $]\tau,T[$ as functions of t, which together with (HB1) then implies the statements (6.16) and (6.17).

(HB3) In light of (6.15), $\mathrm{pr}_1\left(E^m_{r_0}(\underline{\Phi})\cap(M\times\{\tau\})\right) = \underline{B}^\tau_{R^m_{r_0}(\tau-T)}(X)$. On the other hand,

$$|\underline{\phi}(\cdot,\tau)| = \left|\frac{\underline{r}^2}{4(\tau-T)} - \frac{m}{2}\log\left(4\pi(T-\tau)\right)\right| \le \frac{R^m_{r_0}(\tau-T)^2}{4(T-\tau)} + \frac{m}{2}\left(-\log(4\pi(T-\tau))\right)$$

on $\underline{B}^\tau_{R^m_{r_0}(\tau-T)}(X)$. Therefore, making use of Theorem 6.3 again as in (HB2), we see that it suffices to show that

$$\lim_{\tau\nearrow T}\frac{R^m_{r_0}(\tau-T)^{m+2}}{T-\tau} = 0$$

and

$$\lim_{\tau\nearrow T} R^m_{r_0}(\tau-T)\log\left(4\pi(T-\tau)\right) = 0$$

or, more explicitly,

$$\lim_{\tau\nearrow T}\sqrt{(\tau-T)^m\left[\log\left(\frac{4\pi(T-\tau)}{r_0^2}\right)\right]^{m+2}} = 0$$

and

$$\lim_{\tau\nearrow T}\sqrt{(\tau-T)^m\left[\log(4\pi(T-\tau))\right]^{m+2} - (\tau-T)^m\log(r_0^2)} = 0.$$

We know, however, by making the same change of variables as in Example 3.5 (HB3) that both of these statements hold true, i.e. by noting that

$$\lim_{\tau \nearrow T} (\tau - T)^m \left[\log \left(\frac{4\pi(T-\tau)}{r_0^2} \right) \right]^{m+2} = \left(\frac{r_0^2}{4\pi} \right)^m \lim_{s\to\infty} s^{m+2} \exp(-ms) = 0. \quad \square$$

Whilst not quite being a generalization of the heat balls introducted by Ecker [7], the following example provides an adaptation of his construction to the case where F evolves by MCF in a static compact background.

Example 6.5 (Heat Balls on Static Compact Manifolds Pulled Back by MCF)**.** Suppose (M, g) is compact and static and let ${}^m\mathrm{P}_{(X,T)} : \mathcal{D} \to \mathbb{R}^+$ be as in Example 3.7, assuming the setup used there and in Example 6.4. We claim that $E_r^m(\underline{\mathrm{P}})$ is a heat ball for

$$r < r_0 := \min \left\{ 9^{-1/m}, \left(1 + 2\varrho^{-m}\right)^{-1/m}, \right\},$$

where $\varrho = \min \left\{ \sqrt{4\pi(T-\tau_0)} \sqrt{\frac{2\pi e}{m}} \sup\{y \in \mathbb{R}^+ : B_y(X) \Subset \Omega\},\ r_0 \text{ of Example 6.4} \right\}$.

We note the bound from Example 3.7 pulled back to $N \times]\tau_0, T[$:

$$\frac{1}{2}\underline{\Phi} - 1 \le \underline{\mathrm{P}} \le 2\underline{\Phi} + 1,$$

on $(F, \mathrm{pr}_2)^{-1}(\mathcal{D})$, where Φ is as in Example 6.4.

(HB1) The above bound immediately implies that

$$E_{r_0}^m(\underline{\mathrm{P}}) = (\underline{\mathrm{P}} > \frac{1}{(r_0)^m}) \subset (F, \mathrm{pr}_2)^{-1}(\mathcal{D}) \cap E_{\widetilde{r}_0}^m(\underline{\Phi}),$$

where $\widetilde{r}_0 = \left(\frac{2}{\frac{1}{(r_0)^m} - 1} \right)^{1/m}$. By Example 6.4,

$$E_{\widetilde{r}_0}^m(\underline{\Phi}) = \bigcup_{t \in\ ]T - \frac{\widetilde{r}_0^2}{4\pi}, T[} \underline{B}^t_{R^m_{\widetilde{r}_0}(t-T)}(X) \times \{t\} \subset (F, \mathrm{pr}_2)^{-1} \left(B_{\varrho_0}(X) \times\ ]T - \frac{\widetilde{r}_0^2}{4\pi}, T[\right),$$

where $\varrho_0 = \sqrt{\frac{m}{2\pi e}}\widetilde{r}_0$ (cf. Example 3.5). In view of the choice of r_0 above, it is easily seen that $B_\varrho(X) \Subset \Omega$. Thus, we see that

$$E_{\widetilde{r}_0}^m(\underline{\mathrm{P}}) \cap \mathrm{pr}_2^{-1}(]\tau_0, \tau[) \subset (F, \mathrm{pr}_2)^{-1}(\Omega \times]\tau_0, \tau[),$$

whence properness implies that the right-hand set is relatively compact in $N \times]0, T[$.

(HB2) As in Example 6.4 (HB2), we note that

$$|\nabla_{\mathrm{I}} \underline{\varrho}| \le |\underline{\underline{\nabla_g \rho}}| \text{ and}$$
$$|\partial_t \underline{\varrho}| \le |\underline{\partial_t \rho}| + \frac{1}{2}\left(|\underline{\underline{\mathrm{H}}}|^2 + |\underline{\underline{\nabla_g \rho}}|^2 \right)$$

with $\underline{\varrho} = \log \underline{\mathrm{P}}$. Firstly, since $E_{r_0}^m(\underline{\mathrm{P}}) \subset E_{\widetilde{r}_0}^m(\underline{\Phi})$ and r_0 was chosen such that $\widetilde{r}_0$ does not exceed the r_0 of Example 6.4, it is clear that $|\underline{\underline{\mathrm{H}}}|^2$ is $L^1\left(E_{r_0}^m(\underline{\mathrm{P}})\right)$. Secondly, since

$E^m_{r_0}(\underline{\mathrm{P}}) = (F, \mathrm{pr}_2)^{-1}\left(E^m_{r_0}({}^m\mathrm{P}_{(X,T)})\right)$ with ${}^m\mathrm{P}_{(X,T)}$ as in Example 3.7, it follows from Example 3.7 (HB2) that the bounds

$$\underline{|\nabla\rho|^2} \leq \frac{C}{T-t}\left(\log B - \frac{m}{2}\log(4\pi(T-t))\right),$$
$$\underline{\partial_t\rho} \geq -\frac{F}{T-t}\left(1+\log B - \frac{m}{2}\log(4\pi(T-t))\right) + \frac{m-n}{2(T-t)}$$

and

$$\underline{\partial_t\rho} \leq e^{2K(T-t)}\frac{n\left(e^{2K(T-t)}-1\right)+m}{2(T-t)} - e^{-2K(T-t)}\underline{|\nabla\rho|^2}$$

hold on $E^m_{r_0}(\underline{\mathrm{P}})$, where we retain the notation of Example 3.7 (HB2). It therefore suffices to show that $(x,t)\mapsto\frac{1}{T-t}$ and $(x,t)\mapsto\frac{\log(4\pi(T-t))}{T-t}$ are in $L^1\left(E^m_{\widetilde{r}_0}(\underline{\Phi})\right)$ and thus in $L^1\left(E^m_{r_0}(\underline{\mathrm{P}})\right)$. An inspection of the computation in Example 6.4 (HB2) establishes that $\left((x,t)\mapsto\frac{1}{T-t}\right)\in L^1\left(E^m_{\widetilde{r}_0}(\underline{\Phi})\right)$, whereas it suffices to show that $\left((x,t)\mapsto\frac{\log(4\pi(T-t))}{T-t}\right)\in L^1\left(E^m_{\widetilde{r}_0}(\underline{\Phi})\cap\mathrm{pr}_2^{-1}(]\tau,T[)\right)$ for fixed $\tau\in\,]0,T[$ as in Example 6.4 (HB2), whence the integral we are left to establish the finiteness of may, by Lemma 6.3, be estimated

$$\int_\tau^T\int_{B_{R^m_{\widetilde{r}_0}(t-T)}(X)}\frac{\log(4\pi(T-t))}{T-t}\,\mathrm{d}x\,\mathrm{d}t$$
$$\leq\frac{\widetilde{C}_1}{r^m}\int_{\underline{B}_\kappa^{T-\frac{\kappa^2}{4\gamma}}(X)}\mathrm{dvol}_{\mathrm{I}_{T-\frac{\kappa^2}{4\gamma}}}\cdot\int_\tau^T\frac{R^m_r(t-T)^m\log(4\pi(T-t))}{T-t}\,\mathrm{d}t;$$

a straightforward computation then shows the finiteness of the right-hand integral.

(HB3) Since $r_0 < 9^{-1/m}$ and hence $\widetilde{r}_0 < 4^{-1/m}$, we have that $\underline{\Phi} > 4$ on $E^m_{\widetilde{r}_0}(\underline{\Phi})$ so that

$$1\leq\underline{\mathrm{P}}\leq\frac{5}{4}\underline{\Phi}\Rightarrow 0\leq\underline{\rho}\leq\log\frac{5}{4}+\log\underline{\Phi}$$

on $E^m_{r_0}(\underline{\mathrm{P}})$. Hence, to establish (HB3) it suffices to show that

$$\lim_{\tau\nearrow T}\int_{\mathrm{pr}_1(E^m_{\widetilde{r}_0}(\underline{\Phi})\cap(N\times\{\tau\}))}|\log\underline{\Phi}|\mathrm{dvol}_{g_\tau} = 0,$$

but this was shown to hold in Example 6.4. □

Remark 6.6. Note that the approach taken in Examples 6.4 and 6.5 is different from that taken by Ecker [7] in that heat balls were considered as subsets of the *parameter space* $N\times\,]T-\delta_1,T[$ as opposed to being subsets of $M\times\,]T-\delta_1,T[$. In our setting, both approaches are equivalent. However, Ecker's approach more readily generalizes to the varifold setting of Brakke [3].

6.4 Local monotonicity formulæ. We now turn our attention to local monotonicity formulæ for the mean curvature flow. To this end, suppose that $\Phi\in C^{2,1}(\mathcal{D},\to\mathbb{R}^+)$ with $\mathcal{D}\subset M\times]0,T[$ is such that $E^m_r(\underline{\Phi})$ is a heat ball for $r<r_0$ and set $\phi=\log\Phi$ and $\phi^m_r=\log(r^m\Phi)$. The following theorem, which also applies in the case $T=T_\infty$, should be considered a local analogue of Magni, Mantegazza and Tsatis' generalization [22] of Huisken's monotonicity formula (Theorem 6.2).

Theorem 6.7. *If $u \in C^{2,1}(E^m_{r_0}(\underline{\Phi}))$ and $\frac{u}{T-t}\underline{\phi} \in L^1\left(E^m_{r_0}(\underline{\Phi})\right)$, then*

$$\left[\frac{1}{r^m}\iint\limits_{E^m_r(\underline{\Phi})} u\left[|\nabla_{\mathrm{I}}\underline{\phi}|^2 + \left(|\underline{\underline{\mathrm{H}}}|^2 - \frac{1}{2}\mathrm{tr}^T_g\partial_t g\right)\underline{\phi^n_r}\right]\mathrm{dvol}_{\mathrm{I}_t}\,\mathrm{d}t\right]_{r=r_1}^{r=r_2}$$
$$= \int_{r_1}^{r_2}\left(\frac{m}{r^{m+1}}\iint\limits_{E^m_r(\underline{\Phi})} -u\cdot\left(\underline{\partial_t\phi + \Delta_g\phi + |\nabla_g\phi|^2 + \frac{1}{2}\mathrm{tr}_g\partial_t g} + \frac{n-m}{2(T-t)}\right)\right.$$
$$- \underline{\phi^m_r}\cdot(\partial_t - \Delta_{\mathrm{I}})u + u\left|\underline{\underline{\mathrm{H}}} - \nabla^\perp_g\phi\right|^2$$
$$\left. + u\cdot\mathrm{tr}^\perp_g\left(\nabla^2_g\phi + \frac{1}{2}\partial_t g + \frac{g}{2(T-t)}\right)\mathrm{dvol}_{\mathrm{I}_t}\,\mathrm{d}t\right)\mathrm{d}r \quad (6.18)$$

for $0 < r_1 < r_2 < r_0$ provided both spacetime integrands are in $L^1\left(E^m_{r_0}(\underline{\Phi})\right)$. If $u \geq 0$, then the condition $\frac{u}{T-t}\underline{\phi} \in L^1\left(E^m_{r_0}(\underline{\Phi})\right)$ may be lifted so that the the identity (6.18) holds with $\geq$ in place of $=$.

Remark 6.8. As with Theorem 5.1 (cf. Remark 5.2), this identity implies a monotonicity formula if u is a nonnegative subsolution to the heat equation on $(N, \{\mathrm{I}_t\})$, $\Phi(\cdot,t) = (T-t)^{\frac{n-m}{2}}\mathrm{P}(\cdot,t)$ for a positive subsolution P of the backward heat equation and if the matrix Harnack term $\nabla^2_g\phi + \frac{1}{2}\partial_t g + \frac{g}{2(T-t)}$ is nonnegative definite, which in particular holds on gradient shrinking Ricci solitons and, more specifically, for $(M, g_t) \equiv (\mathbb{R}^n, \delta)$ taken with $\Phi(\cdot,t) = (T-t)^{\frac{n-m}{2}}\mathrm{P}(\cdot,t)$ with P the standard heat kernel on $\mathbb{R}^n$. Note also that the expression $\underline{\underline{\mathrm{H}}} - \nabla^\perp_g\phi$ vanishes if F is suitably sense self similar (cf. [7, Introduction] for the case $(M, g_t) \equiv (\mathbb{R}^n, \delta)$).

Remark 6.9. If u is bounded on $E^m_{r_0}(\underline{\Phi})$ and Φ is either the the pulled-back Euclidean backward heat kernel of Example 6.4 or, if M is static and compact, the pulled-back canonical backward heat kernel of Example 6.5, then the estimates of Examples 6.4 (HB2) and 6.5 (HB2) immediately imply that the integrals of (6.18) are finite (cf. Remark 6.10). In particular, if $(M, g_t) \equiv (\mathbb{R}^n, \delta)$, we recover Ecker's formula.

Proof of Theorem 6.7. We first assume that $u(\cdot,t) \equiv 0$ for $t \in [\tau_0, T[$. Just as in the proof of Theorem 5.1, we first approximate, using the notation of §3.2

$$\left[\frac{1}{r^m}J^r_q\left(u\cdot\left[|\nabla_{\mathrm{I}}\underline{\phi}|^2 + \underline{\phi^m_r}\cdot\left(|\underline{\underline{\mathrm{H}}}|^2 - \frac{1}{2}\mathrm{tr}^T_g\partial_t g\right)\right]\right)\right]_{r=r_1}^{r=r_2}$$
$$= \int_{r_1}^{r_2}\frac{m}{r^{m+1}}J^r_q\left(-u\cdot\left[|\nabla_{\mathrm{I}}\underline{\phi}|^2 + \underline{\phi^m_r}\cdot\left(|\underline{\underline{\mathrm{H}}}|^2 - \frac{1}{2}\mathrm{tr}^T_g\partial_t g\right)\right]\right)\mathrm{d}r$$
$$+ \int_{r_1}^{r_2}\frac{1}{r^m}\frac{\mathrm{d}}{\mathrm{d}r}J^r_q\left(u\cdot\left[|\nabla_{\mathrm{I}}\underline{\phi}|^2 + \underline{\phi^m_r}\cdot\left(|\underline{\underline{\mathrm{H}}}|^2 - \frac{1}{2}\mathrm{tr}^T_g\partial_t g\right)\right]\right)\mathrm{d}r.$$

We note that, since u vanishes near T, each individual term in the approximate integrals is summable over $E^m_{r_0}(\underline{\Phi})$, thus allowing us to freely separate these integrals. To calculate the latter integral on the right-hand side, we note that by Proposition 3.10(i) and (iii),

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}r} J^{\cdot}_q \left(\langle u\nabla_{\mathrm{I}}\underline{\phi}, \nabla_{\mathrm{I}}\underline{\phi}\rangle + \underline{\phi^m_r} \cdot \left(|\underline{\underline{\mathrm{H}}}|^2 - \frac{1}{2}\mathrm{tr}^T_g \partial_t g \right) \right) \\
&= \frac{m}{r} J^r_q (|\underline{\underline{\mathrm{H}}}|^2 - \frac{1}{2}\mathrm{tr}^T_g \partial_t g - \langle \nabla_{\mathrm{I}} u, \nabla_{\mathrm{I}}\underline{\phi}\rangle - u\Delta_{\mathrm{I}}\underline{\phi}) + o(1)
\end{aligned}$$

as $q \to \infty$, where the remainder may be bounded from above uniformly in r. Thus, by Lemma 3.9 and the dominated convergence theorem, taking the limit $q \to \infty$ in the above yields

$$\begin{aligned}
&\left[\frac{1}{r^m} \iint\limits_{E^m_r(\underline{\Phi})} u \cdot \left[|\nabla_{\mathrm{I}}\underline{\phi}|^2 + \underline{\phi^m_r} \cdot \left(|\underline{\underline{\mathrm{H}}}|^2 - \frac{1}{2}\mathrm{tr}^T_g \partial_t g \right) \right] \mathrm{dvol}_{\mathrm{I}_t}\, \mathrm{d}t \right]_{r=r_1}^{r=r_2} \\
&\quad = \int_{r_1}^{r_2} \left(\frac{m}{r^{m+1}} \iint\limits_{E^m_r(\underline{\Phi})} -u \cdot \left(|\nabla_{\mathrm{I}}\underline{\phi}|^2 + \underline{\phi^m_r} \cdot \left(|\underline{\underline{\mathrm{H}}}|^2 - \frac{1}{2}\mathrm{tr}^T_g \partial_t g \right) \right) - \langle \nabla_{\mathrm{I}} u, \nabla_{\mathrm{I}}\underline{\phi}\rangle \right. \\
&\qquad\qquad \left. - u\Delta_{\mathrm{I}}\underline{\phi} + u \cdot \left(|\underline{\underline{\mathrm{H}}}|^2 - \frac{1}{2}\mathrm{tr}^T_g \partial_t g \right) \mathrm{dvol}_{\mathrm{I}_t}\, \mathrm{d}t \right) \mathrm{d}r, \qquad (6.19)
\end{aligned}$$

A straightforward computation using the identities (6.3) and $\mathrm{tr}^{\perp}_g g = n - m$ shows that the right-hand spacetime integrand of (6.19) is equal to

$$\begin{aligned}
&- u \left(\underline{\partial_t \phi + |\nabla_g \phi|^2 + \Delta_g \phi + \frac{1}{2}\mathrm{tr}_g \partial_t g} + \frac{n-m}{2(T-t)} \right) - \underline{\phi^m_r} \cdot (\partial_t - \Delta_{\mathrm{I}}) u + u \left| \underline{\underline{\mathrm{H}}} - \nabla^{\perp}_g \phi \right|^2 \\
&\quad + u \cdot \mathrm{tr}^{\perp}_g (\nabla^2_g \phi + \frac{1}{2}\partial_t g + \frac{g}{2(T-t)}) \\
&\qquad - \mathrm{div}_{\mathrm{I}}(\underline{\phi^m_r} \nabla_{\mathrm{I}} u) + \partial_t(\underline{\phi^m_r} u) + \underline{\phi^m_r} \cdot u \cdot \left(|\underline{\underline{\mathrm{H}}}|^2 - \frac{1}{2}\mathrm{tr}^T_g \partial_t g \right)
\end{aligned} \qquad (6.20)$$

Since $\mathrm{tr}_{\mathrm{I}} \partial_t \mathrm{I} = |\underline{\underline{\mathrm{H}}}|^2 - \frac{1}{2}\mathrm{tr}^T_g \partial_t g$, Proposition 3.10(iv) and (v) imply that the functions in the last line of (6.20) do not contribute to the right-hand integral of (6.19), thus establishing the result in the case where u vanishes close to T.

Now, consider $u_l : E^m_{r_0}(\underline{\Phi}) \to \mathbb{R}$ defined by $u_l(x,t) = \chi_l(T-t) \cdot u(x,t)$ for $l \in \mathbb{N}$, where χ_l is as defined in §2.1. Denoting the right-hand spacetime integrand of (6.18) by $i(u)$, the above implies that

$$\begin{aligned}
&\left[\frac{1}{r^m} \iint\limits_{E^m_r(\underline{\Phi})} \chi_l(T-t) \cdot u \left[|\nabla_{\mathrm{I}}\underline{\phi}|^2 + \left(|\underline{\underline{\mathrm{H}}}|^2 - \frac{1}{2}\mathrm{tr}^T_g \partial_t g \right) \underline{\phi^m_r} \right] \mathrm{dvol}_{\mathrm{I}_t}\, \mathrm{d}t \right]_{r=r_1}^{r=r_2} \\
&= \int_{r_1}^{r_2} \left(\frac{m}{r^{m+1}} \iint\limits_{E^m_r(\underline{\Phi})} \chi_l(T-t) \cdot i(u) \mathrm{dvol}_{\mathrm{I}_t}\, \mathrm{d}t \right) \mathrm{d}r \\
&\quad + \int_{r_1}^{r_2} \left(\frac{m}{r^{m+1}} \iint_{E^m_r(\underline{\Phi})} u \cdot \frac{1}{T-t} \underline{\phi^m_r} \cdot \chi_l'(T-t) \cdot (T-t) \mathrm{dvol}_{\mathrm{I}_t}\, \mathrm{d}t \right) \mathrm{d}r.
\end{aligned} \qquad (6.21)$$

Since $0 \leq \chi_l(T-t) \leq 1$, the first two integrands may be bounded in absolute value from above by the absolute values of the corresponding integrands occurring in the statement of this theorem, which are assumed summable. Thus, we may pass to the limit $l \to \infty$ in the first two integrals of identity (6.21) as in the proof of Theorem 5.1, and if $\frac{1}{T-t}\underline{\phi_r^m} \in L^1\left(E_{r_0}^m(\underline{\Phi})\right)$, then, also as in the proof of Theorem 5.1,

$$\left|u \cdot \frac{1}{T-t}\underline{\phi_r^m} \cdot \chi_l'(T-t) \cdot (T-t)\right| \leq C \cdot \left|\frac{u}{T-t}\underline{\phi_r^m}\right| \in L^1\left(E_{r_0}^m(\underline{\Phi})\right),$$

allowing us to apply the dominated convergence theorem, which implies that the last integral on the right-hand side vanishes in the limit $l \to \infty$, since $\chi_l'(T-t)\cdot(T-t) \xrightarrow{l\to\infty} 0$. Finally, if $u \geq 0$, we may discard the latter integral on the right-hand side by estimating it from below by 0, since $\chi_l' \geq 0$, wherefore the aforementioned limits involving the remaining integrals may be taken, thus establishing the result. $\square$

Remark 6.10. Just as in §5, if the right-hand spacetime integrand of (6.18) is not non-negative, then we may still obtain a monotonicity formula by modifying the heat ball (cf. Example 3.8) for certain u if Φ satisfies the inequalities (5.7) and (5.8) with $k = \frac{n-m}{2}$ on $E_{r_0}^m(\Phi)$, viz. if $u \geq 0$ is bounded on $E_{r_0}^m(\underline{\Phi})$ and $(\partial_t - \Delta_{\mathrm{I}})u \leq 0$, there holds

$$\begin{aligned}\left[\frac{1}{r^m}\iint\limits_{E_r^m(\widetilde{\Phi})} u\left[|\nabla_{\mathrm{I}}\underline{\widetilde{\phi}}|^2 + \left(|\underline{\underline{\mathrm{H}}}|^2 - \frac{1}{2}\mathrm{tr}_g^T\partial_t g\right)\underline{\widetilde{\phi_r^n}}\right]\mathrm{dvol}_{\mathrm{I}_t}\,\mathrm{d}t\right]_{r=r_1}^{r=r_2} \\ \geq \int_{r_1}^{r_2}\left(\frac{m}{r^{m+1}}\iint\limits_{E_r^m(\widetilde{\Phi})} u\left|\underline{\underline{\mathrm{H}}} - \nabla_g^\perp\widetilde{\phi}\right|^2\mathrm{dvol}_{\mathrm{I}_t}\,\mathrm{d}t\right)\mathrm{d}r\end{aligned} \tag{6.22}$$

for $0 < r_1 < r_2 < \widetilde{r_0}$ whenever the spacetime integrands are in $L^1(E_{r_0}^m(\underline{\Phi}))$, where $\widetilde{\Phi}(x,t) = e^{\xi_m(t)}\Phi(x,t)$, $\widetilde{\phi} = \log\widetilde{\Phi}$, $\widetilde{\phi_r^m} = \log(r^m\widetilde{\Phi})$, $\xi_m(t) = \int_t^T a - (n-m)b$ and $\widetilde{r_0} = e^{-\frac{\sup|\xi_m|}{m}}r_0$. Hence, more concretely, for $\Phi = {}^{n-2k}\Gamma_{(X,T)}$ (or $\Phi = {}^{n-2k}\mathrm{P}_{(X,T)}$ if M is compact and static), (6.22) holds with a and b given by (5.12) and (5.13) (resp. $a(t) = 0$ and (5.14)) with $k = \frac{n-m}{2}$ and r_0 as in Example 6.4 (resp. Example 6.5), where we note that the only term whose summability on $E_{r_0}^m(\underline{\widetilde{\Phi}})$ is called into question is $|\underline{\underline{\mathrm{H}}}|^2\underline{\widetilde{\phi_r^m}}$ since all the others were handled in Example 6.4, but, by (6.1) and the proof of Lemma 3.10(iv),

$$J_q^{r_0}(|\underline{\underline{\mathrm{H}}}|^2\underline{\phi}_r^m) = J_q^{r_0}(\partial_t\underline{\phi} + \underline{\phi}_r^m \cdot \frac{1}{2}\mathrm{tr}_g^T\partial_t g) + o(1)$$

as $q \to \infty$ so that by using the monotone convergence theorem for the left-hand integral and Lemma 3.9 together with (HB2) and the local geometry bound (2.3) for the right-hand one, we establish the summability of $|\underline{\underline{\mathrm{H}}}|^2\underline{\phi}_r^m$.

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