REGULARITY AND VANISHING THEOREMS FOR YANG-MILLS-HIGGS PAIRS

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Abstract. We obtain a vanishing theorem for Yang-Mills-Higgs pairs on Euclidean and hyperbolic spaces in dimensions greater than 4, as well as a regularity theorem more generally on Riemannian manifolds, and show how the latter may be used to establish a partial compactness theorem. Along the way, we investigate the monotonicity and scaling properties of such pairs, obtaining a generalisation of the well-known monotonicity formula for Yang-Mills connections.

1. Introduction

The purpose of this note is to establish vanishing and regularity theorems for Yang-Mills-Higgs pairs on Riemannian spaces of dimension $n > 4$. Such results have been known to hold for Yang-Mills fields and harmonic maps. We show that they carry over naturally to solutions to the Yang-Mills-Higgs system, which is in some sense a coupling of the Yang-Mills and harmonic map systems. We first briefly outline the geometric setup underlying Yang-Mills-Higgs theory, the details of which may be found in [6, §I] and [3, §IV.2]. We shall adopt the differential geometric notation of [10].

1.1. **Notation.** Let (M, g) be a Riemannian manifold of dimension $n > 4$, G a compact Lie group with semisimple Lie algebra $\mathfrak{g}$, $G \to P \to M$ a principal bundle and V an orthogonal representation of G.

Associated with P and V is a Riemannian vector bundle $V \to E_V \to M$. Likewise, the *adjoint representation* of G on $\mathfrak{g}$ gives rise to a Riemannian vector bundle $\mathfrak{g} \to E_{\mathfrak{g}} \to M$. We shall write $\langle \cdot, \cdot \rangle$ (resp. $|\cdot|$) for the fibrewise inner products (resp. norms) on all of these bundles as well as the naturally induced inner products (resp. norms) on their tensor and exterior products. Moreover, recall that a connection on P is given by a section of $\mathfrak{g} \otimes T^*P$, and such a connection naturally induces metric-compatible covariant derivatives on vector bundles associated with P and an orthogonal representation of G. Such a covariant derivative shall be denoted by ∇, and we shall use this symbol to denote the covariant derivatives induced on tensor product bundles.

In our considerations, we shall write $\{\varepsilon_i\}_{i=1}^n$ for a local frame for TM and $\{\varepsilon^i\}_{i=1}^n$ for the dual coframe for T^*M. Moreover, we shall say it is *adapted at* $x \in M$ if $\langle \varepsilon_i, \varepsilon_j \rangle = \delta_{ij}$ and $\nabla_{\varepsilon_i} \varepsilon_j = 0$ at x. Moreover, unless otherwise stated, all integrals are evaluated with respect to the Riemannian volume measure $\sqrt{g}$ induced by g (or an induced hypersurface measure), and we shall usually omit it for notational brevity.

2000 *Mathematics Subject Classification.* 35B45, 35B53, 35B65, 53C07.

Key words and phrases. Yang-Mills-Higgs pair, regularity, vanishing theorem.

Acknowledgements. The author gratefully acknowledges financial support from the Freie Universität Berlin.

1.2. **Background and results.** A *Yang-Mills-Higgs pair* with (smooth) symmetric potential $W : \mathbb{R} \to [0, \infty[$ with $W(0) \neq 0$ consists of a connection $\mathbf{A} : P \to \mathfrak{g} \otimes T^*P$ on P (*gauge field*) and a section $u : M \to E_V$ (*Higgs field*) solving the system

$$\begin{aligned} \operatorname{div} \underline{\mathbf{F}} &= \sum_{i=1}^{n} (u \odot \nabla_{\varepsilon_i} u) \otimes \varepsilon^i \\ \Delta u &= 2\left(W' \circ |u|^2\right) u, \end{aligned} \tag{1.1}$$

where $\underline{\mathbf{F}} : M \to E_{\mathfrak{g}} \otimes \Lambda^2 T^*M$ denotes the *curvature two-form* of $\mathbf{A}$, div the divergence operator induced by the Levi-Cività connection on TM and $\mathbf{A}$, $\Delta = \operatorname{div} \circ \nabla$ is the connection Laplacian induced by the Levi-Cività connection on TM and $\mathbf{A}$, and $\odot$ is defined fibrewise as the mapping $(E_V)_x \times (E_V)_x \to (E_{\mathfrak{g}})_x$ such that for all $v_1, v_2 \in (E_V)_x$ and $X \in (E_{\mathfrak{g}})_x$,

$$\langle X, v_1 \odot v_2 \rangle = \langle X \cdot v_1, v_2 \rangle \,,$$

where $\cdot$ denotes the fibrewise action of $E_{\mathfrak{g}}$ on E_V induced by that of $\mathfrak{g}$ on V.

Yang-Mills-Higgs pairs were introduced by Higgs [5] as a generalisation of *Yang-Mills fields* (connections $\mathbf{A}$ such that div $\underline{\mathbf{F}} \equiv 0$) in order to reconcile the theory of electroweak forces with the notion of a massive gauge field and have since played a key rôle in the *Standard Model* of particle physics. They naturally arise as critical points of the *energy*

$$\int_M e(\mathbf{A}, u; g, W), \tag{1.2}$$

where $e(\mathbf{A}, u; g, W) = \frac{1}{2}|\underline{\mathbf{F}}|^2 + \frac{1}{2}|\nabla u|^2 + W \circ |u|^2$ denotes the *Yang-Mills-Higgs energy density*. Note that both the energy (1.2) and system of equations (1.1) are *gauge invariant*, i.e. if we let $(\mathbf{g} \cdot \mathbf{A}, \mathbf{g} \cdot u)$ denote the pair obtained from $(\mathbf{A}, u)$ by means of the natural action of a bundle automorphism $\mathbf{g}$ of P, then $e(\mathbf{g} \cdot \mathbf{A}, \mathbf{g} \cdot u; g, W) = e(\mathbf{A}, u; g, W)$ *and* $(\mathbf{g} \cdot \mathbf{A}, \mathbf{g} \cdot u)$ is again a Yang-Mills-Higgs pair.

It was shown by Jaffe and Taubes [6, Corollary 2.3] in the case where $W(x) = \frac{\lambda}{4}(1-x)^2$ with $\lambda > 0$ and $M = \mathbb{R}^n$ that any Yang-Mills-Higgs pair $(\mathbf{A}, u)$ for which (1.2) is finite must actually be trivial in the sense that $\mathbf{A}$ must be the canonical flat connection and u a parallel section of unit norm. More generally, we have the following vanishing theorem on Euclidean and hyperbolic space assuming a growth condition on the Yang-Mills-Higgs energy over geodesic balls (denoted by $B(X, R)$), which was established for Yang-Mills fields by Price [12]:

Theorem A (Vanishing theorem)**.** *Suppose* $(\mathbf{A}, u)$ *is a Yang-Mills-Higgs pair on* $M = \mathbb{R}^n$ *or* $S^n(-\kappa^2)$ *(hyperbolic* n*-space of constant sectional curvature* $-\kappa^2$*). If*

$$\int_{B(X,R)} e(A, u; g, W) = o(R^{n-4}) \text{ as } R \to \infty \tag{1.3}$$

for some $X \in M$*, then* $\mathbf{A}$ *must be gauge equivalent to the canonical flat connection,* $\nabla u \equiv 0$ *and* $|u| \in \mathcal{W}$*, where*

$$\mathcal{W} = \{\sigma \in \mathbb{R}^+ : W(\sigma^2) = W'(\sigma^2) = 0\}.$$

Yang-Mills-Higgs pairs as in the conclusion of this theorem are known as *Higgs equilibria*. On the one hand, this theorem states that the only pairs on Euclidean and hyperbolic spaces of dimension $n > 4$ satisfying the growth condition (1.3) must be

Higgs equilibria. On the other hand, if $\mathcal{W} = \emptyset$, then it implies the *nonexistence* of solutions under these conditions. Motivated by this and the considerations of Higgs [5], we shall call W a *Higgs-like potential* if $\mathcal{W} \neq \emptyset$ and $W'' \geq 0$. On the one hand, the latter condition, assumed to hold as a strict inequality in a neighbourhood of a Higgs equilibrium point by Higgs [5], guarantees that the quantised Higgs particle has a *real-valued* mass. On the other hand, while it is not essential to the proof of Theorem A, it shall play a rôle in its applications (Theorems B and C).

The proof of Theorem A relies on a certain monotonicity property enjoyed by the Yang-Mills-Higgs energy on geodesic balls. In the case where $M = \mathbb{R}^n$ or $S^n(-\kappa^2)$, it reads

$$\frac{\mathrm{d}}{\mathrm{d}R}\left(\frac{1}{R^{n-4}}\int_{B(X,R)} e(A,u;g,W)\right) \geq 0 \tag{1.4}$$

for any $R > 0$ and $X \in M$. This inequality is well known for Yang-Mills fields, having been established by Price [12], but has not been exploited for Yang-Mills-Higgs pairs. We provide a streamlined proof in §2 by utilising the *energy-momentum tensor* associated with the energy (1.2).

In addition to investigating the large-scale behaviour of Yang-Mills-Higgs pairs, we also use the monotonicity property (1.4) to establish the following small-energy regularity theorem, which essentially says that if the monotone quantity in (1.4) is sufficiently small on a small enough geodesic ball, then we obtain a uniform bound on the energy density in a slightly smaller ball.

Theorem B (Regularity theorem)**.** *Fix $X \in M$ and $R_0 < \mathrm{inj}_X$ positive with inj_X the injectivity radius at X, and suppose W is a Higgs-like potential. There exist constants $\varepsilon, C > 0$ depending only on n, R_0, W, the geometry of $B(X, \mathrm{inj}_X)$ and the structure constants of $\mathfrak{g}$ such that if $(\mathbf{A}, u)$ is a Yang-Mills-Higgs pair with*

$$\frac{1}{R^{n-4}}\int_{B(X,R)} e(\mathbf{A},u;g,W) < \varepsilon \tag{1.5}$$

for some $R < R_0$, then

$$R^4 \sup_{B(X,\frac{R}{4})} e(\mathbf{A},u;g,W) \leq \frac{C}{R^{n-4}}\int_{B(X,R)} e(\mathbf{A},u;g,W).$$

The analogue of this theorem for Yang-Mills fields was established by Nakajima [9], whose proof was ultimately an adaptation of an analogous result for harmonic maps (also known as *nonlinear σ-models*) due to Schoen [14]. In some sense, the Yang-Mills-Higgs system is a coupling of both of these systems, and this so-called *ε-regularity property* bears a resemblance to those for these systems. One application of Theorem B is in the study of the compactness of spaces of Yang-Mills-Higgs pairs of uniformly bounded energy and Higgs field on a compact M, in analogy with what has been carried out for harmonic maps and Yang-Mills connections [14, 9]. This relies on an application of Uhlenbeck's Coulomb gauge theorem [15] and elliptic regularity theory.

Theorem C (Partial compactness)**.** *Let M be compact and $\{(\mathbf{A}_i, u_i)\}_{i=1}^{\infty}$ a sequence of Yang-Mills-Higgs pairs satisfying the uniform bounds $|u_i| \leq K$ and $\int_M e(\mathbf{A}_i, u_i; g, W) \leq E_0$. Then there exist a closed set $\mathcal{S} \subset M$ of finite $(n-4)$-dimensional Hausdorff measure, a sequence of bundle automorphisms $\{\mathbf{g}_i\}_{i=1}^{\infty}$ and a Yang-Mills-Higgs pair*

$(\mathbf{A}_\infty, u_\infty)$ *on* $G \to P|_{M\backslash\mathcal{S}} \to M\backslash\mathcal{S}$ *such that* $\{(\mathbf{g}_i \cdot \mathbf{A}_i, \mathbf{g}_i \cdot u_i)\}$ *subconverges to* $(\mathbf{A}_\infty, u_\infty)$ *on* $M\backslash\mathcal{S}$ *locally uniformly in* C^∞.

Throughout this paper, we assume that the Yang-Mills-Higgs pair in question is *smooth*. More generally, one might consider pairs of measurable $(\mathbf{A}, u)$ satisfying (1.1) in the distributional sense with $\mathbf{A}$ and u lying in suitable Sobolev spaces. The methods of [12] would then apply to yield a monotonicity formula and thus a vanishing theorem. However, it is unclear whether an analogue of Theorem B may be established in this weaker setting; indeed, while such a result is known for *stationary* and *minimising* harmonic maps [13, 2], there is no analogue for weak Yang-Mills fields.

2. Monotonicity formula

We shall now establish the monotonicity principle (1.4). To streamline the subsequent proofs, we shall write $\underline{\mathbf{F}}_{ij} = (\underline{\mathbf{F}}, \varepsilon_i \wedge \varepsilon_j)$, $(\cdot,\cdot)$ being the canonical fibrewise bilinear pairing of $E_{\mathfrak{g}} \otimes \Lambda^2 T^*M$ with $\Lambda^2 TM$, and we shall write ∇_i for ∇_{ε_i}.

Theorem 2.1 (Monotonicity formula)**.** *Let* $X \in M$*, write* r *for the distance function* $\mathrm{dist}(X,\cdot)$ *and suppose the Hessian estimate*

$$\Lambda_- r^2 g_r \leq g - \nabla^2(\frac{1}{2}r^2) \leq \Lambda_+ r^2 g_r \tag{2.1}$$

holds on $B(X,R_0)$ *with* $0 < R_0 \leq \mathrm{inj}_X$*,* $\pm\Lambda_\pm \geq 0$ *and* $g_r = g - \mathrm{d}r \otimes \mathrm{d}r$*. There exists a constant* $\Lambda \geq 0$ *depending only on* $\Lambda_\pm$ *and* n *such that if* $(\mathbf{A}, u)$ *is a Yang-Mills-Higgs pair, the inequality*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}R}\left(\frac{e^{\Lambda R^2}}{R^{n-4}}\int_{B(X,R)} e(\mathbf{A},u;g,W)\right) \geq \frac{e^{\Lambda R^2}}{R^{n-4}}\int_{\partial B(X,R)} |\nabla r \llcorner \underline{\mathbf{F}}|^2 + |\nabla r \llcorner \nabla u|^2 \\ + \frac{e^{\Lambda R^2}}{R^{n-3}}\int_{B(X,R)} |\nabla u|^2 + nW \circ |u|^2 \geq 0\end{aligned} \tag{2.2}$$

holds for all $R < R_0$*, where* $\llcorner$ *denotes an interior product. If* $M = \mathbb{R}^n$ *or* $S^n(-\kappa^2)$*, then* $\Lambda = 0$ *and (2.2) holds for all* $R > 0$*.*

Remark 2.2. The Hessian estimate (2.1) is always valid for a suitable choice of $\Lambda_\pm$. In particular, if the sectional curvatures κ of $B(X, \mathrm{inj}_X)$ satisfy the inequality $\kappa_- \leq \kappa \leq \kappa_+$ for $\kappa_\pm \in \mathbb{R}$, then $\Lambda_\pm$ may be explicitly determined in terms of $\kappa_\pm$ on $B(X,R_0)$ whenever $R_0 < i_0(X) := \min\{\frac{\pi}{2\sqrt{\kappa_+}}, \frac{\mathrm{inj}_X}{2}\}$, cf e.g. [10, Theorem 27, p. 175].

Proof of Theorem 2.1. We introduce the *energy-momentum tensor* $T = \sum_{i,j=1}^n T_{ij}\varepsilon^i \otimes \varepsilon^j : M \to T^*M \otimes T^*M$ with

$$T_{ij} = \sum_{k=1}^n \langle \underline{\mathbf{F}}_{ik}, \underline{\mathbf{F}}_{jk}\rangle + \langle \nabla_i u, \nabla_j u\rangle - e(\mathbf{A},u;g,W)g_{ij},$$

where $g_{ij} = \langle \varepsilon_i, \varepsilon_j\rangle$. It may be readily computed in an adapted frame with the help of the Bianchi identity for $\underline{\mathbf{F}}$ (cf. [6, §II.2]) that

- $\mathrm{tr}\, T = \langle g, T\rangle = (4-n)e(\mathbf{A},u;g,W) - |\nabla u|^2 - nW \circ |u|^2$, and
- $\mathrm{div}\, T = 0$.

Let $Z = \nabla(\frac{1}{2}r^2) = r\nabla r$ and define the local vector field $Y = (Z \llcorner T)^\sharp$, where $\sharp : T^*M \to TM$ is the canonical musical isomorphism associated with g with inverse given by $\flat$. On the one hand, we have that $\langle Y, \nabla r\rangle = r\,\langle T, \mathrm{d}r \otimes \mathrm{d}r\rangle$ and on the other,

$$\text{div } Y = (\text{div } T, Z) + \left\langle \nabla Z^\flat, T\right\rangle = -\left\langle g - \nabla^2(\frac{1}{2}r^2), T\right\rangle + \text{tr } T.$$

By the divergence theorem, $\int_{\partial B(X,R)} \langle Y, \nabla r\rangle = \int_{B(X,R)} \text{div } Y$ so that substituting in the above expressions, multiplying by R^{3-n}, rearranging and applying the coarea formula, we obtain the equality

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}R}\left(\frac{1}{R^{n-4}}\int_{B(X,R)} e(\mathbf{A}, u; g, W)\right) &= \frac{1}{R^{n-4}}\int_{\partial B(X,R)} |\nabla r \llcorner \underline{\mathbf{F}}|^2 + |\nabla r \llcorner \nabla u|^2 \\ &\quad + \frac{1}{R^{n-3}}\int_{B(X,R)} |\nabla u|^2 + nW \circ |u|^2 \\ &\quad + \frac{1}{R^{n-3}}\int_{B(X,R)} \left\langle g - \nabla^2(\frac{1}{2}r^2), T\right\rangle. \end{aligned} \tag{2.3}$$

A straightforward computation now shows that the Hessian estimate (2.1) implies that $\left\langle g - \nabla^2(\frac{1}{2}r^2), T\right\rangle \geq R^2 \cdot (4\Lambda_- - (n-1)\Lambda_+)e(\mathbf{A}, u; g, W)$ on $B(X, R)$. Substituting this into (2.3), multiplying by $e^{\Lambda R^2}$ with $\Lambda = \frac{1}{2}(4\Lambda_- - (n-1)\Lambda_+)$ and rearranging, we obtain (2.2). In the case where $M = \mathbb{R}^n$, $\text{inj}_X = \infty$ and $g - \nabla^2(\frac{1}{2}r^2) \equiv 0$ so that the last integral in (2.3) drops out, whereas for $M = S^n(-\kappa^2)$, $\text{inj}_X = \infty$ and $g - \nabla^2(\frac{1}{2}r^2) = 1 - \kappa r \coth(\kappa r)$ so that a quick computation establishes that $\left\langle g - \nabla^2(\frac{1}{2}r^2), T\right\rangle \geq 0$. $\square$

Remark 2.3 (Scaling properties)**.** An alternative to the method we employed here, which was briefly indicated in [1], is to exploit the *scaling properties* of the Yang-Mills-Higgs system. Indeed, let $\{g_{ij} : B(0, \text{inj}_X) \to \mathbb{R}\}$ be the components of g in geodesic normal coordinates about $X \in M$, $\sigma : B(X, \text{inj}_X) \to P$ a local section and $\{A_i : B(0, \text{inj}_X) \to \mathfrak{g}\}_{i=1}^n$ and $\widetilde{u} : B(0, \text{inj}_X) \to V$ local representatives of $\mathbf{A}$ and u with respect to σ in these geodesic normal coordinates. Now, for fixed $\lambda > 0$, if we let $(A^X_\lambda)_i(z) = \lambda A_i(\lambda z)$ and $\widetilde{u}^X_\lambda(z) = \lambda\widetilde{u}(\lambda z)$, we naturally obtain a corresponding connection $\mathbf{A}^X_\lambda$ on the trivial bundle $G \to B(0, \frac{\text{inj}_X}{\lambda}) \times G \to B(0, \frac{\text{inj}_X}{\lambda})$ and section u^X_λ of $V \to B(0, \frac{\text{inj}_X}{\lambda}) \times V \to B(0, \frac{\text{inj}_X}{\lambda})$. It may now be readily checked that $(\mathbf{A}^X_\lambda, u^X_\lambda)$ is a Yang-Mills-Higgs pair with potential W_λ defined by $W_\lambda(x) = \lambda^4 W(\frac{x}{\lambda^2})$ and underlying metric g^X_λ with components $(g^X_\lambda)_{ij}(z) = g_{ij}(\lambda z)$. Moreover, if W is Higgs-like, then so is W_λ, and we have the following *scale-invariance* property of the local Yang-Mills-Higgs energy appearing in Theorem 2.1:

$$\frac{1}{R^{n-4}}\int_{B(X,R)} e(\mathbf{A}, u; g, W) = \int_{B(0,1)} e(\mathbf{A}^X_R, u^X_R; g^X_R, W_R)\sqrt{g^X_R}.$$

3. Proof of Theorem A

Fix $R_0 > 0$. By Theorem 2.1 and (1.3), we see that for $R > R_0$,

$$\frac{1}{R_0^{n-4}}\int_{B(X,R_0)} e(\mathbf{A}, u; g, W) \leq \frac{1}{R^{n-4}}\int_{B(X,R)} e(\mathbf{A}, u; g, W) \xrightarrow{R\to\infty} 0.$$

Therefore, we must have $e(\mathbf{A}, u; g, W) \equiv 0$ on M so that $\underline{\mathbf{F}} \equiv 0$, $\nabla u = 0$ and $W \circ |u|^2 \equiv 0$. On the one hand, we also have $\Delta u = 0$ so that $u \neq 0$ and the system (1.1) imply that $W' \circ |u|^2 \equiv 0$. On the other hand, $\underline{\mathbf{F}} \equiv 0$ expresses the fact that $\mathbf{A}$ is *flat*. Since M is simply connected, a now classical theorem (see [7, Corollary 9.2]) implies that there is a global trivialisation of P with respect to which $\mathbf{A}$ is the canonical flat connection, i.e. $\mathbf{A}$ is obtainable from the canonical flat connection via a bundle automorphism. This establishes the claim. □

4. Proof of Theorem B

We first begin by deriving a differential inequality for $e(\mathbf{A}, u; g, W)$. Writing Δ for the Laplace-Beltrami operator acting on functions, it follows from a straightforward computation in an adapted frame that

$$\begin{aligned} -\Delta e(\mathbf{A}, u; g, W) = &-|\nabla \underline{\mathbf{F}}|^2 - |\nabla^2 u|^2 - \langle \Delta \nabla u, u\rangle - \langle \Delta \underline{\mathbf{F}}, \underline{\mathbf{F}}\rangle \\ &- 4(W'' \circ |u|^2) \cdot \left| \sum_{i=1}^n \langle u, \nabla_i u\rangle \, \varepsilon^i \right|^2 - 4(W' \circ |u|^2)^2 \cdot |u|^2 \\ &- 2(W' \circ |u|^2) \cdot |\nabla u|^2. \end{aligned} \tag{4.1}$$

Now, by the Weitzenböck trick [11, §4.22] and the Bianchi identity for $\underline{\mathbf{F}}$, we have the relations

$$\begin{aligned} \Delta \nabla u &= \nabla \Delta u + \mathrm{div}\ (\underline{\mathbf{F}} \cdot u) + \underline{\mathbf{F}} * \nabla u + \mathrm{Rm} * \nabla u \\ \Delta \underline{\mathbf{F}} &= \mathrm{d}^\nabla (\mathrm{div}\ \underline{\mathbf{F}}) + \underline{\mathbf{F}} * \underline{\mathbf{F}} + \mathrm{Rm} * \underline{\mathbf{F}}, \end{aligned} \tag{4.2}$$

where $\mathrm{d}^\nabla = \sum_{i=1}^n \varepsilon^i \wedge \nabla_i$ is the *exterior covariant differential* associated with ∇, Rm is the Riemann curvature tensor of g and $*$ denotes a tensor contraction involving g, the structure constants of $\mathfrak{g}$ and the induced representation of $\mathfrak{g}$ on V so that any expression of the form $A * B$ satisfies an estimate of the form $|A * B| \leq C|A| \cdot |B|$ with C depending only on n and the structure constants of $\mathfrak{g}$ and the representation V. We therefore readily compute using (1.1) that

$$\nabla \Delta u = 4(W'' \circ |u|^2) \cdot u \otimes \left(\sum_{i=1}^n \langle u, \nabla_i u\rangle \, \varepsilon^i \right) + 2(W' \circ |u|^2) \cdot \nabla u$$

$$\mathrm{d}^\nabla (\mathrm{div}\ \underline{\mathbf{F}}) = \sum_{i,j=1}^n (\nabla_i u \odot \nabla_j u) \otimes \varepsilon^i \wedge \varepsilon^j + \frac{1}{2} \sum_{i,j=1}^n (u \odot (\underline{\mathbf{F}}_{ij} \cdot u)) \otimes \varepsilon^i \wedge \varepsilon^j.$$

Substituting this into (4.2) and (4.2) into (4.1), applying the Cauchy-Schwarz inequality and discarding nonpositive terms, we see that

$$-\Delta e(\mathbf{A}, u; g, W) \leq c_0 e(\mathbf{A}, u; g, W) + c_1 e(\mathbf{A}, u; g, W)^{3/2}, \tag{4.3}$$

where $c_0 = -8W'(0) + 4CR_\infty$ with C as above and R_∞ an upper bound for $|\mathrm{Rm}|$, and $c_1 = 3 \cdot 2^{3/2} C$. Therefore, $e(\mathbf{A}, u; g, W)$ satisfies a *nonlinear* elliptic partial differential inequality. As it turns out, a standard lemma [8, Theorem 5.3.1] states that if (4.3) had been linear, then the conclusion of Theorem (B) would hold true. We recall this and provide an elementary proof for completeness' sake.

Lemma 4.1. *Let $R_0 < \mathrm{inj}_X$, assume the Hessian estimate (2.1) and suppose $v : B(X,R_0) \to [0,\infty[$ is a smooth function such that $-\Delta v \leq cv$ for some constant $c > 0$. Then for all $R \leq R_0$,*

$$v(X) \leq \frac{e^{\frac{1}{4}(c_g+c)R^2}}{\omega_n R^n} \int_{B(X,R)} v,$$

where $c_g \geq 0$ depends on n and Λ_+, and ω_n is the volume of $B(0,1) \subset \mathbb{R}^n$.

Proof. Let $r = \mathrm{dist}(X,\cdot)$ and $f(R) = R^{-n} \int_{B(X,R)} v(y)$. The coarea formula implies that

$$f'(R) = -\frac{n}{R^{n+1}} \int_{B(X,R)} v + R^{-n} \int_{\partial B(X,R)} v. \tag{4.4}$$

Now, since $\left\langle \nabla \left(\frac{1}{2}r^2\right), \nabla r \right\rangle \equiv R$ on $\partial B(X,R) = \{r = R\}$, the divergence theorem implies that

$$\begin{aligned}\int_{\partial B(X,R)} v &= R^{-1} \int_{\partial B(X,R)} \left\langle v \nabla \left(\frac{1}{2}r^2\right), \nabla r \right\rangle \\ &= R^{-1} \int_{B(X,R)} \left\langle \nabla v, \nabla \left(\frac{1}{2}r^2\right), \nabla r \right\rangle + v \cdot \Delta \left(\frac{1}{2}r^2\right),\end{aligned}$$

and using the fact that $\nabla \left(\frac{1}{2}r^2\right) = \nabla \left(\frac{1}{2}\left(r^2 - R^2\right)\right)$, which vanishes on $\partial B(X,R)$, we further obtain the equality

$$\int_{B(X,R)} \left\langle \nabla v, \nabla \left(\frac{1}{2}r^2\right) \right\rangle = \int_{B(X,R)} \Delta v \cdot \frac{1}{2}\left(R^2 - r^2\right).$$

Incorporating all of this into (4.4), we obtain

$$f'(R) = R^{-n-1} \int_{B(X,R)} v \cdot \left(\Delta \left(\frac{1}{2}r^2\right) - n\right) + R^{-n-1} \int_{B(X,R)} \Delta v \cdot \frac{1}{2}(R^2 - r^2).$$

Taking the trace of (2.1), we see that the first of these integrals is bounded from below by $-(n-1)\Lambda_+ R \cdot f(R)$ and using the differential inequality for v as well as the fact that $0 \leq R^2 - r^2 \leq R^2$ on $B(X,R)$, the second is bounded from below by $-\frac{cR}{2} \cdot f(R)$. Altogether, setting $c_g = 2(n-1)\Lambda_+$, f satisfies the differential inequality $\frac{\mathrm{d}}{\mathrm{d}R}\left(e^{\frac{1}{4}(c_g+c)R^2} f(R)\right) \geq 0$. Integrating this and using the 'initial condition' $\lim_{R \searrow 0} f(R) = \omega_n v(X)$ then implies the claim. □

We shall now show that there are geometric constants $\varepsilon, C > 0$ such that if the ε-regularity condition (1.5) holds, then

$$\sup_{\alpha \in]0,1[} \left(\frac{\alpha R}{2}\right)^4 \sup_{B(X,\frac{1}{2}(1-\alpha)R)} e(\mathbf{A},u;g,W) \leq \frac{C}{R^{n-4}} \int_{B(X,R)} e(\mathbf{A},u;g,W);$$

the claim will then follow immediately for $\alpha = \frac{1}{2}$. The key here will be to establish that (1.5) essentially allows us to treat (4.3) as if it were linear. We shall employ a scaling technique due to Schoen [14].

By continuity and compactness, we may find an $\alpha_0 \in [0,1]$ and $X_0 \in \overline{B(X, \frac{1}{2}(1-\alpha_0)R)}$ such that

$$\beta_0^4 := \left(\frac{\alpha_0 R}{2}\right)^4 \cdot e(\mathbf{A}, u; g, W)(X_0) = \sup_{\alpha \in]0,1[} \left(\frac{\alpha R}{2}\right)^4 \sup_{B(X, \frac{1}{2}(1-\alpha)R)} e(\mathbf{A}, u; g, W).$$

We may assume without loss of generality that $\alpha_0 > 0$. Now, note that since $B(X_0, \frac{\alpha_0 R}{4}) \subset B(X, \frac{1}{2}(1 - \frac{\alpha_0}{2})R)$, we have the inequality

$$\sup_{B(X_0, \frac{\alpha_0 R}{4})} e(\mathbf{A}, u; g, W) \leq \left(\frac{4\beta_0}{\alpha_0 R}\right)^4 =: \lambda_0^{-4}.$$

We now consider the *rescaled* Yang-Mills-Higgs pair $(\mathbf{A}^X_{\lambda_0}, u^X_{\lambda_0})$ on $B(0, \beta_0)$ (cf. Remark 2.3). By definition, we have $e(\mathbf{A}^X_{\lambda_0}, u^X_{\lambda_0}; g^X_{\lambda_0}, W_{\lambda_0})(z) = \lambda_0^4 e(\mathbf{A}, u; g, W)(\theta_X(\lambda_0 z))$ so that $e(\mathbf{A}^X_{\lambda_0}, u^X_{\lambda_0}; g^X_{\lambda_0}, W_{\lambda_0}) \leq 1$ and $e(\mathbf{A}^X_{\lambda_0}, u^X_{\lambda_0}; g^X_{\lambda_0}, W_{\lambda_0})(0) = \left(\frac{1}{2}\right)^4$. Setting $v = e(\mathbf{A}^X_{\lambda_0}, u^X_{\lambda_0}; g^X_{\lambda_0}, W_{\lambda_0})$, the inequality (4.3) implies that

$$-\Delta v \leq \lambda_0^2 c_0 v + c_1 v^{3/2} \leq (\lambda_0^2 c_0 + c_1) v$$

so that Lemma 4.1 together with the fact that $c_{g^X_{\lambda_0}} = \lambda_0^2 c_g$ yields

$$\left(\frac{1}{2}\right)^4 = v(0) \leq \frac{\exp\left(\frac{1}{4}\left(\lambda_0^2(c_0 + c_g) + c_1\right) r^2\right)}{\omega_n r^n} \int_{B(0,r)} v \cdot \sqrt{g^X_{\lambda_0}} \tag{4.5}$$

for all $r \leq \beta_0$. We now claim that if (1.5) holds with $\varepsilon = 2^{-n-1} c_2^{-1} e^{-\frac{\Lambda R_0^2}{4}}$, where $c_2 = \omega_n^{-1} \exp(\frac{R_0^2}{256}(c_0 + c_g) + \frac{1}{4} c_1)$, then $\beta_0 < 1$. Indeed, if $\beta_0 \geq 1$, then $\lambda_0 \leq \frac{\alpha_0 R}{4} \leq \frac{R_0}{8}$ so that (4.5) with $r = 1$, scale-invariance and Theorem 2.1 imply that

$$\begin{aligned}
\left(\frac{1}{2}\right)^4 \leq c_2 \int_{B(0,1)} v \cdot \sqrt{g^X_{\lambda_0}} &= \frac{c_2}{\lambda_0^{n-4}} \int_{B(X_0, \lambda_0)} e(\mathbf{A}, u; g, W) \\
&\leq \frac{2^{n-4} c_2 e^{\Lambda R^2/4}}{R^{n-4}} \int_{B(X_0, \frac{R}{2})} e(\mathbf{A}, u; g, W) \\
&\leq \frac{2^{n-4} c_2 e^{\Lambda R^2/4}}{R^{n-4}} \int_{B(X,R)} e(\mathbf{A}, u; g, W) \\
&\leq 2^{n-4} c_2 e^{\Lambda R_0^2/4} \varepsilon = \left(\frac{1}{2}\right)^5,
\end{aligned}$$

which is impossible. Therefore, we must have that $\beta_0 < 1$ so that $\beta_0 \lambda_0 = \frac{\alpha_0 R}{4} \leq \frac{R_0}{8}$. Using (4.5) with $r = \beta_0$ and again applying scale-invariance and Theorem 2.1 now yields

$$\begin{aligned}
(\beta_0)^4 \leq \frac{2^4 c_2}{\beta_0^{n-4}} \int_{B(0,\beta_0)} v \sqrt{g^X_{\lambda_0}} &= \frac{2^4 c_2}{(\beta_0 \lambda_0)^{n-4}} \int_{B(X_0, \beta_0 \lambda_0)} e(\mathbf{A}, u; g, W) \\
&\leq \frac{2^n c_2 e^{\Lambda R^2/4}}{R^{n-4}} \int_{B(X_0, \frac{R}{2})} e(\mathbf{A}, u; g, W) \\
&\leq \frac{2^n c_2 e^{\Lambda R_0^2/4}}{R^{n-4}} \int_{B(X,R)} e(\mathbf{A}, u; g, W),
\end{aligned}$$

which establishes the claim with $C = 2^n c_2 e^{\Lambda R_0^2/4}$. □

5. Proof of Theorem C

We now turn our attention to the proof of Theorem C. By the compactness of M, the injectivity radii and sectional curvatures at all points are uniformly bounded from above and below so that the geometric constants appearing in the monotonicity principle (Theorem 2.1) and regularity theorem (Theorem B) may be determined independently of $X \in M$ provided we restrict our attention to balls of radii less than $R_0 := \min\{\mathrm{inj}_M, \frac{\pi}{2\sqrt{\kappa_+}}\}$, κ_+ being an upper bound on the sectional curvatures of M and inj_M the injectivity radius of M (cf. Remark 2.2).

We first introduce the so-called *singular set* of the sequence as

$$\mathcal{S} = \bigcap_{0<R<R_0} \{x \in M : \liminf_{i\to\infty} \frac{1}{R^{n-4}} \int_{B(X,R)} e(\mathbf{A}_i, u_i; g, W) \geq \varepsilon\}.$$

Now, it follows immediately that $\mathcal{S}$ is closed. To see this, note that

$$\liminf_{i\to\infty} \frac{1}{(R-\delta)^{n-4}} \int_{B(X_j,R-\delta)} e(\mathbf{A}_i, u_i; g, W) \geq \varepsilon$$

for any $0 < R < R_0$ and $0 < \delta < R$ with $X_j \xrightarrow{j\to\infty} X \in M$ implies that for sufficiently large j, $B(X_j, R-\delta) \subset B(X,R)$ so that for such j

$$\begin{aligned}&\frac{1}{(R-\delta)^{n-4}} \liminf_{i\to\infty} \int_{B(X,R)} \frac{1}{e(\mathbf{A}_i, u_i; g, W)}\\ &\geq \liminf_{i\to\infty} \frac{1}{(R-\delta)^{n-4}} \int_{B(X_j,R-\delta)} e(\mathbf{A}_i, u_i; g, W) \geq \varepsilon;\end{aligned}$$

taking the limit $\delta \searrow 0$, we conclude that $X \in \mathcal{S}$. Now, consider the collection $\{B(X,\delta)\}_{X\in\mathcal{S}}$. By compactness and the Vitali covering lemma, we may pass to a finite pairwise disjoint subcollection $\{B(X_j,\delta)\}_{j=1}^N$ such that $\mathcal{S} \subset \bigcup_{j=1}^N B(X_j, 5\delta)$. Therefore, $X_j \in \mathcal{S}$ implies the inequality

$$\sum_{j=1}^N (5\delta)^{n-4} \leq \frac{5^{n-4}}{\varepsilon} \sum_{j=1}^N \liminf_{i\to\infty} \int_{B(X_j,\delta)} e(\mathbf{A}_i, u_i; g, W) \leq \frac{5^{n-4}}{\varepsilon} E_0.$$

Since this holds for any small $\delta > 0$, we immediately obtain that $\mathscr{H}^{n-4}(\mathcal{S}) \leq \frac{5^{n-4}}{\varepsilon} E_0$, where $\mathscr{H}^{n-4}$ denotes the $(n-4)$-dimensional Hausdorff measure on M.

If $X \in M \backslash \mathcal{S}$, then there exists an $R \in]0, R_0[$ and a subsequence, which we again denote by $\{(\mathbf{A}_i, u_i)\}$, such that the ε-regularity condition (1.5) holds on this sequence. Therefore, by Theorem B, we obtain uniform bounds on both ∇u_i and $\underline{\mathbf{F}}_{\mathbf{A}_i}$ on $B(X,R)$, $\underline{\mathbf{F}}_{\mathbf{A}_i}$ being the curvature of $\mathbf{A}_i$. On the one hand, Uhlenbeck's theorem [15] states that there exists a sequence of local sections $\sigma_i : B(X, \theta R) \to P$, $\theta \in]0,1[$ depending only on the geometry of M, with respect to which the $W^{1,p}$ norm of the local representative A_i of $\mathbf{A}_i$ is uniformly bounded on $B(X,\theta R)$ for any $p > \frac{n}{2}$ and the *Coulomb gauge condition* $\sum_{j=1}^n \partial_j (A_i)_j = 0$ holds in geodesic normal coordinates about X. After decreasing θ if necessary, the Coulomb gauge condition together with the system (1.1) for $(\mathbf{A}_i, u_i)$ in the σ_i frame forms an *elliptic system* on $B(X,\theta R)$. Standard techniques [8, Ch. 6] now apply and yield uniform C^∞ estimates on A_i and u_i on $B(X, \frac{\theta R}{2})$.

By the Arzelà-Ascoli theorem, we may pass to a further subsequence of $\{(\mathbf{A}_i, u_i)\}$ whose local representatives converge smoothly on $B(X, \frac{\theta R}{2})$ to a Yang-Mills-Higgs pair on $B(X, \frac{\theta R}{2})$. Since $M \backslash \mathcal{S}$ may be covered by balls for which $(\mathbf{A}_i, u_i)$ can be made to locally smoothly subconverge, it follows from a well-known patching theorem [4, Corollary 4.4.8] that there exists a sequence of bundle automorphisms $\{\mathbf{g}_i\}_{i=1}^{\infty}$, a further subequence $\{(\mathbf{A}_i, u_i)\}_{i=1}^{\infty}$ and a globally defined pair $(\mathbf{A}_\infty, u_\infty)$ on all of $M \backslash \mathcal{S}$, where the underlying bundles are the restrictions of those of $(\mathbf{A}_i, u_i)$ to $M \backslash \mathcal{S}$, such that *any* local representative of $\{(\mathbf{g}_i \cdot \mathbf{A}_i, \mathbf{g}_i \cdot u_i)\}$ converges to $(\mathbf{A}_\infty, u_\infty)$ on $M \backslash \mathcal{S}$ locally uniformly in C^∞. □

Remark 5.1. Note that in the proof of Theorem C, crucial use was made of the uniform bound $|u_i| \leq K$. Comparison with the case of harmonic maps [14], where the sequence is uniformly bounded *ab initio* due to the target manifold being a compact submanifold of Euclidean space, shows that this condition is not unnatural. However, if we assume the growth condition $W(x) \xrightarrow{x \to \infty} \infty$, then we may drop this condition and still obtain partial compactness; this may be seen in the course of the proof by noting that for $X \in M \backslash \mathcal{S}$, Theorem B furnishes a uniform bound for $e(\mathbf{A}_i, u_i; g, W)$ on $B(X, R)$ for some $R > 0$, which then yields a uniform bound on $W \circ |u_i|^2$ on this set and thus also u_i by the growth condition above, whence the proof may be carried out as before.

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