LOCAL REGULARITY FOR THE HARMONIC MAP AND YANG-MILLS HEAT FLOWS

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Abstract. We establish new local regularity theorems for the Yang-Mills and harmonic map heat flows on complete Riemannian manifolds of dimension greater than 4 and 2 respectively, obtaining a necessary and sufficient condition for the smooth local extensibility of these flows. As a corollary, we obtain a new characterisation of singularity formation and use this to obtain an estimate on the Hausdorff measure of the singular sets of these flows at the first singular time. Finally, we show that smooth blow-ups at rapidly forming singularities of these flows are necessarily non-trivial and admit a positive lower bound on their heat ball energies. These results depend heavily on some local monotonicity formulæ for these flows recently established by Ecker [12] and the author [1, 3].

1. Introduction

The main result of this paper is the following local regularity theorem for the harmonic map and Yang-Mills heat flows on complete Riemannian manifolds (see §2 for the setup):

Theorem A. *Let (M, g) be a complete Riemannian manifold of dimension $n > 2k$, $\Omega \subset M$ open and bounded and $c_{n,k} = \sqrt{\frac{n-2k}{2\pi e}}$. Then there exist geometric constants $\varepsilon, C > 0$ such that if $u : Q_R(X,T) \subset \Omega \times]0,T[\to E \otimes \Lambda^{k-1}T^*M$ is a harmonic map ($k = 1$) or Yang-Mills ($k = 2$) heat flow, then the implication*

$$\sup_{(x,t)\in Q_{R/2}(X,T)} \frac{1}{\left(R/(2c_{n,k})\right)^{n-2k}} \iint_{E^{n-2k}_{R/(2c_{n,k})}(x,t)} e^{\xi_k(t-s)}[u,g]_{(x,t)}(y,s)\mathrm{dvol}_g(y)\mathrm{d}s < \varepsilon$$
$$\Rightarrow \left(\frac{7}{32}R\right)^{2k} \sup_{Q_{R/4}(X,T)} \frac{1}{2}|\psi|^2 \le C$$

holds, where ψ is the differential of u or the curvature two-form respectively.

The harmonic map heat flow was first introduced by Eells and Sampson [14] to smoothly deform smooth maps $(M^n, g) \to N \hookrightarrow \mathbb{R}^K$ of Riemannian manifolds M and N into harmonic ones. Key to their work was the fact that the target manifold N had nonpositive sectional curvatures. Without this condition, the harmonic map heat flow does not necessarily exist for all time, which was shown by Coron and Ghidaglia [10] in the case where $n \geq 3$ and Chang, Ding and Ye [7] in the case where $n = 2$. Given that singularities are inevitable, one might ask how big the set of singularities—

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i.e. the *singular set*— is at the maximal time of a smooth harmonic map heat flow. This question was answered by Grayson and Hamilton [18] in the case of *compact* M of dimension at least 3 (and implied by the work of Struwe [28] for Euclidean M under suitable global restrictions on u), where they showed that the singular set is of codimension 2; moreover, they established the existence and nontriviality of smooth blow-ups of rapidly-forming singularities. The crucial quantity in their analysis was a weighted scale-invariant energy of the form

$$\int_M \frac{1}{2}|\psi|^2(\cdot,t)\cdot\Phi^k_{(X,T)}(\cdot,t), \tag{1.1}$$

where $\Phi^k_{(X,T)}(x,t) = (T-t)^k\cdot\Phi_{(X,T)}$ for fixed $(X,T)\in M\times\mathbb{R}$, $\Phi_{(X,T)}$ being the canonical backward heat kernel on M with singularity at (X,T), ψ equal to the differential of u and $k=1$. In particular, they showed that if (1.1) is smaller than a geometric constant $\varepsilon>0$ close to the maximal time T for some fixed X, then a supremum bound on the differential $\mathrm{d}u$ of u holds on a parabolic cylinder terminating at (X,T) (see §2 for the definition), which in turn implies that u may be smoothly extended up to the maximal time T in a neighbourhood of X; from this result, a characterisation of the singular set of u at the maximal time is given in terms of a positive lower bound on (1.1) close to T is given, which then leads to an estimate on the $(n-2)$-dimensional Hausdorff measure of the singular set, as well as a strictly positive lower bound on the weighted energy (1.1) of smooth blow-ups at rapidly-forming singularities. Ultimately, all of these results rely upon the scale-invariance and monotonicity properites of (1.1), which were established by Struwe [28] and Hamilton [20].

Likewise, the Yang-Mills heat flow of connections on a principal G-bundle over (M^n,g), introduced by Atiyah and Bott [4] to study the Morse theory of Yang-Mills connections, also tends to develop singularities in finite time on manifolds of dimension at least 5, which was shown by Naito [21] in the case where M is spherical and Grotowski [19] in the case where M is Euclidean. It has likewise been shown that the singular set is of codimension 4 in the case of compact M of dimension at least 5 by Chen, Shen and Zhou [8], and an analysis of rapidly-forming singularities was carried out by Weinkove [31] in this case. The key ingredient here is also a weighted energy of the form (1.1) with ψ equal to the curvature two-form of the flow of connections and $k=2$, which again leads to an ε-regularity result as with the harmonic map heat flow from which an estimate on the singular set at the maximal time as well as a lower bound on the weighted energy of smooth blow-ups at rapidly-forming singularities readily follows. Again, these results rely upon scale-invariance and monotonicity properties of (1.1) established by Chen & Shen [9] and Hamilton [20].

Theorem A is an analogue of both of these ε-regularity results; however, in contrast to the quantity (1.1), ours, which is expressed as a supremum of a collection of so-called heat ball energies, depends only on *local* data, as these heat balls completely lie within a parabolic cylinder, which allows us to state a criterion (in fact, a necessary and sufficient condition) for the local extensibility (modulo gauge) of u up to the maximal time T valid more generally for *complete* Riemannian manifolds without imposing any extra conditions on u; this closely parallels the analogous regularity results in the static case, which are stated in terms of suitable rescaled energies on balls (see [26, Proposition 2.4] and [25, Theorem 1]). Moreover, the proof rests on a fairly simple blow-up argument as well as suitable local monotonicity formulæ established by Ecker [12] for the harmonic map heat flow on Euclidean domains and the author [1, 3] more generally for both the harmonic map and Yang-Mills heat flows on complete Riemannian manifolds. The blow-up argument we use has been employed by White [32] in establishing a similar local regularity result for the mean curvature flow (see also [11]), which ultimately motivated our approach here.

Theorem A likewise gives rise to a characterisation of the singular set $\mathcal{S}$ of u at the maximal time, which is roughly the set of all points where the antecedent of the implication in Theorem A fails to hold, as well as an estimate on the Hausdorff measure of the singular set at the maximal time under the assumption of summability of $|\psi|^2$ on $\Omega \times [0, T[$, which is the content of the following theorem.

Theorem B. *Let ψ be as in Theorem A and suppose $|\psi|^2 \in L^1(\Omega \times [0, T[)$. Then the singular set $\mathcal{S}$ is closed, and for any $K \Subset \Omega$ and $\delta_0 \in \,]0, \operatorname{dist}(K, \partial\Omega)[$, the estimate*

$$\mathscr{H}^{n-2k}(\mathcal{S} \cap K) \leq \frac{5^{n-2k}\gamma_{n,k}}{\varepsilon} \limsup_{t \nearrow T} \int_{B_{\delta_0}(\mathcal{S} \cap K)} \frac{1}{2}|\psi|^2(\cdot, t)\mathrm{dvol}_g$$

holds, where ε is as in Theorem A, $\gamma_{n,k}$ is a constant depending only on n, k and the geometry of Ω and $\mathscr{H}^{n-2k}$ denotes the $(n-2k)$-dimensional Hausdorff measure on M.

Moreover, as another corollary of Theorem A, we obtain the following non-triviality result for smooth blow-ups at rapidly-forming singularities in the form of a lower bound on the heat ball energy of the blow-up.

Theorem C. *Let u and ψ be as in Theorem A and suppose u has a rapidly-forming singularity at (X, T) in the sense that*

$$\sup_{(x,t) \in Q_{R_0}(X,T)} (T - t)^k \frac{1}{2}|\psi|^2(x, t) \leq C_0$$

holds for some $C_0 > 0$ and $R_0 > 0$. Then u admits a sequence of rescalings converging locally uniformly in C^∞ (modulo gauge) to a smooth self-similar harmonic map or Yang-Mills heat flow u_∞ with corresponding differential or curvature ψ_∞ defined on $\mathbb{R}^n \times \,]-\infty, 0[$ satisfying the estimate

$$\frac{1}{R^{n-2k}} \iint_{E_R^{n-2k}(0,0)} \frac{1}{2}|\psi_\infty|^2 \cdot \left(\frac{n-2k}{-2t}\right) \mathrm{d}x\,\mathrm{d}t \geq \varepsilon$$

for all $R > 0$, where ε is as in Theorem A.

We note that an analogue of Theorem A has been established for the Ricci flow by Ni [22], though in that case its application to the study of singularities is far more subtle and has only been carried out in the case of rapidly-forming singularities (see [15]).

The structure of this paper is as follows. In §2 we describe the underlying geometric setup and introduce the harmonic map and Yang-Mills heat flows, as well as some of their important properties, stating well-known properties of them in the context of our considerations, as well as the local monotonicity formulæ of these flows on heat balls which shall play an important rôle in our considerations. In §3, we prove Theorem A and show how the ε-regularity condition yields a necessary and sufficient condition for the local smooth extensibility of these flows up to the maximal time. In §4 the local regularity theorem is used to provide a definition of the singular set at the maximal times of these flows, which we then analyse, leading to a proof of Theorem B. Finally, in §5, we turn our attention to rapidly-forming singularities of these flows and show that they admit smooth, nontrivial blow-ups, culminating in a proof of Theorem C. Throughout this paper, we have attempted to treat both the Yang-Mills and harmonic map heat flows simultaneously, abstracting away their common properties. The possibility of this approach suggests that there might be a more general principle at play.

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2. Setup

2.1. **Geometry.** Throughout this paper we will be dealing with a complete Riemannian manifold (M^n, g) of dimension $n > 2k > 0$ with k to be fixed shortly according to the heat flow under consideration. We shall adopt the notation of [23] for all Riemannian geometric quantities and operators. We shall write $\langle\cdot,\cdot\rangle$ and $|\cdot|$ for the inner product and norm associated with g respectively and shall append the metric as a subscript when ambiguity must be avoided. Furthermore, when considering tensor products (and exterior products) of bundles constructed from the tangent bundle TM (or the cotangent bundle T^*M and Riemannian vector bundles, we shall simply write $\langle\cdot,\cdot\rangle$ and $|\cdot|$ for the naturally induced inner product and norm. We shall write $(\cdot,\cdot)$ for the canonical fibrewise bilinear pairings of elements of $E \otimes \bigotimes T^*M$ with $\bigotimes TM$ as well as $E \otimes \Lambda T^*M$ with of ΛTM, where E is any vector bundle, and the fibrewise interior product of a vector field X with a section α of $E \otimes \Lambda T^*M$ shall be denoted by $X \llcorner \alpha$. The Euclidean metric on $\mathbb{R}^n$ and its subsets shall be denoted by g_δ.

In all of our considerations, Ω will denote a fixed, open bounded subset of M. Thus, we may find constants $i_0 > 0$, $\kappa_{-\infty}, \kappa_\infty > 0$ and $\{\Lambda_i : i \in \mathbb{N}\} \subset [0, \infty[$ such that $\mathrm{inj}_\Omega > i_0$ and the sectional curvature bounds

$$-\kappa_{-\infty} \leq \sec \leq \kappa_\infty \tag{2.1}$$

hold on Ω for all $i \in \mathbb{N}$. For simplicity, we will assume that $i_0 < \frac{\pi}{2\sqrt{\kappa_\infty}}$ if $\kappa_\infty > 0$. Note that for fixed $x \in \Omega$, we obtain the estimate

$$\Lambda_{-\infty}\mathfrak{r}(\cdot,t)^2 g^{\mathfrak{r}}(\cdot,x) \leq g - \nabla^2\left(\frac{1}{2}\mathfrak{r}(\cdot,x)^2\right) \leq \Lambda_\infty \mathfrak{r}(\cdot,x)^2 g^{\mathfrak{r}}(\cdot,x) \tag{2.2}$$

in $\Omega \cap B_{i_0}(x)$, where $\mathfrak{r}(\cdot,x)$ denotes the distance function measured from x, $g^{\mathfrak{r}} = g - \mathrm{d}\mathfrak{r}(\cdot,x) \otimes \mathrm{d}\mathfrak{r}(\cdot,x)$ and $\Lambda_{\pm\infty}$ are constants which may be determined explicitly in terms of $\kappa_\pm$ by means of a Hessian comparison theorem (cf. [23, Theorem 27, p.175]).

For $x \in M$ and $r > 0$, we denote the geodesic ball of radius r centred at x by $B_r(x) := \{\mathfrak{r}(\cdot,x) < r\}$ and for $t \in]0,T]$, we introduce the *parabolic cylinder* of radius r terminating at (x,t) as

$$Q_r(x,t) := B_r(x) \times \left]t - r^2, t\right[.$$

For later purposes, we introduce for each $\lambda > 0$ and $X \in M$ the (crudely) *rescaled metric* $g_\lambda^X : B_{\mathrm{inj}_X/\lambda}(0) \subset \mathbb{R}^n \to T^*\mathbb{R}^n \otimes T^*\mathbb{R}^n$ defined by

$$g_\lambda^X(y) = \sum_{i,j=1}^n g_{ij}(\lambda y)\ \mathrm{d}y^i \otimes \mathrm{d}y^j,$$

where $\{g_{ij}\}$ are the components of g in geodesic normal coördinates about X and $\{y^i\}$ denote Euclidean coördinates.

2.2. **Flows.** We shall now proceed to describe the flows we are interested in. In both cases, we have a Riemannian vector bundle $V \to E \to M$ with fibrewise inner product $\langle \cdot, \cdot \rangle$, a one-parameter family of connections ∇, and a one-parameter family of E-valued $(k-1)$-forms

$$\{u(\cdot, t) : M \to E \otimes \Lambda^{k-1} T^*M\}_{t \in [0,T[}$$

with a distinguished *fundamental k-form* $\{\psi(\cdot, t) : M \to E \otimes \Lambda^k T^*M\}_{t\in[0,T[}$ satisfying an equation of the form

$$\partial_t \psi - \Delta \psi = B, \tag{2.3}$$

where Δ denotes the connection Laplacian associated with the family of connections on E and the Levi-Cività connection on TM, and B is a suitable polynomial expression in ψ and $\nabla\psi$.

2.2.1. *Harmonic map heat flow.* The *harmonic map heat flow* is given by a one-parameter family of smooth maps $\{u(\cdot,t) : M \to N \subset \mathbb{R}^K\}_{t\in[0,T[}$ to a compact Riemannian submanifold N of $\mathbb{R}^K$ such that the equation

$$(\partial_t - \Delta_g)u = -\sum_{i,j} g^{ij}(b \circ u, \partial_i u \otimes \partial_j u) \tag{2.4}$$

holds, where b denotes the second fundamental form of N extended to an $(\mathbb{R}^K)^* \otimes (\mathbb{R}^K)^*$-valued mapping by setting it equal to zero when paired with vectors normal to N. For this flow, $k = 1$, $\psi = \mathrm{d}u$, $V = \mathbb{R}^K$, $E = \mathbb{R}^K$, the trivial vector bundle with standard fibre $\mathbb{R}^K$, and ∇ is the canonical trivial connection on E. Moreover, by differentiating (2.4) and carefully interchanging derivatives (or equivalently employing a Weizenböck-type argument [24, Theorem 4.22]), we see that $\mathrm{d}u$ satisfies an equation of the form (2.3) with

$$\begin{aligned} B = &-\sum_{i,j=1}^n g^{ij}(\nabla b \circ u, \partial_i u \otimes \partial_j u \otimes \partial_k u) \otimes \mathrm{d}x^k \\ &-\sum_{i,j=1}^n 2(b \circ u, (\nabla \mathrm{d}u, \partial_k \otimes \partial_i) \otimes \partial_j u) \otimes \mathrm{d}x^k \\ &-\sum_{i,j,k=1}^n R_{ij} g^{ik} \partial_k u \otimes \mathrm{d}x^j, \end{aligned} \tag{2.5}$$

where ∇b is the covariant differential of the second fundamental form of N extended as above to an $\bigotimes_{i=1}^3 (\mathbb{R}^K)^*$-valued mapping, its last slot corresponding to the direction of covariant differentiation, and R_{ij} are the components of the Ricci curvature tensor of g.

Given a harmonic map heat flow u, we may *rescale* it as follows: Let $\vartheta_x : B_{\mathrm{inj}_x}(0) \subset \mathbb{R}^n \to B_{\mathrm{inj}_x}(x) \subset M$ denote the geodesic normal coördinate parametrisation centred at $x \in M$. Defining

$$u_\lambda^{(x,t)}(y,s) := (\vartheta_x^* u)(\lambda y, t + \lambda^2 s) = u(\vartheta_x(\lambda y), t + \lambda^2 s),$$

we obtain a one-parameter family of maps $\{u_\lambda^{(x,t)}(\cdot,s) : B_{\mathrm{inj}_x/\lambda}(0) \to N\}_{s \in]-\frac{t}{\lambda^2}, \frac{T-t}{\lambda^2}[}$ solving the equation (2.4) *with respect to the metric* g_λ^x. In the case where (M,g) is Euclidean space, we say that u is *self similar* about (x,t) if $u(y,s) = u_\lambda^{(x,t)}(y,s)$ for all $\lambda > 0$ and $(y,s) \in \mathbb{R}^n \times \mathbb{R}$ for which both sides are defined.

2.2.2. *Yang-Mills heat flow.* Let $G \to P \to M$ be a principal bundle with compact semi-simple Lie group G as its structure group, and write $E_{\mathfrak{g}}$ for the vector bundle associated with P and the adjoint representation of G on its Lie algebra $\mathfrak{g}$. The (negative) Killing form on $\mathfrak{g}$ endows $E_{\mathfrak{g}}$ with the structure of a Riemannian vector bundle; moreover, any connection ω on P induces a unique covariant derivative operator ∇ on $E_{\mathfrak{g}}$ compatible with this Riemannian structure. The *Yang-Mills heat flow* is given by a one-parameter family of connections $\{\omega(\cdot,t) : P \to \mathfrak{g} \otimes T^*P\}_{t \in [0,T[}$ such that the equation

$$\underline{\partial_t \omega} = \operatorname{div} \underline{\Omega}^\omega \tag{2.6}$$

holds, where $\underline{\partial_t \omega}(\cdot,t)$ is the unique section of $E_{\mathfrak{g}} \otimes T^*M$ having $\partial_t \omega : P \times]0,T[\to \mathfrak{g} \otimes T^*P$ as its horizontal lift, $\underline{\Omega}^\omega : M \times [0,T[\to E_{\mathfrak{g}} \otimes \Lambda^2 T^*M$ is the curvature two-form of ω, and the divergence operator div is induced by the Levi-Cività connection on TM and $\omega(\cdot,t)$. For background material on this setup in accordance with our viewpoint, we refer the reader to [6, IV.2].

In order to study $\{\omega(\cdot,t)\}$ on M, fix a connection ω_0 on P. The difference $\omega(\cdot,t) - \omega_0$ is then a horizontal G-equivariant $\mathfrak{g}$-valued one-form on P and therefore corresponds to a $E_{\mathfrak{g}}$-valued one-form $u(\cdot,t) : M \to E_{\mathfrak{g}} \otimes T^*M$ for each t. The choice of ω_0 is *not* canonical. Thus, for this flow, $k = 2$, $V = \mathfrak{g}$, $E = E_{\mathfrak{g}}$, u is as above, $\psi = \underline{\Omega}^\omega$ and ∇ is the one-parameter family of connections on $E_{\mathfrak{g}}$ induced by ω. We note that ψ does *not* depend on the choice of ω_0. Analogously to the case of the harmonic map heat flow, using the fact that

$$\partial_t \underline{\Omega}^\omega = \mathrm{d}^\nabla \underline{\partial_t \omega},$$

d^∇ denoting the exterior covariant derivative associated with the connection on $E_{\mathfrak{g}}$ arising from ω and the Levi-Cività connection on TM (cf. [24, 2.75]), differentiating (2.6) and employing a Weitzenböck-type argument, we deduce that $\underline{\Omega}^\omega$ satisfies an equation of the form (2.3) with

$$\begin{aligned} B = &- \sum_{i,j,k,p=1}^n g^{jk} [(\underline{\Omega}^\omega, \partial_k \wedge \partial_i), (\underline{\Omega}^\omega, \partial_j \wedge \partial_p)] \otimes \mathrm{d}x^i \wedge \mathrm{d}x^p \\ &- \sum_{i,j,k,l=1}^n g^{il} R_{lj} (\underline{\Omega}^\omega, \partial_i \wedge \partial_k) \otimes \mathrm{d}x^j \wedge \mathrm{d}x^k \\ &+ \sum_{i,j,k,l,m=1}^n g^{jk} R^m_{\cdot\ lik} (\underline{\Omega}^\omega, \partial_m \wedge \partial_j) \otimes \mathrm{d}x^i \wedge \mathrm{d}x^l, \end{aligned} \tag{2.7}$$

where $[\cdot,\cdot]$ is the Lie bracket on $E_{\mathfrak{g}}$ naturally induced by that of $\mathfrak{g}$ and $R^{m}_{\cdot\, lik}$ are the components of the Riemann curvature tensor of g, where the sign convention is that of [29].

We now more explicitly describe $\omega(\cdot,t)$ locally. To this end, we fix a local section $\sigma : U \to P$ and let $\Psi_\sigma : U \times \mathfrak{g} \to E_{\mathfrak{g}}$ be the local bundle parametrisation induced by σ. Then $u(x,t) = \sum_{i=1}^n \Psi_\sigma(x, A_i(x,t) - (A_0)_i(x,t)) \otimes \mathrm{d}x^i$ for $(x,t) \in U \times [0,T[$, where

$$A(\cdot,t) := \sum_{i=1}^n A_i(\cdot,t) \otimes \mathrm{d}x^i := \sigma^*\omega(\cdot,t)$$

is the *local connection form* and

$$A_0 := \sum_{i=1}^n (A_0)_i(x) \otimes \mathrm{d}x^i := \sigma^*\omega_0$$

The curvature two-form is then locally given as $F(\cdot,t) = \sum_{i<j} F_{ij}(\cdot,t) \otimes \mathrm{d}x^i \wedge \mathrm{d}x^j : U \to \mathfrak{g} \otimes \Lambda^2 T^*M$ with

$$F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j].$$

We may therefore locally describe a Yang-Mills heat flow by means of a smooth one-parameter family of local $\mathfrak{g}$-valued one-forms $\{A(\cdot,t)\}_{t\in[0,T[}$ solving the system of equations

$$\partial_t A_i = \sum_{p,q} g^{pq} \left(\partial_p F_{qi} + [A_p, F_{qi}] - \sum_{r=1}^n \left(\Gamma^r_{pq} F_{ri} + \Gamma^r_{pi} F_{qr} \right) \right) \tag{2.8}$$

on $U \times]0,T[$ for each $i \in \{1,\ldots,n\}$.

Similarly to the harmonic map heat flow, we may *rescale* a Yang-Mills heat flow as follows: Fix $(x,t) \in M \times]0,T]$ and a local section $\sigma : B_{\mathrm{inj}_x}(x) \to P$. As before, we let $\vartheta_x : B_{\mathrm{inj}_x}(0) \to B_{\mathrm{inj}_x}(x)$ denote the geodesic normal coördinate parametrisation. Defining

$$A_\lambda^{(x,t)}(y,s) := (\vartheta_x^* A)(\lambda y, t+\lambda^2 s) = \sum_{i=1}^n \lambda A_i(\vartheta_x(\lambda y), t+\lambda^2 s)\mathrm{d}y^i,$$

we obtain a smooth one-parameter family $\{A_\lambda^{(x,t)}(\cdot,s) : B_{\mathrm{inj}_x/\lambda}(0) \to \mathfrak{g}\otimes T^*\mathbb{R}^n\}_{s\in]-\frac{t}{\lambda^2},\frac{T-t}{\lambda^2}[}$ solving the system of equations (2.8) *with respect to the metric* g_λ^x. This then gives rise to a family of connections $\{u_\lambda^{(x,t)}(\cdot,s)\}_{s\in]-\frac{t}{\lambda^2},\frac{T-t}{\lambda^2}[}$ on the trivial bundle $B_{\mathrm{inj}_x/\lambda}(0)\times G$ evolving by the Yang-Mills heat flow, where the metric tensor on the base manifold is given by g_λ^x. As with the harmonic map heat flow, if (M,g) is Euclidean, we call u *self similar* about (x,t) if $A(y,s) \equiv A_\lambda^{(x,t)}(y,s)$ for all $\lambda > 0$ and $(y,s) \in \mathbb{R}^n \times \mathbb{R}$ for which both sides are defined.

In contrast to the case of the harmonic map heat flow (with isometrically embedded target manifold), there is no preferred representation of a connection ω as an $E_{\mathfrak{g}}$-valued one-form; indeed, from a differential geometric point of view connections are usually considered equivalent if they are related by the action of the *gauge group*: Given a connection ω on P with local one-forms $\{A^\sigma = \sum_{i=1}^n A_i^\sigma \otimes \mathrm{d}x^i : \sigma \text{ a local section of } P\}$

and a section $g : U \to E_G$ of the bundle $G \to E_G \to M$ associated to P and the action of G on itself by conjugation, which we may write locally as

$$g(x) = \Phi_\sigma(x, g^\sigma(x))$$

with σ a local section of P and Φ_σ the local bundle parametrisation of E_G associated with σ, the collection of local one-forms

$$\{\widetilde{A}^\sigma = \sum_{i=1}^n \widetilde{A}_i^\sigma \otimes \mathrm{d}x^i\}$$

with $\widetilde{A}_i^\sigma := \mathrm{Ad}_{g^\sigma}(A_i^\sigma) - \partial_i g^\sigma \cdot (g^\sigma)^{-1}$ gives rise to a connection $g \cdot \omega$ on P, where Ad denotes the adjoint representation of G on $\mathfrak{g}$ and $\cdot$ the natural right action of G on TG. The connection corresponding to $g \cdot \omega$ satisfies $\underline{\Omega}^{g\cdot\omega} = \mathrm{Ad}_g \underline{\Omega}^\omega$, where Ad is the natural adjoint action of the gauge group on $E_\mathfrak{g}$. A computation readily shows that if ω is a Yang-Mills heat flow, then so is $g \cdot \omega$, albeit on a smaller subset of M if $U \neq M$, and the norm of the curvature form $\underline{\Omega}^\omega$ (as well as its covariant derivatives) is *invariant* under the action of the gauge group.

When speaking of the convergence of a sequence $\{\omega_i\}$ of connections (with corresponding $E_\mathfrak{g}$-valued one-forms $\{u_i\}$), we shall say that $\{u_i\}$ (or $\{\omega_i\}$) converges locally uniformly on $U \subset M$ *modulo gauge* if there exists a sequence $\{g_i : U \to E_G\}$ of elements of the gauge group for which the $E_\mathfrak{g}$-valued one-form $\{\widetilde{u}_i\}$ corresponding to $\{g_i \cdot \omega_i\}$ (and some fixed connection ω_0) converges locally uniformly on $U \subset M$.

When working locally, the action of the gauge group is essentially equivalent to choosing a different representative local one-form of the connection; indeed, if $\sigma : U \to P$ and $\sigma' : U' \to P$ are two local sections of P, then there exists a smooth mapping $g : U \cap U' \to G$ such that $\sigma' \cdot g = \sigma$, $\cdot$ being the right action of G on P, and the associated local connection forms A^σ and $A^{\sigma'}$ are related by

$$A_i^{\sigma'} = \mathrm{Ad}_g(A_i^\sigma) - \partial_i g \cdot g^{-1}.$$

We shall take this view point for the sake of concreteness.

2.2.3. *Common properties.* We recall some well-known facts common to the Yang-Mills and harmonic map heat flow; these results are essentially contained in [18] for the harmonic map heat flow and [31] for the Yang-Mills heat flow, albeit in a slightly different form. For the convenience of the reader, we sketch their proofs.

In both cases, local control on the fundamental form ψ ensures local control on the derivatives of u of all orders in the following sense:

Lemma 2.1 (Higher regularity). *If the estimate $r^{2k} \sup_{Q_r(X,T)} \frac{1}{2}|\psi|^2 \leq c_0$ holds for some $0 < r < \mathrm{inj}_X$ and $X \in M$, then for each $i \in \mathbb{N}$ there exists a constant $c_i > 0$ depending only on c_0, bounds on the covariant derivatives of the curvature tensor in $B_r(X)$ up to order i and the geometry of the target of the underlying flow such that*

$$r^{2(k+i)} \sup_{Q_{r/2}(X,T)} \frac{1}{2}|\nabla^i \psi|^2 \leq c_i. \tag{2.9}$$

A variant of this lemma was established in [18, Theorem 2.2] for the case of the harmonic map heat flow and [31, Theorem 2.2] for the case of the Yang-Mills heat flow. In the former case, the $\{c_i\}$ depend on bounds on the sectional curvature of N

and in the latter the structure constants of $\mathfrak{g}$. For completeness' sake, the proof of this version shall now be sketched.

We first recall the following localised weak maximum principle due to Ecker and Huisken [13]; its proof may be found in [11, Prop. 3.17] under the assumption that M evolves by mean curvature flow, though the proof carries over *mutatis mutandis* to our setting.

Theorem 2.2 (Localised maximum principle)**.** *Fix $t_1, t_2 > 0$ and let $\phi : M \times [t_1, t_2[\to [0, \infty[$ be a C^2 function such that $\phi(\cdot, t) \in C^2_0(M)$ for all $t \in [t_1, t_2[$ and*

$$\phi + |\nabla\phi| + |\partial_t\phi| + |\nabla^2\phi| + \frac{|\nabla\phi|^2}{\phi} \leq c_\phi \tag{2.10}$$

for some constant $c_\phi > 0$. Furthermore suppose that $f : M \times [t_1, t_2[\to [0, \infty[$ is a C^2 function such that

$$(\partial_t - \Delta)f \leq -a_0 f^2 - a_1 \cdot f + \langle X, \nabla f\rangle + a_2 \tag{2.11}$$

on $U := \{(x, t) : \phi(x, t) > 0\}$ with $a_0 > 0$, $a_2 \in \mathbb{R}$, $a_1 : U \to [0, \infty[$ and $X : U \to TM$ a time-dependent vector field such that $a_3 := \sup_U \frac{|X|}{\sqrt{1+a_1}} < \infty$. Then for all $t \in [t_1, t_2[$, we have that

$$\max_M (f \cdot \phi)(\cdot, t) \leq \max_M (f \cdot \phi)(\cdot, s) + c_1,$$

where $c_1 > 0$ depends on n and the constants a_0, a_2, a_3 and c_0.

Proof of Lemma 2.1. By rescaling the flow and crudely rescaling the metric, we may without loss of generality suppose that $r = 1$ and $(x, t) = (0, 0) \in \mathbb{R}^n$, since the scaling behaviour of g, ψ, the Riemann curvature tensor and its covariant derivatives would then establish the desired estimate. Set $f_k = \frac{1}{2}|\nabla^k\psi|^2$. By successively differentiating (2.3) in both cases, we see that whenever

$$\sup_Q f_i \leq c_i \tag{2.12}$$

for $i \in \{0, \ldots, k-1\}$, $\{c_i\}_{i=0}^{k-1} \subset \mathbb{R}$ and $Q \subset Q_1(0, 0)$ open, then

$$(\partial_t - \Delta)f_k \leq -2f_{k+1} + C_k(1 + f_k) \tag{2.13}$$

with C_k depending (polynomially) on $\{c_i\}_{i=0}^{k-1}$ and $\{\sup_Q |\nabla^i R|^2\}_{i=0}^{k}$, as well as the structure constants of $\mathfrak{g}$ in the case of the Yang-Mills heat flow, and bounds $\{\sup_Q |\nabla^i b|^2\}_{i=0}^{k+1}$ on the second fundamental form of N in the case of the harmonic map heat flow (cf. [18, Theorem 2.2] and [31, Theorem 2.2]). We shall employ a variant of the Shi trick (cf. [27, Lemma 4.2]) in order to apply Theorem 2.2. To this end, assuming the bounds (2.12), set $g_k := f_k \cdot (\alpha_k + f_{k-1})$, with $\alpha_k > 0$ to be determined. We compute, writing L for the heat operator $\partial_t - \Delta$, that

$$\begin{aligned}
\mathrm{L}g_k &= \mathrm{L}f_k \cdot (\alpha_k + f_{k-1}) + f_k \cdot \mathrm{L}f_{k-1} - 2\langle \nabla f_k, \nabla f_{k-1}\rangle \\
&\leq (4\varepsilon - 2)f_{k+1}(\alpha_k + f_{k-1}) + C_k(\alpha_k + f_{k-1}) + C_k g_k \\
&\qquad + C_{k-1}f_k(1 + f_{k-1}) + \left(-2 + \frac{2}{\varepsilon} \cdot \frac{f_{k-1}}{\alpha_k + f_{k-1}}\right) f_k^2,
\end{aligned}$$

for every $\varepsilon > 0$, where we have used the evolution inequality (2.13) as well as the inequality

$$\begin{aligned}|\langle \nabla f_k, \nabla f_{k-1}\rangle| &\leq |\nabla^k \psi|^2 \cdot |\nabla^{k+1}\psi| \cdot |\nabla^{k-1}\psi| \\ &\leq 2\varepsilon(\alpha_k + f_{k-1}) f_{k+1} + \frac{2}{\varepsilon(\alpha_k + f_{k-1})} f_k^2 f_{k-1},\end{aligned}$$

where the former inequality follows from Kato's inequality and the latter Young's inequality. Setting $\varepsilon = \frac{1}{2}$ eliminates the first term, and since $x \mapsto \frac{x}{\alpha_k + x}$ is monotone increasing, (2.12) together with another application of Young's inequality in the form

$$C_{k-1} f_k (1 + f_{k-1}) \leq \frac{C_{k-1}^2}{2}(1 + c_{k-1})^2 + \frac{1}{2} f_k^2$$

yields the inequality

$$\mathrm{L} g_k \leq \left(-\frac{3}{2} + 4\frac{c_{k-1}}{\alpha_k + c_{k-1}}\right) g_k^2 + C_k g_k + \widetilde{c}_k$$

with $\widetilde{c}_k = C_k(\alpha_k + c_{k-1}) + \frac{C_{k-1}^2}{2}(1 + c_{k-1})^2$. Choosing $\alpha_k = \frac{13}{3} c_{k-1}$ which then implies that $g_k \leq \frac{16}{3} c_{k-1} f_k$, we arrive at the evolution inequality

$$\begin{aligned}\mathrm{L} g_k &\leq -\frac{27}{1024 c_{k-1}^2} g_k^2 + C_k g_k + \widetilde{c}_k \\ &\leq -\frac{27}{2048 c_{k-1}^2} g_k^2 + \left(\widetilde{c}_k + \frac{512 c_{k-1}^2 C_k^2}{27}\right),\end{aligned} \tag{2.14}$$

which is of the form (2.11), where in the last line we have applied Young's inequality. Now, for each $k \in \mathbb{N} \cup \{0\}$, set $r_k = \frac{1}{2} + \left(\frac{1}{4}\right)^k$ and define $\varphi_k : \mathbb{R}^n \times \left[-r_{k-1}^2, 0\right[\to [0, \infty[$ such that

$$\varphi_k(x,t) = (t + r_{k-1}^2) \cdot \max\{0, (r_{k-1}^2 - |x|^2)^3\}.$$

Then $\varphi_k(\cdot, -r_{k-1}^2) \equiv 0$, for each $t \in \left]-r_{k-1}^2, 0\right[$, $\operatorname{supp} \varphi_k(\cdot, t) = \overline{B_{r_{k-1}}(0)}$, φ_k satisfies the inequality (2.10) with c_{φ_k} depending on k as well as the constants $\Lambda_{\pm\infty}$ arising in the geometry bounds (2.2), and we have the inequality

$$\varphi_k|_{Q_{r_k}(0,0)} \geq (r_{k-1}^2 - r_k^2)^4 =: \gamma_k \tag{2.15}$$

Now, suppose that $\sup_{Q_1(0,0)} f_0 \leq c_0$. Applying Theorem 2.2 to $f = g_1$ and $\phi = \varphi_1$, we immediately obtain a bound $\beta_1 > 0$ depending on n and the coëfficients appearing on the right-hand side of (2.14), viz. c_0, C_0 and C_1, the latter of which depend only on the underlying geometry, such that

$$\sup_{B_1(0)} (g_1 \cdot \varphi_1)(\cdot, t) \leq \beta_1$$

for each $t \in [-1, 0[$. Using (2.15), we obtain that $\sup_{Q_{r_1}(0,0)} g_1 \leq \frac{\beta_1}{\gamma_1}$. Proceeding by induction, we see that

$$\sup_{Q_{r_k}(0,0)} g_k \leq \frac{\beta_k}{\gamma_k}$$

for each $k \in \mathbb{N}$ with $\beta_k > 0$ depending only on n, c_0 and the underlying geometry. Since $g_k \geq \alpha_k f_k$, we may proceed again by induction to obtain estimates

$$\sup_{Q_{r_k}(0,0)} f_k \leq c_k$$

for each $k \in \mathbb{N}$, where c_k depends on n, c_0 and the underlying geometry as before. Since $\frac{1}{2} < r_k \leq 1$ for all k, this establishes the claim. $\square$

Since we are restricting our attention to the harmonic map heat flow with compact target manifold, we automatically have boundedness of u and, by Lemma 2.1, once we have a bound on $\mathrm{d}u$ in a parabolic cylinder, we obtain local bounds on $\nabla^{l-1}\mathrm{d}u$ for all $l \in \mathbb{N}$ in a slightly smaller parabolic cylinder; this fact enables us to conclude that u is smoothly extensible up to the end of the smaller parabolic cylinder and allows us to apply the Arzelà-Ascoli theorem in certain applications (e.g. in the proof of Lemma 3.2). The situation is more complicated in the case of the Yang-Mills heat flow, where a bound on the curvature two-form does not equate to a bound on the differential of u owing to gauge invariance. However, given a bound on the curvature two-form in a parabolic cylinder, we may by suitably choosing a local section (or equivalently by acting an element of the gauge group) deduce the boundedness of the local connection form and all of its derivatives in a slightly smaller parabolic cylinder, the key ingredient being Uhlenbeck's Coulomb gauge theorem [30].

Lemma 2.3 (Nice gauge lemma). *If* $r^4 \sup_{Q_r(X,T)} \frac{1}{2}|\underline{\Omega}^\omega|^2 \leq c_0$ *for some* $0 < r < i_0$ *and* $c_0 > 0$, *then there exist constants* $\theta \in \left]0, \frac{1}{4}\right]$ *and* $\{\alpha_i\}_{i=0}^\infty \subset \left]0, \infty\right[$ *depending only on* c_0, n, *the geometry of* $B_{i_0}(X)$ *and the structure constants of* $\mathfrak{g}$, *and a local section* $\sigma : B_{2\theta r}(X) \to P$ *such that for all* $i \in \mathbb{N} \cup \{0\}$,

$$r^{2(1+i)} \sup_{Q_{\theta r}(X,T)} \frac{1}{2}|\nabla^i \sigma^*\omega|^2 \leq \alpha_i. \tag{2.16}$$

Proof. Let $t_0 = T - \left(\frac{r}{2}\right)^2$ and write g_δ for the Euclidean metric on $B_r(X)$ defined in geodesic normal coördinates at X. Note that we have a bound of the form

$$|g - g_\delta|_{g_\delta}(\vartheta_X(x)) \leq \alpha(n) \cdot \sup_{B_{i_0}(X)} |R|_{g_\delta} \cdot |x|^2 \leq \beta|x|^2,$$

where $\vartheta_X : B_{\mathrm{inj}_X}(0) \subset \mathbb{R}^n \to B_{\mathrm{inj}_X}(X) \subset M$ is the canonical geodesic normal parametrisation of $B_{\mathrm{inj}_X}(X)$, $\alpha(n)$ is a constant depending only on n, R is the Riemann curvature tensor of g and β is any upper bound for the coëfficient of $|x|^2$ in the middle expression of this inequality. Since $\int_{B_{\lambda r}(X)} \frac{1}{2}|\underline{\Omega}^\omega|^{\frac{n}{2}}(\cdot, t_0) \xrightarrow{\lambda \searrow 0} 0$, it follows from Uhlenbeck's Coulomb gauge theorem that there exists a $\theta \in \left]0, \frac{1}{4}\right]$ depending only on c_0, n, β and the structure constants of $\mathfrak{g}$, and a local section $\sigma : B_{2\theta r}(X) \to P$ such that the *Coulomb gauge* condition $\sum_{i=1}^n \partial_i(\sigma^*\omega(\cdot, t_0), \partial_i) = 0$ holds as well as a scale-invariant bound on the $W^{1,p}$ norm of $\sigma^*\omega(\cdot, t_0)$ on $B_{2\theta r}(X)$ in terms of the L^p norm of $\underline{\Omega}^\omega$ for all $p \geq \frac{n}{2}$. Choosing θ smaller depending on β and n if necessary, the Coulomb gauge condition together with (2.8) implies that $\sigma^*\omega(\cdot, t_0)$ solves an elliptic system so that by standard techniques, scale-invariant estimates

$$r^{2(1+i)} \sup_{B_{\theta r}(X)} \frac{1}{2}|\nabla^i \sigma^*\omega(\cdot, t_0)|^2 \leq \beta_i \tag{2.17}$$

hold, where the $\{\beta_i\}_{i=0}^{\infty}$ depend only on c_0, the geometry of $B_r(X)$ and the structure constants of $\mathfrak{g}$. Now, it follows from the bound on $\underline{\Omega}^{\omega}$ and Lemma 2.1 that on $Q_{\frac{r}{2}}(X,T)$,

$$\partial_t \frac{1}{2}|\sigma^*\omega|^2 \leq \frac{2n\sqrt{c_1}}{r^3}\cdot\sqrt{\frac{1}{2}|\sigma^*\omega|^2}.$$

An integration and application of (2.17) then yield the inequality

$$r^2 \sup_{B_{\theta r}(X)} \frac{1}{2}|\sigma^*\omega(\cdot,t)|^2 \leq \alpha_0 := (\beta_0 + n\sqrt{c_1})^2$$

for all $t \in [t_0, T[$, which implies (2.16) in the case $i = 0$. The remaining estimates follow similarly by induction. $\square$

Remark 2.4. We may leverage Lemma 2.3 to obtain bounds on the covariant derivatives of the section u associated with the Yang-Mills heat flow as follows: Let $\varphi : M \to [0,\infty[$ be any smooth function such that $\varphi|_{B_{\theta r}(X)} \equiv 1$ and $\operatorname{supp}\varphi \Subset B_{2\theta r}(0)$. Setting $\omega_0 = \varphi\circ\pi\cdot\omega_\sigma + (1-\varphi\circ\pi)\omega_1$, where $\pi : P \to M$ is the projection map of the bundle, ω_σ is the trivial connection associated with the local section σ and ω_1 is any other smooth connection on P. Then we have the equality

$$u(x,t) = \Psi_\sigma(x, \sigma^*\omega(x,t))$$

for $(x,t) \in Q_{\theta r}(X,T)$, where $\Psi_\sigma : B_{2\theta r}(X)\times\mathfrak{g} \to E_{\mathfrak{g}}$ is the local bundle parametrisation of $E_{\mathfrak{g}}$ associated with σ acting on the former part of the tensor product $\mathfrak{g}\otimes T_x^*M$. This immediately implies that

$$r^{2(1+i)} \sup_{Q_{\theta r}(X,T)} \frac{1}{2}|\nabla^i u|^2 \leq \alpha_i$$

under the hypotheses of the lemma.

2.3. **Heat balls and local monotonicity.** We now turn our attention to the monotonicity properties of the harmonic map and Yang-Mills heat flows.

For fixed $(x,t) \in \Omega\times\,]0,T[$ and $k < \frac{n}{2}$, the *generalised Euclidean $(n-2k)$-heat ball* at (x,t) associated to (Ω, g) is defined by

$$\begin{aligned} E_r^{n-2k}(x,t) &= \left\{(y,s)\in\Omega\times\,]0,T] : \frac{1}{(4\pi(t-s))^{\frac{n-2k}{2}}}\exp\left(\frac{\operatorname{dist}_g(y,x)^2}{4(s-t)}\right) > \frac{1}{r^{n-2k}}\right\} \\ &= \bigcup_{s\in\left]t-\frac{r^2}{4\pi},t\right[\cap]0,T]} \left(B_{R_r^{n-2k}(s-t)}(x)\cap\Omega\right)\times\{s\}, \end{aligned}$$

where $R_r^{n-2k}(\sigma) = \sqrt{2(n-2k)\sigma\log\left(-\frac{4\pi\sigma}{r^2}\right)}$ for $\sigma \in \left]-\frac{r^2}{4\pi}, 0\right[$. It may be seen from the definition of $E_r^{n-2k}(x,t)$ that it is relatively compact in $M\times[0,T[$. Moreover, we have the inequality $R_r^{n-2k}(\sigma) \leq \sqrt{\frac{n-2k}{2\pi e}}r =: c_{n,k}r$ so that

$$E_r^{n-2k}(x,t) \subset Q_{c_{n,k}r}(x,t).$$

For technical reasons, we shall restrict our attention to small r so that $E_r^{n-2k}(x,t) \subset \Omega\times\,]0,T[$.

We now state the local monotonicity principle for these flows. For simplicity, we introduce the shorthand notation

$$[u,g]_{(x,t)} = \frac{1}{2}|\psi|^2 \cdot \left(\partial_t \phi_{(x,t)} + |\nabla \phi_{(x,t)}|^2\right) - \left\langle \nabla \phi_{(x,t)} \llcorner \psi, \partial_t u + \nabla \phi_{(x,t)} \llcorner \psi \right\rangle,$$

where $\phi_{(x,t)}(y,s) = \frac{\mathrm{dist}(x,y)^2}{4(s-t)} - \frac{n-2k}{2} \log\left(4\pi(t-s)\right)$. Note that the right-hand side implicitly depends on the choice of metric g. The following was established by Ecker [12] in the case of the harmonic map heat flow with M Euclidean and more generally for the Yang-Mills and harmonic map heat flows on complete Riemannian manifolds by the author [1, 3].

Theorem 2.5 (Local monotonicity). *Fix $x \in \Omega$ and suppose u is a harmonic map or Yang-Mills heat flow on $Q_R(x,t) \subset \Omega \times \mathbb{R}$. Then there exist $r_0 > 0$ and $\xi_k \in C^\infty(]0,\infty[)$ with $\lim_{s \searrow 0} \xi_k(s) = 0$ depending on the geometry of Ω such that the quantity*

$$r \mapsto \frac{1}{r^{n-2k}} \iint_{E_r^{n-2k}(x,t)} e^{\xi_k(t-s)} [u,g]_{(x,t)}(y,s) \,\mathrm{dvol}_g(y)\,\mathrm{d}s \tag{2.18}$$

is monotone nondecreasing for $r < r_0$ whenever the integrand is defined and summable over $E_{r_0}^{n-2k}(x,t)$. If $(M,g) = (\mathbb{R}^n, g_\delta)$, then $\xi_k \equiv 0$ and (2.18) is constant iff u is self-similar about (x,t) in $E_{r_0}^{n-2k}(x,t)$ (modulo gauge).

Remark 2.6. The geometric quantities r_0 and ξ_k are explicitly given by the following expressions:

$$r_0 = \frac{i_0}{2c_{n,k}};$$
$$\xi_k(\sigma) = \sigma \left[\left((n-1)\Lambda_\infty^+ - 2k\Lambda_{-\infty}^-\right) \log\left(\frac{e}{4\pi\sigma}\right)^{\frac{n-2k}{2}} \right],$$

where $\Lambda_\pm$ arise from the geometry bounds (2.2).

Remark 2.7 (Scale-invariance)**.** We may more explicitly write $[u,g]_{(x,t)}(y,s)$ in geodesic normal coördinates about (x,t) as

$$\frac{1}{2}|\psi|^2 \cdot \frac{n-2k}{2(t-s)} - \left\langle \frac{y}{2(s-t)} \llcorner \psi, \partial_t u + \frac{y}{2(s-t)} \llcorner \psi \right\rangle.$$

In particular, using this local expression, we have that

$$\begin{aligned}
&\frac{1}{r^{n-2k}} \iint_{E_r^{n-2k}(x,t)} e^{\xi_k(t-s)} [u,g]_{(x,t)}(y,s) \,\mathrm{dvol}_g(y)\,\mathrm{d}s \\
&= \frac{1}{(r/\lambda)^{n-2k}} \iint_{E_{r/\lambda}^{n-2k}(0,0)} e^{\xi_k(-\lambda^2 s)} [u_\lambda^{(x,t)}, g_\lambda^{(x,t)}]_{(0,0)}(y,s) \cdot \sqrt{g_\lambda^{(x,t)}}(y) \,\mathrm{d}y\,\mathrm{d}s
\end{aligned}$$

for any $\lambda > 0$. We shall use this (approximate) *scale-invariance* property in the sequel.

Remark 2.8 (Nonnegativity)**.** By monotonicity and scale invariance, if u is smooth on $\overline{Q_R(x,t)}$, then for all $r < r_0$,

$$\frac{1}{r^{n-2k}} \iint_{E_r^{n-2k}(x,t)} e^{\xi_k(t-s)} [u,g]_{(x,t)}(y,s) \,\mathrm{dvol}_g(y)\,\mathrm{d}s$$

$$\geq \lim_{\lambda \searrow 0} \frac{1}{\lambda^{n-2k}} \iint_{E_\lambda^{n-2k}(x,t)} e^{\xi_k(t-s)}[u,g]_{(x,t)}(y,s)\ \mathrm{dvol}_g(y)\,\mathrm{d}s$$

$$= \lim_{\lambda \searrow 0} \iint_{E_1^{n-2k}(0,0)} e^{\xi_k(-\lambda^2 s)}[u_\lambda^{(x,t)}, g_\lambda^{(x,t)}]_{(0,0)}(y,s) \cdot \sqrt{g_\lambda^{(x,t)}}(y)\ \mathrm{d}y\mathrm{d}s = 0,$$

where the last equality follows from the dominated convergence theorem. Therefore, the quantity (2.18) is nonnegative in the case where u is smooth in a neighbourhood of x up to and including time t.

It was shown in [3] that the quantity (2.18) is finite for $r = r_0$ whenever $\frac{1}{2}|\psi|^2 \in L^1(Q_{2c_{n,k}r_0}(x,t))$. In fact, we have the following estimate.

Lemma 2.9 (L^2 estimate)**.** *Suppose $\{u(\cdot,t)\}_{t\in[0,T[}$ is a harmonic map or Yang-Mills heat flow on Ω such that $|\psi|^2 \in L^1(Q_{2c_{n,k}r_0}(x,t))$. Then the estimate*

$$\frac{1}{r^{n-2k}} \iint_{E_r^{n-2k}(x,t)} |[u,g]_{(x,t)}|(y,s)e^{\xi_k(t-s)}\mathrm{dvol}_g(y)\mathrm{d}s \leq \frac{\gamma_{n,k}}{r^{n-2k+2}} \iint_{Q_{4c_{n,k}r}(x,t)} \frac{1}{2}|\psi|^2 \tag{2.19}$$

for $r < \frac{r_0}{2}$, where $\gamma_{n,k}$ depends on n,k and the geometry of Ω.

Proof. In [3, Remark 5.8], it was shown that

$$\begin{aligned} &\frac{1}{r^{n-2k}} \iint_{E_r^{n-2k}(x,t)} |[u,g]_{(x,t)}|(y,s)\mathrm{dvol}_g(y)\mathrm{d}s \\ &\leq c_1 \left(\frac{1}{r^{n-2k+2}} \iint_{Q_{2c_{n,k}r}(x,t)} \frac{1}{2}|\psi|^2 + \frac{1}{r^{n-2k}} \int_{B_{2c_{n,k}r_0}(x)} \frac{1}{2}|\psi|^2(\cdot, t - \frac{r^2}{4\pi}) \right), \end{aligned} \tag{2.20}$$

where c_1 depends on n, k and the geometry of Ω. Now, it may be shown using the methods of [2] that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \int_M \frac{1}{2}|\psi|^2\varphi(\cdot,t) \\ &= -\int_M |\partial_t u|^2\varphi + \int_M \frac{1}{2}|\psi|^2(\partial_t - \Delta)\varphi + \left\langle \nabla^2\varphi, \sum_{i,j} \langle \partial_i \llcorner \psi, \partial_j \llcorner \psi \rangle\ \mathrm{d}x^i \otimes \mathrm{d}x^j \right\rangle \end{aligned} \tag{2.21}$$

for a smooth function $\varphi : M \times [0,T[\to [0,\infty[$ such that $\varphi(\cdot,t)$ is compactly supported in M for each t. We fix a smooth function $\chi : \mathbb{R} \times \mathbb{R} \to [0,\infty[$ such that $\chi(t) = 1$ for $t < \frac{1}{2}$ and $\chi(t) = 0$ for $t > 1$. Setting $\varphi(y,s) = \chi\left(\left(\frac{\mathfrak{r}(y,x)}{4c_{n,k}r}\right)^2 + \frac{t-s}{(4c_{n,k}r)^2}\right)$, it follows that supp $\varphi(\cdot,s) \subset B_{4c_{n,k}r}(x)$, $\varphi(\cdot,s) = 0$ for $s \leq t - (4c_{n,k}r)^2$, and $\varphi \equiv 1$ on $Q_{2c_{n,k}r}(x,t)$. Moreover, by the Hessian estimate (2.2), (2.21) implies that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_M \frac{1}{2}|\psi|^2\varphi(\cdot,t) \leq \frac{c_2}{r^2} \int_{B_{4c_{n,k}r}(x)} \frac{1}{2}|\psi|^2(\cdot,t),$$

where c_2 depends on n, k and the geometry of Ω. Integrating from $t - (4c_{n,k}r)^2$ to $t - \frac{r^2}{4\pi}$, we obtain

$$\int_{B_{2c_{n,k}r}(x)} \frac{1}{2}|\psi|^2(\cdot, t - \frac{r^2}{4\pi}) \leq \int_M \frac{1}{2}|\psi|^2\varphi(\cdot, t - (2c_{n,k}r)^2) \leq \frac{c_2}{r^2} \iint_{Q_{4c_{n,k}r}(x,t)} \frac{1}{2}|\psi|^2,$$

where we have used the fact that $t-\frac{r^2}{4\pi} > t-(2c_{n,k}r)^2$. Substituting this into (2.20) and using the boundedness of ξ_k on bounded subsets of $]0,\infty[$ then yields (2.19). $\square$

Remark 2.10 (Local energy estimate)**.** Using (2.21) with

$$\varphi(y,s) = \chi\left(\left(\frac{\mathfrak{r}(y,x)}{2r}\right)^2 + \frac{t-s}{(2r)^2}\right)$$

and $x \in \Omega$, we may similarly establish the local energy bound

$$\int_{B_r(x)} \frac{1}{2}|\psi|^2(\cdot,s) \le \frac{c}{r^2}\int_{Q_{2r}(x,t)} \frac{1}{2}|\psi|^2$$

for $r < \frac{i_0}{2}$ with $Q_r(x,t) \subset \Omega \times]0,T[$, $s \in \left]t-r^2,t\right[$ and $c>0$ depending only on the geometry of Ω, which leads to the bound

$$\int_K \frac{1}{2}|\psi|^2(\cdot,t) \le \widetilde{c}\cdot \iint_{\Omega\times]0,T[} \frac{1}{2}|\psi|^2$$

for $t \in \left]T-\min\{\frac{i_0}{4},\frac{1}{2}\mathrm{dist}(K,\Omega)\}^2,T\right[$ and $K \Subset \Omega$, where $\widetilde{c}$ depends on $\mathrm{dist}(K,\Omega)$ and the geometry of Ω.

3. The local regularity theorem

We now turn our attention to the promised local regularity theorem and shall henceforth write

$$I_k(u,g;x,t,r) := \frac{1}{r^{n-2k}}\iint_{E_r^{n-2k}(x,t)} e^{\xi_k(t-s)}[u,g]_{(x,t)}(y,s)\,\mathrm{dvol}_g(y)\,\mathrm{d}s$$

for the *heat ball energy* of Theorem 2.5.

Theorem 3.1 (Local regularity)**.** *There exist geometric constants $\varepsilon, C>0$ such that for any $R \le i_0$ and Yang-Mills or harmonic map heat flow u on $Q_R(X,T) \subset \Omega\times\mathbb{R}$, the implication*

$$\sup_{(x,t)\in Q_{R/2}(X,T)} I_k(u,g;x,t,\frac{R}{2c_{n,k}}) < \varepsilon \Rightarrow \left(\frac{7}{32}R\right)^{2k}\sup_{Q_{R/4}(X,T)}\frac{1}{2}|\psi|^2 \le C$$

holds.

We first establish the following lemma.

Lemma 3.2. *There exist constants $\varepsilon, C>0$ such that for any $R \le i_0$ and Yang-Mills or harmonic map heat flow u on $\overline{Q_R(X,T)} \subset \Omega\times\mathbb{R}$, the implication*

$$\sup_{(x,t)\in Q_{R/2}(X,T)} I_k(u,g;x,t,\frac{R}{2c_{n,k}}) < \varepsilon \Rightarrow \sup_{\alpha\in]0,1[}\left(\frac{\alpha R}{2}\right)^{2k}\sup_{Q_{(1-\alpha)R/2}(X,T)}\frac{1}{2}|\psi|^2 \le C$$

holds.

Proof. We proceed by contradiction. Suppose this result is false. Then there exists a sequence $\{R_j\}_{j\in\mathbb{N}} \subset]0,i_0]$ and a sequence of Yang-Mills or harmonic map heat flow pairs $\{(u_j,\psi_j)\}_{j\in\mathbb{N}}$ defined on $\overline{Q_{R_j}(X_j,T_j)}\subset\Omega\times\mathbb{R}$ for each $j\in\mathbb{N}$ such that the ε-regularity condition holds with $\varepsilon=\frac{1}{j}$, $R=R_j$, $u=u_j$ and $\psi=\psi_j$, but the conclusion is false, i.e.

$$(\beta_j)^{2k} := \sup_{\alpha\in]0,1[}\left(\frac{\alpha R_j}{2}\right)^{2k}\sup_{Q_{(1-\alpha)\frac{R_j}{2}}(X,T)}\frac{1}{2}|\psi_j|^2 \xrightarrow{j\to\infty}\infty.$$

By smoothness, there exist $\alpha_j\in]0,1]$ and $(x_j,t_j)\in\overline{Q_{(1-\alpha_j)\frac{R_j}{2}}(X_j,T_j)}$ such that

$$(\beta_j)^{2k}=\left(\frac{\alpha_j R_j}{2}\right)^{2k}\cdot\frac{1}{2}|\psi_j|^2(x_j,t_j). \tag{3.1}$$

By passing to a subsequence if necessary, we may assume that $\{x_j\}_{j\in\mathbb{N}}\subset\overline{\Omega}$ is a convergent sequence. Now, note that:

(1) There holds

$$\sup_{Q_{(1-\frac{\alpha_j}{2})\frac{R_j}{2}}(X,T)}\frac{1}{2}|\psi|^2=\left(\frac{4}{\alpha_j R_j}\right)^{2k}\cdot\left(\left(\frac{\alpha_j R_j}{4}\right)^{2k}\sup_{Q_{(1-\frac{\alpha_j}{2})\frac{R_j}{2}}(X,T)}\frac{1}{2}|\psi|^2\right)\le\left(\frac{4\beta_j}{\alpha_j R_j}\right)^{2k}$$

(2) $Q_{\frac{\alpha_j}{2}\cdot\frac{R_j}{2}}(x_j,t_j)\subset Q_{\left(1-\frac{\alpha_j}{2}\right)\frac{R_j}{2}}(X,T)$.

Altogether, we have

$$\sup_{Q_{\frac{\alpha_j R_j}{4}}(x_j,t_j)}\frac{1}{2}|\psi|^2\le\left(\frac{4\beta_j}{\alpha_j R_j}\right)^{2k}=:\lambda_j^{-2k}. \tag{3.2}$$

We now rescale u_j appropriately about (x_j,t_j). Set

$$u_j'(x,t)=(u_j)_{\lambda_j}^{(x_j,t_j)}.$$

This defines a smooth harmonic map (resp. Yang-Mills) heat flow

$$u_j':\overline{Q_{\beta_j}(0,0)}\to V\otimes\Lambda^{k-1}T^*\mathbb{R}^n.$$

Moreover, (3.1) implies that

$$\frac{1}{2}|\psi_j'|^2(0,0)=\lambda_j^{2k}\cdot\frac{1}{2}|\psi_j'|^2(x_j,t_j)=\left(\frac{1}{2}\right)^{2k} \tag{3.3}$$

and, in light of (3.2), we have $\sup_{\overline{Q_{\beta_j}(0,0)}}\frac{1}{2}|\psi_j'|^2\le 1$. Since $\beta_j\ge 1$ for sufficiently large j, we also have that $\sup_{\overline{Q_1(0,0)}}\frac{1}{2}|\psi_j'|^2\le 1$ for sufficiently large j and therefore, by Lemma 2.1, we have that

$$\sup_{\overline{Q_{\frac{1}{2}}(0,0)}}\frac{1}{2}|\nabla^m\psi_j'|^2\le c_m$$

for all $m\in\mathbb{N}$, where c_m depends only on n, bounds on the covariant derivatives of the Riemann curvature tensor of g up to order m in Ω, the structure constants of $\mathfrak{g}$ in the case of the Yang-Mills heat flow and bounds on the covariant derivatives of the second fundamental form of the target manifold up to order $m+1$ in the case of the harmonic map heat flow. Using Lemma 2.3 in the case of the Yang-Mills heat flow and the compactness of the target manifold in the case of the harmonic map heat flow, we furthermore obtain bounds of the form

$$\sup_{Q_\theta(0,0)} \frac{1}{2}|\nabla^m u_j'|^2 \leq \widetilde{c}_m$$

for all $m \in \mathbb{N} \cup \{0\}$, the $\{\widetilde{c}_m\}_{m=0}^\infty$ having the same dependence as the $\{c_m\}_{m=0}^\infty$. Thus, by the Arzelà-Ascoli theorem, there exists a subsequence of $\{u_j'\}$, which we again denote by $\{u_j'\}$, and a smooth Yang-Mills or harmonic map heat flow $u_\infty : \overline{Q_\theta(0,0)} \to V \otimes \Lambda^{k-1}T^*\mathbb{R}^n$ such that $u_j' \xrightarrow{j\to\infty} u_\infty$ locally uniformly in C^∞. From (3.3), we see that we must also have

$$\frac{1}{2}|\psi_\infty|^2(0,0) = \left(\frac{1}{2}\right)^{2k}. \tag{3.4}$$

Now, for $(x,t) = (x_j, t_j)$, the ε-regularity condition reads

$$\frac{1}{\left(\frac{R_j}{2c_{n,k}}\right)^{n-2k}} \iint_{E^{n-2k}_{\frac{R_j}{2c_{n,k}}}((x_j,t_j))} [u_j, g]_{(x_j,t_j)}(x,t) \cdot e^{\xi(t-t_j)} \mathrm{dvol}_g(x)\, \mathrm{d}t < \frac{1}{j}.$$

For fixed $\overline{R} > 0$ and sufficiently large j, we have that $\frac{\overline{R}\alpha_j c_{n,k}}{2\beta_j} < 1 \Leftrightarrow \lambda_j \overline{R} < \frac{R_j}{2c_{n,k}}$ so that by monotonicity, we also have

$$\frac{1}{(\lambda_j\overline{R})^{n-2k}} \iint_{E^{n-2k}_{\lambda_j\overline{R}}((x_j,t_j))} [u_j, g]_{(x_j,t_j)}(x,t) \cdot e^{\xi(t-t_j)} \mathrm{dvol}_g(x)\, \mathrm{d}t < \frac{1}{j}.$$

Using scale invariance and nonnegativity (Remarks 2.7 and 2.8), we therefore have

$$0 \leq \frac{1}{\overline{R}^{n-2k}} \iint_{E^{n-2k}_{\overline{R}}((0,0))} [u_j', g^{x_j}_{\lambda_j}]_{(0,0)}(x,t) \cdot e^{\xi(-\lambda_j^2 t)} \sqrt{g^{x_j}_{\lambda_j}(x)}\, \mathrm{d}x\, \mathrm{d}t < \frac{1}{j} \tag{3.5}$$

in geodesic normal coördinates about x_j. Choosing $\overline{R}$ so that $E_{\overline{R}}(0,0) \subset Q_\theta(0,0)$ and taking the limit $j \to \infty$ in (3.5), we obtain that

$$\frac{1}{\overline{R}^{n-2k}} \iint_{E_{\overline{R}}(0,0)} [u_\infty, \delta]_{(0,0)} \equiv 0,$$

i.e. u_∞ is a *smooth* self-similar flow on $E_{\overline{R}}(0,0)$ by Theorem 2.5, but then self-similarity implies that

$$\frac{1}{2}|\psi_\infty|^2(x,t) = r^{2k} \cdot \frac{1}{2}|\psi_\infty|^2(rx, r^2 t)$$

for all $(x,t) \in E_{\overline{R}}(0,0)$ and $r \in \,]0,1[$. Since u_∞ is smooth on $\overline{E_{\overline{R}}(0,0)} \subset \overline{Q_\theta(0,0)}$, taking the limit $r \searrow 0$ yields $\frac{1}{2}|\psi_\infty|^2(x,t) = 0$ for $(x,t) \in E_{\overline{R}}(0,0)$. Taking the limit $(x,t) \to (0,0)$ then yields the equality $\frac{1}{2}|\psi_\infty|^2(0,0) = 0$, which contradicts (3.4). □

Proof of Theorem 3.1. Let $u : Q_R(X,T) \to E \otimes \Lambda^{k-1}T^*M$ be a harmonic map or Yang-Mills heat flow and consider $u_\delta : \overline{Q_{\frac{15}{16}R}(X,T)} \to E \otimes \Lambda^{k-1}T^*M$ defined by $u_\delta(y,s) = u(y, s-\delta)$ for $\delta \in \left]0, (1-(\frac{15}{16})^2)\frac{R^2}{4}\right[$. u_δ solves the same flow equation as u so that Lemma 3.2 applies. Choosing $\alpha = \frac{7}{15}$ then yields the desired estimate. □

Remark 3.3 (Extensibility up to $t = T$)**.** The significance of the ε-regularity condition

$$\sup_{(x,t)\in Q_{R/2}(X,T)} I_k(u,g;x,t,\frac{R}{2c_{n,k}}) < \varepsilon$$

is the following: Theorem 3.1 implies a scale-invariant bound on $|\psi|^2$ in $Q_{\frac{R}{4}}(X,T)$ so that after applying Lemma 2.1, as well as Lemma 2.3 and a suitable gauge transformation in the case of the Yang-Mills heat flow, we obtain bounds on $\nabla^m u$ for all $m \in \mathbb{N} \cup \{0\}$ on $Q_{\frac{\theta R}{2}}(X,T)$ for some $\theta \in \left]0, \frac{1}{4}\right]$ so that $\lim_{t\nearrow T} u(\cdot,t)$ exists uniformly in C^∞ on $\overline{B_{\frac{\theta R}{2}}(X)}$ (modulo gauge), i.e. u is smoothly extensible to all of $\overline{Q_{\frac{\theta R}{2}(X,T)}}$. Conversely, if u is smooth on $\overline{Q_{\frac{5R}{2}}(X,T)}$ for some $R > 0$, then by Lemma 2.9 and Remark 2.8, we have for all $\lambda \le R$ that

$$\begin{aligned}
0 &\le \sup_{(x,t)\in Q_{\lambda/2}(X,T)} I_k(u,g;x,t,\frac{\lambda}{2c_{n,k}}) \\
&\le \frac{\gamma_{n,k}\cdot(2c_{n,k})^{n-2k}}{\lambda^{n-2k}} \iint_{Q_{\frac{5\lambda}{2}}(X,T)} \frac{1}{2}|\psi|^2 \\
&= \gamma_{n,k}\cdot(2c_{n,k})^{n-2k} \iint_{Q_{\frac{5}{2}}(0,0)} \frac{1}{2}|\psi_\lambda^{(X,T)}|^2\cdot\sqrt{g_\lambda^X} \xrightarrow{\lambda\searrow 0} 0,
\end{aligned}$$

where in the last line we have used the fact that

$$|\psi_\lambda^{(X,T)}|^2(y,s) = \lambda^{2k}|\psi|^2(\vartheta_X(\lambda y), T+\lambda^2 s) \xrightarrow{\lambda\searrow 0} 0,$$

ϑ_X being the geodesic normal parametrisation of $B_{\mathrm{inj}_X}(X)$, as well as the dominated convergence theorem. Therefore, there exists a $\lambda_0 \le i_0$ for which the ε-regularity condition

$$\sup_{(x,t)\in Q_{\lambda_0/2}(X,T)} I_k(u,g;x,t,\frac{\lambda_0}{2c_{n,k}}) < \varepsilon$$

holds. Theorem 3.1 therefore gives us a necessary and sufficient condition for the local extensibility of $\{u(\cdot,t)\}$ up to time $t = T$ (modulo gauge).

4. The singular set

We shall now make use of the local regularity theorem to draw conclusions about the singular set at the maximal time of a smooth Yang-Mills or harmonic map heat flow. For $(X,T) \in \Omega\times\mathbb{R}$, set

$$R_0(X,T) = \min\{i_0, \mathrm{dist}(X,\partial\Omega), \frac{\sqrt{T}}{2}\}$$

and let u be a harmonic map or Yang-Mills heat flow on $\Omega\times[0,T[$. First of all, it follows from Theorem 3.1 and Remark 3.3 that if a singularity forms at $(X,T)\in\Omega\times\mathbb{R}$, then $X \in \mathcal{S}$, where

$$\mathcal{S} = \left\{ y\in\Omega : \forall R < R_0(y,T)\ \exists (x,t)\in Q_{\frac{R}{2}}(y,T) \text{ s.t. } I_k(u,g;x,t,\frac{R}{2c_{n,k}}) \ge \varepsilon \right\}.$$

The following lemma tells us that we can actually measure $\mathcal{S}$ and gives us another necessary condition for singularity formation.

Lemma 4.1. *$\mathcal{S}$ is closed in Ω and we have the inclusion*

$$\mathcal{S} \subset \{y \in \Omega : \ \forall R < R_0(y) \ \frac{1}{R^{n-2k+2}} \iint_{Q_R(y,T)} \frac{1}{2}|\psi|^2 \geq \frac{\varepsilon}{\gamma_{n,k}}\} \tag{4.1}$$

with $\gamma_{n,k}$ as in Lemma 2.9.

Proof. If $X \in \Omega \backslash \mathcal{S}$, then there exists an $R < R_0(X)$ such that for all $(x,t) \in Q_{\frac{R}{2}}(X,T)$

$$\frac{1}{(R/(2c_{n,k}))^{n-2k}} \iint_{E^{n-2k}_{R/(2c_{n,k})}(x,t)} e^{\xi_k(t-s)}[u,g]_{(x,t)}(y,s)\mathrm{dvol}_g(y)\,\mathrm{d}s < \varepsilon.$$

If $\widetilde{X} \in B_{\widetilde{R}/2}(X)$ with $\widetilde{R} = \frac{R}{2}$, then $Q_{\widetilde{R}/2}(\widetilde{X},T) \subset Q_{R/2}(X,T)$ so that by monotonicity,

$$\frac{1}{\left(\widetilde{R}/(2c_{n,k})\right)^{n-2k}} \iint_{E^{n-2k}_{\widetilde{R}/(2c_{n,k})}(x,t)} e^{\xi_k(t-s)}[u,g]_{(x,t)}(y,s)\mathrm{dvol}_g(y)\,\mathrm{d}s < \varepsilon$$

for all $(x,t) \in Q_{\widetilde{R}/2}(\widetilde{X},T)$. Since $\widetilde{R} < R_0(\widetilde{X})$, we therefore have that $B_{\widetilde{R}/2}(X) \subset \Omega \backslash \mathcal{S}$, i.e. $\mathcal{S}$ is closed in Ω.

As for the inclusion (4.1), suppose $y \in \mathcal{S}$ and fix a sequence $R_i \searrow 0$. Then for sufficiently large i, there is a sequence $(x_i,t_i) \in Q_{\frac{R_i}{4}}(X,T)$ such that

$$I_k(u,g;x,t,\frac{R_i}{4c_{n,k}}) \geq \varepsilon.$$

For fixed $R < R_0(y,T)$ and sufficiently large i local monotonicity and the L^2-estimate (Lemma 2.9) give us

$$\frac{\gamma_{n,k}}{R^{n-2k+2}} \iint_{Q_{R/2}(x_i,t_i)} \frac{1}{2}|\psi|^2 \geq I_k(u,g;x,t,\frac{R}{8c_{n,k}}) \geq \varepsilon.$$

For sufficiently large i, $Q_{R/2}(x_i,t_i) \subset Q_R(X,T)$, which yields the desired inclusion. □

Knowing that $\mathcal{S}$ is measurable, we shall now estimate its $(n-2k)$-dimensional Hausdorff measure (denoted $\mathscr{H}^{n-2k}$) under the assumption of summability of $|\psi|^2$.

Corollary 4.2 (Singular set estimate)**.** *Suppose $|\psi|^2 \in L^1(\Omega \times [0,T[)$. Then for any $K \Subset \Omega$ and $\delta_0 \in]0, \mathrm{dist}(K,\partial\Omega)[$, the estimate*

$$\mathscr{H}^{n-2k}(\mathcal{S} \cap K) \leq \frac{5^{n-2k}\gamma_{n,k}}{\varepsilon} \limsup_{t \nearrow T} \int_{B_{\delta_0}(\mathcal{S}\cap K)} \frac{1}{2}|\psi|^2(\cdot,t)\mathrm{dvol}_g \tag{4.2}$$

holds, where ε is as in Theorem 3.1 and $\gamma_{n,k}$ as in Lemma 2.9.

Proof. Fix $\delta_1 > 0$ and $0 < \delta \ll \min\{\delta_1,\delta_0\}$, and cover $\mathcal{S} \cap K$ with the family of balls $\{B_\delta(x)\}_{x \in \mathcal{S}\cap K}$. By compactness and the Vitali covering lemma, we may pass to a finite pairwise disjoint subfamily $\{B_\delta(x_i)\}_{i \in I_0}$ such that $\mathcal{S} \cap K \subset \bigcup_{i \in I_0} B_{5\delta}(x_i)$ so that Lemma 4.1 implies that

$$\begin{aligned}
\sum_{i \in I_0} (5\delta)^{n-2k} &\leq \frac{5^{n-2k}\gamma_{n,k}}{\varepsilon} \sum_{i \in I_0} \delta^{-2} \int_{T-\delta^2}^{T} \int_{B_\delta(x_i)} \frac{1}{2}|\psi|^2 \\
&\leq \frac{5^{n-2k}\gamma_{n,k}}{\varepsilon} \sup_{t \in]T-\delta_1^2,T[} \int_{B_{\delta_0}(\mathcal{S}\cap K)} \frac{1}{2}|\psi|^2(\cdot,t).
\end{aligned}$$

Passing to the limit $\delta \searrow 0$ then $\delta_1 \searrow 0$ yields the desired estimate. □

Remark 4.3. Note that the right-hand side is finite by virtue of Remark 2.10. Moreover, in contrast to the case of the corresponding statement on the singular set of stationary harmonic and energy-minimising maps (cf. [26, Corollary 2.7] and [5, §VII]), it is not possible in general to take a further limit $\delta_0 \searrow 0$ in (4.2). However, if we assume that ψ satisfies the energy continuity hypothesis

$$\lim_{t\nearrow T}\int_M \frac{1}{2}|\psi|^2(\cdot,t)\cdot\varphi = \int_M f\cdot\varphi$$

for some $f \in L^1_{\mathrm{loc}}(M)$ and all $\varphi \in C_0^\infty(M)$, then we may choose for each small $\delta_0 > 0$ a test function φ with $B_{\delta_0}(\mathcal{S}\cap K) \subset \operatorname{supp}\varphi \subset B_{2\delta_0}(\mathcal{S}\cap K)$ so that

$$0 \le \limsup_{t\nearrow T}\int_{B_{\delta_0}(\mathcal{S}\cap K)} \frac{1}{2}|\psi|^2(\cdot,t) \le \int_M \frac{1}{2}|\psi|^2(\cdot,t)\cdot\varphi \le \int_{B_{2\delta_0}(\mathcal{S}\cap K)} f,$$

which then implies after taking the limit $\delta_0 \searrow 0$, by virtue of the fact that $\mathcal{S}\cap K$ is of Lebesgue measure zero, that $\mathscr{H}^{n-2k}(\mathcal{S}\cap K) = 0$ for any $K \Subset \Omega$. This should be compared with the case of the mean curvature flow, where a similar continuity hypothesis is necessary to deduce a measure-zero singular set (see [11, Ch. 5]).

5. Rapidly-forming singularities

We now turn our attention to rapidly-forming (or type-I) singularities of the harmonic map and Yang-Mills heat flows. The flow u is said to have a *rapidly-forming singularity* at (X,T) if its fundamental form ψ satisfies the scale-invariant estimate

$$\sup_{(x,t)\in Q_R(X,T)} (T-t)^k \frac{1}{2}|\psi|^2(x,t) \le C_0 \tag{5.1}$$

for some $C_0 > 0$ and $R > 0$. We first show that rapidly-forming singularities admit smooth blow-ups. All of the results of this section have been established in the case of a compact base manifold M by Grayson and Hamilton [18] for the harmonic map heat flow and Weinkove [31] for the Yang-Mills heat flow, and examples of heat flows admitting such singularities may be found in [16], [17] and [31].

Lemma 5.1 (Existence of blow-ups)**.** *Suppose (u,ψ) arises from a Yang-Mills or harmonic map heat flow defined on $\Omega\times[0,T[$ and has a rapidly-forming singularity at $(X,T)\in\Omega\times\mathbb{R}$. Then there exists a Yang-Mills (resp. harmonic map) heat flow $\{u_\infty(\cdot,t)\}_{t<0}$ on $(\mathbb{R}^n,g_\delta)$ such that $u_r^{(X,T)} \xrightarrow{r\searrow 0} u_\infty$ subsequentially and locally uniformly in $C^\infty(\mathbb{R}^n\times\,]-\infty,0[)$ (modulo gauge).*

Proof. We shall assume without loss of generality that $R < i_0$. Fix $(x_0,t_0) \in Q_{\frac{R}{2}}(X,T)$. We have that $Q_{\widetilde{R}}(x_0,t_0) \subset Q_R(X,T)$ with $\widetilde{R} := \sqrt{T-t_0}$. Moreover, the estimate (5.1) implies that

$$\sup_{Q_{\widetilde{R}}(x_0,t_0)} \frac{1}{2}|\psi|^2 \le \frac{C_0}{\widetilde{R}^{2k}}.$$

Lemma 2.1 then implies that for each $i\in\mathbb{N}$,

$$\sup_{Q_{\frac{\widetilde{R}}{2}}(x_0,t_0)} \frac{1}{2}|\nabla^i\psi|^2 \le \frac{C_i}{\widetilde{R}^{2(k+i)}},$$

where C_i depends on i, the local geometries of Ω and $E \to \Omega$, and C_0. Therefore, we have the estimate

$$\sup_{(x,t)\in Q_{\frac{R}{2}}(X,T)} (T-t)^{k+i}\frac{1}{2}|\nabla^i\psi|^2(x,t) \leq C_i.$$

Using the compactness of the target manifold of the harmonic map heat flow and arguing as in Lemma 2.3 for the Yang-Mills heat flow and choosing a suitable fixed connection as in Remark 2.4, we also obtain a bound

$$\sup_{Q_{\theta R}(X,T)} (T-t)^{k-1}\frac{1}{2}|u|^2 \leq C_{-1}$$

with $\theta \in \left]0, \frac{1}{4}\right]$ and C_{-1} depending on C_0, and the geometries of Ω and the flow under consideration. Altogether, these estimates imply that for any $r > 0$,

$$\sup_{(x,t)\in Q_{\frac{\theta R}{r}}(0,0)} (-t)^{k+i}\frac{1}{2}|\nabla^i u_r^{(X,T)}|^2(x,t) \leq \widetilde{C}_{i-1} \tag{5.2}$$

for constants $\{\widetilde{C}_i\}_{i=-1}^{\infty}$ depending on $\{C_i\}_{i=-1}^{\infty}$. Thus, considering a sequence $\{u_i = u_{r_i}^{(X,T)}\}_{i\in\mathbb{N}}$ of rescalings of u with $r_i \searrow 0$ and using (5.2), we obtain, for large enough i, bounds on u_i and all of its derivatives on compact subsets of $\mathbb{R}^n \times]-\infty, 0[$. Therefore, by the Arzelà-Ascoli theorem, we may pass to a subsequence that converges locally uniformly in C^∞ to a smooth solution $u_\infty : \mathbb{R}^n \times]-\infty, 0[\to V \otimes \Lambda^{k-1}T^*\mathbb{R}^n$ to the corresponding flow equation on $(\mathbb{R}^n, g_\delta)$. $\square$

The following corollary establishes that smooth blow-ups of rapidly forming singularities are in fact self-similar and gives us a strictly positive lower bound on its *heat ball energy*; the latter guarantees that such blow-ups are non-trivial.

Corollary 5.2 (Self-similarity and non-triviality of blow-ups)**.** *Let $\{u_{r_i}^{(X,T)}\}_{i\in\mathbb{N}}$ be a sequence of rescalings ($r_i \searrow 0$) of a Yang-Mills (or harmonic map) heat flow with rapidly forming singularity at (X,T). Moreover, suppose $u_{r_i}^{(X,T)} \xrightarrow{i\to\infty} u_\infty$ locally uniformly in $C^\infty(\mathbb{R}^n)$. Then u_∞ is self similar. Moreover, u_∞ satisfies the estimate*

$$I_k(u_\infty, g; 0, 0, R) = \frac{1}{R^{n-2k}} \iint_{E_R^{n-2k}(0,0)} \frac{1}{2}|\psi_\infty|^2 \cdot \left(\frac{n-2k}{-2t}\right) \mathrm{d}x\mathrm{d}t \geq \varepsilon \tag{5.3}$$

for all $R > 0$.

Proof. We first establish self-similarity. By virtue of the higher-order estimates derived in Lemma 5.1, the monotone quantity (2.18) is finite on u for sufficiently small R. Therefore, we may take its limit as $R \searrow 0$ and freely apply scale-invariance (Remark 2.7) so that for any $r > 0$,

$$\begin{aligned}
&\lim_{R\searrow 0} I_k(u, g; X, T, R) = \lim_{i\to\infty} I_k(u, g; X, T, r_i r)\\
&= \lim_{i\to\infty} \frac{1}{r^{n-2k}} \iint_{E_r^{n-2k}(0,0)} e^{\xi_k(-\lambda_i^2 s)}[u_{r_i}^{(X,T)}, g_{r_i}^X]_{(0,0)}(y,s)\ \sqrt{g_{r_i}^X}(y,s)\mathrm{d}y\mathrm{d}s\\
&= \frac{1}{r^{n-2k}} \iint_{E_r^{n-2k}(0,0)} [u_\infty, \delta]_{(0,0)}(y,s)\mathrm{d}y\mathrm{d}s = I_k(u_\infty, \delta; 0, 0, r).
\end{aligned}$$

Thus, this last quantity is independent of r and so, by Theorem 2.5, u_∞ must be self similar.

We now turn our attention to the estimate (5.3). We first fix a sequence $\lambda_i \searrow 0$. Since $X \in \mathcal{S}$, there exists a sequence $(x_i, t_i) \in Q_{c_{n,k}r_i r}(X, T)$ such that for sufficiently large i and small fixed $r > 0$, we have

$$I_k(u, g; x_i, t_i, \lambda_i r) \geq \varepsilon.$$

Using monotonicity and translation-invariance (Theorem 2.5 and Remark 2.7), we arrive at the inequality

$$\frac{1}{r^{n-2k}} \iint_{E_r^{n-2k}(0,0)} [u_1^{(x_i,t_i)}, g_1^{x_i}]_{(0,0)}(y,s) e^{\xi(-s)} \sqrt{g_1^{x_i}}(y) \,\mathrm{d}y\,\mathrm{d}s \geq \varepsilon.$$

Taking the limit $i \to \infty$, noting that $(x_i, t_i) \xrightarrow{i\to\infty} (X, T)$ and using the higher order estimates of Lemma 5.1, we obtain the inequality

$$I_k(u, g; X, T, r) \geq \varepsilon.$$

We now let $r = \lambda_i$, use scale-invariance and pass to the limit $i \to \infty$ to obtain (5.3). □

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