LOCAL ENERGY INEQUALITIES FOR MEAN CURVATURE FLOW INTO EVOLVING AMBIENT SPACES

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Abstract. We establish a local monotonicity formula for mean curvature flow into a curved space whose metric is also permitted to evolve simultaneously with the flow, extending the work of Ecker [4], Huisken [9], Lott [10], Magni, Mantegazza and Tsatis [11] and Ecker, Knopf, Ni and Topping [6]. This formula gives rise to a monotonicity inequality in the case where the target manifold's geometry is suitably controlled, as well as in the case of a gradient shrinking Ricci soliton. Along the way, we establish suitable local energy inequalities to deduce the finiteness of the local monotone quantity.

1. Introduction

Let $\{(M, g(\cdot,t))\}_{t\in[0,T_\infty[}$ be an evolving Riemannian manifold, S a smooth m-manifold, $\{x(\cdot,t) : S \to (M, g(\cdot,t))\}_{t\in[0,T[}$ a smooth family of embeddings evolving by mean curvature flow with $0 < T \leq T_\infty$ such that the corresponding space-time mapping $F : S \times [0,T[\to M \times [0,T[$ is proper, and $E_r(F^*\Phi)$ a heat ball in $S \times [0,T[$ for $r < r_0$ with $\Phi = e^{\phi}$ and $\phi_r^m = \log(r^m\Phi)$ defined on an open subset of $M \times]0,T[$ (see §2 and §3 for definitions and setup). Write $a(\cdot,t)$ and $b(\cdot,t)$ for the first and second fundamental form respectively of $x(\cdot,t)$, and set $\underline{\underline{\mathrm{H}}}(\cdot,t) = \mathrm{tr}_{a(\cdot,t)} b(\cdot,t)$.

The main result of this paper is the following local monotonicity identity for mean curvature flow in this setting:

Theorem A. *If* $\frac{F^*\phi}{T-t} \in L^1(E_{r_0}(F^*\Phi))$, *then*

$$\left[\frac{1}{r^m}\iint\limits_{E_r(F^*\Phi)} |\nabla_a F^*\phi|^2 + \left(|\underline{\underline{\mathrm{H}}}|^2 - \frac{1}{2}\mathrm{tr}_a x^*\partial_t g\right) F^*\phi_r^m \ \mathrm{dvol}_a(\cdot,t)\,\mathrm{d}t\right]_{r=r_1}^{r=r_2} \tag{1.1}$$

$$= \int_{r_1}^{r_2}\left(\frac{m}{r^{m+1}}\iint\limits_{E_r(F^*\Phi)} -F^*\left(\frac{\mathrm{L}_g^*\Phi + \frac{n-m}{2(T-t)}\Phi}{\Phi}\right) + |\mathcal{S}_\phi|^2 + \mathrm{tr}_g^\perp\,\mathcal{Q}_T(\phi,g)\ \mathrm{dvol}_a(\cdot,t)\,\mathrm{d}t\right)\mathrm{d}r$$

for $0 < r_1 < r_2 < r_0$ *provided both spacetime integrands are in* $L^1(E_{r_0}(F^*\Phi))$; *moreover, the condition* $\frac{F^*\phi}{T-t} \in L^1(E_{r_0}(F^*\Phi))$ *may be lifted so that the the identity (1.1) holds with* $\geq$ *in place of* $=$. *Here,* L_g^* *denotes the backward heat operator with*

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respect to g, $\mathcal{S}_\phi = \underline{\underline{\mathrm{H}}} - \nabla^\perp\phi$ is an expression characterising certain solitons, and $\mathcal{Q}_T(\phi, g) = \nabla_g^2\phi + \frac{1}{2}\partial_t g + \frac{g}{2(T-t)}$ is the matrix Harnack form.

Mean curvature flow was introduced by Mullins [12] as a model for the motion of grain boundaries. It was studied by Brakke [3] in the context of varifolds and subequently considered in the smooth setting by Huisken, Ecker and various others (see [5] and the references therein). At its core, it is a geometric manifestation of a reaction-diffusion system and therefore tends to develop singularities in finite time.

In his study of rapidly forming singularities, Huisken [9] made crucial use of a monotonicity formula taking the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left((T-t)\int_S F^*\Gamma_{(X,T)}\mathrm{dvol}_a(\cdot,t)\right) \leq 0 \tag{1.2}$$

in the case where $M = \mathbb{R}^{m+1}$, where $\Gamma_{(X,T)}$ is the canonical backward heat kernel on Euclidean $(m+1)$-space with singularity at $(X,T) \in \mathbb{R}^{m+1} \times]0,\infty[$. On the one hand, the monotone quantity characterises homothetic solutions in the sense that it is constant if and only if x homothetically shrinks. On the other hand, it plays a crucial role in the regularity theory of the flow at the first singular time as was shown by White [17]. As a result, it makes sense to look for analogues of (1.2) valid in more general settings, particularly in the case of a non-Euclidean target M. This has been accomplished by Hamilton [8] for compact M, Lott [10] when the target is a Ricci soliton, and Magni, Mantegazza and Tsatis [11] more generally.

In some sense, the quantity occurring in (1.2) plays the role of the area ratio in minimal surface theory, which is the corresponding *static* problem ($x(\cdot,t) \equiv x(\cdot)$); in this case, the monotonicity principle takes the form

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r^m}\int_{x^{-1}B_r(X)}\mathrm{dvol}_a\right) \geq 0 \tag{1.3}$$

for a Euclidean target [2], where equality holds if and only if the submanifold is conical about X. A key difference however is that (1.3) is *localised* in the target, whereas (1.2) is not. A natural analogue of (1.3) was discovered by Ecker [4], also in the case of a Euclidean target:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r^m}\iint\limits_{E_r(F^*\widetilde{\Gamma}_{(X,T)})} |\nabla_a F^*\widetilde{\gamma}_{(X,T)}|^2 + |\underline{\underline{\mathrm{H}}}|^2 F^*\left(\widetilde{\gamma}_{(X,T)}\right)^m_r \mathrm{dvol}_a(\cdot,t)\,\mathrm{d}t\right) \geq 0. \tag{1.4}$$

Here, $\widetilde{\Gamma}_{(X,T)}(x,t) = (4\pi(T-t))^{\frac{n-m}{2}}\,\Gamma_{(X,T)}(x,t)$, $\widetilde{\gamma}_{(X,T)} = \log\widetilde{\Gamma}_{(X,T)}$ and $(\widetilde{\gamma}_{(X,T)})^m_r = \log(r^m\widetilde{\Gamma}_{(X,T)})$. In some sense, this formula is a nonlinear analogue of the *heat ball formula* for solutions to the heat equation due to Watson [16], since the domain of integration is essentially a pulled-back heat ball. Moreover, just like (1.2), the quantity in (1.4) is constant if and only if x homothetically shrinks. The main theorem of this paper is a generalisation of this formula to the setting where the target manifold is curved.

Although somewhat complicated in appearance, the more general formula in the main theorem shares many features of the aforementioned monotonicity formulæ: If $\Phi(x,t) = (4\pi(T-t))^{\frac{n-m}{2}}\Gamma(x,t)$ with $\mathrm{L}_g^*\Gamma \equiv 0$, the first term in the integrand of the right-hand side vanishes. Moreover, the second term vanishes if and only if x is a gradient soliton (up to tangential diffeomorphisms) with potential $-\phi$; if $M = \mathbb{R}^n$ and Γ is the canonical backward heat kernel, this amounts to x being a homothetic shrinker. The last term, though not generally having a sign, occurs in Hamilton's matrix Harnack estimate [7] and vanishes if $T = T_\infty$ and g is a gradient Ricci soliton with potential -ϕ. All of these features are shared with the formula derived by Magni, Mantegazza and Tsatis [11], who restricted their attention to metric tensors g evolving by Ricci or backward Ricci flow. Here we shall go one step further and show that if the (local) geometry of the *target* manifold $(M, g(\cdot,t))$ is suitably controlled, then for a suitable choice of Φ, (1.1) leads to a monotone quantity; in particular, we suppose that the injectivity radius of $(M, g(\cdot,t))$ is bounded from below close to (X,T), that the (spatial) Hessian of the distance function $(y,t) \mapsto d^{g(\cdot,t)}(X,y)$ is suitably bounded near (X,T) and that $\partial_t g$ is bounded near (X,T). This culminates in the following monotonicity inequality, which readily generalises that of Ecker [4].

Theorem B. *Suppose $\{x(\cdot,t) : S \to (M, g(\cdot,t))\}_{t\in[0,T[}$ is a family of embeddings evolving by mean curvature flow and M has locally controlled geometry about (X,T). Then there exists a constant $r_0 > 0$ and smooth function $\eta :]0,T[\to]0,\infty[$ with $\lim_{t\nearrow T}\eta(t) = 1$ depending only on the local geometry of $(M, g(\cdot,t))$ near (X,T) such that the monotonicity inequality*

$$\begin{aligned}
&\left[\frac{1}{r^m}\iint\limits_{E_r(F^*\Phi)} |\nabla_a F^*\phi|^2 + \left(|\underline{\underline{\mathrm{H}}}|^2 - \frac{1}{2}\mathrm{tr}_a x^*\partial_t g\right) F^*\phi_r^m \,\mathrm{dvol}_a(\cdot,t)\,\mathrm{d}t\right]_{r=r_1}^{r=r_2} \\
&\qquad \geq \int_{r_1}^{r_2}\left(\frac{m}{r^{m+1}}\iint\limits_{E_r(F^*\Phi)} \left|\underline{\underline{\mathrm{H}}} - \nabla_g^\perp\phi\right|^2 \mathrm{dvol}_a(\cdot,t)\,\mathrm{d}t\right)\mathrm{d}r
\end{aligned} \tag{1.5}$$

holds for $0 < r_1 < r_2 < r_0$, where Φ is given by the expression

$$\Phi(y,t) = \eta(t)\cdot\frac{1}{(4\pi(T-t))^{\frac{m}{2}}}\exp\left(\frac{d^{g(\cdot,t)}(y,X)^2}{4(t-T)}\right).$$

If $(M, g(\cdot,t)) = (\mathbb{R}^n, \delta)$, then (1.5) holds with equality for all $0 < r_1 < r_2$, and the right-hand side vanishes if and only if x homothetically shrinks.

The structure of the paper is as follows. In §2, we introduce our setup and the relevant properties of mean curvature flow in the setting where the target manifold is equipped with an evolving metric tensor. In particular, we establish a local area estimate analogous to that in [3] in the case where g is suitably locally controlled, and also outline how Magni, Mantegazza and Tsatis' formula [11] may be localised in this setting. In §3, we recall the theory of heat balls as in [4], [6] and [1] and discuss examples. Finally, in §4 we first establish Theorem A, then discuss its implications in various settings and conclude with a proof of Theorem B. In fact, we shall establish a version of (1.1) and thus also (1.5) including an additional function, as was done in Ecker [4], which more closely illustrates the connection to a heat ball formula due to Ecker, Knopf, Ni and Topping [6], thus furnishing a generalisation of both their formula and that of Ecker [4].

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2. Mean curvature flow in an evolving background

As in the introduction, suppose $\{(M, g(\cdot, t))\}_{t\in[0,T_\infty[}$ is an evolving Riemannian manifold, S a smooth m-manifold and $\{x(\cdot, t) : S \to (M, g(\cdot, t))\}_{t\in[0,T[}$ a smooth family of embeddings, where $0 < T \leq T_\infty$. We introduce the corresponding *space-time mapping* $F : S \times [0, T[\to M \times [0, T[$ defined by $F(z, t) = (x(z, t), t)$ and call it *proper* if $F^{-1}(K)$ is compact whenever K is. Throughout this paper, we will always assume this to be the case.

2.1. **Intrinsic geometry.**

We shall use the notational conventions of [15] and [14]. We denote the tangent bundle of M by TM and the cotangent bundle by T^*M. All of the usual operators on smooth functions and sections of tensor bundles carry over to operators on *time-dependent* functions and (local) sections in the setting of evolving manifolds with the added requirement that they be defined with respect to the metric tensor at a fixed value of t at which the corresponding function or section is evaluated; for example, given a smooth function $f : \mathcal{D} \to \mathbb{R}$ with $\mathcal{D} \subset M \times [0, T[$ open, the gradient is defined as the time dependent local section $\nabla_g f : \mathcal{D} \to TM$ with

$$\nabla_g f(x, t) = \left(\nabla_{g(\cdot,t)} f(\cdot, t)\right)(x),$$

where the right-hand side consists of the usual gradient of $f(\cdot, t)$ with respect to to $g(\cdot, t)$ evaluated at x. We similarly obtain the *divergence* operator div_g, Laplacian Δ_g, Hessian ∇^2_g, Riemannian volume measure dvol_g and Levi-Civita connection ∇. If the underlying metric is clear from the context, we shall omit mention of g when using the aforementioned operators. Furthermore, whenever we shall have to deal with sections of tensor products of bundles with naturally defined connections, we shall denote the induced connection by ∇. We shall use the notation h^*T for the *pullback* of a function or more generally $(0, p)$-tensor T by a smooth mapping h whenever this operation is defined.

For $\mathcal{D} \subset M \times]0, T_\infty[$ open, write $C^{k,l}(\mathcal{D})$ for the space of all functions k-times continuously differentiable in space and l-times continuously differentiable in time. The family of Riemannian volume measures $\{\mathrm{dvol}_g(\cdot, t)\}_{t\in[0,T_\infty[}$ together with Lebesgue measure on $[0, T_\infty[$ naturally induces a Borel measure μ on $M \times [0, T_\infty[$. When integrating against the volume measure $\mathrm{dvol}_g(\cdot, t)$ of $g(\cdot, t)$, we will *always* restrict the functions on $M \times [0, T_\infty[$ being integrated to the *time slice* $M \times \{t\}$. We shall write $L^p(\mathcal{D})$ for the space of all μ-measurable functions such that $\iint |f|^p \,\mathrm{d}\mu < \infty$. When context allows for it, we shall simply write $f \in L^p$ or $f \in C^{k,l}$, and simply write C^k for $C^{k,k}$. We write pr_i for the natural projection of $M \times [0, T_\infty[$ onto its ith component.

We define the *heat operator* L_g such that for $f \in C^{2,1}(\mathcal{D})$,

$$\mathrm{L}_g f = \partial_t f - \Delta_g f.$$

We also introduce the *backward heat operator* L_g^* such that for $f \in C^{2,1}(\mathcal{D})$,

$$\mathrm{L}_g^* f = \partial_t f + \Delta_g f + \frac{1}{2}\mathrm{tr}_g \partial_t g \cdot f;$$

this operator naturally arises in the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_M u \cdot v \ \mathrm{dvol}_g(\cdot, t) = \int_M \mathrm{L}_g u \cdot v + u \cdot \mathrm{L}_g^* v \ \mathrm{dvol}_g(\cdot, t),$$

which naturally leads to the *fundamental solution representation formula* for solutions to the heat equation.

We shall write $B_r^{g(\cdot,t)}(X)$ for the geodesic ball of radius r centred at $X \in M$ taken with respect to $g(\cdot, t)$ and introduce the *space-time cylinder*

$$\mathcal{D}_{r_1,r_2}(X,T) = \bigcup_{t \in](T-r_2)^+, T[} B_{r_1}^{g(\cdot,t)}(X) \times \{t\}$$

for $r_1, r_2 > 0$ and $(X,T) \in M \times]0, T_\infty]$. Moreover, we write $\mathrm{inj}_X^{g(\cdot,t)}$ for the injectivity radius of $g(\cdot,t)$ at X and $d^{g(\cdot,t)}$ for the distance function arising from $g(\cdot,t)$.

We shall say that M has *locally controlled geometry* about (X,T) if there are constants $\delta \in]0,T[$ and $j_0, \Lambda_{\pm\infty}, \lambda_{\pm\infty}$ such that $\mathrm{inj}_X^{g(\cdot,t)} > j_0$ and for $t \in]T-\delta, T[$, we have the following bounds on $B_{j_0}^{g(\cdot,t)}(X)$:

$$\begin{aligned} \Lambda_{-\infty}\mathfrak{r}(\cdot,t)^2 g^{\mathfrak{r}}(\cdot,t) \le g(\cdot,t) - \nabla_g^2\left(\frac{1}{2}\mathfrak{r}(\cdot,t)^2\right) &\le \Lambda_\infty \mathfrak{r}(\cdot,t)^2 g^{\mathfrak{r}}(\cdot,t) \\ \lambda_{-\infty} g(\cdot,t) \le \partial_t g(\cdot,t) &\le \lambda_\infty g(\cdot,t) \end{aligned} \tag{2.1}$$

Here, $\mathfrak{r}(\cdot,t) = d^{g(\cdot,t)}(\cdot,X)$ and $g^{\mathfrak{r}}(\cdot,t) = g(\cdot,t) - \mathrm{d}\mathfrak{r}(\cdot,t) \otimes \mathrm{d}\mathfrak{r}(\cdot,t)$. Such bounds naturally arise if for instance $T_\infty > T$ by virtue of the Hessian comparison theorem (cf. [13, Theorem 27, p. 175]).

2.2. **Kernels.** Assume for the moment that M has locally controlled geometry about (X,T) with the bounds assumed just before and introduce the *Euclidean backward heat kernel* concentrated at (X,T) as the map $\Gamma_{(X,T)} : M \times [0,T[\to \mathbb{R}^+$ such that

$$\Gamma_{(X,T)}(y,t) = \frac{1}{(4\pi(T-t))^{n/2}} \exp\left(\frac{d^{g(\cdot,t)}(y,X)^2}{4(t-T)}\right).$$

Although not smooth everywhere, it is smooth on $\mathcal{D}_{j_0,\delta}(X,T)$ and approximates the canonical backward heat kernel well enough for our purposes (cf. [1, §2.3]). We recall the following proposition from [1], using the notation $a \sim a_{\pm\infty}$ to express the inequality $a_{-\infty} \le a \le a_\infty$ and $a \sim a_{\mp\infty}$ to express $a_\infty \le a \le a_{-\infty}$.

Proposition 2.1. *If $(X,T) \in M \times]0, T_\infty[$, then $\Gamma_{(X,T)}$ is smooth on $\mathcal{D}_{j_0,\delta}(X,T)$ with j_0 and δ as before and, setting $\gamma_{(X,T)}(x,t) = \log \Gamma_{(X,T)}(x,t)$, the relations*

$$\partial_t \gamma_{(X,T)}(y,t) \sim \frac{n}{2(T-t)} - \frac{\mathfrak{r}(y,t)^2}{4(t-T)^2} + \frac{\lambda_{\mp\infty}\mathfrak{r}(y,t)^2}{4(t-T)}; \tag{2.2}$$

$$\nabla \gamma_{(X,T)}(y,t) = \frac{\mathfrak{r}(y,t)}{2(t-T)} \nabla \mathfrak{r}(y,t); \tag{2.3}$$

$$\left(\nabla^2\gamma_{(X,T)} + \frac{g}{2(T-t)}\right)(y,t) \sim \Lambda_{\pm\infty}\left[\log\left(\frac{1}{(4\pi(T-t))^{\frac{n}{2}}\,\Gamma_{(X,T)}}\right)g^{\mathrm{r}}\right](y,t);\ \text{and} \tag{2.4}$$

$$\begin{aligned}&\mathrm{L}_g^*\Gamma(y,t)\\ &\sim\left[\left([(n-1)\Lambda_{\pm\infty}+\lambda_{\mp\infty}]\log\left(\frac{1}{(4\pi(T-t))^{\frac{n}{2}}\,\Gamma_{(X,T)}}\right)+\frac{n}{2}\lambda_{\pm\infty}\right)\Gamma_{(X,T)}\right](y,t)\end{aligned} \tag{2.5}$$

hold for $(y,t)\in\mathcal{D}_{j_0,\delta}(X,T)$, where $\lambda_\pm$ and $\Lambda_\pm$ are as in (2.1).

We note in particular that the map $\Phi:\mathcal{D}_{j_0,\delta}(X,T)\to\mathbb{R}^+$ defined by $\Phi(y,t)=(4\pi(s-t))^{\frac{n-m}{2}}\,\Gamma_{(X,s)}(y,t)$ satisfies inequalities of the form

$$\begin{aligned}\left(\mathrm{L}_g^*+\frac{n-m}{2(s-t)}\right)\Phi(y,t)&\le a_0+a_1(t)\Phi(y,t)\\ \Phi\cdot\mathcal{Q}_s(\log\Phi,g)(y,t)&\ge(b_0+b_1(t)\Phi)g(y,t)\end{aligned} \tag{2.6}$$

for all $s\ge T$, where $a_0,b_0\in\mathbb{R}$ and a_1,a_2 are continuous, summable functions on $]T-\delta,T[$ and $\mathcal{Q}_T$ is as in Theorem A; explicitly,

$$\begin{aligned}a_0&=\max\{\frac{n}{2}\lambda^+_{+\infty},[(n-1)\Lambda_{+\infty}+\lambda_{-\infty}]^+\},\\ a_1(t)&=-\frac{ma_0}{2}\log\left(4\pi(T-t)\right),\\ b_0&=\min\{\frac{\lambda^-_{-\infty}}{2},\Lambda^-_{-\infty}\},\\ b_1(t)&=-\frac{mb_0}{2}\log\left(4\pi(T-t)\right).\end{aligned} \tag{2.7}$$

It was shown by Hamilton [7] that similar estimates hold for the canonical backward heat kernel in the case where M is compact and static. It is unclear whether an analogous estimate for the Harnack form $\mathcal{Q}_T$ holds for the canonical backward heat kernel more generally, which is why we have opted to make use of the Euclidean backward heat kernel.

2.3. **Extrinsic geometry.** We now turn our attention to the embeddings $\{x(\cdot,t)\}_{t\in[0,T[}$. To streamline our computations and make taking t-derivatives hassle-free, we first pull TM back by x obtain the bundle

$$\mathbb{R}^n\to x^{-1}TM\to S\times[0,T[\,,$$

over space-time, where $x^{-1}TM$ is realised as the point set

$$x^{-1}TM=\bigcup_{(z,t)\in S\times[0,T[}\{(z,t)\}\times T_{x(z,t)}M.$$

The more customary pullback bundle of the embedding $x(\cdot,t)$ for $t\in[0,T[$, i.e. $\mathbb{R}^n\to x(\cdot,t)^{-1}TM\to S$, may then be realised as the point set

$$x(\cdot,t)^{-1}TM=\bigcup_{z\in S}\{(z,t)\}\times T_{x(z,t)}M$$

so that $x^{-1}TM = \bigcup_{t\in[0,T[} x(\cdot,t)^{-1}TM$. Under this identification, we have a one-to–one correspondence between smooth one-parameter families of sections $\{s(\cdot,t) : S \to x(\cdot,t)^{-1}TM\}_{t\in I}$, $I \subset [0,T[$ an interval, and smooth local sections $s : S\times I \to x^{-1}TM$; in particular, we view $\partial_t x$ as the section $S\times]0,T[\to x^{-1}TM$ given by

$$\partial_t x(z,t) = (z,t,\sum_{j=1}^{n} \partial_t x^i(z,t)\ \partial_i|_{x(z,t)}),$$

where $\{x^i\}_{i=1}^n$ is a coördinate representation of x in a local coördinate system $U \subset M$ centred at $x(z,t)$ and $\{\partial_i|_y\}_{i=1}^n$ is the corresponding local basis of T_yM for each $y \in U$, and this definition is independent of the local coördinate system chosen. We shall view the (spatial) differential of x as the one-parameter family of sections $\{\mathrm{d}x(\cdot,t) : S \to x(\cdot,t)^{-1}TM \otimes T^*S\}_{t\in[0,T[}$ given by

$$\mathrm{d}x(z,t) = \sum_{i=1}^{n}\sum_{j=1}^{m} \frac{\partial x^i}{\partial z^j}(z,t)(z,t,\partial_i|_{x(z,t)}) \otimes\ \mathrm{d}z^j\big|_z\,,$$

where $\{z^j\}_{j=1}^m$ are local coördinates on S in a neighbourhood of z and we use local coördinates on M as before.

We obtain a Riemannian structure on $x(\cdot,t)^{-1}TM$ and $x^{-1}TM$ from g in a natural manner, viz. by setting

$$\langle (z,t,v),(z,t,w)\rangle = (g(x(z,t)), v\otimes w)\,,$$

where $(z,t) \in S\times[0,T[$ and $v,w \in T_{x(z,t)}$; this then gives rise to the fibrewise orthogonal decomposition

$$x(\cdot,t)^{-1}TM = \mathrm{im}\ \mathrm{d}x(\cdot,t) \oplus (T^{\perp}S)_t \tag{2.8}$$

into subbundles *tangent* and *normal* to TS, where im $\mathrm{d}x(\cdot,t)$ denotes the image of $\mathrm{d}x(\cdot,t)$ viewed as a vector bundle morphism $TS \to x(\cdot,t)^{-1}TM$, i.e.

$$\mathrm{im}\ \mathrm{d}x(\cdot,t) = \bigcup_{z\in S}\{v \llcorner \mathrm{d}x(z,t) : v \in T_zS\}$$

with $\llcorner$ the interior product acting on the latter part of the tensor product $(x(\cdot,t)^{-1}TM)_z\otimes T_z^*S$, and $(T^{\perp}S)_t$ denotes the orthogonal complement of im $\mathrm{d}x(\cdot,t)$.

Given a time-dependent vector field $X(\cdot,t) : M \to TM$, we may define the *total section* $\underline{\underline{X}}(\cdot,t) : S \to x(\cdot,t)^{-1}TM$ such that $z \mapsto (z,t,X(F(z,t)))$, the *tangential pullback* of $X(\cdot,t)$ to S as the unique time-dependent vector field $\underline{X}(\cdot,t) : S \to TS$ such that $\underline{X}(z,t)\llcorner \mathrm{d}x(z,t) = \underline{\underline{X}}(z,t)^T$, where $(\cdot)^T$ denotes orthogonal projection onto im $\mathrm{d}x(\cdot,t)$, and the *normal part* $X^{\perp}(\cdot,t) : S \to x(\cdot,t)^{-1}TM$ of $X(\cdot,t)$ as $(z,t) \mapsto \underline{\underline{X}}(z,t)^{\perp}$, where $(\cdot)^{\perp}$ denotes orthogonal projection onto $(T^{\perp}S)_t$.

We introduce the *first fundamental form*

$$a(\cdot,t) := x(\cdot,t)^*g(\cdot,t)$$

of $x(\cdot,t)$, which endows S with the structure of an evolving manifold. The *second fundamental form* of $x(\cdot,t)$ is the section $b(\cdot,t) : S \to x(\cdot,t)^{-1}TM \otimes T^*S\otimes T^*S$ defined as

$$b(\cdot,t) = \nabla \mathrm{d}x(\cdot,t) = \sum_{i=1}^{m} \nabla_{\frac{\partial}{\partial z^i}} \mathrm{d}x(\cdot,t)\otimes \mathrm{d}z^i,$$

where ∇ is the natural covariant derivative for sections of $x(\cdot,t)^{-1}TM \otimes T^*S$ induced by the Levi-Civita connections of $(M, g(\cdot,t))$ and $(S, a(\cdot,t))$. The *mean curvature* of $x(\cdot,t)$ is then the unique section $\underline{\underline{\mathrm{H}}} : S \times [0,T[\to x^{-1}TM$ defined by

$$\underline{\underline{\mathrm{H}}}(\cdot,t) = \mathrm{tr}_{a(\cdot,t)} b(\cdot,t) = \sum_{i,j=1}^{m} a^{ij}(\cdot,t) \cdot (b(\cdot,t), \frac{\partial}{\partial z^i} \otimes \frac{\partial}{\partial z^j}),$$

where $(a^{ij}(\cdot,t))_{i,j=1}^m$ is the inverse matrix of $(a_{ij}(\cdot,t))_{i,j=1}^m := ((a(\cdot,t), \frac{\partial}{\partial z^i} \otimes \frac{\partial}{\partial z^j}))_{i,j=1}^m$.

If $X(\cdot,t) = \nabla_g f(\cdot,t)$ for a differentiable function f, we write $\nabla^\perp f(\cdot,t)$ for $X^\perp(\cdot,t)$. Furthermore, the *normal trace* $\mathrm{tr}_g^\perp T$ of a time-dependent $(0,2)$-tensor $T(\cdot,t)$ on M is defined so as to satisfy the equality

$$x(\cdot,t)^* \mathrm{tr}_{g(\cdot,t)} T(\cdot,t) = \mathrm{tr}_{a(\cdot,t)} x(\cdot,t)^* T(\cdot,t) + \left(\mathrm{tr}_g^\perp T\right)(\cdot,t).$$

Moreover, given such a time-dependent $(0,2)$-tensor, we shall simply write $\mathrm{tr}_a x^* T$ for the function $(z,t) \mapsto \mathrm{tr}_{a(z,t)}(x(\cdot,t)^* T(\cdot,t))(z)$ to avoid clutter.

Finally, we recall the divergence identity

$$\mathrm{div}_a \underline{X}(\cdot,t) = x(\cdot,t)^* \mathrm{div}_g X(\cdot,t) - (\mathrm{tr}_g^\perp \nabla X^\flat)(\cdot,t) + \left\langle \underline{\underline{\mathrm{X}}}, \underline{\underline{\mathrm{H}}} \right\rangle (\cdot,t). \tag{2.9}$$

for time-dependent vector fields $X(\cdot,t)$ on M, where $X^\flat(\cdot,t) : S \to T^*S$ is the time-dependent one-form obtained by lowering the index of $X(\cdot,t)$ by means of the metric tensor $g(\cdot,t)$. This identity follows from a local computation in an orthonormal frame that respects the decomposition (2.8); this formula is derived in [8, §2]) with $X(\cdot,t) = \nabla f(\cdot,t)$ for smooth f.

2.4. **Mean curvature flow.** We say that the family $\{x(\cdot,t)\}_{t \in [0,T[}$ evolves by *mean curvature flow* if the equation

$$\partial_t x = \underline{\underline{\mathrm{H}}}$$

holds on $S \times]0,T[$, where this equation is to be interpreted as one involving sections $S \times]0,T[\to x^{-1}TM$.

As alluded to earlier, $\{(S, a(\cdot,t))\}_{t \in [0,\infty[}$ is an evolving Riemannian manifold. We recall from [11] that under mean curvature flow, we have the evolution equation

$$\partial_t a(\cdot,t) = \sum_{i,j=1}^{m} \left[(x(\cdot,t)^* \partial_t g(\cdot,t), \frac{\partial}{\partial z^i} \otimes \frac{\partial}{\partial z^j}) - 2 \left\langle (b(\cdot,t), \frac{\partial}{\partial z^i} \otimes \frac{\partial}{\partial z^j}), \underline{\underline{\mathrm{H}}}(\cdot,t) \right\rangle \right] \mathrm{d}z^i \otimes \mathrm{d}z^j \tag{2.10}$$

so that the volume measure satisfies the evolution equation

$$\partial_t \mathrm{dvol}_a(\cdot,t) = \left(\frac{1}{2} \mathrm{tr}_a x^* \partial_t g - |\underline{\underline{\mathrm{H}}}|^2 \right) \mathrm{dvol}_a(\cdot,t). \tag{2.11}$$

An immediate consequence is *volume decay*: If S is compact and M is static ($\partial_t g \equiv 0$), then

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_S \mathrm{dvol}_a(\cdot,t) = - \int_S |\underline{\underline{\mathrm{H}}}|^2 \mathrm{dvol}_a(\cdot,t). \tag{2.12}$$

Note also that for functions $f : \mathcal{D} \to \mathbb{R}$ defined on an open subset $\mathcal{D}$ of $M \times [0,T[$, the divergence identity (2.9) implies the following relations between the heat operators on M and S:

$$\begin{aligned} \mathrm{L}_a(F^*f) &= F^*\mathrm{L}_g f + \mathrm{tr}_g^{\perp}\nabla^2 f; \\ \mathrm{L}_a^*(F^*f) &= F^*\mathrm{L}_g^* f - \mathrm{tr}_g^{\perp}\nabla^2 f + 2\left\langle \underline{\underline{\mathrm{H}}}, \nabla^{\perp} f\right\rangle. \end{aligned} \tag{2.13}$$

2.5. **Monotonicity and energy inequalities.** Suppose $\{x(\cdot,t)\}_{t\in[0,T[}$ evolves by mean curvature flow. We shall assume throughout this section that we have a locally controlled geometry about (X,T). To simplify notation, we shall henceforth set $\underline{B}_r^t(X) = x(\cdot,t)^{-1}B_r^{g(\cdot,t)}(X)$ and $\underline{\mathfrak{r}} = F^*\mathfrak{r}$.

We first establish a local analogue of volume decay (see (2.12) above) which is an adaptation of a result due to Brakke [3, §3.6].

Lemma 2.2. *There exists a constant $\gamma \geq 0$ depending only on m, n and the constants appearing in these bounds such that for $R \in \left]0, \min\{j_0, \sqrt{4\gamma\delta}\right[$ and $t \in \left[T - \frac{R^2}{4\gamma}, T\right[$ the inequality*

$$\int_{\underline{B}^t_{R/2}(X)} \mathrm{dvol}_a(\cdot,t) \leq 16 e^{\frac{m\lambda_\infty R^2}{8\gamma}} \int_{\underline{B}_R^{T-\frac{R^2}{4\gamma}}(X)} \mathrm{dvol}_a(\cdot, T - \frac{R^2}{4\gamma}) - \int_{T-\frac{R^2}{4\gamma}}^t \int_{\underline{B}^s_{R/2}(X)} |\underline{\underline{\mathrm{H}}}|^2 \mathrm{dvol}_a(\cdot,s)\,\mathrm{d}s$$

holds.

Proof. Let $\eta : \mathbb{R} \to \mathbb{R}^+$ be defined by $\eta(x) = \left((1-x)^+\right)^4$ and define $\psi_R : S \times \left[T - \frac{R^2}{4\gamma}, T\right[\to \mathbb{R}^+$ by

$$\psi_R(x,t) := \eta\left(\frac{\underline{\mathfrak{r}}(x,t)^2 + \gamma\left(t - \left(T - \frac{R^2}{4\gamma}\right)\right)}{R^2}\right),$$

where $\gamma \geq 0$ is soon to be fixed. It is clear that ψ_R is twice differentiable, and $\operatorname{supp}\psi_R(\cdot,t) \subset \underline{B}_R^t(X) \subset \underline{B}_{j_0}^t(X)$, since

$$1 - \frac{\underline{\mathfrak{r}}(x,t)^2 + \gamma\left(t - \left(T - \frac{R^2}{4\gamma}\right)\right)}{R^2} \geq 0 \Leftrightarrow R^2 \geq \underline{\mathfrak{r}}^2 + \gamma\left(t - \left(T - \frac{R^2}{4\gamma}\right)\right) \geq \underline{\mathfrak{r}}^2.$$

Also, $0 \leq \psi_R \leq 1$. Now, note that by (2.13) and the local geometry bounds (2.1), we have that

$$\mathrm{L}_a \underline{\mathfrak{r}}^2 \geq (2m\Lambda^-_{-\infty} + \lambda^-_{-\infty}) j_0^2 - 2m \tag{2.14}$$

on $\mathcal{D}_{j_0,\delta}(X,T)$. Hence, setting $\gamma = 2m - \left(2m\Lambda^-_{-\infty} + \lambda^-_{-\infty}\right) j_0^2$, it follows from the chain rule that $\mathrm{L}_a\psi_R \leq 0$.

Now, since $x(\cdot,t)$ is a proper embedding for each t, both $\psi_R(\cdot,t)$ and $\partial_t\psi_R(\cdot,t)$ are compactly supported in S so that we may compute using (2.11) that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_S \psi_R\,\mathrm{dvol}_a(\cdot,t) &= \int_S \left(\partial_t\psi_R - |\underline{\underline{\mathrm{H}}}|^2\psi_R + \frac{1}{2}x(\cdot,t)^*\partial_t g(\cdot,t)\cdot\psi_R\right)\mathrm{dvol}_a(\cdot,t) \\ &\leq -\int_S |\underline{\underline{\mathrm{H}}}|^2\psi_R\,\mathrm{dvol}_a(\cdot,t) + \frac{m\lambda_\infty}{2}\int_S \psi_R \mathrm{dvol}_a(\cdot,t), \end{aligned}$$

whence

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-\frac{m\lambda_\infty}{2}\left(t-\left(T-\frac{R^2}{4\gamma}\right)\right)}\int_S \psi_R \mathrm{dvol}_a(\cdot,t)\right)$$
$$\leq -\exp\left(-\frac{m\lambda_\infty R^2}{8\gamma}\right)\int_S |\underline{\underline{\mathrm{H}}}|^2\psi_R \ \mathrm{dvol}_a(\cdot,t)$$

so that, integrating from $T-\frac{R^2}{4\gamma}$ to t and estimating $\exp\left(-\frac{m\lambda_\infty}{2}\left(t-\left(T-\frac{R^2}{4\gamma}\right)\right)\right)$ from below, we obtain

$$e^{-\frac{m\lambda_\infty R^2}{8\gamma}}\cdot\left(\int_S \psi_R \ \mathrm{dvol}_a(\cdot,t)+\int_{T-\frac{R^2}{4\gamma}}^t\int_S |\underline{\underline{\mathrm{H}}}|^2\psi_R \ \mathrm{dvol}_a(\cdot,s)\,\mathrm{d}s\right)$$
$$\leq \int_S \psi_R \ \mathrm{dvol}_a(\cdot,T-\frac{R^2}{4\gamma}). \quad (2.15)$$

Now, since $\psi_R(\cdot,t)\leq \chi_{\underline{B}^t_R(X)}$, the right-hand side of (2.15) may be bounded from above by

$$\int_{\underline{B}_R^{T-\frac{R^2}{4\gamma}}(X)} \mathrm{dvol}_a(\cdot,T-\frac{R^2}{4\gamma}).$$

On the other hand, since

$$\underline{\mathfrak{r}}<\frac{R}{2}\Rightarrow 1-\frac{\underline{\mathfrak{r}}^2+\gamma\left(t-\left(T-\frac{R^2}{4\gamma}\right)\right)}{R^2}\geq 1-\frac{\frac{R^2}{4}+\gamma\cdot\frac{R^2}{4\gamma}}{R^2}=\frac{1}{2}$$

so that

$$\psi_R(\cdot,t)|_{\underline{B}^t_{R/2}(X)}\geq\left(\frac{1}{2}\right)^4=\frac{1}{16}. \quad (2.16)$$

Since the left-hand integrands of (2.15) are nonnegative, we may estimate their (spatial) integrals from below by the respective integrals on $\underline{B}^t_{R/2}(X)$, whence the result follows from (2.16). $\square$

We now recall Magni, Mantegazza and Tsatis' formula [11] which we shall need to derive certain estimates guaranteeing the finiteness of the heat ball integrals we shall consider.

Theorem 2.3 (Monotonicity Formula)**.** *If* $u\in C^{2,1}(S\times[0,T[\,,\mathbb{R})$ *is such that* $\operatorname{supp} u(\cdot,t)\Subset S$ *for each* $t\in[0,T[$ *and* $\Phi\in C^2(\mathcal{D},\mathbb{R}^+)$ *with* $\mathcal{D}\subset M\times\,]0,T[$ *open with* $\operatorname{supp} u(\cdot,t)\times\{t\}\subset(F,\mathrm{pr}_2)^{-1}\left(\mathcal{D}\cap\mathrm{pr}_2^{-1}(\{t\})\right)$*, then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_S u\cdot F^*\Phi\ \mathrm{dvol}_a(\cdot,t)\right)$$
$$=\int_S F^*\Phi\cdot \mathrm{L}_a u+u\cdot\left(F^*\mathrm{L}_g^*\Phi+\frac{n-m}{2(s-t)}F^*\Phi\right)$$
$$-u\cdot F^*\Phi\cdot\mathrm{tr}_g^\perp\mathcal{Q}_s(\log\Phi,g)-u\cdot F^*\Phi\cdot\left|\underline{\underline{\mathrm{H}}}-\nabla_g^\perp\log\Phi\right|^2\mathrm{dvol}_a(\cdot,t)$$

on $]0,T[$ *for every* $s\geq T$.

Proof sketch. It may be shown using (2.11) and (2.13) that the identity

$$\begin{aligned}\partial_t\left(u\cdot F^*\Phi\ \mathrm{dvol}_a\right)(\cdot,t) = \Bigg[&\mathrm{div}_a\left(F^*\Phi\nabla_a u - u\nabla_a F^*\Phi\right) + F^*\Phi\cdot \mathrm{L}_a u\\ &+u\cdot\left(F^*\left(\mathrm{L}_g^*\Phi + \frac{n-m}{2(s-t)}\Phi\right)\right) - u\cdot F^*\Phi\cdot \mathrm{tr}_g^\perp \mathcal{Q}_s(\log\Phi)\\ &- u\cdot F^*\Phi\cdot\left|\underline{\underline{\mathrm{H}}} - \nabla_g^\perp \log\Phi\right|^2\Bigg]\mathrm{dvol}_a(\cdot,t)\end{aligned} \tag{2.17}$$

holds, whence an application of Gauß' theorem implies the result. Indeed, by (2.9), we have that

$$\begin{aligned}\mathrm{div}_a\left(F^*\Phi\cdot\nabla_a u - u\cdot\nabla_a F^*\Phi\right) &= F^*\Phi\cdot\Delta_a u - u\cdot\Delta_a F^*\Phi\\ &= F^*\Phi\cdot\Delta_a u - u\cdot\left[F^*\Delta_g\Phi - \mathrm{tr}_g^\perp\nabla_g^2\Phi + \left\langle\nabla^\perp\Phi,\underline{\underline{\mathrm{H}}}\right\rangle\right],\end{aligned}$$

and since $\partial_t(F^*\Phi) = F^*\partial_t\Phi + \left\langle\nabla^\perp\Phi,\underline{\underline{\mathrm{H}}}\right\rangle$ and $\mathrm{tr}_{a(\cdot,t)}x(\cdot,t)^*\partial_t g(\cdot,t) = \left(F^*\mathrm{tr}_g\partial_t g - \mathrm{tr}_g^\perp\partial_t g\right)(\cdot,t)$ for all $t\in\,]0,T[$, we immediately compute using (2.11) that

$$\begin{aligned}&\partial_t(u\cdot F^*\Phi\mathrm{dvol}_a)(\cdot,t) - \mathrm{div}_a(F^*\Phi\cdot\nabla_a u - u\cdot\nabla_a F^*\Phi)\mathrm{dvol}_a(\cdot,t)\\ &=\Bigg(F^*\Phi\cdot\mathrm{L}_a u + u\cdot F^*\left(\partial_t\Phi + \Delta_g\Phi + \frac{1}{2}\mathrm{tr}_g\partial_t g\cdot\Phi\right)\\ &\qquad - u\cdot F^*\Phi\cdot\mathrm{tr}_g^\perp\left(\frac{\nabla_g^2\Phi}{\Phi} + \frac{1}{2}\partial_t g\right)\\ &\qquad -|\underline{\underline{\mathrm{H}}}|^2\cdot u\cdot F^*\Phi + 2u\cdot F^*\Phi\left\langle\nabla^\perp\log\Phi,\underline{\underline{\mathrm{H}}}\right\rangle\Bigg)\mathrm{dvol}_a(\cdot,t),\end{aligned} \tag{2.18}$$

where standard integration theorems allow us to justify interchanging the derivative and (implicit) integral. Noting that $\nabla_g^2\log\Phi = \frac{\nabla_g^2\Phi}{\Phi} - \frac{\nabla_g\Phi\otimes\nabla_g\Phi}{\Phi^2}$, $\mathrm{tr}_g^\perp\frac{\nabla_g\Phi\otimes\nabla_g\Phi}{\Phi^2} = |\nabla^\perp\log\Phi|^2$ and $\mathrm{tr}_g^\perp\frac{g}{2(s-t)} = \frac{n-m}{2(s-t)}$ and incorporating these relations into (2.18) then yields (2.17). $\square$

If $R\in\left]0,\min\{\frac{j_0}{2},\sqrt{\gamma\delta}\}\right[$ with γ as in Lemma 2.2 and Φ satisfies inequalities of the form (2.6) with $k=\frac{n-m}{2}$ for $(x,t)\in\mathcal{D}_{R,\delta}(X,T)$, Theorem 2.3 immediately implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{\xi_m(t)}\int_S\psi_R\cdot F^*\Phi\ \mathrm{dvol}_a(\cdot,t)\right] \le (a_0-(n-m)b_0)e^{\sup|\xi_m|}\cdot\int_{\underline{B}_R^t(X)}\mathrm{dvol}_a(\cdot,t) \tag{2.19}$$

on $]T-\delta,T[$, where ψ_R is as in the proof of Lemma 2.2 and $\xi_m(t)=\int_t^T a_1-(n-m)b_1$. Using this, we may utilise Lemma 2.2 to uniformly bound the right-hand side of 2.19 in terms of the volume of a pulled-back ball at an earlier time, thus obtaining a local version of Magni, Mantegazza and Tsatis' formula. This approach was taken by Ecker [5, Proposition 4.17] in the case of a Euclidean target.

We now use (2.19) to establish an estimate that shall prove useful in guaranteeing the finiteness of the integrals in our heat ball formulæ; such an estimate was established by Ecker [4, Lemma 1.2] in the case of a Euclidean target.

Lemma 2.4. *Fix* $\kappa \in \left]0, \min\{j_0, \sqrt{4\gamma\delta}\}\right[$ *with* γ *as in Lemma 2.2 and set* $\alpha = \sqrt{\frac{2\gamma}{\pi}}$. *For all* $r < \min\{1, \frac{\kappa}{2\alpha}\}$, *the estimate*

$$\int_{\underline{B}^t_{R_r(t-T)}(X)} \mathrm{dvol}_a(\cdot, t) \leq \widetilde{C}_1 \frac{(R_r(t-T))^m}{r^m} \int_{\underline{B}_\kappa^{T-\frac{\kappa^2}{4\gamma}}(X)} \mathrm{dvol}_a(\cdot, T - \frac{\kappa^2}{4\gamma}) \tag{2.20}$$

holds for $t \in \left]T - \frac{e^{-\frac{1}{2m}}}{4\pi} r^2, T\right[$, *where* $\widetilde{C}_1 \geq 0$ *is a constant depending only on* m *and the local geometry of* M *about* (X, T) *and* $R_r(s) = \sqrt{2ms \log\left(\frac{-4\pi s}{r^2}\right)}$.

Proof. Let $\Phi : \mathcal{D}_{j_0,\delta}(X, T) \to \mathbb{R}$ be defined by $\Phi(z, t) = (4\pi(\sigma - t))^{\frac{n-m}{2}} \Gamma_{(X,\sigma)}(z, t)$ with $\sigma \geq T$ to be fixed. We recall the estimates (2.6) together with the explicit constants and functions in (2.7). Using these in (2.19) with $R = \alpha r$ gives us

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{\xi_m(t)} \int_S \psi_{\alpha r} \cdot F^*\Phi \ \mathrm{dvol}_a(\cdot, t)\right] \\
&\leq (a_0 - (n-m)b_0) e^{\sup|\xi_m|} \cdot \int_{\underline{B}^t_{\alpha r}(X)} \mathrm{dvol}_a(\cdot, t) \\
&\leq 16(a_0 - (n-m)b_0) e^{\sup|\xi_m| + \frac{m\lambda_\infty \kappa^2}{8\gamma}} \int_{\underline{B}_\kappa^{T-\frac{\kappa^2}{4\gamma}}(X)} \mathrm{dvol}_a(\cdot, T - \frac{\kappa^2}{4\gamma})
\end{aligned}$$

on $]T - \delta, T[$, where the last line follows from an application of Lemma 2.2, noting that $\alpha r < \frac{\kappa}{2}$. Integrating on $\left]T - \frac{R^2}{4\pi}, t_0\right[$ for $t_0 \in \left]T - \frac{R^2}{4\pi}, T\right[$ fixed and crudely estimating e^{ξ_m} from above and below then yields the inequality

$$\begin{aligned}
&\int_S \psi_{\alpha r} \cdot F^*\Phi \ \mathrm{dvol}_a(\cdot, t_0) \\
&\leq e^{2\sup|\xi_m|}\left(\int_S \psi_{\alpha r} \cdot F^*\Phi \ \mathrm{dvol}_a(\cdot, T - \frac{R^2}{4\pi}) + \widetilde{c}_1 \int_{\underline{B}_\kappa^{T-\frac{\kappa^2}{4\gamma}}(X)} \mathrm{dvol}_a(\cdot, T - \frac{\kappa^2}{4\gamma})\right)
\end{aligned} \tag{2.21}$$

with $\widetilde{c}_1 = \frac{4}{\pi}(a_0 - (n-m)b_0) e^{\frac{m\lambda^+_\infty j_0^2}{8\gamma}}$. On the one hand, we note that since $\sigma \geq T$,

$$\Phi(z, T - \frac{r^2}{4\pi}) = \frac{1}{\left(4\pi(\sigma - T + \frac{r^2}{4\pi})\right)^{\frac{m}{2}}} \exp\left(\frac{d^{g(\cdot, T - \frac{r^2}{4\pi})}(z, X)^2}{4(T - \frac{r^2}{4\pi} - \sigma)}\right) \leq \frac{1}{r^m}$$

for all $z \in M$, whence, since $\psi_{\alpha r}(\cdot, t) \leq \chi_{\underline{B}^t_{\frac{\kappa}{2}}(X)}$, Lemma 2.2 implies that the right-hand side of (2.21) may be bounded from above by

$$\frac{e^{2\sup|\xi_m|}}{r^m} \cdot \left(\max\{16 e^{\frac{m\lambda^+_\infty j_0^2}{8\gamma}}, \widetilde{c}_1\}\right) \int_{\underline{B}_\kappa^{T-\frac{\kappa^2}{4\gamma}}(X)} \mathrm{dvol}_a(\cdot, T - \frac{\kappa^2}{4\gamma}).$$

On the other hand, since

$$\sup_{\left]-\frac{r^2}{4\pi}, 0\right[} R_r \leq \sqrt{\frac{m}{2\pi e}} r, \tag{2.22}$$

we have that $\alpha r \geq 2\sqrt{\frac{m}{\pi}} r \geq 2R_r$ and $\psi_{\alpha r}|_{\underline{B}^t_{\frac{\alpha r}{2}}(X)} \geq \frac{1}{16}$ so that the left-hand side of (2.21) may be bounded from below by

$$\frac{1}{16} \int_{\underline{B}^{t_0}_{R_r(t_0-T)}(X)} F^*\Phi \ \mathrm{dvol}_a(\cdot, t_0). \tag{2.23}$$

Finally, we restrict our attention to $t_0 \in \left]T - e^{-\frac{1}{2m}} \cdot \frac{r^2}{4\pi}, T\right[$. For such t_0, it is clear that

$$(R_r(t_0 - T))^2 \geq T - t_0.$$

We now fix the parameter σ introduced above by setting $\sigma = t_0 + (R_r(t_0 - T))^2$ which, in light of the above, satisfies the requirement $\sigma \geq T$; this implies that for $z \in \underline{B}^{t_0}_{R_r(t_0-T)}(X)$,

$$\frac{d^{g(\cdot,t_0)}(X, x(z, t_0))^2}{4(t_0 - \sigma)} \geq -\frac{1}{4}$$

so that we have the estimate

$$F^*\Phi(\cdot, t_0)|_{\underline{B}^{t_0}_{R_r(t_0-T)}(X)} \geq \frac{e^{-\frac{1}{4}}}{(4\pi)^{\frac{m}{2}} (R_r(t_0 - T))^m}.$$

Using this in (2.23) then implies the result. □

3. Heat balls

We now turn our attention to heat balls, the sets on which the integrals occurring in our local monotonicity formulæ will be evaluated. The approach taken here was first introduced in [6] and subsequently adapted by the author of the present paper in [1] to obtain local monotonicity formulæ for the harmonic map and Yang-Mills heat flows.

Fix $\Phi \in C^{1,1}(\mathcal{D}, \mathbb{R}^+)$ with $\mathcal{D} \subset M \times]0, T[$ open. Setting

$$E_r(\Phi) = \{\Phi > r^{-m}\} = \{\log(r^m\Phi) > 0\} \subset \mathcal{D}$$

for $r > 0$ and writing $\phi = \log \Phi$ and $\phi^m_r = \log(r^m\Phi)$, we say that $E_r(\Phi)$ is an (m, Φ)-*heat ball* (or simply a *heat ball*) for $r < r_0$ if there exists an $r_0 \in]0, 1[$ such that the following properties hold:

(HB1) $E_{r_0}(\Phi) \cap (M \times]0, \tau[) \Subset \mathcal{D}$ for every $\tau \in]0, T[$;
(HB2) $|\nabla\phi|^2, \partial_t\phi \in L^1(E_{r_0}(\Phi))$; and
(HB3) $\displaystyle\lim_{\tau \nearrow T} \int_{\{\Phi(\cdot,\tau) > \frac{1}{r_0^m}\}} |\phi| \ \mathrm{dvol}_g(\cdot, \tau) = 0.$

The following was established in [1, §3.1]:

Lemma 3.1. *Suppose M has locally controlled geometry about (X, T) as in §2.1 and let $\Phi : \mathcal{D}_{j_0,\delta}(X, T) \to \mathbb{R}^+$ be defined such that $\Phi(x, t) = (4\pi(T - t))^{\frac{n-m}{2}} \Gamma_{(X,T)}(x, t)$. Then $E_r(\Phi)$ is a heat ball for $r < r_0 = \frac{1}{2} \min\{j_0\sqrt{\frac{2\pi e}{m}}, \sqrt{4\pi\delta}, 1\}$. Moreover, it is given* *explicitly by*

$$E_r(\Phi) = \bigcup_{t\in \left]T-\frac{r^2}{4\pi},T\right[} B^{g(\cdot,t)}_{R_r(t-T)}(X)\times\{t\} \tag{3.1}$$

with $R_r(s)=\sqrt{2ms\log\left(\frac{-4\pi s}{r^2}\right)}$.

The heat ball described in Lemma 3.1 is known as a *Euclidean heat ball.* We now show that the pull-back of a Euclidean heat ball by mean curvature flow is itself a heat ball in $S\times]0,T[$ under the same assumptions on the local geometry of M. This generalises a result due to Ecker [4] for Euclidean targets.

Lemma 3.2. *Suppose M and Φ are as in Lemma 3.1 and fix $\kappa\in \left]0,\min\{j_0,\sqrt{4\gamma\delta}\}\right[$ with γ as in Lemma 2.2. Then $F^{-1}E_r(\Phi)=E_r(F^*\Phi)$ is a heat ball in $S\times]0,T[$ for $r<r_0:=\min\{\frac{\kappa}{2\alpha},1\}$.*

Proof. We verify the conditions and freely use the notation of §2.5. First note that r_0 does not exceed the value of r_0 given in Lemma 3.1.

(HB1) By Lemma 3.1,

$$\overline{E_{r_0}(\Phi)\cap \mathrm{pr}_2^{-1}\left(]T-\delta,\tau[\right)}\Subset \mathcal{D}_{j_0,\delta}(X,T),$$

for $\tau\in]0,T[$ which thus implies that

$$\overline{E_{r_0}(F^*\Phi)\cap \mathrm{pr}_2^{-1}(]T-\delta,\tau[)}\subset F^{-1}\left(\overline{E_{r_0}(\Phi)\cap \mathrm{pr}_2^{-1}\left(]T-\delta,\tau[\right)}\right)\Subset S\times]0,T[$$

by properness.

(HB2) It is clear from (3.1) that

$$\begin{aligned} E_{r_0}(F^*\Phi) &= F^{-1}\left(\bigcup_{t\in \left]T-\frac{r_0^2}{4\pi},T\right[} B^{g(\cdot,t)}_{R_{r_0}(t-T)}(X)\times\{t\}\right)\\ &= \bigcup_{t\in \left]T-\frac{r_0^2}{4\pi},T\right[} \underline{B}^{t}_{R_{r_0}(t-T)}(X)\times\{t\}. \end{aligned}\tag{3.2}$$

Now, we note that, by the chain rule, the Cauchy-Schwarz inequality and Young's inequality, the inequality

$$|\partial_t F^*\phi| = |F^*\partial_t\phi+\left\langle \underline{\underline{\mathrm{H}}},\underline{\underline{\nabla_g\phi}}\right\rangle| \le |F^*\partial_t\phi|+\frac{1}{2}\left(|\underline{\underline{\mathrm{H}}}|^2+F^*|\nabla_g\phi|^2\right)$$

holds. Moreover, it is clear that $|\nabla_a F^*\phi|\le F^*|\nabla_g\phi|$; hence, in view of these two inequalities, (3.2) and the gradient and time-derivative bounds in Proposition 2.1, it suffices to show that

$$\int_{T-\frac{r_0^2}{4\pi}}^{T}\int_{\underline{B}^t_{R_{r_0}(t-T)}(X)} \frac{\underline{\mathfrak{r}}^2}{(T-t)^2}\mathrm{dvol}_a(\cdot,t)\,\mathrm{d}t<\infty, \tag{3.3}$$

where we set $\mathfrak{r}(x,t) := d^{g(\cdot,t)}(X,x)$ and $\underline{\mathfrak{r}} := F^*\mathfrak{r}$, and

$$\int_{T-\frac{r_0^2}{4\pi}}^{T} \int_{\underline{B}^t_{R_{r_0}(t-T)}(X)} \frac{1}{T-t} \mathrm{dvol}_a(\cdot,t)\,\mathrm{d}t < \infty \tag{3.4}$$

since, by Lemma 2.2,

$$\int_{T-\frac{R^2}{4\gamma}}^{T} \int_{\underline{B}^t_{\sqrt{\frac{m}{2\pi e}}r_0}(X)} |\underline{\underline{\mathrm{H}}}|^2\,\mathrm{dvol}_a(\cdot,t)\,\mathrm{d}t \le 16 e^{\frac{m\lambda_\infty \kappa^2}{8\gamma}} \left(\int_{\underline{B}_R(X)} \mathrm{dvol}_a \right) (T - \frac{R^2}{4\gamma}) < \infty,$$

which establishes that $|\underline{\underline{\mathrm{H}}}|^2 \in L^1\left(E_{r_0}(F^*\Phi)\right)$ due to the inclusion

$$E_{r_0}(F^*\Phi) \cap \mathrm{pr}_2^{-1}(]T - \frac{R^2}{4\gamma}, T[) \subset \bigcup_{t \in\ ]T-\frac{R^2}{4\gamma},T[} \underline{B}^t_{\sqrt{\frac{m}{2\pi e}}r_0}(X) \times \{t\}$$

and the fact that $E_{r_0}(F^*\Phi) \cap \mathrm{pr}_2^{-1}(]T - \frac{r_0^2}{4\pi}, T - \frac{R^2}{4\gamma}])$ is relatively compact in the domain of F by (HB1).

Now, in light of Lemma 2.4, the estimate

$$\int_{\underline{B}^t_{R_{r_0}(t-T)}(X)} \frac{\underline{\mathfrak{r}}^2}{(T-t)^2} \mathrm{dvol}_a(\cdot,t) \le \widetilde{C}_1 \int_{\underline{B}_\kappa^{T-\frac{\kappa^2}{4\gamma}}(X)} \mathrm{dvol}_a(\cdot, T - \frac{\kappa^2}{4\gamma}) \cdot \frac{(R_{r_0}(t-T))^{m+2}}{(T-t)^2},$$

holds for $t \in]\tau, T[$ for $\tau = T - \frac{\exp(-\frac{1}{2m})}{4\pi} r_0^2$ with $\widetilde{C}_1$ a geometric constant, and likewise the estimate

$$\int_{\underline{B}^t_{R_{r_0}(t-T)}(X)} \frac{1}{T-t} \mathrm{dvol}_a(\cdot,t) \le \widetilde{C}_1 \int_{\underline{B}_\kappa^{T-\frac{\kappa^2}{4\gamma}}(X)} \mathrm{dvol}_a(\cdot, T - \frac{\kappa^2}{4\gamma}) \cdot \frac{(R_{r_0}(t-T))^{m}}{T-t}$$

for $t \in]\tau, T[$, but both right-hand sides are summable on $]\tau, T[$ as functions of t, which together with (HB1) then implies the statements (3.3) and (3.4).

(HB3) In light of (3.2), $E_{r_0}(F^*\Phi) \cap (M \times \{\tau\}) = \underline{B}^\tau_{R_{r_0}(\tau-T)}(X) \times \{\tau\}$. On the other hand,

$$\begin{aligned} |F^*\phi(\cdot,\tau)| &= \left| \frac{\underline{r}^2}{4(\tau - T)} - \frac{m}{2}\log\left(4\pi(T-\tau)\right) \right| \\ &\le \frac{\left(R_{r_0}(\tau-T)\right)^2}{4(T-\tau)} + \frac{m}{2}\left(-\log(4\pi(T-\tau))\right) \end{aligned}$$

on $\underline{B}^\tau_{R_{r_0}(\tau-T)}(X)$. Therefore, making use of Lemma 2.4 again as in (HB2), we see that it suffices to show that

$$\lim_{\tau \nearrow T} \frac{\left(R_{r_0}(\tau-T)\right)^{m+2}}{T-\tau} = 0$$

and

$$\lim_{\tau \nearrow T} \left(R_{r_0}(\tau - T)\right)^m \log\left(4\pi(T-\tau)\right) = 0$$

or, more explicitly,

$$\lim_{\tau \nearrow T} \sqrt{(\tau - T)^m \left[\log\left(\frac{4\pi(T-\tau)}{r_0^2} \right) \right]^{m+2}} = 0$$

and

$$\lim_{\tau \nearrow T} \sqrt{(\tau - T)^m \left[\log(4\pi(T-\tau))\right]^{m+2} - (\tau - T)^m \log(r_0^2)} = 0.$$

These assertions follow from a suitable change of variables, i.e. by noting that

$$\lim_{\tau \nearrow T} (\tau - T)^m \left[\log\left(\frac{4\pi(T-\tau)}{r_0^2}\right)\right]^{m+2} = \left(\frac{r_0^2}{4\pi}\right)^m \lim_{s\to\infty} s^{m+2}\exp(-ms) = 0. \quad \square$$

$\square$

Remark 3.3. We note that the approach taken to heat balls in the preceding lemma is different from that taken by Ecker [4] in that heat balls were considered as subsets of the *parameter space* $S \times]0,T[$ as opposed to being subsets of $M \times]0,T[$. In our setting, both approaches are equivalent. However, Ecker's approach more readily generalises to the varifold setting of Brakke [3].

For later purposes, we shall need to suitably modify the heat ball introduced in Lemma 3.2 in order to extract a monotone quantity from (1.1). The following lemma guarantees that our modifications again yield heat balls.

Lemma 3.4. *Suppose $E_r(\Phi)$ is a heat ball in $M \times]0,T[$ for $r < r_0$ and let $\eta \in L^\infty(E_{r_0}(\Phi)) \cap C^1(E_{r_0}(\Phi))$ with $|\nabla_g \eta|^2, \partial_t \eta \in L^1(E_{r_0}(\Phi))$. Define $\widetilde{\Phi} : E_{r_0}(\Phi) \to \mathbb{R}^+$ by $\widetilde{\Phi} = e^\eta \cdot \Phi|_{E_{r_0}(\Phi)}$. Then $E_r(\widetilde{\Phi})$ is a heat ball for $r < \widetilde{r}_0 := r_0 \exp(-\sup|\eta|/m)$ and the inclusions*

$$E_{re^{\frac{-\sup|\eta|}{m}}}(\Phi) \subset E_r(\widetilde{\Phi}) \subset E_{re^{\frac{\sup|\eta|}{m}}}(\Phi) \tag{3.5}$$

hold for all $r < \widetilde{r}_0$.

Proof. We first note that the inclusions (3.5) follow from the inequality

$$e^{-\sup|\eta|}\Phi \le \widetilde{\Phi} \le e^{\sup|\eta|}\Phi.$$

Moreover, we have $E_{\widetilde{r}_0}(\widetilde{\Phi}) \subset E_{r_0}(\Phi)$. We now verify (HB1)-(HB3).

(HB1) (3.5) immediately implies that

$$E_{\widetilde{r}_0}(\widetilde{\Phi}) \cap \mathrm{pr}_2^{-1}(]0,\tau[) \subset E_{r_0}(\Phi) \cap \mathrm{pr}_2^{-1}(]0,\tau[) \Subset \mathcal{D}$$

for every $\tau \in]0,T[$.

(HB2) If $\widetilde{\phi} := \log \widetilde{\Phi}$, then $\widetilde{\phi} = \phi + \eta$, whence, in view of (3.5) and the following remark, the assumptions on ϕ and η imply that $\partial_t \widetilde{\phi} = \partial_t \phi + \partial_t \eta \in L^1(E_{\widetilde{r}_0}(\widetilde{\Phi}))$ and, since $|\nabla(\phi+\eta)|^2 \le 2\left(|\nabla\phi|^2 + |\nabla\eta|^2\right)$, we also have that $|\nabla(\widetilde{\phi})|^2 \in L^1(E_{\widetilde{r}_0}(\widetilde{\Phi}))$.

(HB3) By (3.5), it suffices to show that

$$\lim_{\tau \nearrow T} \int_{\mathrm{pr}_1\left(E_{r_0}(\Phi) \cap \mathrm{pr}_2^{-1}(\{\tau\})\right)} |\widetilde{\phi}| \, \mathrm{dvol}_g(\cdot,\tau) = 0,$$

but $|\widetilde{\phi}| \le |\phi| + |\eta| \le |\phi| + \underbrace{\sup|\eta|}_{=:G}$, whence, writing $\mathrm{Vol}_{g(\cdot,t)}$ for the Riemannian volume measure of $g(\cdot,t)$ considered as a set function,

$$\lim_{\tau \nearrow T} \int_{\mathrm{pr}_1\left(E_{r_0}(\Phi) \cap \mathrm{pr}_2^{-1}(\{\tau\})\right)} |\widetilde{\phi}| \, \mathrm{dvol}_g(\cdot,\tau)$$

$$\leq \lim_{\tau \nearrow T} \int_{\mathrm{pr}_1\left(E_{r_0}(\Phi)\cap \mathrm{pr}_2^{-1}(\{\tau\})\right)} |\phi| \,\mathrm{dvol}_g(\cdot,\tau) + G \lim_{\tau\nearrow T} \mathrm{Vol}_{g(\cdot,\tau)}\left(\mathrm{pr}_1\left(E_{r_0}(\Phi)\cap (M\times\{\tau\})\right)\right)$$
$$= 0,$$

where the last step follows from [1, Remark 3.4] and the fact that Φ satisfies (HB3).

□

Suppose $E_r(\Phi)$ is a heat ball for $r < r_0$. We now recall some integration-by-parts formulæ that shall be required for the derivation of our local monotonicity formulæ. To that end, we introduce the following approximation scheme: Fix a function $\chi \in C^2(\mathbb{R}, [0,1])$ such that $\chi|_{]-\infty,\frac{1}{2}]} \equiv 0$, $\chi' \geq 0$ and $\chi|_{[1,\infty[} \equiv 1$, and define for each $q \in \mathbb{N}$ the function $\chi_q(x) = \chi(2^q x)$. It then follows that χ_q approximates the characteristic function $\chi_{]0,\infty[}$ and $|x\chi_q'(x)| \leq C\chi_{]2^{-(q+1)},2^{-q}[} \xrightarrow{q\to\infty} 0$ for all $x \in \mathbb{R}$ and some absolute constant $C > 0$. Now, for any $f \in L^1(\mathcal{D})$, we introduce the *approximate* integral

$$J_q^r(f) := \iint f \cdot (\chi_q \circ \phi_r^m) \cdot (\chi_{]0,T-q^{-1}[} \circ \mathrm{pr}_2)\,\mathrm{d}\mu,$$

where we write μ for the Borel measure induced by $\mathrm{dvol}_{g(\cdot,t)}$ and Lebesgue measure; this integral approximates the heat ball integral

$$I^r(f) := \iint_{E_r(\Phi)} f\,\mathrm{d}\mu$$

as well as its derivative with respect to r for $q \to \infty$ (see [6] and [1]).

The integration-by-parts formulæ we shall need are summarised in the following proposition.

Proposition 3.5. *Suppose $X : E_{r_0}(\Phi) \to TM$ is a C^1 time-dependent vector field and $|X|^2 \in L^1$, $f \in C^1(E_{r_0}(\Phi))$ with $\partial_t f \in L^1$ and $\mathrm{tr}_g \partial_t g \in L^1$. The following implications hold for almost every $r \in]0, r_0[$:*

(i) $\mathrm{div} X \in L^1 \Rightarrow \dfrac{\mathrm{d}}{\mathrm{d}r} \displaystyle\iint_{E_r(\Phi)} \langle X, \nabla\phi\rangle \,\mathrm{d}\mu = -\frac{m}{r} \iint_{E_r(\Phi)} \mathrm{div}\ X\,\mathrm{d}\mu;$

(ii) $f, f\phi_r^m \in L^1 \Rightarrow \dfrac{\mathrm{d}}{\mathrm{d}r} \displaystyle\iint_{E_r(\Phi)} f\,\phi_r^m\,\mathrm{d}\mu = \frac{m}{r} \iint_{E_r(\Phi)} f\,\mathrm{d}\mu;$

(iii) $f \in L^\infty,\ \phi_{r_0}^m \cdot \mathrm{tr}_g \partial_t g \in L^1 \Rightarrow \displaystyle\iint_{E_r(\Phi)} \partial_t\left(f\cdot\phi_r^m\right)\mathrm{d}\mu = -\iint_{E_r(\Phi)} f\cdot\phi_r^m\cdot\frac{1}{2} tr_g \partial_t g\,\mathrm{d}\mu;$

and

(iv) $\mathrm{div}\ X \in L^\infty \Rightarrow \displaystyle\iint_{E_r(\Phi)} \mathrm{div}(X\cdot\phi_r^m)\,\mathrm{d}\mu = 0.$

Proof. (i) This was established in [1, Proposition 3.9] and amounts to proving that

$$\frac{\mathrm{d}}{\mathrm{d}r} J_q^r(\langle X, \nabla\phi\rangle) = -\frac{m}{r} J_q^r(\mathrm{div}\ X)$$

and noting that the limit and derivative may be interchanged due to [1, Lemma 3.8].

We proceed similarly to establish the remaining assertions:

(ii) We first compute that

$$\frac{\mathrm{d}}{\mathrm{d}r}J_q^r(f\cdot\phi_r^m)=\frac{m}{r}J_q^r(f)+\frac{m}{r}\iint f\cdot(\phi_r^m\cdot\chi_q'\circ\phi_r^m)\cdot(\chi_{]0,T-q^{-1}[}\circ\mathrm{pr}_2)\,\mathrm{d}\mu. \tag{3.6}$$

It follows from the definition of χ_q that

$$|\phi_r^m\cdot\chi_q'\circ\phi_r^m|\leq C\chi_{]2^{-(q+1)},2^{-q}[}\circ\phi_r^m\xrightarrow{q\to\infty}0$$

for an absolute constant $C>0$ so that the latter term on the right-hand side tends to 0 as $q\to\infty$ uniformly in $r\in[r_1,r_2]$. Taking limits and noting the summability conditions on f and $f\cdot\phi_r^m$ then yields the result.

(iii) We integrate by parts with respect to t to obtain

$$\begin{aligned}J_q^r(\partial_t(f\cdot\phi_r^m))=&-\iint f\cdot\phi_r^m\cdot\chi_q'\circ\phi_r^m\cdot\chi_{]0,T-q^{-1}[}\circ\mathrm{pr}_2\cdot\partial_t\phi\,\mathrm{d}\mu\\&-\iint f\cdot\phi_r^m\cdot\chi_q\circ\phi_r^m\cdot\chi_{]0,T-q^{-1}[}\circ\mathrm{pr}_2\cdot\frac{1}{2}\mathrm{tr}_g\partial_t g\ \mathrm{d}\mu\\&+\int f\cdot\phi_r^m\cdot(\chi_q\circ\phi_r^m)\mathrm{dvol}_g(\cdot,T-q^{-1})\\&-\int f\cdot\phi_r^m\cdot(\chi_q\circ\phi_r^m)\mathrm{dvol}_g(\cdot,0)\end{aligned} \tag{3.7}$$

As in the proof of (ii), the definition of χ_q immediately implies that the first integral on the right-hand side tends to 0 as $q\to\infty$. Furthermore, the third integral tends to 0 as $q\to\infty$ due to (HB3), and the fourth integral is equal to 0 due to (HB1). Finally, the second integral is equal to $-J_q^r(f\cdot\phi_r^m\cdot\frac{1}{2}\mathrm{tr}_g\partial_t g)$ which tends to $-I_r(f\cdot\phi_r^m\cdot\frac{1}{2}\mathrm{tr}_g\partial_t g)$ as $q\to\infty$ due to the summability conditions imposed.

(iv) Similarly to the preceding proof, we integrate by parts to compute that

$$J_q^r(\mathrm{div}(X\cdot\phi_r^m))=-\iint\langle X,\nabla\phi\rangle\cdot(\phi_r^m\cdot\chi_q'\circ\phi_r^m)\cdot\chi_{]0,T-q^{-1}[}\circ\mathrm{pr}_2\ \mathrm{d}\mu=o(1)$$

$$\begin{aligned}J_q^r(\mathrm{div}(X\cdot\phi_r^m))&=-\iint\langle X,\nabla(\chi_q\circ\phi_r^m)\rangle\cdot\phi_r^m\cdot(\chi_{]t_0-\delta_0,t_0-q^{-1}[}\circ\mathrm{pr}_2)\,\mathrm{d}\mu\\&=-\iint\langle X,\nabla\phi\rangle\cdot(\phi_r^m\cdot\chi_q'\circ\phi_r^m)\cdot\chi_{]t_0-\delta_0,t_0-q^{-1}[}\circ\mathrm{pr}_2\ \mathrm{d}\mu\\&=o(1)\end{aligned}$$

as $q\to\infty$ on account of the definition of χ_q and the summability conditions on X and $\nabla\phi$.

□

4. Monotonicity formulæ and inequalities

We now turn our attention to local monotonicity formulæ for the mean curvature flow. To this end, suppose that $\Phi\in C^{2,1}(\mathcal{D},\mathbb{R}^+)$ with $\mathcal{D}\subset M\times]0,T[$ is such that $E_r(F^*\Phi)$ is a heat ball for $r<r_0$ and set $\phi=\log\Phi$ and $\phi_r^m=\log(r^m\Phi)$. The following

theorem amounts to the main result with the addition of an auxiliary function u. We assume that $\mathcal{Q}_T$ and $\mathcal{S}_\phi$ are as in Theorem A and write μ for the Borel measure on $S \times [0,T[$ induced by $\mathrm{dvol}_a(\cdot,t)$ and Lebesgue measure.

Theorem 4.1. *Suppose the family of embeddings* $\{x(\cdot,t) : S \to (M, g(\cdot,t))\}_{t\in[0,T[}$ *evolves by mean curvature flow. If* $u \in C^{2,1}(E_{r_0}(F^*\Phi))$ *and* $\frac{u}{T-t}F^*\phi \in L^1(E_{r_0}(F^*\Phi))$, *then*

$$\left[\frac{1}{r^m}\iint\limits_{E_r(F^*\Phi)} u\left[|\nabla_a F^*\phi|^2 + \left(|\underline{\underline{\mathrm{H}}}|^2 - \frac{1}{2}\mathrm{tr}_a x^*\partial_t g\right)F^*\phi_r^m\right]\,\mathrm{d}\mu\right]_{r=r_1}^{r=r_2}$$
$$= \int_{r_1}^{r_2}\frac{m}{r^{m+1}}\iint\limits_{E_r(F^*\Phi)}\left(-u\cdot F^*\left(\frac{(\mathrm{L}_g^* + \frac{n-m}{2(T-t)})\Phi}{\Phi}\right) - F^*\phi_r^m\cdot \mathrm{L}_a u\right.$$
$$\left. + u\,|\mathcal{S}_\phi|^2 + u\cdot \mathrm{tr}_g^{\perp}\mathcal{Q}_T(\phi,g)\right)(z,t)\;\mathrm{d}\mu(z,t)\,\mathrm{d}r \quad (4.1)$$

for $0 < r_1 < r_2 < r_0$ *provided both spacetime integrands are in* $L^1(E_{r_0}(F^*\Phi))$. *If* $u \geq 0$, *then the condition* $\frac{u}{T-t}F^*\phi \in L^1(E_{r_0}(F^*\Phi))$ *may be lifted so that the the identity (4.1) holds with* $\geq$ *in place of* $=$.

Remark 4.2. In light of the remarks in §1 regarding Theorem A, which we may recover by setting $u = 1$, this identity implies a monotonicity formula if u is a nonnegative subsolution to the heat equation on $\{(S, a(\cdot,t))\}_{t\in[0,T[}$, $\Phi(\cdot,t) = (T-t)^{\frac{n-m}{2}}\mathrm{P}(\cdot,t)$ for a positive subsolution P of the backward heat equation and if the matrix Harnack form $\mathcal{Q}_T(\phi,g)$ is nonnegative definite, which in particular holds on gradient shrinking Ricci solitons with potential $-\phi$ and $T_\infty = T$, which includes $(M,g_t) \equiv (\mathbb{R}^n,\delta)$ with $\Phi(\cdot,t) = (T-t)^{\frac{n-m}{2}}\Gamma_{(X,T)}(\cdot,t)$ as a special case. Moreover, $\mathcal{S}_\phi$ vanishes if and only if x is a gradient soliton (up to tangential diffeomorphisms) with potential $-\phi$; in particular, if $(M, g(\cdot,t)) \equiv (\mathbb{R}^n,\delta)$, then $\mathcal{S}_\phi \equiv 0$ if and only if x is a homothetic shrinker (see [5, Ch. 2]).

Remark 4.3. If u is bounded on $E_{r_0}(F^*\Phi)$ and Φ is as in Lemma 3.2, then the estimates derived in that lemma for establishing (HB2) imply that the integrals of (4.1) are finite. In particular, if $(M,g_t) \equiv (\mathbb{R}^n,\delta)$, we recover Ecker's formula [4].

Remark 4.4. In the special case where $S = M$ and $x(\cdot,t)$ is the identity map for all t, we have that $\underline{\underline{\mathrm{H}}} \equiv 0$ so that (4.1) reduces to the identity

$$\left[\frac{1}{r^n}\iint_{E_r(\Phi)} u\left(|\nabla\phi|^2 - \frac{1}{2}\mathrm{tr}_g\partial_t g\right)\mathrm{d}\mu\right]_{r=r_1}^{r=r_2}$$
$$= -\int_{r_1}^{r_2}\frac{m}{r^{m+1}}\iint_{E_r(\Phi)} u\cdot\left(\frac{\mathrm{L}_g^*\Phi}{\Phi}\right) + \phi_r^m\cdot \mathrm{L}_a u\;\mathrm{d}\mu\;\mathrm{d}r,$$

which was established by Ecker, Knopf, Ni and Topping [6]. Therefore, our formula may be viewed as a generalisation of theirs.

Proof of Theorem 4.1. We first assume that $u(\cdot,t)\equiv 0$ for $t\in[\tau_0,T[$ and approximate using the scheme introduced in §3. Set

$$i_0(u)=\left(u\cdot\left[|\nabla_a F^*\phi|^2+F^*\phi_r^m\cdot\left(|\underline{\underline{\mathrm{H}}}|^2-\frac{1}{2}\mathrm{tr}_a x^*\partial_t g\right)\right]\right).$$

By the fundamental theorem of calculus, we have that

$$\left[\frac{1}{r^m}J_q^r\left(i_0(u)\right)\right]_{r=r_1}^{r=r_2}=\int_{r_1}^{r_2}\frac{m}{r^{m+1}}J_q^r\left(-i_0(u)\right)\mathrm{d}r+\int_{r_1}^{r_2}\frac{1}{r^m}\frac{\mathrm{d}}{\mathrm{d}r}J_q^r\left(i_0(u)\right)\mathrm{d}r.$$

We note that, since u vanishes near T, each individual term in the approximate integrals is summable over $E_{r_0}(F^*\Phi)$, thus allowing us to freely separate these integrals. To calculate the latter integral on the right-hand side, we apply Proposition 3.5(i) with $X=u\nabla_a F^*\phi$ and Proposition 3.5(ii) with $f(z,t)=u\cdot\left(|\underline{\underline{\mathrm{H}}}|^2-\frac{1}{2}\mathrm{tr}_a x^*\partial_t g\right)(z,t)$ to obtain

$$\frac{\mathrm{d}}{\mathrm{d}r}J_q^r\left(i_0(u)\right)=\frac{m}{r}J_q^r\left(u\cdot\left(|\underline{\underline{\mathrm{H}}}|^2-\frac{1}{2}\mathrm{tr}_a x^*\partial_t g\right)-\langle\nabla_a u,\nabla_a F^*\phi\rangle-u\Delta_a F^*\phi\right)+o(1)$$

as $q\to\infty$, where the remainder term may be bounded from above uniformly in $r\in\,]r_1,r_2[$. Thus, by the dominated convergence theorem, taking the limit $q\to\infty$ in the above yields

$$\begin{aligned}&\left[\frac{1}{r^m}\iint_{E_r(F^*\Phi)}u\cdot\left[|\nabla_a F^*\phi|^2+F^*\phi_r^m\cdot\left(|\underline{\underline{\mathrm{H}}}|^2-\frac{1}{2}\mathrm{tr}_a x^*\partial_t g\right)\right]\mathrm{d}\mu\right]_{r=r_1}^{r=r_2}\\
&\quad=\int_{r_1}^{r_2}\Bigg(\frac{m}{r^{m+1}}\iint_{E_r(F^*\Phi)}-u\cdot\left(|\nabla_a F^*\phi|^2+F^*\phi_r^m\cdot\left(|\underline{\underline{\mathrm{H}}}|^2-\frac{1}{2}\mathrm{tr}_a x^*\partial_t g\right)\right)\\
&\qquad\qquad-\langle\nabla_a u,\nabla_a F^*\phi\rangle-u\Delta_a F^*\phi\\
&\qquad\qquad+u\cdot\left(|\underline{\underline{\mathrm{H}}}|^2-\frac{1}{2}\mathrm{tr}_a x^*\partial_t g\right)\mathrm{d}\mu\Bigg)\mathrm{d}r,\end{aligned}\tag{4.2}$$

A straightforward computation using the relations (2.13) and the fact that $\mathrm{tr}_g^\perp g=n-m$ shows that on $S\times\{t\}$ the right-hand space-time integrand of (4.2) is equal to

$$\begin{aligned}&-u\cdot F^*\left(\frac{(\mathrm{L}_g^*+\frac{n-m}{2(T-t)})\Phi}{\Phi}\right)-F^*\phi_r^m\cdot\mathrm{L}_a u+u\,|\mathcal{S}_\phi|^2+u\cdot\mathrm{tr}_g^\perp\mathcal{Q}_T(\phi,g)\\
&\qquad-\mathrm{div}_a(F^*\phi_r^m\cdot\nabla_a u)+\partial_t(F^*\phi_r^m u)+F^*\phi_r^m\cdot u\cdot\left(|\underline{\underline{\mathrm{H}}}|^2-\frac{1}{2}\mathrm{tr}_a x^*\partial_t g\right)\end{aligned}\tag{4.3}$$

On the one hand, the divergence theorem-type identity in Proposition 3.5 (iv) implies that the integral over $E_r(F^*\Phi)$ of the first term on the second line of (4.3) vanishes. On the other hand, since $\frac{1}{2}\mathrm{tr}_a\partial_t a=|\underline{\underline{\mathrm{H}}}|^2-\frac{1}{2}\mathrm{tr}_a x^*\partial_t g$, Proposition 3.5 (iii) implies that the integral over $E_r(F^*\Phi)$ of the following terms is zero. This therefore establishes the result in the case where u vanishes close to T.

Now, consider $u_l:E_{r_0}(F^*\Phi)\to\mathbb{R}$ defined by $u_l(z,t)=\chi_l(T-t)\cdot u(z,t)$ for $l\in\mathbb{N}$, where χ_l is as defined in §3. Denoting the right-hand spacetime integrand of (4.1) by

$i_1(u)$, (4.1) holds with u_l in place of u and

$$
\begin{aligned}
&\left[\frac{1}{r^m}\iint\limits_{E_r(F^*\Phi)} \chi_l(T-t)\cdot i_0(u)(z,t)\,\mathrm{d}\mu(z,t)\right]_{r=r_1}^{r=r_2} \\
&= \int_{r_1}^{r_2}\left(\frac{m}{r^{m+1}}\iint\limits_{E_r(F^*\Phi)} \chi_l(T-t)\cdot i_1(u)(z,t)\,\mathrm{d}\mu(z,t)\right)\mathrm{d}r \\
&+ \int_{r_1}^{r_2}\left(\frac{m}{r^{m+1}}\iint_{E_r(F^*\Phi)} u(z,t)\cdot\frac{1}{T-t}F^*\phi_r^m(z,t)\cdot\chi_l'(T-t)\cdot(T-t)\,\mathrm{d}\mu(z,t)\right)\mathrm{d}r.
\end{aligned}
\tag{4.4}
$$

Since $0\le\chi_l(T-t)\le 1$, the first two integrands may be bounded in absolute value from above by the absolute values of the corresponding integrands occurring in the statement of this theorem, which are assumed summable. Thus, we may pass to the limit $l\to\infty$ in the first two integrals of identity (4.4), and if $\frac{1}{T-t}F^*\phi_r^m\in L^1\left(E_{r_0}(F^*\Phi)\right)$, then

$$
\left|u\cdot\frac{1}{T-t}F^*\phi_r^m\cdot\chi_l'(T-t)\cdot(T-t)\right|\le C\cdot\left|\frac{u}{T-t}F^*\phi_r^m\right|\in L^1\left(E_{r_0}(F^*\Phi)\right),
$$

allowing us to apply the dominated convergence theorem, which implies that the last integral on the right-hand side vanishes in the limit $l\to\infty$, since $\chi_l'(T-t)\cdot(T-t)\xrightarrow{l\to\infty}0$. Finally, if $u\ge 0$, we may discard the latter integral on the right-hand side by estimating it from below by 0, since $\chi_l'\ge 0$, wherefore the aforementioned limits involving the remaining integrals may be taken, thus establishing the result. $\square$

We now proceed to show that even if right-hand space-time integrand of (4.1) is not nonnegative, but u is a bounded nonnegative subsolution to the heat equation and Φ satisfies suitable inequalities, then we may nevertheless obtain a monotonicity formula by modifying the heat ball as in Lemma 3.4. For this purpose, suppose that Φ satisfies inequalities of the form

$$
\begin{aligned}
\frac{\left(\mathrm{L}_g^*+\frac{n-m}{2(T-t)}\right)\Phi}{\Phi}(y,t)&\le a(t)\\
\mathcal{Q}_T(\phi,g)(y,t)&\ge b(t)g(y,t)
\end{aligned}
\tag{4.5}
$$

for $(y,t)\in E_{r_0}(\Phi)$, where $a,b\in C^0(]0,T[)\cap L^1(]0,T[)$. Thus, the function $\eta: E_{r_0}(\Phi)\to\mathbb{R}$ defined by

$$
\eta(z,t)=\exp\left(\int_t^T a-(n-m)b\right)
\tag{4.6}
$$

gives rise to the bounded, once differentiable function $F^*\eta$ on $E_{r_0}(F^*\Phi)$ so that by Lemma 3.4, $E_r(F^*\widetilde{\Phi})$ is a heat ball for $r<\widetilde{r}_0$, where $\widetilde{\Phi}:=\eta\cdot\Phi|_{E_{r_0}(\Phi)}$ and

$$
\widetilde{r}_0=r_0\exp\left(-\int_0^T|a-(n-m)b|/m\right)
\tag{4.7}
$$

On the other hand, it follows from (4.5) that the inequality

$$\left(\operatorname{tr}_g^{\perp} \mathcal{Q}_T(\log \widetilde{\Phi}, g) - \frac{\left(\mathrm{L}_g^* + \frac{n-m}{2(T-t)}\right)\widetilde{\Phi}}{\widetilde{\Phi}}\right)(z,t) \geq 0$$

holds for $(z,t) \in E_{\widetilde{r}_0}(F^*\widetilde{\Phi})$. Altogether, setting $\widetilde{\phi} = \log \widetilde{\Phi}$ and $\widetilde{\phi}_r^m = \log(r^m \widetilde{\Phi})$, we have the inequality

$$\begin{aligned} &\left[\frac{1}{r^m} \iint\limits_{E_r(F^*\widetilde{\Phi})} u \left[|\nabla_a F^* \widetilde{\phi}|^2 + \left(|\underline{\underline{\mathrm{H}}}|^2 - \frac{1}{2}\operatorname{tr}_a x^* \partial_t g\right) F^* \widetilde{\phi}_r^m\right] \mathrm{d}\mu \mathrm{d}t\right]_{r=r_1}^{r=r_2} \\ &\qquad \geq \int_{r_1}^{r_2} \left(\frac{m}{r^{m+1}} \iint\limits_{E_r(F^*\widetilde{\Phi})} u \cdot \left|\underline{\underline{\mathrm{H}}} - \nabla_g^{\perp} \widetilde{\phi}\right|^2 \mathrm{d}\mu\right) \mathrm{d}r \end{aligned} \tag{4.8}$$

for all $r < \widetilde{r}_0$ whenever both integrands are summable over $E_{\widetilde{r}_0}(F^*\widetilde{\Phi})$.

Now suppose M has locally controlled geometry about (X,T) with constants as in §2.1. It immediately follows from inequalities (2.4) and (2.5) of Proposition 2.1, that the kernel Φ of Lemma 3.1 satisfies inequalities of the form (4.5) with

$$\begin{aligned} a(t) &= ((n-1)\Lambda_\infty + \lambda_{-\infty})^+ \log\left(\frac{1}{(4\pi(T-t))^{m/2}}\right) + \frac{n}{2}\lambda_\infty \\ b(t) &= \Lambda_{-\infty}^- \log\left(\frac{1}{(4\pi(T-t))^{m/2}}\right) + \frac{\lambda_{-\infty}}{2}. \end{aligned} \tag{4.9}$$

Moreover, since $E_{\widetilde{r}_0}(F^*\widetilde{\Phi}) \subset E_{r_0}(F^*\Phi)$ with r_0 as in Lemma 3.1, we see in light of the boundedness of u and $(z,t) \mapsto (\operatorname{tr}_a x(\cdot,t)^* \partial_t g(\cdot,t))(z)$ on $E_{r_0}(F^*\Phi)$ that the space-time integrands of (4.8) are summable if $F^*\phi_r^m$, $|\underline{\underline{\mathrm{H}}}|^2$, $|\nabla_a F^*\phi|^2$ and $F^*|\nabla_g \phi|^2$ are summable on $E_{r_0}(F^*\Phi)$, but this follows immediately from the defining properties (HB2) and (HB3) of a heat ball and the estimates in the proof of Lemma 3.2 (see also [1, Remark 3.4]). Finally, if $(M, g(\cdot,t)) \equiv (\mathbb{R}^n, \delta)$ for all t, then $\frac{F^*\phi}{T-t)} \in L^1(E_r(F^*\Phi))$ for all $r > 0$, the heat ball axioms (HB1)-(HB3) in fact hold for any $r_0 > 0$, $\eta \equiv 1$ and, by Remark 4.2, (4.8) holds with equality; moreover, in the case where $u \equiv 1$, the right-hand side vanishes if and only if x is a homothetic shrinker. This immediately leads to the following statement, which reduces to Theorem B in the case where $u \equiv 1$.

Corollary 4.5. *Suppose $\{x(\cdot,t) : S \to (M, g(\cdot,t))\}_{t\in[0,T[}$ is a family of embeddings evolving by mean curvature flow, M has locally controlled geometry about (X,T), and $u \in C^{2,1}(E_{r_0}(F^*\Phi))$ is a bounded, nonnegative subsolution to the heat equation. Then the monotonicity inequality (4.8) holds for $0 < r_1 < r_2 < \widetilde{r}_0$ with $\widetilde{\Phi} = \eta \cdot \Phi|_{E_{r_0}(\Phi)}$, r_0 and Φ and r_0 as in Lemma 3.2, and $\widetilde{r}_0$ and η determined by the functions (4.9) according to the relations (4.7) and (4.6) respectively. If $(M, g(\cdot,t)) \equiv (\mathbb{R}^n, \delta)$, then $\eta \equiv 1$ and (4.8) holds with equality for all $0 < r_1 < r_2$; furthermore, if $u \equiv 1$, then the right-hand side vanishes if and only if x is a homothetic shrinker.*

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