ENERGY IDENTITIES AND MONOTONICITY FOR EVOLVING k-FORMS ON MOVING RIEMANNIAN SPACES

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Abstract. In this note we establish energy identities for vector bundle-valued k-forms satisfying suitable heat-type equations, where the ambient space is equipped with a simultaneously evolving metric tensor. We show that if this metric is suitably controlled, or a gradient shrinking Ricci soliton, then this energy identity reduces to a monotonicity formula. As special cases, we obtain new montonicity formulæ for the harmonic map and Yang-Mills heat flows in this more general setting, unifying and generalising results due to Struwe, Chen and Struwe, Hamilton, and Chen and Shen in the process.

1. Introduction

In the study of singularity formation in nonlinear geometric heat flows, such as the Yang-Mills, harmonic map heat and mean curvature flows, energy identities play a crucial rôle. In each case, the monotonicity of a suitably weighted energy has implications on the size of the singular set. In the case of the harmonic map heat flow, this approach was taken by Struwe [17], Chen and Struwe [4], Cheng [6], Hamilton [11], and Grayson and Hamilton [9]; in the case of the Yang-Mills flow by Chen and Shen [5], Chen, Shen and Zhou [3], and Hong and Tian [12]; and in the case of the mean curvature flow by Brakke [2], White [19] and Ecker [7]. For the former two flows, monotonicity formulæ are known to hold when the base manifold is either Euclidean space, or a compact manifold with a fixed metric tensor. In the case of the mean curvature flow, such formulæ have been established without any such topological restrictions and, more recently in the case where the target manifold is equipped with a simultaneously evolving metric tensor, which was accomplished by Lott [14] and Magni, Mantegazza and Tsatis [15]; this includes, in particular, evolution by Ricci flow. In the following, we shall show that an analogue of these last results holds for suitable vector bundle-valued k-forms on Riemannian manifolds equipped with simultaneously evolving metric tensors. The generality of this result permits us to immediately obtain the now well-known monotonicity formulæ for the Yang-Mills and harmonic map heat flows, as well as generalise them. It is our hope that this general formula may find use in the study of singularity formation in more general geometric settings, as well as be applicable to related flows.

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2. Main results

Let M be a smooth n-manifold without boundary equipped with a smooth one-parameter family of *complete* metric tensors $\{g(\cdot,t)\}_{t\in[0,T[}$ and $V\to M$ a smooth finite-dimensional Riemannian vector bundle equipped with a one-parameter family $\{\nabla^t\}_{t\in[0,T[}$ of compatible connections. Write $\Lambda^k T^*M$ for the kth exterior product of the cotangent bundle of M, ∇^t for the covariant derivative on $V\otimes\Lambda^k T^*M$ induced by the Levi-Civita connection associated with $g(\cdot,t)$ and the connection ∇^t on V, d^{∇^t} for the corresponding covariant exterior derivative (cf. §2.75 of [16]), and δ^{∇^t} for its formal adjoint with respect to the L^2-inner product arising from the metric $\langle\cdot,\cdot\rangle_t$ induced by $g(\cdot,t)$ and the Riemannian structure on V; the Hodge Laplacian is then given by $\Delta^{\nabla^t} := \mathrm{d}^{\nabla^t}\delta^{\nabla^t}+\delta^{\nabla^t}\mathrm{d}^{\nabla^t}$. As is customary, we shall write $|\cdot|_t$ for the (fibrewise) norm associated with the (fibrewise) inner product $\langle\cdot,\cdot\rangle_t$ and $\llcorner$ for the interior product symbol. For brevity, we shall henceforth omit the parameter t in these symbols and indicate its value at the ends of expressions whenever necessary.

Our main result is then the following weighted energy identity:

Theorem 2.1. *Suppose $\{\psi(\cdot,t) : M\to V\otimes\Lambda^k T^*M\}_{t\in[0,T[}$ is a smooth one-parameter family of sections for some fixed $k\in\mathbb{N}\cup\{0\}$ and $\Gamma : M\times[0,T[\,\to\,]0,\infty[$ is smooth. Set*

$$E_k(\psi,\Gamma,g)(t) = (4\pi(T-t))^k\int_M \frac{1}{2}|\psi|^2\Gamma\ \mathrm{dvol}_g(\cdot,t).$$

If the integrals

$$\int_M\left(\left|\frac{\mathrm{d}\Gamma}{\Gamma}\right|+\left|\frac{\mathrm{d}\Gamma}{\Gamma}\right|^2+\left|\frac{\nabla\mathrm{d}\Gamma}{\Gamma}\right|\right)|\psi|^2\Gamma\ \mathrm{dvol}_g(\cdot,t)$$

and

$$\int_M\left\{\left(1+\left|\frac{\mathrm{d}\Gamma}{\Gamma}\right|^2\right)\left|\mathrm{d}^\nabla\psi\right|^2+\left|\delta^\nabla\psi\right|^2+\left|\Delta^\nabla\psi\right|^2\right\}\Gamma\ \mathrm{dvol}_g(\cdot,t)$$

are finite for each $t\in\,]0,T[$, then the following identity holds on $]0,T[$:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}E_k(\psi,\Gamma,g)\\
&= (4\pi(T-t))^k\int_M\left(\left\langle\psi,\mathrm{H}^\nabla\psi\right\rangle-\left|\frac{\nabla\Gamma}{\Gamma}\llcorner\psi-\delta^\nabla\psi\right|^2\right)\Gamma\ \mathrm{dvol}_g(\cdot,t)\\
&\quad+(4\pi(T-t))^k\int_M\left(\frac{1}{2}|\psi|^2\mathrm{H}^*\Gamma-\left|\mathrm{d}^\nabla\psi\right|^2\Gamma\right)\mathrm{dvol}_g(\cdot,t)\\
&\quad-(4\pi(T-t))^k\int_M\left\langle\mathcal{Q}_T(\log\Gamma,g),\sum_{i,j=1}^n\langle\partial_i\llcorner\psi,\partial_j\llcorner\psi\rangle\,\mathrm{d}x^i\otimes\mathrm{d}x^j\right\rangle\Gamma\ \mathrm{dvol}_g(\cdot,t)
\end{aligned}\tag{1}$$

Here, $\mathrm{H}^\nabla := \partial_t+\Delta^\nabla$ is the heat operator induced by the Hodge Laplacian, $\mathrm{H}^*\Gamma := \left(\partial_t+\Delta_g+\frac{1}{2}\mathrm{tr}_g\partial_t g\right)\Gamma$ the backward heat operator induced by the Laplace-Beltrami operator Δ_g acting on Γ, and for any smooth function $f : M\times[0,T[\,\to\mathbb{R}$, $\mathcal{Q}_T(f,g)(\cdot,t) := \left(\nabla\mathrm{d}f+\frac{1}{2}\partial_t g+\frac{g}{2(T-t)}\right)(\cdot,t) : M\to T^*M\otimes T^*M$ is the so-called matrix Harnack form.

The matrix Harnack form $\mathcal{Q}_T(\log\Gamma,g)$ was explicitly introduced by Hamilton [10] in the case where Γ solves the (backward) heat equation and M is *static*, i.e. $\partial_t g\equiv 0$; he showed that it is nonnegative definite in the case where M is compact, the sectional curvatures of g are nonnegative and (M, g) is Ricci-parallel, in which case it leads to a Harnack inequality for Γ in the classical sense. Another notable instance where it arises is when g evolves by *Ricci flow*, i.e. $\partial_t g = -2\mathrm{Ric}$; in this case, it reads

$$\mathcal{Q}_T(\log\Gamma, g)(\cdot, t) = \left(-\mathrm{Ric} + \nabla \mathrm{d}\log\Gamma + \frac{g}{2(T-t)}\right)(\cdot, t),$$

and this expression vanishes if and only if g is a gradient shrinking soliton with potential $-\log\Gamma$ (cf. Appendix C of [13]); this includes Euclidean space as a special case, where Γ is the usual backward heat kernel concentrated at a point. We may therefore immediately deduce the following from the identity (1):

Corollary 2.2. *If $\psi : M \times [0, T[\to V \otimes \Lambda^k T^*M$ is a subsolution to the heat equation in the sense that $\left\langle \psi, \mathrm{H}^\nabla \psi \right\rangle \leq 0$, Γ is a positive subsolution to the backward heat equation, i.e. $\mathrm{H}^*\Gamma \leq 0$, and the matrix Harnack form $\mathcal{Q}_T(\log\Gamma, g)$ is nonnegative definite, then $E_k(\psi, \Gamma, g)$ is monotone nonincreasing whenever the integrals in Theorem 2.1 are finite. In particular, $E_k(\psi, \Gamma, g)$ is monotone nonincreasing if g is a gradient shrinking Ricci soliton with potential $-\log\Gamma$.*

We now turn our attention to two concrete examples where Theorem 2.1 leads to a monotone quantity. In both of these cases, formulæ in the case where M is compact, $\partial_t g \equiv 0$ and $\mathcal{Q}_T(\log\Gamma, g) \geq 0$ were obtained by Hamilton in [11].

2.1. **Yang-Mills heat flow.** Let $G \to P \to M$ be a principal fibre bundle with compact semi-simple structure group G and denote its Lie algebra by $\mathfrak{g}$. We take V to be the vector bundle associated to P and the adjoint representation of G on $\mathfrak{g}$, which is a Riemannian vector bundle with the fibrewise inner product induced by minus the Killing form of $\mathfrak{g}$.

Fixing a background connection $\omega_0 : P \to \mathfrak{g} \otimes T^*P$, we say that a smooth one-parameter family of connections $\{\omega(\cdot, t) = \omega_0 + \overline{a(\cdot, t)}\}_{t \in [0, T[}$ evolves by the *Yang-Mills heat flow* if

$$\left(\partial_t a + \delta^\nabla \underline{\Omega}^\omega\right)(\cdot, t) = 0 \tag{2}$$

holds on M for all $t \in]0, T[$, where for each $t \in [0, T[$ $a(\cdot, t) : M \to V \otimes T^*M$ is the unique section with lift $\overline{a(\cdot, t)}$, the section $\underline{\Omega}^\omega(\cdot, t) : M \to V \otimes \Lambda^2 T^*M$ is the *curvature form* of $\omega(\cdot, t)$ and ∇^t arises from the natural (Riemannian) connection on V induced by $\omega(\cdot, t)$ and the Levi-Civita connection of $g(\cdot, t)$.

A straightforward computation shows that the curvature form $\underline{\Omega}^\omega$ satisfies the *Bianchi identity* $\mathrm{d}^\nabla \underline{\Omega}^\omega = 0$ and the heat equation $\mathrm{H}^\nabla \underline{\Omega}^\omega = 0$. Hence, if Γ is a positive subsolution to the backward heat equation and the matrix Harnack form $\mathcal{Q}_T(\log\Gamma, g)$ is nonnegative definite, then provided the integrals in Theorem 2.1 are finite, Corollary 2.2 applies with $k = 2$ and we have the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} E_2(\underline{\Omega}^\omega, \Gamma, g) \leq -\left(4\pi(T-t)\right)^2 \int_M \left| \frac{\nabla\Gamma}{\Gamma} \lrcorner \underline{\Omega}^\omega - \delta^\nabla \underline{\Omega}^\omega \right|^2 \Gamma \, \mathrm{dvol}_g(\cdot, t); \tag{3}$$

moreover, equality holds whenever Γ solves the backward heat equation and the matrix Harnack form vanishes. The expression within the norm on the right-hand side of (3) may be written more suggestively as $\partial_t a + \nabla\log\Gamma \lrcorner \underline{\Omega}^\omega$ which, when lifted to the principal bundle P, takes the form $\partial_t \omega + \mathcal{L}_{X^\omega}\omega$, where $\mathcal{L}$ the component-wise Lie derivative operator and $X^\omega(\cdot, t)$ is the $\omega(\cdot, t)$-horizontal lift of $\nabla\log\Gamma(\cdot, t)$. Therefore, the right-hand side of (3) vanishes if and only if ω is a *Yang-Mills gradient soliton* with the same potential function $-\log\Gamma$ as the gradient shrinking soliton. We therefore obtain the following integral of the flow:

Corollary 2.3. *If Γ is a positive solution to the backward heat equation, ω is a Yang-Mills gradient soliton with potential $-\log\Gamma$ and g is a gradient shrinking Ricci soliton also with potential $-\log\Gamma$, then $E_2(\underline{\Omega}^\omega,\Gamma,g)$ is constant on $]0,T[$ whenever the integrals in Theorem 2.1 are finite.*

In the case where (M,g) is Euclidean space and Γ is the canonical backward heat kernel concentrated at (X,T) with $X\in\mathbb{R}^n$, $\log\Gamma(x,t)=\frac{x-X}{2(t-T)}$ so that ω is a Yang-Mills gradient soliton with potential $-\log\Gamma$ whenever ω is scale-invariant about (X,T) in a suitable gauge (cf. [18]).

2.2. **Harmonic map heat flow.** Let $(N,g_N)\subset(\mathbb{R}^K,\delta)$ be a Riemannian submanifold of Euclidean space. A smooth one-parameter family of maps $\{u(\cdot,t):M\to N\}$ is said to evolve by the *harmonic map heat flow* if for each $(x,t)\in M\times]0,T[$ there holds

$$(\partial_t u-\Delta_g u)(x,t)\perp T_{u(x,t)}N\subset\mathbb{R}^K,$$

where u is viewed as a map into $\mathbb{R}^K$ and the Laplace-Beltrami operator Δ_g is applied componentwise.

In this case, the differential form to consider is the section $\psi=\mathrm{d}u:M\to\mathbb{R}^K\otimes T^*M$ given by the differential of u, and V is taken to be the trivial Riemannian vector bundle over M with standard fibre $\mathbb{R}^K$ and fibrewise inner product given by the Euclidean inner product. Although $\mathrm{d}u$ does not solve the heat equation $\mathrm{H}^\nabla\psi=0$, it nevertheless satisfies the Pythagoras-type identity

$$\left\langle \mathrm{H}^\nabla \mathrm{d}u,\mathrm{d}u\right\rangle-\left|\partial_X u+\Delta_g u\right|^2=-\left|\partial_t u+\partial_X u\right|^2$$

for all vector fields $X:M\to TM$, the left-hand side of which coïncides with the parenthetical expression in the first integrand on the right-hand side of (1), where we take $k=1$ and $\psi=\mathrm{d}u$. We may therefore conclude as before that if u evolves by the harmonic map heat flow, Γ is a subsolution to the backward heat equation and $\mathcal{Q}_T(\log\Gamma,g)$ is nonnegative definite, then whenever the integrals in Theorem 2.1 are finite, a straightforward modification of Corollary 2.2 applies with $k=1$ and we have the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}E_1(\mathrm{d}u,\Gamma,g)\leq-4\pi(T-t)\int_M\left|\partial_t u+\partial_{\frac{\nabla\Gamma}{\Gamma}}u\right|^2\Gamma\,\mathrm{dvol}_g(\cdot,t),\tag{4}$$

with equality whenever Γ solves the backward heat equation and the matrix Harnack form vanishes. In this case, it is evident that the right-hand side of (4) vanishes if and only if u solves the transport equation $\partial_t u+\partial_{\nabla\log\Gamma}u=0$, which amounts to u being a gradient soliton with potential $-\log\Gamma$. We therefore obtain an integral of motion for u:

Corollary 2.4. *If Γ is a solution to the backward heat equation, u is a gradient soliton with potential $-\log\Gamma$ evolving by the harmonic map heat flow and g is a gradient shrinking Ricci soliton also with potential $-\log\Gamma$, then $E_1(\mathrm{d}u,\Gamma,g)$ is constant on $]0,T[$ whenever the integrals in Theorem 2.1 are finite.*

As with the Yang-Mills heat flow, if (M,g) is Euclidean space and Γ is the canonical backward heat kernel concentrated at (X,T) with $X\in\mathbb{R}^n$, then u is a gradient soliton with potential $-\log\Gamma$ whenever u is scale-invariant about (X,T).

3. Proof of Theorem 2.1

We first note that for sections $\psi_1, \psi_2 : M \to V \otimes \Lambda^k T^*M$ the identity

$$\partial_t \langle \psi_1, \psi_2 \rangle = -\left\langle \partial_t g, \sum_{i,j=1}^n \langle \partial_i \lrcorner \psi, \partial_j \lrcorner \psi \rangle \, \mathrm{d}x^i \otimes \mathrm{d}x^j \right\rangle$$

holds, as may be verified by a straightforward computation in local coördinates. Using this and the relations $\partial_t \mathrm{dvol}_g = \frac{1}{2}\mathrm{tr}_g \partial_t g \ \mathrm{dvol}_g$ and $\left\langle g, \sum_{i,j=1}^n \langle \partial_i \lrcorner \psi, \partial_j \lrcorner \psi \rangle \, \mathrm{d}x^i \otimes \mathrm{d}x^j \right\rangle = k|\psi|^2$, we compute the t-derivative of the integrand of $E_k(\psi, \Gamma, g)(t)$ together with the factor of $(T-t)$ to obtain

$$\begin{aligned}
&\partial_t \left[(4\pi(T-t))^k \, \frac{1}{2}|\psi|^2 \Gamma \ \mathrm{dvol}_g \right] \\
&= (4\pi(T-t))^k \left[\left\langle \mathrm{H}^\nabla \psi, \psi \right\rangle \Gamma + \frac{1}{2}|\psi|^2 \mathrm{H}^* \Gamma \right. \\
&\qquad - \left\langle \frac{1}{2}\partial_t g + \frac{g}{2(T-t)}, \sum_{i,j=1}^n \langle \partial_i \lrcorner \psi, \partial_j \lrcorner \psi \rangle \, \mathrm{d}x^i \otimes \mathrm{d}x^j \right\rangle \Gamma \\
&\qquad\qquad \left. - \frac{1}{2}|\psi|^2 \Delta_g \Gamma - \left\langle \Delta^\nabla \psi, \psi \right\rangle \Gamma \right] \mathrm{dvol}_g.
\end{aligned} \tag{5}$$

Now, the final parenthetical term on the right-hand side of (5) may by means of the divergence identity

$$\left\langle \mathrm{d}^\nabla \psi_1, \psi_2 \right\rangle - \left\langle \psi_1, \delta^\nabla \psi_2 \right\rangle = \mathrm{div} \left(\sum_{i=1}^n \langle \psi_1, \partial_i \lrcorner \psi_2 \rangle \, \mathrm{d}x^i \right)$$

for sections ψ_1, ψ_2 of the bundle $V \otimes \left(\bigoplus_{i=1}^n \Lambda^i T^*M \right)$ be rewritten as

$$\begin{aligned}
&- \left\langle \Delta^\nabla \psi, \psi \right\rangle \Gamma \\
&\quad = -\left(|\delta^\nabla \psi|^2 + |\mathrm{d}^\nabla \psi|^2 \right) \Gamma + \left\langle \delta^\nabla \psi, \nabla\Gamma \lrcorner \psi \right\rangle - \left\langle \psi, \nabla\Gamma \lrcorner \mathrm{d}^\nabla \psi \right\rangle \\
&\qquad + \mathrm{div} \left(\sum_{i=1}^n \left(\left\langle \delta^\nabla \psi, \partial_i \lrcorner \psi \right\rangle - \left\langle \psi, \partial_i \lrcorner \mathrm{d}^\nabla \psi \right\rangle \right) \mathrm{d}x^i \right).
\end{aligned}$$

For the penultimate parenthetical term, we recall the *energy-momentum tensor* $T : M \to T^*M \otimes T^*M$ associated with ψ given by

$$T = \sum_{i,j=1}^n \langle \partial_i \lrcorner \psi, \partial_j \lrcorner \psi \rangle \, \mathrm{d}x^i \otimes \mathrm{d}x^j - \frac{1}{2}|\psi|^2 g,$$

which a somewhat tedious computation (cf. [1]) shows satisfies the divergence identity

$$\mathrm{div}\, T = -\sum_{i=1}^n \left(\left\langle \delta^\nabla \psi, \partial_i \lrcorner \psi \right\rangle + \left\langle \partial_i \lrcorner \mathrm{d}^\nabla \psi, \psi \right\rangle \right) \mathrm{d}x^i.$$

Using this, we may rewrite the penultimate term as

$$
\begin{aligned}
-\frac{1}{2}|\psi|^2\Delta_g\Gamma &= \langle \nabla \mathrm{d}\Gamma, T\rangle - \left\langle \nabla\mathrm{d}\Gamma, \sum_{i,j=1}^{n}\langle \partial_i \llcorner \psi, \partial_j\llcorner\psi\rangle\, \mathrm{d}x^i\otimes \mathrm{d}x^j\right\rangle \\
&= \operatorname{div}(\nabla\Gamma\llcorner T) - \nabla\Gamma\llcorner \operatorname{div} T - \left\langle \nabla\mathrm{d}\Gamma, \sum_{i,j=1}^{n}\langle \partial_i \llcorner \psi, \partial_j\llcorner\psi\rangle\, \mathrm{d}x^i\otimes \mathrm{d}x^j\right\rangle .
\end{aligned}
$$

After substituting these expressions into (5) and simplifying, we integrate over M and apply the divergence theorem [8], which is valid in this context due to the summability condition imposed in the theorem, whence the result follows. $\square$

4. Approximate monotonicity

In this section, we shall show that for the heretofore discussed flows, a slightly modified monotonicity formula holds under weaker integrability conditions than those given earlier at the cost of more regularity on the part of the evolving metric tensor $g(\cdot,t)$, which we henceforth assume is smooth up to and including $t=T$.

In this section we shall take Γ to be the Euclidean backward heat kernel concentrated at a fixed $(X,T)\in M\times\mathbb{R}$, i.e. $\Gamma(x,t) = \frac{1}{(4\pi(T-t))^{n/2}}\exp\left(\frac{d_{g(\cdot,t)}(x,X)^2}{4(t-T)}\right)$, where $d_{g(\cdot,t)}$ is the geodesic distance function associated with $g(\cdot,t)$. We assume that either $\mathrm{H}^{\nabla}\psi=0$ or $\psi=\mathrm{d}u$ and u evolves by the harmonic map heat flow. Moreover, we assume that there exists an $E_0>0$ with $\int_M |\psi|^2\mathrm{dvol}_g(\cdot,t)\le E_0$ for all $t\in[0,T[$. To simplify notation, we set

$$
\mathcal{S} = \begin{cases} \frac{\nabla\Gamma}{\Gamma}\llcorner\psi - \delta^{\nabla}\psi & \text{if } \mathrm{H}^{\nabla}\psi = 0 \\ \partial_t u + \partial_{\nabla\log\Gamma} u & \text{if } \psi=\mathrm{d}u \text{ and } u \text{ solves the harmonic map heat flow.}\end{cases}
$$

We also fix a smooth function $\chi:\mathbb{R}\to[0,1]$ such that $\chi(x)=0$ for $x<\frac{1}{2}$, $\chi(x)=1$ for $x>1$ and $\chi'\ge 0$, and set $\varphi(x,t)=\chi\left(\frac{d_{g(\cdot,t)}(x,X)+r-2j_0}{2(r-j_0)}\right)$, where $j_0=\frac{1}{2}\inf\{\text{injectivity radius of } g(\cdot,t) \text{ at } X : t\in[0,T]\}$ and $r\in\,]0,j_0[$ is fixed; thus, this defines a smooth function $\varphi: M\times[0,T[\,\to[0,1]$ such that $\operatorname{supp}\varphi(\cdot,t)$ is compact and contained in $\{d_{g(\cdot,t)}(\cdot,X)<j_0\}$.

We note that although $\Gamma(\cdot,t)$ is not smooth on all of M, it is smooth on the support of $\varphi(\cdot,t)$, whence, computing $\frac{\mathrm{d}}{\mathrm{d}t}E_k(\varphi\psi,\Gamma,g)$ as in the proof of Theorem 2.1, we obtain the identity

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left((4\pi(T-t))^k\int_M \frac{1}{2}|\psi|^2\Gamma\varphi^2\mathrm{dvol}_g\right) \\
&= (4\pi(T-t))^k\int_M \frac{1}{2}|\psi|^2\varphi^2\mathrm{H}^*\Gamma - \varphi^2\Gamma\left\langle \mathcal{Q}_T(\log\Gamma,g), \sum_{i,j=1}^n \langle\partial_i\llcorner\psi,\partial_j\llcorner\psi\rangle\,\mathrm{d}x^i\otimes\mathrm{d}x^j\right\rangle \\
&\qquad - \varphi^2\Gamma\left(|\mathcal{S}|^2+|\mathrm{d}^{\nabla}\psi|^2\right) + |\psi|^2\Gamma\varphi\left(\partial_t\varphi + \left\langle\nabla\varphi,\frac{\nabla\Gamma}{\Gamma}\right\rangle\right) \\
&\qquad + 2\varphi\Gamma\left(\langle\nabla\varphi\llcorner\psi,\mathcal{S}\rangle - \left\langle\nabla\varphi\llcorner\mathrm{d}^{\nabla}\psi,\psi\right\rangle\right)\mathrm{dvol}_g(\cdot,t).
\end{aligned} \tag{6}
$$

First of all, although Γ does not solve the backward heat equation, it satisfies the inequalities

$$\mathrm{H}^*\Gamma \leq c_0 - c_1 \log(T-t)\Gamma$$
$$\mathcal{Q}_T(\log \Gamma, g) \geq -\left(c_2 + c_3 \log(T-t)\Gamma\right) g,$$

where $c_0, \ldots, c_3$ are nonnegative constants depending on the geometry of $(M, g(\cdot, t))$ in a neighbourhood of (X, T). Moreover, we note that all derivatives of φ are smooth and supported in $\{(x,t) \in M \times [0, T[\, : r \leq d_{g(\cdot,t)}(x, X) \leq j_0\}$ so that Γ and $|\nabla \Gamma|$ are bounded on this set and we may bound the last two terms of the right-hand integrand in (6) from above by

$$C \cdot \frac{1}{2}|\psi|^2 + \frac{1}{2}\left(|\mathcal{S}|^2 + |\mathrm{d}^\nabla \psi|^2\right)\Gamma\varphi^2,$$

where C is a positive constant depending on the local geometry of $(M, g(\cdot, t))$ about (X, T) and use was made of Young's inequality. Incorporating all of these observations into (6), we now have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t} E_k(\varphi\psi, \Gamma, g) \\ &\qquad \leq \left(c_0 - 2kc_2 - (c_1 + 2kc_3)\log(T-t)\right) E_k(\varphi\psi, \Gamma, g)(t) + C\int_M |\psi|^2 \mathrm{dvol}_g(\cdot, t).\end{aligned}$$

We have therefore established the following monotonicity principle:

Corollary 4.1. *Suppose Γ is the Euclidean backward heat kernel concentrated at $(X, T) \in M \times \mathbb{R}$, $g(\cdot, t)$ is smooth up to and including $t = T$, φ is the cutoff function defined above and ψ is either a solution to the heat equation $\mathrm{H}^\nabla \psi = 0$ or $\psi = \mathrm{d}u$ and u evolves by the harmonic map heat flow. If $\int_M |\psi|^2 \mathrm{dvol}_g(\cdot, t) \leq E_0$ for all $t \in [0, T[$, then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{l(t)} E_k(\varphi\psi, \Gamma, g)(t) + \widetilde{C} E_0 (T-t)\right) \leq 0,$$

where $l(t) = (T-t)\left(c_0 + c_1 + 2k(c_3 - c_2) - (c_1 + 2kc_3)\log(T-t)\right)$, $\widetilde{C} = C \cdot \sup e^l$, and $c_0, \ldots, c_3$ and C are the geometric constants introduced above.

In the case where M is compact, $\partial_t g \equiv 0$ and $\psi = \mathrm{d}u$ with u evolving by the harmonic map heat flow, we recover the formula of Chen and Struwe [4], whereas when $\psi = \underline{\Omega}^\omega$ with ω evolving by the Yang-Mills heat flow, we obtain Chen and Shen's formula [5].

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