HEAT BALL FORMULÆ FOR k-FORMS ON EVOLVING MANIFOLDS

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ABSTRACT. We establish a local monotonicity identity for vector bundle-valued differential k-forms on superlevel sets of appropriate heat kernel-like functions. As a consequence, we obtain new local monotonicity formulæ for the harmonic map and Yang-Mills heat flows on evolving manifolds. We also show how these methods yield corresponding local monotonicity formulæ for the Yang-Mills-Higgs flow.

1. INTRODUCTION

It is well known that the fundamental solution representation formula, a consequence of the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n} u\cdot\Gamma_{(X,T)}(\cdot,t)=0 \tag{1.1}$$

for suitable $u:\mathbb{R}^n\times\,]0,\infty[\,\to\mathbb{R}$ solving the heat equation and

$$\Gamma_{(X,T)}(x,t)=\frac{1}{(4\pi(T-t))^{\frac{n}{2}}}\exp\left(\frac{|x-X|^2}{4(t-T)}\right)$$

the canonical backward heat kernel concentrated at $(X,T)\in\mathbb{R}^n\times\,]0,\infty[$, is ubiquitous in the study of the behaviour of solutions to the heat equation; less well known however is the so-called *heat ball formula* [10, 22]

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r^n}\iint\limits_{\{\Gamma_{(X,T)}>r^{-n}\}} u(x,t)\cdot\frac{|x-X|^2}{4(t-T)^2}\,\mathrm{d}x\,\mathrm{d}t\right)=0 \tag{1.2}$$

for sufficiently small $r>0$ which, in contrast to (1.1), has as region of integration a relatively compact set. Analogues of (1.1) have been used with much success in the study of solutions to nonlinear heat-type equations such as the harmonic map heat equation in higher dimensions: Struwe [21] showed that if $u:\mathbb{R}^n\times\,]0,\infty[\,\to N\subset\mathbb{R}^n$ with N a Riemannian submanifold of $\mathbb{R}^n$ solves the *harmonic map heat equation*, i.e.

$$\forall (x,t)\in\mathbb{R}^n\times\,]0,\infty[\;\;(\partial_t-\Delta)u(x,t)\perp T_{u(x,t)}N$$

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and is in a suitable growth class at infinity, then

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left((T-t)\int_{\mathbb{R}^n}\frac{1}{2}|\mathrm{d}u|^2\cdot\Gamma_{(X,T)}(x,t)\,\mathrm{d}x\right)\\&=-(T-t)\int_{\mathbb{R}^n}\left|\partial_t u+\sum_{i=1}^n\frac{(x-X)^i}{2(t-T)}\partial_i u\right|^2\cdot\Gamma_{(X,T)}(x,t)\,\mathrm{d}x;\end{aligned}\tag{1.3}$$

in this case, the right-hand side is nonpositive and zero iff u is parabolically scale-invariant, a fact that has deep implications in the study of the formation of singularities. Similar formulæ have been used by Huisken to study the mean curvature flow [15] as well as Chen and Shen and Hong and Tian to study the Yang-Mills flow [5, 14], though formulæ akin to (1.2) have only recently been exploited in the nonlinear setting, most notably by Ecker for the mean curvature flow with Euclidean background, a certain reaction-diffusion equation and the harmonic map heat equation on Euclidean space [6, 8]; for the last, the formula takes the form

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r^{n-2}}\iint\limits_{E_r(X,T)}\frac{1}{2}|\mathrm{d}u|^2(x,t)\cdot\frac{n-2}{2(T-t)}-\left\langle\sum_{i=1}^n\frac{(x-X)^i}{2(t-T)}\partial_i u,\mathcal{S}_u(x,t)\right\rangle\mathrm{d}x\,\mathrm{d}t\right)\\&\qquad=\frac{n-2}{r^{n-1}}\iint\limits_{E_r(X,T)}|\mathcal{S}_u(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}\tag{1.4}$$

where $E_r(X,T)=\{(x,t)\in\mathbb{R}^n\times\,]0,T[\;:4\pi(T-t)\Gamma_{(X,T)}(x,t)>r^{-n}\}$ and $\mathcal{S}_u(x,t)$ is the expression under the absolute value on the right-hand side of (1.3). In a sense, this formula is a more natural analogue of the well-known monotonicity formula for *harmonic maps* due to Schoen and Uhlenbeck [20] and Price [19], viz. solutions u to the harmonic map heat equation with $\partial_t u\equiv 0$, which reads

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r^{n-2}}\int_{B_r(X)}\frac{1}{2}|\mathrm{d}u|^2(x)\,\mathrm{d}x\right)\geq 0.$$

It was recently shown by Ecker, Knopf, Ni and Topping [9] that an analogue of (1.2) holds on *evolving* Riemannian manifolds: If M^n is a smooth oriented manifold equipped with a one-parameter family of metric tensors $\{g(\cdot,t)\}_{t\in[0,T_\infty[}$, then the identity

$$\begin{aligned}&\left[\frac{1}{r^n}\iint\limits_{\{\Phi>r^{-n}\}}u(x,t)\cdot|\nabla\log\Phi|^2(x,t)\mathrm{dvol}_g(x,t)\,\mathrm{d}t\right]_{r=r_1}^{r=r_2}\\&=-\int_{r_1}^{r_2}\left(\frac{n}{r^{n+1}}\iint\limits_{\{\Phi>r^{-n}\}}\left(\log(r^n\Phi)\cdot\mathrm{H}u+u\cdot\frac{\mathrm{H}^*\Phi}{\Phi}\right)(x,t)\;\mathrm{dvol}_g(x)\,\mathrm{d}t\right)\mathrm{d}r,\end{aligned}\tag{1.5}$$

holds for sufficiently small positive $r_1<r_2$ whenever these integrals make sense, where $\mathrm{H}=\partial_t-\Delta_g$ is the heat operator induced by the Laplace-Beltrami operator Δ_g, $\mathrm{H}^*\Phi(\cdot,t)=\left(\partial_t+\Delta_{g(\cdot,t)}+\frac{1}{2}\mathrm{tr}_g\partial_t g\right)\Phi(\cdot,t)$ is the *conjugate* heat operator acting on Φ, $u:M\times\,]0,\infty[\,\to\mathbb{R}$ is smooth, and $\Phi:M\times\,]0,T[\,\to\mathbb{R}^+$ is in some sense backward heat kernel-like; thus, it may be seen that the right-hand side vanishes if u solves the

heat equation and Φ solves the so-called backward heat equation $\mathrm{H}^*\Phi = 0$, where this equation arises naturally from the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_M u(x,t)\Phi(x,t)\mathrm{dvol}_g(\cdot,t)(x) = \int_M \left(\mathrm{H}u\cdot\Phi + u\cdot\mathrm{H}^*\Phi\right)(x,t)\mathrm{dvol}_g(x,t).$$

The main purpose of this paper is to establish the following analogue of (1.5) for vector bundle-valued differential k-forms (Theorem 5.1):

Main Theorem. *If $E \to M^n$ is a finite-dimensional Riemannian vector bundle equipped with a smooth one-parameter family of Riemannian connections $\{\nabla^t\}_{t\in]0,T[}$ and $\{\psi(\cdot,t) : M \to E\otimes\Lambda^k T^*M\}_{t\in[0,T[}$ is a smooth one-parameter family of sections, then the identity*

$$\left[\frac{1}{r^{n-2k}}\iint\limits_{E_r^{n-2k}(\Phi)} \frac{1}{2}|\psi|^2\cdot\left(\partial_t\phi + |\nabla\phi|^2\right) - \langle\nabla\phi\llcorner\psi, \mathcal{S}_{\psi,\phi}\rangle - \left\langle\nabla\phi\llcorner\mathrm{d}^\nabla\psi,\psi\right\rangle\mathrm{dvol}_g\,\mathrm{d}t\right]_{r=r_1}^{r=r_2}$$
$$= \int_{r_1}^{r_2}\left(\frac{n-2k}{r^{n-2k+1}}\iint\limits_{E_r^{n-2k}(\Phi)} -\frac{1}{2}|\psi|^2\cdot\frac{\mathrm{H}^*\Phi + \frac{k}{T-t}\Phi}{\Phi}(\cdot,t) - \left\langle\mathrm{H}^\nabla\psi,\psi\right\rangle + |\mathcal{S}_{\psi,\phi}|^2\right.$$
$$\left. + |\mathrm{d}^\nabla\psi|^2 + \left\langle\mathcal{Q}_T(\phi,g), \sum_{i,j}\langle\varepsilon_i\llcorner\psi,\varepsilon_j\llcorner\psi\rangle\,\omega^i\otimes\omega^j\right\rangle\mathrm{dvol}_g\,\mathrm{d}t\right)\mathrm{d}r \tag{1.6}$$

holds for sufficiently small $r_1 < r_2$ whenever $(x,t)\mapsto\frac{|\psi(x,t)|^2}{T-t}$ is summable over $E_{r_2}^{n-2k}(\Phi)$, $n > 2k$ and these integrals make sense, where $E_r^{n-2k}(\Phi) = \{\Phi > r^{2k-n}\}$, $\Phi = e^\phi$ is weighted backward heat kernel-like in an appropriate sense, H^∇ is the heat operator associated with the naturally defined Hodge Laplacian, $\mathcal{S}_{\psi,\phi} = \nabla\phi\llcorner\psi - \delta^\nabla\psi$ is an expression characterising certain solitons, and $\mathcal{Q}_T(\phi,g) = \nabla^2\phi + \frac{1}{2}\partial_t g + \frac{g}{2(T-t)}$ is the matrix Harnack form (see §2 for details).

This formula shares many of the features of (1.5): The first term in the integrand of the right-hand side vanishes if $\Phi(x,t) = (4\pi(T-t))^k\Gamma(x,t)$ and Γ solves the backward heat equation, and the second term vanishes if ψ solves the heat equation $\mathrm{H}^\nabla\psi = 0$; however, though the following two terms are nonnegative, the expression $\mathcal{Q}_T(\phi,g)$ occurring in the last term, which in the case $\partial_t g \equiv 0$ reduces to the expression occurring in Hamilton's matrix Harnack estimate [11] and, in the case of Ricci flow ($\partial_t g = -2\mathrm{Ric}$), is an expression characterising certain solitons, does not in general have a sign. Nevertheless, in certain cases (cf. Corollary 5.6), it may be controlled in such a way as to allow us to derive monotonicity formulæ from (1.6).

Besides being an interesting formula in its own right, (1.6) may be used to prove local monotonicity formulæ for solutions to nonlinear heat-type equations possessing a canonical differential form satisfying a heat-type equation, such as is the case for the Yang-Mills heat equation, where ψ is taken to be the curvature of the connection, and the harmonic map heat equation, where ψ is taken to be the differential of the map (cf. Examples 2.5 and 2.6); in both cases, we are led to new formulæ in the curved setting, where in the latter case we obtain a generalisation of Ecker's formula (1.4), and in the former case one of our own [3].

The structure of this paper is as follows. In §2, we fix notation, describe our geometric setup, give a brief exposition of the backward heat kernel-like functions and heat-type equations to be considered in the sequel. A generalised notion of heat ball adapted from that in [9] is then introduced in §3, where examples and integration-by-parts formulæ à la [6] are given. In §4 we shall establish a local energy inequality along the lines of that in [8] that provides us with natural conditions under which the integrals of (1.6) are finite for suitable ψ and Φ. Finally, in §5 we derive (1.6) and apply it to ψ solving the heat-type equations introduced in §2 and suitable Φ.

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2. Setup

2.1. **Notation.** We denote by $\mathbb{R}^+$ the positive reals and $\mathbb{N}$ the positive integers. For $x \in \mathbb{R}$, we set $x^+ = \max\{0, x\}$ and $x^- = \min\{0, x\}$. Moreover, given smooth manifolds M, N and P, we write $C^{k,l}(M \times N, P)$ for the set of all maps $M \times N \to P$ k-times differentiable in the first entry and l-times differentiable in the second, writing $C^k(M \times N, P)$ whenever $k = l$. By $\mathrm{pr}_1 : M \times N \to M$ and $\mathrm{pr}_2 : M \times N \to N$ we denote the respective (smooth) projections. Furthermore, if $E \to M$ is any vector bundle, $\Gamma(E)$ shall denote the $C^\infty(M)$-module of (smooth) sections of E. Finally, we fix a function $\chi \in C^2(\mathbb{R}, [0, 1])$ such that $\chi|_{]-\infty,\frac{1}{2}]} \equiv 0$, $\chi' \geq 0$ and $\chi|_{[1,\infty[} \equiv 1$ and define for each $q \in \mathbb{N}$ the function $\chi_q(x) = \chi(2^q x)$. Writing χ_A for the characteristic function of a set A, we may verify that $\chi_q \xrightarrow{q\to\infty} \chi_{]0,\infty[}$ and $x\chi_q'(x) \xrightarrow{q\to\infty} 0$ pointwise.

2.2. **Geometry.** Throughout this paper, $(M, \{g(\cdot, t)\}_{t\in[0,T_\infty[})$ shall be an *evolving* oriented manifold, i.e. M is to be assumed oriented and equipped with a smooth one-parameter family $\{g(\cdot, t)\}_{t\in[0,T_\infty[}$ of Riemannian metrics; if $\partial_t g \equiv 0$, we call M *static* and set $T_\infty = \infty$. We denote by TM its tangent bundle, T^*M its cotangent bundle, $\Lambda^k T^*M$ the kth exterior product bundle of T^*M and $(\Lambda T^*M, \wedge)$ the exterior algebra bundle of T^*M. We shall append $g(\cdot, t)$ (or t for brevity) to all of the usual quantities and operators of Riemannian geometry and $x \in M$ as a subscript whenever appropriate, whence we shall write $\mathrm{tr}_{g(\cdot,t)}$ for the trace with respect to $g(\cdot, t)$, $\nabla_{g(\cdot,t)}$ for the gradient of a function, $\mathrm{div}_{g(\cdot,t)}$ for the divergence, $\nabla^2_{g(\cdot,t)}$ for the Hessian, $\Delta_{g(\cdot,t)} := \mathrm{tr}_{g(\cdot,t)} \circ \nabla^2_{g(\cdot,t)}$ for the Laplace-Beltrami operator on functions, $d^{g(\cdot,t)}(x, \cdot)$ for the geodesic distance from x, $\mathrm{inj}_x^{g(\cdot,t)}$ for the injectivity radius at x, $\mathrm{dvol}_g(\cdot, t)$ for the volume form and $\mathrm{Vol}_{g(\cdot,t)}$ for the volume of a set with respect to the Borel measure induced by $\mathrm{dvol}_g(\cdot, t)$. We furthermore introduce a Borel measure μ on $M \times [0, T_\infty[$ induced by $\mathrm{d}\mu := \mathrm{dvol}_g(\cdot, t) \wedge \mathrm{d}t$ and introduce the shorthand notation

$$\iint_{\mathcal{D}} f \,\mathrm{d}\mu(x, t) := \iint_{\mathcal{D}} f(x, t) \,\mathrm{d}\mu(x, t)$$

and

$$\int_\Omega f\,\mathrm{dvol}_g(\cdot,t) := \int_\Omega f(x,t)\mathrm{dvol}_g(\cdot,t)(x),$$

where $\mathcal{D} \subset M \times [0, T_\infty[$ and $\Omega \subset M$ are Borel-measurable sets; thus, the latter notation indicates that the integrand is to be restricted to the time slice $\Omega \times \{t\}$. If

$$\iint_{\mathcal{D}} |f|^p \,\mathrm{d}\mu(x,t) < \infty$$

for a μ-measurable function $f : \mathcal{D} \to \mathbb{R}$ and $p \geq 1$, we write $f \in L^p(\mathcal{D})$ (or simply $f \in L^p$) and furthermore write $f \in L^\infty$ if f is essentially bounded.

We denote the $g(\cdot,t)$-geodesic ball of radius r centered at $X \in M$ by $B_r^t(X)$ and define for $(X,T) \in M \times]0, T_\infty]$ and $r_1, r_2 > 0$ the *spacetime cylinder*

$$\mathcal{D}_{r_1,r_2}(X,T) = \bigcup_{t \in](T-r_2)^+, T[} B_{r_1}^t(X) \times \{t\}.$$

In applications, we shall be interested in spacetime points (X,T) about which there is a spacetime cylinder where the injectivity radius at X may be suitably controlled; to this end, whenever $T < T_\infty$, we may fix $\delta \in]0, T[$ such that for all $t \in]T-\delta, T[$

$$\mathrm{inj}_X^{g(\cdot,t)} > \frac{\mathrm{inj}_X^{g(\cdot,T)}}{2} =: j_0 \tag{2.1}$$

which furnishes us with local geometry bounds of the form

$$\begin{gathered}\Lambda_{-\infty}\mathfrak{r}(\cdot,t)^2 g^{\mathfrak{r}}(\cdot,t) \leq g(\cdot,t) - \nabla^2\left(\frac{1}{2}\mathfrak{r}(\cdot,t)^2\right) \leq \Lambda_\infty \mathfrak{r}(\cdot,t)^2 g^{\mathfrak{r}}(\cdot,t) \\ \lambda_{-\infty}\mathfrak{r}(\cdot,t)^2 \leq \partial_t \mathfrak{r}(\cdot,t)^2 \leq \lambda_\infty \mathfrak{r}(\cdot,t)^2 \\ \vartheta_{(X,t)}^* \mathrm{dvol}_g(\cdot,t) \leq \sigma\,\mathrm{d}x\end{gathered} \tag{2.2}$$

on $B_{j_0}^t(X)$ for all $t \in]T-\delta, T[$, where $\mathfrak{r}(x,t) := d^{g(\cdot,t)}(x,X)$, $\Lambda_{\pm\infty}, \sigma \in \mathbb{R}$ are constants depending on the geometry of M in $\mathcal{D}_{j_0,\delta}(X,T)$; $\lambda_{\pm\infty} \in \mathbb{R}$ are such that

$$\lambda_{-\infty} g(\cdot,t) \leq \partial_t g(\cdot,t) \leq \lambda_\infty g(\cdot,t) \tag{2.3}$$

on $B_{j_0}^t(X)$ for all $t \in]T-\delta, T[$; $g^{\mathfrak{r}}(\cdot,t) = g(\cdot,t) - \mathrm{d}\mathfrak{r}(\cdot,t) \otimes \mathrm{d}\mathfrak{r}(\cdot,t)$; $\vartheta_{(X,t)} : B_{j_0}(0) \to B_{j_0}^t(X)$ defines $g(\cdot,t)$-exponential coördinates at X; and $\mathrm{d}x = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$ is the canonical volume form on $\mathbb{R}^n$. The constants $\Lambda_{\pm\infty}$ may be given more explicitly in terms of sectional curvature bounds (cf. [18, Theorem 27, p. 175]).

We shall also *always* suppose given a finite-dimensional Riemannian vector bundle $E \to M$ equipped with a smooth one-parameter family of Riemannian connections $\{\nabla^t\}_{t \in]0,T[}$, $T \leq T_\infty$. The Riemannian metrics on TM and E induce a metric on $E \otimes \Lambda^k T^*M$ for each $k \in \mathbb{N}$ and $t \in [0, T_\infty[$ and thus on $E \otimes \Lambda T^*M$ which we denote simply by $\langle\cdot,\cdot\rangle_t$ (with norm $|\cdot|_t$) and make the identification $E \otimes \Lambda^0 T^*M \cong E$; moreover, we write $\llcorner$ for the (fibrewise) interior product $TM \times (E \otimes \Lambda T^*M) \to E \otimes \Lambda T^*M$ and $(\cdot,\cdot)$ for the (fibrewise) canonical nondegenerate bilinear pairing $(E \otimes \Lambda T^*M) \times \Lambda TM \to E$.

We denote by d^{∇^t} and δ^{∇^t} the exterior covariant differential and codifferential respectively; these are induced by ∇^t and the Levi-Civita connection on TM induced by $g(\cdot,t)$, and given in any local $g(\cdot,t)$-orthonormal frame $\{\varepsilon_i(\cdot,t)\}$ for TM with dual coframe $\{\omega^i(\cdot,t)\}$ by

$$\mathrm{d}^{\nabla^t}\psi = \sum_{i=1}^n \omega^i(\cdot,t)\wedge\nabla^t_{\varepsilon_i(\cdot,t)}\psi$$

and

$$\delta^{\nabla^t}\psi = -\sum_{i=1}^n \varepsilon_i(\cdot,t)\llcorner\nabla_{\varepsilon_i(\cdot,t)}\psi,$$

where $\psi : M \to E\otimes\Lambda T^*M$ is a smooth section of $E\otimes\Lambda T^*M$ and ∇^t denotes the connection on $E\otimes\Lambda T^*M$ induced by the connection ∇^t on E and the Levi-Civita connection on TM induced by $g(\cdot,t)$. These operators satisfy the relation

$$\left\langle \mathrm{d}^{\nabla^t}\psi_1,\psi_2\right\rangle = \left\langle \psi_1,\delta^{\nabla^t}\psi_2\right\rangle + \operatorname{div}\left(\sum_{i=1}^n \langle\psi_1,\varepsilon_i\llcorner\psi_2\rangle\,\omega^i\right) \tag{2.4}$$

for all $\psi_1,\psi_2\in\Gamma(E\otimes\Lambda T^*M)$, which motivates the so-called *Hodge Laplacian* $\Delta^{\nabla^t} := \mathrm{d}^{\nabla^t}\delta^{\nabla^t}+\delta^{\nabla^t}\mathrm{d}^{\nabla^t}$ and the associated heat operator $\mathrm{H}^\nabla\psi(\cdot,t) = (\partial_t+\Delta^{\nabla^t})\psi(\cdot,t)$ acting on a one-parameter family of sections $\{\psi(\cdot,t)\in\Gamma(E\otimes\Lambda T^*M)\}_{t\in[0,T[}$.

In the sequel, we shall omit t from all of the symbols introduced above for brevity and write e.g. $\mathrm{d}^\nabla\psi$ for $(x,t)\mapsto\left(\mathrm{d}^{\nabla^t}\psi(\cdot,t)\right)(x)$ and $|\psi|$ for $(x,t)\mapsto|\psi(\cdot,t)|_t(x)$.

2.3. **Kernels.** We proceed to introduce the heat kernel-type objects to be used in our considerations.

If the metrics $\{g(\cdot,t)\}_{t\in[0,T_\infty[}$ are complete, it is well known that the system

$$\begin{aligned} \mathrm{H}^*v = 0 \text{ on } M\times\,]0,T[\\ \lim_{t\nearrow T} v(\cdot,t) = \delta_X, \end{aligned}$$

where $\mathrm{H}^*v(\cdot,t) = \partial_t+\Delta_{g(\cdot,t)}+\frac{1}{2}\mathrm{tr}_g\partial_t g(\cdot,t)\cdot$ is the *backward heat operator* induced by the Laplace-Beltrami operator Δ_g acting on v, and δ_X is the delta distribution and the limit is to be interpreted in the distributional sense, admits a unique minimal positive solution $\mathrm{P}_{(X,T)} : M\times[0,T[\,\to\mathbb{R}^+$ for each $(X,T)\in M\times\,]0,T_\infty]$ which we refer to as the *canonical backward heat kernel* concentrated at (X,T). A related function defined independently of the completeness of $\{g(\cdot,t)\}$ that shall be of use to us in the sequel is the *Euclidean backward heat kernel* concentrated at (X,T); this function is defined as the (locally Lipschitz) map $\Gamma_{(X,T)} : M\times[0,T[\,\to\mathbb{R}^+$ such that

$$\Gamma_{(X,T)}(x,t) = \frac{1}{(4\pi(T-t))^{n/2}}\exp\left(\frac{d^{g(\cdot,t)}(x,X)}{4(t-T)}\right).$$

If $(M,\{g(\cdot,t)\})\equiv(\mathbb{R}^n,\delta)$, then $\mathrm{P}_{(X,T)}\equiv\Gamma_{(X,T)}$. More generally, we have the following asymptotics.

Theorem 2.1. [9, 23] *Suppose M is complete. For all $\varepsilon>0$ there exist a relatively compact neighbourhood Ω of X; $\tau_0\in]0,T[$; and $\xi\in C^\infty(\Omega\times[\tau_0,T],\mathbb{R}^+)$ with $\xi(X,T)=1$ such that on $\Omega\times[\tau_0,T[$,*

$$\left|\mathrm{P}_{(X,T)}-\xi\cdot\Gamma_{(X,T)}\right|\le\varepsilon.$$

Though $\Gamma_{(X,T)}$ is not a kernel per se, we shall make heavy use of it in the sequel as its definition, together with the local geometry bounds (2.2), enables us to explicity compute and estimate its derivatives rather easily; we summarize the relevant consequences in the following proposition, which follows from a direct computation. In order to tame the unwieldy inequalities that are to follow, we adopt the notation $a \sim (a_1, a_2)$ to express the inequality $a_1 \leq a \leq a_2$ whenever $a \in \mathbb{R}$ and $(a_1, a_2) \in \mathbb{R}^2$, here considered an $\mathbb{R}$-vector space; moreover, if $b_{\pm\infty} \in \mathbb{R}$, we shall write $b_{\pm\infty}$ for $(b_{-\infty}, b_{\infty}) \in \mathbb{R}^2$ and $b_{\mp\infty}$ for $(b_{\infty}, b_{-\infty}) \in \mathbb{R}^2$ in such relations.

Proposition 2.2. *If $(X,T) \in M \times]0, T_\infty[$, then $\Gamma_{(X,T)}$ is smooth on $\mathcal{D}_{j_0,\delta}(X,T)$ with j_0 and δ as before and, setting $\gamma_{(X,T)}(x,t) = \log \Gamma_{(X,T)}(x,t)$, the identities*

$$\partial_t \gamma_{(X,T)}(x,t) \sim \frac{n}{2(T-t)} - \frac{\mathfrak{r}(x,t)^2}{4(t-T)^2} + \frac{\lambda_{\mp\infty}\mathfrak{r}(x,t)^2}{4(t-T)}; \tag{2.5}$$

$$\nabla \gamma_{(X,T)}(x,t) = \frac{\mathfrak{r}(x,t)}{2(t-T)} \nabla \mathfrak{r}(x,t); \tag{2.6}$$

$$\left(\nabla^2 \gamma_{(X,T)}(x,t) + \frac{g}{2(T-t)}\right)(x,t) \sim \Lambda_{\pm\infty}\left[\log\left(\frac{1}{(4\pi(T-t))^{\frac{n}{2}}\Gamma_{(X,T)}}\right) g^{\mathfrak{r}}\right](x,t); \text{ and} \tag{2.7}$$

$$\begin{aligned} &\mathrm{H}^*\Gamma(x,t) \\ &\sim \left[\left([(n-1)\Lambda_{\pm\infty} + \lambda_{\mp\infty}] \log\left(\frac{1}{(4\pi(T-t))^{\frac{n}{2}}\Gamma_{(X,T)}(x,t)}\right) + \frac{n}{2}\lambda_{\pm\infty}\right)\Gamma_{(X,T)}\right](x,t) \end{aligned} \tag{2.8}$$

hold for $(x,t) \in \mathcal{D}_{j_0,\delta}(X,T)$, where $\lambda_\pm$ and $\Lambda_\pm$ are as in (2.2).

Analogous estimates are also at our disposal for the canonical backward heat kernel in the case where M is compact and static; we shall be content with those in the following theorem.

Theorem 2.3. [17, 11] *Set $\rho_{(X,T)} = \log \mathrm{P}_{(X,T)}$. If M is closed and static, then there exist $B, C, F \in \mathbb{R}^+$ depending on the geometry of M such that the inequalities*

$$(T-t)\left|\nabla \rho_{(X,T)}\right|^2(x,t) \leq C \log\left(\frac{B}{(4\pi(T-t))^{n/2}\mathrm{P}_{(X,T)}}\right)(x,t);$$

$$\partial_t \rho_{(X,T)}(x,t) + e^{-2K(T-t)}\left|\nabla \rho_{(X,T)}\right|^2(x,t) - e^{2K(T-t)}\frac{n}{2(T-t)} \leq 0;$$

$$(T-t)\partial_t \rho_{(X,T)}(x,t) \geq -F\left(1 + \log\left(\frac{B}{(4\pi(T-t))^{n/2}\mathrm{P}_{(X,T)}}\right)\right)(x,t); \text{ and}$$

$$\left(\nabla^2 \rho_{(X,T)} + \frac{g}{2(T-t)}\right)(x,t) \geq -F\left[\left(1 + \log\left(\frac{B}{(4\pi(T-t))^{n/2}\mathrm{P}_{(X,T)}}\right)\right) g\right](x,t)$$

hold for $(x,t) \in M \times [T-1, T[$, where $K \geq 0$ is such that $\mathrm{Ric} \geq -Kg$, *where* Ric *denotes the Ricci curvature tensor of g. If M is of nonnegative sectional curvature and $\nabla\mathrm{Ric} \equiv 0$, then*

$$\left(\nabla^2 \rho_{(X,T)} + \frac{g}{2(T-t)}\right)(x,t) \geq 0.$$

2.4. **Flows of interest.** We now turn our attention to evolution equations that may be cast as certain heat equations involving differential forms.

Let $\{\psi(\cdot,t) \in \Gamma(E\otimes\Lambda^k T^*M)\}_{t\in[0,T[}$ be a smooth one-parameter family of E-valued k-forms with $T \leq T_\infty$ and $k \in \mathbb{N}\cup\{0\}$. In applications, we shall typically assume that $\{\psi(\cdot,t)\}$ solves a problem of one of the following two types:

(I) $\psi(\cdot,t)$ solves the *heat equation*, viz. $(\partial_t + \Delta^\nabla)\psi = 0$ on $M\times]0,T[$.

(II) $\psi(\cdot,t)$ is d^{∇^t}-closed ($\mathrm{d}^{\nabla^t}\psi(\cdot,t)=0$) and there exist a smooth one-parameter family of sections $\{u(\cdot,t)\in\Gamma(E\otimes\Lambda^{k-1}T^*M)\}_{t\in[0,T[}$ and a one-parameter family of vector subbundles $\{E_0^t\}_{t\in[0,T[}$ of E such that

(II.1) $\partial_t\psi = \mathrm{d}^\nabla\partial_t u$ on $M\times[0,T[$;

(II.2) for all $t\in]0,T[$, $\partial_t u(\cdot,t)\in\Gamma(E_0^t\otimes\Lambda^{k-1}T^*M)$ and $\psi(\cdot,t)\in\Gamma(E_0^t\otimes\Lambda^k T^*M)$; and

(II.3) $(\partial_t u+\delta^\nabla\psi)(x,t)\perp(E_0^t\otimes\Lambda^{k-1}T^*M)_x$ for all $(x,t)\in M\times]0,T[$.

Though a family $\{\psi(\cdot,t)\}$ solving a problem of type (II) does not in general solve a problem of type (I), the following Pythagoras-type identity suffices for our purposes.

Lemma 2.4. *If $\{\psi(\cdot,t)\in\Gamma(E\otimes\Lambda^kT^*M)\}_{t\in]0,T[}$ solves a problem of type* (II)*, then the identity*

$$\left\langle\left(\partial_t+\Delta^\nabla\right)\psi,\psi\right\rangle = \left|X\llcorner\psi-\delta^\nabla\psi\right|^2-\left|\partial_t u+X\llcorner\psi\right|^2$$

holds for all $X\in\Gamma(TM)$.

Proof. Note that $\Delta^\nabla\psi=\mathrm{d}^\nabla\delta^\nabla\psi$ so that, by (II.1) and (2.4), we have

$$\begin{aligned}\left\langle\left(\partial_t+\Delta^\nabla\right)\psi,\psi\right\rangle &= \left\langle \mathrm{d}^\nabla\left(\partial_t u+\delta^\nabla\psi\right),\psi\right\rangle\\ &= \left\langle\partial_t u+\delta^\nabla\psi,\delta^\nabla\psi\right\rangle+\operatorname{div}\underbrace{\left(\sum_{i=1}^n\left\langle\partial_t u+\delta^\nabla\psi,\varepsilon_i\llcorner\psi\right\rangle\omega^i\right)}_{=0\text{ by (II.2) and (II.3)}},\end{aligned}$$

and since $\left\langle\partial_t u,\delta^\nabla\psi\right\rangle=-|\partial_t u|^2$ by (II.2) and (II.3), we obtain the result by adding and subtracting $|X\llcorner\psi|^2-2\left\langle X\llcorner\psi,\delta^\nabla\psi\right\rangle=|X\llcorner\psi|^2+2\left\langle X\llcorner\psi,\partial_t u\right\rangle$ to the right-hand side. ☐

We now turn our attention to some examples which shall serve to illustrate these problems' worthiness of our consideration.

Example 2.5 (Yang-Mills heat equation)**.** Suppose $G\to P\to M$ is a principal fibre bundle with compact connected semi-simple structure group G with Lie algebra $\mathfrak{g}$ and write E for the vector bundle associated to P and the adjoint representation of G on $\mathfrak{g}$; together with the Riemannian metric induced by minus the Killing form, E is a Riemannian vector bundle. We say that a smooth one-parameter family of connections $\{\omega(\cdot,t)=\omega_0+\overline{a(\cdot,t)}\in\Gamma(\mathfrak{g}\otimes T^*P)\}_{t\in[0,T[}$ on P, ω_0 some fixed connection, solves the *Yang-Mills heat equation* if

$$\partial_t a(\cdot,t)+\delta^{\nabla^t}\underline{\Omega}^{\omega(\cdot,t)}=0$$

on M for all $t\in]0,T[$, where for each $t\in[0,T[$ $a(\cdot,t)\in\Gamma(E\otimes T^*M)$ is the unique section with lift $\overline{a(t)}\in\Gamma(\mathfrak{g}\otimes T^*P)$, $\underline{\Omega}^{\omega(\cdot,t)}\in\Gamma(E\otimes\Lambda^2T^*M)$ is the *curvature* of ω^t

and ∇^t is the (Riemannian) connection on E induced by $\omega(\cdot,t)$ and the Levi-Civita connection of $g(\cdot,t)$; this equation naturally arises as the negative gradient flow of the energy density $\frac{1}{2}|\underline{\Omega}^\omega|^2$ when M is static. Taking $\psi(\cdot,t)=\underline{\Omega}^{\omega(\cdot,t)}$, it follows from the Bianchi identity $\mathrm{d}^{\nabla^t}\underline{\Omega}^{\omega(\cdot,t)}=0$ and $\partial_t\underline{\Omega}^{\omega(\cdot,t)}=\mathrm{d}^{\nabla^t}\partial_t a(\cdot,t)$ that $\{\psi(\cdot,t)\}$ solves a problem of type (II) with $u=a$ and $E_0^t=E$. On the other hand, it readily follows from the aforementioned identities that $(\partial_t+\Delta^{\nabla^t})\underline{\Omega}^{\omega(\cdot,t)}=0$ so that $\{\psi(\cdot,t)\}$ solves a problem of type (I). □

Example 2.6 (Harmonic map heat equation)**.** Suppose $(N,g_N)\subset(\mathbb{R}^K,\delta)$ is a Riemannian submanifold. A smooth one-parameter family of maps $\{u(\cdot,t):(M,g(\cdot,t))\to(N,g_N)\}_{t\in[0,T[}$ solves the *harmonic map heat equation* if for each $(x,t)\in M\times\,]0,T[$,

$$(\partial_t u-\Delta_g u)\,(x,t)\perp T_{u(x,t)}N\subset\mathbb{R}^K, \tag{HM}$$

where u is viewed as a map into $\mathbb{R}^K$ so that the Laplacian is applied componentwise. By taking E to be the trivial $\mathbb{R}^K$-bundle $M\times\mathbb{R}^K\to M$ equipped with the canonical Euclidean metric and flat connection, and setting $v=(M\times[0,T[\,\ni(x,t)\mapsto(x,u(x,t))\in E)$ so that $v(\cdot,t)$ is $u(\cdot,t)$ viewed as a section of E, we may write (HM) in the form

$$\left(\partial_t v+\delta^{\nabla}\mathrm{d}u\right)(x,t)\perp\left(u(\cdot,t)^{-1}TN\right)_x,$$

where ∇^t is induced by the connection on E and the Levi-Civita connection of $g(\cdot,t)$, $\mathrm{d}u$ denotes the differential of u as a section of $E\otimes T^*M$ and $u(\cdot,t)^{-1}TN$ is the pullback bundle of TN by $u(\cdot,t)$, here viewed as a subbundle of E; this equation arises as the negative gradient flow of the energy density $\frac{1}{2}|\mathrm{d}u|^2$ when M is static. A direct computation shows that $\mathrm{d}^\nabla\mathrm{d}v=0$ whence, setting $E_0^t=u(\cdot,t)^{-1}TN$, $\psi(\cdot,t):=\mathrm{d}u(\cdot,t)$ solves a problem of type (II). □

3. Generalised heat balls

We define a notion of heat ball in an attempt to unify those of Ecker [7, 6] and Ecker, Knopf, Ni and Topping [9], in particular allowing for different powers of r, whilst also accommodating for kernels not globally defined in spacetime.

3.1. **Definition and examples.** Suppose we are given $\Phi\in C^{1,1}(\mathcal{D},\mathbb{R}^+)$, where $\mathcal{D}\subset M\times]0,T[$ $(T\in\,]0,T_\infty])$ is open. Set

$$E_r^m(\Phi)=\left\{\Phi>\frac{1}{r^m}\right\}=\{\log(r^m\Phi)>0\}\subset\mathcal{D}$$

for $r>0$ and $0<m\le\dim M$ and write $\phi=\log(\Phi)$ and $\phi_r^m:=\log(r^m\Phi)$.

We assume that there exists an $r_0\in\,]0,1[$ such that

(HB1) $E_{r_0}^m(\Phi)\cap(M\times\,]0,\tau[)\Subset\mathcal{D}$ for every $\tau\in]0,T[$;
(HB2) $|\nabla\phi|^2,\partial_t\phi\in L^1(E_{r_0}^m(\Phi))$; and
(HB3) $\displaystyle\lim_{\tau\nearrow T}\int_{\{\Phi(\cdot,\tau)>\frac{1}{r_0^m}\}}|\phi|\,\mathrm{dvol}_g(\cdot,\tau)=0.$

Definition 3.1. With the above definition and assumptions, $E_r^m(\Phi)$ is said to be an (m,Φ)-*heat ball*.

Remark 3.2. Since $r_1<r_2\Rightarrow E_{r_1}^m(\Phi)\subset E_{r_2}^m(\Phi)$, if r_0 satisfies the above properties then so does $r\in]0,r_0[$.

Remark 3.3. In view of (HB1) and (HB3), $\phi \in L^1(E^m_{r_0}(\Phi))$.

Remark 3.4. Note that by Remark 3.2 and (HB3), if $r < r_0 < 1$, then $\phi > -m \log r > 0$ on $E^m_r(\Phi)$ and

$$\begin{aligned} 0 &= \lim_{\tau \nearrow T} \int_{\{\Phi(\cdot,\tau) > \frac{1}{r^m}\}} |\phi| \mathrm{dvol}_g(\cdot,\tau) \geq \lim_{\tau \nearrow T} (-m \log r) \int_{\{\Phi(\cdot,\tau) > \frac{1}{r^m}\}} \mathrm{dvol}_g(\cdot,\tau) \\ &= (-m \log r) \lim_{\tau \nearrow T} \mathrm{Vol}_{g(\cdot,\tau)} \left(\{\Phi(\cdot,\tau) > \frac{1}{r^m}\} \right) \end{aligned}$$

so that

$$\begin{aligned} &\int_{\{\Phi(\cdot,\tau) > \frac{1}{r^m}\}} |\phi^m_r| \mathrm{dvol}_g(\cdot,\tau) \\ &\quad \leq \int_{\{\Phi(\cdot,\tau) > \frac{1}{r^m}\}} |\phi| \mathrm{dvol}_g(\cdot,\tau) + m|\log r| \cdot \mathrm{Vol}_{g(\cdot,\tau)} \left(\{\Phi(\cdot,\tau) > \frac{1}{r^m}\} \right) \xrightarrow{\tau \nearrow T} 0. \end{aligned}$$

We now proceed to give examples of heat balls. For all of the following considerations, we fix $X \in M$ and suppose the geometry bounds (2.1) and (2.2) of §2.2 hold on $\mathcal{D}_{j_0,\delta}(X,T)$ with j_0 and δ as defined there.

The first example is an analogue of the Euclidean heat balls of Watson [22] and Ecker [8], the idea being to mimic their constructions with the Euclidean heat kernel.

Example 3.5 (Euclidean m-heat balls)**.** Set

$$\begin{aligned} \Phi = {}^m\Gamma_{(X,T)} &:= \left(\mathcal{D}_{j_0,\delta}(X,T) \ni (x,t) \mapsto [4\pi(T-t)]^{\frac{n-m}{2}} \right) \cdot \Gamma_{(X,T)} \\ &= \left(\mathcal{D}_{j_0,\delta}(X,T) \ni (x,t) \mapsto \frac{1}{(4\pi(T-t))^{m/2}} \exp\left(\frac{d^t(X,x)^2}{4(t-T)} \right) \right) \end{aligned}$$

for fixed $m > 0$.

Note that

$$\phi^m_r(x,t) > 0 \Leftrightarrow d^t(x,X)^2 < 2m(t-T) \log\left(\frac{4\pi(T-t)}{r^2} \right) =: R^m_r(t-T)^2,$$

where we suppose $R^m_r(t-T) \geq 0$ whenever it is defined. On the other hand, since $t - T < 0$ in $\mathcal{D}_{j_0,\delta}(X,T)$, we see that

$$\begin{aligned} R^m_r(t-T)^2 > 0 &\Leftrightarrow \log\left(\frac{4\pi(T-t)}{r^2} \right) < 0 \\ &\Leftrightarrow t > T - \frac{r^2}{4\pi}, \end{aligned}$$

whence it is clear that

$$E^m_r(\Phi) = \left(\bigcup_{t \in]T - \frac{r^2}{4\pi}, T[} B^t_{R^m_r(t-T)}(X) \times \{t\} \right) \bigcap \mathcal{D}_{j_0,\delta}(X,T).$$

Let

$$r_0 = \frac{1}{2} \min\left\{ j_0 \cdot \sqrt{\frac{2\pi e}{m}}, \sqrt{4\pi\delta}, 1 \right\}.$$

We claim that $E^m_r(\Phi)$ is an (m,Φ)-heat ball for $r < r_0$ and now proceed to verify the conditions (HB1)-(HB3).

(HB1) A quick computation shows that

$$R^m_r \leq \sqrt{\frac{m}{2\pi e}}\, r \tag{3.1}$$

wherever R^m_r is defined. Thus, we have that $R^m_{r_0} < \frac{j_0}{2}$ and, from the definition of r_0, $T - \frac{r_0^2}{4\pi} > T - \delta$, whence

$$E^m_r(\Phi) = \bigcup_{t \in]T - \frac{r^2}{4\pi}, T[} B^t_{R^m_r(t-T)}(X) \times \{t\} \tag{3.2}$$

and

$$\overline{E^m_r(\Phi) \cap \mathrm{pr}_2^{-1}(]T-\delta, \tau[)} = \bigcup_{t \in [T - \frac{r_0^2}{4\pi}, \tau]} \overline{B^t_{R^m_{r_0}(t-T)}(X)} \times \{t\} \Subset \mathcal{D}_{j_0,\delta}(X,T)$$

for every $\tau \in]T-\delta, T[$.

(HB2) Using exponential coördinates about X with respect to $g(\cdot, t)$ ($t \in]T-\delta, T[$ fixed), we see by virtue of Proposition 2.2 and the volume bound of (2.2) that

$$\begin{aligned}\int_{B^t_{R^m_{r_0}(t-T)}(X)} |\nabla\phi|^2 \mathrm{dvol}_g(\cdot, t) &\leq \sigma \int_{B_{R^m_{r_0}(t-T)}(0)} \frac{|x|^2}{4(T-t)^2}\,\mathrm{d}x \\ &= \frac{n\omega_n \sigma}{4(n+2)} \cdot \frac{R^m_{r_0}(t-T)^{n+2}}{(T-t)^2}\end{aligned}$$

and similarly,

$$\begin{aligned}&\int_{B^t_{R^m_{r_0}(t-T)}(X)} |\partial_t \phi| \mathrm{dvol}_g(\cdot, t) \\ &\quad \leq \frac{n\omega_n\sigma}{2}\left\{\frac{R^m_{r_0}(t-T)^n}{T-t} + \frac{R^m_{r_0}(t-T)^{n+2}}{2(n+2)(T-t)^2} + \frac{\lambda R^m_{r_0}(t-T)^{n+2}}{2(n+2)(T-t)}\right\},\end{aligned}$$

where ω_n is the volume of $B_1(0) \subset \mathbb{R}^n$ and $\lambda \geq 0$ depends on $\lambda_\pm$. A straightforward but lengthy computation then shows that the bounding functions of t are summable over $\left]T - \frac{r_0^2}{4\pi}, T\right[$.

(HB3) Passing to exponential coördinates as in the verification of (HB2) and using the fact that $\{\Phi(\cdot, \tau) > \dfrac{1}{r^m}\} = B_{R^m_{r_0}(\tau-T)}(X)$, we see that it suffices to show that

$$\lim_{\tau \nearrow T} \int_{B_{R^m_{r_0}(\tau-T)}(0)} \left(|\phi| \circ \vartheta_{(X,\tau)}\right)(x)\,\mathrm{d}x = 0. \tag{3.3}$$

However, for $\tau \in \left]T - \frac{1}{4\pi}, T\right[$, we may bound this integral as

$$\begin{aligned}&\int_{B_{R^m_{r_0}(\tau-T)}(0)} \left(|\phi| \circ \vartheta_{(X,\tau)}\right)(x)\,\mathrm{d}x \\ &\quad \leq \int_{B_{R^m_{r_0}(\tau-T)}(0)} \frac{|x|^2}{4(T-\tau)} - \frac{m}{2}\log\left(4\pi(T-\tau)\right)\mathrm{d}x\end{aligned}$$

$$= n\omega_n \left(\frac{R^m_{r_0}(\tau - T)^{n+2}}{4(n+2)(T-\tau)} - \frac{m}{2}\log\left(4\pi(T-\tau)\right) R^m_{r_0}(\tau - T)^n \right) \xrightarrow{\tau \nearrow T} 0. \quad \square$$

Remark 3.6. If $(M, g(\cdot, t)) \equiv (\mathbb{R}^n, \delta)$, the preceding example reduces to the heat balls of Watson [22] for $m = n$ and to those of Ecker [8] for $m = n - \gamma$ with $\gamma \in \,]0, n[$ fixed.

Following [9], we turn our attention to heat balls constructed from the canonical backward heat kernel on M. However, for later purposes, we shall require that the backward heat kernel satisfy certain differential inequalities in heat balls which are only at our disposal on compact static manifolds. For this reason, we now consider heat balls on static compact manifolds. Strictly speaking, these do not generalize the Euclidean heat balls of Watson and Ecker, but they provide an adaptation different from that of Example 3.5 in this setting.

Example 3.7 (m-heat balls on static compact manifolds)**.** We suppose that (M^n, g) is compact and static, $m \in \,]0, n]$, and let $\mathrm{P}_{(X,T)}$ denote the canonical backward heat kernel on M centred at $(X, T) \in M \times \mathbb{R}$. By Theorem 2.1, there exists a neighbourhood $\Omega \subset M$ of $X \in M$ and $\tau_0 \in \,]T - 1, T[\, \cap \,]0, T[$ such that

$$\frac{1}{2}\Phi_{(X,T)} - 1 \leq \mathrm{P}_{(X,T)} \leq 2\Phi_{(X,T)} + 1 \tag{3.4}$$

on $\Omega \times [\tau_0, T[$. Hence, we set $\mathcal{D} = \Omega \times [\tau_0, T[$ and define the map

$$\begin{aligned} {}^m\mathrm{P}_{(X,T)} &: \mathcal{D} \to \mathbb{R}^+ \\ (x, t) &\mapsto (4\pi(T-t))^{\frac{n-m}{2}}\, \mathrm{P}_{(X,T)}(x, t). \end{aligned}$$

We claim that $E^m_r({}^m\mathrm{P}_{(X,T)})$ is a heat ball for

$$r < r_0 := \frac{1}{2}\min\left\{9^{-1/m}, \left(1 + 2\varrho^{-m}\right)^{-1/m}\right\},$$

where $\varrho = \min\left\{\sqrt{4\pi(T-\tau_0)}, \sqrt{\frac{2\pi e}{m}}\sup\{y \in \mathbb{R}^+ : B_y(X) \Subset \Omega\},\ r_0 \text{ of Example 3.5}\right\}$.

To simplify notation, we write P for ${}^m\mathrm{P}_{(X,T)}$ and ρ for $\log \mathrm{P}$. Now, multiplying (3.4) by $(T-t)^{\frac{n-m}{2}}$ and noting that $T - t < 1$ for $t \in [\tau_0, T[$, it is clear that

$$\frac{1}{2}\Phi - 1 \leq \mathrm{P} \leq 2\Phi + 1, \tag{3.5}$$

on $\mathcal{D}$, where Φ is as in Example 3.5.

(HB1) The inequality (3.5) immediately implies that

$$E^m_{r_0}(\mathrm{P}) \subset \mathcal{D} \cap E^m_{\widetilde{r}_0}(\Phi)$$

with $\widetilde{r}_0 = \left(\frac{2}{\frac{1}{(r_0)^m} - 1}\right)^{1/m}$ and, by (3.2),

$$E^m_{\widetilde{r}_0}(\Phi) = \bigcup_{t \in \,]T - \frac{\widetilde{r}_0^2}{4\pi}, T[} B^t_{R^m_{\widetilde{r}_0}(t-T)}(X) \times \{t\} \subset B_{\varrho_0}(X) \times \,]T - \frac{\widetilde{r}_0^2}{4\pi}, T[$$

with $\varrho_0 = \sqrt{\frac{m}{2\pi e}}\widetilde{r}_0$ (cf. Example 3.5). In view of the choice of r_0 above, it is easily verified that $B_{\varrho_0}(X) \Subset \Omega$ and $]T - \frac{\widetilde{r}_0^2}{4\pi}, T[\subset \,]\tau_0, T[$, whence

$$E^m_{r_0}(\mathrm{P}) \subset E^m_{\widetilde{r}_0}(\Phi) \tag{3.6}$$

and

$$\overline{E^m_{\widetilde{r}_0}(\mathrm{P}) \cap \mathrm{pr}_2^{-1}(]\tau_0, \tau[)} \subset \overline{B_{\varrho_0}(X)} \times [T - \frac{\widetilde{r}_0^2}{4\pi}, \tau] \subset \Omega \times]\tau_0, T[= \mathcal{D}.$$

(HB2) By Theorem 2.3, the summability of $|\partial_t \rho|$ and $|\nabla\rho|^2$ on $E^m_{r_0}(\mathrm{P})$ immediately follows from the summability of $(x,t) \mapsto \frac{1}{T-t}$ and $(x,t) \mapsto \frac{\log(4\pi(T-t))}{T-t}$ on $E^m_{\widetilde{r}_0}(\Phi)$ by the inclusion (3.6).

(HB3) Since $r_0 < 9^{-1/m}$ and hence $\widetilde{r}_0 < 4^{-1/m}$, it is clear that $\Phi > 4$ on $E^m_{\widetilde{r}_0}(\Phi)$, whence

$$1 \le \mathrm{P} \le \frac{5}{4}\Phi \Rightarrow 0 \le \rho \le \log\frac{5}{4} + \log\Phi$$

on $E^m_{r_0}(\mathrm{P})$. Thus, to establish (HB3) it suffices to show that

$$\lim_{\tau \nearrow T} \int_{\mathrm{pr}_1(E^m_{\widetilde{r}_0}(\Phi) \cap (M \times \{\tau\}))} |\log\Phi| \,\mathrm{dvol}_g(\cdot, \tau) = 0,$$

but this was established in Example 3.5 (HB3). □

3.2. **Integration formulæ.** We now derive integration formulæ for integrals over heat balls in the spirit of [6] and [8]. These shall be used repeatedly in the sequel. To this end, we shall consider the *approximate* integrals

$$J^r_q(f) := \iint f \cdot (\chi_q \circ \phi^m_r) \cdot (\chi_{]0, T-q^{-1}[} \circ \mathrm{pr}_2)\,\mathrm{d}\mu,$$

where χ_q is as in §2, and analyze them, as well as their derivatives with respect to r, in the limit $q \to \infty$. The idea here is that these approximate the heat ball integrals

$$I^r(f) := \iint_{E^m_r(\Phi)} f\,\mathrm{d}\mu$$

which would, with the right conditions on Φ, yield an integral over $\partial E^m_r(\Phi)$ upon differentiation with respect to r (cf. [6]). However, without additional information about Φ, we wouldn't be able to utilize this technique, which is why we follow the approach of [9].

To streamline the proofs of the integration formulæ to follow, we summarize the relevant properties of these approximate integrals in the following lemma, which readily follows from standard integration theorems.

Lemma 3.8. *Let $f \in L^1\left(E^m_{r_0}(\Phi)\right)$ and suppose J^r_q and I^r are as above. Then*

(1) Whenever $0 < r \le r_0$, we have $|J^r_q(f)| \le I^{r_0}(|f|)$ and $r \mapsto J^r_q(f)$ is smooth.

(2) For every $r \in]0, r_0]$, $J^r_q(f) \xrightarrow{q\to\infty} I^r(f)$.

(3) Whenever $0 < r_1 < r_2 < r_0$ and $\int_{r_1}^{r_2} \frac{\mathrm{d}}{\mathrm{d}r} J^r_q(f) \xrightarrow{q\to\infty} \int_{r_1}^{r_2} J$ with $J \in L^1(]r_1, r_2[)$, the identity

$$I^{r_2}(f) - I^{r_1}(f) = \int_{r_1}^{r_2} J$$

holds. In particular, $\frac{\mathrm{d}}{\mathrm{d}r} I^r(f) = J$ almost everywhere on $]0, r_0[$.

Proposition 3.9. *Suppose $X \in C^1(E^m_{r_0}(\Phi), TM)$ is such that $X(\cdot,t)$ is a local section of TM for each $t \in \mathrm{pr}_2(E^m_{r_0}(\Phi))$ and $|X|^2 \in L^1$, $f \in C^1(E^m_{r_0}(\Phi))$ with $\partial_t f \in L^1$ and $\mathrm{tr}_g \partial_t g \in L^1$. The following implications hold for almost every $r \in]0, r_0[$:*

(i) $\mathrm{div} X \in L^1 \Rightarrow \dfrac{\mathrm{d}}{\mathrm{d}r} \displaystyle\iint_{E^m_r(\Phi)} \langle X, \nabla\phi\rangle \,\mathrm{d}\mu = -\frac{m}{r} \iint_{E^m_r(\Phi)} \mathrm{div}\ X \,\mathrm{d}\mu$;

(ii) $f \in L^\infty \Rightarrow \dfrac{\mathrm{d}}{\mathrm{d}r} \displaystyle\iint_{E^m_r(\Phi)} f \cdot \partial_t \phi \ \mathrm{d}\mu = -\frac{m}{r} \iint_{E^m_r(\Phi)} \partial_t f + \frac{1}{2} f tr_g \partial_t g \ \mathrm{d}\mu.$

Proof. (i) Using Gauß' theorem andd the fact that $\nabla \phi^m_r = \nabla \phi$, we see that

$$\frac{\mathrm{d}}{\mathrm{d}r} J^r_q(\langle X, \nabla\phi\rangle) = -\frac{m}{r} J^r_q(\mathrm{div}\ X),$$

whence the summability conditions div $X, \langle X, \nabla\phi\rangle \in L^1(E^m_{r_0}(\Phi))$ permit us to appeal to Lemma 3.8.

(ii) By integrating by parts with respect to t and using (HB1), we see that

$$\frac{\mathrm{d}}{\mathrm{d}r} J^r_q(f \cdot \partial_t \phi) = \frac{m}{r} \left(\int f \cdot (\chi_q \circ \phi^m_r) \mathrm{dvol}_g(\cdot, T - q^{-1}) - J^r_q(\partial_t f + \frac{1}{2} \mathrm{tr}_g \partial_t g \cdot f) \right)$$

The former term on the right-hand side tends to 0 as $q \to \infty$ uniformly in $r \in [r_1, r_2]$ by the boundedness of f and the fact that $\mathrm{Vol}_{g(\cdot, T-q^{-1})}(\{\Phi(\cdot, T - q^{-1}) > \frac{1}{r^m}\}) = o(1)$ as $q \to \infty$ (cf. Remark 3.4), whence an application of Lemma 3.8 yields the result.

□

4. A local energy inequality

In this section we shall establish a local energy inequality for problems of type (I) and (II) expressing the integral of $\frac{1}{2}|\psi|^2$ over cross-sections of the Euclidean $(n-2k)$-heat ball close to T in terms of appropriate L^2-norms of ψ; this will allow us to deduce a natural condition under which the quantities occurring in the heat ball formulæ to be derived in the sequel are finite.

We first recall an energy identity from [2]. As in §1, we write

$$\mathcal{Q}_s(f, g) = \nabla^2 f + \frac{1}{2} \partial_t g + \frac{g}{2(s-t)}$$

for the matrix Harnack form associated with a parameter $s > 0$, a smooth function f on an open subset of $M \times]0, T_\infty[$ and the evolving metric g, and set $\mathcal{S}_{\psi,f} = \nabla f \llcorner \psi - \delta^\nabla \psi$ for a one-parameter family $\{\psi(\cdot,t) \in \Gamma(E \otimes \Lambda^k T^* M)\}_{t \in [0,T[}$.

Theorem 4.1. *If $\varphi \in C^{2,1}(M \times [0, T[, \mathbb{R})$ with $\varphi(\cdot, t) \in C^2_0(M)$ for each $t \in [0, T[$, $\Phi \in C^2(\mathcal{D}, \mathbb{R}^+)$ with $\mathcal{D}$ open such that* supp $\varphi(\cdot,t) \subset \mathrm{pr}_1(\mathcal{D} \cap (M \times \{t\}))$ *for each $t \in]0, T[$, and $\{\psi(\cdot, t) \in \Gamma(E \otimes \Lambda^k T^* M)\}_{t \in]0,T[}$ is a smooth one-parameter family of* *sections, then*

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_M \frac{1}{2}|\psi|^2\Phi\varphi^2\mathrm{dvol}_g(\cdot,t)\right)\\
&=\int_M \Bigg\{\left[\left\langle\psi,\mathrm{H}^\nabla\psi\right\rangle\Phi+\frac{1}{2}|\psi|^2\left(\mathrm{H}^*+\frac{k}{T-t}\right)\Phi\right]-\Phi\cdot\left[\left|\mathrm{d}^\nabla\psi\right|^2+|\mathcal{S}_{\psi,\log\Phi}|^2\right]\\
&\qquad\qquad -\Phi\left\langle\mathcal{Q}_T(\log\Phi,g),\sum_{i,j}\langle\varepsilon_i\llcorner\psi,\varepsilon_j\llcorner\psi\rangle\,\omega^i\otimes\omega^j\right\rangle\Bigg\}\varphi^2 \qquad (4.1)\\
&\qquad\qquad +|\psi|^2\Phi\varphi\left(\partial_t\varphi+\left\langle\nabla\varphi,\frac{\nabla\Phi}{\Phi}\right\rangle\right)\\
&\qquad\qquad\quad +2\varphi\Phi\left[\langle\nabla\varphi\llcorner\psi,\mathcal{S}_{\psi,\log\Phi}\rangle-\left\langle\nabla\varphi\llcorner\mathrm{d}^\nabla\psi,\psi\right\rangle\right]\mathrm{dvol}_g(\cdot,t).
\end{aligned}
$$

on $]0,T[$.

Proof. This is an immediate consequence of Theorem 2.1 of [2] with $\varphi\psi$ in place of ψ and Φ in place of $(4\pi(T-t))^k\Gamma$. ☐

We now restrict our attention to problems of type (I) and (II). Let $s\geq T$ and note that

$$
\begin{aligned}
&\frac{1}{2}|\psi|^2\frac{k}{T-t}\Phi-\Phi\left\langle\mathcal{Q}_T(\log\Phi,g),\sum_{i,j}\langle\varepsilon_i\llcorner\psi,\varepsilon_j\llcorner\psi\rangle\,\omega^i\otimes\omega^j\right\rangle\\
&=\frac{1}{2}|\psi|^2\frac{k}{s-t}\Phi-\Phi\left\langle\mathcal{Q}_s(\log\Phi,g),\sum_{i,j}\langle\varepsilon_i\llcorner\psi,\varepsilon_j\llcorner\psi\rangle\,\omega^i\otimes\omega^j\right\rangle
\end{aligned}
$$

so that by Lemma 2.4 (4.1) reduces to

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_M \frac{1}{2}|\psi|^2\Phi\varphi^2\mathrm{dvol}_g\right)\\
&=\int_M \Bigg\{\left[\frac{1}{2}|\psi|^2\left(\mathrm{H}^*+\frac{k}{s-t}\right)\Phi\right]-\Phi\cdot\left[\left|\mathrm{d}^\nabla\psi\right|^2+|\mathcal{T}_{\psi,\phi}|^2\right]\\
&\qquad\qquad -\Phi\left\langle\mathcal{Q}_s(\log\Phi,g),\sum_{i,j}\langle\varepsilon_i\llcorner\psi,\varepsilon_j\llcorner\psi\rangle\,\omega^i\otimes\omega^j\right\rangle\Bigg\}\varphi^2 \qquad (4.2)\\
&\qquad\qquad +|\psi|^2\Phi\varphi\left(\partial_t\varphi+\left\langle\nabla\varphi,\frac{\nabla\Phi}{\Phi}\right\rangle\right)\\
&\qquad\qquad -2\varphi\Phi\left[\langle\nabla\varphi\llcorner\psi,\mathcal{T}_{\psi,\phi}\rangle+\left\langle\nabla\varphi\llcorner\mathrm{d}^\nabla\psi,\psi\right\rangle\right]\mathrm{dvol}_g(\cdot,t),
\end{aligned}
$$

where $\mathcal{T}_{\psi,\phi}$ is given by

$$
\mathcal{T}_{\psi,\phi}=\begin{cases}\nabla\phi\llcorner\psi-\delta^\nabla\psi, & \{\psi(\cdot,t)\}\text{ solves a problem of type (I)}\\ \partial_t u+\nabla\phi\llcorner\psi, & \{\psi(\cdot,t)\}\text{ solves a problem of type (II).}\end{cases} \qquad (4.3)
$$

Note that the last two terms in the integrand of the right-hand integral in (4.2) may, by the Cauchy-Schwarz and Young inequalities, be estimated as

$$
\begin{aligned}
&|\psi|^2\Phi\varphi\left(\partial_t\varphi+\left\langle\nabla\varphi,\frac{\nabla\Phi}{\Phi}\right\rangle\right)-2\varphi\Phi\left(\langle\nabla\varphi\llcorner\psi,\mathcal{S}_{\psi,\phi}\rangle+\left\langle\nabla\varphi\llcorner\mathrm{d}^{\nabla}\psi,\psi\right\rangle\right)\\
&\quad\le|\psi|^2\left(\varphi\left(|\partial_t\varphi|\Phi+|\nabla\varphi|\cdot|\nabla\Phi|\right)+2|\nabla\varphi|^2\Phi\right)+\frac{1}{2}\left(|\mathcal{S}_{\psi,\phi}|^2+|\mathrm{d}^{\nabla}\psi|^2\right)\Phi\varphi^2; \qquad (4.4)
\end{aligned}
$$

this observation immediately leads to the following monotonicity-type identity:

Lemma 4.2. *Let $\varphi\in C^{2,1}(M\times]T-\delta_0,T[\,,[0,1])$ be such that*

$$\varphi|_{\mathcal{D}_{r_1,\delta_0}(X,T)}\equiv 1 \text{ and } \varphi|_{(M\times]T-\delta_0,T[)\setminus\mathcal{D}_{r_2,\delta_0}(X,T)}\equiv 0$$

for $0<r_1<r_2<R$ with $R>0$ fixed, $\delta_0\in]0,T[$ and $\Phi\in C^{2,1}(\mathcal{D}_{R,\delta_0}(X,T),\mathbb{R}^+)$, and suppose $\{\psi(\cdot,t)\in\Gamma(E\otimes\Lambda^kT^*M)\}_{t\in]0,T[}$ solves a problem of type (I) or (II).

If Φ, $|\nabla\Phi|$ are bounded on $\mathcal{D}_{r_2,\delta_0}(X,T)\setminus\mathcal{D}_{r_1,\delta_0}(X,T)$ and the inequalities

$$
\begin{aligned}
\left(\mathrm{H}^*+\frac{k}{s-t}\right)\Phi(x,t)&\le a_0+a_1(t)\Phi(x,t)\\
\Phi\cdot\mathcal{Q}_s(\log\Phi,g)(x,t)&\ge\left(b_0+b_1(t)\Phi(x,t)\right)g(x,t)
\end{aligned}
\qquad (4.5)
$$

hold for $(x,t)\in\mathcal{D}_{r_2,\delta_0}(X,T)$ with $a_1,b_1\in C(]T-\delta_0,T[)\cap L^1(]T-\delta_0,T[)$, $a_0,b_0\in\mathbb{R}$ and some $s\ge T$, then

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\int_t^T l}\int_M\frac{1}{2}|\psi(\cdot,t)|^2\Phi\varphi^2(\cdot,t)\mathrm{dvol}_g(\cdot,t)\right)\\
&\le e^{\int_t^T l}\left(\int_M-\frac{1}{2}\Phi\varphi^2(\cdot,t)\cdot\left(|\mathrm{d}^{\nabla}\psi|^2+|\mathcal{T}_{\psi,\phi}|^2\right)\mathrm{dvol}_g+C_0\int_{B^t_{r_2}(X)}\frac{1}{2}|\psi|^2\mathrm{dvol}_g\right)(\cdot,t),
\end{aligned}
$$

where $l_k(t)=a_1(t)-2kb_1(t)$ and $C_0=C_0(l_k,a_0,b_0,\Phi,\varphi,r_1,r_2)>0$ is given by

$$C_0=\frac{\exp(\int_{T-\delta_0}^T|l|)}{2}\left(a_0-2kb_0+2C_1\right). \qquad (4.6)$$

with $C_1=\sup_{\mathcal{D}_{r_2,\delta}(X,T)\setminus\mathcal{D}_{r_1,\delta}(X,T)}\left[(|\partial_t\varphi|+2|\nabla\varphi|^2)\Phi+|\nabla\varphi|\cdot|\nabla\Phi|\right]$.

Remark 4.3. We note that such an inequality was exploited by Chen and Struwe [4] and Chen and Shen [5] with $\Phi={}^{n-2k}\Gamma_{(X,T)}$ to establish monotonicity formulæ for the harmonic map and Yang-Mills heat flows on static compact manifolds (cf. [2]). A similar idea was used by Hamilton [12] to establish monotonicity formulæ for these flows on static compact manifolds, where he instead considered $\Phi={}^{n-2k}\mathrm{P}_{(X,T)}$ and $\varphi\equiv 1$.

We now make use of Lemma 4.2 to derive the promised local energy inequality.

Lemma 4.4. *Suppose $\{\psi(\cdot,t)\in\Gamma\left(E\otimes\Lambda^kT^*M\right)\}_{t\in]0,T[}$ with $n>2k$, $T<T_\infty$, solves a problem of type (I) or (II), and R^m_r is as in Example 3.5. For every $0<r<\min\left\{1,\frac{j_0}{2c_{n,k}},\sqrt{4\pi\delta}\right\}$ with $c_{n,k}:=\sqrt{\frac{n-2k}{2\pi e}}$ and $t\in\left]T-r^2\exp\left(-\frac{1}{2(n-2k)}\right)/4\pi,T\right[$,* *the estimates*

$$\frac{1}{R_r^{n-2k}(t-T)^{n-2k}}\int_{B^t_{R_r^{n-2k}(t-T)}(X)}\frac{1}{2}|\psi|^2\mathrm{dvol}_g(\cdot,t)$$
$$\leq \frac{\widetilde{C}_0}{r^{n-2k}}\left(r^{-2}\iint\limits_{\mathcal{D}_{2c_{n,k}r,\frac{r^2}{4\pi}}(X,T)}\frac{1}{2}|\psi|^2\,\mathrm{d}\mu+\int_{B^{T-\frac{r^2}{4\pi}}_{2c_{n,k}r}(X)}\frac{1}{2}|\psi|^2\mathrm{dvol}_g(\cdot,T-\frac{r^2}{4\pi})\right) \tag{4.7}$$

and

$$\iint\limits_{E_r^{n-2k}\left({}^{n-2k}\Gamma_{(X,T)}\right)}|\mathcal{T}_{\psi,\log{}^{n-2k}\Gamma_{(X,T)}}|^2+|\mathrm{d}^\nabla\psi|^2\,\mathrm{d}\mu$$
$$\leq \widetilde{C}_0\left(r^{-2}\iint\limits_{\mathcal{D}_{2c_{n,k}r,\frac{r^2}{4\pi}}(X,T)}\frac{1}{2}|\psi|^2\,\mathrm{d}\mu+\int_{B^{T-\frac{r^2}{4\pi}}_{2c_{n,k}r}(X)}\frac{1}{2}|\psi|^2\mathrm{dvol}_g(\cdot,T-\frac{r^2}{4\pi})\right) \tag{4.8}$$

hold, where $\widetilde{C}_0$ is a constant depending only on n, k, χ and local geometry bounds (2.2).

Proof. (cf. [8, Appendix]) We first apply Lemma 4.2 by taking

$$\varphi(x,t)=\begin{cases}\chi\left(\frac{c_{n,k}r}{\mathfrak{r}(x,t)}\right), & x\neq X\\ 1, & x=X\end{cases}$$

with χ and $\mathfrak{r}$ as in §2.1 and §2.2, and $r<\frac{j_0}{2c_{n,k}}$, whence it is easily verified that φ satisfies the hypotheses of Lemma 4.2 with $r_1=c_{n,k}r$ and $r_2=2r_1$. We moreover take $\Phi={}^{n-2k}\Gamma_{(X,s)}$, here considered a function $\mathcal{D}_{j_0,\delta}(X,T)\to\mathbb{R}$, with $s\geq T$ to be fixed. It may be shown exactly as in Proposition 2.2 that the inequalities (4.5) hold with a_0, $a_1(t)$, b_0 and $b_1(t)$ given by

$$\begin{aligned}a_0&=\max\{\frac{n}{2}\lambda^+_{+\infty},[(n-1)\Lambda_{+\infty}+\lambda_{-\infty}]^+\}\\ a_1(t)&=-\frac{(n-2k)a_0}{2}\log\left(4\pi(T-t)\right)\\ b_0&=\min\{\frac{\lambda^-_{-\infty}}{2},\Lambda^-_{-\infty}\}\\ b_1(t)&=-\frac{(n-2k)b_0}{2}\log\left(4\pi(T-t)\right);\end{aligned}$$

moreover, a straightforward computation shows that

$$\max\{\Phi,r|\nabla\Phi|\}\leq\frac{d_{n,k}}{r^{n-2k}}$$

on $\mathcal{D}_{2c_{n,k}r,\delta}(X,T)\backslash\mathcal{D}_{c_{n,k}r,\delta}(X,T)$, where $d_{n,k}$ is a positive constant depending only on n and k. Setting $\lambda=\max\{|\lambda^-_{-\infty}|,|\lambda^+_{\infty}\}$ and restricting our attention to $r<1$, the constant C_0 of Lemma 4.2 defined by (4.6) may be written as

$$C_0 = \frac{\exp\left(\int_0^T |l_k|\right)}{2}\left(a_0 - 2kb_0 + d_{n,k}\sup|\chi'|\cdot\left[\frac{\frac{4\sup|\chi'|}{c_{n,k}^2} + \frac{2}{c_{n,k}} + r^2\lambda}{r^{n-2k+2}}\right]\right)$$

$$\leq \exp\left(\int_0^T |l_k|\right)\cdot\frac{\varsigma_0}{r^{n-2k+2}}$$

where ς_0 is a positive constant depending only on n, k, χ and the local geometry of M about (X,T). Hence, after an integration, Lemma 4.2 implies that

$$\int_M \frac{1}{2}|\psi|^2\Phi\varphi^2\mathrm{dvol}_g(\cdot,t_0) + \int_{T-\frac{r^2}{4\pi}}^{t_0}\int_M \frac{1}{2}\Phi\varphi^2\cdot\left(|\mathrm{d}^\nabla\psi|^2 + |\mathcal{T}_{\psi,\phi}|^2\right)\mathrm{dvol}_g\,\mathrm{d}t$$

$$\leq e^{2\int_0^T|l_k|}\cdot\left[\frac{\varsigma_0}{r^{n-2k+2}}\iint\limits_{\mathcal{D}_{2c_{n,k}r,\frac{r^2}{4\pi}}(X,T)}\frac{1}{2}|\psi|^2\mathrm{d}\mu + \int_M\frac{1}{2}|\psi|^2\Phi\varphi^2\mathrm{dvol}_g(\cdot,T-\frac{r^2}{4\pi})\right]$$

for $t_0 \in \left]T-\frac{r^2}{4\pi},T\right[$. Now, using $\chi_{B^t_{c_{n,k}r}(X)} \leq \varphi(\cdot,t) \leq \chi_{B^t_{2c_{n,k}r}(X)}$ and noting that

$$\Phi(\cdot,T-\frac{r^2}{4\pi}) \leq \frac{1}{(4\pi(s-T+\frac{r^2}{4\pi})^{\frac{n-2k}{2}}} \leq \frac{1}{r^{n-2k}},$$

we obtain

$$\int_{B^{t_0}_{c_{n,k}r}(X)}\frac{1}{2}|\psi|^2\Phi\mathrm{dvol}_g(\cdot,t_0) + \int_{T-\frac{r^2}{4\pi}}^{t_0}\int_{B^t_{c_{n,k}r}(X)}\frac{1}{2}\Phi\cdot\left(|\mathrm{d}^\nabla\psi|^2 + |\mathcal{T}_{\psi,\phi}|^2\right)\mathrm{dvol}_g\,\mathrm{d}t$$

$$\leq \frac{e^{2\int_0^T|l_k|}}{r^{n-2k}}\cdot\left[\frac{\varsigma_0}{r^2}\iint\limits_{\mathcal{D}_{2c_{n,k}r,\frac{r^2}{4\pi}}(X,T)}\frac{1}{2}|\psi|^2\mathrm{d}\mu + \int_{B^{T-\frac{r^2}{4\pi}}_{2c_{n,k}r}(X)}\frac{1}{2}|\psi|^2\mathrm{dvol}_g(\cdot,T-\frac{r^2}{4\pi})\right]. \tag{4.9}$$

We first take $s=T$, discard the first term on the left-hand side of (4.9), use the inclusion $B^t_{R_r^{n-2k}(t-T)}(X) \subset B^t_{c_{n,k}r}(X)$ as well as the identity

$$\Phi(\cdot,t)|_{B^t_{R_r^{n-2k}(t-T)}(X)} \geq \frac{1}{r^{n-2k}},$$

and take the limit $t_0 \nearrow T$ on the left hand side to obtain

$$\frac{1}{r^{n-2k}}\iint\limits_{E_r^{n-2k}\left({}^{n-2k}\Gamma_{(X,T)}\right)}\frac{1}{2}\cdot\left(|\mathrm{d}^\nabla\psi|^2 + |\mathcal{T}_{\psi,\phi}|^2\right)\mathrm{d}\mu$$

$$\leq \frac{\varsigma_0\cdot e^{2\int_0^T|l_k|}}{r^{n-2k+2}}\int_{T-\frac{r^2}{4\pi}}^T\int_{B^t_{2c_{n,k}r}(X)}\frac{1}{2}|\psi|^2\mathrm{dvol}_g\,\mathrm{d}t$$

$$+\frac{e^{2\int_0^T|l|}}{r^{n-2k}}\int_{B^{T-\frac{r^2}{4\pi}}_{2c_{n,k}r}(X)}\frac{1}{2}|\psi|^2\mathrm{dvol}_g(\cdot,T-\frac{r^2}{4\pi}),$$

thus establishing (4.8). As for (4.7), we discard the second term on the left-hand side of (4.9) and take $s = t_0 + R_r^{n-2k}(t_0 - T)^2$, which, for $t_0 \in \left]T - e^{-\frac{1}{2(n-2k)}} \frac{r^2}{4\pi}, T\right[$, is greater than T, so that

$$\Phi(\cdot, t_0)|_{B^{t_0}_{R_r^{n-2k}(t_0-T)}(X,T)} \geq \frac{\exp(-\frac{1}{4})}{(4\pi)^{\frac{n-2k}{2}} R_r^{n-2k}(t_0 - T)^{n-2k}}.$$

Therefore, using that $R_r^{n-2k}(t_0 - t) \leq c_{n,k} r$, (4.9) implies that

$$\begin{aligned} &\frac{e^{-\frac{1}{4}}}{(4\pi)^{\frac{n-2k}{2}} R_r^{n-2k}(t_0 - T)^{n-2k}} \int_{B_{R_r^{n-2k}(t_0-T)}(X)} \frac{1}{2}|\psi|^2 \mathrm{dvol}_g(\cdot, t_0) \\ &\leq \frac{e^{2\int_0^T |l_k|}}{r^{n-2k}} \cdot \left[\frac{\varsigma_0}{r^2} \iint\limits_{\mathcal{D}_{2c_{n,k}r, \frac{r^2}{4\pi}}(X,T)} \frac{1}{2}|\psi|^2 \,\mathrm{d}\mu + \int_{B^{T-\frac{r^2}{4\pi}}_{2c_{n,k}r}(X)} \frac{1}{2}|\psi|^2 \mathrm{dvol}_g(\cdot, T - \frac{r^2}{4\pi}) \right] \end{aligned} \tag{4.10}$$

for $t_0 \in \left]T - e^{-\frac{1}{2(n-2k)}} \frac{r^2}{4\pi}, T\right[$ which establishes (4.7). □

5. Local monotonicity formulæ

We now proceed to establish the main theorem of this paper from which we shall derive local monotonicity formulæ for evolving k-forms. Throughout this section, we suppose $\dim M = n > 2k$, that $E_r^{n-2k}(\Phi)$ is an $(n-2k, \Phi)$ heat ball for $r < r_0$, set $\phi = \log \Phi$ and retain the notation of §4.

Theorem 5.1. *If $\left\{\psi(\cdot, t) \in \Gamma(E \otimes \Lambda^k T^* M)\right\}_{t \in [0,T[}$ is a smooth one-parameter family of sections, then the identity (1.6) holds with '=' replaced with '≥' whenever $0 < r_1 < r_2 < r_0$ provided both spacetime integrands are in $L^1\left(E_{r_0}^{n-2k}(\Phi)\right)$. If*

$$\left((x,t) \mapsto \frac{|\psi(\cdot,t)(x)|^2}{T-t}\right) \in L^1\left(E_{r_0}^{n-2k}(\Phi)\right), \tag{5.1}$$

then (1.6) holds with equality.

Remark 5.2. The right-hand integrand of (1.6) is equal to $-\frac{1}{\Phi}$ times that of the right-hand side of (4.1) with φ formally equal to 1, so that this formula bears a resemblence to its nonlocal counterpart. In particular, if we write $\Phi(x,t) = (4\pi(T-t))^k\, \Gamma(x,t)$, we may immediately deduce analogously to [2] that the left-hand quantity is monotone nondecreasing provided:

1. $\{\psi(\cdot,t)\}_{t\in[0,T[}$ solves a problem of type (I) or (II), in which case $-\left\langle \mathrm{H}^\nabla \psi, \psi \right\rangle + |\mathrm{d}^\nabla \psi|^2 + |\mathcal{S}_{\psi,\log\Gamma}|^2 = |\mathrm{d}^\nabla \psi|^2 + |\mathcal{T}_{\psi,\log\Gamma}|^2$;
2. Γ is a (nonnegative) subsolution to the backward heat equation, i.e. $\mathrm{H}^*\Gamma \leq 0$; and
3. The matrix Harnack form $\mathcal{Q}_T(\log\Gamma, g)$ is nonnegative definite. This holds e.g. if M is static, of nonnegative sectional curvature and Ricci parallel, and Γ is a nonnegative solution to the backward heat equation on M. Another notable example is when g evolves by the Ricci flow, i.e. $\partial_t g = -2\mathrm{Ric}$, and Φ is a nonnegative solution to the backward heat equation, in which case this expression *vanishes* whenever g is a *gradient shrinking soliton* with potential

$-\log\Gamma$ (cf. [16, Appendix C]). Note that if $k=0$, i.e. $\psi(\cdot,t)\in\Gamma(E)$ for all $t\in[0,T[$, this term is absent from (1.6).

In particular, if ψ solves a problem of type (I) or (II), the identity (1.6) reads

$$\left[\frac{1}{r^{n-2k}}\iint_{E_r^{n-2k}(\Phi)}\frac{1}{2}|\psi|^2\cdot\left(\partial_t\phi+|\nabla\phi|^2\right)-\langle\nabla\phi\llcorner\psi,\mathcal{T}_{\psi,\phi}\rangle-\left\langle\nabla\phi\llcorner\mathrm{d}^\nabla\psi,\psi\right\rangle\mathrm{d}\mu\right]_{r=r_1}^{r=r_2}$$
$$=\int_{r_1}^{r_2}\left(\frac{n-2k}{r^{n-2k+1}}\iint_{E_r^{n-2k}(\Phi)}-\frac{1}{2}|\psi|^2\cdot\frac{\mathrm{H}^*\Gamma}{\Gamma}+|\mathcal{T}_{\psi,\phi}|^2+|\mathrm{d}^\nabla\psi|^2\right.$$
$$\left.+\left\langle\mathcal{Q}_T(\log\Gamma,g),\sum_{i,j}\langle\varepsilon_i\llcorner\psi,\varepsilon_j\llcorner\psi\rangle\,\omega^i\otimes\omega^j\right\rangle\mathrm{d}\mu(x,t)\right)\mathrm{d}r. \tag{5.2}$$

Therefore, we may immediately deduce the following:

Corollary 5.3. *If $\{\psi(\cdot,t)\in\Gamma(E\otimes\Lambda^kT^*M)\}_{t\in[0,T[}$ is a solution to a problem of type (I) or (II) satisfying the symmetry condition $|\mathcal{T}_{\psi,\phi}|=|\mathrm{d}^\nabla\psi|=0$ and the summability condition (5.1), $\Phi(x,t)=(4\pi(T-t))^k\,\Gamma(x,t)$ with $\mathrm{H}^*\Gamma=0$ and the matrix Harnack form $\mathcal{Q}_T(\log\Gamma,g)$ vanishes, then the quantity $r\mapsto\frac{1}{r^{n-2k}}\iint_{E_r^{n-2k}(\Phi)}\frac{1}{2}|\psi|^2(\partial_t\phi+|\nabla\phi|^2)\mathrm{d}\mu$ is constant for $r<r_0$.*

In the special case where ψ arises from a solution to the Yang-Mills or harmonic map heat equation (Examples 2.5 and 2.6), the symmetry condition $\mathcal{T}_{\psi,\log\Gamma}$ equates to the solution under consideration being a *gradient soliton* with potential $-\log\Gamma$ (cf. [2]); thus, if g is a gradient shrinking Ricci soliton with potential $-\log\Gamma$ evolving by Ricci flow, Corollary 5.3 yields an integral of motion for the flow. In the particular case where $(M,g)\equiv(\mathbb{R}^n,\delta)$ and Γ is the Euclidean backward heat kernel, the symmetry condition equates to scale-invariance and we recover the results of [8] and [3].

Proof of Theorem 5.1. We first assume that $\psi(\cdot,t)\equiv 0$ for $\tau<t<T$, whence, since ϕ and ψ are smooth on $E_r^{n-2k}(\Phi)\cap\mathrm{pr}_2^{-1}(]0,\tau[)\Subset M\times]0,T[$, all terms occurring in the integrands of (1.6) are summable over $E_{r_0}^{n-2k}(\Phi)$. Now, note that the left-hand integrand may be written as

$$\begin{aligned}i_1(\psi)&:=\frac{1}{2}|\psi|^2(\partial_t\phi+|\nabla\phi|^2)-\langle\nabla\phi\llcorner\psi,\nabla\phi\llcorner\psi-\delta^\nabla\psi\rangle-\left\langle\nabla\phi\llcorner\mathrm{d}^\nabla\psi,\psi\right\rangle\\&=-\left\langle(\nabla\phi\llcorner\Psi)^\sharp+\sum_{i=1}^n\left(\left\langle\varepsilon_i\llcorner\mathrm{d}^\nabla\psi,\psi\right\rangle-\langle\varepsilon_i\llcorner\psi,\delta^\nabla\psi\rangle\right)\varepsilon_i,\nabla\phi\right\rangle+\frac{1}{2}|\psi|^2\partial_t\phi,\end{aligned}$$

where $\Psi\in\Gamma(T^*M\otimes T^*M)$ is the *energy-momentum tensor* associated to ψ given by

$$\Psi(\cdot,t)=\sum_{i,j=1}^n\langle\varepsilon_i\llcorner\psi(\cdot,t),\varepsilon_j\llcorner\psi(\cdot,t)\rangle\,\omega^i\otimes\omega^j-\frac{1}{2}|\psi(\cdot,t)|^2g(\cdot,t);$$

a straightforward albeit lengthy computation (cf. [1]) shows that it satisfies the identity

$$\operatorname{div}\,\Psi(\cdot,t) = -\sum_{i=1}^{n}\left(\left\langle \delta^{\nabla^t}\psi(\cdot,t), \varepsilon_i \llcorner \psi(\cdot,t)\right\rangle + \left\langle \varepsilon_i \llcorner \mathrm{d}^{\nabla^t}\psi(\cdot,t), \psi(\cdot,t)\right\rangle\right)\omega^i. \tag{5.3}$$

Hence, adopting the approximate integral notation of §3.2 and using the proofs of Propositions 3.9(i) and 3.9(ii), we have that

$$\begin{aligned}\left[\frac{J_q^r(i_1(\psi))}{r^{n-2k}}\right]_{r=r_1}^{r=r_2} &= \int_{r_1}^{r_2} \frac{2k-n}{r^{n-2k+1}} J_q^r(i_1(\psi)) + \frac{1}{r^{n-2k}}\frac{\mathrm{d}}{\mathrm{d}r}J_q^r(i_1(\psi))\,\mathrm{d}r \\ &= \int_{r_1}^{r_2} \frac{n-2k}{r^{n-2k+1}} J_q^r\left(i_2(\psi)-i_1(\psi)\right)\mathrm{d}r + o(1) \text{ as } q\to\infty,\end{aligned} \tag{5.4}$$

where

$$i_2(\psi) = \operatorname{div}\left((\nabla\phi\llcorner\Psi)^\sharp + \sum_{i=1}^{n}\left(\left\langle \varepsilon_i\llcorner \mathrm{d}^\nabla\psi,\psi\right\rangle - \left\langle \varepsilon_i\llcorner\psi, \delta^\nabla\psi\right\rangle\right)\varepsilon_i\right) - \partial_t\left(\frac{1}{2}|\psi|^2\right) - \frac{1}{2}|\psi|^2\cdot\frac{1}{2}\mathrm{tr}_g\partial_t g.$$

Taking the limit $q\to\infty$ in (5.4), we obtain

$$\left[\frac{1}{r^{n-2k}}\iint\limits_{E_r^m(\Phi)} i_1(\psi)\,\mathrm{d}\mu\right]_{r=r_1}^{r=r_2} = \int_{r_1}^{r_2}\left[\frac{n-2k}{r^{n-2k+1}}\iint\limits_{E_r^m(\Phi)}\left(i_2(\psi)-i_1(\psi)\right)\mathrm{d}\mu\right]\mathrm{d}r. \tag{5.5}$$

Thus, it suffices to show that $i_2(\psi)-i_1(\psi)$ is equal to the innermost integrand of the right-hand side of (1.6). Using (5.3) and (2.4), it is easily computed that

$$\begin{aligned} i_2(\psi)-i_1(\psi) = &-\frac{1}{2}|\psi|^2\frac{\mathrm{H}^*\Phi}{\Phi} - \left\langle \mathrm{H}^\nabla\psi,\psi\right\rangle \\ &+ |\mathcal{S}_{\psi,\phi}|^2 + |\mathrm{d}^\nabla\psi|^2 + \left\langle \nabla^2\phi + \frac{1}{2}\partial_t g, \sum_{i,j=1}^{n}\langle\varepsilon_i\llcorner\psi,\varepsilon_j\llcorner\psi\rangle\,\omega^i\otimes\omega^j\right\rangle,\end{aligned}$$

whence adding and subtracting

$$(x,t)\mapsto\left\langle \frac{g}{2(T-t)}, \sum_{i,j}\langle\varepsilon_i\llcorner\psi,\varepsilon_j\llcorner\psi\rangle\,\omega^i\otimes\omega^j\right\rangle(x,t) = \frac{k}{T-t}\cdot\frac{1}{2}|\psi|^2(x,t)$$

on the right-hand side then implies the claim.

Now let $\{\psi(\cdot,t)\}$ be arbitrary. We apply (5.5) to the time dependent section $x\mapsto\psi(\cdot,t)(x)\cdot\chi_q(T-t)$ with $q\in\mathbb{N}$, which clearly vanishes near T, to obtain

$$\begin{aligned}&\left[\frac{1}{r^{n-2k}}\iint\limits_{E_r^{n-2k}(\Phi)}\chi_q(T-t)^2\cdot i_1(\psi)\,\mathrm{d}\mu(x,t)\right]_{r=r_1}^{r=r_2} \\ &\quad= \int_{r_1}^{r_2}\left(\frac{n-2k}{r^{n-2k+1}}\iint\limits_{E_r^{n-2k}(\Phi)}\chi_q(T-t)^2\cdot\left[i_2(\psi)-i_1(\psi)\right]\mathrm{d}\mu(x,t)\right)\mathrm{d}r\end{aligned}$$

$$+\int_{r_1}^{r_2}\left(\frac{n-2k}{r^{n-2k+1}}\iint\limits_{E_r^{n-2k}(\Phi)}\chi_q(T-t)\chi_q'(T-t)|\psi|^2\,\mathrm{d}\mu(x,t)\right)\mathrm{d}r.$$

If the condition (5.1) holds, we may pass to the limit $q\to\infty$, where the latter integral on the right-hand side tends to 0 due to (5.1), the observation that

$$\chi_q(T-t)\chi_q'(T-t)|\psi|^2=\chi_q(T-t)\cdot(T-t)\chi_q'(T-t)\cdot\frac{|\psi|^2}{T-t}$$

and the fact that $(T-t)\chi_q'(T-t)\xrightarrow{q\to\infty}0$; moreover, the other interchangings of limit and integral are justified by the summability over $E_{r_0}^{n-2k}(\Phi)$ of both (innermost) integrands of (1.6). If however (5.1) doesn't hold, we may simply discard the latter integral on the right-hand side and then take limits since $\chi_q'\geq 0$. $\square$

Analogously to Lemma 4.2, even if the matrix Harnack expression and backward heat operator applied to Φ do not have the right signs, we may nevertheless obtain a local monotonicity formula provided they satisfy certain inequalities.

Corollary 5.4. *If $\left\{\psi(\cdot,t)\in\Gamma(E\otimes\Lambda^kT^*M)\right\}_{t\in[0,T[}$ is a smooth one-parameter family of sections and the kernel estimates $\frac{\mathrm{H}^*\Phi}{\Phi}+\frac{k}{T-t}\leq a(t)$ and $\mathcal{Q}_T(\phi,g)\geq b(t)g$ hold on $E_{r_0}^{n-2k}(\Phi)$ with $a,b\in C(\mathrm{pr}_2(E_{r_0}^{n-2k}(\Phi)))\cap L^1\left(\mathrm{pr}_2(E_{r_0}^{n-2k}(\Phi))\right)$, then*

$$\left[\frac{1}{r^{n-2k}}\iint\limits_{E_r^{n-2k}(\Phi)}\frac{1}{2}|\psi|^2(\partial_t\phi+|\nabla\phi|^2)-\langle\nabla\phi\llcorner\psi,\mathcal{S}_{\psi,\phi}\rangle-\left\langle\nabla\phi\llcorner\mathrm{d}^\nabla\psi,\psi\right\rangle\mathrm{d}\widetilde{\mu}_k\right]_{r=r_1}^{r=r_2}$$
$$\geq\int_{r_1}^{r_2}\left(\frac{n-2k}{r^{n-2k+1}}\iint\limits_{E_r^{n-2k}(\Phi)}\left|\mathrm{d}^\nabla\psi\right|^2+|\mathcal{S}_{\psi,\phi}|^2-\left\langle\mathrm{H}^\nabla\psi,\psi\right\rangle\mathrm{d}\widetilde{\mu}_k(x,t)\right)\mathrm{d}r$$

for $0<r_1<r_2<r_0$ whenever the spacetime integrands are in $L^1(E_{r_0}^{n-2k}(\Phi),\mu)$, where

$$\xi_k(t)=\int_t^T a-2kb$$

and $\widetilde{\mu}_k$ is the Borel measure on the domain of Φ induced by $\mathrm{d}\widetilde{\mu}_k:=\mathrm{dvol}_g(\cdot,t)\wedge e^{\xi_k(t)}\,\mathrm{d}t$. If only the left-hand spacetime integrand is known to be summable over $E_{r_0}^{n-2k}(\Phi)$ and

$$|\mathrm{d}^\nabla\psi|^2+|\mathcal{S}_{\psi,\phi}|^2-\left\langle\mathrm{H}^\nabla\psi,\psi\right\rangle\geq 0\tag{5.6}$$

on $E_{r_0}^{n-2k}(\Phi)$, then

$$\left[\frac{1}{r^{n-2k}}\iint\limits_{E_r^{n-2k}(\Phi)}\frac{1}{2}|\psi|^2(\partial_t\phi+|\nabla\phi|^2)-\langle\nabla\phi\llcorner\psi,\mathcal{S}_{\psi,\phi}\rangle-\left\langle\nabla\phi\llcorner\mathrm{d}^\nabla\psi,\psi\right\rangle\mathrm{d}\widetilde{\mu}_k\right]_{r=r_1}^{r=r_2}\geq 0$$

for $0<r_1<r_2<r_0$, i.e. the parenthetical quantity is monotone nondecreasing.

Proof. We apply Theorem 5.1 to $\psi_q(x,t) := e^{\xi_k(t)/2}\chi_q(T-t)\psi(\cdot,t)(x)$ to obtain

$$\left[\frac{1}{r^{n-2k}}\iint\limits_{E_r^{n-2k}(\Phi)} \chi_q(T-t)^2 e^{\xi_k(t)} i_1(\psi)\,\mathrm{d}\mu(x,t)\right]_{r=r_1}^{r=r_2}$$
$$= \int_{r_1}^{r_2}\left(\frac{n-2k}{r^{n-2k+1}}\iint\limits_{E_r^{n-2k}(\Phi)}\left(-\frac{1}{2}|\psi|^2\cdot\left(\frac{\mathrm{H}^*\Phi}{\Phi}+\frac{k}{T-t}\right)+|\mathcal{S}_{\psi,\phi}|^2+|\mathrm{d}^\nabla\psi|^2\right.\right.$$
$$\left.-\left\langle \mathrm{H}^\nabla\psi,\psi\right\rangle + \left\langle \mathcal{Q}_T(\phi,g),\sum_{i,j=1}^n \langle\varepsilon_i\llcorner\psi,\varepsilon_j\llcorner\psi\rangle\,\omega^i\otimes\omega^j\right\rangle\right) e^{\xi_k(t)}\chi_q(T-t)^2$$
$$\left.+\chi_q'(T-t)\chi_q(T-t)e^{\xi_k(t)}|\psi|^2 - \frac{\partial_t\xi_k}{2}\cdot e^{\xi_k(t)}\chi_q(T-t)^2|\psi|^2\,\mathrm{d}\mu(x,t)\right)\mathrm{d}r, \tag{5.7}$$

where $i_1(\psi)$ is as in the proof of Theorem 5.1. Making use of the kernel estimates and noting that $\chi_q'(T-t)\geq 0$, $\partial_t\xi_k = 2kb-a$ and $\left\langle g,\sum_{i,j=1}^n\langle\varepsilon_i\llcorner\psi,\varepsilon_j\llcorner\psi\rangle\,\omega^i\otimes\omega^j\right\rangle = k|\psi|^2$, we may estimate the r-integrand of the right-hand integral of equation (5.7) from below by

$$\frac{n-2k}{r^{n-2k+1}}\iint\limits_{E_r^{n-2k}(\Phi)}\chi_q(T-t)^2e^{\xi_k(t)}\left(-\frac{1}{2}|\psi|^2a+|\mathcal{S}_{\psi,\phi}|^2+|\mathrm{d}^\nabla\psi|^2\right.$$
$$\left.-\left\langle\mathrm{H}^\nabla\psi,\psi\right\rangle+kb|\psi|^2-(2kb-a)\cdot\frac{1}{2}|\psi|^2\right)\mathrm{d}\mu(x,t)$$
$$= \frac{n-2k}{r^{n-2k+1}}\iint\limits_{E_r^{n-2k}(\Phi)}\left(|\mathcal{S}_{\psi,\phi}|^2+\left|\mathrm{d}^\nabla\psi\right|^2-\left\langle\mathrm{H}^\nabla\psi,\psi\right\rangle\right)\chi_q(T-t)^2e^{\xi_k(t)}\,\mathrm{d}\mu(x,t). \tag{5.8}$$

Since $t\mapsto e^{\xi_k(t)}$ is bounded on $\mathrm{pr}_2(E_{r_0}^{n-2k}(\Phi))$, we may take limits exactly as in the preceding theorem, thus establishing the first claim. For the second, we bound the right-hand side of (5.8) from below by 0 and then take limits. ☐

Remark 5.5. It is clear from Remark 5.2 that if $\{\psi\}_{t\in[0,T[}$ solves a problem of type (I) or (II), the positivity condition (5.6) holds.

We now turn our attention to concrete $(n-2k)$-heat balls, viz. Examples 3.5 and 3.7 for $m=n-2k$, fix $X\in M$ and suppose $T<T_\infty$, assuming the local geometry bounds (2.2) and (2.3); in both cases, the corresponding kernel satisfies kernel estimates of the form supposed in Corollary 5.4 on the respective heat ball as may be seen either by appealing to Remark 4.3 or by computing directly from Proposition 2.2 and Theorem 2.3. For $\Phi = {}^{n-2k}\Gamma_{(X,T)}$, the latter approach yields kernel estimates as in Corollary 5.4 with

$$a(t) = [(n-1)\Lambda_\infty+\lambda_{-\infty}]^+\log\left(\frac{1}{(4\pi(T-t))^{\frac{n-2k}{2}}}\right)+\frac{n}{2}\lambda_\infty \tag{5.9}$$

and

$$b(t) = \Lambda^-_{-\infty} \log\left(\frac{1}{(4\pi(T-t))^{\frac{n-2k}{2}}}\right) + \frac{\lambda_{-\infty}}{2}, \tag{5.10}$$

and similarly for $\Phi = {}^{n-2k}\mathrm{P}_{(X,T)}$ when M is compact and static, in which case we obtain $a(t) = 0$ and

$$b(t) = -F\left[1 + \log\left(\frac{B}{(4\pi(T-t))^{\frac{n-2k}{2}}}\right)\right] \tag{5.11}$$

with F and B as in Theorem 2.3.

Corollary 5.6. *Fix $X \in M$, suppose $T < T_\infty$ and assume the local geometry bounds (2.2) and (2.3). If $\{\psi(\cdot,t) \in \Gamma(E \otimes \Lambda^k T^*M)\}_{t\in[0,T[}$ solves a problem of type (I) or (II), then the monotonicity formula*

$$\left[\frac{1}{r^{n-2k}} \iint\limits_{E_r^{n-2k}(\Phi)} \frac{1}{2}|\psi|^2(\partial_t\phi + |\nabla\phi|^2) - \langle \nabla\phi \llcorner \psi, \mathcal{T}_{\psi,\phi}\rangle - \left\langle \nabla\phi \llcorner \mathrm{d}^\nabla\psi, \psi\right\rangle \mathrm{d}\widetilde{\mu}_k\right]_{r=r_1}^{r=r_2}$$
$$\geq \int_{r_1}^{r_2}\left(\frac{n-2k}{r^{n-2k+1}} \iint\limits_{E_r^{n-2k}(\Phi)} \left|\mathrm{d}^\nabla\psi\right|^2 + |\mathcal{T}_{\psi,\phi}|^2 \,\mathrm{d}\widetilde{\mu}_k\right)\mathrm{d}r \geq 0 \tag{5.12}$$

holds for $0 < r_1 < r_2 < r_0$ whenever $|\psi|^2 \in L^1\left(\mathcal{D}_{2c_{n,k}r_0, \frac{r_0^2}{4\pi}}(X,T), \mu\right)$, where $\Phi = {}^{n-2k}\Gamma_{(X,T)}$, $c_{n,k} = \sqrt{\frac{n-2k}{2\pi e}}$, r_0 is as in Example 3.5 with $m = n-2k$, $\widetilde{\mu}_k$ is the Borel measure on the domain of Φ induced by $\mathrm{d}\widetilde{\mu}_k = \mathrm{dvol}_g(\cdot,t) \wedge e^{\xi_k(t)}\,\mathrm{d}t$ with

$$\xi_k(t) = (T-t)\left[([(n-1)\Lambda_\infty + \lambda_{-\infty}]^+ - 2k\Lambda^-_{-\infty})\log\left(\frac{e}{4\pi(T-t)}\right)^{\frac{n-2k}{2}} + \frac{n\lambda_\infty - 2k\lambda_{-\infty}}{2}\right]$$

and $\mathcal{T}_{\psi,\phi}$ is defined by (4.3).

If M is static and compact, then the monotonicity formula (5.12) holds for $0 < r_1 < r_2 < \widetilde{r_0}$ whenever $|\psi|^2 \in L^1(E_{r_0}^{n-2k}(\Phi), \mu)$, where instead $\Phi = {}^{n-2k}\mathrm{P}_{(X,T)}$, r_0 is as before, $\widetilde{r_0}$ is chosen such that $E_{\widetilde{r_0}}^{n-2k}({}^{n-2k}\mathrm{P}_{(X,T)}) \subset E_{r_0}^{n-2k}({}^{n-2k}\Gamma_{(X,T)})$ (cf. r_0 of Example 3.7) and $\widetilde{\mu}_k$ is the Borel measure on the domain of Φ induced by $\mathrm{d}\widetilde{\mu}_k = \mathrm{dvol}_g(\cdot,t) \wedge e^{\xi_k(t)}\,\mathrm{d}t$ with

$$\xi_k(t) = 2kF(T-t)\left[\log\left(\frac{e}{4\pi(T-t)}\right)^{\frac{n-2k}{2}} + 1 + \log B\right]$$

and F and B as in Theorem 2.3.

Remark 5.7. Note that this corollary immediately yields local monotonicity formulæ for solutions to the Yang-Mills and harmonic map heat equations, viz. taking $k = 2$

and assuming the setup of Example 2.5 and that ω solves the Yang-Mills heat equation, there holds

$$\left[\frac{1}{r^{n-4}}\iint\limits_{E_r^{n-4}(\Phi)}\frac{1}{2}|\underline{\Omega}^\omega|^2(\partial_t\phi+|\nabla\phi|^2)-\langle\nabla\phi\llcorner\underline{\Omega}^\omega,\partial_t a+\nabla\phi\llcorner\underline{\Omega}^\omega\rangle\,\mathrm{d}\widetilde{\mu}_2\right]_{r=r_1}^{r=r_2}$$
$$\geq\int_{r_1}^{r_2}\left(\frac{n-4}{r^{n-3}}\iint\limits_{E_r^{n-4}(\Phi)}|\partial_t a+\nabla\phi\llcorner\underline{\Omega}^\omega|^2\,\mathrm{d}\widetilde{\mu}_2\right)\mathrm{d}r, \tag{5.13}$$

and likewise taking $k=1$ and assuming the setup of Example 2.6 and that u solves the harmonic map heat equation, there holds

$$\left[\frac{1}{r^{n-2}}\iint\limits_{E_r^{n-2}(\Phi)}\frac{1}{2}|\mathrm{d}u|^2(\partial_t\phi+|\nabla\phi|^2)-\langle\partial_{\nabla\phi}u,\partial_t u+\partial_{\nabla\phi}u\rangle\,\mathrm{d}\widetilde{\mu}_1\right]_{r=r_1}^{r=r_2}$$
$$\geq\int_{r_1}^{r_2}\left(\frac{n-2}{r^{n-1}}\iint\limits_{E_r^{n-2}(\Phi)}|\partial_t u+\partial_{\nabla\phi}u|^2\,\mathrm{d}\widetilde{\mu}_1\right)\mathrm{d}r. \tag{5.14}$$

Proof of Corollary 5.6. We wish to apply Corollary 5.4. To this end, take $\Phi={}^{n-2k}\Gamma_{(X,T)}$. In light of Remark 5.5 and the inequalities (5.9)-(5.10), it suffices to check that both spacetime integrands of (5.12) are summable over $E_{r_0}^{n-2k}(\Phi)$ with respect to $\widetilde{\mu}_k$ or equivalently μ, since ξ_k is bounded. Now, it is clear from Lemma 4.4 that the right-hand spacetime integrand is summable over $E_{r_0}^{n-2k}(\Phi)$, whereas the left-hand integrand may by the Cauchy-Schwarz inequality and Young's inequality be bounded from above in modulus by

$$\frac{1}{2}|\psi|^2\left(|\partial_t\phi|+3|\nabla\phi|^2\right)+\frac{1}{2}\left(|\mathcal{T}_{\psi,\phi}|^2+|\mathrm{d}^\nabla\psi|^2\right)$$

and since by Proposition 2.2

$$\left(|\partial_t\phi|+3|\nabla\phi|^2\right)(x,t)\leq\mathrm{const}\cdot\left(\frac{1}{T-t}+\frac{R_r^{n-2k}(t-T)^2}{(t-T)^2}+\frac{R_r^{n-2k}(t-T)^2}{T-t}\right)$$

for $(x,t)\in\mathcal{D}_{j_0,\delta}(X,T)$ with R_r^{n-2k} as defined in Example 3.5 and 'const' some constant depending only on n, k, the local geometry of M about (X,T) and the auxiliary function χ, it suffices to show that

$$\iint\limits_{E_{r_0}^{n-2k}(\Phi)}\frac{1}{2}|\psi|^2\cdot\left(\frac{1}{T-t}+\frac{R_r^{n-2k}(t-T)^2}{(t-T)^2}+\frac{R_r^{n-2k}(t-T)^2}{T-t}\right)\mathrm{d}\mu(x,t)<\infty.$$

Note that since the integrand is bounded away from $t=T$ from above by a constant times $|\psi|^2$, it suffices to establish summability over $E_{r_0}^{n-2k}(\Phi)\cap\mathrm{pr}_2^{-1}\left(]\tau,T[\right)$ for τ

sufficiently close to T. By Lemma 4.4, we have that

$$\int_{B^t_{R^{n-2k}_{r_0}(t-T)}(X)} \frac{1}{2}|\psi|^2 \mathrm{dvol}_g(\cdot,t)$$

$$\leq \frac{\widetilde{C}_0 R^{n-2k}_{r_0}(t-T)^{n-2k}}{r_0^{n-2k}} \left(r_0^{-2} \iint_{\mathcal{D}_{2c_{n,k}r_0,\frac{r_0^2}{4\pi}}(X,T)} \frac{1}{2}|\psi|^2 \,\mathrm{d}\mu + \int_{B^{T-\frac{r_0^2}{4\pi}}_{2c_{n,k}r_0}(X)} \frac{1}{2}|\psi|^2 \mathrm{dvol}_g(\cdot, T-\frac{r_0^2}{4\pi}) \right)$$

for $t \in \left]T - r_0^2 \exp\left(-\frac{1}{2(n-2k)}\right)/4\pi, T\right[$ so that, by Tonelli's theorem, it suffices to show that

$$\int_{T-\frac{r_0^2}{4\pi}}^{T} \frac{R^{n-2k}_{r_0}(t-T)^{n-2k}}{T-t} + R^{n-2k}_{r_0}(t-T)^{n-2k+2}\left(\frac{1}{T-t} + \frac{1}{(T-t)^2}\right) \mathrm{d}t < \infty,$$

but this follows from a straightforward computation as in Example 3.5. Hence, Corollary 5.4 applies, implying the monotonicity formula for $\Phi = {}^{n-2k}\Gamma_{(X,T)}$. The static compact case with $\Phi = {}^{n-2k}\mathrm{P}_{(X,T)}$ follows similarly. $\square$

Remark 5.8. A careful computation of the t-integrals in the preceding proof shows that we in fact have an estimate of the form

$$\frac{1}{r^{n-2k}} \iint_{E_r^{n-2k}(\Phi)} \frac{1}{2}|\psi|^2(\partial_t\phi + |\nabla\phi|^2) - \langle \nabla\phi \llcorner \psi, \mathcal{T}_{\psi,\phi}\rangle - \left\langle \nabla\phi \llcorner \mathrm{d}^{\nabla}\psi, \psi \right\rangle \mathrm{d}\widetilde{\mu}_k$$

$$\leq \mathrm{const}\cdot \left(\frac{1}{r_0^{n-2k+2}} \iint_{\mathcal{D}_{2c_{n,k}r_0,\frac{r_0^2}{4\pi}}(X,T)} \frac{1}{2}|\psi|^2 \,\mathrm{d}\mu + \frac{1}{r_0^{n-2k}} \int_{B^{T-\frac{r_0^2}{4\pi}}_{2c_{n,k}r_0}(X)} \frac{1}{2}|\psi|^2 \mathrm{dvol}_g(\cdot, T-\frac{r_0^2}{4\pi}) \right)$$

under the hypotheses of the theorem, where 'const' depends only on n, k, the local geometry of M about (X,T) and the auxiliary function χ.

Remark 5.9. The rather abstract nature of the techniques used here suggests that similar results should be obtainable by similar means for related geometric heat equations. In particular, assuming the setup of Example 2.5, suppose $(V, \langle\cdot,\cdot\rangle)$ is an inner product space and $\rho : G \to \mathrm{O}(V)$ is a unitary representation. The vector bundle E_0 associated to ρ and P may thus be equipped with a Riemannian metric, again denoted $\langle\cdot,\cdot\rangle$ and, as before, a one-parameter family of connections on P gives rise to a one-parameter family of connections on E_0 which we simply denote by ∇^t. A one-parameter family of pairs $\{(\omega(\cdot,t) = \omega_0 + \underline{a(\cdot,t)}, u(\cdot,t)\}_{t\in[0,T[}$ consisting of a smooth family of connections $\omega(\cdot,t)$ on P (ω_0 a fixed connection) and sections $u(\cdot,t)$ of E_0 is said to solve the *Yang-Mills-Higgs heat equation* with (radial) potential $W \in C^\infty(\mathbb{R}, [0,\infty[)$ if the equations

$$\begin{gathered} \partial_t a + \delta^{\nabla^t}\underline{\Omega}^{\omega(\cdot,t)} + u(\cdot,t)\odot \mathrm{d}^{\nabla^t}u(\cdot,t) = 0 \\ \partial_t u + \delta^{\nabla^t}\mathrm{d}^{\nabla^t}u(\cdot,t) + 2(W'\circ|u(\cdot,t)|^2)u(\cdot,t) = 0 \end{gathered} \tag{5.15}$$

hold, where $\odot : E_0 \times E_0 \otimes \Lambda T^*M \to E_0 \otimes \Lambda T^*M$ is a fibrewise bilinear map defined such that

$$\langle X, e_1 \odot e_2 \rangle = \langle X \cdot e_1, e_2 \rangle \,,$$

for all $X \in \Gamma(E)$ and $e_1, e_2 \in \Gamma(E_0)$, where $\cdot$ is the natural action of E on E_0 induced by the derivative of ρ at the identity of G and extended to the rest of $E_0 \times E_0 \otimes \Lambda T^*M$ such that for all $\eta \in \Gamma(\Lambda T^*M)$ and $e_1, e_2 \in \Gamma(E_0)$, $e_1 \odot (e_2 \otimes \eta) = (e_1 \odot e_2) \otimes \eta$. Naturally associated to such families of pairs is the energy density $e(\omega, u) = \frac{1}{2}\left(|\underline{\Omega}^\omega|^2 + |\mathrm{d}^\nabla u|^2\right) + W \circ |u|^2$ from which the equations (5.15) arise in the case where M is static. Setting $\mathcal{S}_{\omega,\phi} = \partial_t a + \nabla\phi \llcorner \underline{\Omega}^\omega$ and $\mathcal{S}_{u,\phi} = \partial_t u + \nabla_{\nabla\phi} u$, the local monotonicity identity

$$\begin{aligned}
&\left[\frac{1}{r^{n-4}} \iint\limits_{E_r^{n-4}(\Phi)} e^{\xi(t)} \left(e(\omega,u)\left(\partial_t\phi + |\nabla\phi|^2\right) - \langle \nabla\phi \llcorner \underline{\Omega}^\omega, \mathcal{S}_{\omega,\phi}\rangle - \langle \nabla_{\nabla\phi} u, \mathcal{S}_{u,\phi}\rangle\right) \mathrm{d}\mu(x,t)\right]_{r=r_1}^{r=r_2} \\
&\geq \int_{r_1}^{r_2} \frac{n-4}{r^{n-3}} \iint\limits_{E_r^{n-4}(\Phi)} e^{\xi(t)} \Bigg[-e(\omega,u)\left(\partial_t\phi + \Delta\phi + |\nabla\phi|^2 + \frac{1}{2}\mathrm{tr}_g \partial_t g + \frac{2}{T-t} + \xi'(t)\right) \\
&\qquad + \frac{|\mathrm{d}^\nabla u|^2 + 4W \circ u}{2(T-t)} + |\mathcal{S}_{\omega,\phi}|^2 + |\mathcal{S}_{u,\phi}|^2 \\
&\qquad + \left\langle \mathcal{Q}_T(\phi, g), \sum_{i,j} \left(\langle \varepsilon_i \llcorner \underline{\Omega}^\omega, \varepsilon_j \llcorner \underline{\Omega}^\omega\rangle + \langle \nabla_{\varepsilon_i} u, \nabla_{\varepsilon_j} u\rangle\right) \omega^i \otimes \omega^j \right\rangle\Bigg] \mathrm{d}\mu(x,t)\,\mathrm{d}r
\end{aligned} \tag{5.16}$$

may be established for $0 < r_1 < r_2 < r_0$ whenever $E_r^{n-4}(\Phi)$ is a heat ball for $r < r_0$ and both spacetime integrands are in $L^1(E_{r_0}^{n-4}(\Phi)$ with equality holding whenever $\left((x,t) \mapsto \frac{e(\omega,u)(x,t)}{T-t}\right) \in L^1(E_{r_0}^{n-4}(\Phi))$; the proof is identical to that of Theorem 5.1, where instead the energy-momentum tensor associated to e, viz.

$$\widetilde{\Psi}_t = \sum_{i,j=1}^n \left(\left\langle \varepsilon_i \llcorner \underline{\Omega}^{\omega(\cdot,t)}, \varepsilon_j \llcorner \underline{\Omega}^{\omega(\cdot,t)}\right\rangle + \left\langle \nabla^t_{\varepsilon_i} u, \nabla^t_{\varepsilon_j} u\right\rangle\right) \omega^i \otimes \omega^j - e(\omega,u)(\cdot,t),$$

together with the divergence identity

$$\mathrm{div}\ \widetilde{\Psi}_t = \sum_{j=1}^n \left(\left\langle \partial_t a, \varepsilon_j \llcorner \underline{\Omega}^{\omega(\cdot,t)}\right\rangle + \left\langle \partial_t u, \nabla^t_j u(\cdot,t)\right\rangle\right) \omega^j$$

is used. Furthermore, if the kernel estimates for Φ hold as in Corollary 5.4, then by setting $\xi(t) = \int_t^T a - 4b$, we obtain

$$\begin{aligned}
&\left[\frac{1}{r^{n-4}} \iint\limits_{E_r^{n-4}(\Phi)} e(\omega,u)\left(\partial_t\phi + |\nabla\phi|^2\right) - \langle \nabla\phi \llcorner \underline{\Omega}^\omega, \mathcal{S}_{\omega,\phi}\rangle - \langle \nabla_{\nabla\phi} u, \mathcal{S}_{u,\phi}\rangle \,\mathrm{d}\widetilde{\mu}\right]_{r=r_1}^{r=r_2} \\
&\qquad \geq \int_{r_1}^{r_2} \frac{n-4}{r^{n-3}} \iint\limits_{E_r^{n-4}(\phi)} |\mathcal{S}_{\omega,\phi}|^2 + |\mathcal{S}_{u,\phi}|^2 \mathrm{d}\widetilde{\mu}\,\mathrm{d}r
\end{aligned}$$

with $\widetilde{\mu}$ the Borel measure on the domain of Φ induced by $\mathrm{d}\widetilde{\mu} = \mathrm{dvol}_g(\cdot,t) \wedge e^{\xi(t)}\,\mathrm{d}t$ whenever both spacetime integrands are summable. Thus, for $T < T_\infty$, we may take $\Phi = {}^{n-4}\Gamma_{(X,T)}$ (or $\Phi = {}^{n-4}\mathrm{P}_{(X,T)}$ if M is static) as in Corollary 5.6 to obtain a local monotonicity formula on Euclidean (resp. static compact) heat balls whenever $0 < r_1 < r_2 < r_0$ (resp. $\widetilde{r_0}$) and $e(\omega, u) \in L^1\left(\mathcal{D}_{2c_{n,k}r_0, \frac{r_0^2}{4\pi}}(X,T), \mu\right)$, this summability condition arising from estimates obtained similarly to (4.7) and (4.8) by appealing to a nonlocal monotonicity formula due to Hong [13]; the details in the case $(M, g(\cdot,t)) = (\mathbb{R}^n, \delta)$ may be found in [3].

References

[1] A. Afuni, "Monotonicity for p-harmonic vector bundle-valued k-forms," *ArXiv e-prints* (June, 2015) , `arXiv:1506.03439 [math.DG]`.
[2] A. Afuni, "Energy identities and monotonicity for evolving k-forms on moving Riemannian spaces," Submitted (08, 2016) . `http://ax0.github.io/preprints/evform.pdf`.
[3] A. Afuni, "Local monotonicity for the Yang–Mills–Higgs flow," *Calculus of Variations and Partial Differential Equations* **55** no. 1, (2016) 1–14.
[4] Y. M. Chen and M. Struwe, "Existence and partial regularity results for the heat flow for harmonic maps," *Math. Z.* **201** no. 1, (1989) 83–103.
[5] Y. Chen and C. Shen, "Monotonicity formula and small action regularity for Yang-Mills flows in higher dimensions.," *Calc. Var. Partial Differ. Equ.* **2** no. 4, (1994) 389–403.
[6] K. Ecker, "A local monotonicity formula for mean curvature flow," *Ann. of Math. (2)* **154** no. 2, (2001) 503–525.
[7] K. Ecker, *Regularity theory for mean curvature flow*. Progress in Nonlinear Differential Equations and their Applications, 57. Birkhäuser Boston, Inc., Boston, MA, 2004.
[8] K. Ecker, "Local monotonicity formulas for some nonlinear diffusion equations.," *Calc. Var. Partial Differ. Equ.* **23** no. 1, (2005) 67–81.
[9] K. Ecker, D. Knopf, L. Ni, and P. Topping, "Local monotonicity and mean value formulas for evolving Riemannian manifolds," *J. Reine Angew. Math.* **616** (2008) 89–130.
[10] W. Fulks, "A mean value theorem for the heat equation.," *Proc. Am. Math. Soc.* **17** (1966) 6–11.
[11] R. S. Hamilton, "A matrix Harnack estimate for the heat equation.," *Commun. Anal. Geom.* **1** no. 1, (1993) 113–126.
[12] R. S. Hamilton, "Monotonicity formulas for parabolic flows on manifolds.," *Commun. Anal. Geom.* **1** no. 1, (1993) 127–137.
[13] M.-C. Hong, *Monotonicity Formula for Heat Flow for Yang-Mills-Higgs Equations*. Mathematics research report. Centre for Mathematics and its Applications, Australian National University, 1998.
[14] M.-C. Hong and G. Tian, "Asymptotical behaviour of the Yang-Mills flow and singular Yang-Mills connections," *Math. Ann.* **330** no. 3, (2004) 441–472.
[15] G. Huisken, "Asymptotic behavior for singularities of the mean curvature flow.," *J. Differ. Geom.* **31** no. 1, (1990) 285–299.
[16] B. Kleiner and J. Lott, "Notes on Perelman's papers," *ArXiv Mathematics e-prints* (May, 2006) , `math/0605667`.
[17] P. Li and S.-T. Yau, "On the parabolic kernel of the Schrödinger operator," *Acta Math.* **156** no. 3-4, (1986) 153–201.
[18] P. Petersen, *Riemannian geometry*, vol. 171 of *Graduate Texts in Mathematics*. Springer, New York, second ed., 2006.
[19] P. Price, "A monotonicity formula for Yang-Mills fields.," *Manuscr. Math.* **43** (1983) 131–166.
[20] R. Schoen and K. Uhlenbeck, "A regularity theory for harmonic maps.," *J. Differ. Geom.* **17** (1982) 307–335.
[21] M. Struwe, "On the evolution of harmonic maps in higher dimensions.," *J. Differ. Geom.* **28** no. 3, (1988) 485–502.

[22] N. A. Watson, “A theory of subtemperatures in several variables,” *Proc. London Math. Soc. (3)* **26** (1973) 385–417.

[23] C. Yu, “Some estimates of fundamental solutions on noncompact manifolds with time-dependent metrics,” *Manuscripta Math.* **139** no. 3-4, (2012) 321–341.

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