Assessment 1

Question 1 (15 points).
Let $\kappa_p(A)$ be the condition number of A induced by the vector p-norm.
(a) Prove that $\kappa_1(A) = \kappa_\infty(A)$ for any non-singular 2×2 matrix A.
(b) Consider

$$A = \begin{pmatrix} -4 & 8 & -7 \\ -5 & 3 & 2 \\ -3 & 3 & -9 \end{pmatrix}, \quad A^{-1} = \frac{1}{234}\begin{pmatrix} 33 & -51 & -37 \\ 51 & -15 & -43 \\ 6 & 12 & -28 \end{pmatrix}.$$

Is $\kappa_1(A) = \kappa_\infty(A)$ true for general $n \times n$ matrices?
(c) Prove that $\kappa_1(A) = \kappa_\infty(A)$ whenever $A \in \mathbb{R}^{n\times n}$ is symmetric and non-singular.

Solution: (a) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2\times 2}$.
We know from lectures that $\kappa_p(A) = \|A\|_p \|A^{-1}\|_p$. Since A is non-singular, the inverse of A is

$$A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Let's compute $\|A\|_1$, $\|A^{-1}\|_1$, $\|A\|_\infty$, and $\|A^{-1}\|_\infty$. We have

$$\begin{aligned}
\|A\|_1 &= \max(|a| + |c|, |b| + |d|);\\
\left\|A^{-1}\right\|_1 &= \frac{1}{|ad - bc|}\max(|a| + |b|, |c| + |d|);\\
\|A\|_\infty &= \max(|a| + |b|, |c| + |d|);\\
\left\|A^{-1}\right\|_\infty &= \frac{1}{|ad - bc|}\max(|a| + |c|, |b| + |d|).
\end{aligned}$$

Without loss of generality, assume that $|a| + |c| > |b| + |d|$ and $|a| + |b| > |c| + |d|$.
Then $\kappa_1(A) = (|a| + |c|)(|a| + |b|)/|ad - bc| = \kappa_\infty(A)$.

(b) Using the given example, we have

$$\begin{aligned}
&\|A\|_1 = |-7| + 2 + |-9| = 18;\\
&\|A^{-1}\|_1 = (|-37| + |-43| + |-28|)/234 = 108/234 = 6/13;\\
&\|A\|_\infty = |-4| + 8 + |-7| = 19;\\
&\|A^{-1}\|_\infty = (33 + |-51| + |-37|)/234 = 121/234;\\
&\kappa_1(A) = 18 \cdot \frac{6}{13} = \frac{108}{13} \approx 8.31;\\
&\kappa_\infty(A) = 19 \cdot \frac{121}{234} = \frac{2299}{234} \approx 9.8.
\end{aligned}$$

Hence $\kappa_1(A) \neq \kappa_\infty(A)$ for this example and we conclude that $\kappa_1(A) \neq \kappa_\infty(A)$ for a general $n \times n$ matrix A.

(c) Let $(A)_{ij} = a_{ij}$. Since A is symmetric and non-singular, $A = A^T$, i.e. $a_{ij} = a_{ji}$ and A^{-1} exists. Also, $A = A^T \Rightarrow A^{-1} = \left(A^T\right)^{-1} \Rightarrow A^{-1} = \left(A^{-1}\right)^T$. Thus, let $\left(A^{-1}\right)_{ij} = b_{ij}$, then

$$\|A^{-1}\|_1 = \max_{1\le j\le n}\sum_{i=1}^n |b_{ij}| = \max_{1\le j\le n}\sum_{i=1}^n |b_{ji}| = \|\left(A^{-1}\right)^T\|_\infty = \|A^{-1}\|_\infty,$$

$$\|A\|_1 = \max_{1\le j\le n}\sum_{i=1}^n |a_{ij}| = \max_{1\le j\le n}\sum_{i=1}^n |a_{ji}| = \|A^T\|_\infty = \|A\|_\infty.$$

Thus, $\kappa_1(A) = \|A\|_1\|A^{-1}\|_1 = \|A\|_\infty\|A^{-1}\|_\infty = \kappa_\infty(A)$.

Question 2 (10 points).
Let

$$A = \begin{pmatrix} 29 & -8 \\ -11 & 3 \end{pmatrix}$$

and $\kappa_1(A)$ be the condition number induced by the vector 1-norm.
(a) Compute $\kappa_1(A)$ and find vectors $b, \Delta b$ such that $Ax = b, A(x + \Delta x) = b + \Delta b$ and

$$\frac{\|\Delta x\|_1}{\|x\|_1} = \kappa_1(A)\frac{\|\Delta b\|_1}{\|b\|_1}. \tag{1}$$

(b) Normalise your vectors so that $\|b\|_1 = 1$ and $\|\Delta b\|_1 = 0.01$. Compute x and $x + \Delta x$.
Hint: For part (a) you may want to use the first question discussed in Monday's drop-in session and $\kappa_1(A) = \|A\|_1 \cdot \|A^{-1}\|_1$.

Solution: (a) We have

$$\begin{aligned} \|A\|_1 &= 29 + 11 = 40; \\ A^{-1} &= \begin{pmatrix} -3 & -8 \\ -11 & -29 \end{pmatrix}; \\ \|A^{-1}\|_1 &= 29 + 8 = 37; \\ \kappa_1(A) &= 37 \cdot 40 = 1480. \end{aligned}$$

To find b and Δb, note that

$$\|b\|_1 = \|A\|_1\|x\|_1, \tag{2}$$
$$\|\Delta x\|_1 = \|A^{-1}\|_1\|\|\Delta b\|_1 \tag{3}$$

implies condition (1).
According to the trick in the drop-in session, let $x = (1, 0)^T$ and $\Delta b = (0, 1)^T$. Then

$$b = \begin{pmatrix} 29 & -8 \\ -11 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 29 \\ -11 \end{pmatrix}.$$

Since $\|x\|_1 = 1$, it is obvious that (2) is satisfied. Now we find Δx by using $\Delta x = A^{-1}\Delta b$,

$$\Delta x = \begin{pmatrix} -3 & -8 \\ -11 & -29 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -8 \\ -29 \end{pmatrix}.$$

Since $\|\Delta b\|_1 = 1$, it is obvious that (3) is satisfied. Thus, we find such b and Δb to be $(29, -11)^T$ and $(-8, -29)^T$.

(b) Normalise the above b and Δb in 1-norm, we have $b = (29/40, -11/40)^T$ and $\Delta b = (0, 0.01)^T$. Compute x and Δx, we have

$$\begin{aligned} x &= \begin{pmatrix} -3 & -8 \\ -11 & -29 \end{pmatrix}\begin{pmatrix} 29/40 \\ -11/40 \end{pmatrix} = \begin{pmatrix} 1/40 \\ 0 \end{pmatrix}; \\ \Delta x &= \begin{pmatrix} -3 & -8 \\ -11 & -29 \end{pmatrix}\begin{pmatrix} 0 \\ 1/100 \end{pmatrix} = \begin{pmatrix} -2/25 \\ -29/100 \end{pmatrix}; \\ x + \Delta x &= \begin{pmatrix} -11/200 \\ -29/100 \end{pmatrix}. \end{aligned}$$

And it is easy to check that (2) and (3) are satisfied.

Question 3.
Consider the root finding problem $P(a,z) = 0$ with $P(a,z) = \sum_{j=0}^{n} a_j z^j$. Let z_a be a solution of $P(a,z) = 0$ for given vector of coefficients a. Show that if we perturb a single coefficient a_j by an amount of Δa_j to first order, the zero z is perturbed by

$$\Delta z = \frac{-z^j \Delta a_j}{\frac{\partial}{\partial z} P(a,z)}.$$

Based on this perturbation result, state the condition number of a zero of P with respect to a perturbation in the coefficient a_j. Note that a formal proof that your stated condition number satisfies the general definition of condition number is not required.

Solution: We want to find Δz such that if $P(a,z) = 0$, then $P(a+\Delta a_j, z+\Delta z) = 0$. We expand $P(a+\Delta a, z+\Delta z) = 0$ at (a,z), where z is a zero of this equation, and ignoring higher order terms gives

$$P(a,z) + \frac{\partial P}{\partial a}(a,z)\Delta a + \frac{\partial P}{\partial z}(a,z)\Delta z = 0.$$

If we consider only a perturbation in a_j and use the fact that $P(a,z) = 0$, this simplifies to

$$z^j \Delta a_j + \frac{\partial P}{\partial z}\Delta z = 0$$

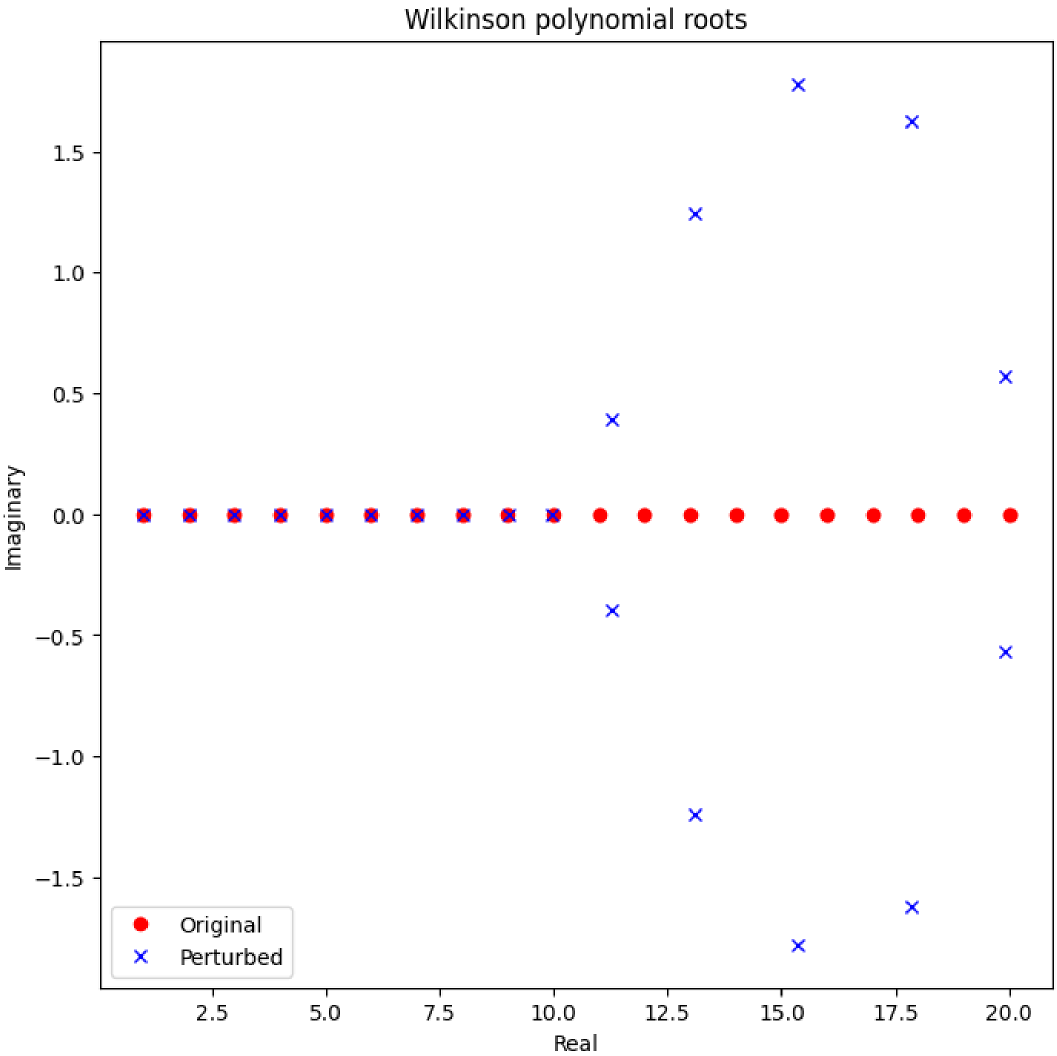
which we can solve for Δz to obtain

$$\Delta z = \frac{-z^j \Delta a_j}{\frac{\partial P}{\partial z}(a,z)}.$$

The condition number is the ratio of the relative forward error to the relative backward error, that is

$$\kappa = \frac{|\Delta z|}{|z|} \Big/ \frac{|\Delta a_j|}{|a_j|} = \frac{|z^{j-1}||a_j|}{|\frac{\partial P}{\partial z}(a,z)|}.$$

Question 4a.
The famous Wilkinson polynomial is defined as $P(z) = \Pi_{j=1}^{20}(z - j)$. hence, its zeros are the integers from 1 to 20. You can find the coefficients for this polynomial in a blog post by Cleve Moler (the inventor of Matlab).
The coefficient of z^{20} of the exact Wilkinson polynomial is 1. Perturb this coefficient to be $1+\epsilon$, where $\epsilon = 10^{-10}$.
(a) Using the roots command of Numpy, compute the corresponding zeros of the perturbed polynomial and plot them with matplotlib.pyplot in the complex plane using blue crosses. Also, plot the exact zeros of the unperturbed Wilkinson polynomial using red circles (for both, use the corresponding options in the plot function of Matplotlib). You want to draw the exact roots first so that the crosses overlay the circles.
Your answer should contain your plot in jpg or png image format.
(b) Interpret the perturbations you see in the plot. What magnitudes of condition numbers for the zeros would you expect from your observation?
Solution: (a)



(b) The higher zeros are more strongly affected. The perturbation of the first coefficient 1 is 10^{-10}. Thus, since the difference between the perturbed roots and original roots are visible to us on this plotting scale, we expect the condition numbers for the zeros to be at least of magnitude 10^9 to 10^{10} or higher so that the resulting forward error leaves no single correct digit.