Project 5

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I. Equations of Motion

Each frame of the robot is defined as:

$$T_1^0 = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 & 0 \\ \sin(\theta_1) & \cos(\theta_1) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_2^1 = \begin{bmatrix} \cos(\theta_2) & -\sin(\theta_2) & 0 & L_1 \\ \sin(\theta_2) & \cos(\theta_2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{tool}^2 = \begin{bmatrix} 1 & 0 & 0 & L_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

A. Newton-Euler Formulation

Starting with the outward iteration, the initial force and torque is set to zero:

$$f_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, n_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

The forces and torques on the center-of-mass of link 1 is:

$$w_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}, \dot{w}_1 = \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \end{bmatrix} \tag{2}$$

$$\dot{v_1} = \begin{bmatrix} -gs_1 \\ -gc_1 \\ 0 \end{bmatrix}, \dot{v_{c_1}} = \begin{bmatrix} -0.5L_1\dot{\theta}_1^{\,2} - gs_1 \\ 0.5L_1\ddot{\theta}_1 - gc_1 \\ 0 \end{bmatrix} \tag{3}$$

$$F_1 = \begin{bmatrix} -m_1\left(0.5L_1\dot{\theta}_1^{\,2} + gs_1\right) \\ m_1\left(0.5L_1\ddot{\theta}_1 - gc_1\right) \\ 0 \end{bmatrix}, N_1 = \begin{bmatrix} 0 \\ 0 \\ 0.0833L_1^2m_1\ddot{\theta}_1 \end{bmatrix} \tag{4}$$

The forces and torques on the center-of-mass of link 2 is:

$$w_2 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 + \dot{\theta}_1 \end{bmatrix}, \dot{w}_2 = \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_2 + \ddot{\theta}_1 \end{bmatrix} \tag{5}$$

$$\dot{v_2} = \begin{bmatrix} s_2\left(L_1\ddot{\theta}_1 - gc_1\right) - c_2\left(L_1\dot{\theta}_1^{\,2} + gs_1\right) \\ c_2\left(L_1\ddot{\theta}_1 - gc_1\right) + s_2\left(L_1\dot{\theta}_1^{\,2} + gs_1\right) \\ 0 \end{bmatrix}, \dot{v_{c_2}} = \begin{bmatrix} s_2\left(L_1\ddot{\theta}_1 - gc_1\right) - 0.5L_1\left(\dot{\theta}_2 + \dot{\theta}_1\right)^2 - c_2\left(L_1\dot{\theta}_1^{\,2} + gs_1\right) \\ c_2\left(L_1\ddot{\theta}_1 - gc_1\right) + s_2\left(L_1\dot{\theta}_1^{\,2} + gs_1\right) + 0.5L_2\left(\ddot{\theta}_2 + \ddot{\theta}_1\right) \\ 0 \end{bmatrix} \tag{6}$$

$$F_2 = \begin{bmatrix} -m_2\left(0.5L_1\left(\dot{\theta}_2 + \dot{\theta}_1\right)^2 - s_2\left(L_1\ddot{\theta}_1 - gc_1\right) + c_2\left(L_1\dot{\theta}_1^{\,2} + gs_1\right)\right) \\ m_2\left(c_2\left(L_1\ddot{\theta}_1 - gc_1\right) + s_2\left(L_1\dot{\theta}_1^{\,2} + gs_1\right) + 0.5L_2\left(\ddot{\theta}_2 + \ddot{\theta}_1\right)\right) \\ 0 \end{bmatrix}, N_2 = \begin{bmatrix} 0 \\ 0 \\ 0.0833L_2^2m_2\left(\ddot{\theta}_2 + \ddot{\theta}_1\right) \end{bmatrix} \tag{7}$$

Now proceeding with the inward iteration:

$$f_2 = \begin{bmatrix} -m_2 \Bigl(0.5L_1 \bigl(\dot{\theta}_2 + \dot{\theta}_1\bigr)^2 - s_2 \bigl(L_1\ddot{\theta}_1 - gc_1\bigr) + c_2 \bigl(L_1\dot{\theta}_1^{\,2} + gs_1\bigr)\Bigr) \\ m_2 \Bigl(c_2 \bigl(L_1\ddot{\theta}_1 - gc_1\bigr) + s_2 \bigl(L_1\dot{\theta}_1^{\,2} + gs_1\bigr) + 0.5L_2 \bigl(\ddot{\theta}_2 + \ddot{\theta}_1\bigr)\Bigr) \\ 0 \end{bmatrix} \tag{8}$$

$$n_2 = \begin{bmatrix} 0 \\ 0 \\ 0.0833L_2^2 m_2 \bigl(\ddot{\theta}_2 + \ddot{\theta}_1\bigr) + 0.5L_2 m_2 \Bigl(c_2 \bigl(L_1\ddot{\theta}_1 - gc_1\bigr) + s_2 \bigl(L_1\dot{\theta}_1^{\,2} + gs_1\bigr) + 0.5L_2 \bigl(\ddot{\theta}_2 + \ddot{\theta}_1\bigr)\Bigr) \end{bmatrix} \tag{9}$$

The forces and torques generated at joint 1 are:

$$f_1 = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} \tag{10}$$

$$\begin{aligned} a = -m_1 \bigl(0.5L_1\dot{\theta}_1^{\,2} + gs_1\bigr) - m_2 s_2 \Bigl(c_2 \bigl(L_1\ddot{\theta}_1 - gc_1\bigr) + s_2 \bigl(L_1\dot{\theta}_1^{\,2} + gs_1\bigr) + 0.5L_2 \bigl(\ddot{\theta}_2 + \ddot{\theta}_1\bigr)\Bigr) \\ - m_2 c_2 \Bigl(0.5L_1 \bigl(\dot{\theta}_2 + \dot{\theta}_1\bigr)^2 - s_2 \bigl(L_1\ddot{\theta}_1 - gc_1\bigr) + c_2 \bigl(L_1\dot{\theta}_1^{\,2} + gs_1\bigr)\Bigr) \end{aligned} \tag{11}$$

$$\begin{aligned} b = m_1 \bigl(0.5L_1\ddot{\theta}_1 - gc_1\bigr) + m_2 c_2 \Bigl(c_2 \bigl(L_1\ddot{\theta}_1 - gc_1\bigr) + s_2 \bigl(L_1\dot{\theta}_1^{\,2} + gs_1\bigr) + 0.5L_2 \bigl(\ddot{\theta}_2 + \ddot{\theta}_1\bigr)\Bigr) \\ - m_2 s_2 \Bigl(0.5L_1 \bigl(\dot{\theta}_2 + \dot{\theta}_1\bigr)^2 - s_2 \bigl(L_1\ddot{\theta}_1 - gc_1\bigr) + c_2 \bigl(L_1\dot{\theta}_1^{\,2} + gs_1\bigr)\Bigr) \end{aligned} \tag{12}$$

$$n_1 = \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix} \tag{13}$$

$$\begin{aligned} c = L_1 \Bigl(m_2 \cos(\theta_2) \Bigl(\cos(\theta_2) \bigl(L_1\ddot{\theta}_1 - g\cos(\theta_1)\bigr) + \sin(\theta_2) \bigl(L_1\dot{\theta}_1^{\,2} + g\sin(\theta_1)\bigr) + 0.5L_2 \bigl(\ddot{\theta}_2 + \ddot{\theta}_1\bigr)\Bigr) \\ -m_2 \sin(\theta_2) \Bigl(0.5L_1 \bigl(\dot{\theta}_2 + \dot{\theta}_1\bigr)^2 - \sin(\theta_2) \bigl(L_1\ddot{\theta}_1 - g\cos(\theta_1)\bigr) + \cos(\theta_2) \bigl(L_1\dot{\theta}_1^{\,2} + g\sin(\theta_1)\bigr)\Bigr)\Bigr) \\ + 0.5L_1 m_1 \bigl(0.5L_1\ddot{\theta}_1 - g\cos(\theta_1) + 0.0833L_2^2 m_2(\ddot{\theta}_2 + \ddot{\theta}_1\bigr) \\ + 0.5L_2 m_2 \Bigl(\cos(\theta_2) \bigl(L_1\ddot{\theta}_1 - g\cos(\theta_1)\bigr) + \sin(\theta_2) \bigl(L_1\dot{\theta}_1^{\,2} + g\sin(\theta_1)\bigr) + 0.5L_2 \bigl(\ddot{\theta}_2 + \ddot{\theta}_1\bigr)\Bigr) + 0.0833L_1^2 m_1 \ddot{\theta}_1 \end{aligned} \tag{14}$$

B. Lagrange Formulation

The link inertia matrices are:

$$^0I_1 = {}^0_1R\,{}^1I_{c1}\,{}^0_1R^T = \begin{bmatrix} 0.0833L_1^2 m_1 s_1^2 & -0.0417L_1^2 m_1 s_{2(1)} & 0 \\ -0.0417L_1^2 m_1 s_{2(1)} & 0.0833L_1^2 m_1 c_1^2 & 0 \\ 0 & 0 & 0.0833L_1^2 m_1 \end{bmatrix}$$

$$^0I_2 = {}^0_2R\,{}^2I_{c2}\,{}^0_2R^T = \begin{bmatrix} 0.0833L_1^2 m_1 s_{12}^2 & -0.0417L_1^2 m_1 s_{2(12)} & 0 \\ -0.0417L_1^2 m_1 s_{2(12)} & 0.0833L_1^2 m_1 c_{12}^2 & 0 \\ 0 & 0 & 0.0833L_1^2 m_1 \end{bmatrix} \tag{16}$$

The link Jacobian matrices are:

$$J_{v1}^0 = \begin{bmatrix} -0.5L_1 s_1 & 0 \\ 0.5L_1 c_1 & 0 \\ 0 & 0 \end{bmatrix} \tag{17}$$

$$J_{\omega1}^0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \tag{18}$$

$$J_{v2}^{0} = \begin{bmatrix} -0.5L_2 s_{12} - L_1 s_1 & -0.5L_2 s_{12} \\ 0.5L_2 c_{12} + L_1 c_1 & 0.5L_2 c_{12} \\ 0 & 0 \end{bmatrix} \tag{19}$$

$$J_{\omega 2}^{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \tag{20}$$

The manipulator inertia matrix is:

$$M = \begin{bmatrix} 0.42L_1^2 m_1 + L_1^2 m_2 + 0.25L_2^2 m_2 + L_1 L_2 m_2 c_2 & 0.08L_1^2 m_1 + 0.25L_2^2 m_2 + 0.50L_1 L_2 m_2 c_2 \\ 0.08L_1^2 m_1 + 0.25L_2^2 m_2 + 0.50L_1 L_2 m_2 c_2 & 0.25L_2^2 m_2 + 0.08L_1^2 m_1 \end{bmatrix} \tag{21}$$

The velocity coupling vector is:

$$V = \begin{bmatrix} -0.5L_1 L_2 m_2 \dot{\theta}_2 s_2 (\dot{\theta}_2 + 2\dot{\theta}_1) \\ 0.5L_1 L_2 m_2 \dot{\theta}_1^2 s_2 \end{bmatrix} \tag{22}$$

The gravitational vector is:

$$G = \begin{bmatrix} g m_2 (0.5L_2 c_{12} + L_1 c_1) + 0.5L_1 g m_1 c_1 \\ 0.5L_2 g m_2 c_{12} \end{bmatrix} \tag{23}$$

The final equation is:

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = M \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_1 \end{bmatrix} + V + G \tag{24}$$

II. Results

To go from $(0.1, 0, 0)$ to $(0.9, 0, 0)$ in 4s with a force of $-10N, 0, 0)$ and a torque of $(0, 0, 10Nm$ acting on the end effector, a linear-parabolic blend was done in joint space to get a trajectory. The maximum acceleration was set to 5 rad/s. Fig. 1 shows the position, velocity, and acceleration of the end effector.

A. Trajectory

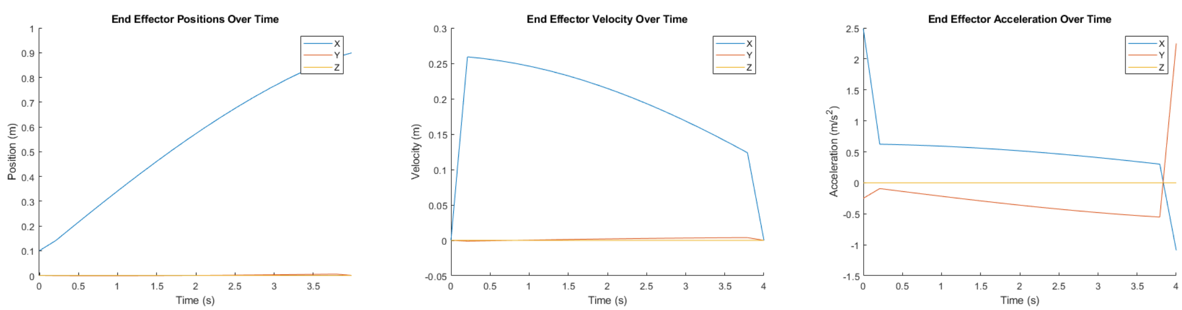

Fig. 1: Position, Velocity, and Acceleration of the End Effector. A liner-parabolic blend was used for joint space interpolation.

B. Joint Torque with Gravity

Fig. 2 shows the joint torques for each joint as well as graphs of each component or torque. The components with the largest effect are the gravity, and applied torques. The inertia has some effect in the start and end which makes sense as the actuator needs to bring the link into motion from a standstill and stop it. The centrifugal and Coriolis torques have an extremely small effect because the link is not moving very quickly.

C. Joint Torque with No Gravity

When gravity is not acting on the links, Fig. 3 shows that the overall joint torque needed to actuate the robot decreases. Now the main component in the joint torque is resisting the applied force acting on the end effector. Gravity, Coriolis, and centrifugal components all play negligible roles.

As shown in Fig. 4, the ratio of joint torque without gravity to the ratio of joint torque with gravity decreases over time. This is because the arm is stretching out in its trajectory which makes the force of gravity exert a larger torque on the links. The gravity torque in J1 can cause the overall torque to be over double the amount as shown in Fig. 4. This shows the importance of robot design as both configurations can accomplish the same task but one will use significantly more torque causing higher power draw and increased wear.

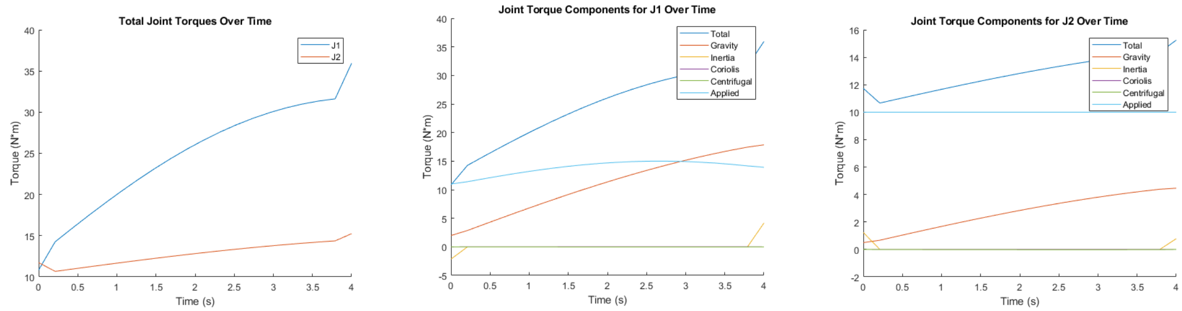

Fig. 2: Joint Torques of Each Joint with Components and Effect of Gravity. The applied torque is the effect of the 10N force and 10 Nm torque on the end effector.

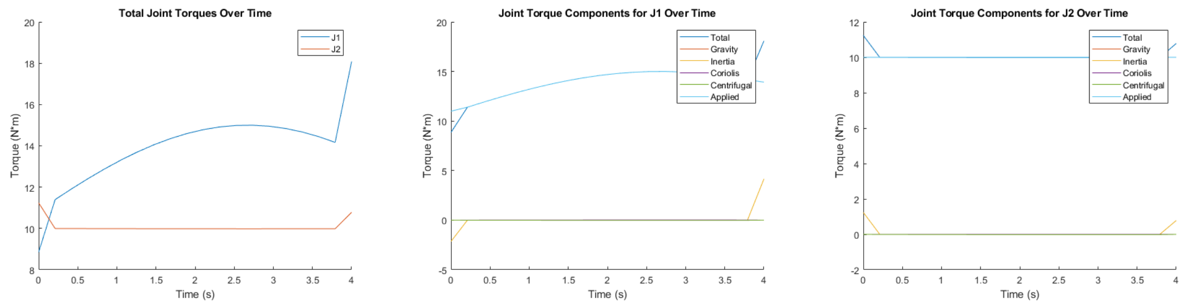

Fig. 3: Joint Torques of Each Joint with Components. The applied torque is the effect of the 10N force and 10 Nm torque on the end effector.

III. Moment of Inertia

The moment of inertia for a uniform-density cube is:

$$I_{cube} = \frac{1}{6}ml^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{25}$$

The moment of inertia for a uniform-density cylinder with radius r, length h, and with its central axis along the z-axis is:

$$I_{cylinder} = \begin{bmatrix} \frac{1}{12}m(3r^2+h^2) & 0 & 0 \\ 0 & \frac{1}{12}m(3r^2+h^2) & 0 \\ 0 & 0 & \frac{1}{2}mr^2 \end{bmatrix} \tag{26}$$

The density is $2710\frac{kg}{m^3}$.

A. Body A

Body A is composed of a cube with sides of 0.1 m and a cylindrical hole with a radius of 0.01 m. Its distance from the center of mass of the link is:

$$P_{CM} = \begin{bmatrix} -0.4 \\ 0 \\ 0 \end{bmatrix} \tag{27}$$

The moment of inertia of the cube according to Eq. (25) is:

$$I_{cube} = \begin{bmatrix} 0.0045 & 0 & 0 \\ 0 & 0.0045 & 0 \\ 0 & 0 & 0.0045 \end{bmatrix} \tag{28}$$

The moment of inertia of the cylindrical hole according to Eq. (29) is:

$$I_{cylinder} = \begin{bmatrix} 0.00007308 & 0 & 0 \\ 0 & 0.00007308 & 0 \\ 0 & 0 & 0.0426 \end{bmatrix} \tag{29}$$

The total moment of inertia is:

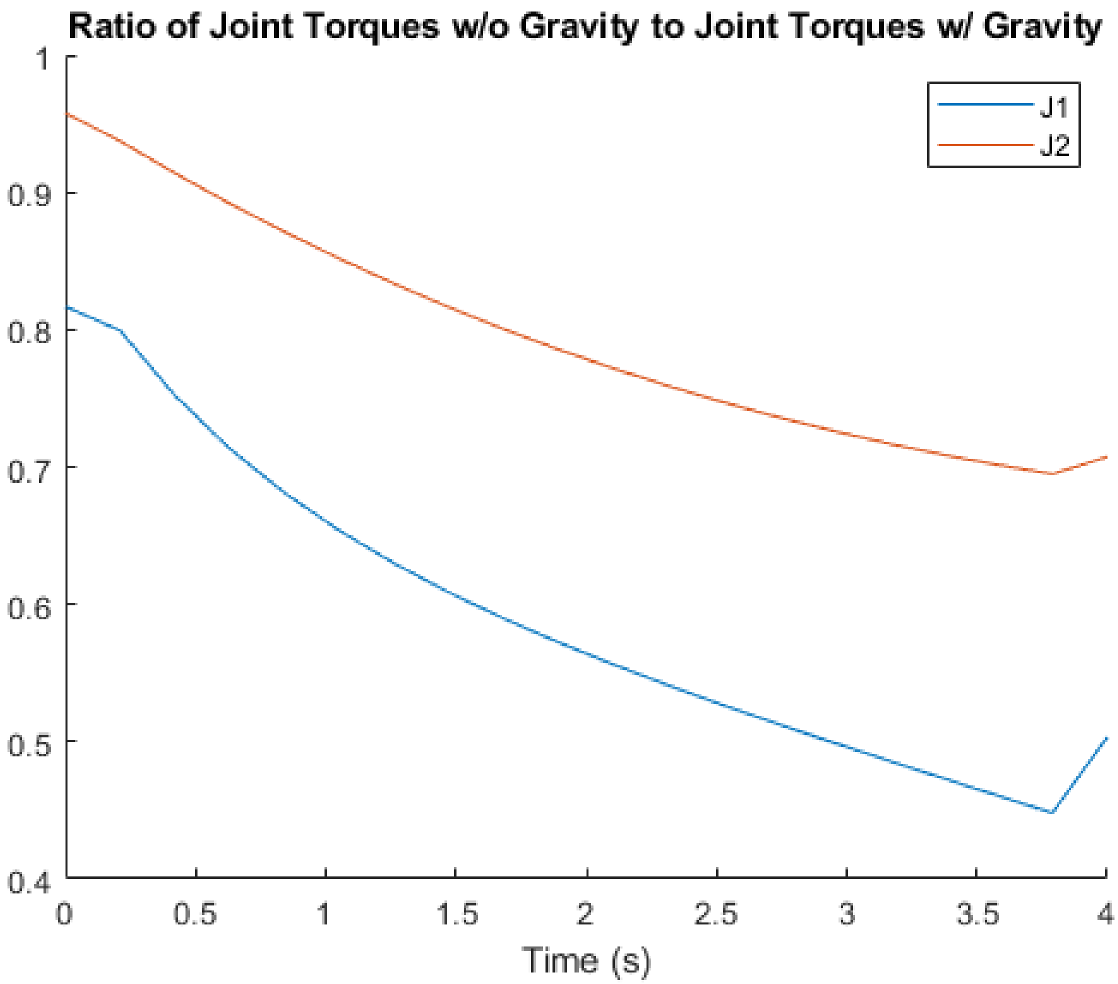

Fig. 4: Ratio of Total Joint Torques on each Joint Defined by $\tau_{no\ gravity}/\tau_{gravity}$.

$$I_A = I_{cube} - I_{cylinder} = \begin{bmatrix} 0.0044 & 0 & 0 \\ 0 & 0.0044 & 0 \\ 0 & 0 & 0.0045 \end{bmatrix} \tag{30}$$

To put the I_A into the frame of the center of mass of the link, we can use the parallel axis theorem:

$$I_{CM,A} = I_A + m_A(P_{CM}^T P_{CM} I_3 - P_{CM} \otimes P_{CM}) = \begin{bmatrix} 0.0044 & 0 & 0 \\ 0 & 0.4244 & 0 \\ 0 & 0 & 0.4245 \end{bmatrix} \tag{31}$$

B. Body B

Body B is just a cylinder, but it's central axis is along the x-axis instead of z. This means that $I_x x$ and $I_z z$ values get swapped.

$$I_B = \begin{bmatrix} 0.0002 & 0 & 0 \\ 0 & 0.0218 & 0 \\ 0 & 0 & 0.0218 \end{bmatrix} \tag{32}$$

C. Body C

Body C is identical to Body A. Its distance from the center of mass of the link is:

$$P_{CM} = \begin{bmatrix} 0.4 \\ 0 \\ 0 \end{bmatrix} \tag{33}$$

Body C is also rotated 45 $\deg .TOputI_C$ into the frame of the center of mass of the link we do:

$$I_{CM,C} = R_x(45) I_C R_x(45)^T + m_C (P_{CM}^T P_{CM} I_3 - P_{CM} \otimes P_{CM}) = \begin{bmatrix} 0.0040 & 0 & 0 \\ 0 & 0.3526 & -0.0002 \\ 0 & -0.0002 & 0.3526 \end{bmatrix} \tag{34}$$

where:

$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\theta) & -sin(\theta) \\ 0 & sin(\theta) & cos(\theta) \end{bmatrix} \tag{35}$$

D. Total

The total moment of inertia of the link is:

$$I_{CM,Link} = I_{CM,A} + I_B + I_{CM,C} = \begin{bmatrix} 0.0086 & 0 & 0 \\ 0 & 0.7989 & -0.0002 \\ 0 & -0.0002 & 0.7989 \end{bmatrix} \tag{36}$$