Métodos de primer orden

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Outline

Motivation

Nicholson equation

$$x'(t) = -dx(t) + px(t-\tau)e^{-x(t-\tau)} \qquad (1)$$

with $p, d > 0$.

The IVP for (1) reads

$$x(t) = \varphi(t) \qquad \varphi \in C([-\tau, 0]).$$

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2. $p > d \Longrightarrow$ unique equilibrium point, which is locally asymptotically stable:
   - for all τ when $p < de^2$,
   - for $\tau < \tau^*(p)$ when $p \geqslant de^2$.

   Moreover if $\varphi > 0$ then

$$\liminf_{t \to +\infty} x(t) \geqslant \min\{\ln\left(\frac{p}{d}\right), e^{-\tau d}\}.$$

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3. $p, d \in C_T \Longrightarrow$ positive T-periodic solutions if $p(t) > d(t)$ for all t and no T-periodic solutions if $p(t) \leqslant d(t)$ for all t (furthermore, 0 is a global attractor).

Generalisation: Nicholson system

$$\begin{cases} x_1'(t) = -d_1 x_1(t) + b_1 x_2(t) + p_1 x_1(t-\tau) e^{-x_1(t-\tau)} \\ x_2'(t) = -d_2 x_2(t) + b_2 x_1(t) + p_2 x_2(t-\tau) e^{-x_2(t-\tau)} \end{cases} \quad (2)$$

1. $p_i + b_i \leqslant d_i \implies 0$ is a global attractor of positive solutions.
2. Uniform persistence if $p_i + b_i > d_i$.
3. T-periodic solutions $b_i(t) < d_i(t) < p_i(t) + d_i(t)$ for all t.

Persistence definitions

Let $X \neq \emptyset$ and $\rho : X \to \mathbb{R}^+$. A semiflow $\Phi : J \times X \to X$ is called

- weakly ρ-persistent, if

$$\limsup_{t\to+\infty} \rho(\Phi(t,x)) > 0 \qquad \forall x \in X, \rho(x) > 0.$$

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- uniformly weakly (strongly) ρ-persistent, if there exists some $\varepsilon > 0$ such that

$$\limsup_{t\to+\infty} \rho(\Phi(t,x)) > \varepsilon \qquad \forall x \in X, \rho(x) > 0 \qquad \text{(resp. liminf)}.$$

In our case $\rho(x) = |x|$ (persistence of some species).

Clearly,

$$\text{(USP)} \Longrightarrow \text{(UWP)} \Longrightarrow \text{(WP) and (SP)} \Longrightarrow \text{(WP)}.$$

Results for X locally compact, $X \backslash \{0\}$ positively invariant:

$$\text{(UWP)} \Longrightarrow \text{(USP), but (WP)} \not\Longrightarrow \text{(SP) and (SP)} \not\Longrightarrow \text{(USP)}.$$

Extra conditions $\Longrightarrow$ all definitions are equivalent (see [2]).

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Fonda [3]:

(USP) $\Longleftrightarrow \exists\, U \ni 0$ open and $V : U \to [0, +\infty)$ continuous such that:

1. $V(x) = 0 \iff x = 0.$
2. $\forall\, x \neq 0\, \exists t_x > 0$ such that $V(\Phi(t_x, x)) > V(x)$.

Outline

Our system reads

$$x'(t) = f(t, x(t), x(t-\tau)) \tag{3}$$

with $f : [0, +\infty) \times [0, +\infty)^{2N} \to \mathbb{R}^N$ continuous.
Initial condition:

$$x(t) = \varphi(t) \qquad -\tau \leqslant t \leqslant 0 \tag{4}$$

$$\varphi \in X := \text{positive cone of } C([-\tau, 0], \mathbb{R}^N),$$

which is **not** locally compact.

Basic assumption:

(H1) If $x_j = 0$ for some j and $y \neq 0$ then $f_j(t, x, y) > 0$ for all $t > 0$.

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(H1) $\Longrightarrow X^\circ$ pos. invariant, i.e.: $\varphi > 0 \Longrightarrow x > 0$, but **(H1)** $\not\Longleftarrow$ (WP).

Consider $V : (0, +\infty)^N \to (0, +\infty)$ smooth such that $V(x) \to 0$ as $x \to 0$ in $(0, +\infty)^N$.

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Easy case:

(H2) There exist $t_0, r > 0$ such that

$$\langle \nabla V(x), f(t,x,y) \rangle > 0 \qquad \text{for } t > t_0, V(x), V(y) < r.$$

$$\textbf{(H1)} + \textbf{(H2)} \Longrightarrow \text{(WP)}$$

(H2) $\Longrightarrow$ (Fonda) when $\tau = 0$. However, **(H2)** is not satisfied e.g. in (1).

More realistic:

(H2') There exist $t_0, r > 0$ such that

$$\langle \nabla V(x), f(t,x,x) \rangle > 0 \qquad \text{for } t > t_0 \text{ and } V(x) < r.$$

(H1) + **(H2')** do not suffice because $x(t) - x(t-\tau)$ may be large.

Proposition: (SP) holds if we also assume monotonicity:

(H3)

$$\langle \nabla V(x), f(t,x,y) \rangle \geqslant \langle \nabla V(x), f(t,x,x) \rangle$$

whenever $V(x) \leqslant V(y)$.

Proof: Suppose $s_n \to +\infty$, $x(s_n) \to 0 \Longrightarrow v(s_n) \to 0$.

Set t_n such that $v(t_n) = \text{mín}_{t \leqslant s_n} v(t)$. For $n \gg 0$,

$$v'(t_n) \leqslant 0 \text{ and } v(t_n - \tau) \geqslant v(t_n).$$

Thus,

$$\begin{aligned} 0 \geqslant v'(t_n) &= \langle \nabla V(x(t_n)), f(t_n, x(t_n), x(t_n - \tau)) \rangle \\ &\geqslant \langle \nabla V(x(t_n)), f(t_n, x(t_n), x(t_n)) \rangle > 0, \end{aligned}$$

a contradiction.

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a contradiction.

Nicholson does not satisfy **(H3)**, then: why is it (USP)?

Local monotonicity

(H3') There exists $\eta > 0$ such that

$$\langle \nabla V(x), f(t,x,y) \rangle \geqslant \langle \nabla V(x), f(t,x,x) \rangle$$

whenever $V(x) \leqslant V(y) \leqslant \eta$.

Nicholson: $f(x,y) = -dx + pye^{-y} \Longrightarrow$ **(H3')** with $\eta = 1$ and $V(x) = x$.

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(H3') is not enough! Reason: $x(t) - x(t-\tau)$ may be large.

A standard assumption in population models

(H4) $\langle \nabla V(x), f(t,x,y)\rangle \geqslant -kV(x)$ for some constant k.

Thus,

$$v'(t) = \langle \nabla V(x(t)), f(t, x(t), x(t-\tau)\rangle \geqslant -kv(t),$$

whence

$$v(t-\tau) \leqslant e^{k\tau} v(t) \text{ for all } t \geqslant \tau.$$

Consequence:

$$\textbf{(H1)} + \textbf{(H2')} + \textbf{(H3')} + \textbf{(H4)} \Longrightarrow \text{(SP)}.$$

But... why is Nicholson (USP)?

Assume **(H1)** + **(H2')** + **(H3')** + **(H4)** and $i = \liminf_{t \to +\infty} v(t) \ll 1$. Then $i > 0$ and there are 3 cases:

1. $v(t) \geqslant i$ for all $t \gg 0$. Then choose as before $t_n \to +\infty$ such that $v(t_n) \to i$, $v(t_n - \tau) \geqslant v(t_n)$ and $v'(t_n) \leqslant 0$ and a contradiction yields.

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2. $v(t)$ oscillates around i. Then we may choose a sequence $t_n \to +\infty$ such that $v(t_n) \to i^-$ and $v'(t_n) \leqslant 0$. However, it might happen that $v(t_n - \tau) < v(t_n)$ for n large, then **(H3')** + **(H4)** are not of any help.

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3. $v(t) \to i^-$ as $t \to +\infty$: same situation.

Wouldn't it be great if

$$\langle \nabla V(x), f(t,x,y)\rangle \geqslant c(i) > 0$$

whenever $V(x)$, $V(y)$ are close to i?

For example, if $v(t) \to i^-$, then

$$v'(t) = \langle \nabla V(x(t)), f(t, x(t), x(t-\tau))\rangle \geqslant c$$

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OK for (1) but too ambitious when $N > 1$: the condition fails for example if $V(x) = |x|^2$ and $f(x,y) = y$.

And yet it works

Let

$$\theta_i := \limsup_{t\to+\infty, V(x), V(y)\to i^-} |f(t,x,y)|$$

and observe that if $v(t) \to i^-$ then

$$|x(t) - x(t-\tau)| \leqslant \tau |x'(\xi)| \qquad \xi \in [t-\tau, t]$$

and hence:

$$\limsup_{t\to+\infty} |x(t) - x(t-\tau)| \leqslant \tau\theta_i.$$

At last we get (USP)

(H2"') $\exists\, r > 0$ such that, $\forall\, i \in (0, r)$ and some $C_i > \tau\theta_i$

$$\liminf_{t\to+\infty, V(x), V(y)\to i^-, |y-x|\leqslant C_i} \langle x, f(t,x,y)\rangle > 0.$$

Teorema

$$\textbf{(H1)} + \textbf{(H2")} + \textbf{(H3')} + \textbf{(H4)} \Longrightarrow \textbf{(USP)}.$$

More precisely, all solutions of (3)-(4) with $\varphi_j(t) > 0$ *for all* j *and* $t \in [-\tau, 0]$ *satisfy*

$$\liminf_{t\to+\infty} V(x(t)) \geqslant \min\{r, e^{-k\tau}\eta\}.$$

Outline

Periodic solutions

Consider a continuous function $a : [0, +\infty) \to (0, +\infty)$ and define

$$K(t, x, y) := \langle \nabla V(x), f(t, x, y)\rangle + a(t)V(x),$$

$$\phi(t, r) := \sup_{V(x), V(y) \leqslant r} \frac{K(t, x, y)}{a(t)}.$$

Autonomous system: **(H1)** + **(H2')** and $\phi(R) < R$ for some R, then there exists a positive equilibrium point in the region

$$V_\varepsilon^R := \{\varepsilon < V(x) < R\}.$$

Indeed, the field $f(x, x)$ points outwards over ∂V_ε^R.

Faces $\overline{V_\varepsilon^R} \cap \{x_i = 0\}$: use **(H1)**.

Bottom cap $\{V = \varepsilon\}$: the outer normal is $-\nabla V$ and **(H2')** guarantees $\langle \nabla V(x), f(x,x)\rangle > 0$.

Top cap $\{V = R\}$: the outer normal is ∇V and the inequality $\phi(R) < R$ implies $\langle \nabla V(x), f(x,x)\rangle < 0$.

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Top cap $\{V = R\}$: the outer normal is ∇V and the inequality $\phi(R) < R$ implies $\langle \nabla V(x), f(x,x)\rangle < 0$.

Important condition: $\chi(V_\varepsilon^R) \neq 0$ (e.g. $\overline{V_\varepsilon^R}$ is a retract of $C \simeq \overline{B}$).

Closed orbits for the non-autonomous system

Let $T \geqslant \tau$ and consider the assumption:

(H5) There exists $R > 0$ such that $\phi(t, R) < R$ for $0 \leqslant t \leqslant T$.

Teorema

(H1) + **(H2')** + **(H3')** + **(H4)** + **(H5)** $\Longrightarrow$ *Positive T-periodic solutions.*

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Idea of the proof: $deg(I - K) = \chi(V_\varepsilon^R)$.

Outline

Conversely...

Assume that

$$\phi^*(r) := \sup_{t \geqslant 0} \phi(t, r).$$

is continuous and

(H6) For every $\varepsilon > 0$ there exists $\mu > 0$ such that $V^{-1}(0, \mu) \subset B_\varepsilon(0)$.

Teorema

Assume **(H6)** *and suppose there exists R_0 such that $\phi^*(r) < r$ for $0 < r < R_0$. Then every solution with initial data $\varphi(t) \in (0, R_0)$ for all $t \in [-\tau, 0]$ tends to 0 as $t \to +\infty$.*

Sketch of the proof

If $v \leqslant r$ on $[t-\tau, t]$ and $v'(t) \geqslant 0$, then

$$v(t) \leqslant \phi(r).$$

Let $R_{j+1} := \phi^*(R_j) < R_j$, then two different situations may occur:

1. There exists $t_j \to +\infty$ such that $v(t) \in (0, R_j]$ for $t \geqslant t_j$. Then $v(t) \to 0$ because

$$\phi^*(\lim_{j\to\infty} R_j) = \lim_{j\to\infty} \phi^*(R_j) = \lim_{j\to\infty} R_j,$$

.

2. There exist j and t_j such that $v(t) \in (R_{j+1}, R_j]$ and decreases strictly for $t \geqslant t_j$. Let $r := \lim_{t\to+\infty} v(t)$, fix $\tilde{r} > r$ such that $\phi^*(\tilde{r}) < r$ and $\tilde{t}$ such that $v(t) \leqslant \tilde{r}$ for $t \geqslant \tilde{t}$. Thus $v(t) \leqslant \phi^*(\tilde{r}) < r$ for $t \geqslant t^* + \tau$, a contradiction.

Outline

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Thanks for your attention!