Project 1. The Finite Volume Method

Consider the one-dimensional *shallow water* equation

$$\begin{bmatrix} h \\ m \end{bmatrix}_t + \begin{bmatrix} m \\ \frac{m^2}{h} + \frac{1}{2}gh^2 \end{bmatrix}_x = \mathbf{S}(x,t), \tag{1}$$

where $h = h(x,t)$ is the *depth* or the height of the water, $m = m(x,t)$ is a quantity – usually called the *discharge* in shallow water theory – which measures the flow rate of the fluid past a point, g is the acceleration due to gravity, and $\mathbf{S}$ is some source term. Note that (1) has the form $\mathbf{q}_t + \mathbf{f}(\mathbf{q})_x = \mathbf{S}(x,t)$ with $\mathbf{q} = (h, m)^T$. In this model, $u = m/h$ is the horizontal velocity. For this project, we choose $g = 1$ and the spatial domain $\Omega = [0, 2]$.

Remark: Remember to discretize all data by cell averages.

1.1 (a) Discretize the domain Ω with N cells. Implement the (local) Lax-Friedrichs method for (1) with
- periodic boundary conditions i.e., $\mathbf{q}_0 = \mathbf{q}_N$ and $\mathbf{q}_{N+1} = \mathbf{q}_1$,
- open boundary conditions i.e., $\mathbf{q}_0 = \mathbf{q}_1$ and $\mathbf{q}_{N+1} = \mathbf{q}_N$.

(b) Test your code on the problem with the initial condition

$$h(x,0) = h_0(x) = 1 + 0.5\sin(\pi x) \qquad m(x,0) = m_0(x) = uh_0(x)\ , \tag{2}$$

and

$$\mathbf{S}(x,t) = \begin{bmatrix} \frac{\pi}{2}\,(u-1)\cos\pi(x-t) \\ \frac{\pi}{2}\cos\pi(x-t)\left(-u + u^2 + gh_0(x-t)\right) \end{bmatrix}\ , \tag{3}$$

where $u = 0.25$. The exact solution to this problem is given by

$$h(x,t) = h_0(x-t) \qquad m = u\,h\ . \tag{4}$$

In your computations use periodic boundary conditions and evaluate the time-step as

$$k = \mathrm{CFL}\frac{\Delta x}{\max_i\left(|u_i| + \sqrt{gh_i}\right)},$$

with CFL $= 0.5$. Plot the numerical solution at the final time $T = 2$. Measure the error of the scheme at $T = 2$ as a function of Δx and plot the results on a log-log graph.

1.2 (a) Set $\mathbf{S} = 0$ and run the program for each of the following initial conditions

$$h(x,0) = 1 - 0.1\sin(\pi x)\ , \qquad m(x,0) = 0, \tag{5}$$

$$h(x,0) = 1 - 0.2\sin(2\pi x)\ , \qquad m(x,0) = 0.5, \tag{6}$$

with periodic boundary conditions. Plot the solution at time $T = 2$. Comment on the regularity of the solution and the performance of the scheme.

(b) For each initial function (5) and (6), measure the error of the scheme at $T = 2$ as a function of Δx and plot the results on a log-log graph. To measure the error, compare each numerical solution with a reference numerical solution computed on a very fine mesh.

1.3 (a) This problem pertains to the application of Godunov's method with Roe linearization to (1). What is the motivation for doing so?

(b) Derive a Roe matrix $\hat{A}$ for (1). (Have a look at the supplementary material titled "Roe_matrix.pdf".)

(c) Implement Godunov's method with a Roe approximate Riemann solver for the initial boundary value problem described in 1.1.

(d) Repeat 1.1 and 1.2 with the Roe solver.

(e) Comment on how the Lax-Friedrichs and Roe methods differ in terms of accuracy.

1.4 (a) For $\mathbf{S} = 0$, implement the following initial condition

$$h(x,0) = 1\ , \qquad m(x,0) = \begin{cases} -1.5 & x < 1 \\ 0.0 & x > 1 \end{cases}\ , \tag{7}$$

with open boundary conditions. Obtain a reference solution using the Lax-Friedrichs flux.

(b) Plot the solutions with the Lax-Friedrichs scheme at time $T = 0.5$ with varying mesh size, and compare them with the reference solution. Comment on the type of waves that can be seen in the solution.

(c) Repeat the same experiment with the Roe scheme. Does the numerical solution look the same as before? Does it converge to the Lax-Friedrichs solution with mesh refinement? If not, can you explain why this is the case?