Project 1. (Notes on constructing the Roe matrix)

The first project requires the construction of a Roe matrix for the one-dimensional shallow water equations. In general, for a conservation law

$$\mathbf{q}_t + \mathbf{f}_x = 0, \tag{1}$$

the Roe matrix $\hat{A}$ must satisfy

$$\hat{A}(\mathbf{q}_r - \mathbf{q}_l) = \mathbf{f}_r - \mathbf{f}_l\ . \tag{2}$$

Suppose we introduce a parameter vector $\mathbf{z}$, such that the the jump in $\mathbf{q}$ and $\mathbf{f}$ can be expressed in terms of the jump in $\mathbf{z}$

$$\mathbf{q}_r - \mathbf{q}_l = B(\mathbf{z}_r - \mathbf{z}_l), \qquad \mathbf{f}_r - \mathbf{f}_l = C(\mathbf{z}_r - \mathbf{z}_l)\ , \tag{3}$$

where the matrices B and C depend on $\mathbf{z}_l, \mathbf{z}_r$. Then clearly we can choose $\hat{A} = CB^{-1}$.

For the shallow water equations, if we choose $\mathbf{z} = h^{-1/2}\mathbf{q}$, we can get the following Roe matrix

$$\hat{A} = \begin{bmatrix} 0 & 1 \\ \overline{(z_1^2)}g - (\overline{z_2})^2\,(\overline{z_1})^{-2} & 2\overline{z_2}\,(\overline{z_1})^{-1} \end{bmatrix} \tag{4}$$

where

$$\mathbf{z} = (z_1, z_2)^\top, \qquad \overline{\phi} = \frac{\phi_r + \phi_l}{2}, \qquad \Delta\phi = \phi_r - \phi_l. \tag{5}$$

Try to derive the above Roe matrix by computing the corresponding matrices B and C. Also check whether the matrix is consistent with the actual flux Jacobian.

You may need the following important identity

$$\Delta(ab) = \overline{a}\Delta b + \overline{b}\Delta a. \tag{6}$$