Project 2. MUSCL reconstruction

Consider the one-dimensional *shallow water* equations

$$\begin{bmatrix} h \\ m \end{bmatrix}_t + \begin{bmatrix} m \\ \frac{m^2}{h} + \frac{1}{2}gh^2 \end{bmatrix}_x = \mathbf{S}(x,t), \tag{1}$$

as done in Project 1. We choose $g = 1$ and the spatial domain $\Omega = [0, 2]$. In order to obtain a high-order accurate scheme, consider the semi-discrete finite volume scheme obtained by integrating the conservation law in each cell $I_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$

$$\frac{\mathrm{d}\mathbf{q}_i}{\mathrm{d}t} = -\frac{1}{h}\left(\mathbf{F}_{i+\frac{1}{2}} - \mathbf{F}_{i-\frac{1}{2}}\right) + \mathbf{S}_i(t) \tag{2}$$

where $\mathbf{q}_i$ and $\mathbf{S}_i$ are cell averages, and $\mathbf{F}_{i+\frac{1}{2}}$ is a consistent numerical flux approximating $\mathbf{f}(\mathbf{q}_{i+\frac{1}{2}})$. Using the MUSCL approach, high spatial accuracy can be achieved. To ensure high temporal accuracy, we can use a suitable time-marching strategy to solve the system of ODEs (2). In particular, we use the third-order *Strong Stability Preserving Runge-Kutta scheme* (SSP-RK3). The algorithm for SSP-RK3 to solve an ODE of the form

$$\frac{\mathrm{d}\mathbf{q}_i}{\mathrm{d}t} = \mathbf{L}(\mathbf{q}, t)$$

is given by

$$\begin{aligned}
\mathbf{q}^{(1)} &= \mathbf{q}^n + k\mathbf{L}(\mathbf{q}^n, t^n) \\
\mathbf{q}^{(2)} &= \frac{3}{4}\mathbf{q}^n + \frac{1}{4}\left(\mathbf{q}^{(1)} + k\mathbf{L}(\mathbf{q}^{(1)}, t^n + k)\right) \\
\mathbf{q}^{(3)} &= \frac{1}{3}\mathbf{q}^n + \frac{2}{3}\left(\mathbf{q}^{(2)} + k\mathbf{L}(\mathbf{q}^{(2)}, t^n + k/2)\right) \\
\mathbf{q}^{n+1} &= \mathbf{q}^{(3)}.
\end{aligned}$$

Remark: Remember to discretize all data by cell averages.

2.1 (a) Implement the MUSCL scheme for (1) with i) the Lax-Friedrichs flux and ii) the Roe flux. Update the solution using SSP-RK3. To evaluate the slopes for the MUSCL scheme use

- Zero slope
- The minmod limiter
- The MUSCL limiter
- The minmod-type TVB limiter (with parameter M)

Implement both open and periodic boundary conditions. Note that you may need two ghost cells on either side of the domain to implement the boundary conditions. Furthermore, you may need to find (by trial and error), the suitable parameter M of the TVB limiter.

(b) Test your code on the problem with the initial condition

$$h(x,0) = h_0(x) = 1 + 0.5\sin(\pi x) \qquad m(x,0) = m_0(x) = uh_0(x)\ , \tag{3}$$

and

$$\mathbf{S}(x,t) = \begin{bmatrix} \frac{\pi}{2}(u-1)\cos\pi(x-t) \\ \frac{\pi}{2}\cos\pi(x-t)\left(-u + u^2 + gh_0(x-t)\right) \end{bmatrix}, \tag{4}$$

where $u = 0.25$. The exact solution to this problem is given by

$$h(x,t) = h_0(x-t) \qquad\qquad m = u\,h\ . \tag{5}$$

In your computations use periodic boundary conditions and evaluate the time-step as

$$k = \text{CFL}\frac{\Delta x}{\max_i\left(|u_i| + \sqrt{gh_i}\right)},$$

with CFL $= 0.5$. Plot the numerical solution at the final time $T = 2$. Measure the error of the schemes at $T = 2$ as a function of Δx and plot the results on a log-log graph.

2.2 (a) Set $\mathbf{S} = 0$ and run the program for each of the following initial conditions

$$h(x,0) = 1 - 0.1\sin(\pi x)\ , \qquad\qquad m(x,0) = 0, \tag{6}$$

$$h(x,0) = 1 - 0.2\sin(2\pi x)\ , \qquad\qquad m(x,0) = 0.5, \tag{7}$$

with periodic boundary conditions. Plot the solution at time $T = 2$.

(b) Does the same value of M for the TVB limiter, give the "best" results for all initial conditions?

(c) For each initial function (6) and (7), measure the error of the schemes at $T = 2$ as a function of Δx and plot the results on a log-log graph. To measure the error, compare each numerical solution with a reference numerical solution computed on a very fine mesh.

2.3 (a) For $\mathbf{S} = 0$, implement the following initial condition

$$h(x,0) = 1\ , \qquad\qquad m(x,0) = \begin{cases} -1.5 & x < 1 \\ 0.0 & x > 1 \end{cases}\ , \tag{8}$$

with open boundary conditions. Obtain a reference solution using the Lax-Friedrichs flux.

(b) In Project 1, you had noticed an entropy violation with the Roe flux for this initial condition. Does the problem disappear for the high-order MUSCL approach?