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Lab session # 2
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1 Question 1

A fully connected graph of n vertices can have a maximum of $\frac{n(n-1)}{2}$ edges. We can count the different edges for each vertices: the first will have a maximum of $n-1$ edges, the second will have $(n-1)-1 = n-2$ edges (we do not count again the edge between the first and the second node). Recursively the maximum number will be:

$$E = \sum_{i=0}^{n-1} n - 1 - i = \frac{n(n-1)}{2} \tag{1}$$

To build a triangle you need to pick three vertices out of the n there are. Hence the maximum number of triangles in a fully vonnected graph will be :

$$T = \binom{n}{3} = \frac{n!}{(n-3)! \times 3!} = \frac{n(n-1)(n-2)}{6} \tag{2}$$

2 Question 2

No, two graphs can have the same degree distribution without being isomorphic to each other. For example, the below graphs have both a degree distribution of $[0, 3, 2, 1]$ but don't share a bijective mapping.

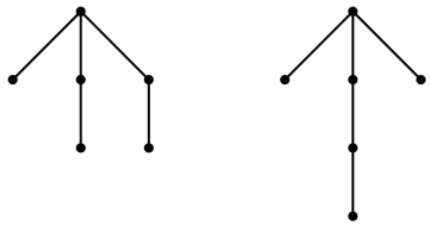


Figure 1: Two non-isomorphic graphs with the same degree distribution.

3 Question 3

For C_3, the answer is immediately $T = 1$. For $n > 3$, the graphs have no closed triplets, they each are connected to two nodes that aren't connected, so we will have for C_n a global clustering coefficient $T = 0$

4 Question 4

Let $L = D - A = DL_{rw}$ We can write for all $f \in \mathbb{R}^n$:

$$\begin{aligned}
f^T L f &= f^T D f - f^T A f \\
&= \sum_{i=1}^{n} D_{ii} f_i^2 + \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} f_i f_j \\
&= \frac{1}{2} \left(\sum_{i=1}^{n} D_{ii} f_i^2 + 2 \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} f_i f_j + \sum_{j=1}^{n} D_{jj} f_j^2 \right)
\end{aligned}$$

Because $D_{ii} = \sum_{j=1}^{n} A_{ij}$, we can write :

$$f^T L f = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} (f_i - f_j)^2$$

If we apply this to the eigenvector of the smallest eigenvalue of L_{rw}, which is u_1 :

$$\begin{aligned} u_1^T D L_{rw} u_1 &= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} ([u_1]_i - [u_1]_j)^2 \\ 2u_1^T D \lambda_1 u_1 &= \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} ([u_1]_i - [u_1]_j)^2 \\ \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} ([u_1]_i - [u_1]_j)^2 &= 2\lambda_1 \sum_{i=1}^{n} D_{ii} [u_1]_i^2 \end{aligned}$$

Hence, 0 is the smallest eigenvalue of L_{rw} (an eigenvector is the vector filled with ones, A multiplied by this vector gives the vector of the degrees which is cancelled by D^{-1} to give the eigenvector again, cancelled itself by the identity matrix).
In the end, we determine :

$$\sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} ([u_1]_i - [u_1]_j)^2 = 0 \tag{3}$$

5 Question 5

For the first, we have $m = 14$, $n_c = 2$, $l_1 = 6$, $l_2 = 6$, $d_1 = 12$, $d_2 = 12$.

$$Q_1 = 2 \left(\frac{6}{14} - \left(\frac{12}{2 \times 14} \right)^2 \right) = 2 \left(\frac{3}{7} - \left(\frac{3}{7} \right)^2 \right) = 2 \times \frac{3}{7} \left(1 - \frac{3}{7} \right) = \frac{24}{49} \quad (= 0.49)$$

For the second, we have $m = 14$, $n_c = 2$, $l_1 = 2$, $l_2 = 5$, $d_1 = 4$, $d_2 = 10$.

$$Q_1 = \frac{1}{7} \left(1 - \frac{1}{7} \right) + \frac{5}{14} \left(1 - \frac{5}{14} \right) = \frac{1}{7} \times \frac{6}{7} + \frac{5}{14} \times \frac{9}{14} = \frac{24 + 45}{14^2} = \frac{69}{14^2} = \frac{69}{196} \quad (= 0.35)$$

6 Question 6

Figure 2: P4

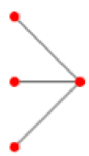


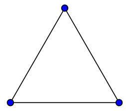
Figure 3: S4

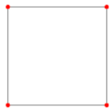
For P_4, we have $\phi(P_4) = [3, 2, 1]$
For S_4, we have $\phi(S_4) = [3, 3, 0]$
Hence,

$$\begin{aligned} k(P_4, P_4) &= \langle \phi(P_4), \phi(P_4) \rangle = 3^2 + 2^2 + 1^2 = 9 + 4 + 1 = 14 \\ k(P_4, S_4) &= \langle \phi(P_4), \phi(S_4) \rangle = 3^2 + 3 \times 2 + 1 \times 0 = 9 + 6 + 0 = 15 \\ k(S_4, S_4) &= \langle \phi(S_4), \phi(S_4) \rangle = 3^2 + 3^2 + 0[2] = 9 + 9 + 0 = 18 \end{aligned}$$

7 Question 7

If $k(G, G') = 0$, it means that G and G' don't share graphlets. It means they don't share subgraphs that are isomorphic. An example of this is if we take G as a triangle and G' as a square





The first vector gives $f_G = [1, 0, 0, 0]$ and the second $f_{G'} = [0, 4, 0, 0]$, hence $k(G, G') = 0$

References