Measurement of Linear Dependence and Feedback Between Multiple Time Series

JOHN GEWEKE*

Measures of linear dependence and feedback for multiple time series are defined. The measure of linear dependence is the sum of the measure of linear feedback from the first series to the second, linear feedback from the second to the first, and instantaneous linear feedback. The measures are nonnegative, and zero only when feedback (causality) of the relevant type is absent. The measures of linear feedback from one series to another can be additively decomposed by frequency. A readily usable theory of inference for all of these measures and their decompositions is described; the computations involved are modest.

KEY WORDS: Multiple time series; Feedback; Dependence; Causality.

1. INTRODUCTION

In discussions of the relations between time series, concepts of dependence and feedback are frequently invoked. These concepts appear to be useful whether one is merely describing an estimated relationship between two time series, or describing the properties of an engineering or econometric model. In the case of linear systems, these concepts have been given exact definition in the literature (Granger 1963), as have their antonyms, "independence" and "unidirectional causality" (Granger 1969). The empirical literature abounds with tests of independence and unidirectional causality for various pairs of time series, but there have been virtually no investigations of the degree of dependence or the extent of various kinds of feedback. The latter approach is more realistic in the typical case in which the hypothesis of independence or unidirectional causality is not literally entertained, but it requires that one be able to measure linear dependence and feedback.

In this article we construct measures of linear dependence and feedback between multiple time series that coincide with those proposed in the literature for more specific cases. In the case of univariate series, our measure of linear dependence is the same as the measure of information proposed by Gel'fand and Yaglom (1959). Our measures of feedback from one series to another and vice versa are monotonic transformations of those suggested by Granger (1963) for multiple time series in the case in which there is no instantaneous feedback. For the special case in which at least one of the time series is univariate, our measures are monotonic transformations of certain "R^2 measures for time series" proposed by Pierce (1979).

This article breaks fresh ground in three directions. First, it is shown that linear dependence and feedback among two multiple time series X and Y can be measured in a rather natural way such that linear dependence is the sum of linear feedback from X to Y, linear feedback from Y to X, and instantaneous linear feedback (for brevity, we henceforth frequently dispense with the adjective "linear"). The decomposition is rather obvious, but to the author's knowledge has not been pointed out; it also leads at once to the well-known results of Sims (1972) on the implications of Wiener-Granger causality (Wiener 1956; Granger 1963, 1967). Second, measures of feedback from X to Y and Y to X can, under circumstances often satisfied by economic time series, be additively decomposed by frequency so that one may speak of "feedback from Y to X (or X to Y) at frequency λ." Third, the measures of dependence and feedback proposed here are simple to estimate. The hypothesis that a certain kind of feedback is absent, either altogether or at particular frequencies, amounts to linear restrictions on the coefficients of a vector autoregression estimated by ordinary least squares. Confidence intervals for these measures may also be constructed from ordinary least squares estimates.

The plan of this article is as follows. In Section 2 we construct a canonical form that is useful in studying linear dependence and feedback between multiple time series. In Section 3 the measures of linear dependence and feedback are set forth, and the relations between them just described are demonstrated. Problems of inference are taken up in Section 4. An empirical example is presented and discussed in Section 5, and conclusions follow.

2. ALTERNATIVE REPRESENTATIONS FOR MULTIPLE TIME SERIES

We focus our attention initially on a wide-sense stationary, purely nondeterministic multiple time series Z

* John Geweke is Associate Professor, Department of Economics, University of Wisconsin-Madison, Madison, WI 53706. Financial support for this work was provided by National Science Foundation grant SES8005606. This paper was presented at the NBER/NSF Seminar on Time Series, Washington, June 13–14, 1980.

© Journal of the American Statistical Association
June 1982, Volume 77, Number 378
Theory and Methods Section

$= \{\mathbf{z}_t, t \text{ real}\}$. By wide-sense stationary, we mean that the mean of $\mathbf{z}_t$ exists and does not depend on t, and for all t and s cov$(\mathbf{z}_t, \mathbf{z}_{t+s})$ exists and depends on s but not on t. By purely nondeterministic, we mean that the correlation of $\mathbf{z}_{t+p}$ and $\mathbf{z}_t$ vanishes as p increases in such a way that in the limit the best linear forecast of $\mathbf{z}_{t+p}$ conditional on $\{\mathbf{z}_{t-s}, s \geq 0\}$ is the unconditional mean of $\mathbf{z}_{t+p}$, which for convenience we take to be $\mathbf{0}$.

We further suppose that there exists a moving average representation for $\mathbf{z}_t$,

$$\mathbf{z}_t = \sum_{s=0}^{\infty} \mathbf{A}_s \boldsymbol{\epsilon}_{t-s}, \quad E(\boldsymbol{\epsilon}_t) = \mathbf{0}, \text{var}(\boldsymbol{\epsilon}_t) = \boldsymbol{\Upsilon}. \tag{2.1}$$

In the moving average representation, $| \sum_{s=0}^{\infty} \mathbf{A}_s z^s | \neq 0$ for all z: $| z | < 1$, the coefficients satisfy the square summability condition $\sum_{s=0}^{\infty} \| \mathbf{A}_s \|^2 < \infty$, and the vector $\boldsymbol{\epsilon}_t$ is serially uncorrelated. (For any square matrix $\mathbf{C}$, $\| \mathbf{C} \|$ denotes the square root of the largest eigenvalue of $\mathbf{C}'\mathbf{C}$ and $| \mathbf{C} |$ denotes the square root of the determinant of $\mathbf{C}'\mathbf{C}$.) The existence of the moving average representation is equivalent to the existence of the spectral density matrix $\mathbf{S}_z(\lambda)$ of $\mathbf{z}_t$ at almost all frequencies $\lambda \in [-\pi, \pi]$ (Doob 1953, pp. 499–500). It provides a lower bound on the mean squared error of one-step-ahead minimum mean squared error linear forecasts, which is

$$| \boldsymbol{\Upsilon} | = \exp\left((1/2\pi) \int_{-\pi}^{\pi} \ln | \mathbf{S}_z(\lambda) | \, d\lambda \right) > 0. \tag{2.2}$$

The condition $| \boldsymbol{\Upsilon} | > 0$ is equivalent to our assumption that Z is purely nondeterministic (Rozanov 1967, p. 72).

We confine our attention to cases in which the relation in (2.1) can be inverted so that $\mathbf{z}_t$ becomes a linear function of $Z_{t-1} = \{\mathbf{z}_{t-s}, s \geq 1\}$ and $\boldsymbol{\epsilon}_t$,

$$\mathbf{z}_t = \sum_{s=1}^{\infty} \mathbf{B}_s \mathbf{z}_{t-s} + \boldsymbol{\epsilon}_t. \tag{2.3}$$

A sufficient condition for invertibility is that there exist a constant $c \geq 1$ such that for almost all λ,

$$c^{-1}\mathbf{I}_n \ \text{\textcircled{$\leq$}}\ \mathbf{S}_z(\lambda) \ \text{\textcircled{$\leq$}}\ c\mathbf{I}_n \tag{2.4}$$

(Rozanov 1967, pp. 77–78), which we henceforth assume. ($\mathbf{A}$ ⊜ $\mathbf{B}$ indicates that $\mathbf{B} - \mathbf{A}$ is positive semidefinite; $\mathbf{A}$ ⧀ $\mathbf{B}$ indicates that $\mathbf{B} - \mathbf{A}$ is positive semidefinite and not null.) This assumption is nontrivial, because processes like $\mathbf{z}_t = \boldsymbol{\epsilon}_t + \boldsymbol{\epsilon}_{t-1}$ are excluded. The requirement (2.4) that the spectral density matrix be bounded uniformly away from zero almost everywhere in $[-\pi, \pi]$ is more restrictive than (2.2). On the other hand (2.4) is less restrictive than the assumption that Z is a moving average, autoregressive process of finite order with invertible moving average and autoregressive parts, which is sometimes taken as the point of departure in the theory of multiple time series.

Suppose now that $\mathbf{z}_t$: $m \times 1$ has been partitioned into $k \times 1$ and $l \times 1$ subvectors $\mathbf{x}_t$ and $\mathbf{y}_t$, $\mathbf{z}_t' = (\mathbf{x}_t', \mathbf{y}_t')$, reflecting an interest in relationships between X and Y. Adopt a corresponding partition of $\mathbf{S}_z(\lambda)$,

$$\mathbf{S}_z(\lambda) = \begin{bmatrix} \mathbf{S}_x(\lambda) & \mathbf{S}_{xy}(\lambda) \\ \mathbf{S}_{yx}(\lambda) & \mathbf{S}_y(\lambda) \end{bmatrix}.$$

From (2.4) X and Y each possess autoregressive representations, which we denote by

$$\mathbf{x}_t = \sum_{s=1}^{\infty} \mathbf{E}_{1s}\mathbf{x}_{t-s} + \mathbf{u}_{1t}, \quad \text{var}(\mathbf{u}_{1t}) = \boldsymbol{\Sigma}_1 \tag{2.5}$$

and

$$\mathbf{y}_t = \sum_{s=1}^{\infty} \mathbf{G}_{1s}\mathbf{x}_{t-s} + \mathbf{v}_{1t}, \quad \text{var}(\mathbf{v}_{1t}) = \mathbf{T}_1, \tag{2.6}$$

respectively. The disturbance $\mathbf{u}_{1t}$ is the one-step-ahead error when $\mathbf{x}_t$ is forecast from its own past alone, and similarly for $\mathbf{v}_{1t}$ and $\mathbf{y}_t$. These disturbance vectors are each serially uncorrelated, but may be correlated with each other contemporaneously and at various leads and lags. Since $\mathbf{u}_{1t}$ is uncorrelated with $X_{t-1} = \{\mathbf{x}_{t-s}, s \geq 1\}$, (2.5) denotes the linear projection of $\mathbf{x}_t$ on its own past, and likewise (2.6) denotes the linear projection of $\mathbf{y}_t$ on $Y_{t-1} = \{\mathbf{y}_{t-s}, s \geq 1\}$.

The linear projection of $\mathbf{x}_t$ on X_{t-1} and Y_{t-1}, and of $\mathbf{y}_t$ on X_{t-1} and Y_{t-1} is given by (2.3), which we partition

$$\mathbf{x}_t = \sum_{s=1}^{\infty} \mathbf{E}_{2s}\mathbf{x}_{t-s} + \sum_{s=1}^{\infty} \mathbf{F}_{2s}\mathbf{y}_{t-s} + \mathbf{u}_{2t}, \qquad \text{var}(\mathbf{u}_{2t}) = \boldsymbol{\Sigma}_2 \tag{2.7}$$

and

$$\mathbf{y}_t = \sum_{s=1}^{\infty} \mathbf{G}_{2s}\mathbf{y}_{t-s} + \sum_{s=1}^{\infty} \mathbf{H}_{2s}\mathbf{x}_{t-s} + \mathbf{v}_{2t}, \qquad \text{var}(\mathbf{v}_{2t}) = \mathbf{T}_2. \tag{2.8}$$

The disturbance vectors $\mathbf{u}_{2t}$ and $\mathbf{v}_{2t}$ are each serially uncorrelated, but since each is uncorrelated with X_{t-1} and Y_{t-1}, they can be correlated with each other only contemporaneously. We shall find the partition

$$\boldsymbol{\Upsilon} = \text{var}\begin{pmatrix} \mathbf{u}_{2t} \\ \mathbf{v}_{2t} \end{pmatrix} = \begin{bmatrix} \boldsymbol{\Sigma}_2 & \mathbf{C} \\ \mathbf{C}' & \mathbf{T}_2 \end{bmatrix}$$

useful.

If the system (2.7) through (2.8) is premultiplied by the matrix

$$\begin{bmatrix} \mathbf{I}_k & -\mathbf{C}\mathbf{T}_2^{-1} \\ -\mathbf{C}'\boldsymbol{\Sigma}_2^{-1} & \mathbf{I}_l \end{bmatrix}$$

then in the first k equations of the new system, $\mathbf{x}_t$ is a linear function of X_{t-1}, Y_t, and a disturbance $\mathbf{u}_{2t} - \mathbf{C}\mathbf{T}_2^{-1}\mathbf{v}_{2t}$. Since the disturbance is uncorrelated with $\mathbf{v}_{2t}$ it is uncorrelated with $\mathbf{y}_t$—as well as with X_{t-1} and Y_{t-1}. Hence the linear projection of $\mathbf{x}_t$ on X_{t-1} and Y_t,

$$\mathbf{x}_t = \sum_{s=1}^{\infty} \mathbf{E}_{3s}\mathbf{x}_{t-s} + \sum_{s=0}^{\infty} \mathbf{F}_{3s}\mathbf{y}_{t-s} + \mathbf{u}_{3t}, \qquad \text{var}(\mathbf{u}_{3t}) = \boldsymbol{\Sigma}_3, \tag{2.9}$$

is provided by the first k equations of the new system. Similarly, the existence of the linear projection of $\mathbf{y}_t$ on Y_{t-1} and X_t,

$$\mathbf{y}_t = \sum_{s=1}^{\infty} \mathbf{G}_{3s}\mathbf{y}_{t-s} + \sum_{s=0}^{\infty} \mathbf{H}_{3s}\mathbf{x}_{t-s} + \mathbf{v}_{3t}, \qquad \operatorname{var}(\mathbf{v}_{3t}) = \mathbf{T}_3, \tag{2.10}$$

follows from the last l equations.

We finally consider the linear projections of $\mathbf{x}_t$ on X_{t-1} and Y, and of $\mathbf{y}_t$ on Y_{t-1} and X. Let $\tilde{\mathbf{D}}(\lambda) = \mathbf{S}_{xy}(\lambda)\mathbf{S}_y(\lambda)^{-1}$, for all $\lambda \in [-\pi, \pi]$ for which the terms are defined and (2.4) is true. Because of (2.4), the inverse Fourier transform

$$\mathbf{D}_s = \frac{1}{2\pi}\int_{-\pi}^{\pi} \tilde{\mathbf{D}}(\lambda)e^{-i\lambda s}\, d\lambda$$

of $\tilde{\mathbf{D}}(\lambda)$ satisfies the condition $\sum_{s=-\infty}^{\infty} \| \mathbf{D}_s \|^2 < \infty$. From the spectral representation of Z it is evident that $\mathbf{w}_t = \mathbf{x}_t - \sum_{s=-\infty}^{\infty} \mathbf{D}_s\mathbf{y}_{t-s}$ is uncorrelated with all $\mathbf{y}_s$, and that

$$\mathbf{x}_t = \sum_{s=-\infty}^{\infty} \mathbf{D}_s\mathbf{y}_{t-s} + \mathbf{w}_t \tag{2.11}$$

therefore provides the linear projection of $\mathbf{x}_t$ on Y. Since

$$\mathbf{S}_w(\lambda) = \mathbf{S}_x(\lambda) - \mathbf{S}_{xy}(\lambda)\mathbf{S}_y(\lambda)^{-1}\mathbf{S}_{yx}(\lambda)$$

consists of the first k rows and columns of $\mathbf{S}_z(\lambda)^{-1}$, $c^{-1}\mathbf{I}_k \leqq \mathbf{S}_w(\lambda) \leqq c\mathbf{I}_k$ for almost all λ. Hence $\mathbf{w}_t$ possesses an autoregressive representation, which we write

$$\mathbf{w}_t = \sum_{s=1}^{\infty} \mathbf{B}_s\mathbf{w}_{t-s} + \mathbf{u}_{4t}. \tag{2.12}$$

Consequently,

$$\mathbf{x}_t = \sum_{r=1}^{\infty} \mathbf{B}_r\mathbf{x}_{t-r} - \sum_{r=0}^{\infty} \mathbf{B}_r \sum_{s=-\infty}^{\infty} \mathbf{D}_s\mathbf{y}_{t-s-r} + \mathbf{u}_{4t} \tag{2.13}$$

where $\mathbf{B}_0 = -\mathbf{I}_k$. Grouping terms, (2.13) may be written

$$\mathbf{x}_t = \sum_{s=1}^{\infty} \mathbf{E}_{4s}\mathbf{x}_{t-s} + \sum_{s=-\infty}^{\infty} \mathbf{F}_{4s}\mathbf{y}_{t-s} + \mathbf{u}_{4t}, \qquad \operatorname{var}(\mathbf{u}_{4t}) = \boldsymbol{\Sigma}_4, \tag{2.14}$$

where $\mathbf{E}_{4s} = \mathbf{B}_s$ and $\mathbf{F}_{4s} = \sum_{r=0}^{\infty} \mathbf{B}_r\mathbf{D}_{s-r}$. Since $\mathbf{u}_{4t}$ is a linear function of W_t, it is uncorrelated with Y; and since X_{t-1} is a linear function of Y and W_{t-1}, $\mathbf{u}_{4t}$ is uncorrelated with X_{t-1}. Hence (2.14) provides the linear projection of $\mathbf{x}_t$ on X_{t-1} and all Y, $\sum_{s=1}^{\infty} \| \mathbf{E}_{4s} \|^2 < \infty$ and $\sum_{s=-\infty}^{\infty} \| \mathbf{F}_{4s} \|^2 < \infty$. The same argument may be used to demonstrate that the linear projection of $\mathbf{y}_t$ on Y_{t-1} and X_t,

$$\mathbf{y}_t = \sum_{s=1}^{\infty} \mathbf{G}_{4s}\mathbf{y}_{t-s} + \sum_{s=-\infty}^{\infty} \mathbf{H}_{4s}\mathbf{x}_{t-s} + \mathbf{v}_{4t}, \qquad \operatorname{var}(\mathbf{v}_{4t}) = \mathbf{T}_4 \tag{2.15}$$

exists and all coefficients are square summable.

The extension of these representations to the kinds of nonstationary time series that are often the focus of applied work is straightforward. The nature of the autoregressions (2.5) through (2.6) is inessential. The analysis could have proceeded beginning with the innovations $\mathbf{u}_{1t}$ and $\mathbf{v}_{1t}$ in lieu of $\mathbf{x}_t$ and $\mathbf{y}_t$, respectively, and nothing (including the numerical measures of feedback for the series in question) would have been changed. So long as there exist $\mathbf{R}(L)$ and $\mathbf{S}(L)$ such that $\mathbf{z}_t^{*\prime} = (\mathbf{x}_t^{*\prime}, \mathbf{y}_t^{*\prime})$, $\mathbf{x}_t^* = \mathbf{R}(L)\mathbf{x}_t$ and $\mathbf{y}_t^* = \mathbf{S}(L)\mathbf{y}_t$, is a wide-sense stationary, purely nondeterministic multiple time series with autoregressive representation, the measures of feedback exist and have the same interpretation. This includes series that can be transformed to stationarity by univariate first differencing, by far the most common form of nonstationarity assumed in applied work.

3. MEASURES OF LINEAR DEPENDENCE AND FEEDBACK

To begin, define the *measure of linear feedback from Y to X*,

$$F_{Y\to X} = \ln(\,|\,\boldsymbol{\Sigma}_1\,|\,/\,|\,\boldsymbol{\Sigma}_2\,|\,).$$

This definition has several motivations. First, since $|\,\boldsymbol{\Sigma}_1\,| \geq |\,\boldsymbol{\Sigma}_2\,|$, it is nonnegative, as any measure must be. Second, the statements that $F_{Y\to X} = 0$, $\boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_2$, $\mathbf{F}_{2s} \equiv 0$, and that Y does not cause X if the universe of information at t is Z_t, are equivalent. Third, this measure is invariant with respect to scaling of X and Y; in fact, it remains unchanged if X and Y are premultiplied by different invertible lag operators. Fourth, $F_{Y\to X}$ is a monotonic transformation of the "strength of causality $Y \Rightarrow X$," $1 - |\,\boldsymbol{\Sigma}_2\,|\,/\,|\,\boldsymbol{\Sigma}_1\,|$, which was proposed by Granger (1963) for the case in which there is no instantaneous causality; Pierce's (1979) definition of R_+^2, which requires $k = 1$, is equivalent to Granger's "strength." Fifth, $\exp(-F_{Y\to X})$ is the reduction in the total predictive variance of X (Granger 1963), which may be achieved using past Y in addition to past X in prediction. Sixth, to anticipate the most obvious result of the next section, when Z is Gaussian the maximum likelihood estimate of $F_{Y\to X}$ is simple to construct; and the asymptotic distribution of $F_{Y\to X}$ is the well-known chi square under the null hypothesis $F_{Y\to X} = 0$, and may be approximated under the alternative. Finally, $F_{Y\to X}$ is one term in the additive decomposition of the measure of dependence to be introduced presently and may itself be decomposed additively by frequency.

Symmetrically, define the *measure of linear feedback from X to Y*,

$$F_{X\to Y} = \ln(\,|\,\mathbf{T}_1\,|\,/\,|\,\mathbf{T}_2\,|\,).$$

The *measure of instantaneous linear feedback*

$$F_{X\cdot Y} = \ln(\,|\,\mathbf{T}_2\,|\cdot|\,\boldsymbol{\Sigma}_2\,|\,/\,|\,\boldsymbol{\Upsilon}\,|\,)$$

has motivation similar to that of the first two measures. It is nonzero if and only if the partial correlation between $\mathbf{x}_t$ and $\mathbf{y}_t$ (conditional on past values of those series) is zero. If the universe of information at t is Z_t then $F_{X\cdot Y}$

$\neq 0$, "X causes Y instantaneously" and "Y causes X instantaneously" are equivalent statements. A concept closely related to the notion of linear feedback is that of linear dependence. From (2.5) through (2.8), X and Y are linearly indedendent if and only if $\mathbf{\Sigma}_1 = \mathbf{\Sigma}_2$, $\mathbf{T}_1 = \mathbf{T}_2$, and $\mathbf{C} = 0$. This suggests the *measure of linear dependence*

$$F_{X,Y} = \ln(|\mathbf{\Sigma}_1| \cdot |\mathbf{T}_1| / |\mathbf{Y}|)$$

which is the "measure of information" proposed by Gel'fand and Yaglom (1959).

Obviously,

$$F_{X,Y} = F_{Y\to X} + F_{X\to Y} + F_{X\cdot Y}.$$

The measure of linear dependence is the sum of the measures of the three types of linear feedback. The following result provides further motivation for these measures.

Theorem 1. In the notation of equations (2.5) through (2.10) and (2.14) through (2.15),

$$\begin{aligned}
\text{(i) } F_{X,Y} &= \ln(|\mathbf{\Sigma}_1| \cdot |\mathbf{T}_1| / |\mathbf{Y}|)\\
&= \ln(|\mathbf{\Sigma}_1| / |\mathbf{\Sigma}_4|) = \ln(|\mathbf{T}_1| / |\mathbf{T}_4|),\\
\text{(ii) } F_{X\to Y} &= \ln(|\mathbf{T}_1| / |\mathbf{T}_2|) = \ln(|\mathbf{\Sigma}_3| / |\mathbf{\Sigma}_4|),\\
\text{(iii) } F_{Y\to X} &= \ln(|\mathbf{\Sigma}_1| / |\mathbf{\Sigma}_2|) = \ln(|\mathbf{T}_3| / |\mathbf{T}_4|),\\
\text{(iv) } F_{X\cdot Y} &= \ln(|\mathbf{T}_2| \cdot |\mathbf{\Sigma}_2| / |\mathbf{Y}|)\\
&= \ln(|\mathbf{\Sigma}_2| / |\mathbf{\Sigma}_3|) = \ln(|\mathbf{T}_2| / |\mathbf{T}_3|).
\end{aligned}$$

Proof.

(i) Since $\mathbf{u}_{4t}$ is the disturbance in the autoregressive representation (2.12) of $\mathbf{w}_t$,

$$\begin{aligned}
\ln|\mathbf{\Sigma}_4| &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln|\mathbf{S}_w(\lambda)|\,d\lambda\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln|\mathbf{S}_x(\lambda) - \mathbf{S}_{xy}(\lambda)\mathbf{S}_y(\lambda)^{-1}\mathbf{S}_{yx}(\lambda)|\,d\lambda\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} (\ln|\mathbf{S}_z(\lambda)| - \ln|\mathbf{S}_y(\lambda)|)d\lambda \qquad (3.1)\\
&= \ln(|\mathbf{Y}| / |\mathbf{T}_1|).
\end{aligned}$$

Hence

$$\ln(|\mathbf{\Sigma}_1| / |\mathbf{\Sigma}_4|) = \ln(|\mathbf{\Sigma}_1| \cdot |\mathbf{T}_1| / |\mathbf{Y}|),$$

and by an argument symmetric in X and Y

$$\ln(|\mathbf{T}_1| / |\mathbf{T}_4|) = \ln(|\mathbf{\Sigma}_1| \cdot |\mathbf{T}_1| / |\mathbf{Y}|).$$

(ii) By construction of (2.9) $\mathbf{\Sigma}_3 = \mathbf{\Sigma}_2 - \mathbf{C}\mathbf{T}_2^{-1}\mathbf{C}'$, so $|\mathbf{\Sigma}_3| \cdot |\mathbf{T}_2| = |\mathbf{Y}|$. Combining this result with $|\mathbf{\Sigma}_4| \cdot |\mathbf{T}_1| = |\mathbf{Y}|$ from (3.1), (ii) is obtained.

(iii) Follows by symmetry with (ii).

(iv) Follows algebraically from (i), (ii), and (iii).

Clearly the partial correlation coefficient between $\mathbf{x}_t$ and $\mathbf{y}_t$ is zero if and only if $\mathbf{x}_t$ does not enter the right side of (2.8) and $\mathbf{y}_t$ does not enter the right side of (2.10). The result shows that instantaneous feedback can be measured equivalently in either equation. A similar interpretation for linear dependence applies to (2.14) and (2.15). Likewise it is a short step from this theorem to Sims's (1972) result that Y does not cause X if, and only if, in the linear projection of Y on future, current, and past X coefficients on future X are zero. The statement "Y does not cause X" is equivalent to $\mathbf{\Sigma}_1 = \mathbf{\Sigma}_2$ and $\mathbf{T}_3 = \mathbf{T}_4$, which is in turn equivalent to $\mathbf{H}_{3s} \equiv \mathbf{H}_{4s}$. From our derivation of (2.14) from (2.11), coefficients on $X - X_t$ in (2.15) are zero if and only if the coefficients on $X - X_t$ in the projection of $\mathbf{y}_t$ on X are zero.

Consider now the decomposition of $F_{Y\to X}$ and $F_{X\to Y}$ by frequency. Formally, we seek nonnegative functions $f_{X\to Y}(\lambda)$ and $f_{Y\to X}(\lambda)$ such that

$$F_{X\to Y} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_{X\to Y}(\lambda)d\lambda$$

and

$$F_{Y\to X} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_{Y\to X}(\lambda)d\lambda.$$

In addition, these functions ought to have an intuitive interpretation if the decomposition of feedback by frequency is to be enlightening in empirical work. Gel'fand and Yaglom (1959) suggested the decomposition of linear feedback

$$F_{X,Y} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln(|\mathbf{S}_z(\lambda)| / |\mathbf{S}_x(\lambda)| \cdot |\mathbf{S}_y(\lambda)|)d\lambda,$$

which when X and Y are univariate may be expressed

$$F_{X,Y} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln(1 - C(\lambda))d\lambda,$$

where $C(\lambda)$ is the coherence of X and Y at frequency λ. Granger (1963) and Pierce (1979) have remarked on the difficulty of decomposing coherence by variety of feedback: heuristically, the problem is that feedback from both X to Y and Y to X can be strong and yet cancel at a given frequency, leaving a coherence of zero. Granger (1969) suggested a decomposition of coherence by feedback, but the components are complex and the decomposition assumes that instantaneous feedback is absent.

The construction of $f_{Y\to X}(\lambda)$ is motivated by consideration of the system

$$\begin{bmatrix} \mathbf{E}_2(L) & \mathbf{F}_2(L)\\ \mathbf{H}_3(L) & \mathbf{G}_3(L)\end{bmatrix}\begin{pmatrix}\mathbf{x}_t\\ \mathbf{y}_t\end{pmatrix} = \begin{pmatrix}\mathbf{u}_{2t}\\ \mathbf{v}_{3t}\end{pmatrix} \qquad (3.2)$$

constructed from (2.7) and (2.10). ($\mathbf{E}_2(L) = \mathbf{I} - \sum_{s=1}^{\infty} \mathbf{E}_{2s}L^s$, where L is the conventional lag operator. The relation between the other elements of the matrices in (3.2) and (3.7) and the terms in (2.7) through (2.10) is the same.) Observe that $\mathrm{cov}(\mathbf{u}_{2t}, \mathbf{v}_{3t}) = 0$. In this system all instantaneous feedback has been combined with feedback from X to Y in the last l equations in (3.2), so that only feedback from Y to X accounts for the relation between X and Y expressed in the first k equations. The existence of the joint autoregressive representation (2.3) for X and Y assures that the matrix of lag operators in (3.2) can be inverted to express $\mathbf{x}_t$ and $\mathbf{y}_t$ as one-sided distributed lags of $\mathbf{u}_{2t}$ and $\mathbf{v}_{3t}$:

$$\begin{pmatrix} \mathbf{x}_t \\ \mathbf{y}_t \end{pmatrix} = \begin{bmatrix} \mathbf{E}^2(L) & \mathbf{F}^2(L) \\ \mathbf{H}^3(L) & \mathbf{G}^3(L) \end{bmatrix} \begin{pmatrix} \mathbf{u}_{2t} \\ \mathbf{v}_{3t} \end{pmatrix}. \tag{3.3}$$

The first k equations of (3.3) provide a decomposition of $\mathbf{x}_t$ into distributed lags on the orthogonal, serially uncorrelated processes $\mathbf{u}_{2t}$ and $\mathbf{v}_{3t}$,

$$\mathbf{x}_t = \mathbf{E}^2(L)\mathbf{u}_{2t} + \mathbf{F}^2(L)\mathbf{v}_{3t}, \tag{3.4}$$

and a corresponding decomposition of the spectral density $\mathbf{S}_x(\lambda)$ into the sum of two positive semidefinite matrices (the convention that the prime (′) denotes conjugation as well as transposition for complex matrices is observed).

$$\mathbf{S}_x(\lambda) = \tilde{\mathbf{E}}^2(\lambda)\boldsymbol{\Sigma}_2\tilde{\mathbf{E}}^2(\lambda)' + \tilde{\mathbf{F}}^2(\lambda)\mathbf{T}_3\tilde{\mathbf{F}}^2(\lambda)', \tag{3.5}$$

where $\tilde{\mathbf{E}}^2(\lambda)$ and $\tilde{\mathbf{F}}^2(\lambda)$ are the Fourier transforms of $\mathbf{E}^2(L)$ and $\mathbf{F}^2(L)$, respectively. When X is univariate ($k = 1$),

$$\tilde{F}^2(\lambda)T_3\tilde{F}^2(\lambda)'/S_x(\lambda)$$

is the fraction of $S_x(\lambda)$ due to v_{3t}. In (2.7) the fraction of Σ_1 accounted for by Y_{t-1} is $1 - \Sigma_2/\Sigma_1$. This relation of $1 - \Sigma_2/\Sigma_1$ and $F_{Y\to X} = \ln(\Sigma_1/\Sigma_2)$ in the univariate case suggests the *measure of linear feedback from Y to X at frequency* λ,

$$f_{Y\to X}(\lambda) = \ln(\,|\,\mathbf{S}_x(\lambda)\,|\,/\,|\,\tilde{\mathbf{E}}^2(\lambda)\boldsymbol{\Sigma}_2\tilde{\mathbf{E}}^2(\lambda)'\,|\,). \tag{3.6}$$

Symmetrically, we can begin with the system

$$\begin{bmatrix} \mathbf{E}_3(L) & \mathbf{F}_3(L) \\ \mathbf{H}_2(L) & \mathbf{G}_2(L) \end{bmatrix} \begin{pmatrix} \mathbf{x}_t \\ \mathbf{y}_t \end{pmatrix} = \begin{pmatrix} \mathbf{u}_{3t} \\ \mathbf{v}_{2t} \end{pmatrix}, \tag{3.7}$$

invert to obtain

$$\begin{pmatrix} \mathbf{x}_t \\ \mathbf{y}_t \end{pmatrix} = \begin{bmatrix} \mathbf{E}^3(L) & \mathbf{F}^3(L) \\ \mathbf{H}^2(L) & \mathbf{G}^2(L) \end{bmatrix} \begin{pmatrix} \mathbf{u}_{3t} \\ \mathbf{v}_{2t} \end{pmatrix}, \tag{3.8}$$

and define the *measure of linear feedback from X to Y at frequency* λ,

$$f_{X\to Y}(\lambda) = \ln(\,|\,\mathbf{S}_y(\lambda)\,|\,/\,|\,\tilde{\mathbf{G}}^2(\lambda)\mathbf{T}_2\tilde{\mathbf{G}}^2(\lambda)'\,|\,). \tag{3.9}$$

These measures have some of the attractions of $F_{Y\to X}$ and $F_{X\to Y}$: invariance with respect to premultiplication of X and Y by invertible lag operators, nonnegativity, and the properties $f_{Y\to X}(\lambda) = 0$ if and only if $\tilde{\mathbf{F}}_2(\lambda) = 0$, and $f_{X\to Y}(\lambda) = 0$ if and only if $\tilde{\mathbf{H}}_2(\lambda) = 0$. In general, it is possible that $|\,\tilde{\mathbf{E}}^2(\lambda)\tilde{\mathbf{E}}^2(\lambda)'\,| = 0$ or $|\,\tilde{\mathbf{G}}^2(\lambda)\tilde{\mathbf{G}}^2(\lambda)'\,| = 0$, in which case $f_{Y\to X}(\lambda)$ or $f_{X\to Y}(\lambda)$, respectively, will not exist. This problem and the relation of $f_{Y\to X}(\lambda)$ to $F_{Y\to X}$ and $f_{X\to Y}(\lambda)$ to $F_{X\to Y}$ are addressed in the following result.

Theorem 2. It is always the case that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f_{Y\to X}(\lambda)d\lambda \le F_{Y\to X}$$

and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f_{X\to Y}(\lambda)d\lambda \le F_{X\to Y}.$$

If all roots of $|\,\mathbf{G}_3(L)\,|$ lie outside the unit circle then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f_{Y\to X}(\lambda)d\lambda = F_{Y\to X},$$

and if all roots of $|\,\mathbf{E}_3(L)\,|$ lie outside the unit circle then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f_{X\to Y}(\lambda)d\lambda = F_{X\to Y}.$$

Proof. From the construction of (2.9) and (2.10),

$$\begin{bmatrix} \mathbf{E}_2(L) & \mathbf{F}_2(L) \\ \mathbf{H}_3(L) & \mathbf{G}_3(L) \end{bmatrix} = \begin{bmatrix} \mathbf{I}_k & \mathbf{0} \\ -\mathbf{C}'\boldsymbol{\Sigma}_2^{-1} & \mathbf{I}_l \end{bmatrix} \begin{bmatrix} \mathbf{E}_2(L) & \mathbf{F}_2(L) \\ \mathbf{H}_2(L) & \mathbf{G}_2(L) \end{bmatrix}. \tag{3.10}$$

The leading term of the matrix lag operator in (3.2)—that is, the matrix of coefficients on contemporaneous $\mathbf{u}_{2t}$ and $\mathbf{v}_{3t}$—is therefore

$$\begin{bmatrix} \mathbf{I}_k & \mathbf{0} \\ -\mathbf{C}'\boldsymbol{\Sigma}_2^{-1} & \mathbf{I}_l \end{bmatrix},$$

and the leading term of the matrix lag operator in (3.3) is

$$\begin{bmatrix} \mathbf{I}_k & \mathbf{0} \\ \mathbf{C}'\boldsymbol{\Sigma}_2^{-1} & \mathbf{I}_l \end{bmatrix}.$$

Since $\mathbf{E}^2(L)$ is a square matrix lag operator with leading term $\mathbf{I}_k$ it follows (Rozanov 1967, Theorem 4.2) that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\,|\,\tilde{\mathbf{E}}^2(\lambda)\,|^2\,d\lambda \ge 0,$$

with

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\,|\,\tilde{\mathbf{E}}^2(\lambda)\,|^2\,d\lambda = 0$$

if $|\,\mathbf{E}^2(L)\,|$ has all roots outside the unit circle. But the determinant of the invertible matrix of lag operators in (3.2) is

$$|\,\mathbf{G}_3(L)\,|\cdot|\,\mathbf{E}_2(L) - \mathbf{F}_2(L)\mathbf{G}_3(L)^{-1}\mathbf{H}_3(L)\,| = |\,\mathbf{G}_3(L)\,|\cdot|\,\mathbf{E}^2(L)\,|,$$

so the condition on the roots of $\mathbf{E}^2(L)$ is equivalent to the condition that the roots of $|\,\mathbf{G}_3(L)\,|$ all lie outside the unit circle. The result for $f_{Y\to X}(\lambda)$ follows immediately from (3.6), and that for $f_{X\to Y}(\lambda)$ is symmetric.

Theorem 2 shows that although $f_{Y\to X}(\lambda)$ may fail to exist for some λ, it exists almost everywhere, and $(1/2\pi)\int_{-\pi}^{\pi} f_{Y\to X}(\lambda)d\lambda$ is finite. The condition on the roots of $\mathbf{G}_3(L)$—which need not be satisfied—can be motivated heuristically. If $\mathbf{G}_3(L)$, equivalently $\mathbf{E}^2(L)$, is invertible, then from (3.4),

$$\mathbf{u}_{2t} = \mathbf{E}^2(L)^{-1}\mathbf{x}_t - \mathbf{E}^2(L)^{-1}\mathbf{F}^2(L)\mathbf{v}_{3t}:$$

(U_{2t}, V_{3t}) and (X_t, V_{3t}) span the same space. (U_{2t}, V_{3t}) and (X_t, Y_t) always span the same space (because (3.2) and (3.3) both exist), so the invertibility condition is equivalent to the stipulation that (X_t, Y_t) and (X_t, V_{3t}) span equivalent spaces. Measures of the marginal explanatory power of $\mathbf{V}_{3t}$ in (3.4) are equivalent to measures of the marginal explanatory power of $\mathbf{Y}_t$ in (2.7) only when this condition is met. When the condition is not met the marginal explanatory power of $\mathbf{V}_{3t}$ is less than that of $\mathbf{Y}_t$: hence

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f_{Y\to X}(\lambda)d\lambda < F_{Y\to X}.$$

The only interesting case in which the condition is certain to be met appears to be $F_{X\to Y} = F_{X\cdot Y} = 0$. From $F_{X\cdot Y} = 0$ and (3.10) it follows that $\mathbf{G}_3(L) = \mathbf{G}_2(L)$, and from $F_{X\to Y} = 0$, $\mathbf{H}_2(L) = 0$ whence $\mathbf{G}_2(L)$ must be invertible. (In empirical work undertaken with macroeconomic time series by the author, point estimates of $\mathbf{E}_3(L)$ and $\mathbf{G}_3(L)$ have usually turned out to be invertible; i.e., the equalities, and not just the inequalities, set forth in Theorem 2 are usually satisfied by estimated measures of feedback. Published examples are provided in Geweke 1982a,b.)

4. INFERENCE

The measures of linear dependence and feedback, and their decomposition by frequency, can be constructed from the parameters of equations (2.7) and (2.8), and $\mathbf{C} = \text{cov}(\mathbf{u}_{2t}, \mathbf{v}_{2t})$. Inference about these measures can be based on estimates of these equations and on their estimated distribution. A central problem in estimating (2.7) and (2.8) is common to all time series work: the number of unknown parameters is countably infinite. We shall begin by supposing that for purposes of estimation, all lag lengths in the canonical form have been truncated at p, and that the equations

$$\mathbf{x}_t = \sum_{s=1}^{p} \mathbf{E}_{1s}\mathbf{x}_{t-s} + \mathbf{u}_{1t}, \tag{4.1}$$

$$\mathbf{x}_t = \sum_{s=1}^{p} \mathbf{E}_{2s}\mathbf{x}_{t-s} + \sum_{s=1}^{p} \mathbf{F}_{2s}\mathbf{y}_{t-s} + \mathbf{u}_{2t}, \tag{4.2}$$

$$\mathbf{y}_t = \sum_{s=1}^{p} \mathbf{G}_{1s}\mathbf{y}_{t-s} + \mathbf{v}_{1t}, \tag{4.3}$$

and

$$\mathbf{y}_t = \sum_{s=1}^{p} \mathbf{G}_{2s}\mathbf{y}_{t-s} + \sum_{s=1}^{p} \mathbf{H}_{2s}\mathbf{x}_{t-s} + \mathbf{v}_{2t} \tag{4.4}$$

have been estimated by ordinary least squres. (The symbols in (4.1) through (4.4) should not really be the same as those in (2.5) through (2.8) unless the truncation is literally true; it was thought that introducing new notation might be confusing at this point.) Let

$$\underset{n\times k}{\hat{\mathbf{U}}_1},\ \underset{n\times k}{\hat{\mathbf{U}}_2},\ \underset{n\times l}{\hat{\mathbf{V}}_1},\ \underset{n\times l}{\hat{\mathbf{V}}_2}$$

denote the respective matrices of residuals from the four equations, and $\hat{\mathbf{\Sigma}}_1 = \hat{\mathbf{U}}_1'\hat{\mathbf{U}}_1/n$, $\hat{\mathbf{\Sigma}}_2 = \hat{\mathbf{U}}_2'\hat{\mathbf{U}}_2/n$, $\hat{\mathbf{T}}_1 = \hat{\mathbf{V}}_1'\hat{\mathbf{V}}_1/n$, $\hat{\mathbf{T}}_2 = \hat{\mathbf{V}}_2'\hat{\mathbf{V}}_2/n$, and $\hat{\mathbf{C}} = \hat{\mathbf{U}}_2'\hat{\mathbf{V}}_2/n$. Let

$$\hat{\mathbf{Y}} = \begin{bmatrix} \hat{\mathbf{\Sigma}}_2 & \hat{\mathbf{C}} \\ \hat{\mathbf{C}}' & \hat{\mathbf{T}}_2 \end{bmatrix}.$$

Then

$$\hat{F}_{Y\to X} \equiv \ln(\,|\,\hat{\mathbf{\Sigma}}_1\,|\,/\,|\,\hat{\mathbf{\Sigma}}_2\,|\,),$$

$$\hat{F}_{X\to Y} \equiv \ln(\,|\,\hat{\mathbf{T}}_1\,|\,/\,|\,\hat{\mathbf{T}}_2\,|\,),$$

$$\hat{F}_{X\cdot Y} \equiv \ln(\,|\,\hat{\mathbf{\Sigma}}_2\,|\cdot|\,\hat{\mathbf{T}}_2\,|\,/\,|\,\hat{\mathbf{Y}}\,|\,),$$

and

$$F_{X,Y} \equiv \hat{F}_{Y\to X} + \hat{F}_{X\to Y} + \hat{F}_{X\cdot Y}.$$

If autoregressions are really of order p and the disturbances in (4.1) through (4.4) are Gaussian, these are maximum likelihood estimates conditional on presample values of $\mathbf{x}_t$ and $\mathbf{y}_t$.

First, consider inference about $F_{Y\to X}$, $F_{X\to Y}$, and $F_{X\cdot Y}$. If the disturbances in (4.1) through (4.4) are independent and identically distributed, the conventional large-sample distribution theory may be used to test the null hypothesis that a given measure of feedback is zero—or, in the language of Granger (1969) or Sims (1972), that causality is unidirectional. If $F_{Y\to X} = 0$, then

$$n\,\hat{F}_{Y\to X} \overset{a}{\sim} \chi^2(klp); \tag{4.5}$$

if $F_{X\to Y} = 0$,

$$n\,\hat{F}_{X\to Y} \overset{a}{\sim} \chi^2(klp); \tag{4.6}$$

if $F_{X\cdot Y} = 0$,

$$n\,\hat{F}_{X\cdot Y} \overset{a}{\sim} \chi^2(kl). \tag{4.7}$$

Since these are tests of nested hypotheses, $\hat{F}_{Y\to X}$, $\hat{F}_{X\to Y}$, and $\hat{F}_{X\cdot Y}$ are asymptotically independent. All three restrictions can be tested at once since

$$n\,\hat{F}_{X,Y} \overset{a}{\sim} \chi^2(kl(2p+1))$$

if $F_{X,Y} = 0$.

If the point hypothesis of unidirectional causality is not taken literally, a more appropriate approach is to construct confidence intervals for the measures of feedback introduced in Section 3. If the disturbances in (4.1) through (4.4) are Gaussian, then the estimated measures of feedback are the usual likelihood ratio test statistics for the hypothesis of no feedback deflated by sample size, and if feedback is present then their asymptotic distributions will be approximately noncentral chi-square under the alternative that feedback is present (Wald 1943). (Wald's asymptotic approximation is better the smaller the measure of linear feedback or dependence.) Specifically,

$$n\,\hat{F}_{Y\to X} \overset{\dot{a}}{\sim} \chi'^2(klp, n\,F_{Y\to X}); \tag{4.8}$$

$$n\,\hat{F}_{X\to Y} \overset{\dot{a}}{\sim} \chi'^2(klp, n\,F_{X\to Y}); \tag{4.9}$$

$$n\,\hat{F}_{X\cdot Y} \overset{\dot{a}}{\sim} \chi'^2(kl, n\,F_{X\cdot Y}); \tag{4.10}$$

and

$$n\,\hat{F}_{X,Y} \overset{\dot{a}}{\sim} \chi'^2(kl(2p+1), n\,F_{X,Y}). \tag{4.11}$$

Since the only unknown parameter in each of these distributions is the measure of linear feedback or dependence in question, inference about these parameters may be conducted in the usual way. Direct numerical evaluation of the noncentral chi-squared distribution is difficult, but several good approximations have been suggested (Johnson and Kotz 1970). One that is convenient for our purposes is due to Sankaran (1963):[1] if $x \sim \chi'^2(r; \lambda)$, then

$$(x - (r-1)/3)^{1/2} \sim N((\lambda + (2r+1)/3)^{1/2}, 1)).$$

(For small r, there is a nonnegligible probability that $x < (r-1)/3$. In this event, one can use $-((r-1)/3 - x)^{1/2}$ in lieu of $(x - (r-1)/3)^{1/2}$. For $r \geq 16$, however, $P[x < (r-1)/3] < .005$ for all $\lambda \geq 0$.) Using this approximation, a $100(1-\alpha)$percent confidence interval for λ is (λ_L, λ_U), where

$$\begin{aligned} \lambda_L &= [(x - (r-1)/3)^{1/2} - z_{\alpha/2}]^2 - (2r+1)/3, \\ \lambda_U &= [(x - (r-1)/3)^{1/2} + z_{\alpha/2}]^2 - (2r+1)/3, \end{aligned} \tag{4.12}$$

and $z_{\alpha/2}$ is such that $\Phi(z_{\alpha/2}) = 1 - \alpha/2$, Φ being the cumulative standard normal distribution function. (Since the confidence intervals are based on the approximation, $\lambda_L < 0$ is possible.) An approximate 95 percent confidence interval for $F_{Y\to X}$, for example, is

$$\left\{\left[\left(\hat{F}_{Y\to X} - \frac{klp-1}{3n}\right)^{1/2} - \frac{1.96}{\sqrt{n}}\right]^2 - \frac{2klp+1}{3n},\right.$$
$$\left.\left[\left(\hat{F}_{Y\to X} - \frac{klp-1}{3n}\right)^{1/2} + \frac{1.96}{\sqrt{n}}\right]^2 - \frac{2klp+1}{3n}\right\}.$$

A similar approach may be used to test nonpoint null hypotheses about measures of linear feedback and independence. For example, since $\hat{F}_{Y\to X}$ and $\hat{F}_{X\to Y}$ are asymptotically independent, the asymptotic distribution of

$$\left(n\,\hat{F}_{Y\to X} - \frac{klp-1}{3}\right)^{1/2} - \left(n\,\hat{F}_{X\to Y} - \frac{klp-1}{3}\right)^{1/2} \tag{4.13}$$

is approximately $N(0, 2)$ if $F_{Y\to X} = F_{X\to Y}$.

Point estimates of $f_{Y\to X}(\lambda)$ and $f_{X\to Y}(\lambda)$ for various values of λ can be constructed using the ordinary least squares estimates of (4.1) through (4.4) in lieu of the unknown lag operators in (3.2), (3.3), and (3.5) through (3.8). Consistent estimates of $\mathbf{S}_y(\lambda)$ and $\mathbf{S}_x(\lambda)$ can be obtained from estimates of (4.1) and (4.3), respectively, but in finite samples the estimates of $f_{X\to Y}(\lambda)$ and $f_{Y\to X}(\lambda)$ so obtained may be negative for some λ. This difficulty can be avoided by constructing $\mathbf{S}_x(\lambda)$ and $\mathbf{S}_y(\lambda)$ directly from the estimates of (4.2) and (4.4).

The hypotheses $f_{Y\to X}(\lambda_0) = 0$ and $f_{X\to Y}(\lambda_0) = 0$ each constitute linear restrictions on the parameters of (2.7) through (2.8), for any specified $\lambda_0 \in [-\pi, \pi]$. The cases $f_{Y\to X}(\lambda_0)$ and $f_{X\to Y}(\lambda_0)$ are symmetric. Since $f_{Y\to X}(\lambda_0) = 0$ if and only if $\tilde{\mathbf{F}}_2(\lambda_0) = 0$, $f_{Y\to X}(\lambda_0) = 0$ is equivalent to

$$\sum_{s=1}^{p} \mathbf{F}_{2s} \cos(\lambda_0 s/2\pi) = 0,$$

and

$$\sum_{s=1}^{p} \mathbf{F}_{2s} \sin(\lambda_0 s/2\pi) = 0.$$

(To the extent that a bivariate autoregression of order p only approximates the true autoregression, these equivalences are also approximate.) Confidence intervals for $f_{Y\to X}(\lambda_0)$ can be constructed, but because of the nonlinearity of each in the estimated parameters in (4.1) through (4.4), the expressions are cumbersome and the asymptotic distribution theory may not be reliable in samples of the size that econometricians usually use. Although restrictions like $f_{Y\to X}(\lambda_0) = 0$ constitute point null hypotheses, tests of these restrictions may well constitute more powerful tests of unidirectional causality than do tests of $F_{Y\to X} = 0$ in situations where sample size is modest and feedback is likely to be greater at one frequency than another under alternatives with high prior probability.

In practice, the lag truncation parameter p must be chosen as part of the estimation procedure. Since the autoregressive representation is not a known function of a finite number of unknown parameters, consistency requires that p be allowed to increase with sample size. While it is clear that there exists a rate of increase such that consistent estimates are obtained, just what the rate should be or (more to the point) what the value of p should be in a given sample depends on the values of the unknown parameters themselves. In applied work the choice of p is likely to be influenced by the features of the functions $f_{Y\to X}(\lambda)$ and $f_{X\to Y}(\lambda)$ of interest to the investigator. The larger is p the greater is the range of possible shapes for these functions. The covariance matrix for the parameter estimates corresponding to (2.1) is of the form $\mathbf{Y} \otimes \mathbf{R}^{-1}$, where $\mathbf{R}$ is an $mp \times mp$ block Toeplitz form, each block being $m \times m$. For large values of p, $\mathbf{R}$ can be factorized approximately as the product of Fourier coefficients and the spectral density matrix $\mathbf{S}_z(\lambda)$ (Grenander and Szego 1958). This implies, for example, that when p is large and even, the estimates $f_{Y\to X}(2\pi j/p)$, $j = 0, \ldots, p/2$, are asymptotically approximately independent, as are the test statistics introduced in the preceding paragraph at those frequencies. This suggests that if the values of $f_{Y\to X}(\lambda_0)$ and $f_{Y\to X}(\lambda_1)$ are each of interest, as is often the case with economic time series, then p should be at least $2\pi/|\lambda_0 - \lambda_1|$. Lag lengths shorter than this will generally not allow sufficient flexibility in the functional form of $f_{Y\to X}(\lambda)$ to permit resolution of $f_{Y\to X}(\lambda_0)$ and $f_{Y\to X}(\lambda_1)$.

[1] Susan Porter-Hudak suggested this use of Sankaran's approximation.

Table 1. Estimated Measures of Feedback Between the Monetary Growth Rate, and Real GNP and the GNP Deflator, 1929–1978[a]

	Money (M) and GNP (Y)		Money (M) and Deflator (P)	
Estimate 1	$\hat{F}_{M\to Y} = .091$,	$\hat{F}_{Y\to M} = .122$	$\hat{F}_{M\to P} = .179^*$,	$\hat{F}_{P\to M} = .055$
Estimate 2	$\hat{F}_{M\to Y} = .170$	$\hat{F}_{Y\to M} = .090$	$\hat{F}_{M\to P} = .176^*$,	$\hat{F}_{P\to M} = .043$
	(−.020, .348)	(−.045, .213)	(−.017, .358	(−.043, .117)
	$\hat{F}_{M\cdot Y} = .091^{**}$		$\hat{F}_{M\cdot P} = .020$	
	(.000, .265)		(−.012, .120)	

Frequency (λ)	Period (Years)	$\hat{f}_{M\to Y}(\lambda)$	$\hat{f}_{Y\to M}(\lambda)$	$\hat{f}_{M\to P}(\lambda)$	$\hat{f}_{P\to M}(\lambda)$
0	∞	.000	.056	.620**	.105
.0625 π	32	.046	.052	.565**	.068
.1250 π	16	.069	.044	.424*	.036
.1875 π	10.67	.084	.033	.258*	.023
.2500 π	8	.102*	.023	.128	.020
.3750 π	5.33	.155**	.018	.033	.037
.5000 π	4	.221**	.045	.053	.105
.6250 π	3.2	.253**	.102	.095	.050
.7500 π	2.67	.236**	.161*	.124	.025
1.0000 π	2	.202**	.205**	.159**	.016

[a] Single asterisks denote estimates significantly different from zero at 10% level, double at 5% level. 90% confidence intervals for maximum likelihood estimates of feedback, computed as described in Section 4, are shown parenthetically.

5. EMPIRICAL EXAMPLE

The example presented here is taken from some empirical work by the author (Geweke 1982b). It illustrates how measures of feedback may be used to describe and interpret relationships between economic time series. In this example the two series are univariate, but multivariate examples may be found in the same source. The series involved are gross national product adjusted for inflation (real GNP), the ratio of GNP to real GNP (GNP deflator) and the narrowly defined money stock M1, for the period 1929 to 1978.[2]

Reasons why economists have been interested in the relationships between these time series are discussed in Geweke (1982b). In most macroeconomic models money does not significantly affect real economic activity in the long run, although in the short run variations in money can have substantial effects, and these effects vary from model to model. In contrast the price level always responds eventually to persistent movements in the money supply, but in quite a few models it is not much affected by money in the short run. The notions of "long run" and "short run" can be made conceptually concise in models where time is an artifact, but their analytical counterparts become elusive once time is taken literally. One of the points of this example is to adduce some evidence on whether the decomposition of feedback by frequency proposed in this article provides a reasonable empirical counterpart of the notions of short and long run. The other is to test the foregoing properties of macroeconomic models, using this interpretation.

A vector autoregression for real GNP and money, and another for the GNP deflator and money, were estimated for the period 1929 to 1978: that is, the first observation on the dependent variables is for 1929 and the last is for 1978. The data involved are first differences of logarithms of the original series, three lags were used, and an intercept was included in each equation, so each equation had seven explanatory variables. The equations were estimated by ordinary least squares. Results were not much affected by incorporating more lags in the autoregressions, or by using levels instead of first differences of the logarithms. Preliminary diagnostic checks showed that variances in the disturbance terms $\mathbf{u}_{2t}$ and $\mathbf{v}_{2t}$ ranged from about 10 to 16 times higher before and during World War II than in the postwar period. (The substantial difference in prewar and postwar variances of disturbances in almost any regression equation using macroeconomic time series is well known to users of these data: e.g., Gordon 1980, Tables 1 and 2.) A simple correction for heteroscedasticity was made by multiplying all observations for the 1929 to 1950 period by .275 before estimation. Once the heteroscedasticity correction was made, goodness-of-fit tests showed no evidence of difference in the coefficients of the autoregressions between the 1929 to 1950 and 1951 to 1978 periods: the test statistics were $\chi^2(15) = 12.6$ for the real GNP and money system and $\chi^2(15) = 12.0$ for the GNP deflator and money system.

Estimated measures of feedback and their decomposition by frequency were computed as described in the previous section. These estimates are presented in Table 1, and Figures 1 and 2. Two estimates of $F_{X\to Y}$ and $F_{Y\to X}$ are provided. The first is the quasi-maximum likelihood estimate described in Section 4. The second is formed

[2] Gross National Product is U.S. Bureau of the Census (1975), series F1, and Real Gross National Product is series F3, through 1970. The corresponding data for 1971 through 1978 were taken from various issues of the *Survey of Current Business*. The money supply is M1, U.S. Bureau of the Census (1975), series X414, extended through 1978 using the *Survey of Current Business*. M1 was discontinued after 1978. For descriptions of primary sources, see U.S. Bureau of the Census (1975), pp. 216 and 990.

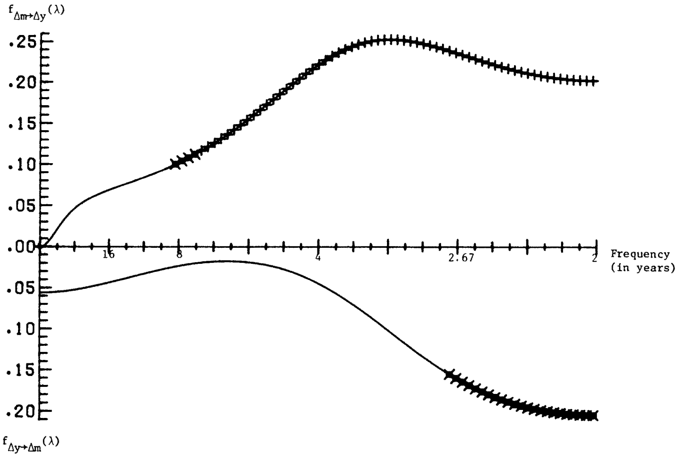

Figure 1. Feedback Between Monetary Growth Rate $\Delta m_t = (\ln(M1_t) - \ln(M1_{t-1}))$ and Real GNP Growth Rate $\Delta y_t = (\ln(RGNP_t) - \ln(RGNP_{t-1}))$, 1929–1978. (Frequency is measured by the corresponding period, in years. Estimated measures of feedback significantly different from zero at the 1% significance level are indicated with interspersed asterisks; at the 5% level, with interspersed crosses; at the 10% level, with interspersed squares)

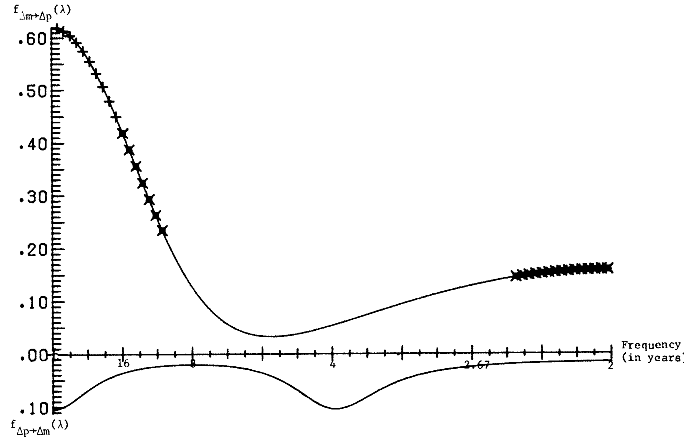

Figure 2. Feedback Between Monetary Growth Rate $\Delta m_t = (\ln(M1_t) - \ln(M1_{t-1}))$ and Growth Rate of GNP Deflator $\Delta p_t = (\ln(P_t) - \ln(P_{t-1}))$, 1929–1978. (Frequency is measured by the corresponding period, in years. Estimated measures of feedback significantly different from zero at the 1% significance level are indicated with interspersed asterisks; at the 5% level, with interspersed crosses; at the 10% level, with interspersed squares)

using the estimates of (4.2) and (4.4) and C. The estimates $\hat{\Sigma}_2$ and $\hat{T}_2$ are taken directly from the estimates of (4.2) and (4.4), respectively, while $\ln|\hat{\Sigma}_1|$ and $\ln|\hat{T}_1|$ are formed by numerical integration of the logarithm of the spectral densities of X and Y implied by the point estimates of (4.2) and (4.4). For the point estimates of both systems the invertibility conditions of Theorem 2 were met and hence numerical integration of $\hat{f}_{Y\to X}(\lambda)$ and $\hat{f}_{X\to Y}(\lambda)$ produced the second estimates of $F_{Y\to X}$ and $F_{X\to Y}$, respectively.

The estimated measures of feedback for the real GNP (Y) and money (M) system are in striking accordance with the proposition of macroeconomic theory that variations in the money supply ought not to lead to variations in real output in the long run, if we take "long run" to mean low frequencies. The estimate $\hat{f}_{M\to Y}(\lambda)$ literally vanishes as $\lambda \to 0$, and is statistically insignificant at the 10 percent level for values of λ corresponding to periods of more than 8 years. The estimate attains its maximum at a frequency corresponding to a period of 3.2 years. Over the range of frequencies corresponding to periods ranging from 2.0 to 4.2 years $\hat{f}_{M\to Y}(\lambda)$ exceeds .20, implying that 18 percent or more of the variation in real output is ascribed to money innovations at these frequencies. The marginal significance level of these point estimates is around 1.4 percent. There is evidence of feedback from Y to M at higher frequencies. After attaining a minimum at $\lambda = \pi/3$, corresponding to a period of 6 years, $\hat{f}_{M\to Y}(\lambda)$ rises steadily, attaining statistical significance at the 10 percent level for periods shorter than 2.7 years. At the two-year frequency, Y innovations explain about 18 percent of the variance in M. One could interpret this kind of feedback as arising from countercyclical monetary policy.

Strong support for the notion that variations in money are an important source of variation in prices over the long run is provided by the estimated frequency decomposition of the measure of feedback from M to P. The estimate $\hat{f}_{M\to P}(\lambda)$ increases sharply as $\lambda \to 0$, implying that in the very long run variations in money innovations explain half the variation in the GNP deflator, itself the most aggregate of all price indices. (Even this estimate is probably low: the sample period includes the period of very stringent World War II wage and price controls, and when the system is reestimated for portions of the postwar period not subject to controls, $\hat{f}_{M\to P}(0)$ usually exceeds 4.0, implying that 98 percent of the variation in prices at very low frequencies is estimated to arise from variations in money at the same low frequencies.) There is weak evidence of feedback from money to prices at higher frequencies, but it is clearly much smaller. Strong feedback from money to prices is concentrated over such a small range of frequencies that it cannot be detected by overall measures and tests: $\hat{F}_{M\to P}$ is not significantly different from 0, and the failure to find any statistical association between money and prices is common (e.g., Feige and Pearce 1976). There is no evidence of feedback from prices to money for this period.

6. CONCLUSION

Measures of linear dependence and feedback for multiple time series that are wide-sense stationary, autoregressive, and purely nondeterministic have been proposed. The measure of linear dependence is the sum of the measure of linear feedback from the first series to the second, linear feedback from the second to the first, and instantaneous linear feedback. The measures are nonnegative and zero only when feedback (or causality) of the relevant type is absent. The measures of linear feedback from one series to another can be additively decomposed by frequency. In the case of univariate series, the measure of feedback from X to Y at a given frequency is a monotonic transformation of the fraction of the spectral density of Y due to the innovation in X in a bivariate autoregressive representation rotated so that all instantaneous feedback has been removed from the X-to-Y relation. There is a readily usable theory of inference for all of these measures and their decompositions, and the computations involved are quite modest.

It would appear that two lines of further investigation are most important. One is the application of this approach to more series, in the manner illustrated in Section 5, to assess its usefulness in the economic interpretation of observed relationships between time series. It appears to be a potentially important part of a bridge between the traditional approach in econometric model construction and the "measurement without theory" of time series analysts. The second question concerns the adequacy of our theory of inference in samples of the size typically used in macroeconomics; some Monte Carlo work would seem to be essential before much confidence can be placed in the confidence intervals and marginal significance levels of the type reported in Section 5.

[*Received November 1980. Revised May 1981.*]

REFERENCES

DOOB, J. (1953), *Stochastic Processes*, New York: John Wiley.
FEIGE, E.L., and PEARCE, D.K. (1976), "Economically Rational Expectations: Are Innovations in the Rate of Inflation Independent of Innovations in Measures of Monetary and Fiscal Policy?" *Journal of Political Economy*, 84, 499–522.
GEL'FAND, I.M., and YAGLOM, A.M. (1959), *American Mathematical Society Translations Series 2*, 12, 199–246.
GEWEKE, J. (1982a), "Causality, Exogeneity, and Inference," invited paper, Fourth World Congress of the Econometric Society, in *Advances in Quantitative Economics*, ed. D. Hildenbrand, Cambridge: Cambridge University Press, forthcoming.
——— (1982b), "The Neutrality of Money in the United States, 1872–1980: An Interpretation of the Evidence," Unpublished manuscript, University of Wisconsin.
GEWEKE, J., MEESE, R., and DENT, W. (1979), "Comparing Alternative Tests of Causality in Temporal Systems: Analytic Results and Experimental Evidence," University of Wisconsin, Social Systems Research Institute Workshop Series 7928.
GORDON, R.J. (1980), "A Consistent Characterization of a Near-Century of Price Behavior," *American Economic Review*, 70, 243–249.
GRANGER, C.W.J. (1963), "Economic Processes Involving Feedback," *Information and Control*, 6, 28–48; also Chapter 7 of GRANGER, C.W.J., and HATANAKA, M., *Spectral Analysis of Economic Time Series*, Princeton, N.J.: Princeton University Press.
——— (1969), "Investigating Causal Relations by Econometric Models and Cross-Spectral Methods," *Econometrica*, 37, 424–438.
GRENANDER, U., and SZEGO, G. (1958), *Toeplitz Forms and Their Applications*, Berkeley, Calf.: University of California Press.
JOHNSON, N.L., and KOTZ, S. (1970), *Distributions in Statistics: Continuous Univariate Distributions* (Vol. 2), New York: John Wiley.
PIERCE, D.A. (1979), "R^2 Measures for Time Series," *Journal of the American Statistical Association*, 74, 901–910.
ROZANOV, Y.A. (1967), *Stationary Random Processes*, San Francisco: Holden-Day.
SANKARAN, M. (1963), "Approximations to the Non-Central Chi-Square Distribution," *Biometrika*, 50, 199–204.
SIMS, C.A. (1972), "Money, Income and Causality," *American Economic Review*, 62, 540–552.
U.S. BUREAU OF THE CENSUS (1975), *Historical Statistics of the United States: Colonial Times to 1970*, Washington, D.C.: U.S. Government Printing Office.
WALD, A. (1943), "Tests of Statistical Hypotheses Concerning Several Parameters When the Number of Observations is Large," *Transactions of the American Mathematical Society*, 54, 426–485.
WIENER, N. (1956), "The Theory of Prediction," in *Modern Mathematics for the Engineer*, ed. E.F. Beckenbach, New York: McGraw-Hill.

Comment

ARNOLD ZELLNER*

This is indeed a technically impressive article that breaks new ground. Geweke develops and interprets new measures of unidirectional and two-way feedback for vector processes and then presents applications of his measures in analyses of two bivariate processes. The analysis and empirical results are clearly presented and should be of interest to many. However, there are a few points that deserve further consideration.

First, the word "causality" is misused throughout the article. At some points, it is identified with feedback or dependence and nearly with correlation. Such loose usage of the word causality detracts from the general value of the article and strikes a discordant note for readers who appreciate the difficulties in identifying causality with feedback or correlation. Further, it is the case that Granger has changed his views regarding his well-known 1969 statistical definition of causality that is employed in Geweke's paper. (See Zellner 1979 for a discussion of various concepts of causality including the Wiener-Granger concept. Information about Granger's more recent views is also provided in this paper.)

Second, while the measures of dependence, $F_{Y \to X}$,

* Arnold Zellner is H.G.B. Alexander Professor of Economics and Statistics, Graduate School of Business, University of Chicago, Chicago, IL 60637. Research financed by the National Science Foundation and by the H.G.B. Alexander Endowment Fund, Graduate School of Business, University of Chicago.

June 1982, Volume 77, Number 378
Theory and Methods Section