Mecánica del Continuo
Trabajo Práctico Nº7
Ecuaciones Constitutivas

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Ejercicio 1

Muestre que la *Ley de Hooke*:

$$\sigma_{ij} = \lambda e_{\alpha\alpha}\delta_{ij} + 2\mu e_{ij}$$

puede escribirse como

$$\begin{aligned} e_{xx} &= \frac{1}{E}(\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{xx})) & e_{xy} &= \frac{1+\nu}{E}\sigma_{xy} \\ e_{yy} &= \frac{1}{E}(\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})) & e_{yz} &= \frac{1+\nu}{E}\sigma_{yz} \\ e_{zz} &= \frac{1}{E}(\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})) & e_{zx} &= \frac{1+\nu}{E}\sigma_{zx} \end{aligned}$$

si E y ν se relacionan con las contantes de *Lamé* de acuerdo a las siguientes relaciones:

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \qquad G = \mu = \frac{E}{2(1+\nu)} \qquad E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \qquad \nu = \frac{\lambda}{2(\lambda+\mu)}$$

....

Cuando $i \neq j$:

$$\begin{aligned} \sigma_{xy} &= \lambda e_{\alpha\alpha}\delta_{xy} + 2\mu e_{xy} & \delta_{xy} &= 0 \\ &= 2\mu e_{xy} & \mu &= \frac{E}{2(1+\nu)} \\ &= \frac{2E}{2(1+\nu)}e_{xy} & &\textit{Simplificando} \\ &= \frac{E}{(1+\nu)}e_{xy} & &\textit{Despejando} \\ \frac{(1+\nu)}{E}\sigma_{xy} &= e_{xy} \end{aligned}$$

Paso de expresar la tensión en función de la deformación a expresar la deformación en función de la tensión:

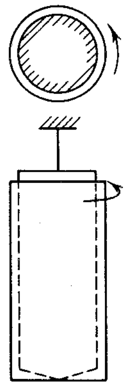
$$
\begin{aligned}
\sigma_{ii} &= \lambda e_{\alpha\alpha}\delta_{ii} + 2\mu e_{ii} && e_{\alpha\alpha} = 3e_{ii} \\
& && \delta_{ii} = 1 \\
&= 3\lambda e_{ii} + 2\mu e_{ii} && \textit{Factor común} \\
&= (3\lambda + 2\mu)e_{ii} && \textit{Despejando} \\
\frac{1}{(3\lambda + 2\mu)}\sigma_{ii} &= e_{ii}
\end{aligned}
$$

Cuando $i = j$:

$$
\begin{aligned}
\sigma_{xx} &= \lambda e_{\alpha\alpha}\delta_{xx} + 2\mu e_{xx} && e_{\alpha\alpha} = \frac{1}{(3\lambda + 2\mu)}\sigma_{\alpha\alpha} \\
& && \delta_{xx} = 1 \\
&= \frac{\lambda}{(3\lambda + 2\mu)}\sigma_{\alpha\alpha} + 2\mu e_{xx} && \sigma_{\alpha\alpha} = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} \\
&= \frac{\lambda}{(3\lambda + 2\mu)}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) + 2\mu e_{xx} && \textit{Despejando} \\
e_{xx} &= \frac{1}{2\mu}\left(\sigma_{xx} - \frac{\lambda}{(3\lambda + 2\mu)}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})\right) && \textit{Factor común} \\
&= \frac{1}{2\mu(3\lambda + 2\mu)}\left((3\lambda + 2\mu)\sigma_{xx} - \lambda(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})\right) && \textit{Distribuyendo} \\
&= \frac{1}{2\mu(3\lambda + 2\mu)}\left((3\lambda + 2\mu)\sigma_{xx} - \lambda\sigma_{xx} - \lambda(\sigma_{yy} + \sigma_{zz})\right) && \textit{Factor común} \\
&= \frac{1}{2\mu(3\lambda + 2\mu)}\left((3\lambda + 2\mu - \lambda)\sigma_{xx} - \lambda(\sigma_{yy} + \sigma_{zz})\right) && \textit{Sumando} \\
&= \frac{1}{2\mu(3\lambda + 2\mu)}\left(2(\lambda + \mu)\sigma_{xx} - \lambda(\sigma_{yy} + \sigma_{zz})\right) && \textit{Factor común} \\
&= \frac{2(\lambda + \mu)}{2\mu(3\lambda + 2\mu)}\left(\sigma_{xx} - \frac{\lambda}{2(\lambda + \mu)}(\sigma_{yy} + \sigma_{zz})\right) && \nu = \frac{\lambda}{2(\lambda + \mu)} \\
&= \frac{(\lambda + \mu)}{\mu(3\lambda + 2\mu)}\left(\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right) && \frac{1}{E} = \frac{\lambda + \mu}{\mu(3\lambda + 2\mu)} \\
&= \frac{1}{E}\left(\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right)
\end{aligned}
$$

Ejercicio 2

Considere un flujómetro de *Couette* de la figura



Obtenga la distribución de velocidades en el canal, la expresión de la viscosidad del fluido, y la relación torque-angular, $T = f(\omega)$. Finalmente, analice cómo cambia el torque para ensayos realizados con distintos fluidos a la misma velocidad angular.

....

$$\begin{aligned} T &= \tau(2\pi r L)r \\ &= 2\pi L r^2 \tau \\ &= 2\pi L r^2 \mu \frac{\partial v_\theta}{\partial r} \end{aligned} \qquad \tau = \mu \frac{\partial v_\theta}{\partial r}$$

$$v_\theta(r = R_1) = 0$$

$$v_\theta(R_1) = \frac{T}{2\pi L \mu}\left(-\frac{1}{R_1} + C\right) = 0 \qquad C = \frac{1}{R_1}$$

$$\frac{\partial v_\theta}{\partial r} = \frac{T}{2\pi L r^2 \mu}$$

$$v_\theta = \int \frac{T}{2\pi L r^2 \mu} dr = \frac{T}{2\pi L \mu}\left(-\frac{1}{r} + C\right)$$

$$v_\theta(r) = \frac{T}{2\pi L \mu}\left(-\frac{1}{r} + \frac{1}{R_1}\right)$$

$$v_\theta(r = R_2) = \omega R_2 = \frac{T}{2\pi L \mu}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$\mu = \frac{T}{2\pi L \omega R_2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$T = \frac{2\pi L \omega R_2 \mu}{\frac{1}{R_1} - \frac{1}{R_2}}$$

Ejercicio 3

Considere un estado bidimensional de tensiones en una placa delgada en la cual $\sigma_z = \tau_{zx} = \tau_{zy} = 0$. Las ecuaciones de equilibrio actuando en la placa con cargas distribuidas de cuerpo X, Y (constantes) son

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + Y = 0$$

Sabemos que estas ecuaciones se satisfacen idénticamente si $\sigma_x, \sigma_y, \tau_{xy}$ se derivan de una función arbitraria $\Phi(x, y)$ en la forma (ver Guía 3):

$$\sigma_x = \frac{\partial^2 \Phi}{\partial y^2}$$
$$\sigma_y = \frac{\partial^2 \Phi}{\partial x^2}$$
$$\tau_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y} - Xy - Yx$$

a. Demostrar que para un material elástico lineal, para el cual se verifica:

$$e_{xx} = \frac{1}{E}(\sigma_x - \nu\sigma_y)$$
$$e_{yy} = \frac{1}{E}(\sigma_y - \nu\sigma_x)$$
$$e_{xy} = \frac{\tau_{xy}}{G} = \frac{(1+\nu)}{E}\tau_{xy}$$

las condiciones de compatibilidad se verifican si y solo si:

$$\nabla^4 \Phi = \frac{\partial^4 \Phi}{\partial x^4} + 2\frac{\partial^4 \Phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \Phi}{\partial y^4} = 0$$

Decimos en este caso que $\Phi(x, y)$ satisface la ecuación *biarmónica*. Una función de estas características es llamada *función de tensión de Airy*.

$$e_{xx} = \frac{1}{E}(\sigma_x - \nu\sigma_y) = \frac{1}{E}\left(\frac{\partial^2 \Phi}{\partial y^2} - \nu\frac{\partial^2 \Phi}{\partial x^2}\right)$$
$$e_{yy} = \frac{1}{E}(\sigma_y - \nu\sigma_x) = \frac{1}{E}\left(\frac{\partial^2 \Phi}{\partial x^2} - \nu\frac{\partial^2 \Phi}{\partial y^2}\right)$$
$$e_{xy} = \frac{\tau_{xy}}{G} = \frac{(1+\nu)}{E}\tau_{xy} = \frac{(1+\nu)}{E}\left(-\frac{\partial^2 \Phi}{\partial x \partial y} - Xy - Yx\right)$$

Ecuación de compatibilidad para el estado plano de tensiones:

$$\frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} = 2\frac{\partial^2 e_{xy}}{\partial x \partial y}$$
$$\frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} - 2\frac{\partial^2 e_{xy}}{\partial x \partial y} = 0$$
$$\frac{1}{E}\left(\frac{\partial^4 \Phi}{\partial y^4} - \nu\frac{\partial^4 \Phi}{\partial x^2 \partial y^2}\right) + \frac{1}{E}\left(\frac{\partial^4 \Phi}{\partial x^4} - \nu\frac{\partial^4 \Phi}{\partial x^2 \partial y^2}\right) + 2\frac{1}{E}(1+\nu)\left(\frac{\partial^4 \Phi}{\partial x^2 \partial y^2}\right) = 0$$
$$\frac{\partial^4 \Phi}{\partial y^4} + \frac{\partial^4 \Phi}{\partial x^4} - 2\nu\frac{\partial^4 \Phi}{\partial x^2 \partial y^2} + 2\frac{\partial^4 \Phi}{\partial x^2 \partial y^2} + 2\nu\frac{\partial^4 \Phi}{\partial x^2 \partial y^2} = 0$$
$$\frac{\partial^4 \Phi}{\partial y^4} + \frac{\partial^4 \Phi}{\partial x^4} + 2\frac{\partial^4 \Phi}{\partial x^2 \partial y^2} = 0 = \nabla^4 \Phi$$

b. Verificar que las funciones del ejercicio 10 de la Guía 3, son funciones de *Airy*.

$$\Phi(x,y) = ax^2 + bxy + cy^2$$

$$\frac{\partial \Phi}{\partial x} = 2ax + by$$
$$\frac{\partial^2 \Phi}{\partial x^2} = 2a$$
$$\frac{\partial^3 \Phi}{\partial x^3} = 0$$

$$\frac{\partial \Phi}{\partial y} = bx + 2cy$$
$$\frac{\partial^2 \Phi}{\partial y^2} = 2c$$
$$\frac{\partial^3 \Phi}{\partial y^3} = 0$$

$$\frac{\partial \Phi}{\partial x} = 2ax + by$$
$$\frac{\partial^2 \Phi}{\partial x^2} = 2a$$
$$\frac{\partial^3 \Phi}{\partial x^2 \partial y} = 0$$

$$0 + 2 \cdot 0 + 0 = 0$$

Es una *función de Airy*.

$$\Phi(x,y) = ax^3 + bx^2y + cxy^2 + dy^3$$

$$\frac{\partial \Phi}{\partial x} = 3ax^2 + 2byx + cy^2$$
$$\frac{\partial^2 \Phi}{\partial x^2} = 6ax + 2by$$
$$\frac{\partial^3 \Phi}{\partial x^3} = 6a$$
$$\frac{\partial^4 \Phi}{\partial x^4} = 0$$

$$\frac{\partial \Phi}{\partial y} = bx^2 + 2cxy + 3dy^2$$
$$\frac{\partial^2 \Phi}{\partial y^2} = 2cx + 6dy$$
$$\frac{\partial^3 \Phi}{\partial y^3} = 6d$$
$$\frac{\partial^4 \Phi}{\partial y^4} = 0$$

$$\frac{\partial^3 \Phi}{\partial x^2 \partial y} = 2b$$
$$\frac{\partial^4 \Phi}{\partial x^2 \partial y^2} = 0$$

$$0 + 2 \cdot 0 + 0 = 0$$

Es una *función de Airy*.

c. ¿Qué condiciones se deben cumplir para que la función:

$$\Phi(x,y) = ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4$$

sea una función de *Airy*?

$$\frac{\partial \Phi}{\partial x} = 4ax^3 + 3byx^2 + 2cy^2x + dy^3$$
$$\frac{\partial^2 \Phi}{\partial x^2} = 12ax^2 + 6byx + 2cy^2$$
$$\frac{\partial^3 \Phi}{\partial x^3} = 24ax + 6by$$
$$\frac{\partial^4 \Phi}{\partial x^4} = 24a$$

$$\frac{\partial \Phi}{\partial y} = bx^3 + 2cx^2y + 3dxy^2 + 4ey^3$$
$$\frac{\partial^2 \Phi}{\partial y^2} = 2cx^2 + 6dxy + 12ey^2$$
$$\frac{\partial^3 \Phi}{\partial y^3} = 6dx + 24ey$$
$$\frac{\partial^4 \Phi}{\partial y^4} = 24e$$

$$\frac{\partial^3 \Phi}{\partial x^2 \partial y} = 6bx + 4cy$$
$$\frac{\partial^4 \Phi}{\partial x^2 \partial y^2} = 4c$$

$$24a + 2 \cdot 4c + 24e = 0$$
$$3a + c + 3e = 0$$

d. ¿Qué condiciones se deben cumplir para que la función:

$$\Phi(x,y) = ax^5 + bx^4y + cx^3y^2 + dx^2y^3 + exy^4 + fy^5$$

sea una función de Airy?

$$\frac{\partial \Phi}{\partial x} = 5ax^4 + 4byx^3 + 3cy^2x^2 + 2dy^3x + ey^4 \qquad \frac{\partial \Phi}{\partial y} = bx^4 + 2cx^3y + 3dx^2y^2 + 4exy^3 + 5fy^4$$

$$\frac{\partial^2 \Phi}{\partial x^2} = 20ax^3 + 12byx^2 + 6cy^2x + 2dy^3 \qquad \frac{\partial^2 \Phi}{\partial y^2} = 2cx^3 + 6dx^2y + 12exy^2 + 20fy^3$$

$$\frac{\partial^3 \Phi}{\partial x^3} = 60ax^2 + 24byx + 6cy^2 \qquad \frac{\partial^3 \Phi}{\partial y^3} = 6dx^2 + 24exy + 60fy^2$$

$$\frac{\partial^4 \Phi}{\partial x^4} = 120ax + 24by \qquad \frac{\partial^4 \Phi}{\partial y^4} = 24ex + 120fy$$

$$\frac{\partial^3 \Phi}{\partial x^2 \partial y} = 12bx^2 + 12cxy + 6dy^2$$

$$\frac{\partial^4 \Phi}{\partial x^2 \partial y^2} = 12cx + 12dy$$

$$\begin{aligned}
120ax + 24by + 2(12cx + 12dy) + 24ex + 120fy &= 0 \\
120ax + 24by + 24cx + 24dy + 24ex + 120fy &= 0 \\
(120a + 24c + 24e)x + (24b + 24d + 120f)y &= 0 \\
24(5a + c + e)x + 24(b + d + 5f)y &= 0 \\
(5a + c + e)x + (b + d + 5f)y &= 0
\end{aligned}$$

Ejercicio 4

Muestre que el polinomio

$$\Phi = \frac{a_2}{2}x^2 + b_2xy + \frac{c_2}{2}y^2$$

es una *función de tensión de Airy*. Examine las condiciones de borde satisfechas por esta función sobre las aristas de una placa rectangular: $x = \pm L$, y $y = \pm C$; de este modo, identifique el problema para cual Φ es la solución.

. . . .

a.

$$\frac{\partial \Phi}{\partial x} = a_2x + b_2y \qquad \frac{\partial \Phi}{\partial y} = b_2x + c_2y$$

$$\frac{\partial^2 \Phi}{\partial x^2} = a_2 \qquad \frac{\partial^2 \Phi}{\partial y^2} = c_2$$

$$\frac{\partial^3 \Phi}{\partial x^3} = 0 \qquad \frac{\partial^3 \Phi}{\partial y^3} = 0$$

$$\frac{\partial^3 \Phi}{\partial x^2 \partial y} = 0 \qquad 0 + 2 \cdot 0 + 0 = 0$$

Es una *función de Airy*.

b.

$$\sigma_{xx} = \frac{\partial^2\Phi}{\partial y^2} = c_2 \qquad \sigma_{yy} = \frac{\partial^2\Phi}{\partial x^2} = a_2 \qquad \sigma_{xy} = -\frac{\partial^2\Phi}{\partial x\partial y} = -b_2 \qquad \sigma = \begin{pmatrix} c_2 & -b_2 \\ -b_2 & a_2 \end{pmatrix}$$

$$\nu_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad t_1 = \sigma\nu_1 = \begin{pmatrix} c_2 \\ -b_2 \end{pmatrix} \qquad p = \begin{pmatrix} t_1 \int_{-C}^{C} c_2 dy \\ -t_1 \int_{-C}^{C} b_2 dy \end{pmatrix} = \begin{pmatrix} 2t_1Cc_2 \\ -2t_1Cb_2 \end{pmatrix}$$

$$\nu_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \qquad t_2 = \sigma\nu_2 = \begin{pmatrix} -c_2 \\ b_2 \end{pmatrix} \qquad p = \begin{pmatrix} -t_2 \int_{-C}^{C} c_2 dy \\ t_2 \int_{-C}^{C} b_2 dy \end{pmatrix} = \begin{pmatrix} -2t_2Cc_2 \\ 2t_2Cb_2 \end{pmatrix}$$

$$\nu_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad t_3 = \sigma\nu_3 = \begin{pmatrix} -b_2 \\ a_2 \end{pmatrix} \qquad p = \begin{pmatrix} -t_3 \int_{-L}^{L} b_2 dx \\ t_3 \int_{-L}^{L} a_2 dx \end{pmatrix} = \begin{pmatrix} -2t_3Lb_2 \\ 2t_3La_2 \end{pmatrix}$$

$$\nu_4 = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \qquad t_4 = \sigma\nu_4 = \begin{pmatrix} b_2 \\ -a_2 \end{pmatrix} \qquad p = \begin{pmatrix} t_4 \int_{-L}^{L} b_2 dx \\ -t_4 \int_{-L}^{L} a_2 dx \end{pmatrix} = \begin{pmatrix} 2t_4Lb_2 \\ -2t_4La_2 \end{pmatrix}$$

Ejercicio 5

Haga lo mismo para

$$\Phi = \frac{a_3}{6}x^3 + \frac{b_3}{2}x^2y + \frac{c_3}{2}xy^2 + \frac{d_3}{6}y^3$$

...

a.

$$\frac{\partial\Phi}{\partial x} = \frac{a_3}{2}x^2 + b_3yx + \frac{c_3}{2}y^2 \qquad \frac{\partial\Phi}{\partial y} = \frac{b_3}{2}x^2 + c_3xy + \frac{d_3}{2}y^2$$

$$\frac{\partial^2\Phi}{\partial x^2} = a_3x + b_3y \qquad \frac{\partial^2\Phi}{\partial y^2} = c_3x + d_3y$$

$$\frac{\partial^3\Phi}{\partial x^3} = a_3 \qquad \frac{\partial^3\Phi}{\partial y^3} = d_3$$

$$\frac{\partial^4\Phi}{\partial x^4} = 0 \qquad \frac{\partial^4\Phi}{\partial y^4} = 0$$

$$\frac{\partial^3\Phi}{\partial x^2\partial y} = b_3 \qquad 0 + 2\cdot 0 + 0 = 0$$

$$\frac{\partial^4\Phi}{\partial x^2\partial y^2} = 0$$

Es una *función de Airy*.

b.

$$\sigma_{xx} = \frac{\partial^2\Phi}{\partial y^2} = c_3x + d_3y \qquad \sigma_{yy} = \frac{\partial^2\Phi}{\partial x^2} = a_3x + b_3y$$

$$\sigma_{xy} = -\frac{\partial^2\Phi}{\partial x\partial y} = -(b_3x + c_3y) \qquad \sigma = \begin{pmatrix} c_3x + d_3y & -(b_3x + c_3y) \\ -(b_3x + c_3y) & a_3x + b_3y \end{pmatrix}$$

$$\nu_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad t_1 = \sigma\nu_1 = \begin{pmatrix} c_3x + d_3y \\ -(b_3x + c_3y) \end{pmatrix} \qquad p = \begin{pmatrix} 2t_1Cxc_3 \\ -2t_1Cxb_3 \end{pmatrix}$$

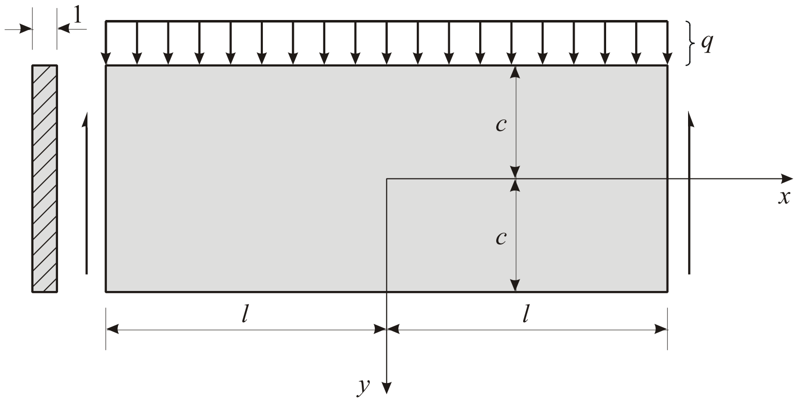
$$\nu_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \qquad t_2 = \sigma\nu_2 = \begin{pmatrix} -(c_3x + d_3y) \\ b_3x + c_3y \end{pmatrix} \qquad p = \begin{pmatrix} -2t_2Cxc_3 \\ 2t_2Cxb_3 \end{pmatrix}$$

$$\nu_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad t_3 = \sigma\nu_3 = \begin{pmatrix} -(b_3x + c_3y) \\ a_3x + b_3y \end{pmatrix} \qquad p = \begin{pmatrix} -2t_3Lyc_3 \\ 2t_3Lyb_3 \end{pmatrix}$$

$$\nu_4 = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \qquad t_4 = \sigma\nu_4 = \begin{pmatrix} b_3x + c_3y \\ -(a_3x + b_3y) \end{pmatrix} \qquad p = \begin{pmatrix} 2t_4Lyc_3 \\ -2t_4Lyb_3 \end{pmatrix}$$

Ejercicio 6

Una placa delgada de material Hookeano isotrópico se carga sobre las fronteras como se muestra en la figura



No hay fuerzas de cuerpo. Se sugiere que la distribución de tensiones sea

$$\begin{aligned}
\sigma_{xx} &= \frac{q}{2I}(l^2 - x^2)y + \frac{q}{2I}\left(\frac{2}{3}y^3 - \frac{2}{5}c^2 y\right) \\
\sigma_{yy} &= -\frac{q}{2I}\left(\frac{1}{3}y^3 - c^2 y + \frac{2}{3}c^3\right) \\
\sigma_{xy} &= -\frac{q}{2I}(c^2 - y^2)x \\
\sigma_{zz} = \sigma_{xz} &= \sigma_{zy} = 0
\end{aligned}$$

donde q es la carga por unidad de área y $I = \frac{2c^3}{3}$ es una constante. Determine si la solución es correcta o no.

....

$$\begin{aligned}
\nabla \bullet \sigma &= 0 \\
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} &= 0 && \Longrightarrow && -\frac{2xyq}{2I} + \frac{2qxy}{2I} = 0 \\
\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} &= 0 && \Longrightarrow && -\frac{q}{2I}(y^2 - c^2) - \frac{q}{2I}(c^2 - y^2) = 0 \\
\sum F_y &= -F_2 - F_1 + F_3 = 0 \\
&= -F_2 - F_1 + 2Lq = 0 && \Longrightarrow && F_2 = -F_1 + 2Lq = -Lq + 2Lq = Lq \\
\sum M_0 &= 2LF_1 - L2Lq = 0 && \Longrightarrow && F_1 = Lq \\
\nu_1 &= \begin{pmatrix} 0 \\ -1 \end{pmatrix} \\
t_1 = \sigma\nu_1 &= \begin{pmatrix} 0 \\ \frac{q}{2I}\left(-\frac{1}{3}c^3 + c^3 + \frac{2}{3}c^3\right) \end{pmatrix} = \begin{pmatrix} 0 \\ q \end{pmatrix} \\
\nu_2 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
t_2 = \sigma\nu_2 &= \begin{pmatrix} \frac{q}{2I}\left(\frac{2}{3}y^3 - \frac{2}{3}c^2 y\right) \\ -\frac{q}{2I}\left(c^2 - y^2\right)L \end{pmatrix} = \begin{pmatrix} 0 \\ -qL \end{pmatrix}
\end{aligned}$$

Apéndice

1. *Fluido no viscoso:*

$$\sigma_{ij} = -p\delta_{ij} \qquad\qquad p = \rho RT \tag{1}$$

p: presión
ρ: densidad
R: constante de los gases
T: temperatura

Si el fluido es incompresible entonces la densidad es constante.

2. *Fluido Newtoniano:*

$$\sigma_{ij} = -p\delta_{ij} + 2\mu V_{ij} \tag{2}$$

3. *Sólido elástico Hookeano:*

$$\text{Ley de Hooke:} \qquad\qquad \sigma_{ij} = \lambda e_{\alpha\alpha}\delta_{ij} + 2\mu e_{ij} \tag{3}$$

También se puede escribir como:

$$\begin{aligned}
\sigma_{xx} &= \lambda(e_{xx} + e_{yy} + e_{zz}) + 2Ge_{xx} \\
\sigma_{yy} &= \lambda(e_{xx} + e_{yy} + e_{zz}) + 2Ge_{yy} \\
\sigma_{zz} &= \lambda(e_{xx} + e_{yy} + e_{zz}) + 2Ge_{xx} \\
\sigma_{xy} &= 2Ge_{xy} \\
\sigma_{yz} &= 2Ge_{yz} \\
\sigma_{zx} &= 2Ge_{zx}
\end{aligned}$$

$$\begin{aligned}
e_{xx} &= \frac{1}{E}(\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{xx})) \\
e_{yy} &= \frac{1}{E}(\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})) \\
e_{zz} &= \frac{1}{E}(\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})) \\
e_{xy} &= \frac{1+\nu}{E}\sigma_{xy} \\
e_{yz} &= \frac{1+\nu}{E}\sigma_{yz} \\
e_{zx} &= \frac{1+\nu}{E}\sigma_{zx}
\end{aligned}$$

λ, μ: *constantes de Lamé*.

E: *módulo de Young*.

ν: coeficiente de Poisson.

$\mu = G$: módulo de corte.

Donde:

$$\begin{aligned}
\lambda &= \frac{E\nu}{(1+\nu)(1-2\nu)} \\
G = \mu &= \frac{E}{2(1+\nu)} \\
E &= \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \\
\nu &= \frac{\lambda}{2(\lambda + \mu)}
\end{aligned}$$

Función de tensión de Airy:

$$\nabla^4\Phi = \frac{\partial^4\Phi}{\partial x^4} + 2\frac{\partial^4\Phi}{\partial x^2\partial y^2} + \frac{\partial^4\Phi}{\partial y^4} = 0$$

Referencias

[1] Y. C. Fung, *A First Course in Continuum Mechanics*, tercera edición, PRENTICE HALL, 1994.