data sampling

Fourier analysis

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spectral power

errors in analysis

linear filters

time-frequency analysis

Fourier series

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Fourier analysis describes a signal as **a sum of periodic sinusoidal functions**

equivalent formulation: $$s(t) \approx \sum_{n=-N}^{N} c_n e^{i2\pi f_n t}$$ $c_n \in \mathbb{C}$ (c_n are complex)

N to be chosen

approximation by different number of Fourier modes

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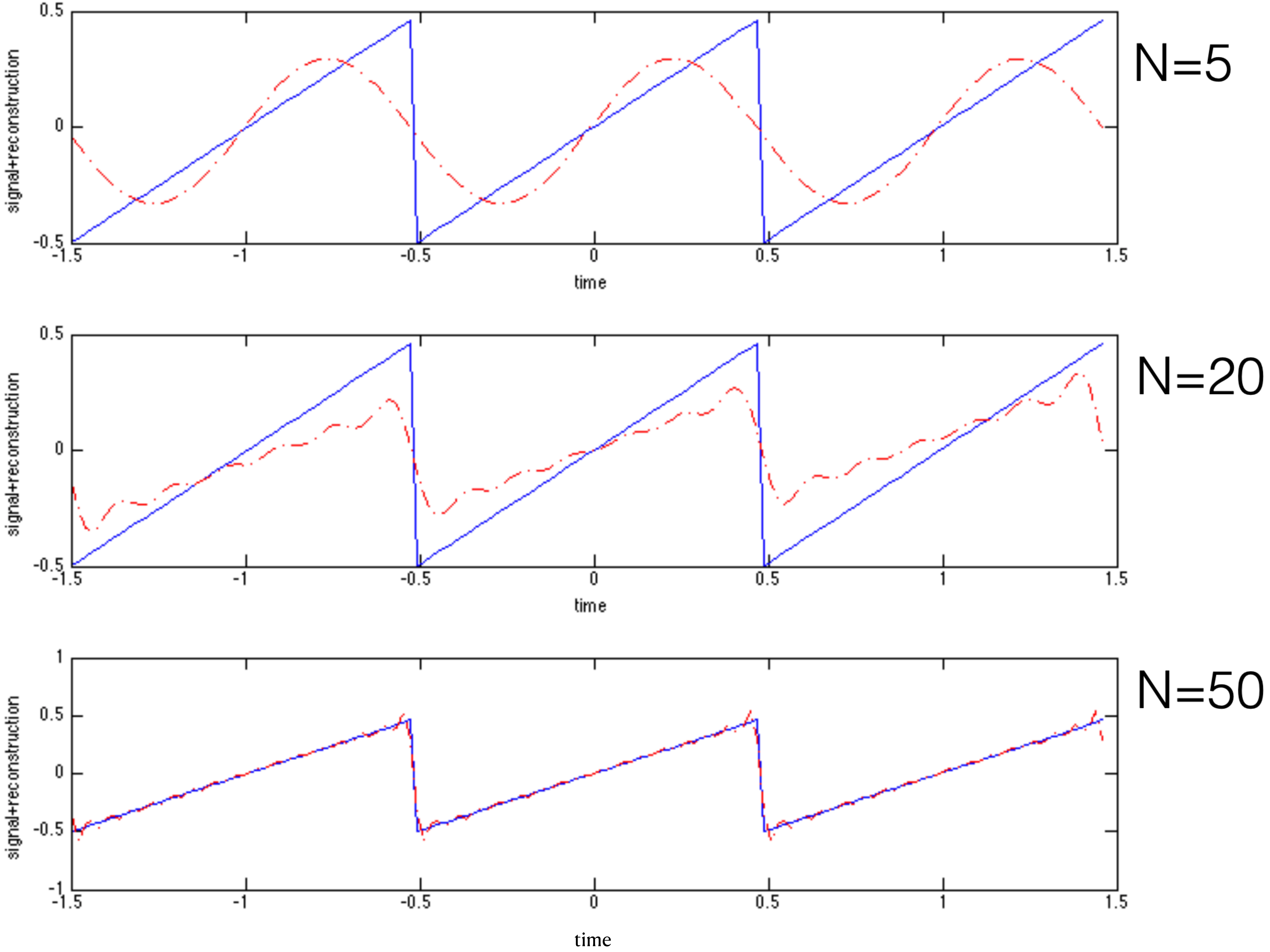

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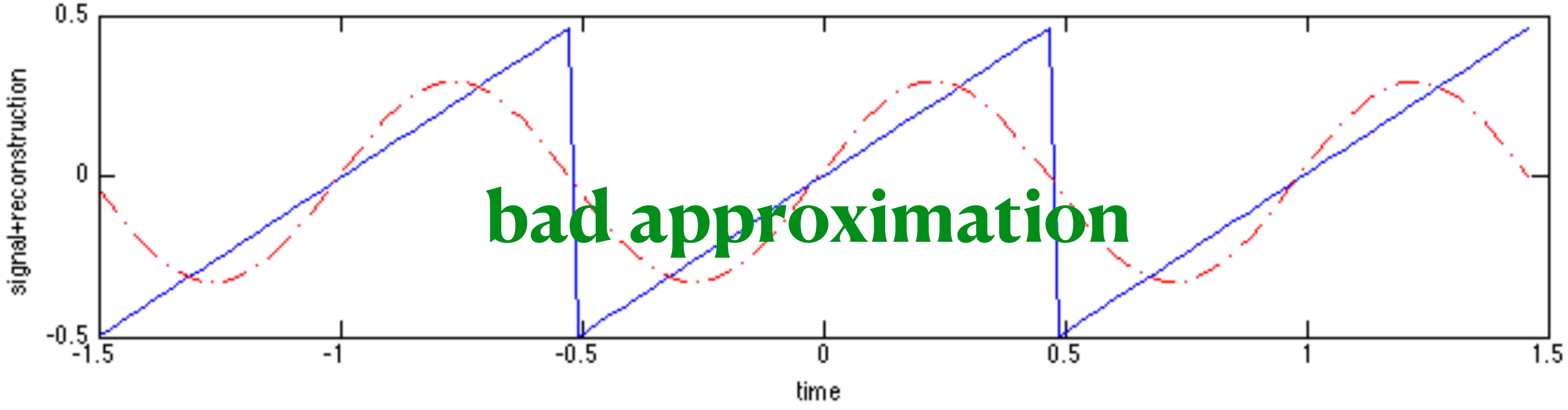
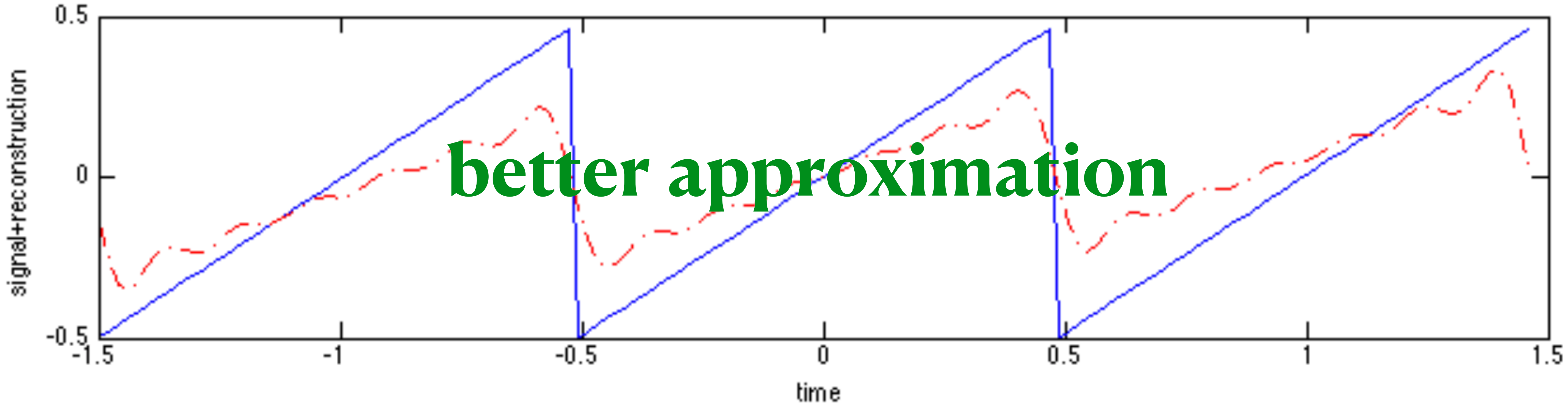
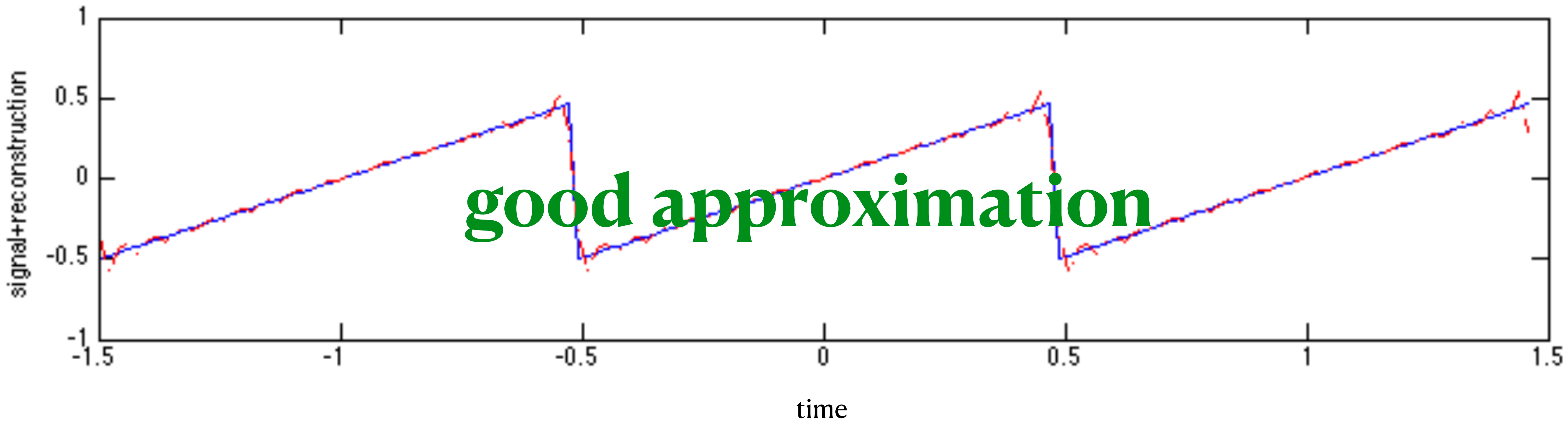


N=5



N=20



N=50

coefficients a_n, b_n or c_n estimates the contribution of Fourier mode n to signal

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time $\rightarrow$ frequency

$$c_n = \sum_{k=1}^{N} s(t_k) e^{-i2\pi nk/N}$$

Discrete Fourier Transform (DFT)

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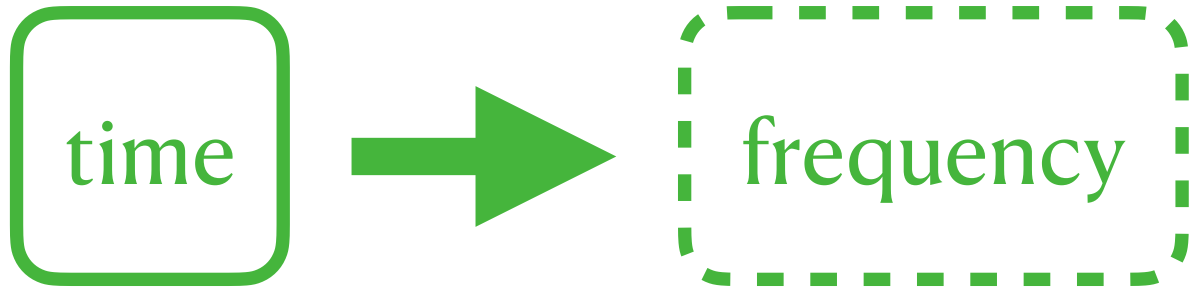

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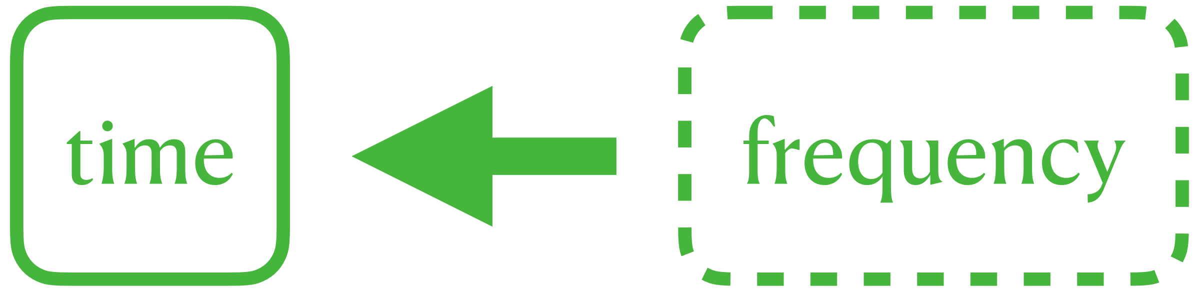

$$s(t_k) = \frac{1}{N} \sum_{n=-N/2}^{N/2} c_n e^{i2\pi nk/N}$$

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$$f_n = n\Delta f = n/T$$

Discrete Fourier Transform (DFT)

time $\leftarrow$ frequency

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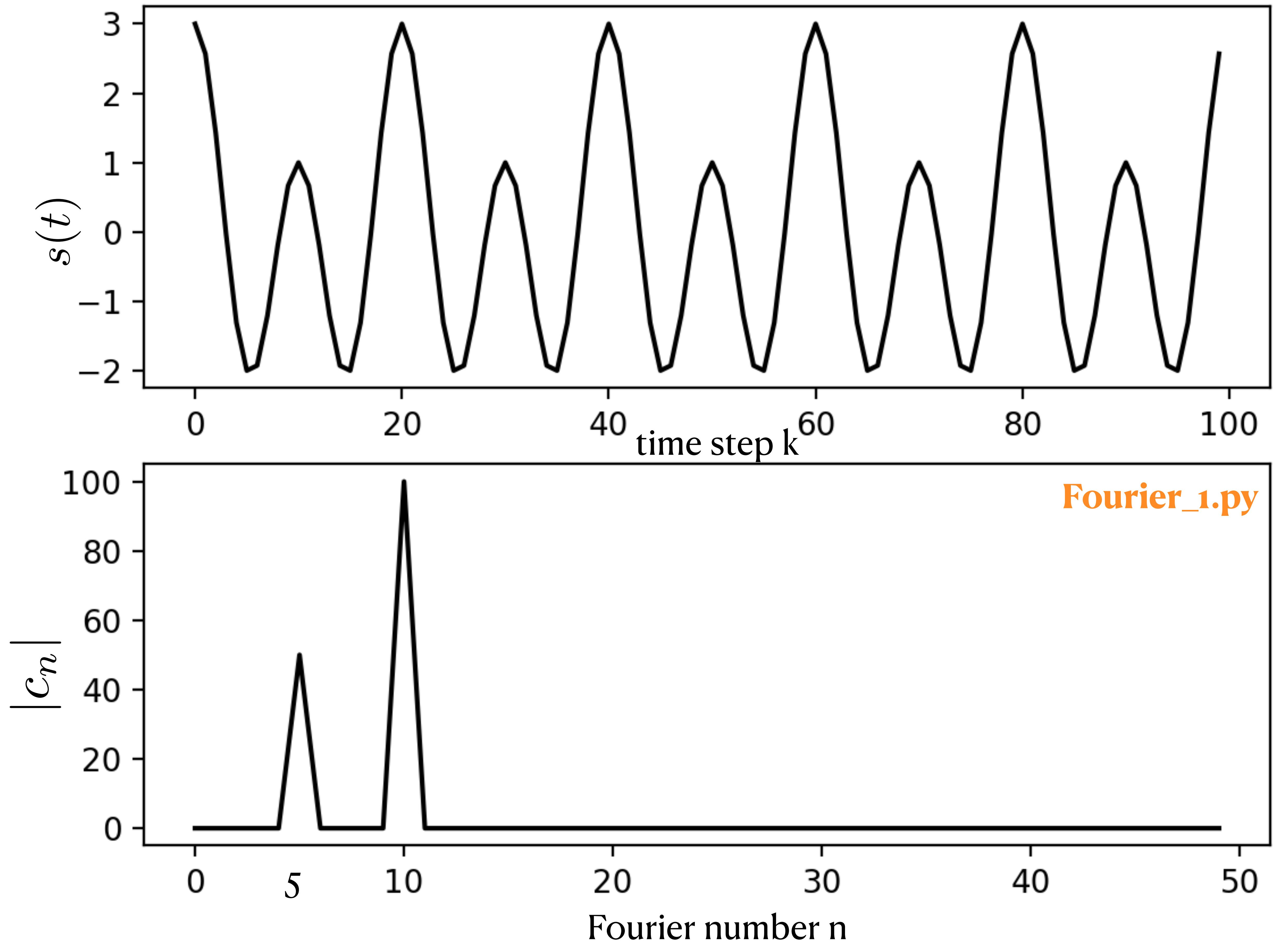
$$f_1 = 0.5Hz \ , \ \ f_2 = 1.0Hz$$



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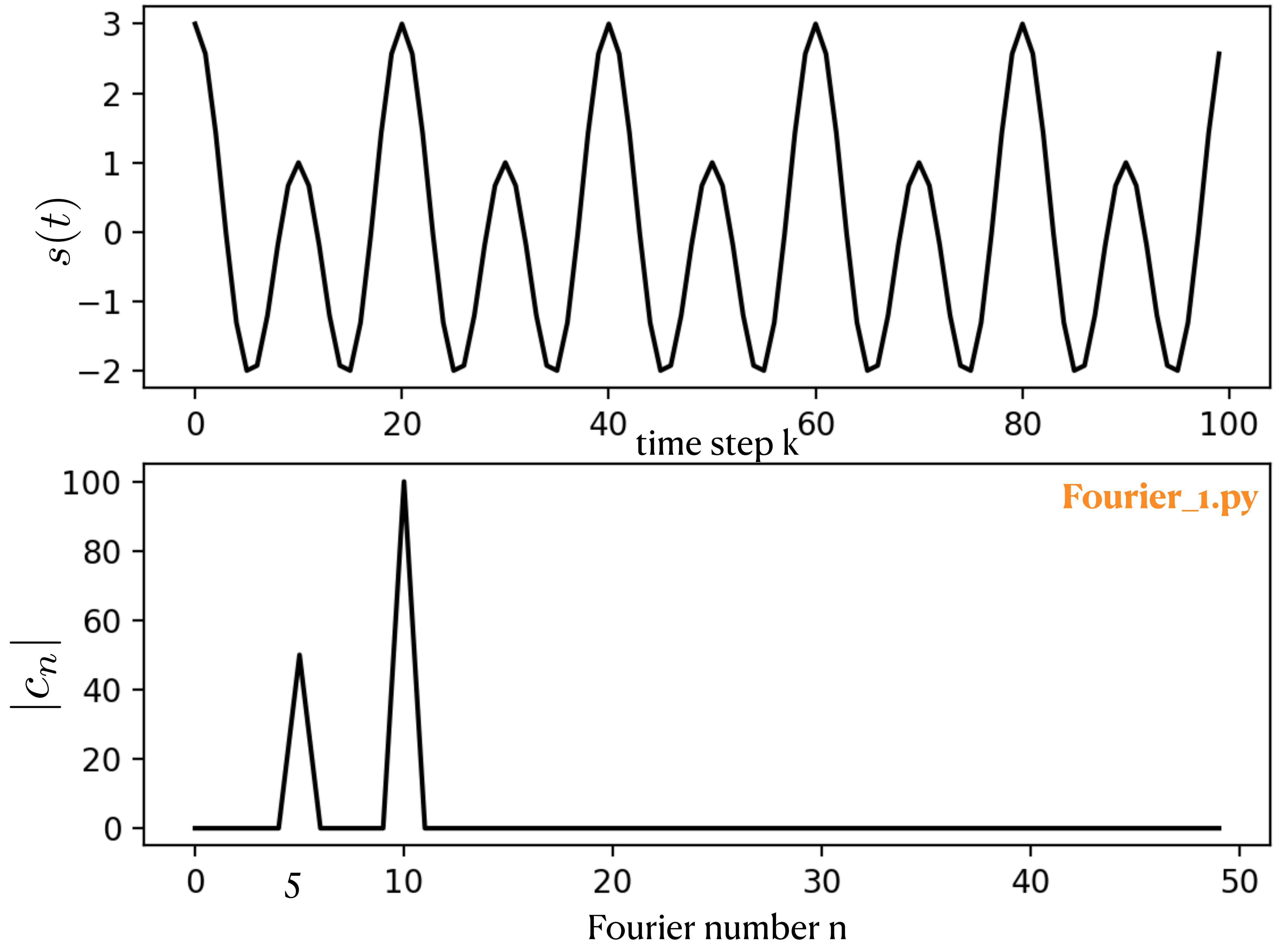

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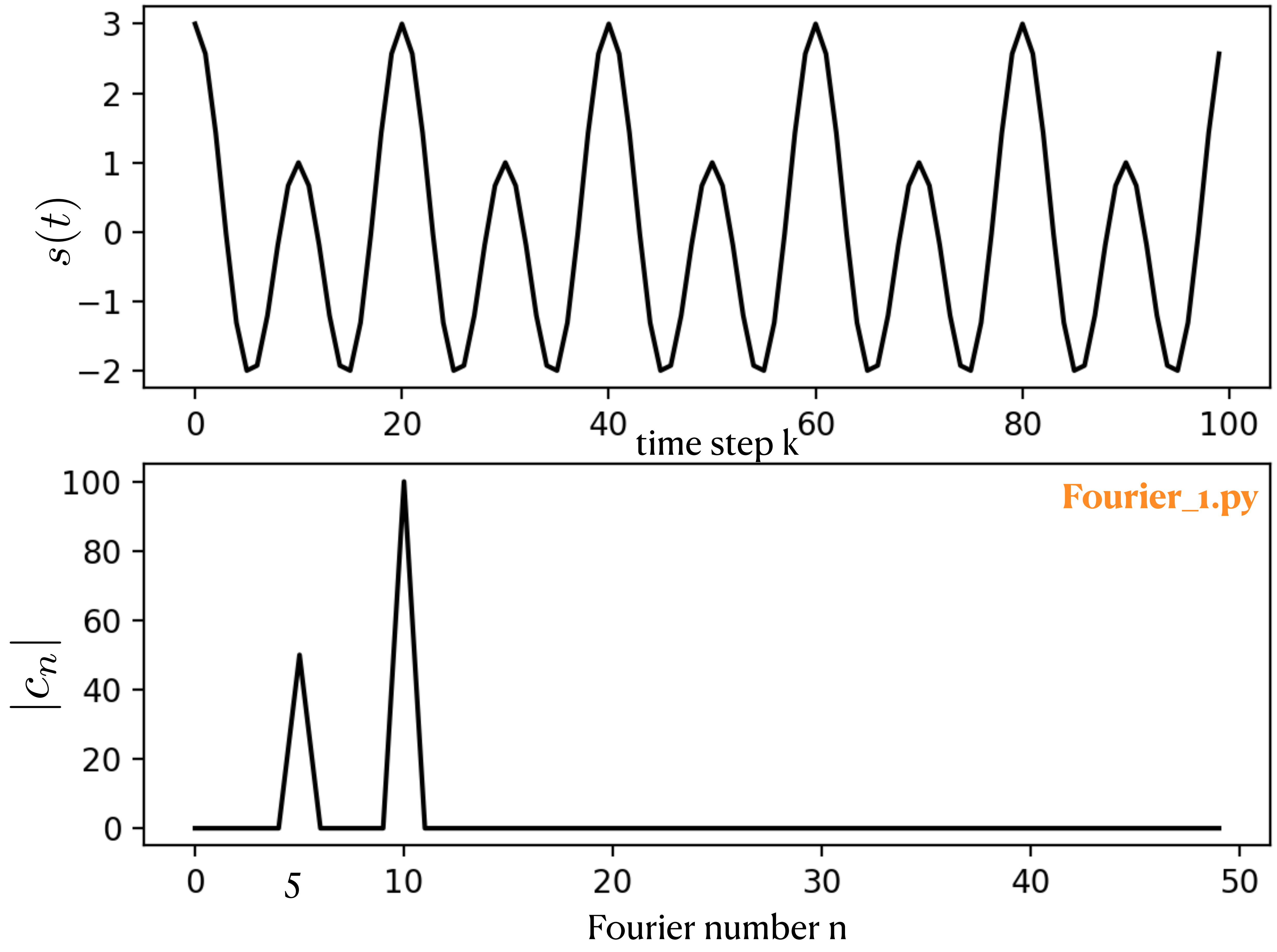

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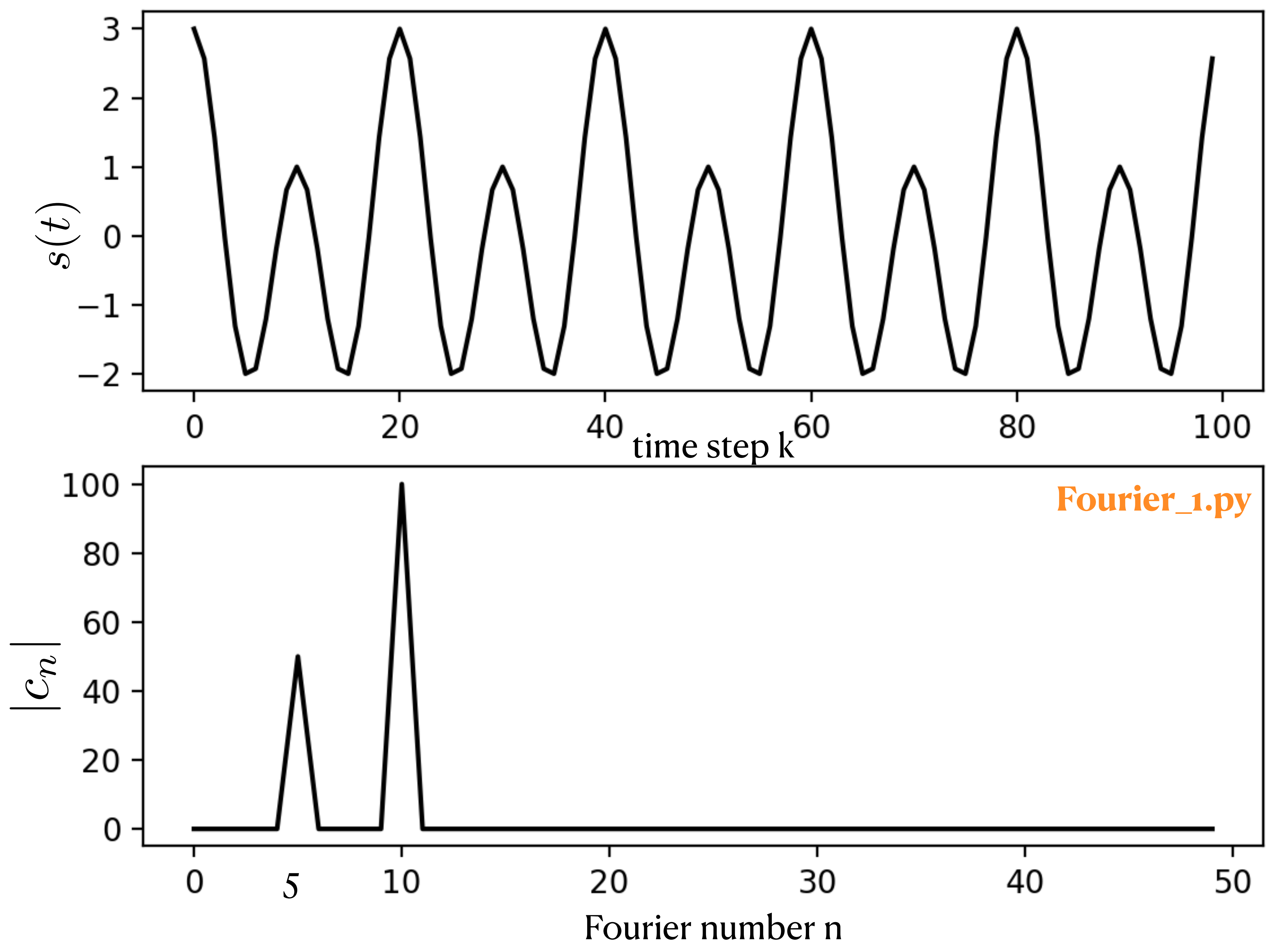

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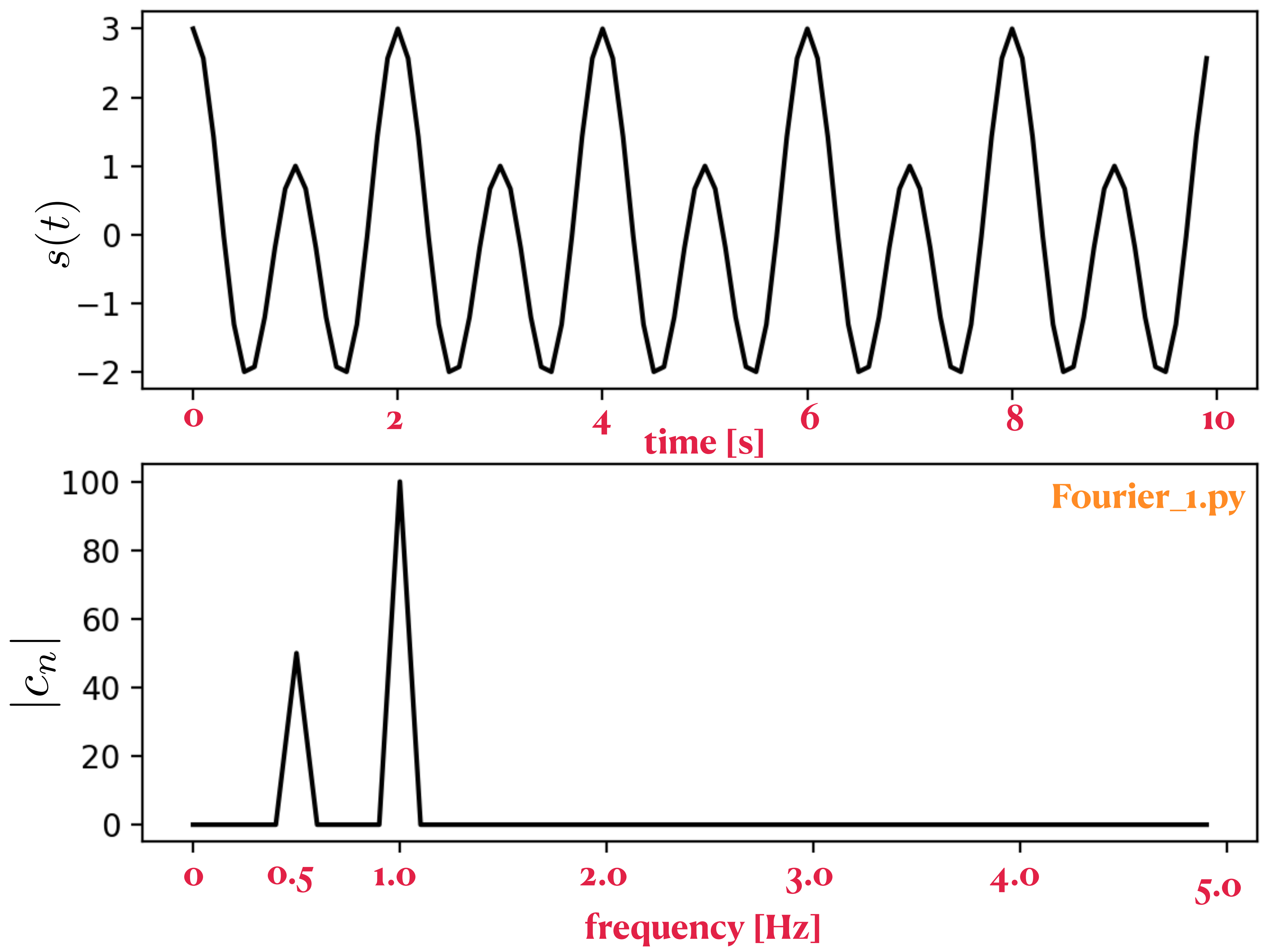
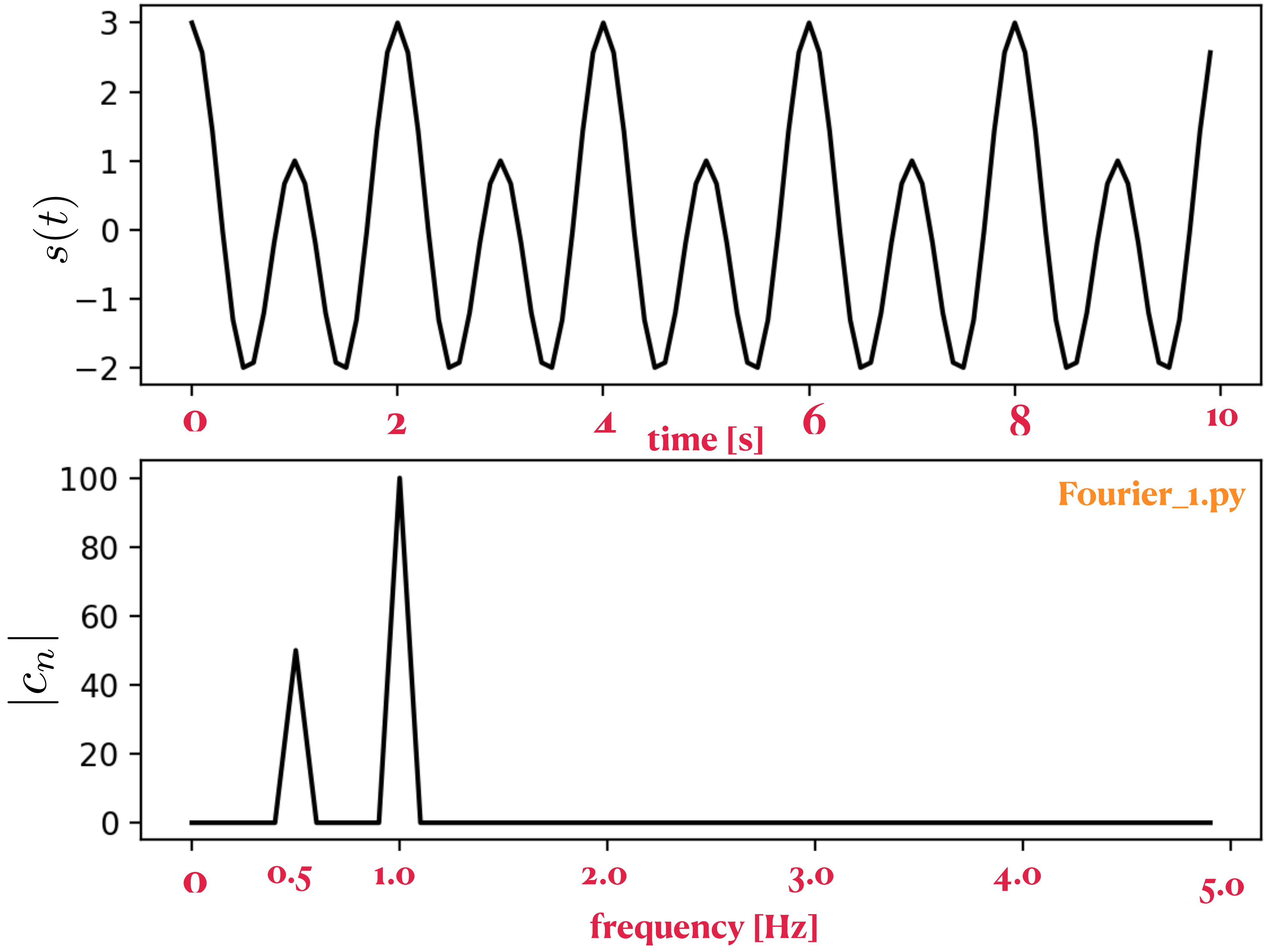


We note:

- for the spectral power it is sufficient to consider

$$0 \leq f \leq f_s/2$$



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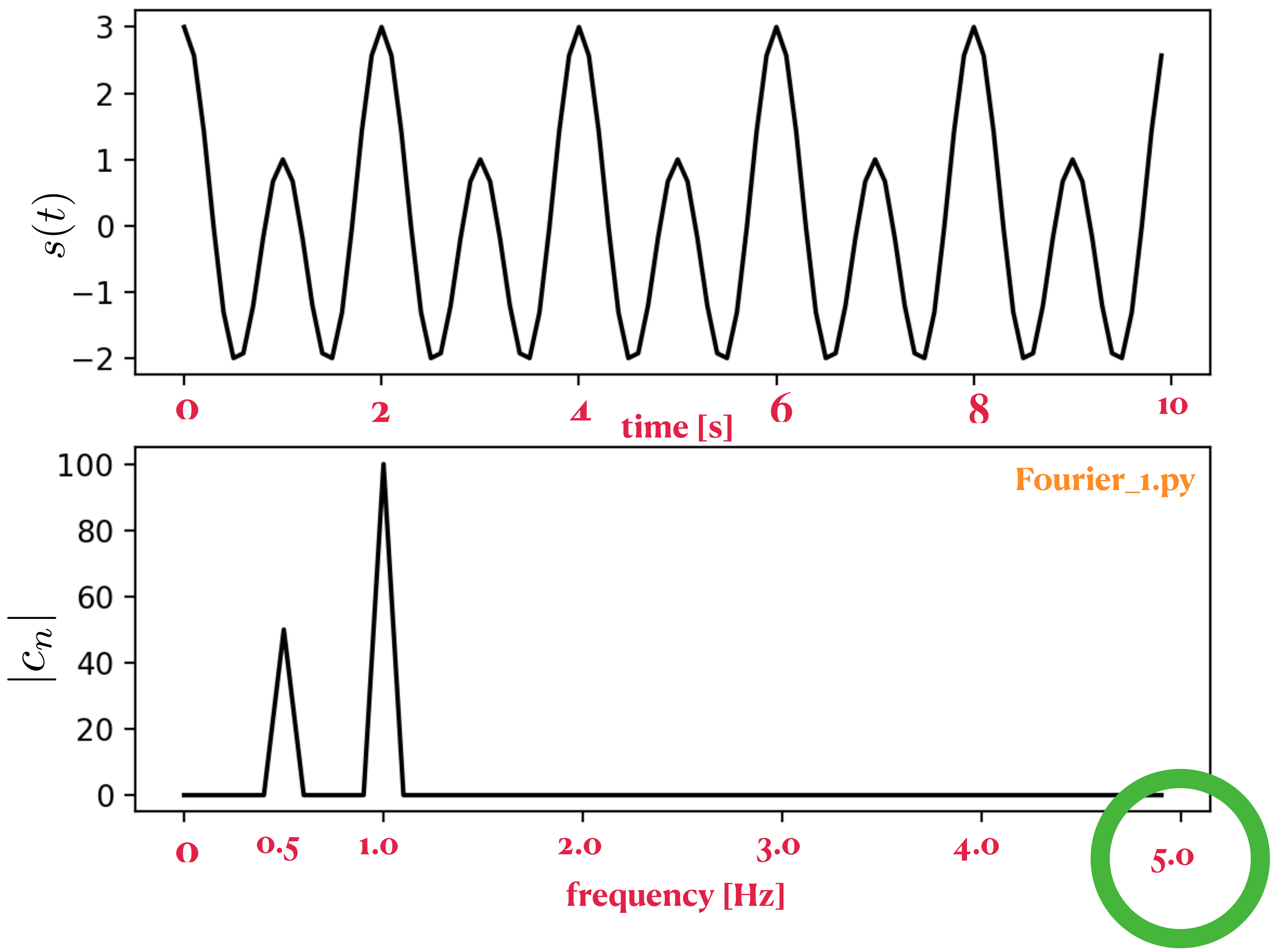
- maximum frequency of DFT is

Nyquist frequency $f_s/2$

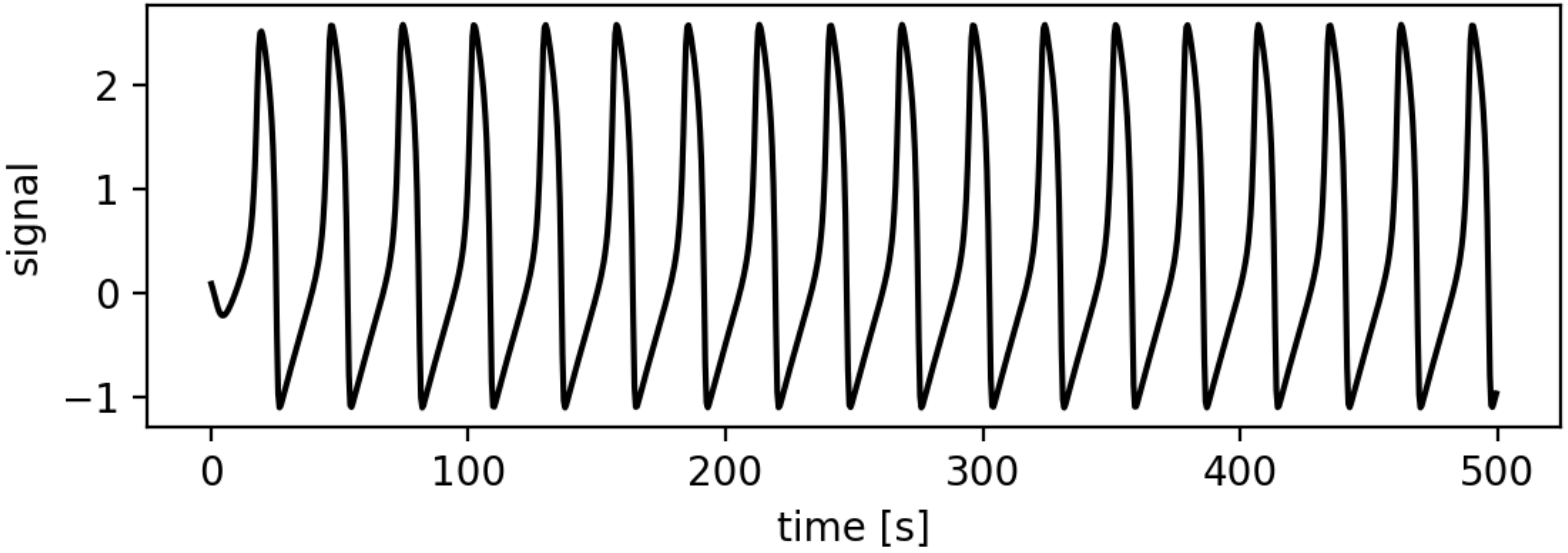


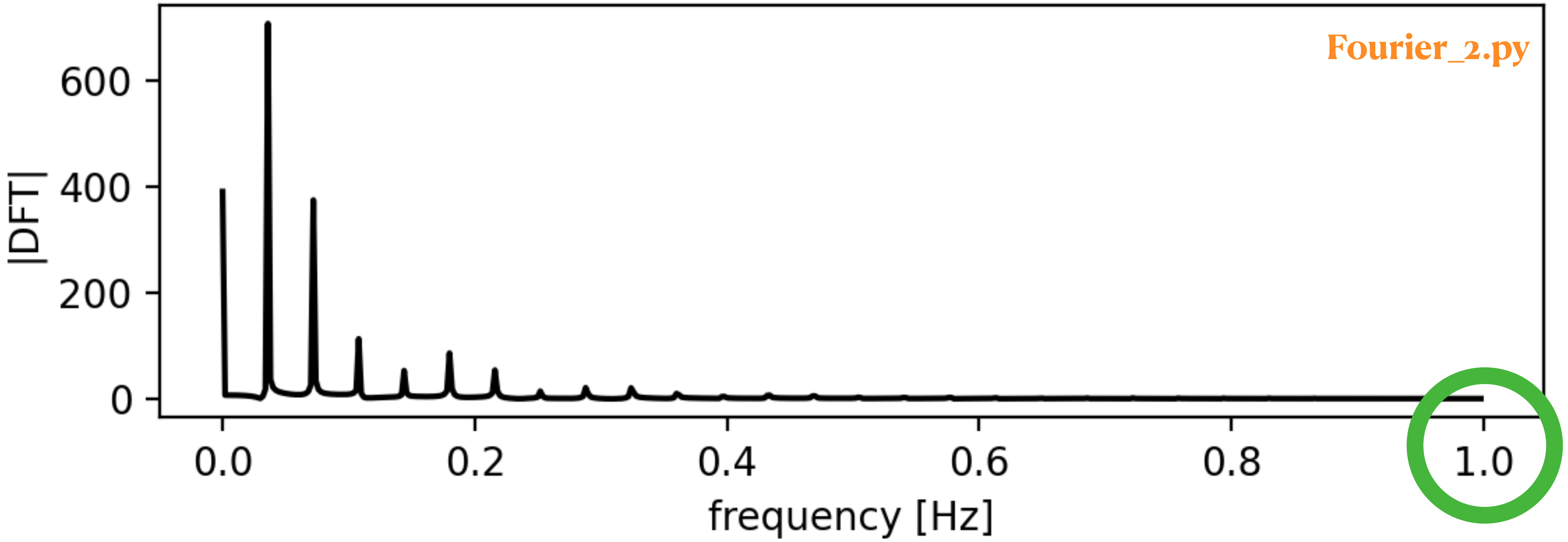
example signal: FitzHugh-Nagumo model activity

- T = 500s
- f_s = 2Hz

$$\Delta f = 0.002Hz$$

Nyquist frequency is 1Hz
= maximum frequency





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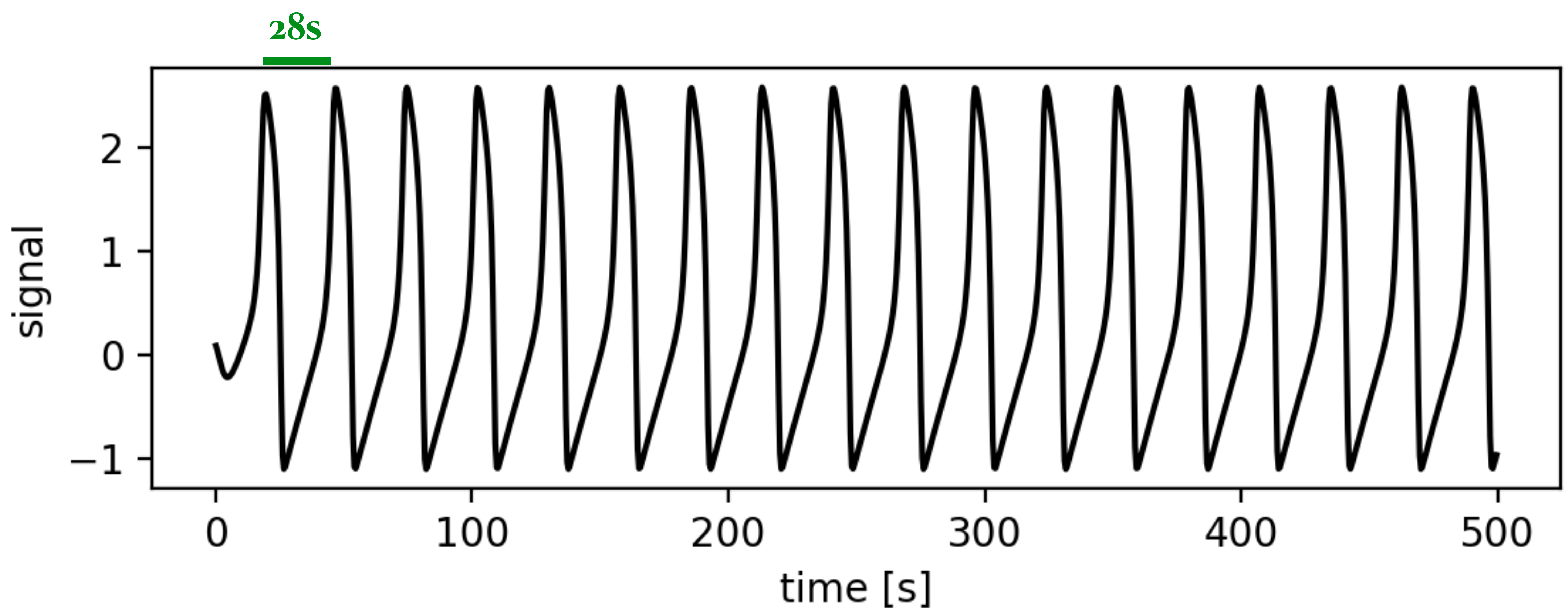
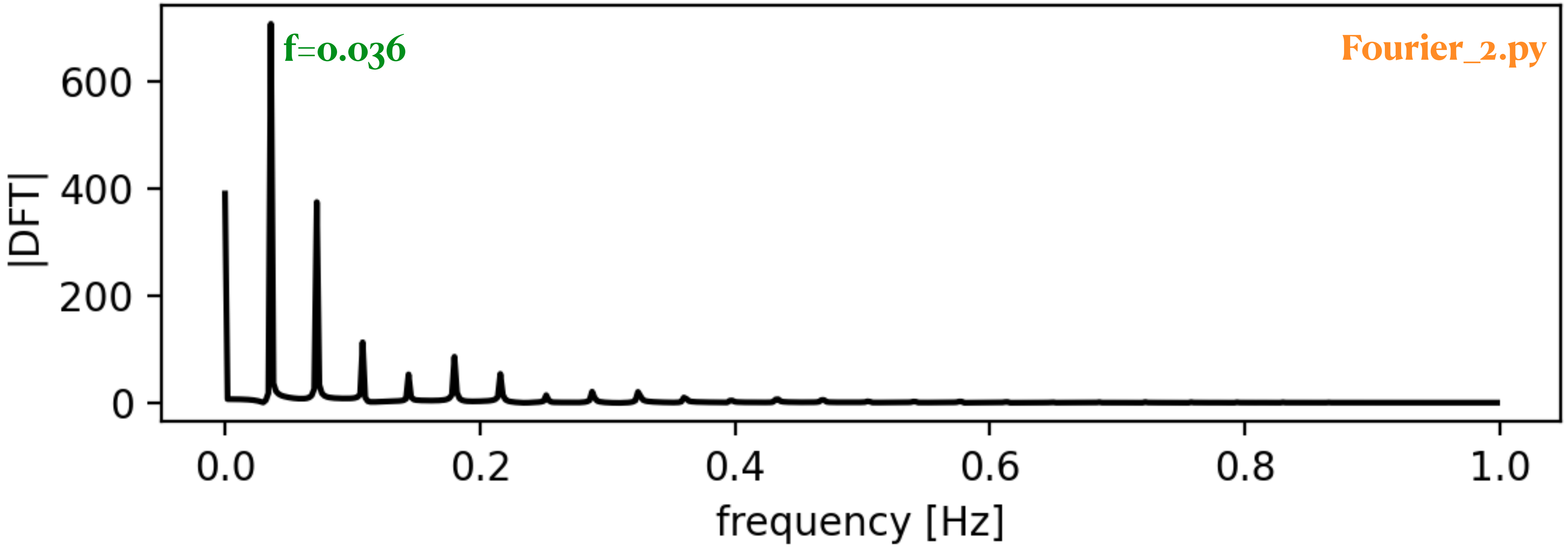


Nyquist frequency is 1Hz
= maximum frequency

principal data frequency
≈ highest power frequency



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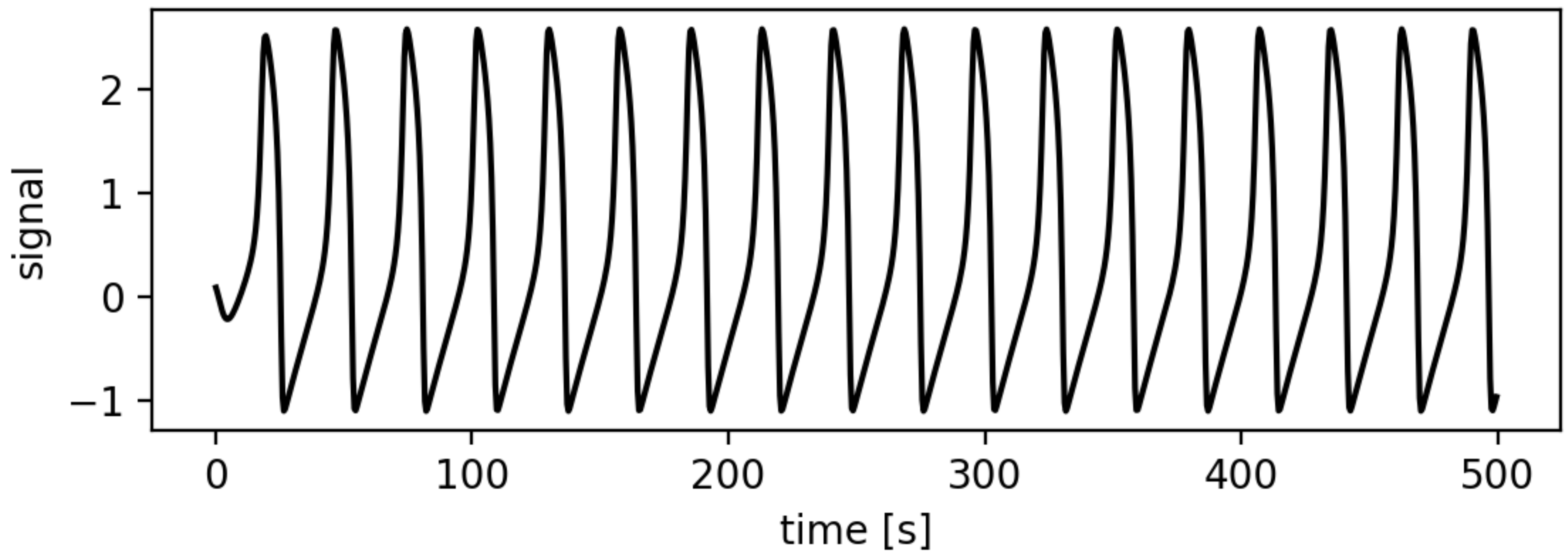

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DFT has multiple power peaks

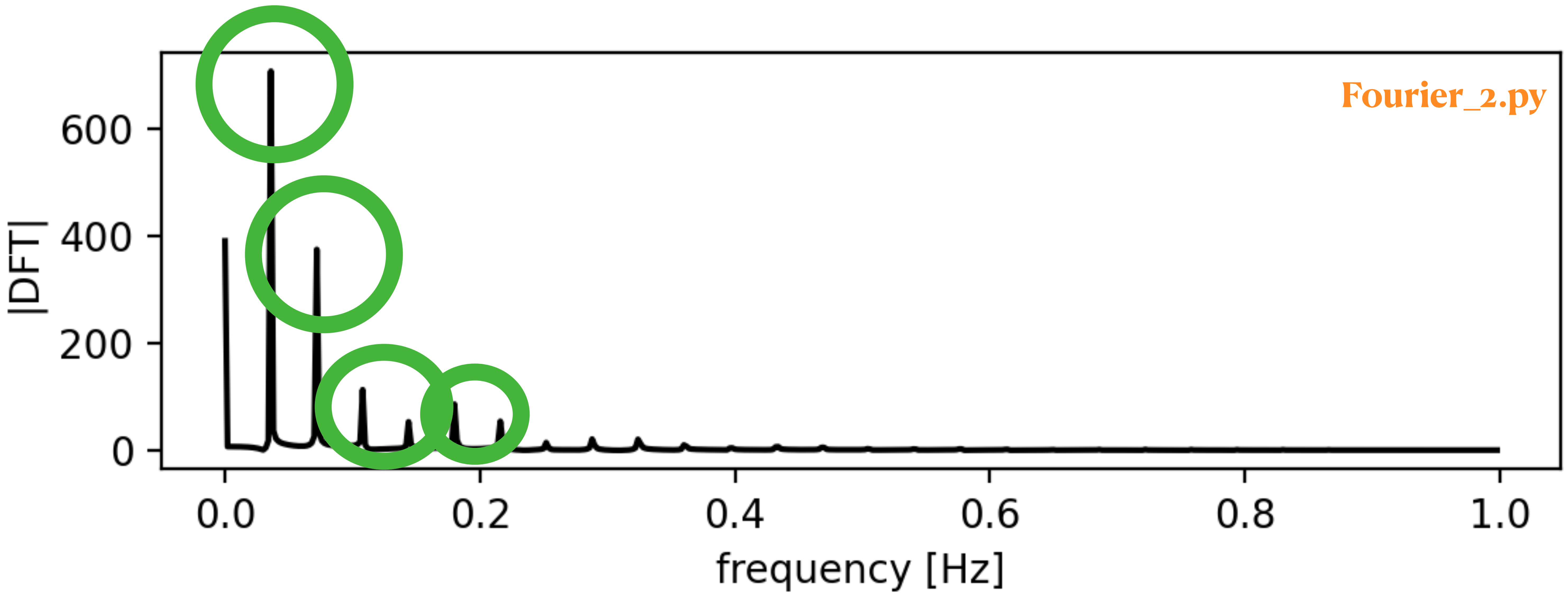
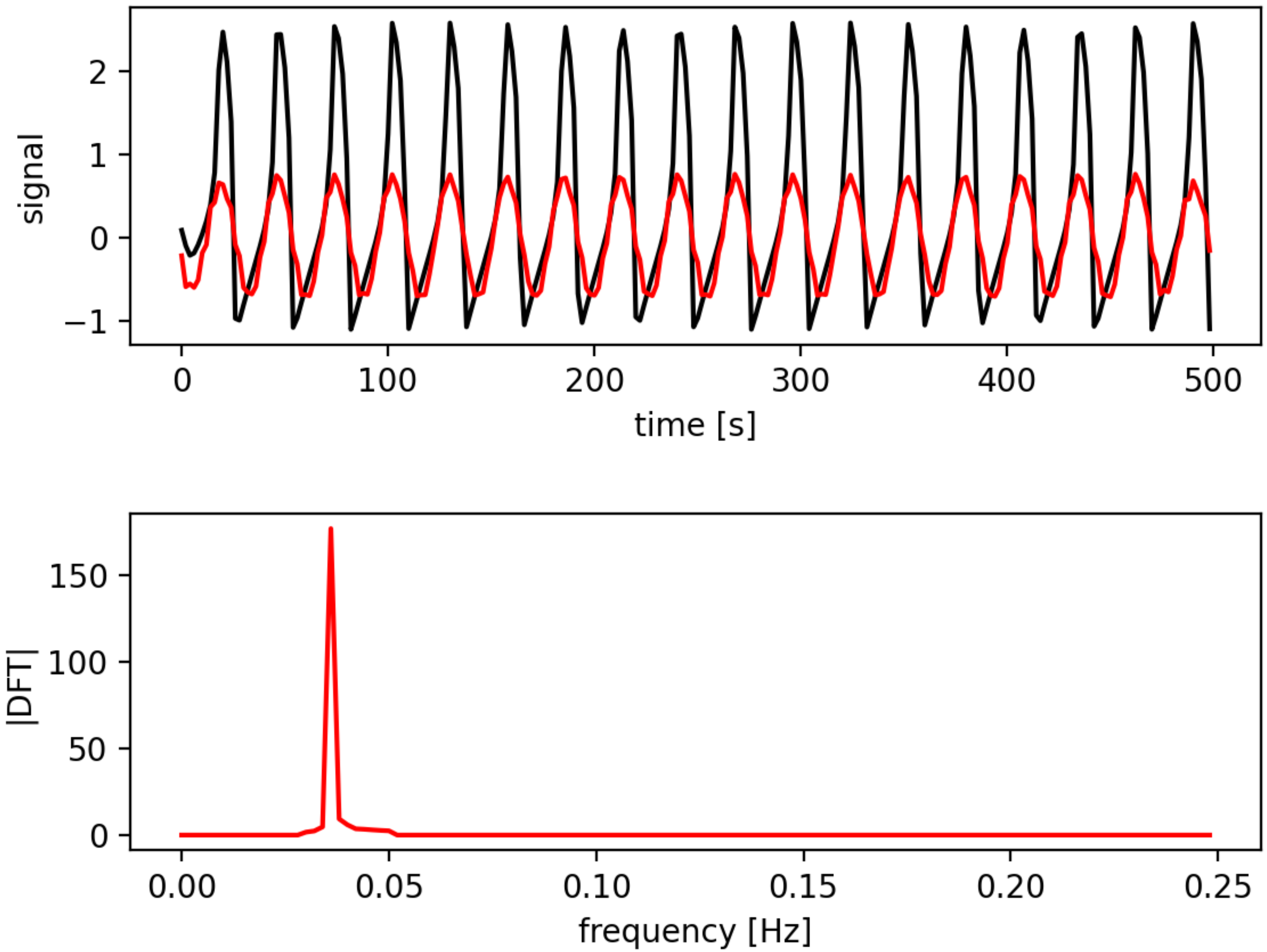
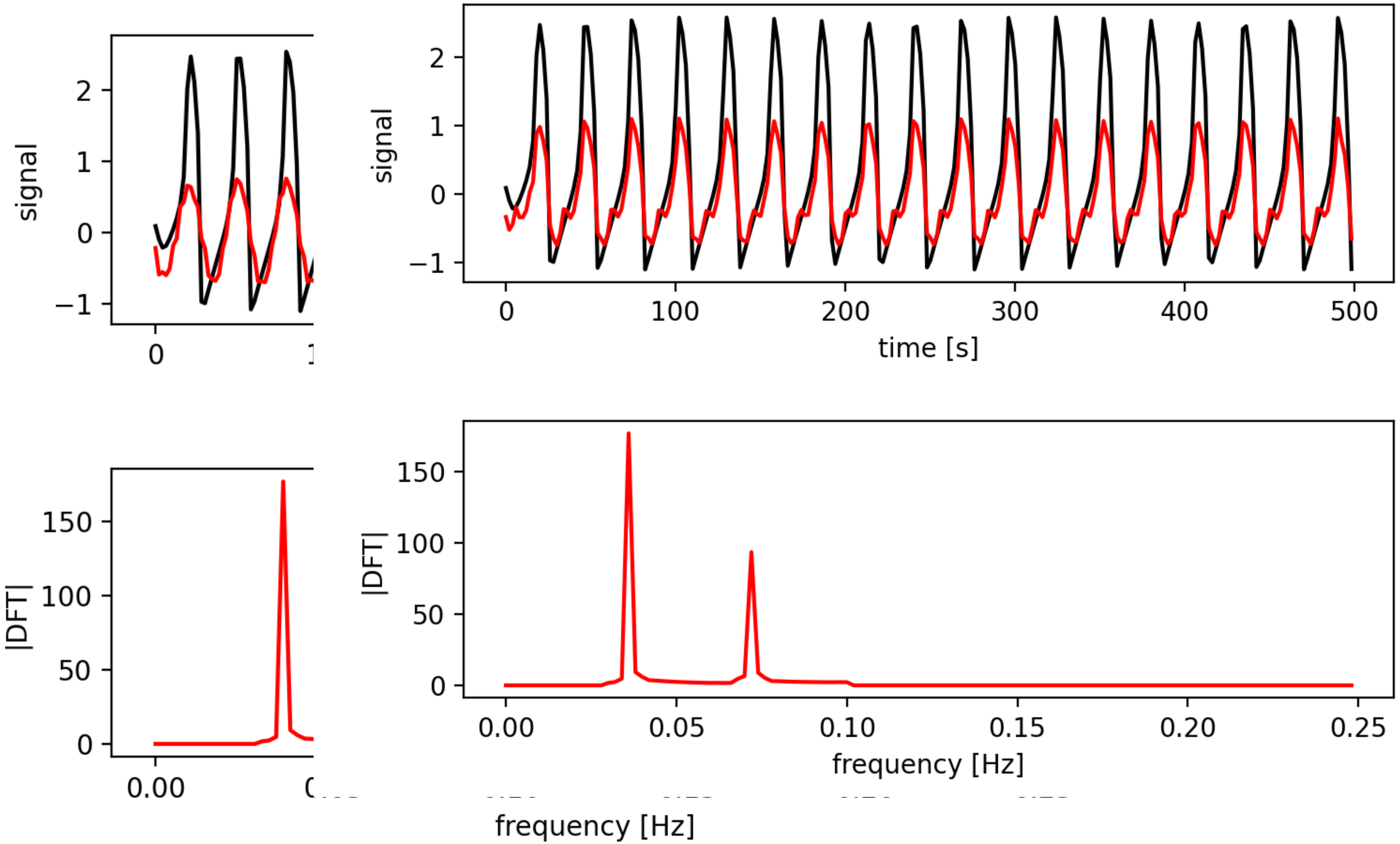
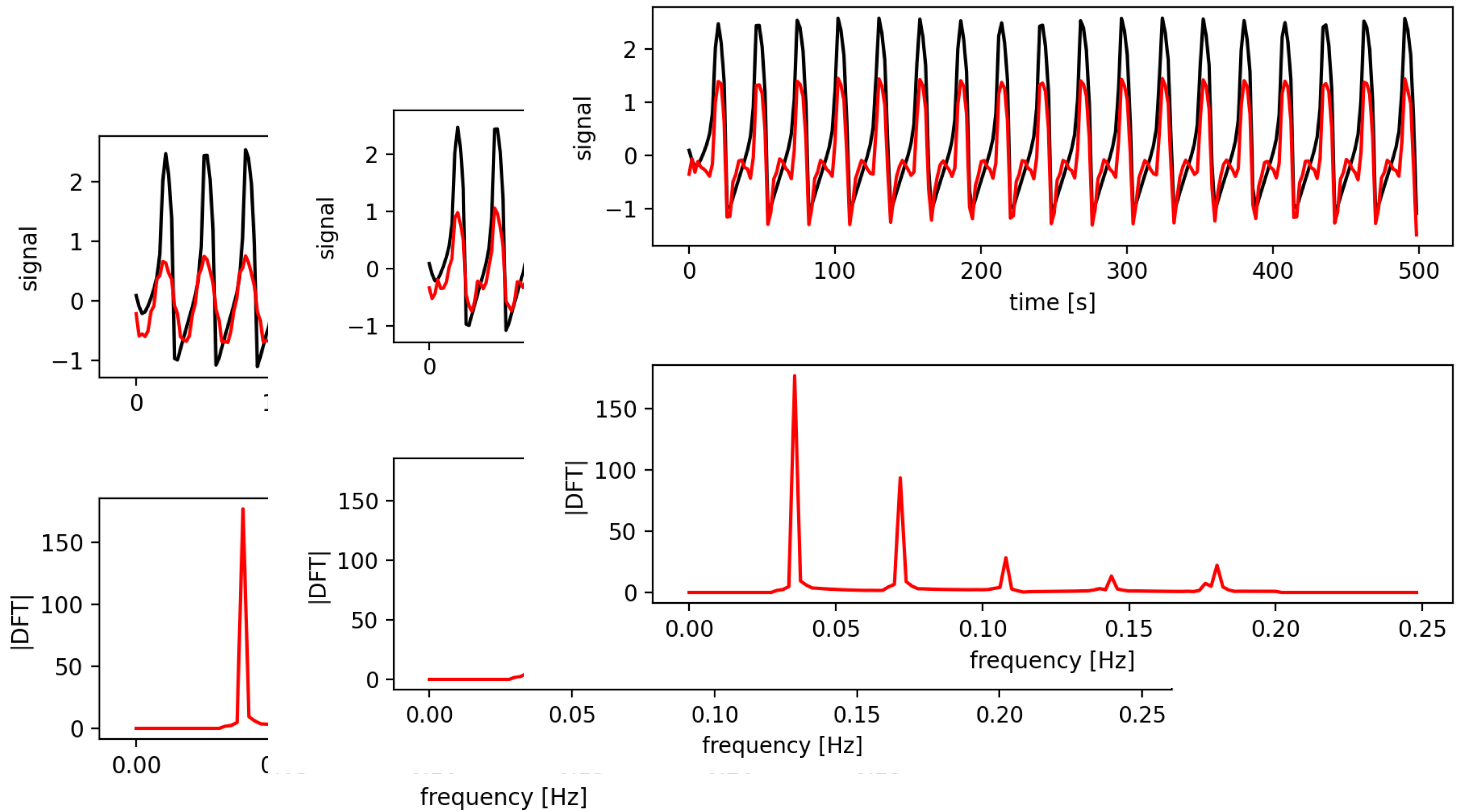


illustration for spectral decomposition (Fourier_2.py)





signal
time [s]
|DFT|
frequency [Hz]


signal
time [s]
|DFT|
frequency [Hz]

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spectral power of evenly sampled data

spectral power of **un**evenly sampled data

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$$
\begin{aligned}
P(f) &= \frac{1}{N}\left|\sum_{n=1}^{N} g_n e^{-2\pi i f t_n}\right|^2 \\
&= \frac{1}{N}\left[\left(\sum_n g_n \cos(2\pi f t_n)\right)^2 + \left(\sum_n g_n \sin(2\pi f t_n)\right)^2\right]
\end{aligned}
$$

Discrete Fourier Transform (DFT)

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Discrete Fourier Transform (DFT)

$$P(f) = \frac{A^2}{2} \left(\sum_n g_n \cos(2\pi f [t_n - \tau]) \right)^2$$

$$+ \frac{B^2}{2} \left(\sum_n g_n \sin(2\pi f [t_n - \tau]) \right)^2$$

generalised DFT

with general constants A, B, τ

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generalised DFT

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<u>Lomb (1976) and Scargle (1982):</u>

$$A = \left(\sum_n \cos^2\left(2\pi f[t_n - \tau]\right)\right)^{-1} \qquad B = \left(\sum_n \sin^2\left(2\pi f[t_n - \tau]\right)\right)^{-1}$$

$$\tau = \frac{1}{4\pi f}\tan^{-1}\left(\frac{\sum_n \sin(4\pi f t_n)}{\sum_n \cos(4\pi f t_n)}\right)$$

(JT VanderPlas. *Understanding the Lomb–Scargle Periodogram*. Astrophys. J. Suppl. 236:16-44 (2018)

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generalised DFT

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Lomb-Scargle Periodogram

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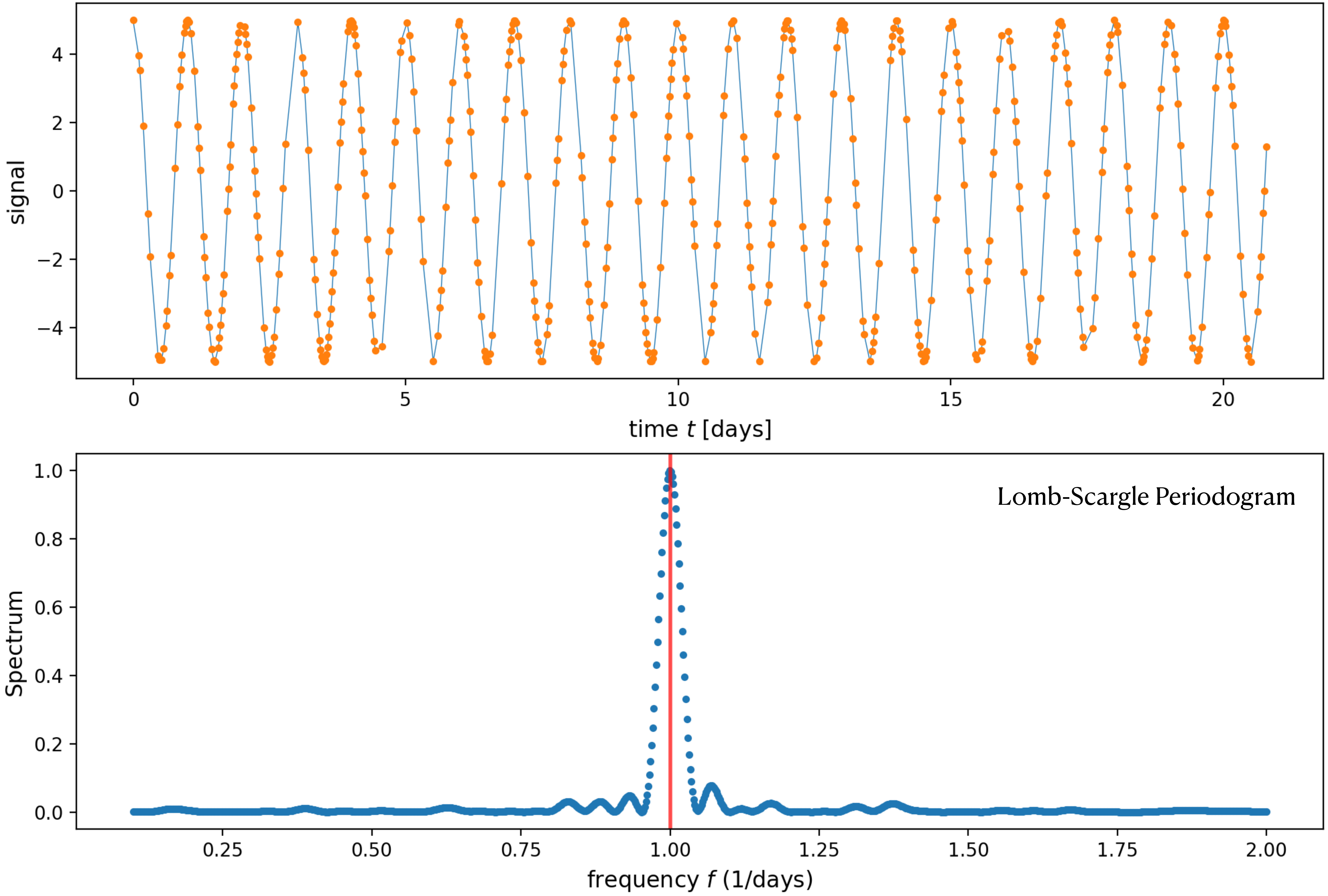
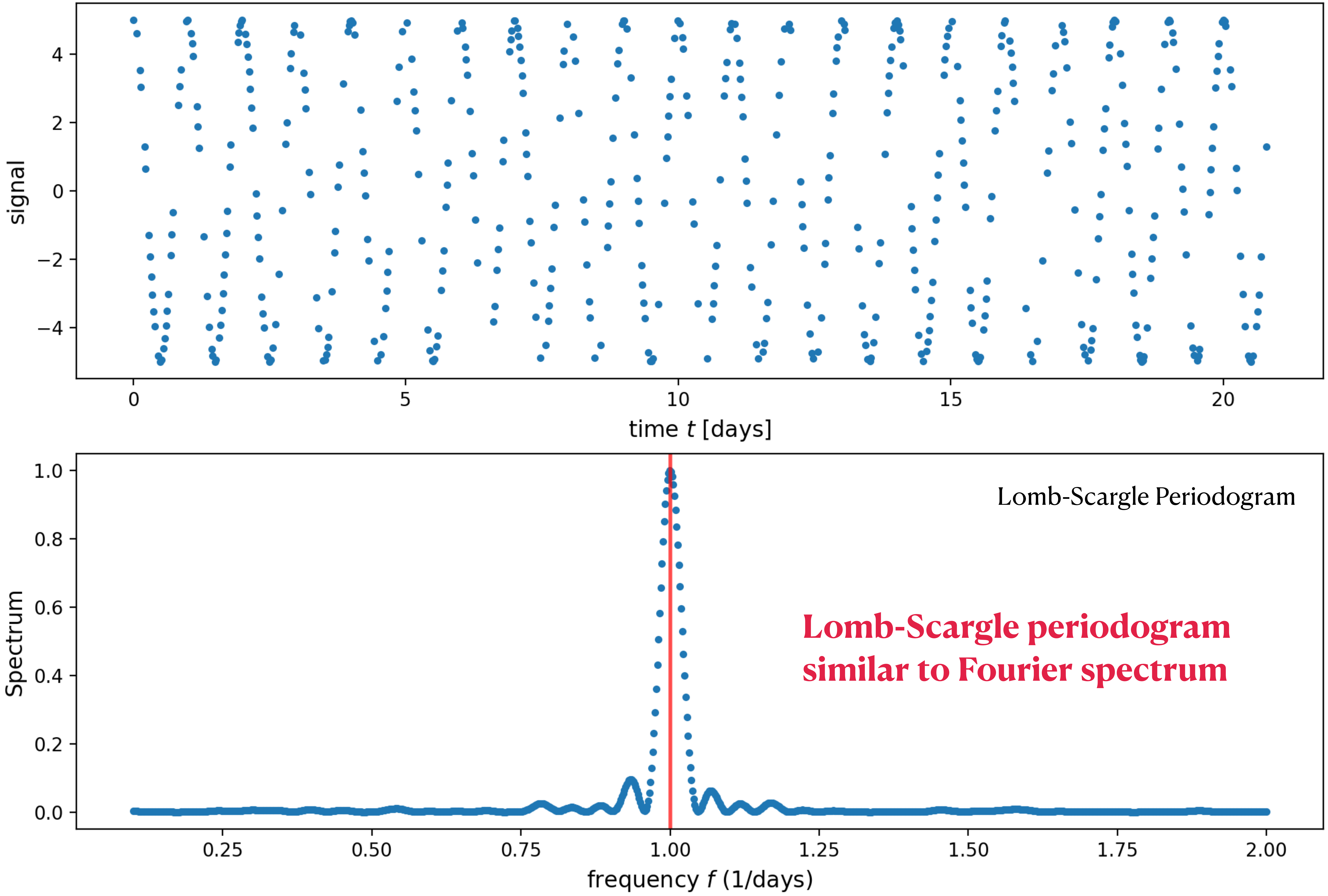
50% of points



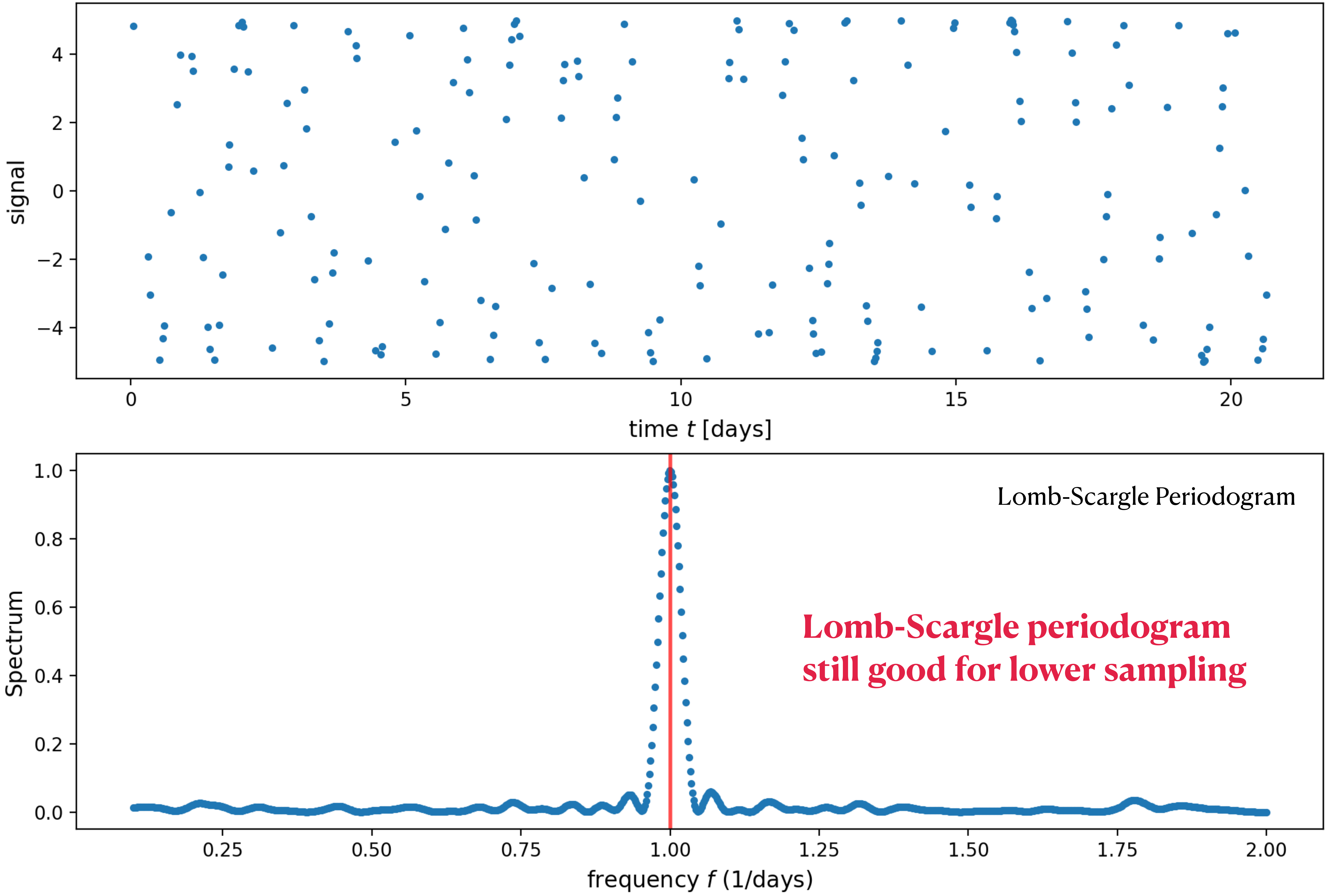
signal
time t [days]
Spectrum
frequency f (1/days)
Lomb-Scargle Periodogram

50% of points

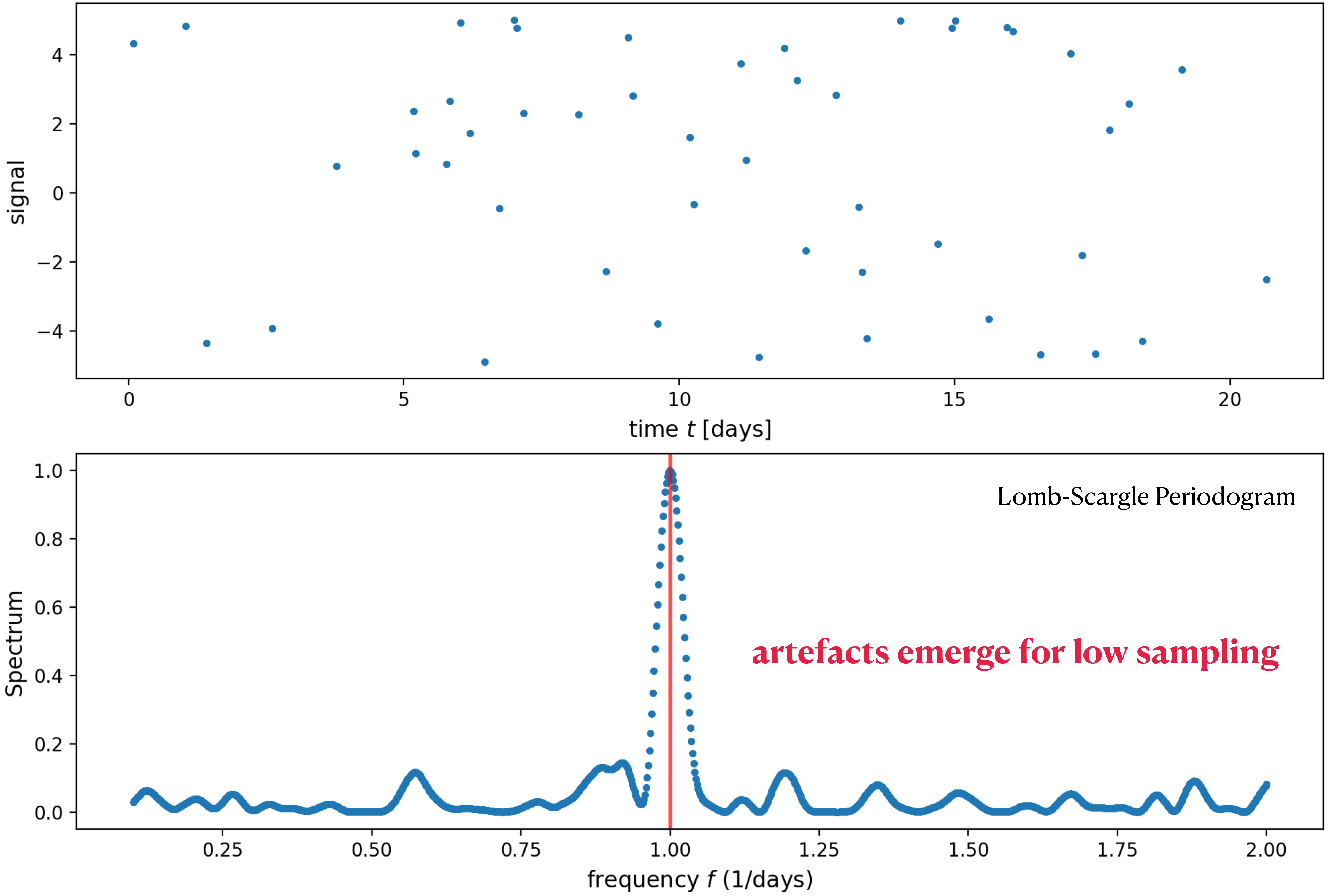


signal
4
2
0
−2
−4
0
5
10
15
20
time t [days]
Spectrum
1.0
0.8
0.6
0.4
0.2
0.0
Lomb-Scargle Periodogram
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
frequency f (1/days)
Lomb-Scargle periodogram
similar to Fourier spectrum

20% of points


signal
time t [days]
Spectrum
frequency f (1/days)
Lomb-Scargle Periodogram
Lomb-Scargle periodogram
still good for lower sampling

5% of points


signal
4
2
0
−2
−4
0
5
10
15
20
time t [days]
Lomb-Scargle Periodogram
artefacts emerge for low sampling
Spectrum
1.0
0.8
0.6
0.4
0.2
0.0
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
frequency f (1/days)

major idea of Lomb and Scargle:

optimal fit of sinusoidal function to signal

$$y(t; f) = A_f \sin(2\pi f(t - \phi_f))$$

with amplitude A_f and phase ϕ_f for each frequency f.

$\Rightarrow$ minimize $\chi^2(f) \equiv \sum_n (y_n - y(t_n; f))^2$