data sampling

Fourier analysis

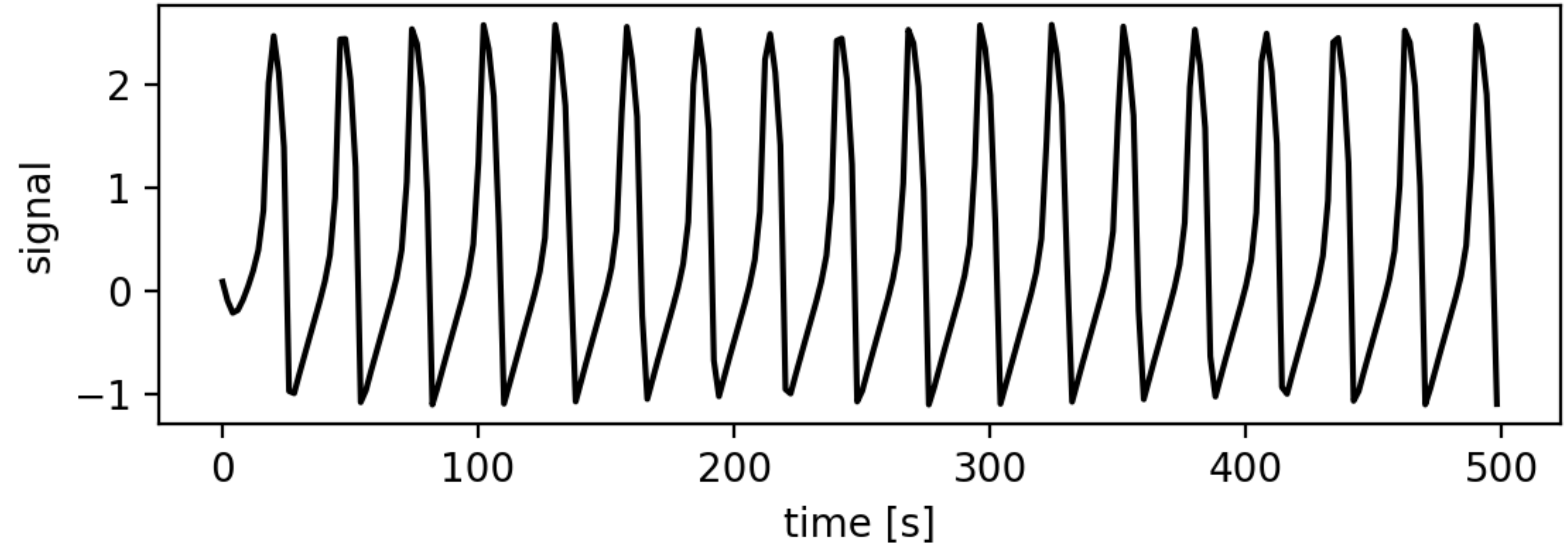
errors in analysis

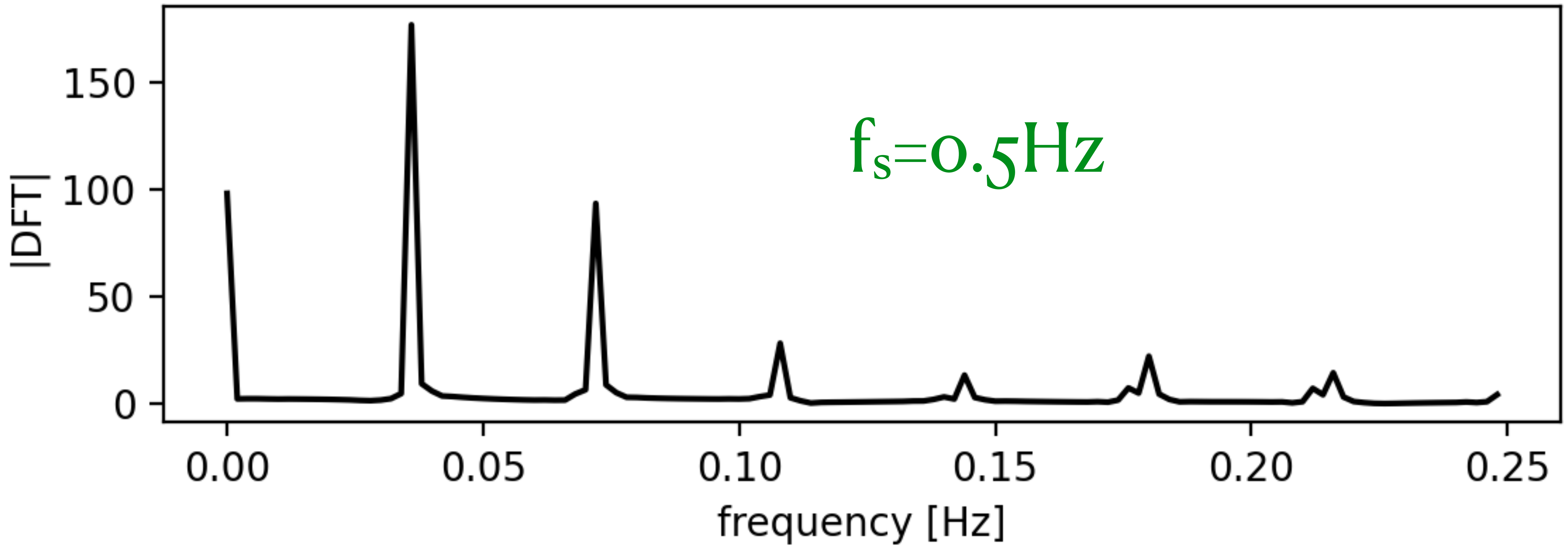
linear filters

time-frequency analysis

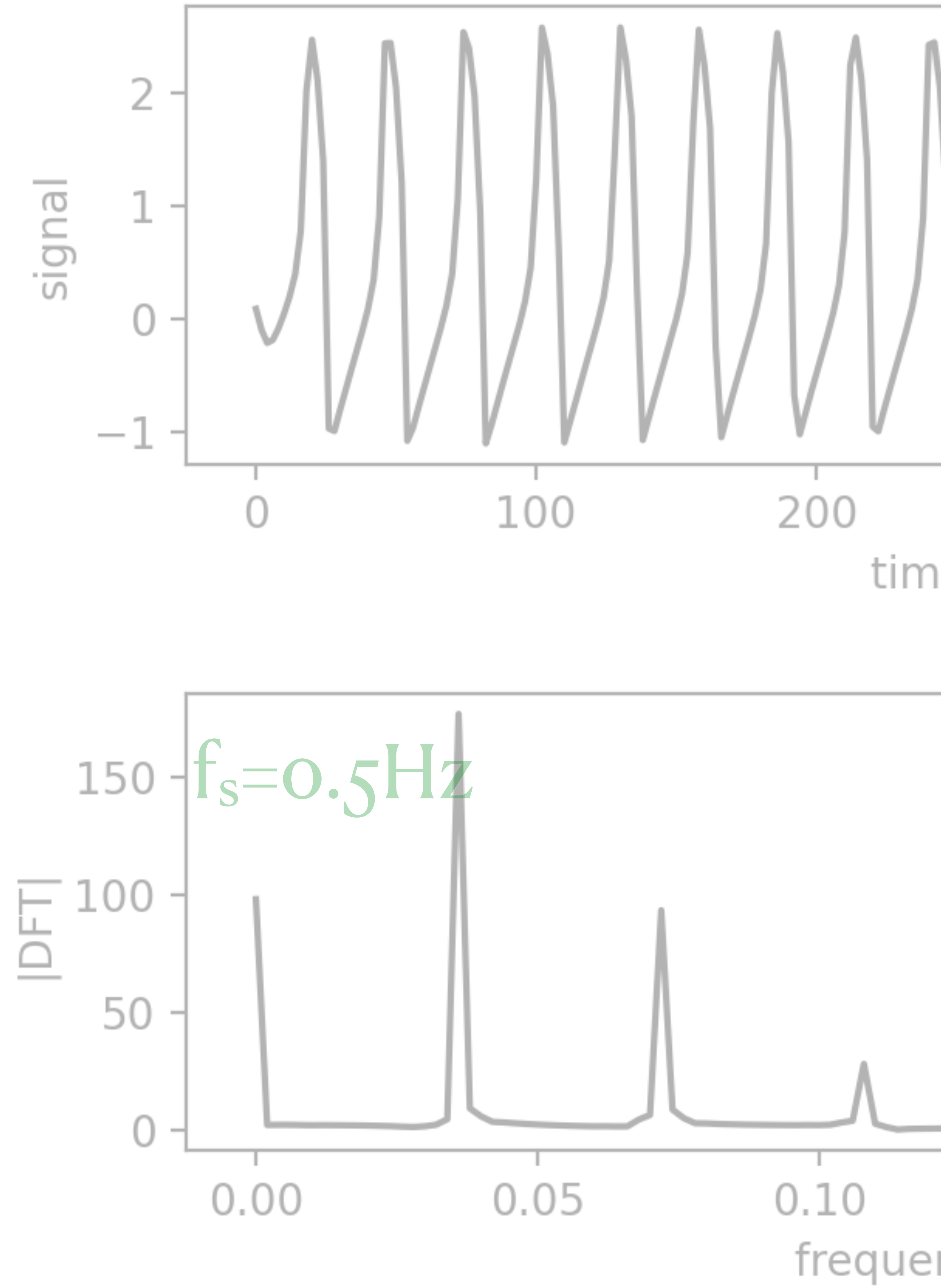
Error: subsampling in evenly sampled data

SamplingError_1.py

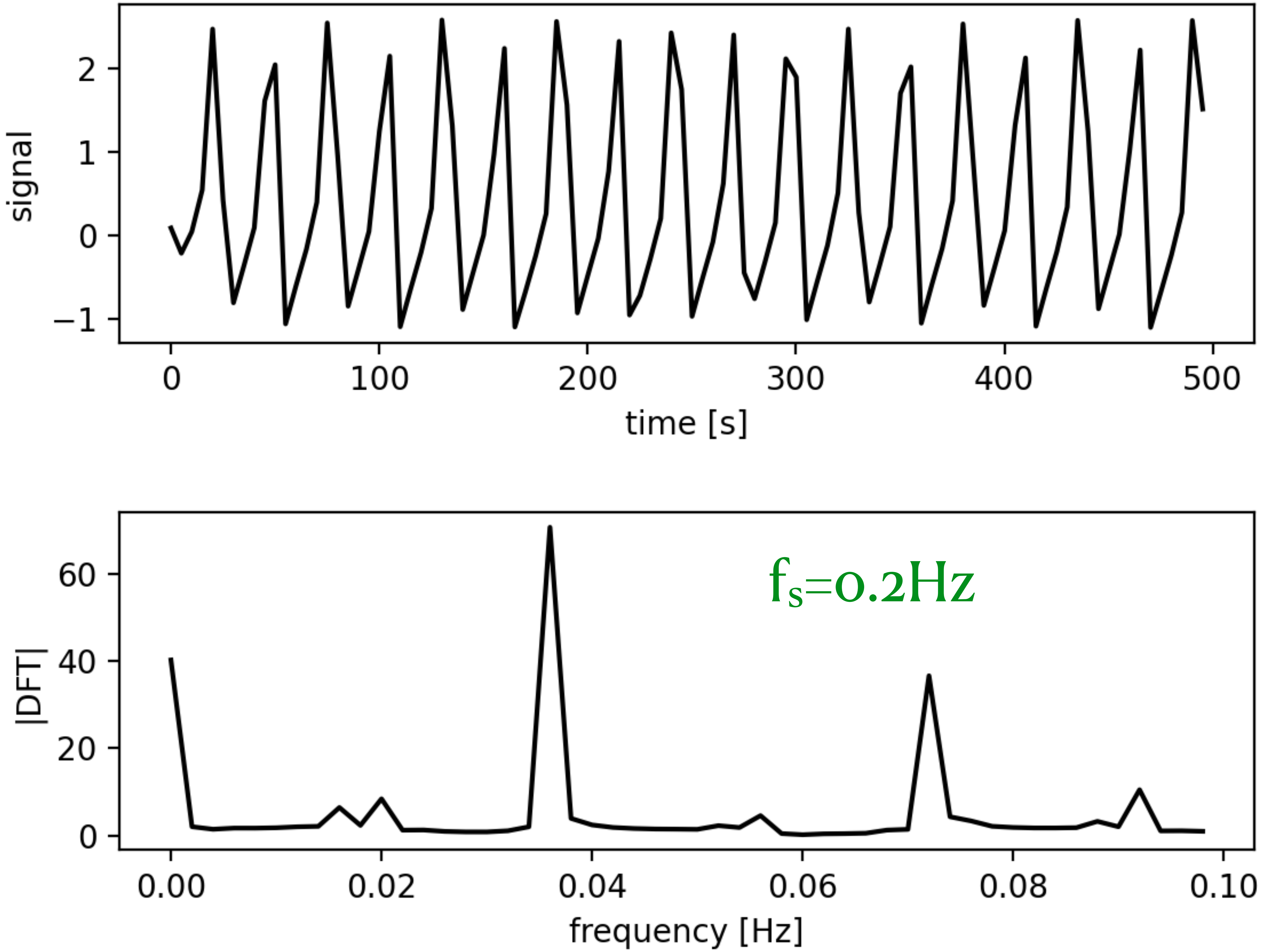


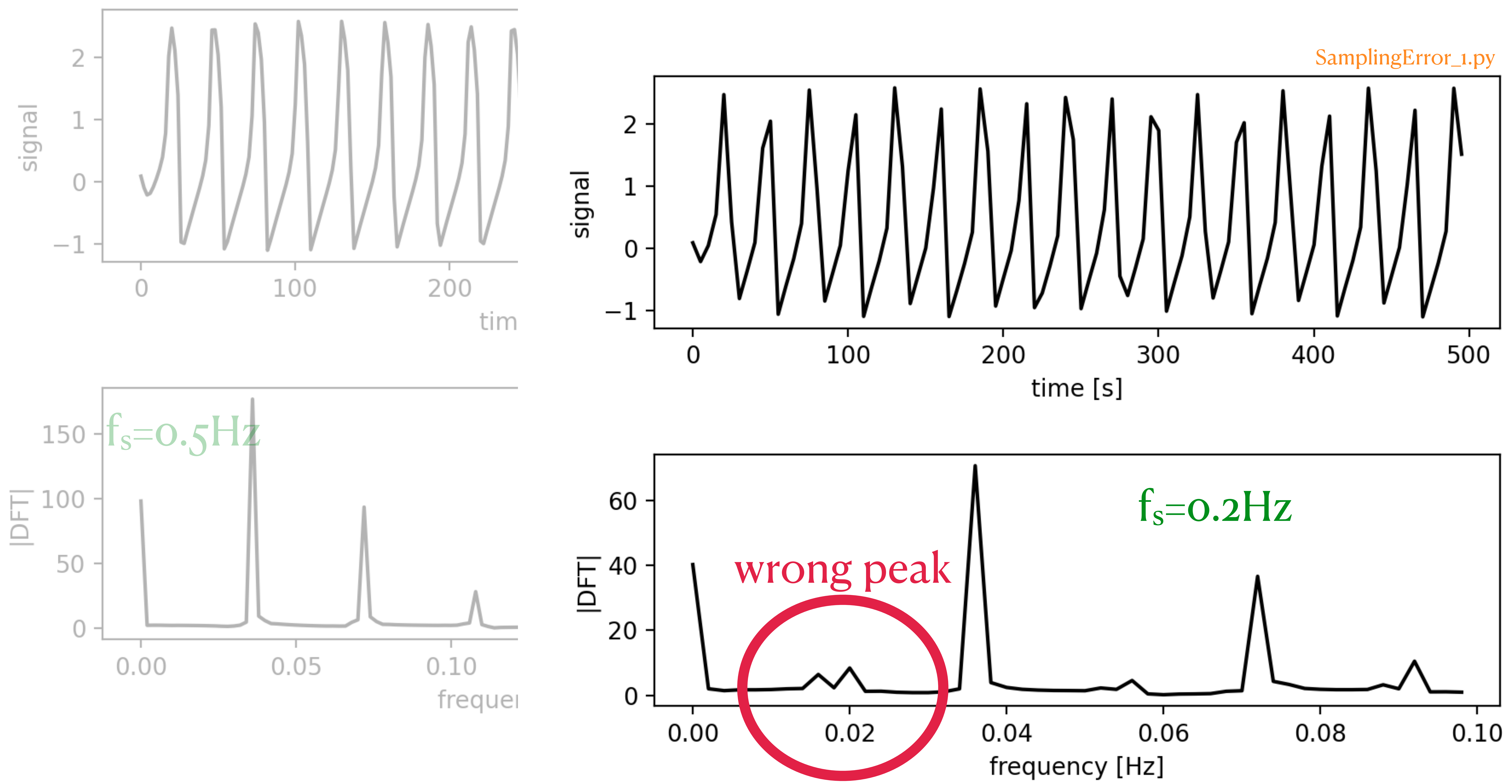


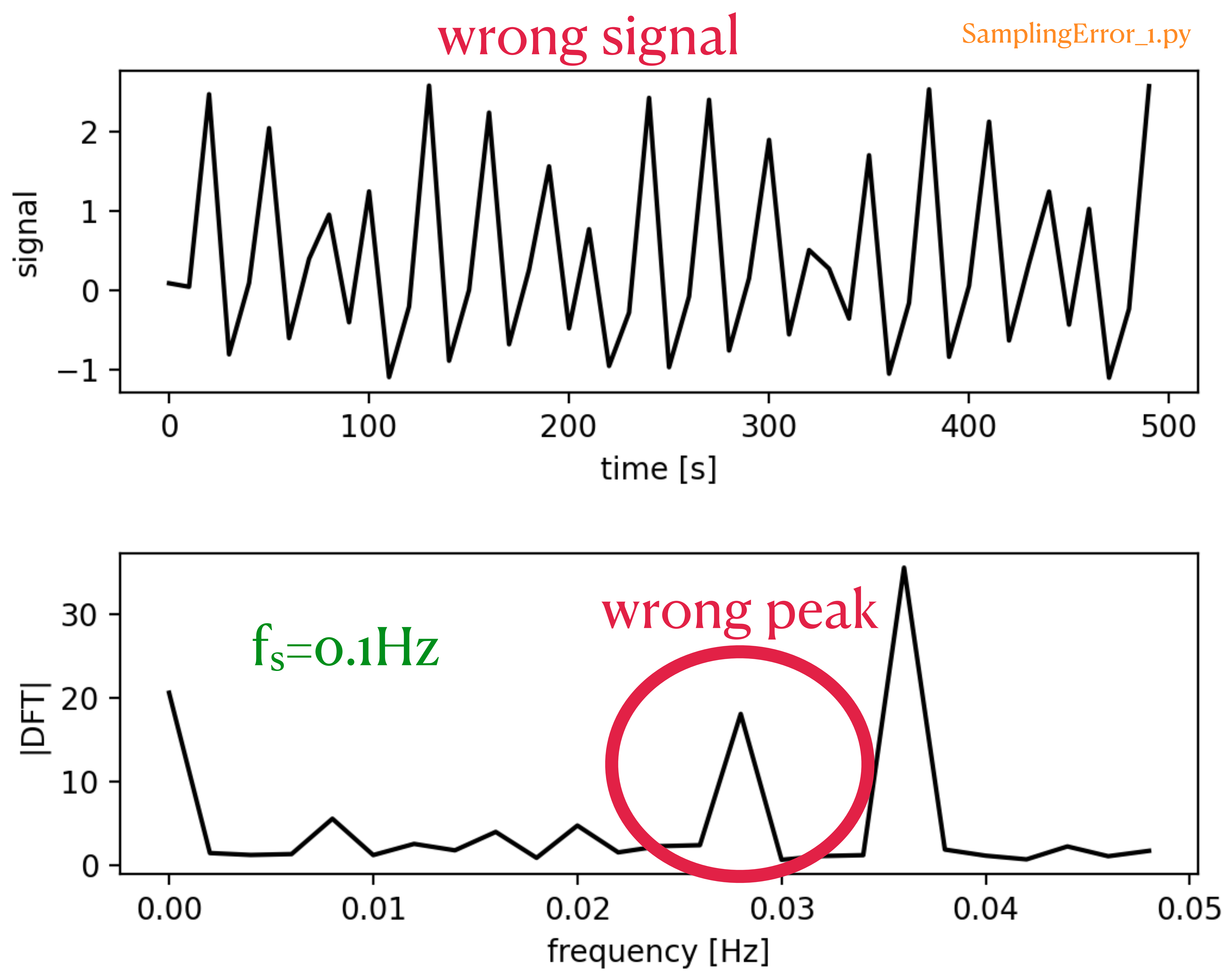
f_s=0.5Hz



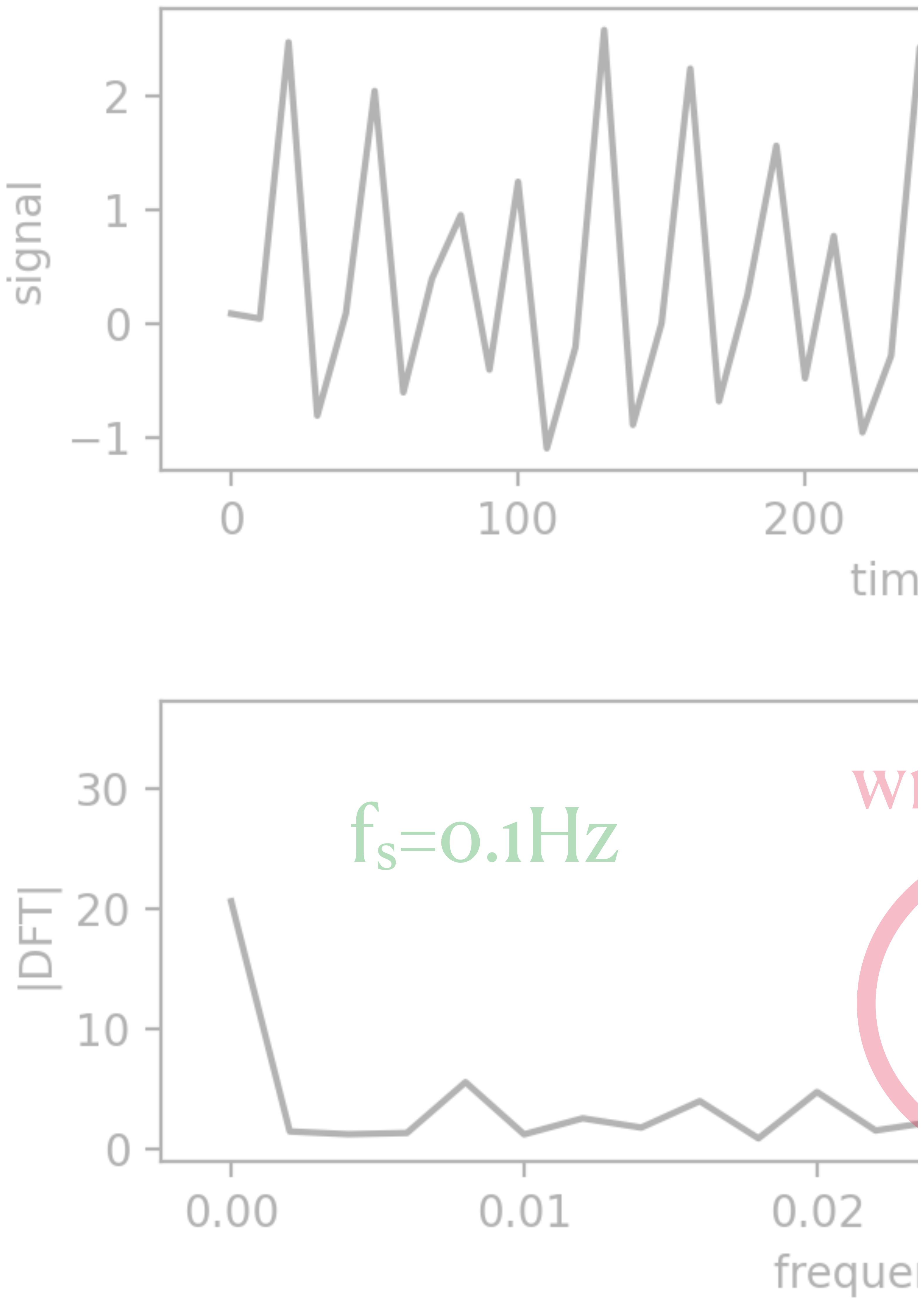
SamplingError_1.py

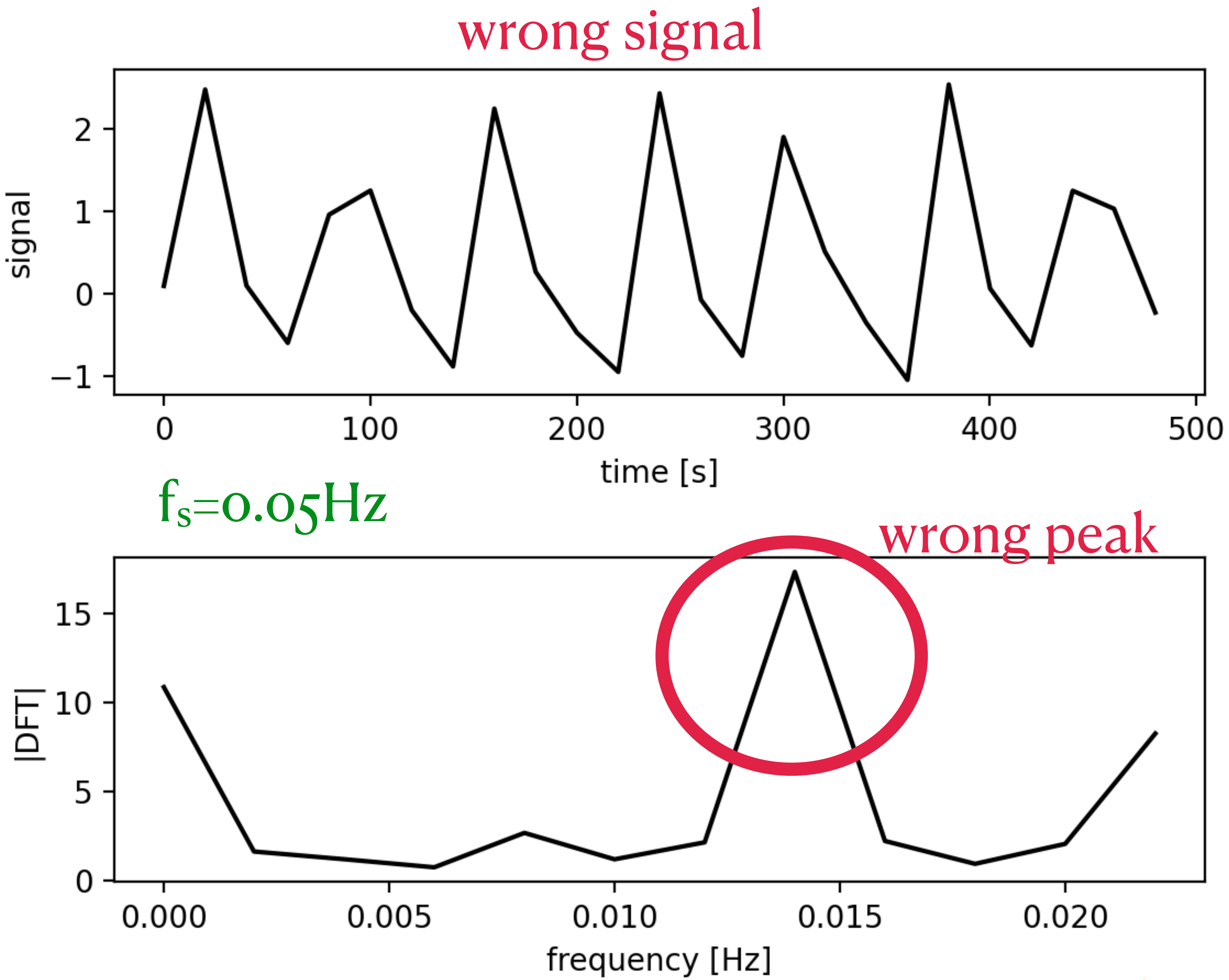



SamplingError_1.py
signal
time [s]
|DFT|
frequency [Hz]
f_s=0.5Hz
f_s=0.2Hz
wrong peak



wrong signal
SamplingError_1.py
signal
time [s]
wrong peak
f_s=0.1Hz
|DFT|
frequency [Hz]





SamplingError_1.py

data sampling

Fourier analysis

errors in analysis **aliasing** **spectral power**

linear filters

time-frequency analysis

data sampling

Fourier analysis

errors in analysis **aliasing for evenly sampled data**

aliasing for unevenly sampled data

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time-frequency analysis

- assume: we have a signal $s(t)$ and its Fourier Transform $X(f)$

- assume: we have a signal s(t) and its Fourier Transform X(f)

- then the Poisson summation formula yields:

$$\boxed{\sum_{k=1}^{N} s(t_k) e^{-i2\pi nk/N} = \frac{1}{\Delta t} \sum_{p=-\infty}^{\infty} X(f_n - p f_s)}$$

DFT

periodic continuation of Fourier transform with period f_s

$$\mathrm{DFT} = X_A \sim X(f_n) + X(f_n - f_s) + X(f_n + f_s) + \cdots$$

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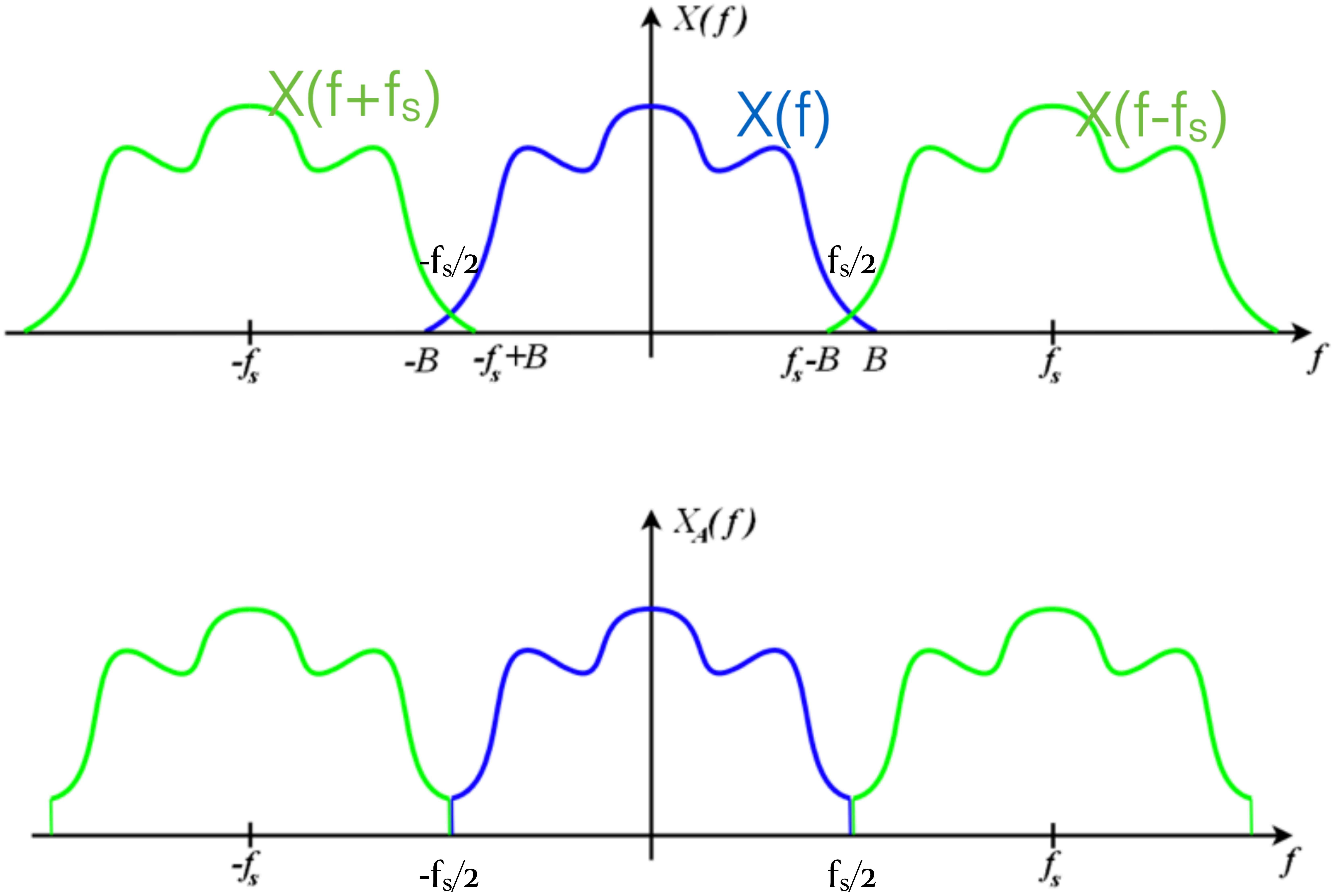
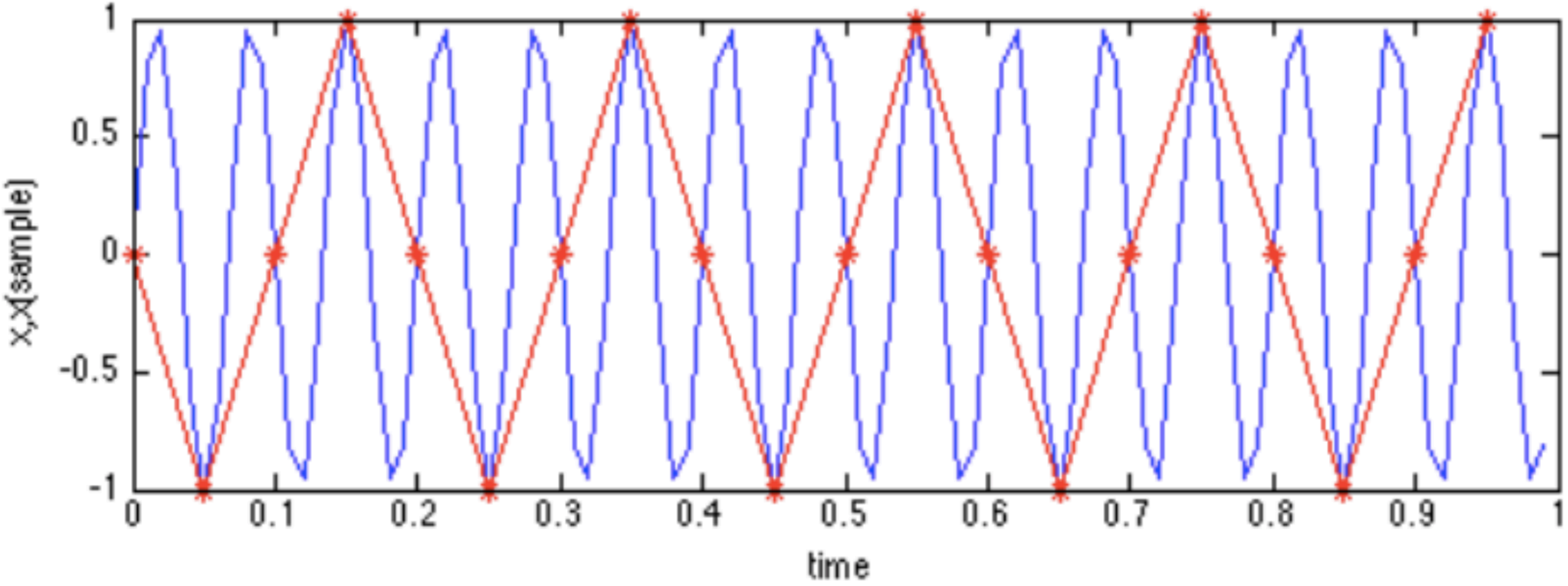
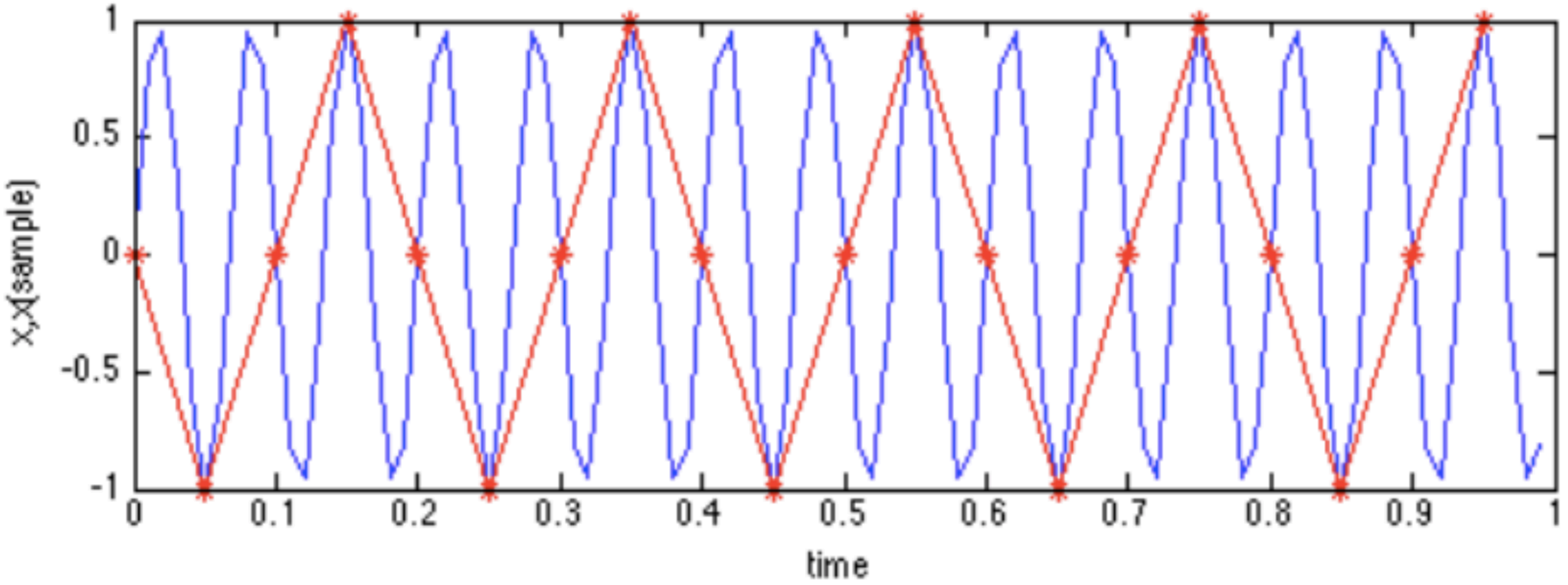


Example: sub-sampled periodic signal



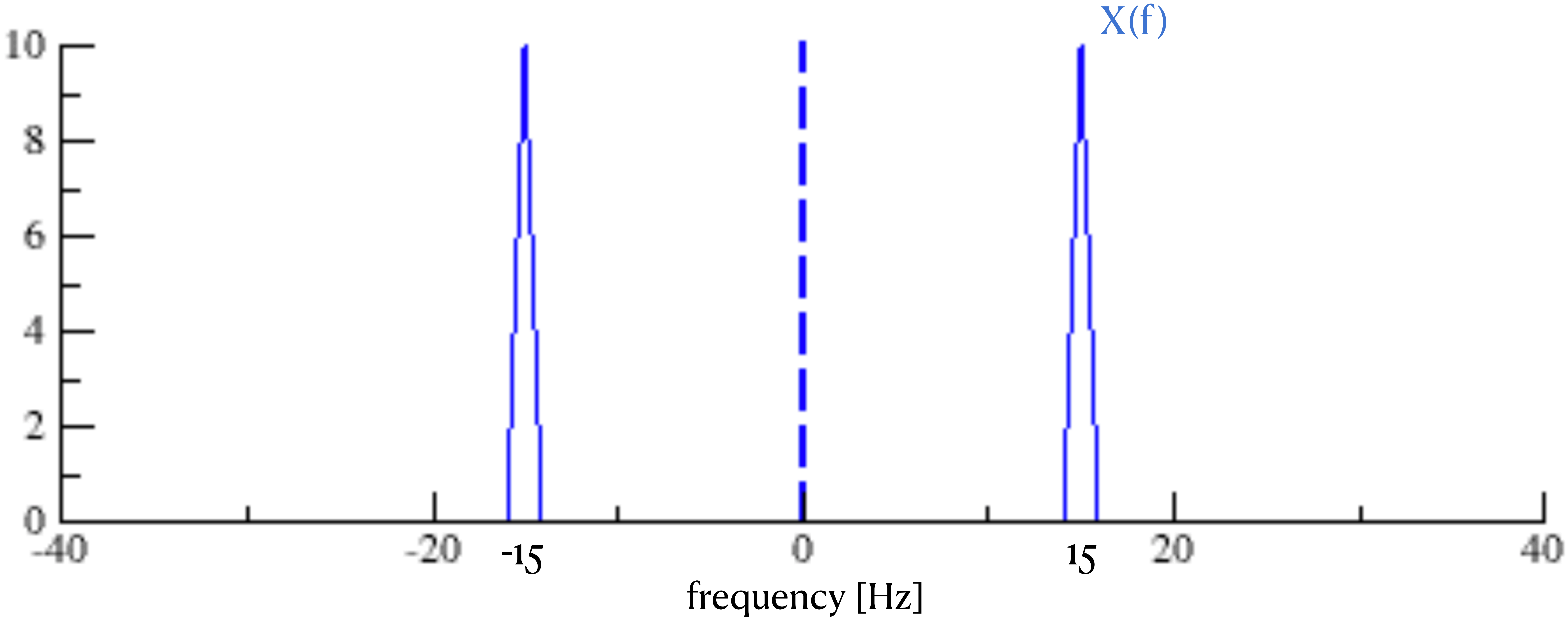
original time series with f_m=15Hz

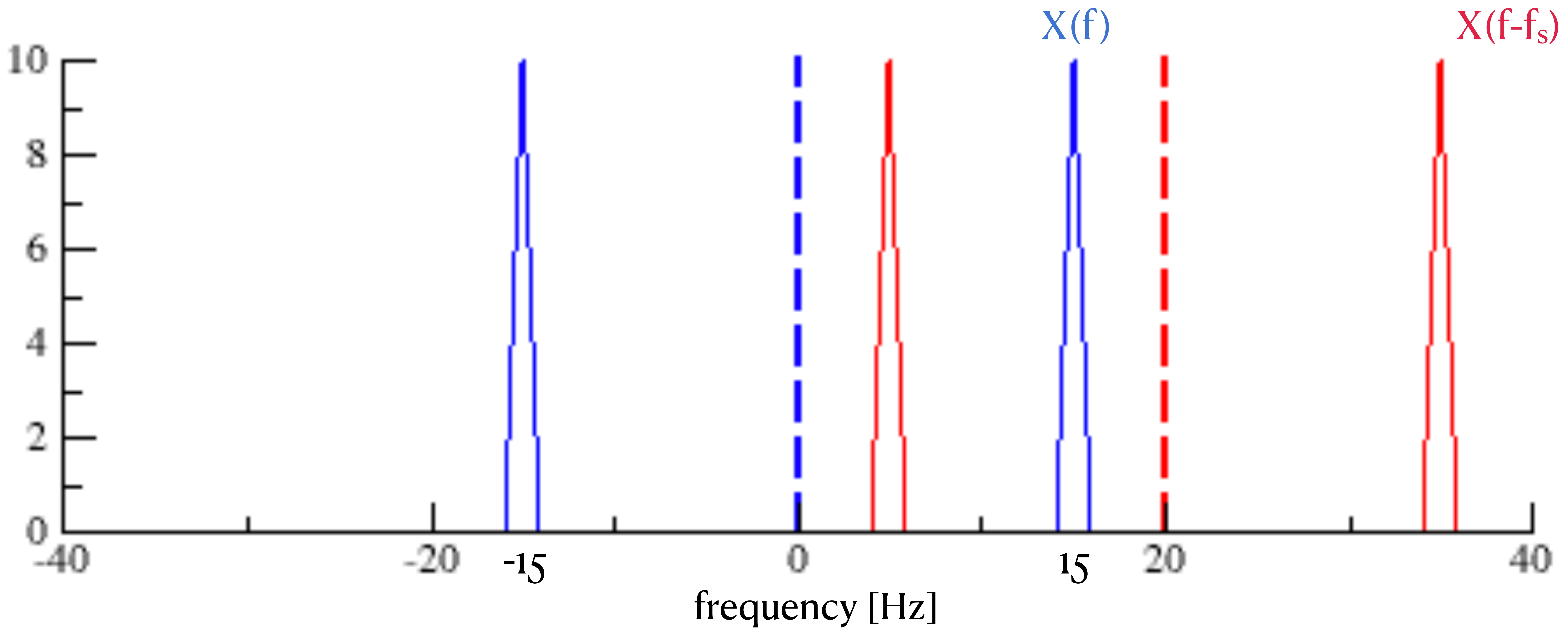
subsampled time series with f_s=20Hz



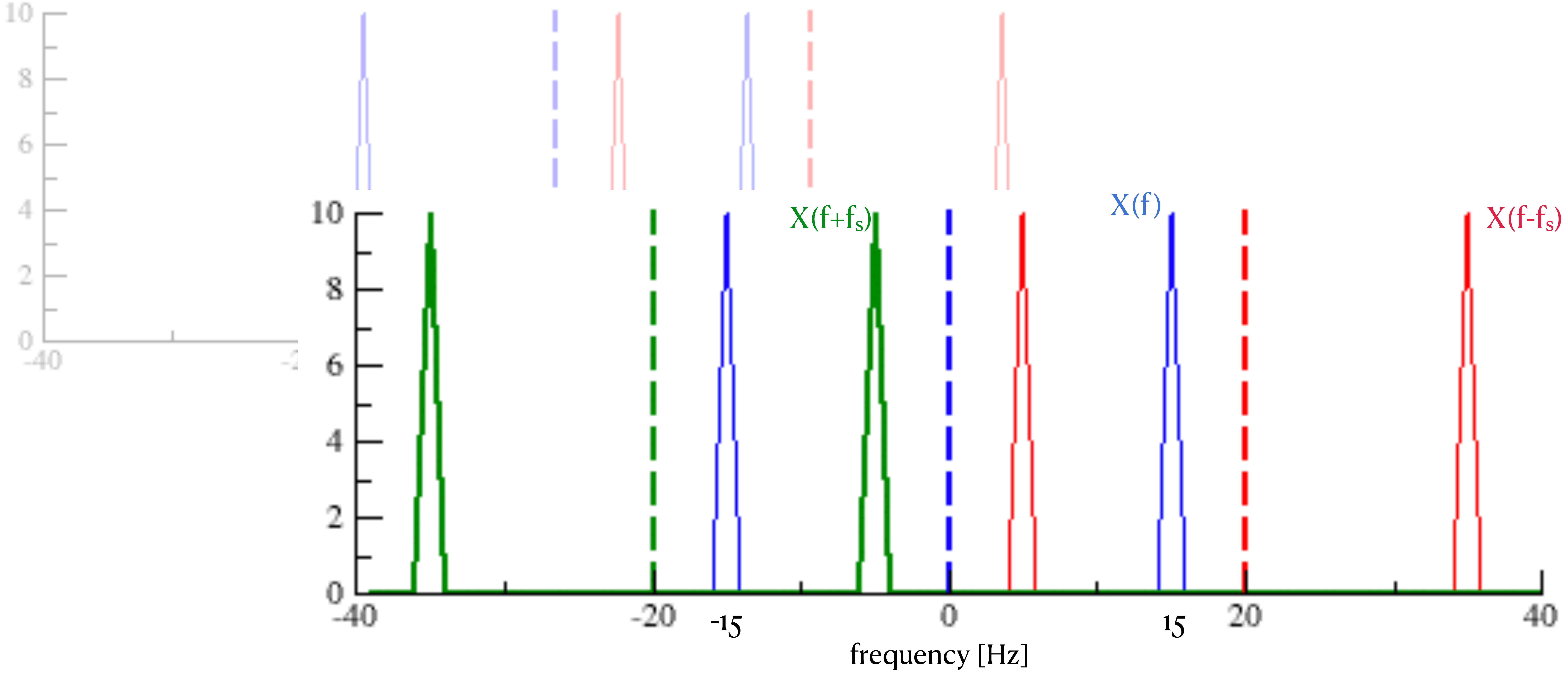
original time series with f_m=15Hz

subsampled time series with f_s=20Hz

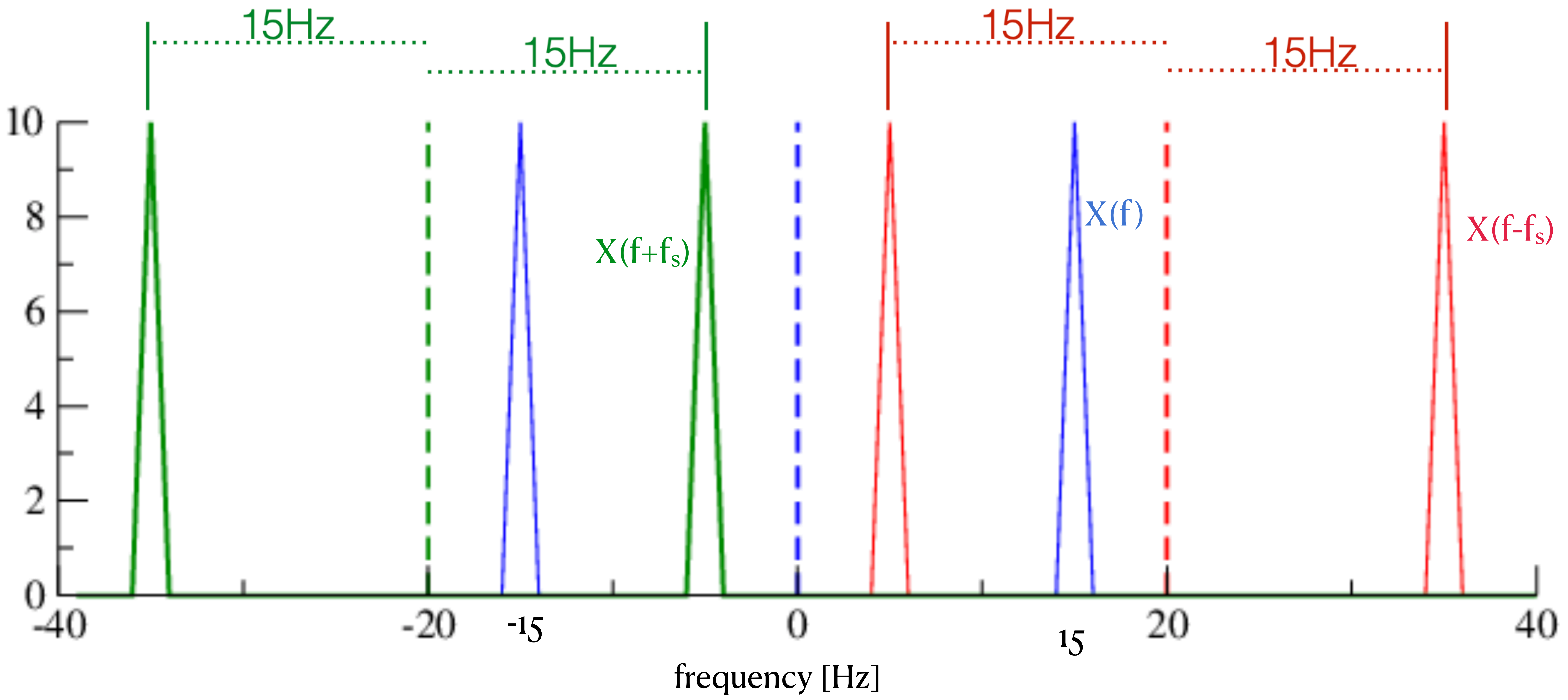




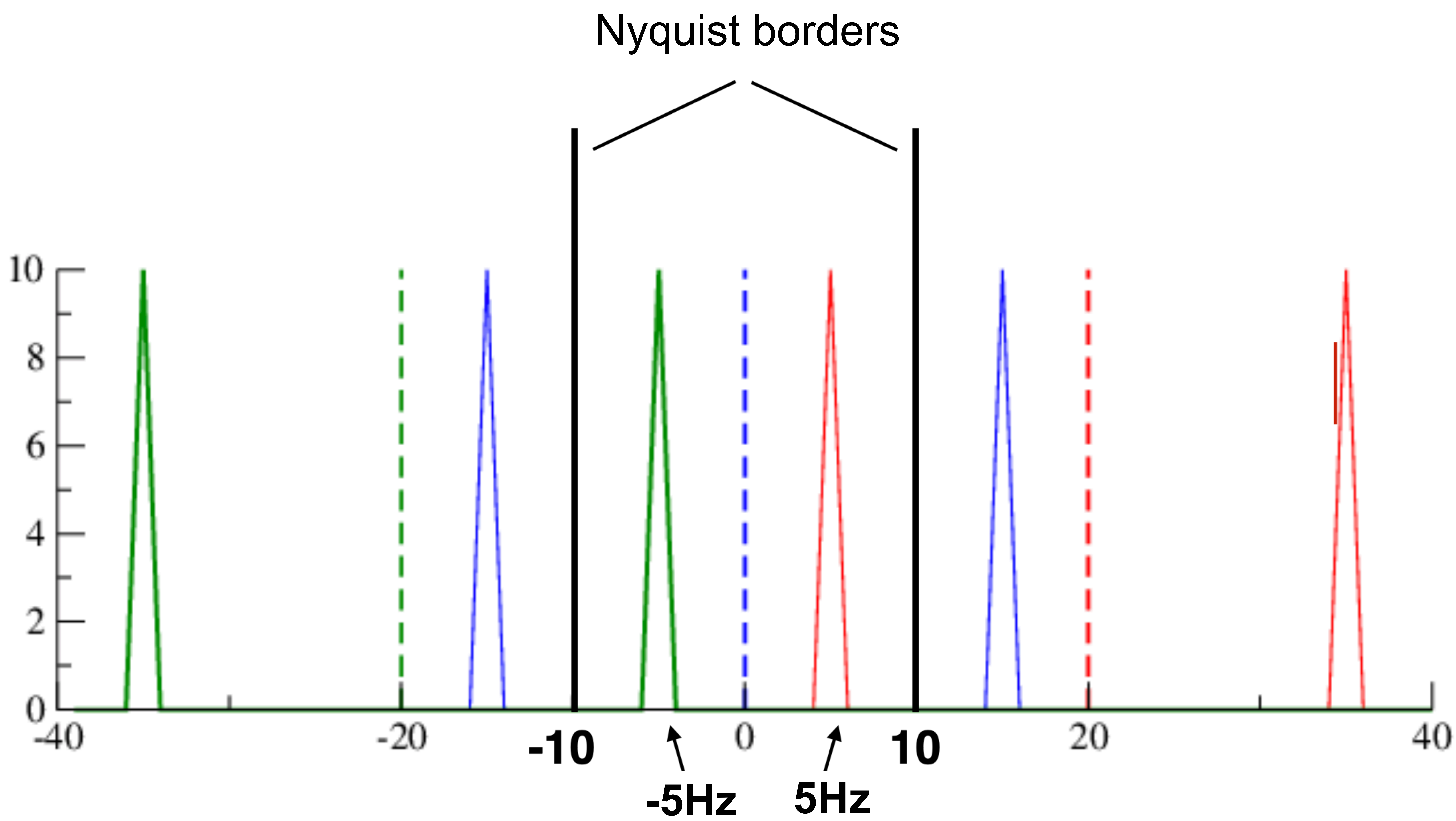
X(f)
X(f-fs)
10
8
6
4
2
0
-40
-20
-15
0
15
20
40
frequency [Hz]



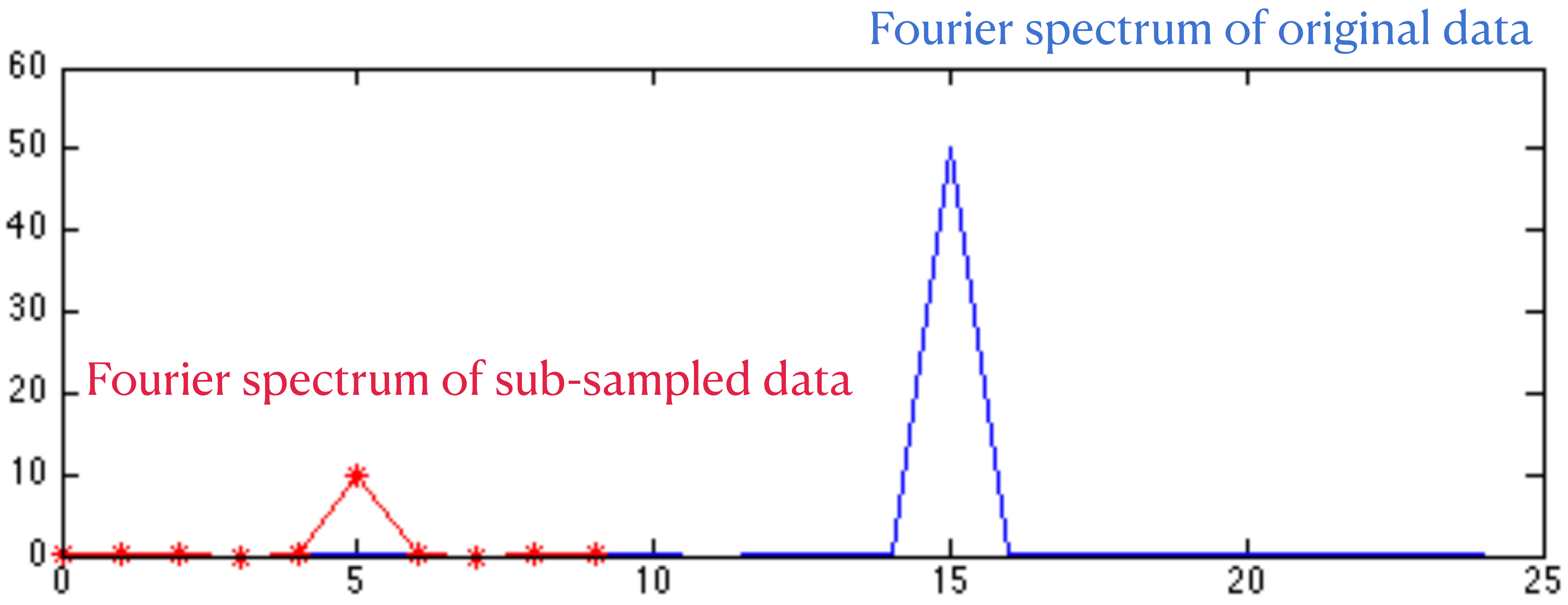
X(f+fs)
X(f)
X(f-fs)
-40
-20
-15
0
15
20
40
frequency [Hz]



15Hz
15Hz
15Hz
15Hz
10
8
6
4
2
0
X(f+fs)
X(f)
X(f-fs)
-40
-20
-15
0
15
20
40
frequency [Hz]

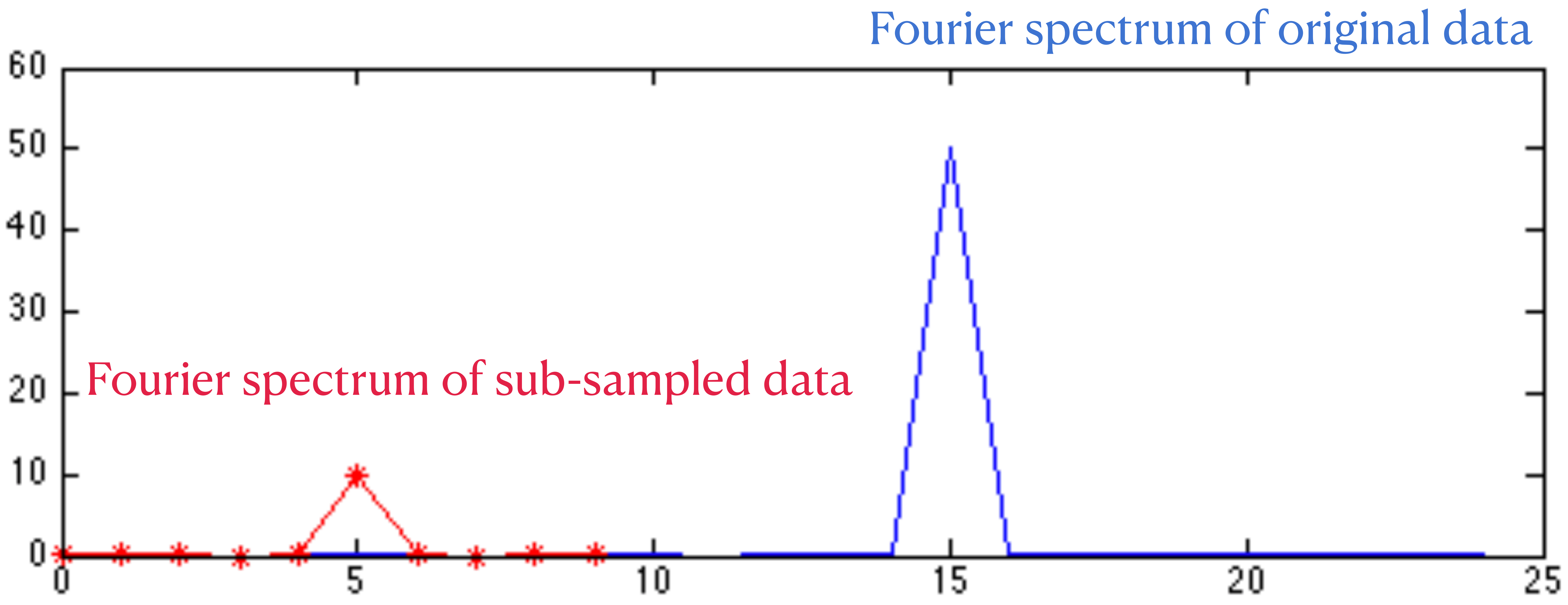


Nyquist borders
10
8
6
4
2
0
-40
-20
-10
0
10
20
40
-5Hz
5Hz



Fourier spectrum of original data
Fourier spectrum of sub-sampled data
60
50
40
30
20
10
0
0
5
10
15
20
25

aliasing



erroneous peak

Importance of aliasing:
resting-state functional Magnetic Resonance Imaging (rs-fMRI)

in resting state, major resting-state physiological rhythms are

- cardiac rhythm of 60-80 beats per minute, i.e. ~ **1.0 - 1.3 Hz**
- respiratory rhythm of 12-15 breaths per minute, i.e. ~ **0.2 - 0.25 Hz**

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in resting state fMRI, sampling rate may be ~ **0.5 - 2.0 Hz**

		cardiac	respiratory
resulting frequencies:	f_s=0.5Hz	f_c=1.3Hz	f_r=0.2Hz
expected frequency peaks in spectrum		0.0Hz+1.3Hz 0.0Hz-1.3Hz	0.0Hz+0.2Hz 0.0Hz-0.2Hz

		cardiac	respiratory
resulting frequencies:	f_s=0.5Hz	f_c=1.3Hz	f_r=0.2Hz

	copy of spectrum	-0.5Hz+1.3Hz -0.5Hz-1.3Hz	-0.5Hz+0.2Hz -0.5Hz-0.2Hz
expected frequency peaks in spectrum		0.0Hz+1.3Hz 0.0Hz-1.3Hz	0.0Hz+0.2Hz 0.0Hz-0.2Hz
	copy of spectrum	0.5Hz+1.3Hz 0.5Hz-1.3Hz	0.5Hz+0.2Hz 0.5Hz-0.2Hz

		cardiac	respiratory
resulting frequencies:	f_s=0.5Hz	f_c=1.3Hz	f_r=0.2Hz
expected frequency peaks in spectrum		-1.5Hz+1.3Hz	-1.5Hz+0.2Hz
		-1.5Hz-1.3Hz	-1.5Hz-0.2Hz
		-1.0Hz+1.3Hz	-1.0Hz+0.2Hz
		-1.0Hz-1.3Hz	-1.0Hz-0.2Hz
		-0.5Hz+1.3Hz	-0.5Hz+0.2Hz
		-0.5Hz-1.3Hz	-0.5Hz-0.2Hz
		0.0Hz+1.3Hz	0.0Hz+0.2Hz
		0.0Hz-1.3Hz	0.0Hz-0.2Hz
		0.5Hz+1.3Hz	0.5Hz+0.2Hz
		0.5Hz-1.3Hz	0.5Hz-0.2Hz
		1.0Hz+1.3Hz	1.0Hz+0.2Hz
		1.0Hz-1.3Hz	1.0Hz-0.2Hz
		1.5Hz+1.3Hz	1.5Hz+0.2Hz
		1.5Hz-1.3Hz	1.5Hz-0.2Hz

		cardiac	respiratory
resulting frequencies:	f_s=0.5Hz	f_c=1.3Hz	f_r=0.2Hz
expected frequency peaks in spectrum		-0.2Hz	-1.3Hz
		-2.8Hz	-1.7Hz
		0.3Hz	-0.8Hz
		-2.3Hz	-1.2Hz
		0.8Hz	-0.3Hz
		-1.8Hz	-0.7Hz
		1.3Hz	0.2Hz
		-1.3Hz	-0.2Hz
		1.8Hz	0.7Hz
		-0.8Hz	0.3Hz
		2.3Hz	1.2Hz
		-0.3Hz	0.8Hz
		2.8Hz	1.7Hz
		0.2Hz	1.3Hz

resulting frequencies:	$f_s=0.5$Hz	cardiac $f_c=1.3$Hz	respiratory $f_r=0.2$Hz
		-0.2Hz	-1.3Hz
		-2.8Hz	-1.7Hz
		0.3Hz	-0.8Hz
		-2.3Hz	-1.2Hz
		0.8Hz	-0.3Hz
		-1.8Hz	-0.7Hz
$0 < f < f_s/2$		1.3Hz	**0.2Hz**
		-1.3Hz	-0.2Hz
		1.8Hz	0.7Hz
		-0.8Hz	0.3Hz
		2.3Hz	1.2Hz
		-0.3Hz	0.8Hz
		2.8Hz	1.7Hz
		0.2Hz	1.3Hz

resulting frequencies: f_s=0.5Hz

	cardiac f_c=1.3Hz	respiratory f_r=0.2Hz
	-0.2Hz	-1.3Hz
	-2.8Hz	-1.7Hz
	0.3Hz	-0.8Hz
	-2.3Hz	-1.2Hz
	0.8Hz	-0.3Hz
	-1.8Hz	-0.7Hz
$0 < f < f_s/2$	1.3Hz	**0.2Hz**
	-1.3Hz	-0.2Hz
	1.8Hz	0.7Hz
	-0.8Hz	0.3Hz
	2.3Hz	1.2Hz
	-0.3Hz	0.8Hz
	2.8Hz	1.7Hz
aliased rhythm	**0.2Hz**	1.3Hz

		cardiac	respiratory
resulting frequencies:	f_s=**2.0**Hz	f_c=1.3Hz	f_r=0.2Hz

expected frequency peaks in spectrum	copy of spectrum	-2.0Hz+1.3Hz	-2.0Hz+0.2Hz
		-2.0Hz-1.3Hz	-2.0Hz-0.2Hz
		0.0Hz+1.3Hz	0.0Hz+0.2Hz
		0.0Hz-1.3Hz	0.0Hz-0.2Hz
	copy of spectrum	2.0Hz+1.3Hz	2.0Hz+0.2Hz
		2.0Hz-1.3Hz	2.0Hz-0.2Hz

cardiac respiratory

resulting frequencies: f_s=**2.0**Hz f_c=1.3Hz f_r=0.2Hz

	cardiac	respiratory
	-2.0Hz+1.3Hz	-2.0Hz+0.2Hz
	-2.0Hz-1.3Hz	-2.0Hz-0.2Hz
	0.0Hz+1.3Hz	**0.2Hz**
	0.0Hz-1.3Hz	0.0Hz-0.2Hz
	2.0Hz+1.3Hz	2.0Hz+0.2Hz
aliased rhythm →	**0.7Hz**	2.0Hz-0.2Hz

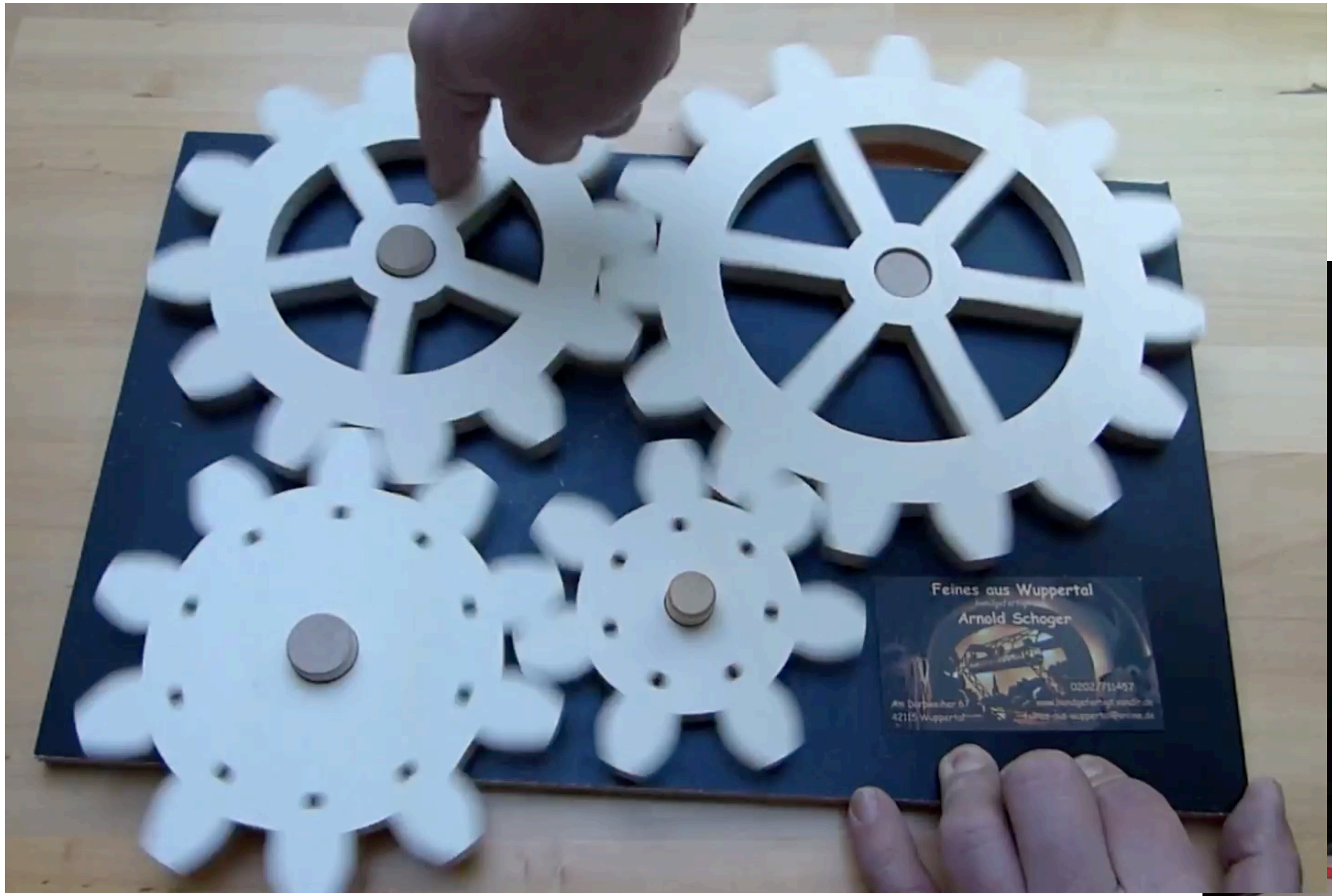
resulting frequencies: f_s=**2.0**Hz

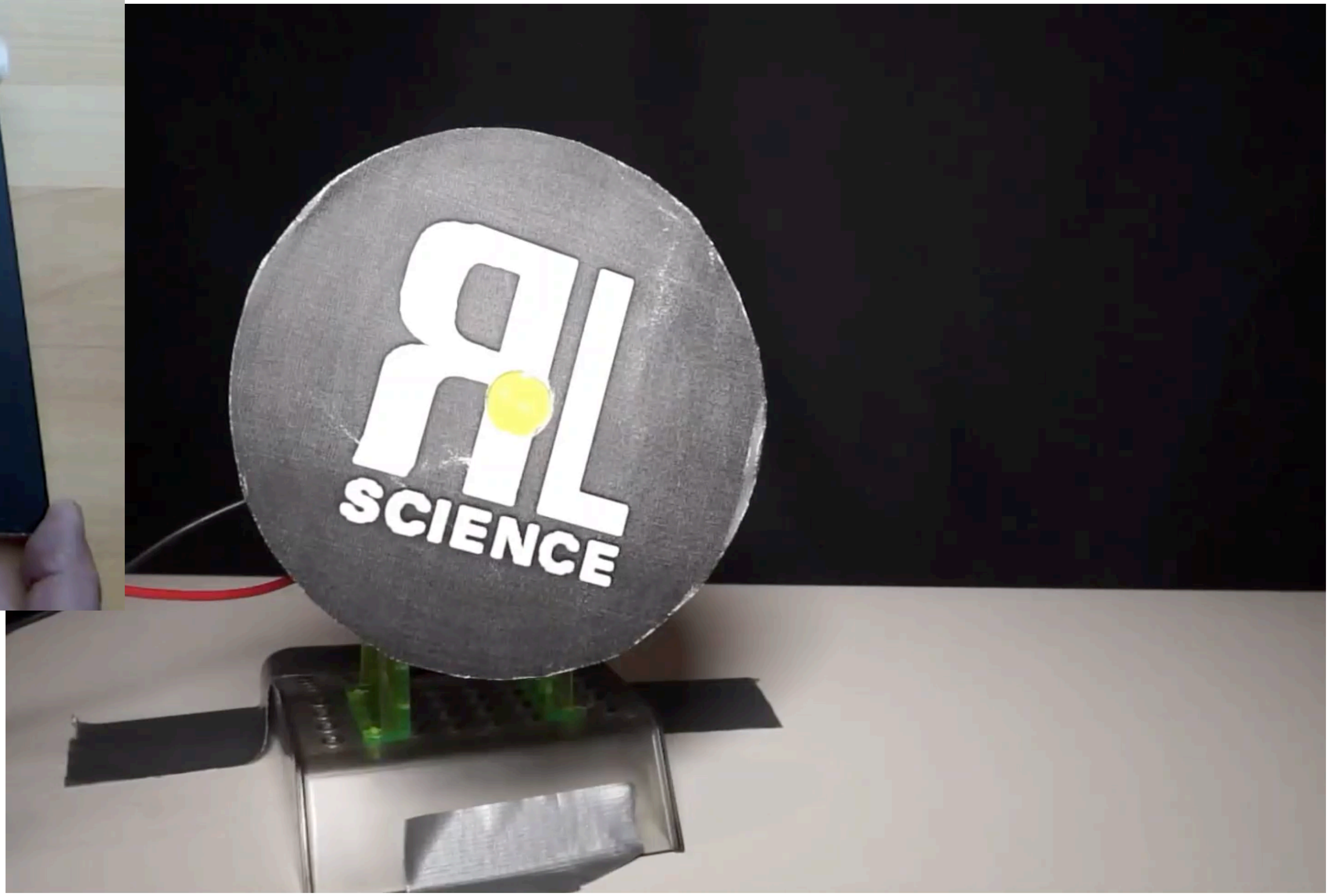
	cardiac	respiratory
	f_c=1.3Hz	f_r=0.2Hz
	-2.0Hz+1.3Hz	-2.0Hz+0.2Hz
	-2.0Hz-1.3Hz	-2.0Hz-0.2Hz
	0.0Hz+1.3Hz	**0.2Hz**
	0.0Hz-1.3Hz	0.0Hz-0.2Hz
	2.0Hz+1.3Hz	2.0Hz+0.2Hz
aliased rhythm →	**0.7Hz**	2.0Hz-0.2Hz

autonomous rhythms affect resting state-fMRI spectrum

(see also Huotari et al. (2019), doi.org/10.3389/fnins.2019.00279)

aliasing = stroboscope effect





data sampling

Fourier analysis

errors in analysis aliasing **spectral power**

linear filters

time-frequency analysis

definition:

power spectral density (PSD)

$$\boxed{S(f_n) = \frac{\Delta t}{N} |\mathrm{DFT}(f_n)|^2}$$

of a sampled signal with duration $T = N\Delta t$

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power spectral density (PSD)

$$\boxed{S(f_n) = \frac{\Delta t}{N} |\mathrm{DFT}(f_n)|^2}$$

of a sampled signal with duration $T = N\Delta t$

For interpretation, the observed system should be **stationary in time (in wider sense)**.

$$E[s(t)]_{t_1}^{t_2} = E[s(t)]_{t_3}^{t_4} = \mathrm{const}$$

temporal average independent of time window

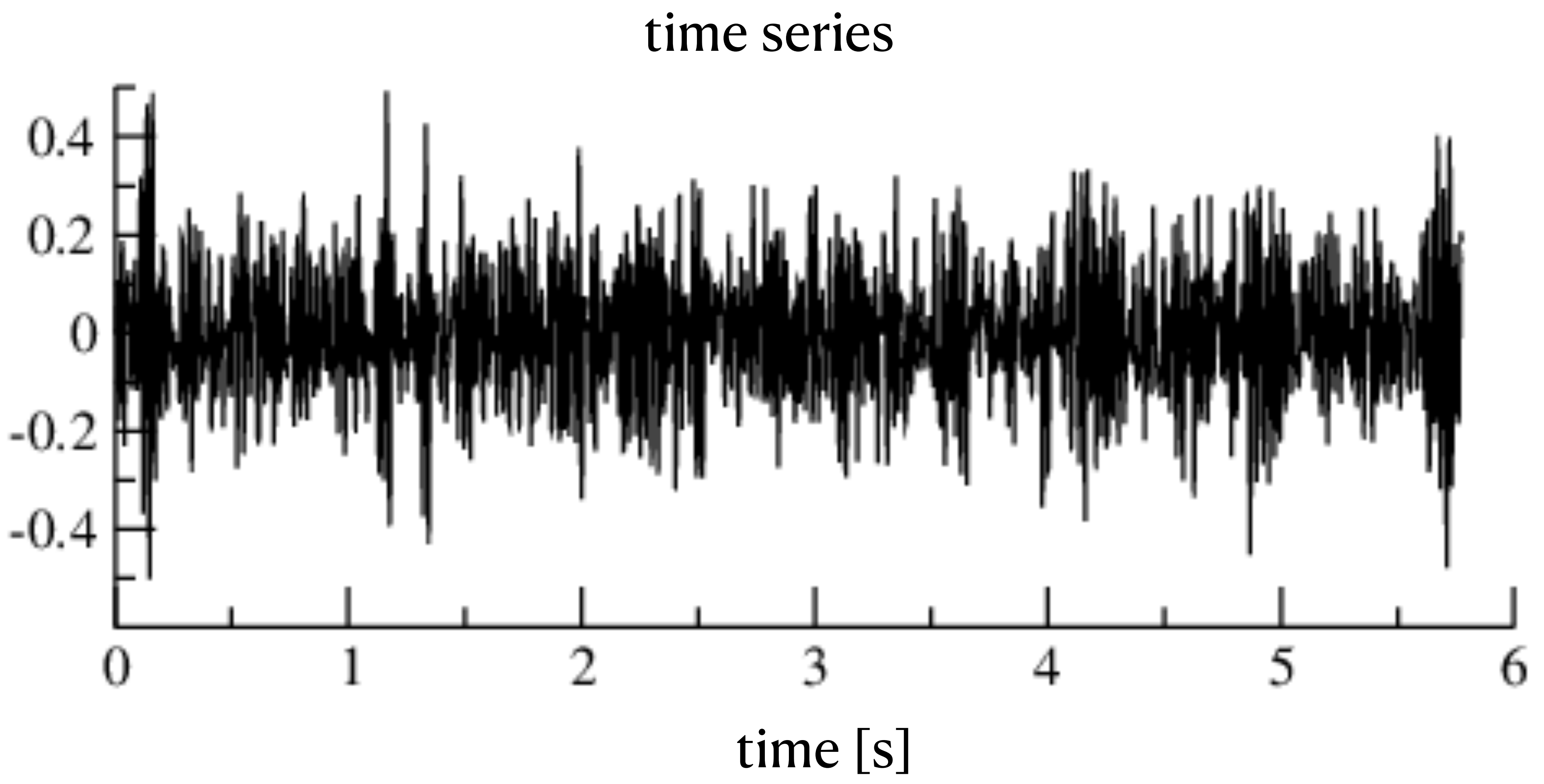
$$Var[s(t)]_{t_1}^{t_2}, Var[s(t)]_{t_3}^{t_4} < \infty$$

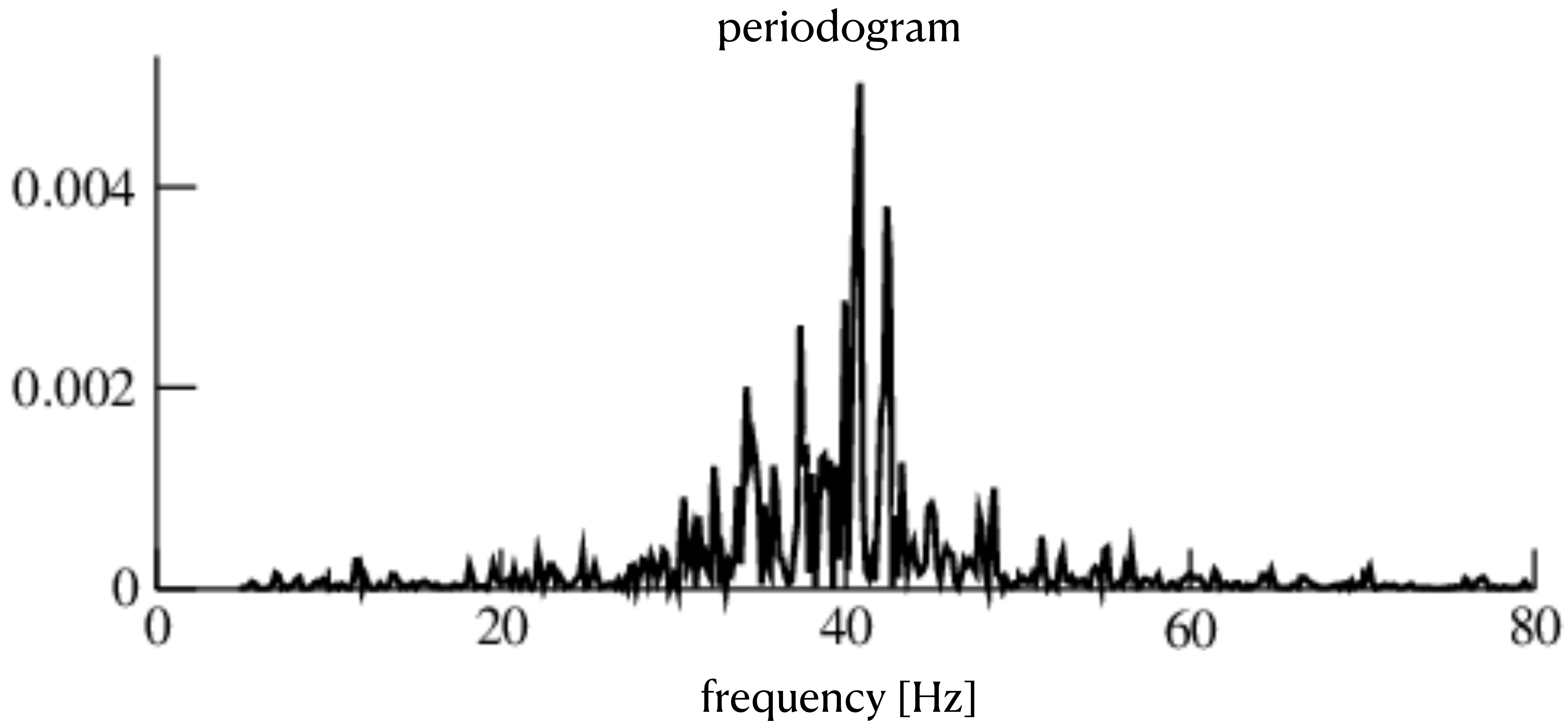
temporal variance is finite

1. Periodogram

$$S(f_n) = \frac{\Delta t}{N} |\mathrm{DFT}(f_n)|^2$$

DFT: computed from full signal





SamplingError_5.py

1. Periodogram

$$S(f_n) = \frac{\Delta t}{N} |\mathrm{DFT}(f_n)|^2$$

DFT: computed from full signal

2. Bartlett method

$$S(f_n) = \frac{\Delta t}{N} |\mathrm{DFT}(f_n)|^2$$

DFT: average over segments

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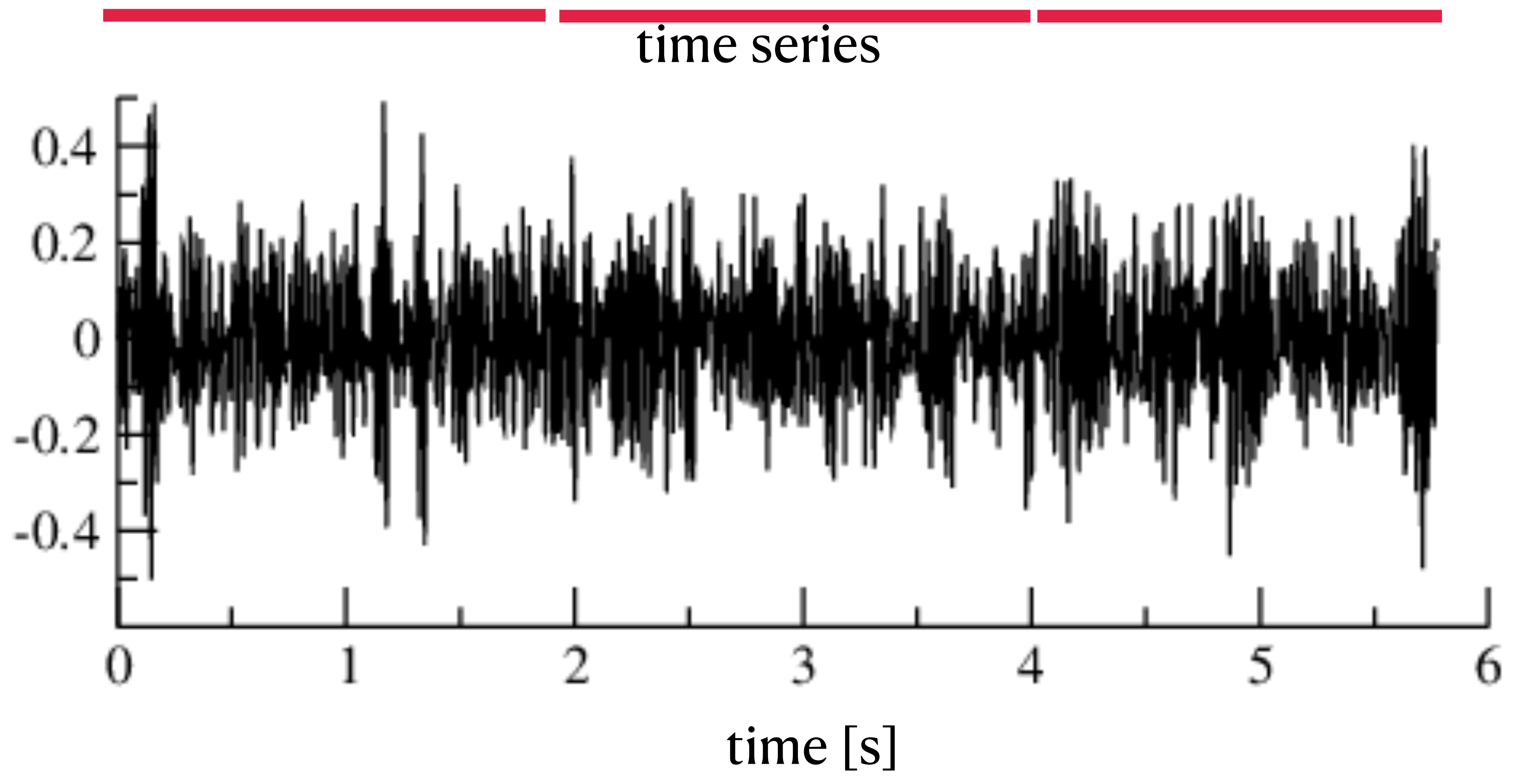

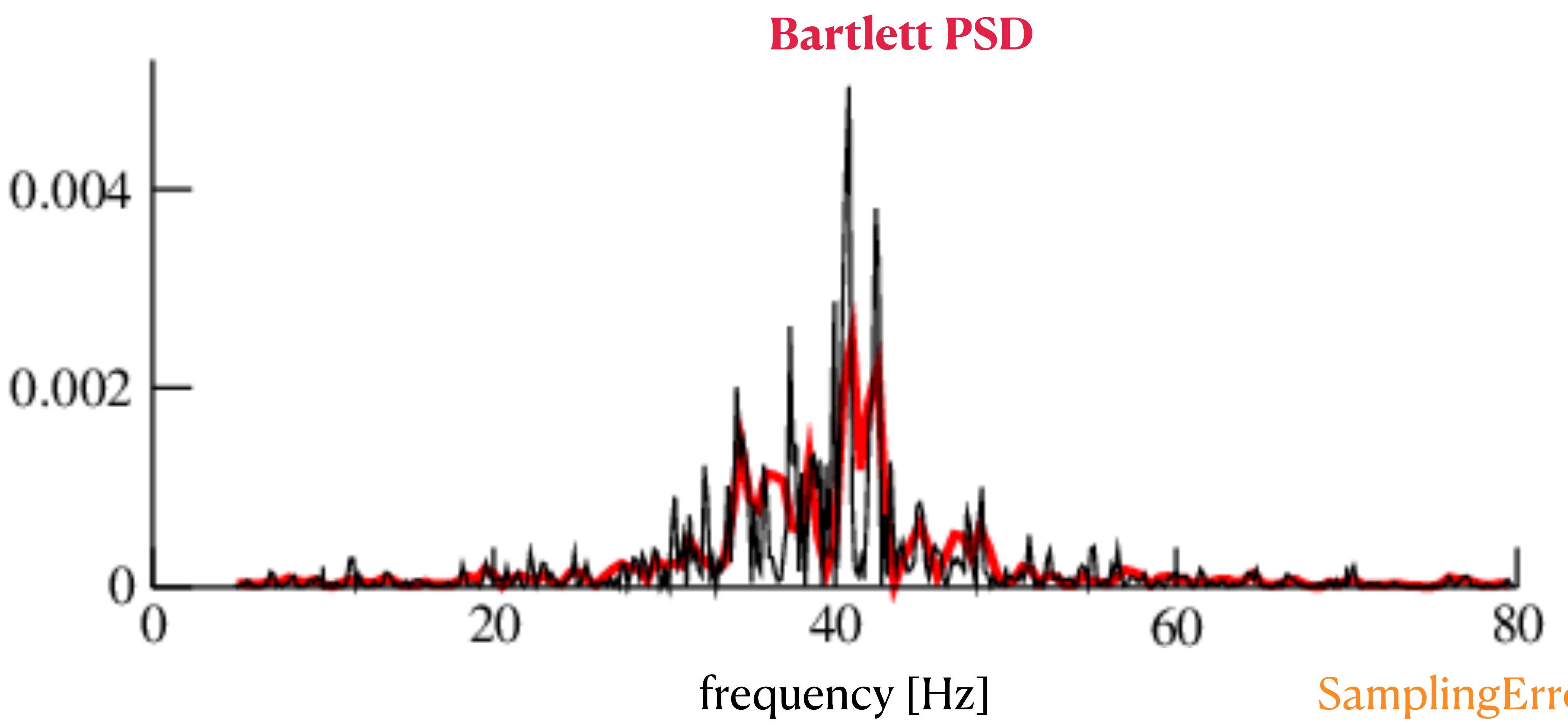

SamplingError_5.py

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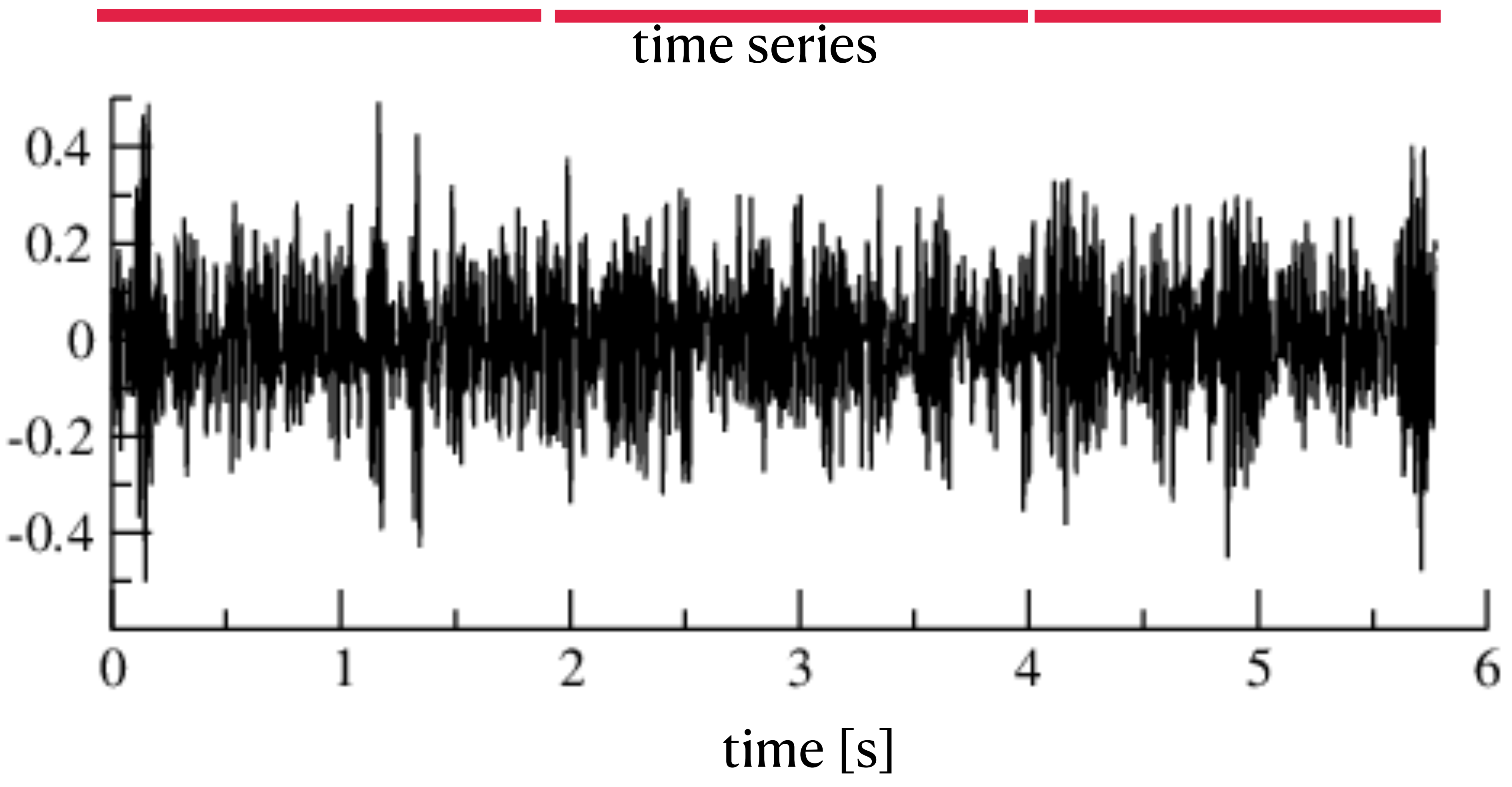
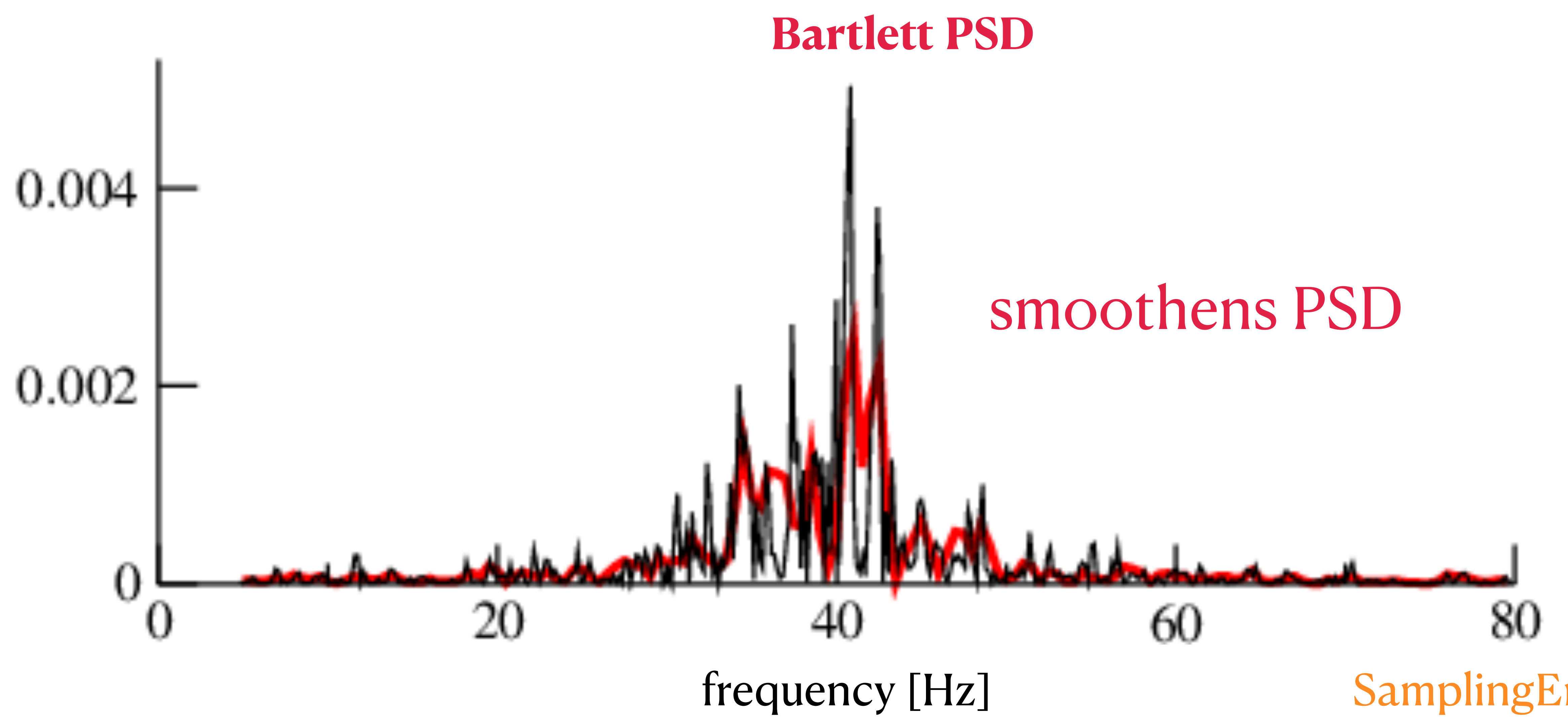


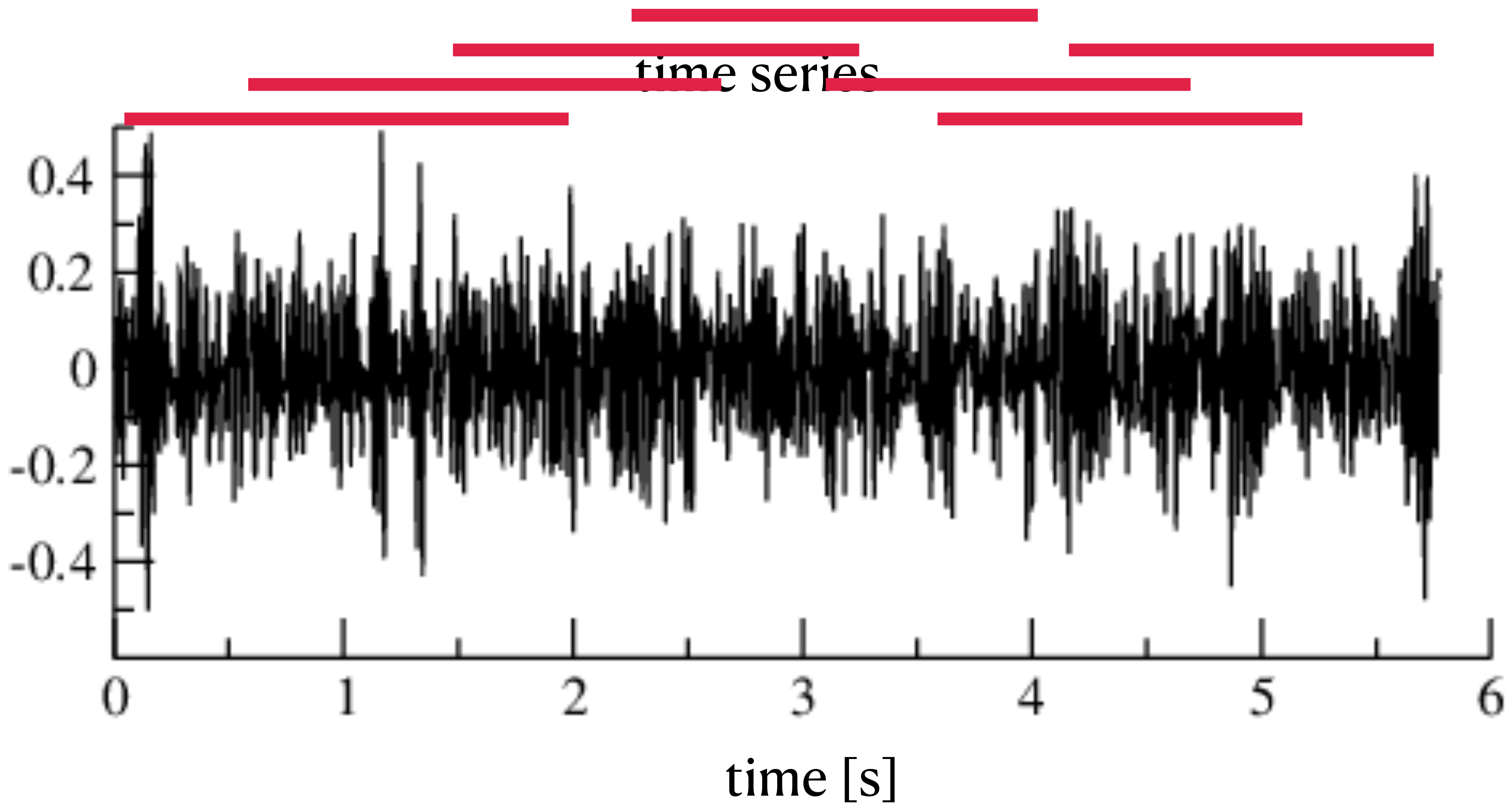


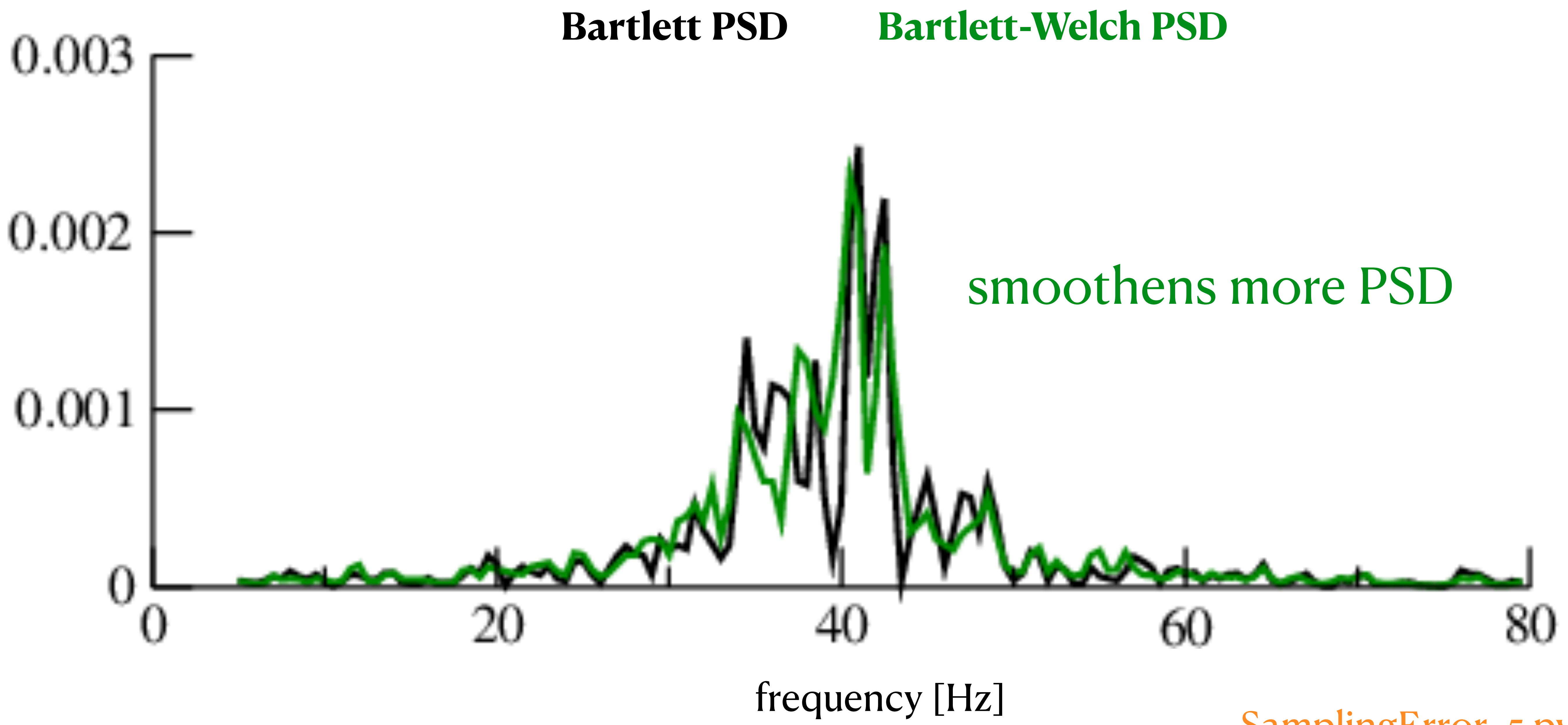
SamplingError_5.py

3. Bartlett-Welch method

$$S(f_n) = \frac{\Delta t}{N} |\mathrm{DFT}(f_n)|^2$$

DFT: average over **overlapping** segments





SamplingError_5.py

4. Multi taper method

$$S(f_n) = \frac{\Delta t}{N} |\mathrm{DFT}(f_n)|^2$$

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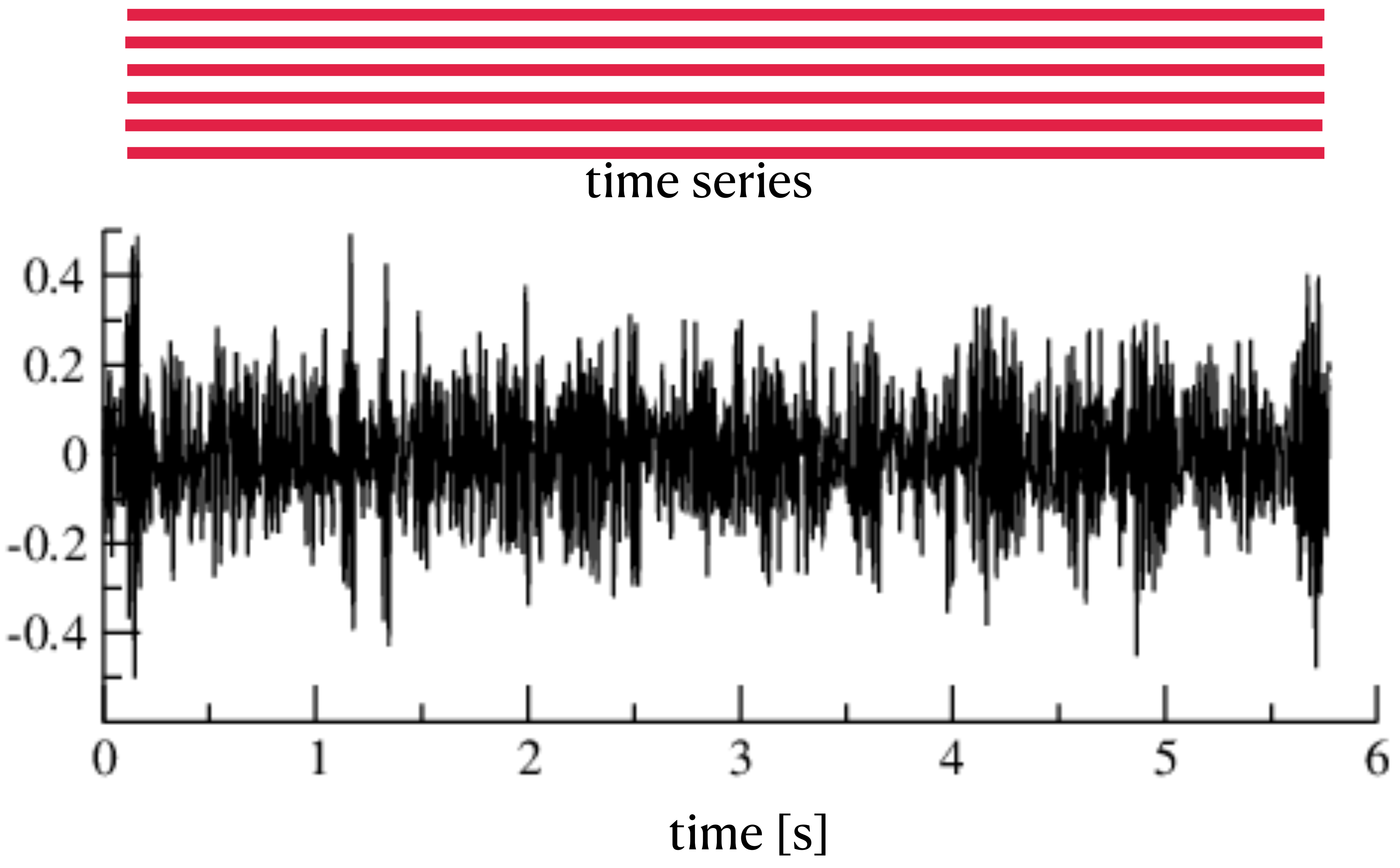
$$S(f_n) = \frac{\Delta t}{N} |\mathrm{DFT}(f_n)|^2$$

$$\mathrm{DFT} = \sum_{k=1}^{N} s(t_k) \underbrace{w(t_k)}_{\text{window function}} e^{-i2\pi nk/N} \sim \sum_{m=-N/2}^{N/2} c_{n-m} d_m$$

4. Multi taper method

$$S(f_n) = \frac{\Delta t}{N} |\mathrm{DFT}(f_n)|^2$$

$$\mathrm{DFT} = \sum_{k=1}^{N} s(t_k) \underbrace{w(t_k)}_{\text{window function}} e^{-i2\pi nk/N} \sim \sum_{m=-N/2}^{N/2} c_{n-m} d_m$$



average over different window functions

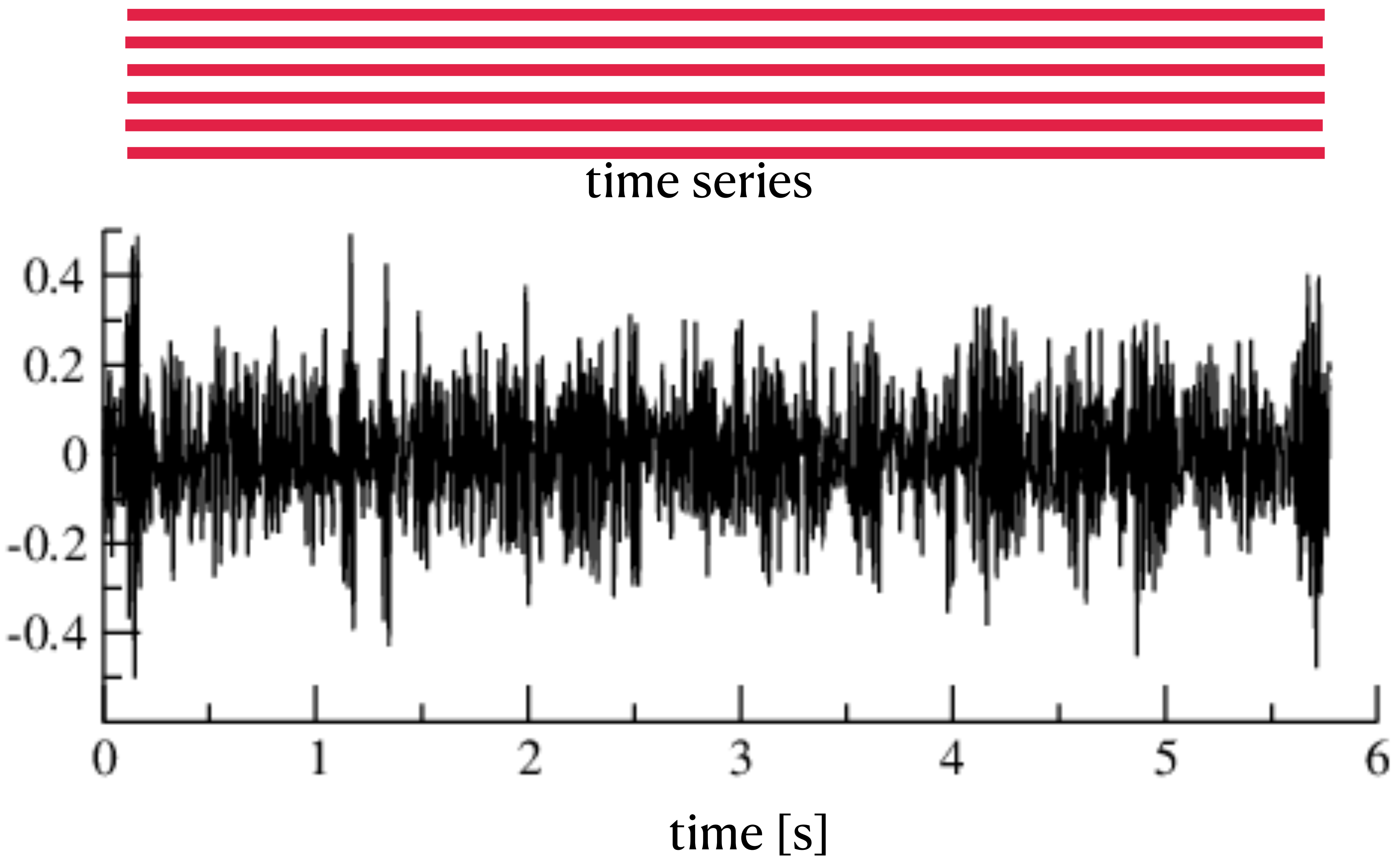
called data tapers

4. Multi taper method

$$S(f_n) = \frac{\Delta t}{N} |\mathrm{DFT}(f_n)|^2$$

$$\mathrm{DFT} = \sum_{k=1}^{N} s(t_k) w(t_k) e^{-i2\pi nk/N} \sim \sum_{m=-N/2}^{N/2} c_{n-m} d_m$$

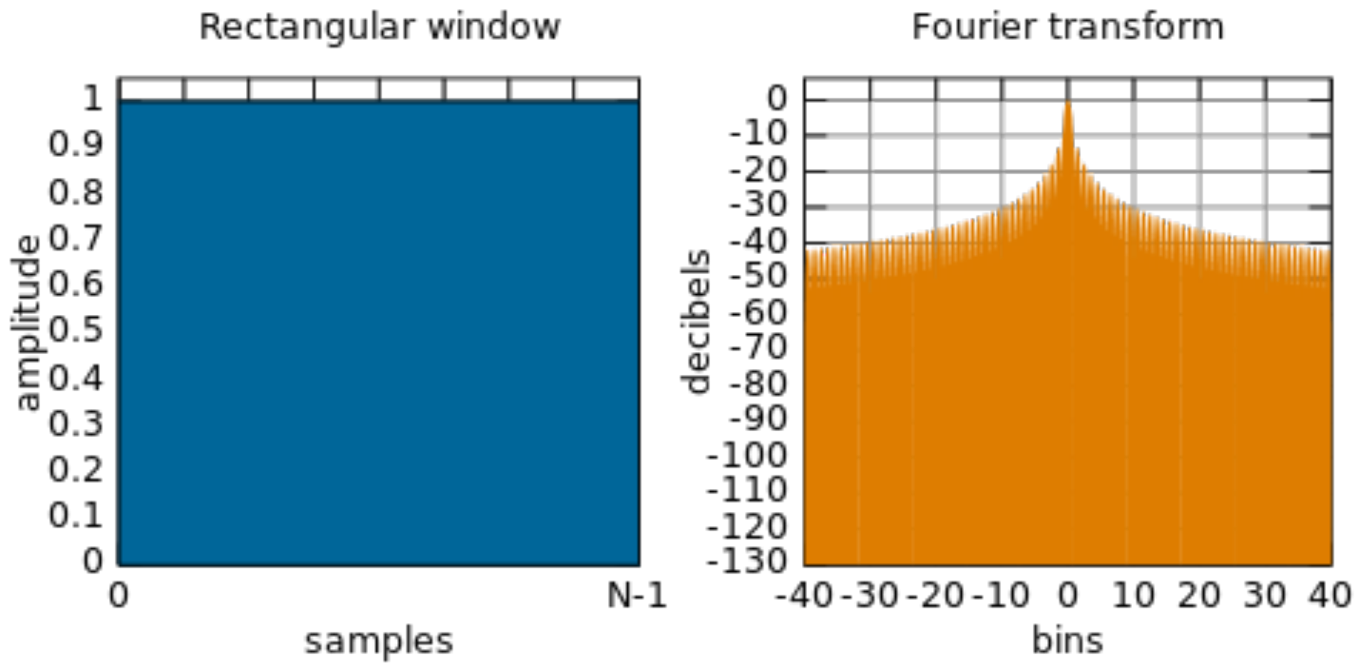
window function



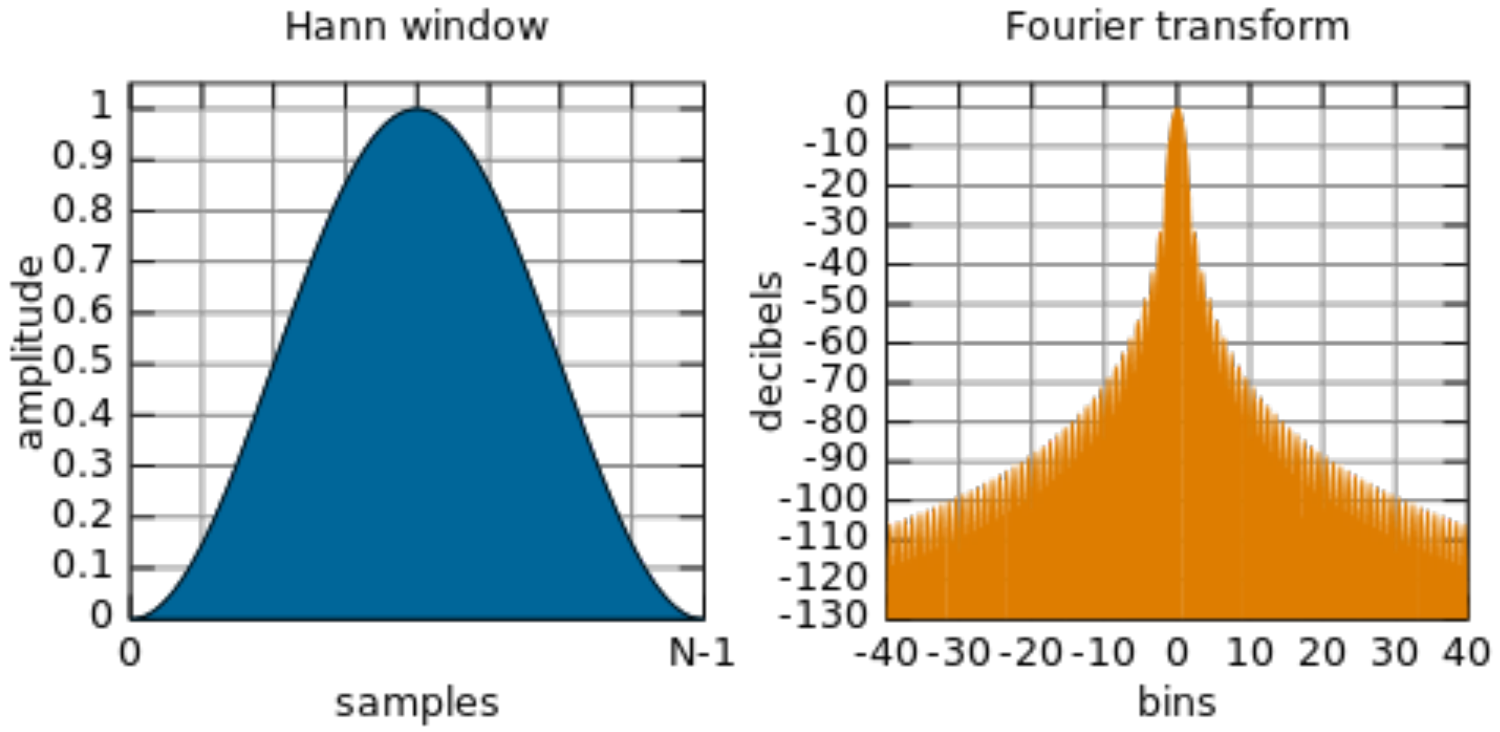
average over different window functions

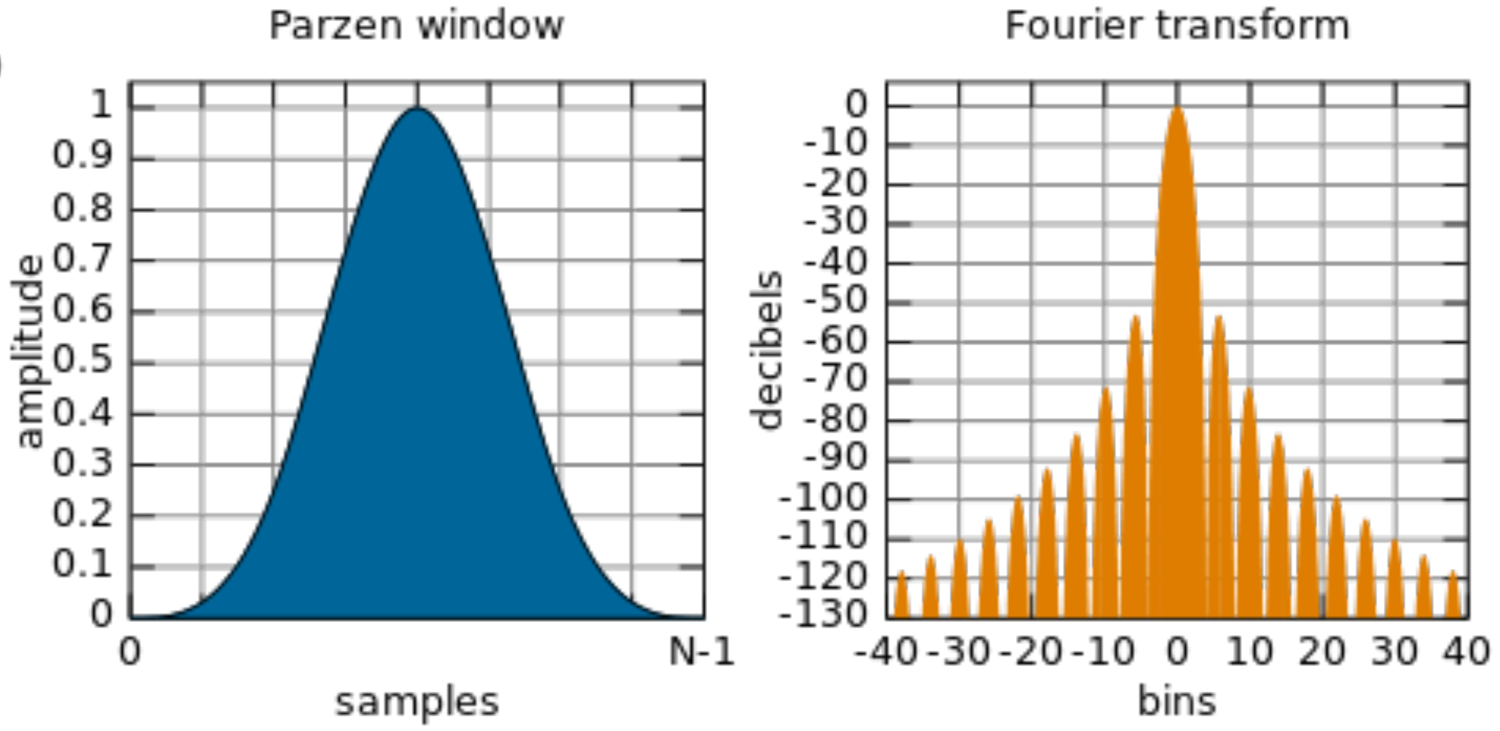
called data tapers

but: which window functions are optimal ?



recall: different window functions possible

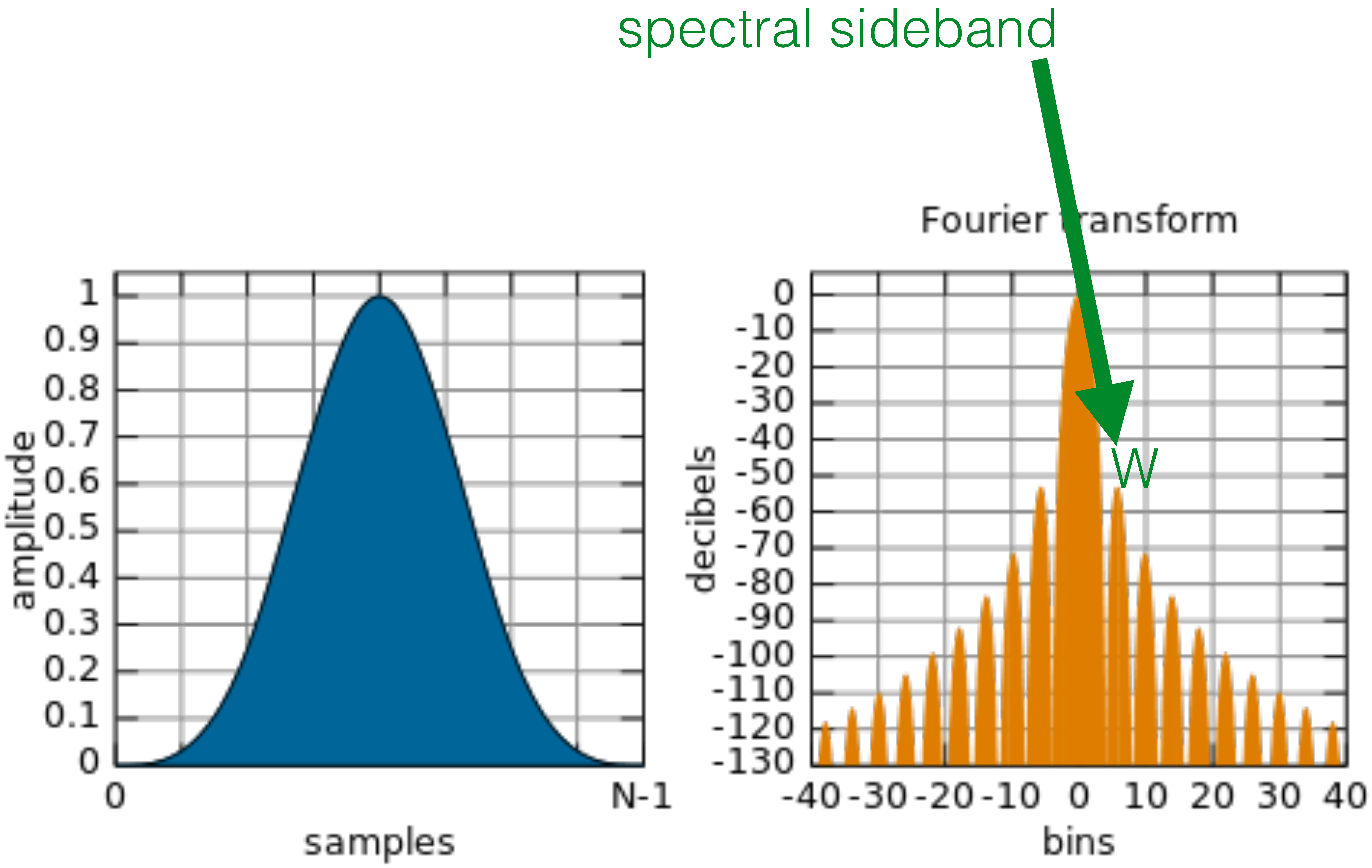




optimization of window function

= minimizing spectral side bands

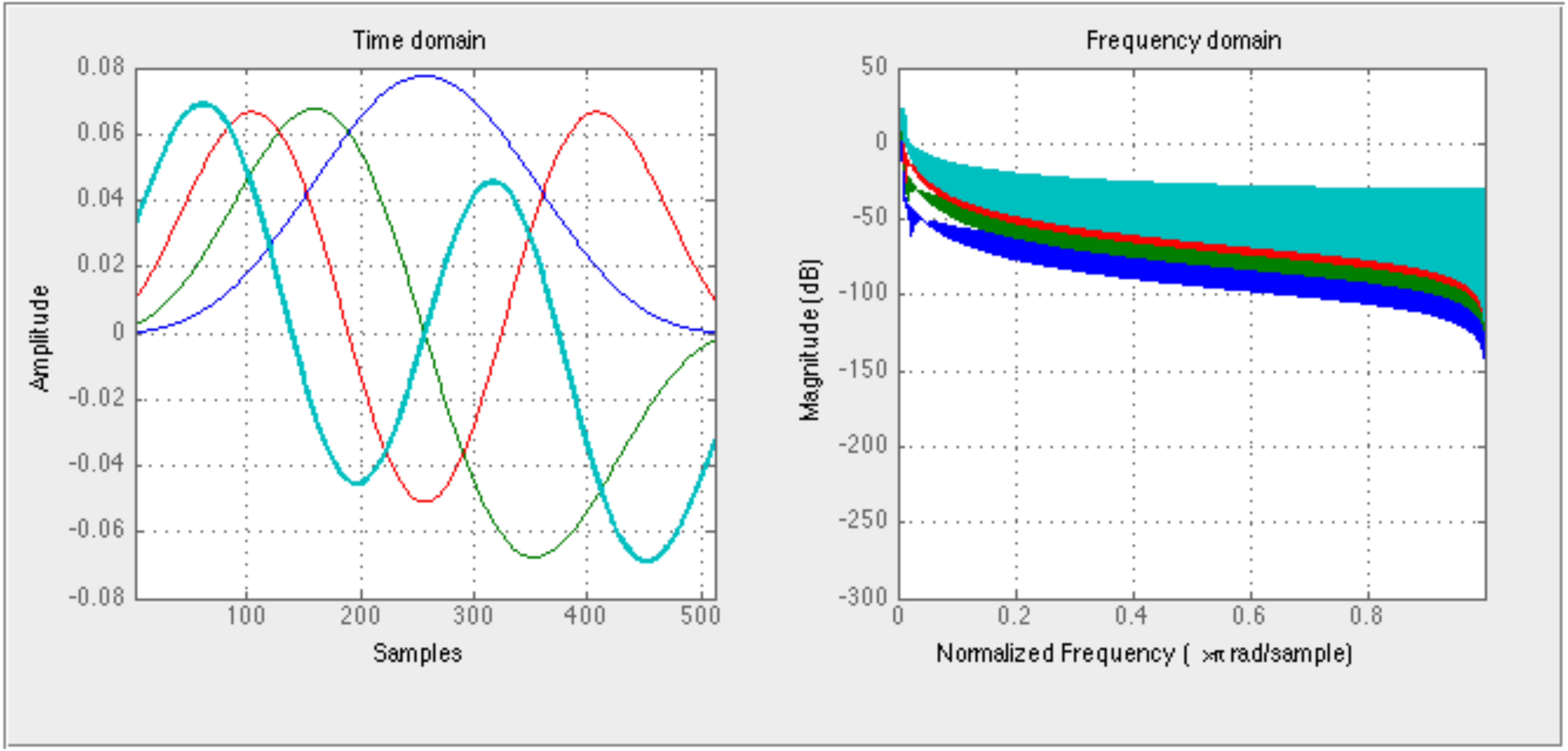
spectral concentration problem



solution:

- given: border frequency W and T data points
- then: there are ***n=2WT*** **orthogonal optimal** window functions
- these *data tapers* are *n* ***Slepian sequences***
- multi taper method:
  weighted average DFT over n *data tapers*

for illustration: 4 Slepian sequences



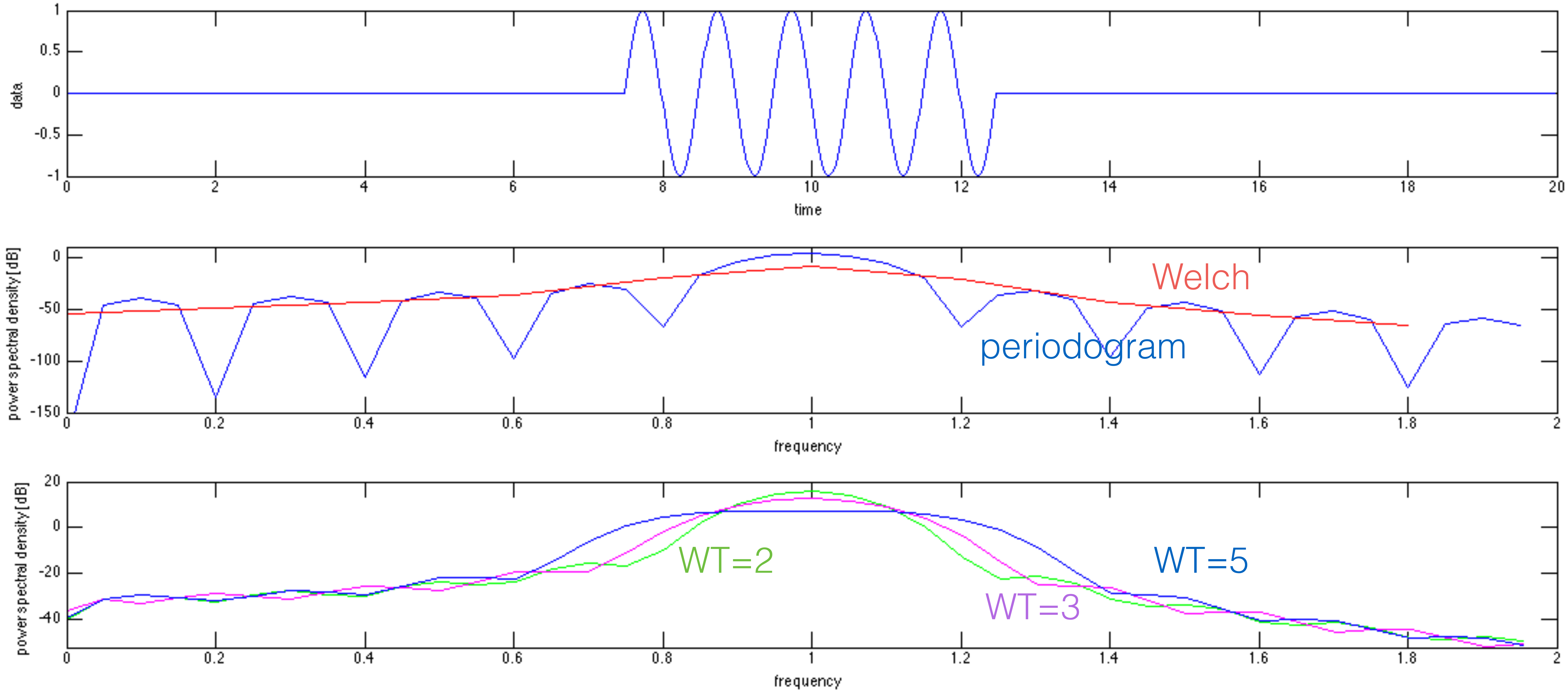
$$W = w\Delta f \ = \frac{w}{T}$$

w: multiple of frequency resolution

$$n = 2\frac{w}{T}T = 2w$$

optimal number of *data taper*

reduction of von *spectral leakage*



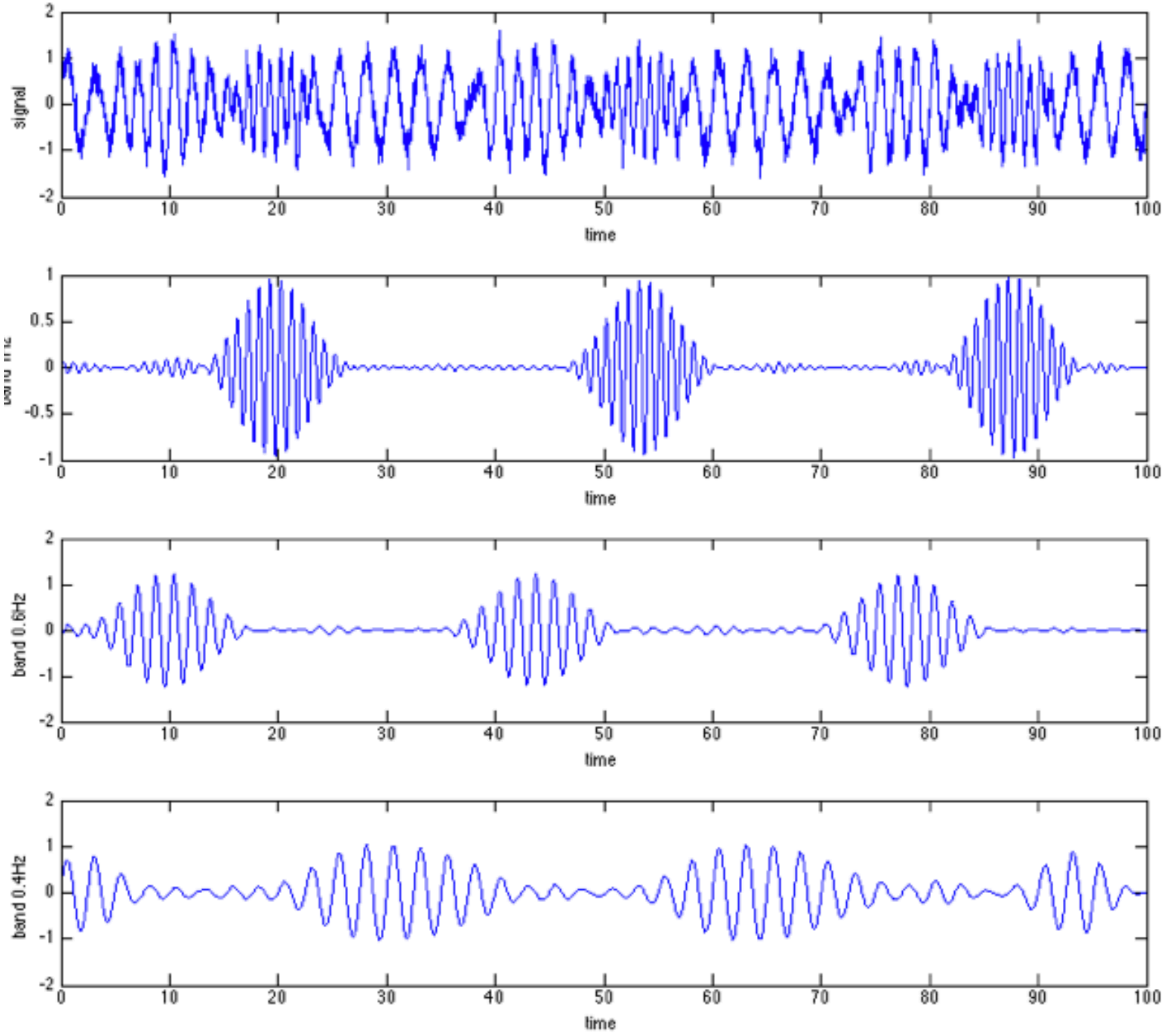
example data set: **transient oscillations**

sum of signals

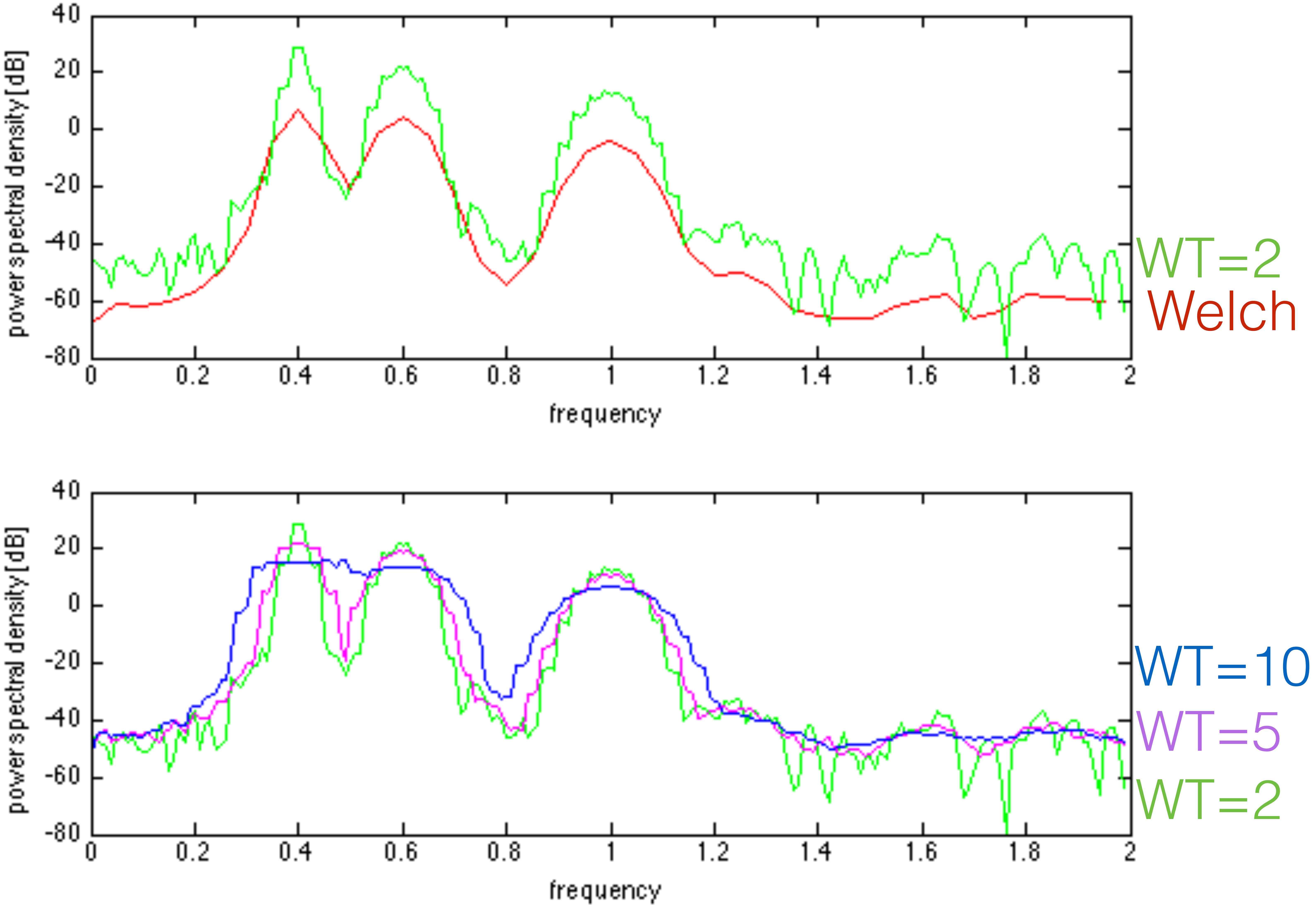
frequency band **1.0**Hz

frequency band **0.6**Hz

frequency band **0.4**Hz



transient oscillations



transient oscillations

multi taper method smoothes power spectrum

Fast Fourier Transform (FFT)

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- efficient implementation of DFT

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- efficient implementation of DFT

- Radix-2 Algorithmus (Cooley und Tukey,1965) for

$$N = 2^n = 2, 4, 8, 16, 32, 64, 128, 256, 512, \ldots$$

- Radix-4 Algorithmus (schneller als Radix-2) for

$$N = 4^n = 4, 16, 64, 256, 1024, 4096, \ldots$$

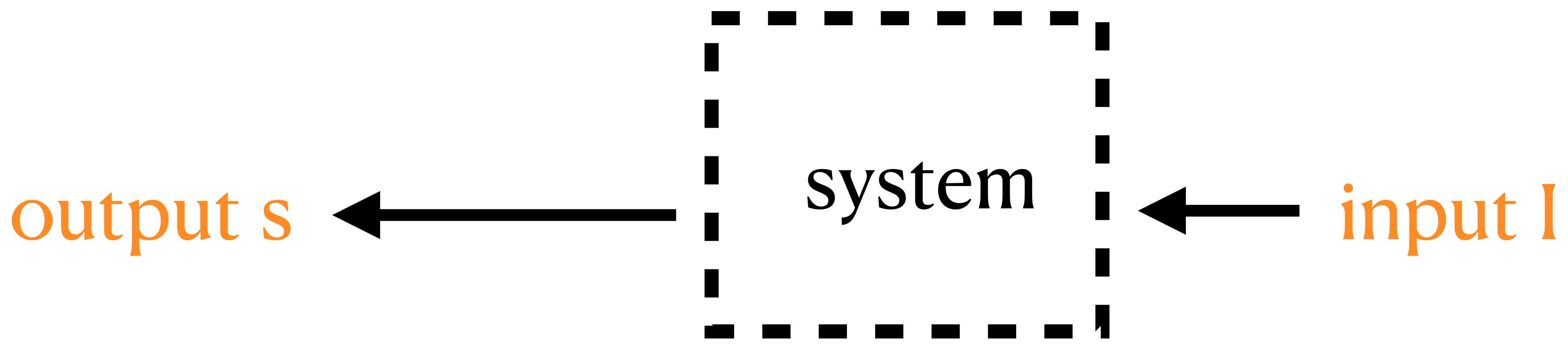
data sampling

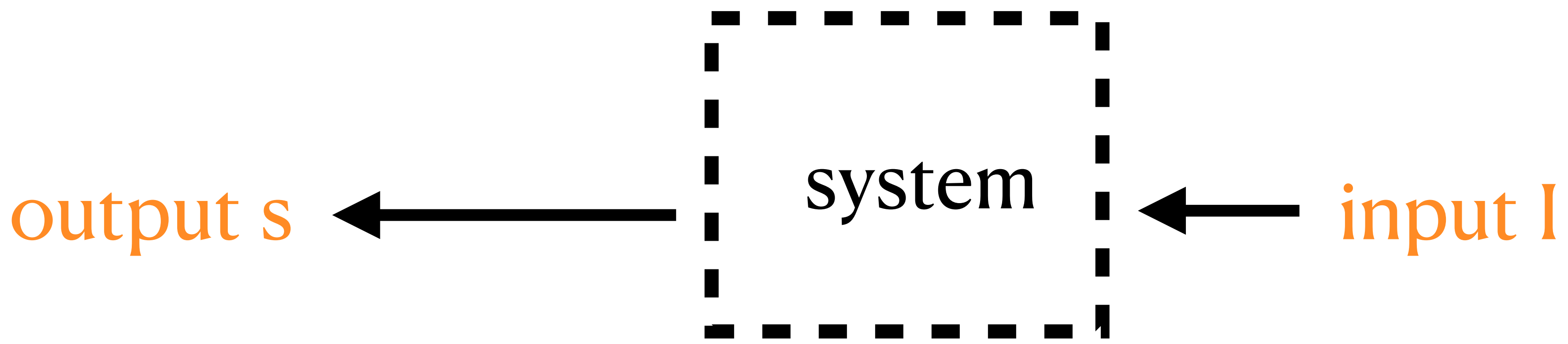
Fourier analysis

errors in analysis

linear filters

time-frequency analysis


output s
system
input I



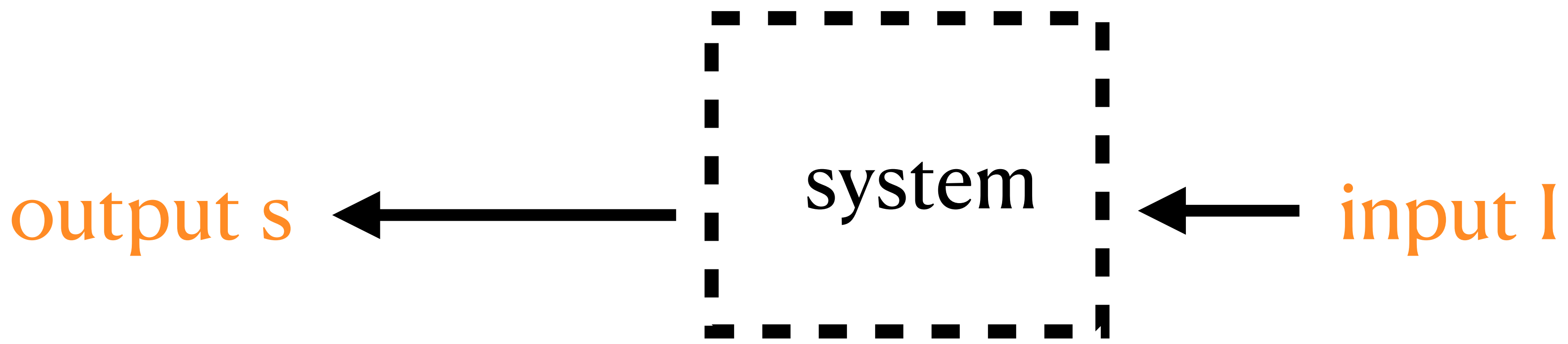
continuous time

$$s(t) = \int_{-\infty}^{t} H(t-\tau) I(\tau) d\tau$$

output

impulse response function

input



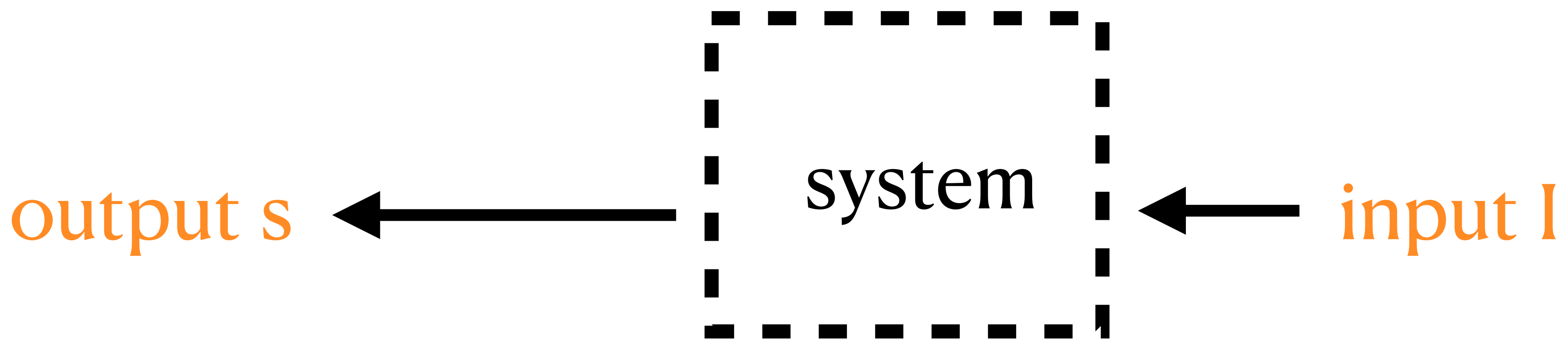
discrete time

$$s(t_k) = \sum_{m=-\infty}^{k} H(t_{k-m}) I(t_m)$$

output

input

impulse
response function



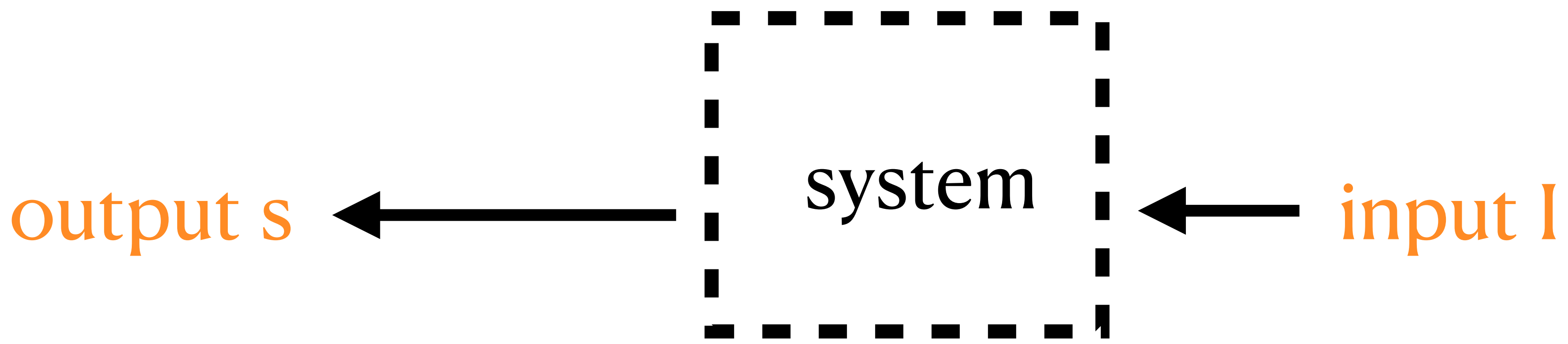
$$s(t_k) = \sum_{m=-\infty}^{k} H(t_{k-m}) I(t_m)$$

output

impulse response function

input

$$\tilde{s}(f) = \tilde{H}(f)\tilde{I}(f)$$



$$s(t_k) = \sum_{m=-\infty}^{k} H(t_{k-m}) I(t_m)$$

output

impulse response function

input

$$\tilde{s}(f) = \tilde{H}(f)\tilde{I}(f)$$

$$\boxed{\mathrm{PSD}(f) = |\tilde{H}(f)|^2 |\tilde{I}(f)|^2}$$

data sampling

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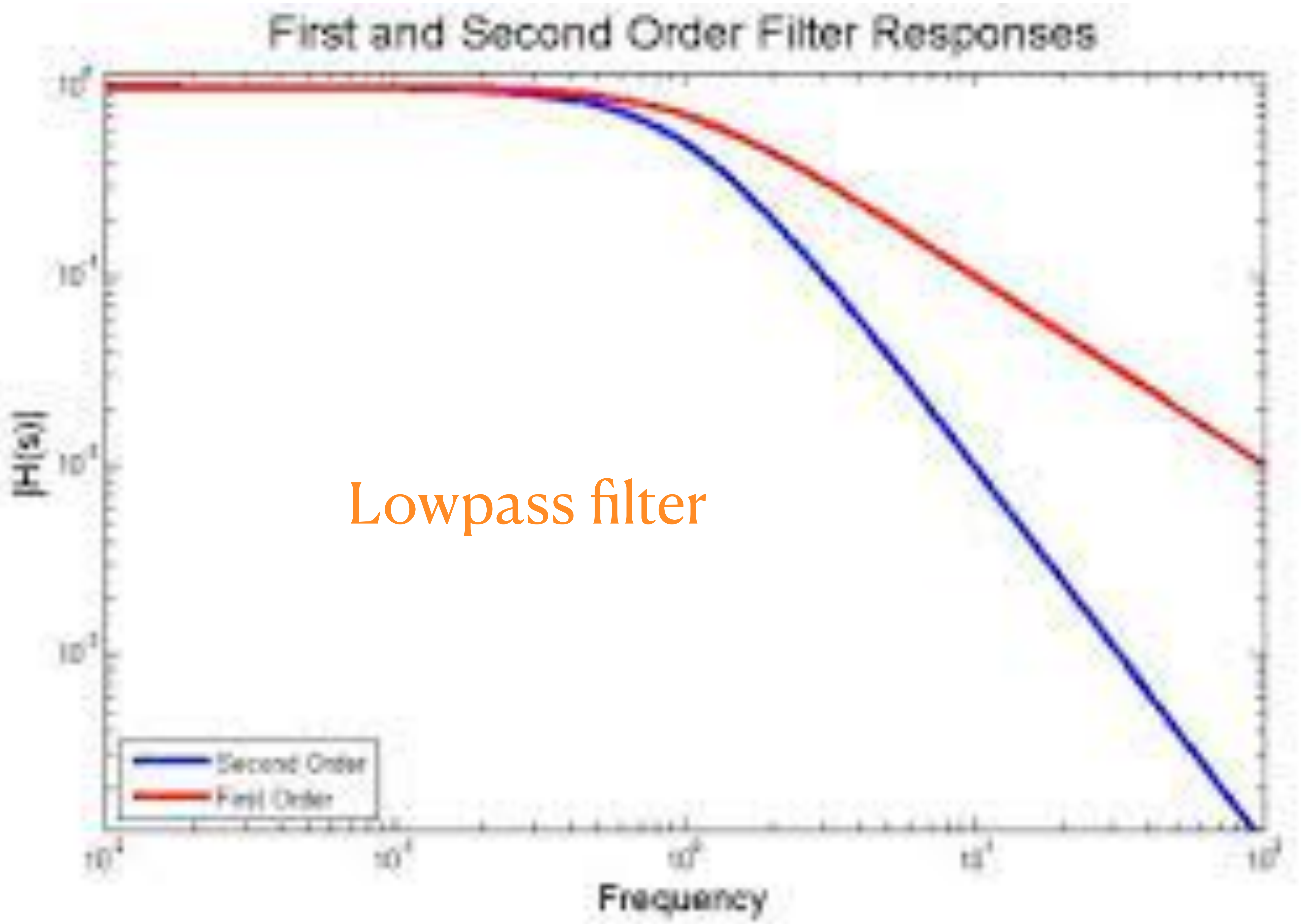
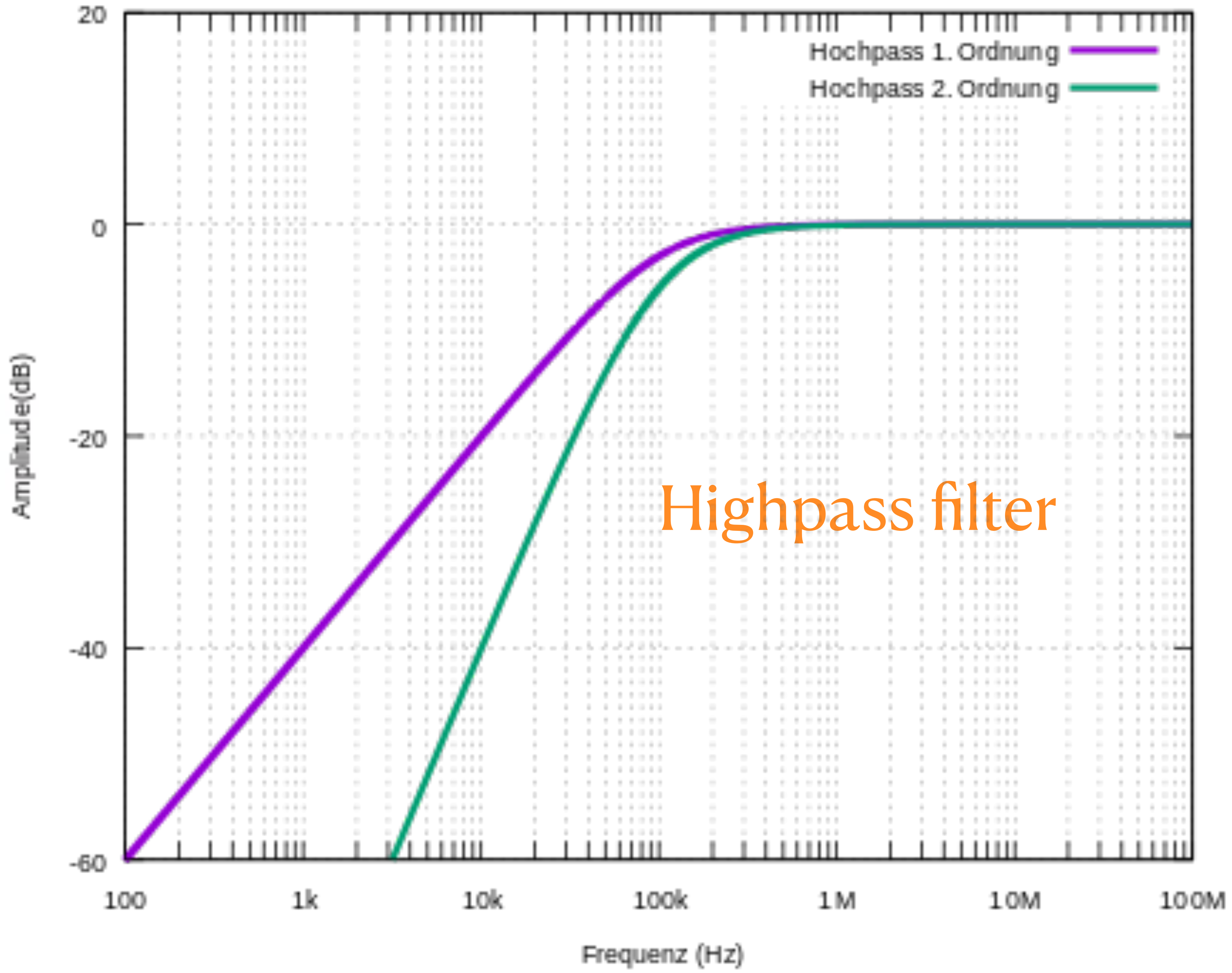
linear filters **frequency pass filter**

time-dependent filters

time-frequency analysis

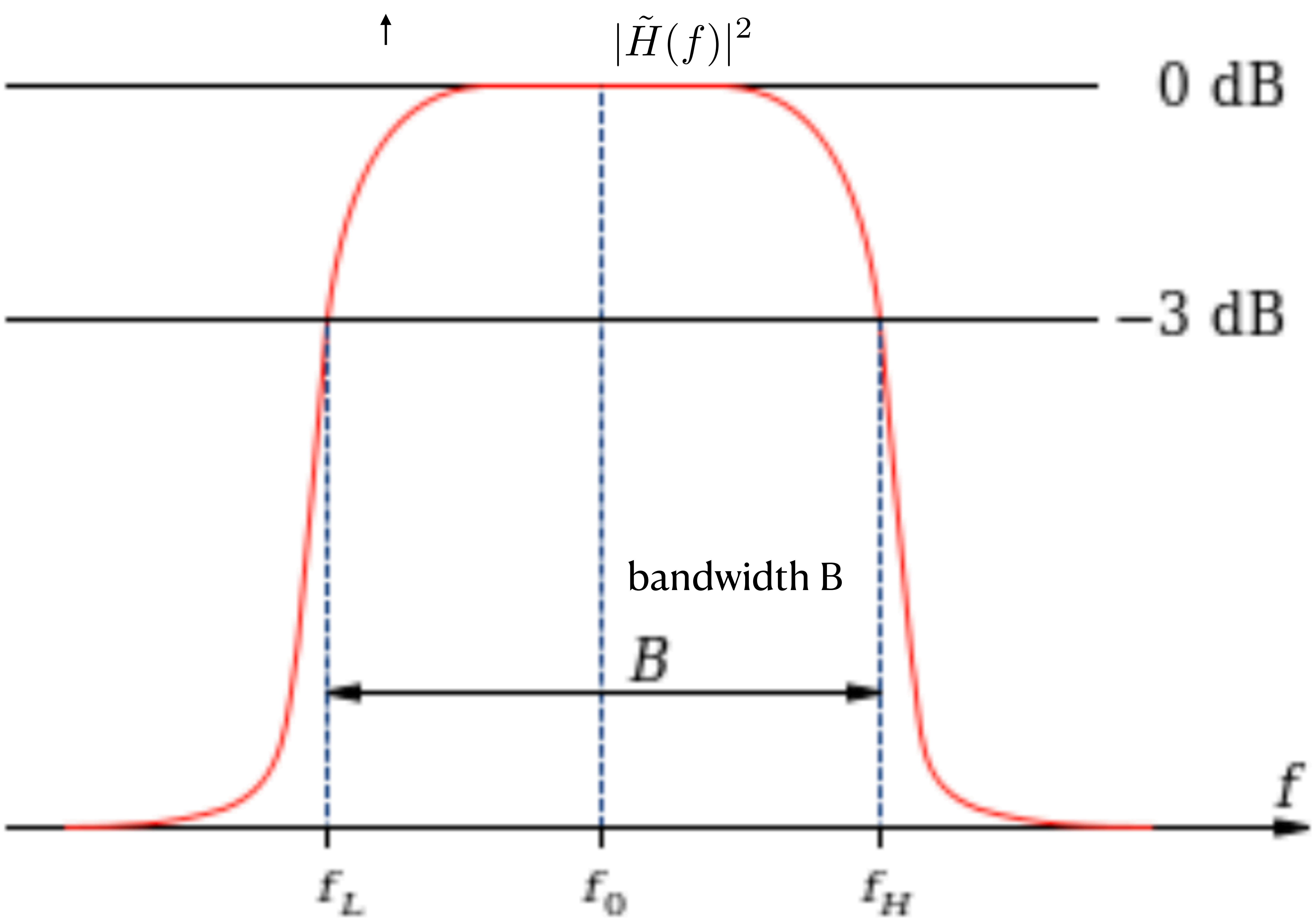
$$\mathrm{PSD}(f) = |\tilde{H}(f)|^2 |\tilde{I}(f)|^2$$





Bandpass filter

$$\mathrm{PSD}(f) = |\tilde{H}(f)|^2 |\tilde{I}(f)|^2$$



data sampling

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Example:

$$\tilde{s}(w) = \frac{1}{iw + 1/\tau_0} \tilde{I}(w)$$

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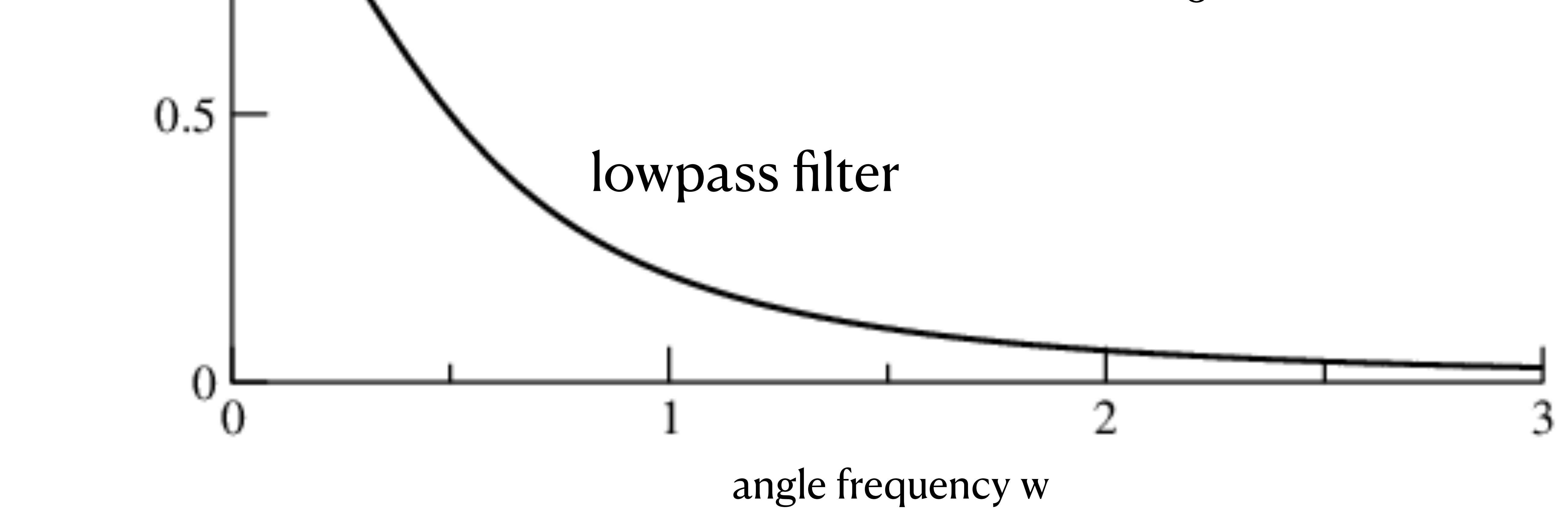
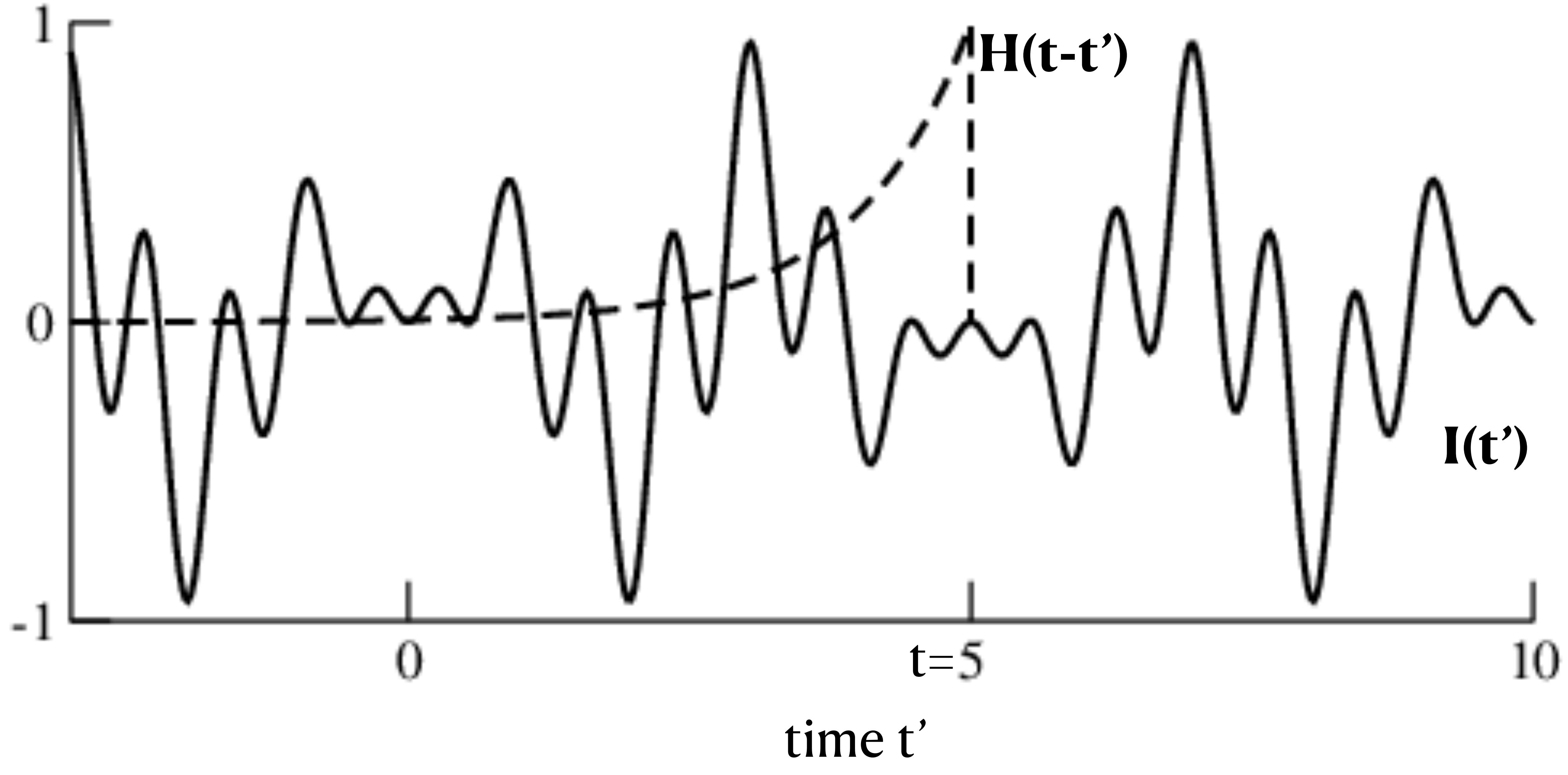
$$\tilde{s}(w) = \frac{1}{iw + 1/\tau_0} \tilde{I}(w)$$

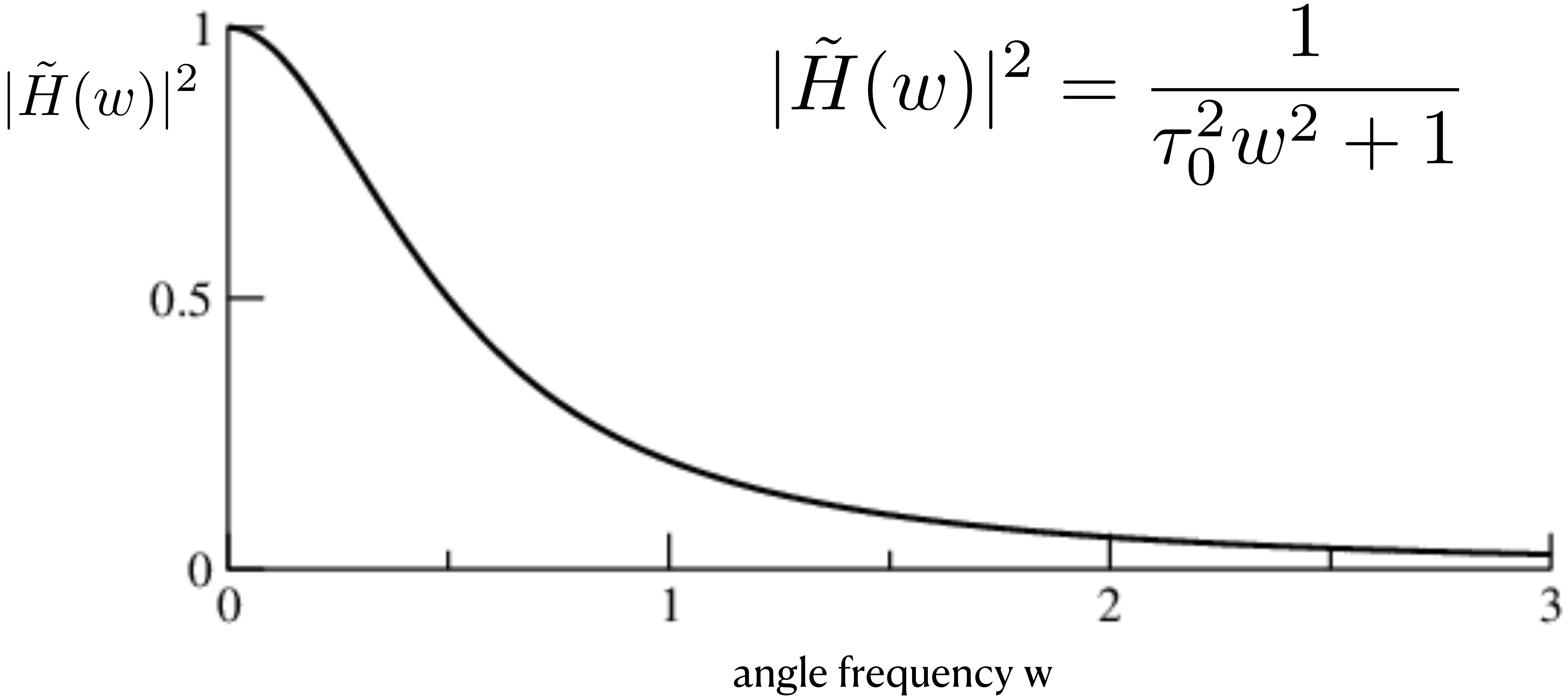
$$|\tilde{H}(w)|^2 = \frac{1}{\tau_0^2 w^2 + 1}$$





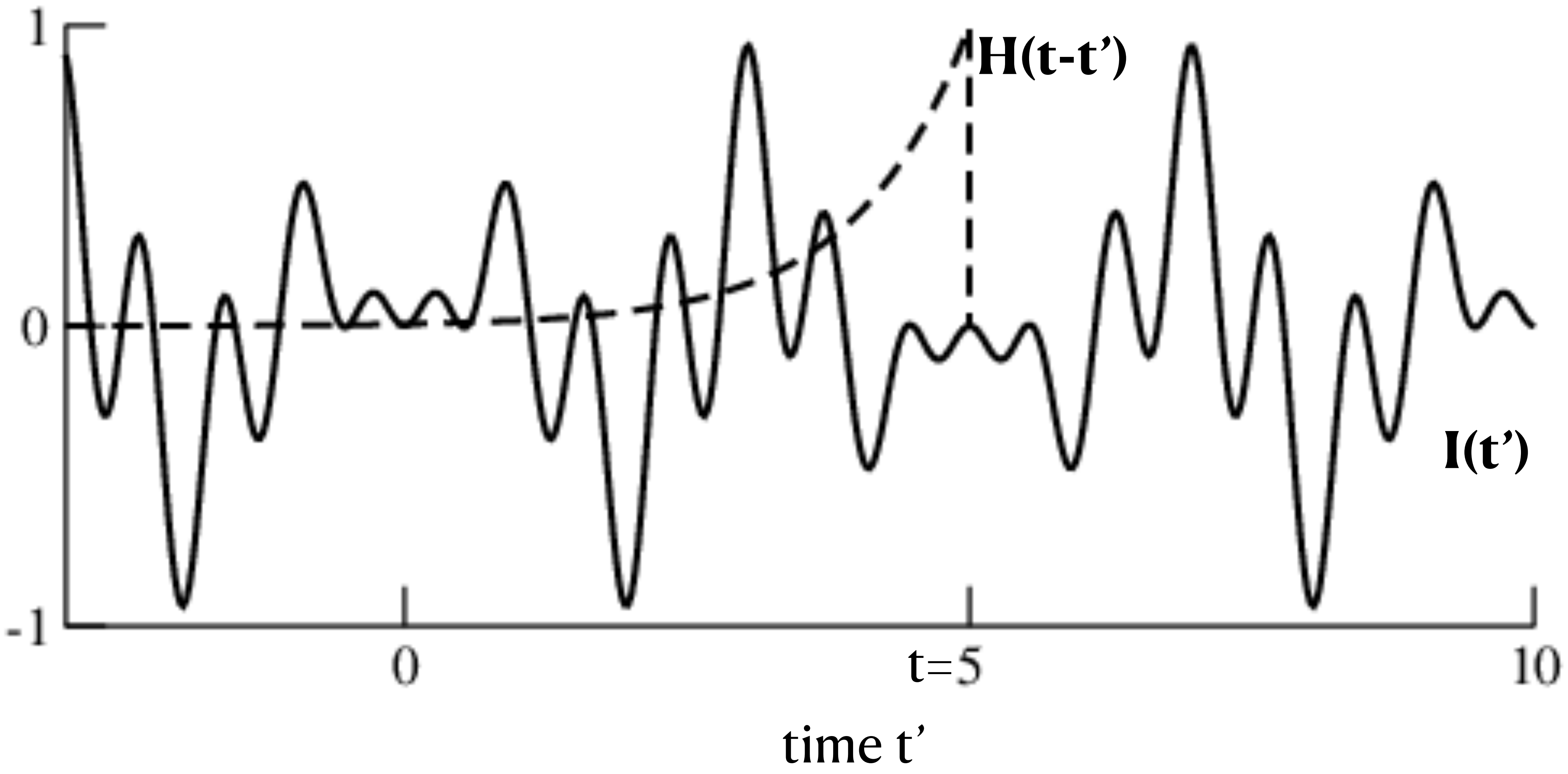
Example:

$$\tilde{s}(w) = \frac{1}{iw + 1/\tau_0}\tilde{I}(w)$$

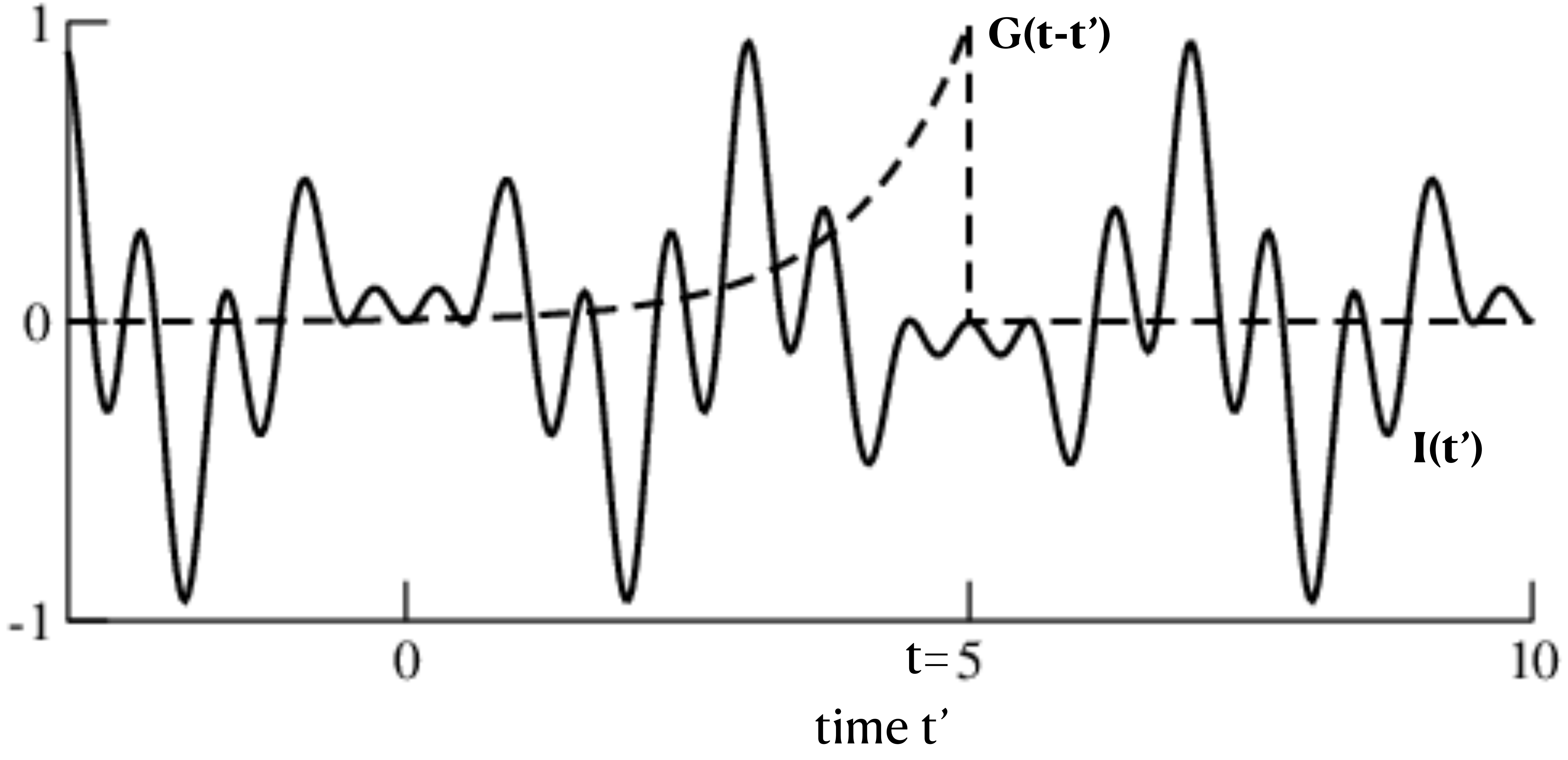


$$H(t) = e^{-t/\tau_0}$$

$$s(t) = \int_{-\infty}^{t} e^{-(t-t')/\tau_0} I(t')dt'$$



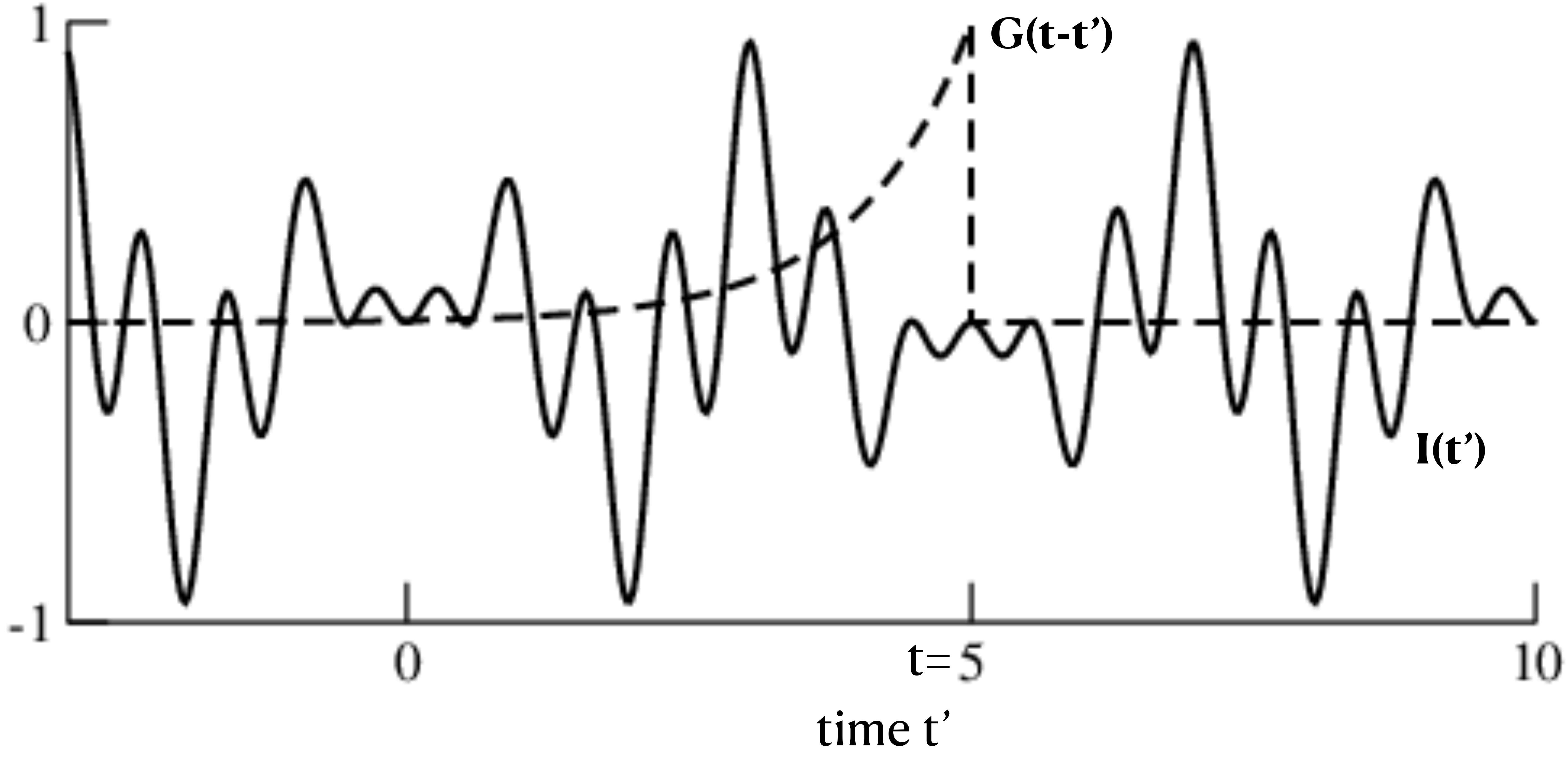
$$s(t) = \int_{-\infty}^{\infty} G(t-t')I(t')dt'$$



$$s(t) = \int_{-\infty}^{\infty} G(t-t')I(t')dt'$$

G(t): filter window = sliding window

s(t): correlation function of G and I



$$s(t) = \int_{-\infty}^{\infty} G(t-t')I(t')dt'$$

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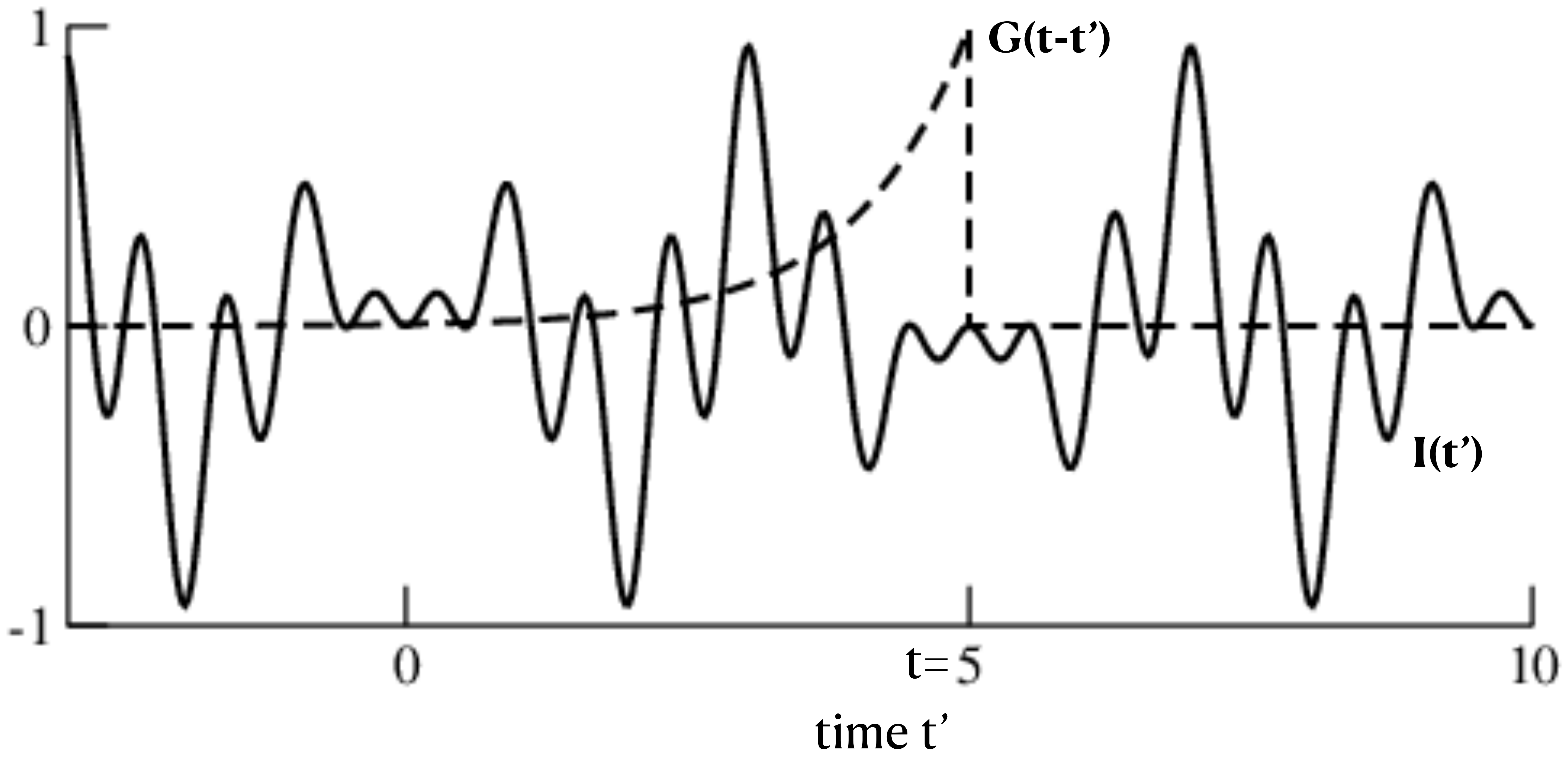

linear filter

can be seen as a time-dependent correlation function

of a signal with a sliding window