A Brief Memo of Key Math Tools

I'm writing here the math concepts we have been using in this class. I'll expand this list as we use more tools.

1 Notation

- The equivalence sign $\equiv$ defines a new variable as a function of other pre-defined variables. In other words, it is used to give a name to a math expression, to simplify notation and to avoid writing many times a complicated expression.

2 Logarithm function

2.1 Calculus

Recall that $\ln(1) = 0$, and $\ln(x)$ is only defined for $x > 0$. We have used three standard rules:

- Multiplication: $\ln(a \times b) = \ln(a) + \ln(b)$ $\forall a, b > 0$.
- Power: $\ln(a^b) = b \ln(a)$ $\forall a, b > 0$.
- Division: $\ln(a/b) = \ln(a) - \ln(b)$ $\forall a, b > 0$.
  - This one is easy to show using the two previous rules:
    $\ln(a/b) = \ln(a \times b^{-1}) = \ln(a) + \ln(b^{-1}) = \ln(a) - \ln(b)$.

2.2 Calculus for exponential function

Sometimes we use the exp function as well. Recall that $\exp(\ln(x))) = x$ and $\ln(\exp(x)) = x$; and $\exp(0) = 1$. Rules for exponential function are:

- Multiplication: $\exp(a)\exp(b) = \exp(a + b)$.
- Power: $\exp(a \times b) = (\exp(a))^b$.
- Division: $\exp(a)/\exp(b) = \exp(a - b)$.

– Again, you can show the division rule by combining the multiplication and the power rules.

- And the power rule also implies that: $a^b = \exp(b \ln(a))$.

  – The proof goes as follows: $a^b = (\exp(\ln(a)))^b = \exp(\ln(a) \times b) = \exp(b \ln(a))$.

2.3 Growth rates and logarithm

Recall two key concepts.

- The derivative of $f(x) \equiv \ln(x)$ is $f'(x) = 1/x$.

- A first-order Taylor expansion is a linear approximation of a function around a point x_0 given as: $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$.

Now, let us apply the Taylor expansion to $x_0 = 0$ and $f(x) = \ln(1 + x)$ to obtain: $\ln(1 + x) \approx \ln(1 + 0) + (1/1) \times (x - 0) = x$, that is, $\ln(1 + x) \approx x$ for $x \approx 0$.

Now, let's apply that to think about the growth rate of a variable y_t. We define its growth rate g_{t+1} as:

$$g_{t+1} \equiv \frac{y_{t+1} - y_t}{y_t}$$

Applying the rule that $\ln(1 + x) \approx x$ for x close enough to 0, we get:

$$\frac{y_{t+1} - y_t}{y_t} \approx \ln\left(1 + \frac{y_{t+1} - y_t}{y_t}\right) = \ln\left(\frac{y_{t+1}}{y_t}\right) = \ln y_{t+1} - \ln y_t.$$

And thus,

$$\Delta \ln y_{t+1} \approx \frac{y_{t+1} - y_t}{y_t}$$

The difference in logs approximates the growth rate of a variable.

3 Power functions

Let $f(x) = x^a$. In principle, you can always rewrite the function f as $f(x) = \exp(a \ln(x))$ and use the calculus rules for ln and exp. But you may as well want to learn by heart that:

- Multiplication: $x^a x^b = x^{a+b}$
- Power: $(x^a)^b = x^{a \times b}$
- Division: $x^a / x^b = x^{a-b}$
- Power revisited: $x^a \times y^a = (x \times y)^a$

And the derivative of $f(x) \equiv x^a$ is $f'(x) = ax^{a-1}$.

4 To be added