# Convergence of Sequences Sheet I

## 1 Real Numbers

### Cauchy Sequences

1. **Remark.** Unfortunately, the inequalities $\varepsilon \leq a_n \leq -\varepsilon$ are not true, since otherwise, $\varepsilon \leq -\varepsilon$, which implies $2\varepsilon \leq 0$ and so $\varepsilon \leq 0$. Instead of a proof by contradiction, we can write a direct proof as the next paragraph shows.

   Since $(a_n)_{n=1}^{\infty}$ is Cauchy, there exists $N \geq 1$ such that $|a_i - a_j| \leq 1$ for all $i, j \geq N$. In particular, $|a_i - a_N| \leq 1$ for all $i \geq N$, or $a_N - 1 \leq a_i \leq a_N + 1$ for all $i \geq N$. Now define

   $$K := \max\{|a_N - 1|, |a_N + 1|\}$$

   (note that $K$ is a non-negative real number). Now $|a_N - 1| \leq K$ and $|a_N + 1| \leq K$. So for any $i \geq N$, $-K \leq -|a_N - 1| \leq a_N - 1 \leq a_i \leq a_N + 1 \leq |a_N + 1| \leq K$. So $|a_i| \leq K$ for all $i \geq N$. If $N = 1$, then this proves that $(a_n)_{n=1}^{\infty}$ is bounded. Assume $N > 1$. By Lemma 5.1.14, the finite sequence $(a_n)_{n=1}^{N-1}$ is bounded by some real number $M \geq 0$, i.e., $|a_n| \leq M$ for all $1 \leq n \leq N - 1$. It's now easily seen that $|a_n| \leq M + K$ for all $n \geq 1$. Hence $(a_n)_{n=1}^{\infty}$ is bounded.

   **Further remark.** I'm assuming that $(a_n)_{n=1}^{\infty}$ is a sequence in $\mathbb{R}$. But if more specifically $(a_n)_{n=1}^{\infty}$ is a sequence in $\mathbb{Q}$, then the $K$ above is rational. In the general case (where $K \in \mathbb{R}$), you can choose a large rational number $K'$ such that $K' \geq K$. Then $K'$ will be a rational bound, if you are seeking rational bounds that is.

2. **Remark.** The solution is correct, just the word "consecutives" is not necessary.

3. **Remark.** The first solution is almost correct, but there are some gaps. It's **not** given that $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are eventually $\varepsilon$-close **for any** $\varepsilon \geq 0$ (in fact, $\varepsilon > 0$); $\varepsilon$ is **fixed** here. The second problem is that you have only shown that $(b_n)_{n=1}^{\infty}$ has an upper bound. Unless it also has a lower bound, it won't be bounded. The solution is okay till this: $|a_n - b_n| \leq \varepsilon$ for all $n \geq N$. This means that $|b_n| = |b_n - a_n + a_n| \leq |b_n - a_n| + |a_n| = |a_n - b_n| + |a_n| \leq \varepsilon + M$ for all $n \geq N$. If $N = 1$, then $(b_n)_{n=1}^{\infty}$ is bounded by $\varepsilon + M$. Assume $N > 1$. By Lemma 5.1.14, the finite sequence $(b_n)_{n=1}^{N-1}$ is bounded by some real number $S \geq 0$, i.e., $|b_n| \leq S$ for all $1 \leq n \leq N - 1$. Hence $(b_n)_{n=1}^{\infty}$ is bounded by $S + \varepsilon + M$.

   I'm not entirely sure about the second solution, especially, this part is confusing: "Sort by values the sequences $(a_n)$ and $(b_n)$, we get that the new sequence $(a'_n)$ converges, since it is increasing and bounded, and it converges to $\sup(a'_n) \leq L$." I'd suggest we stick to the first solution. The second one also looks unduly complicated for a simple exercise.

4. **Remark.** The solution is correct.

5. **Remark.** The first half of the solution is correct, except the inequality

$$M(|b_n - b_m| + |a_n - a_m|) \leq M\varepsilon,$$

which will be $M(|b_n - b_m| + |a_n - a_m|) \leq 2M\varepsilon$. The second half however is wrong. We need to show that $(a_n b_n)_{n=1}^{\infty}$ and $(a'_n b_n)_{n=1}^{\infty}$ are equivalent. This means we need to show that for every $\varepsilon > 0$, there exists a positive integer $N$ such that $|a_n b_n - a'_n b_n| \leq \varepsilon$ for all $n \geq N$. Now $(b_n)_{n=1}^{\infty}$ is bounded. So there exists a real number $K > 0$ such that $|b_n| \leq K$ for all $n \geq 1$. Since $(a_n)_{n=1}^{\infty}$ and $(a'_n)_{n=1}^{\infty}$ are equivalent, there exists a positive integer $N$ such that $|a_n - a'_n| \leq \frac{\varepsilon}{K}$ for all $n \geq N$. So for all $n \geq N$, $|a_n b_n - a'_n b_n| = |a_n - a'_n||b_n| \leq \frac{\varepsilon}{K} K = \varepsilon$. Thus $(a_n b_n)_{n=1}^{\infty}$ and $(a'_n b_n)_{n=1}^{\infty}$ are equivalent. Hence $xy = x'y$.

6. **Remark.** The solution is correct.

7. **Remark.** You can straightaway show convergence like this: Choose $\varepsilon > 0$. By Corollary 5.4.13 (Archimedean property), there exists a positive integer $N$ such that $N \cdot 1 > \frac{1}{\varepsilon}$, i.e., $\frac{1}{N} < \varepsilon$. Now for any integer $n \geq N$, $|\frac{1}{n} - 0| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon$. Hence $\lim_{n\to\infty} \frac{1}{n} = 0$.

8. **Remark.** Since this is just Problem 3, with the only difference that now the sequences have rational terms, that same solution works here also; the small necessary changes are obvious.

9. **Remark.** The main ideas of the proof are correct, but some technical changes need to be made: Change all "$\varepsilon \geq 0$" to "$\varepsilon > 0$;" change "By definition of real number, we know that there exists some lower bound $c \in \mathbb{Q} \ldots$" to "By the definition of positive real numbers, we know that there exists some lower bound $c \in \mathbb{Q} \ldots$"; change "This is absurd, since $a_n$ is a strictly positive real number." to "This is absurd, since $a_n$ is a positive rational number."; change "let $(a_n)$ be a negatively upperly bounded Cauchy sequence" to "let $(a_n)$ be a Cauchy sequence of rational numbers negatively bounded away from 0"; change "positively lowerly bounded" to "positively bounded away from 0"; change "$a_{n_0}$ and $b_{n_0}$ are negative or zero" to "$a_{n_0}$ or $b_{n_0}$ are negative or zero" (note that $-2 + 1 \leq 0$, where $-2$ is negative but 1 is positive).

10. **Remark.** Unfortunately, the solution is wrong. It's not necessary that if $K$ is an integer satisfying $N - 1 \geq K \geq M - 1$, then $y > K \geq x$. This is because we can choose $K = M - 1$. For this choice, $K = M - 1 < x$. See below a solution.

    Since $y - x > 0$, $N(y - x) > 1$ for some positive integer $N$ (by the Archimedean property). Now by Exercise 11, $\lfloor Nx \rfloor \leq Nx < \lfloor Nx \rfloor + 1$. Hence $Nx < \lfloor Nx \rfloor + 1 \leq Nx + 1 < Ny$. Since $N > 0$, this implies $x < \frac{\lfloor Nx \rfloor + 1}{N} < y$; clearly, $\frac{\lfloor Nx \rfloor + 1}{N} \in \mathbb{Q}$.

11. **Remark.** I prefer the second solution.

12. **Remark.** You can also use the Archimedean property. Since $x$ is positive, there exists (by the Archimedean property) a positive integer $N$ such that $Nx > 1$. So $x > \frac{1}{N}$.

13. **Remark.** As a general rule, you should go for a direct proof (as opposed to proof by contradiction) if it's possible and simple. It follows directly from the definition of absolute value that $r \le |r|$ and $-r \le |r|$ for all $r \in \mathbb{R}$. We will use this fact. Suppose $|x - y| < \varepsilon$. Then $x - y \le |x - y| < \varepsilon$, which implies $x < y + \varepsilon$. Also, $y - x = -(x - y) \le |x - y| < \varepsilon$, which implies $y - \varepsilon < x$. So $y - \varepsilon < x < y + \varepsilon$. The converse part is okay. You should also at least mention that the second claim $|x - y| \le \varepsilon \Leftrightarrow y - \varepsilon \le x \le y + \varepsilon$ can be similarly proved.

14. **Remark.** The solution is correct.

15. **Remark.** I prefer the first solution. It's straightforward.

# Least upper bound of a subset of $\mathbb{R}$ / Completeness

1. **Remark.** The solution is correct. The writing however can be made a bit more clear.

2. **Remark.** The first solution is unfortunately incomplete. We do lose generality by assuming $K > 0$, since $K$ may very well be negative or even 0. The other two solutions have also become quite complicated. Consider the solution in the following paragraph.

   Let $A = \{i \in \mathbb{Z} : L < i \le K$ and $\frac{i}{n}$ is an upper bound for $E\}$. $A \ne \emptyset$ because $K \in A$, and $A$ is finite because $A \subseteq \{i \in \mathbb{Z} : L < i \le K\}$ $(\#(\{i \in \mathbb{Z} : L < i \le K\}) = K - L)$. Let $m$ be the smallest element of $A$. Then $L < m \le K$ and $\frac{m}{n}$ is an upper bound for $E$. Also, $m - 1 \notin A$, and so either $\frac{m-1}{n}$ is not an upper bound for $E$ or we don't have $L < m - 1 \le K$ (or both). Assume that we don't have $L < m - 1 \le K$. Since $m - 1 < m \le K$, we must have $m - 1 \le L$, or $\frac{m-1}{n} \le \frac{L}{n}$. Since $\frac{L}{n}$ is not an upper bound for $E$, $\frac{m-1}{n}$ is not an upper bound for $E$ either. So in any case, the number $\frac{m-1}{n}$ is not an upper bound for $E$.

3. **Remark.** Unfortunately, the first solution is incorrect. You don't get $\frac{m-m'}{n} \ge 0 > \frac{m-m'}{n}$ from $\frac{m}{n} \ge x > \frac{m-1}{n}$ and $\frac{m'}{n} \ge x > \frac{m'-1}{n}$. That's because $-\frac{m'}{n} \ngeq -x$; in fact, $-\frac{m'}{n} \le -x$. You can consider the solution in the following paragraph.

   Suppose $m < m'$. Then $m \le m' - 1$ (as $m, m'$ are integers), and so $\frac{m}{n} \le \frac{m'-1}{n}$. But then $\frac{m'-1}{n}$ is an upper bound for $E$, a contradiction. So we cannot have $m < m'$. Similarly, $m' \not< m$. Thus the only possibility is $m = m'$.

4. **Remark.** The solution is correct.

5. **Remark.** I can suggest a very short solution. Since $x < y$, $x - \sqrt{2} < y - \sqrt{2}$. So By Proposition 5.4.14, there exists a rational number $q$ such that $x - \sqrt{2} < q < y - \sqrt{2}$. Hence $x < q + \sqrt{2} < y$. Note that $q + \sqrt{2}$ is irrational.

6. **Remark.** Let me first remark that the problem itself is a little wrong. In Tao's text, $\mathbb{N} = \{0, 1, 2, \ldots\}$. So $1 - \frac{1}{n}$ is undefined when $n = 0$. So change the claim to

$$\sup\left\{1 - \frac{1}{n} : n \in \mathbb{N} - \{0\}\right\} = 1$$

The second solution is correct. The first solution would also be correct, but with these following changes: "$S = \sup\{1 - \frac{1}{n} : n \in \mathbb{N}\}$" to "$S = \{1 - \frac{1}{n} : n \in \mathbb{N} - \{0\}\}$" and "$1 - \frac{1}{n} < s \leq 1$" to "$1 - \frac{1}{n} < s < 1$". So when you ultimately conclude $s = 1$, you have a contradiction. This will prove that no number $s$ less than 1 is an upper bound.

7. **Remark.** The solution is correct.

8. **Remark.** The solution is correct.

9. **Remark.** The solution is correct, just replace "$\leq$" with "$<$" in part (a).

10. **Remark.** The solution is correct, just replace "$\leq$" with "$<$" and "$\geq$" with "$>$".

11. **Remark.** The solution is correct.

12. **Remark.** The solution is correct.

13. **Remark.** The solution is correct.

14. **Remark.** The solution is correct.

15. **Remark.** The solution is correct.

# Real number sequences

1. **Remark.** The solution is correct.

2. **Remark.** The solution is correct, just some little things need to be changed. In part (c), separate the cases $c = 0$ and $c \neq 0$. If $c = 0$, then $(ca_n)_{n=m}^{\infty}$ is the constant sequence $0, 0, \ldots,$ which converges to 0; also, $c \lim_{n\to\infty} a_n = 0$. If $c \neq 0$, your solution stands. In part (d), change the phrase "strictly non-zero" to "bounded away from zero".

3. **Remark.** The solution is correct.

4. **Remark.** This is a strange one! The given statement is itself the definition of convergence. So there is in fact nothing to prove here.

5. **Remark.** Although the idea of the solution is correct, things are looking a bit off. Suppose first that $(a_n)_{n=m}^{\infty}$ is convergent to $c$. Then for any $\varepsilon > 0$, there exists an integer $N \geq m$ such that $|a_n - c| \leq \varepsilon$ for all $n \geq N$. Let $M = \max(N, m')$. Then for all $n \geq M$, $|a_n - c| \leq \varepsilon$ (since $n \geq M \geq N$); note that since $n \geq M \geq m'$, each $a_n$ here is a term of $(a_n)_{n=m'}^{\infty}$. Hence $(a_n)_{n=m'}^{\infty}$ is also convergent to $c$. Conversely, let $(a_n)_{n=m'}^{\infty}$ be convergent to $c$. Then for any $\varepsilon > 0$, there exists an integer $K \geq m'$ such that $|a_n - c| \leq \varepsilon$ for all $n \geq K$. Since $n \geq K \geq m' \geq m$, each $a_n$ here is a term of $(a_n)_{n=m}^{\infty}$. So $(a_n)_{n=m}^{\infty}$ also converges to $c$.

6. **Remark.** You can also use this observation: $(a_{n+k})_{n=m}^{\infty} = (a_j)_{j=m+k}^{\infty}$; since $k \geq 0$, we have $m' = m + k \geq m$, and so the result follows from the previous exercise.

7. **Remark.** This seems to be problematic: "$(c_n) = \frac{n}{4}$, which does not converge." The sequence $(\frac{n}{4})_{n=1}^{\infty}$ does in fact converge (to 0). Also, it's not exactly clear here what "contact point" and "amount of contact point" mean.