Finite Automata Sheet

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Automata theory problems from the book "Automata Theory, Languages, and Computation" by J. Ullman.

DFA and NFA equivalence

1. **(Equivalence theorem.)** If $D = (Q_D, \Sigma, \delta_D, \{q_0\}, F_D)$ is the DFA constructed from NFA $N = (Q_N, \Sigma, \delta_N, q_0, F_N)$ by the subset construction, then $L(D) = L(N)$. Conversely if L is a language that is accepted by some DFA then it is accepted by some NFA.
Proof. By definition of language of both N and a generic DFA

$$L(N) := \{w \in \Sigma^* : \hat{\delta}_N(q_0, w) \cap F_Q \neq \varnothing\}$$

$$L(D) := \{w \in \Sigma^* : \hat{\delta}_D(\{q_0\}, w) \in F_D\}$$

it is obvious from these definitions that

$$\hat{\delta}_D(\{q_0\}, w) \in F_D \implies \hat{\delta}_D(\{q_0\}, w) \cap F_D = \varnothing.$$

In other words, the only thing to prove is that for any $w \in \Sigma^n$

$$\hat{\delta}_D(\{q_0\}, w) = \hat{\delta}_N(q_0, w).$$

We need to rely in the transition function constructed by subsets out of $\hat{\delta}_N$. By induction on $n = |w|$:

Basis. Assume $n = 0$ so that $w = \varepsilon$. Then the construction by subsets transition function will simply be $\delta_D(\{q_0\}, \varepsilon) = \{q_0\}$ then also we have

$$\hat{\delta}_D(\{q_0\}, \varepsilon) = q_0 = \hat{\delta}_N(q_0, \varepsilon).$$

Thus we conclude that $\delta_D = \hat{\delta}_D = \hat{\delta}_N$.

Hypothesis. Assume that for the string x of lenght n it held that

$$\hat{\delta}_D(\{q_0\}, x) = \hat{\delta}_N(q_0, x) = \{p_1, \ldots, p_m\} = S.$$

Thesis. We have to prove that for a string $w = xa$ such that $n = |x|$ it follows that

$$\hat{\delta}_D(\{q_0\}, w) = \hat{\delta}_N(q_0, w).$$

To prove this assertion we recall the definition of $\hat{\delta}_D$ for any DFA and compare it to the one constructed by subsets. We have that

$$\hat{\delta}_D(\{q_0\}, w) \stackrel{def}{=} \delta_D(\hat{\delta}_D(\{q_0\}, x), a) \tag{1}$$

$$\delta_D(S, a) \stackrel{c.b.s.}{=} \bigcup_{p \in S} \delta_N(p, a) = \bigcup_{i=1}^{m} \delta_N(p_i, a). \tag{2}$$

Note that in (2), by construction S is a subset of Q_N such that

$$\hat{\delta}_D(\{q_0\}, w) = \delta_D(\hat{\delta}_D(\{q_0\}, x), a) = \delta_D(S, a) \tag{3}$$

hence, the transition function constructed by subsets is a valid transition function for D. Furthermore by equation (1)

$$\hat{\delta}_D(\{q_0\}, w) = \bigcup_{i=1}^{m} \delta_N(p_i, a).$$

We also note that in the case of the extended transition function of N

$$\hat{\delta}_N(q_0, w) \stackrel{def}{=} \bigcup_{i=1}^{m} \delta_N(p_i, a) = \delta_D(S, a) \stackrel{(3)}{=} \hat{\delta}_D(\{q_0\}, w).$$

Conversely, to prove that a DFA $D = (Q_D, \Sigma, \delta_D, \{q_0\}, F_D)$ induced language is the same as some NFA language, by structural induction on n, the lenght of the input string, we define the NFA N as

Basis. We let the start state q_0 of D be the start state also of N. So that for $w = \varepsilon$, $\hat{\delta}_D(\{q_0\}, \varepsilon) = \hat{\delta}_N(q_0, \varepsilon)$.

Hypothesis. For $|x| = n$ suppose $\hat{\delta}_D(\{q_0\}, x) = p_n$ then we let $\hat{\delta}_N(q_0, x) = \{p_n\}$.

Thesis. For $w = xa$ with $|w| = n + 1$, by hypothesis

$$\hat{\delta}_D(\{q_0\}, w) = \delta_D(\hat{\delta}_D(\{q_0\}, x), a) = \delta_D(p_n, a) = p_{n+1}.$$

We let $\hat{\delta}_N(q_0, w) = \{p_{n+1}\}$.

□

2. **(Worst case.)** Show that there exists a NFA with N states such that equivalent DFA has 2^N states.
Solution.

3. Prove that if N is an NFA that has at most one choice of state for any state and input symbol (i.e., $\delta(q, a)$ never has size greater than 1), then the DFA D constructed from N by the subset construction has exactly the states and transitions of N plus transitions to a new dead state whenever N is missing a transition for a given state and input symbol.