Differential Calculus Sheet

Gallo Tenis / to A Mathematical Room

February 23, 2025

Problems in differential calculus from the books of M. Spivak "Calculus", T. Tao "Analysis I" and R. G. Bartle "Introduction to Real Analysis". These weeks topics will be about theorems of differential calculus.

Differentiation

Definition and Rules

1. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is an **even function** (that is, $f(-x) = f(x)$ for all $x \in \mathbb{R}$) and has a derivative at every point, then the derivative f' is an **odd function** (that is, $f'(-x) = -f'(x)$ for all $x \in \mathbb{R}$). Also prove that if $g : \mathbb{R} \to \mathbb{R}$ is a differentiable odd function, then g' is an even function.
Solution.

   (a) Let $h : \mathbb{R} \to \mathbb{R}$ be defined as $h(x) = -x$. If $f(x) = f(-x)$ then $f(x) = f(h(x))$. This is valid since h is bijective. By chain rule

   $$f'(x) = f'(h(x))h'(x) = -f'(-x)$$

   hence, $-f'(x) = f'(-x)$.

   (b) Let h be the same as the former item. If $-g(x) = g(-x)$ then

   $$(-g(x))' = -g'(x) = (g(h(x)))' = g'(h(x))h(x) = -g'(-x).$$

   We conclude that $g'(x) = g'(-x)$.

   $\square$

2. Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) := x^2 \sin(1/x^2)$ for $x \neq 0$, and $g(0) := 0$. Show that g is differentiable for all $x \in \mathbb{R}$. Also show that the derivative g' is not bounded on the interval $[-1, 1]$.
Solution.

(a) First we show that g is differentiable in $x = 0$. Since $g'(0) = 0$ we need to prove that $g'(x)$ is continuous when approaching $x = 0$. Hence,

$$\lim_{x \to 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x \to 0} \frac{g(x)}{x} = \lim_{x \to 0} x \sin\left(\frac{1}{x^2}\right).$$

We see that since $|\sin(x)| \leq 1$, then

$$\left| x \sin\left(\frac{1}{x^2}\right)\right| \leq |x|.$$

And the right hand side tends to 0 when x tends to 0. So by Squeeze theorem we conclude that

$$\lim_{x \to 0} x \sin\left(\frac{1}{x^2}\right) = 0$$

so it is differentiable in $x = 0$.

For computing the derivative in all of the other points of $\mathbb{R}$ we see that $\frac{1}{x}, x^2$ and $\sin(x)$ are continuous in all of its domain, so their composition and product is continuous in all of its domain as well. So there is not an issue of continuity. The derivative function for any $x \neq 0$ results in

$$g'(x) = 2x \sin\left(\frac{1}{x^2}\right) + x^2 \cos\left(\frac{1}{x^2}\right) \frac{-2}{x^3}$$

$$= 2x \sin\left(\frac{1}{x^2}\right) + \cos\left(\frac{1}{x^2}\right) \frac{-2}{x}.$$

(b) If g is not bounded in $[-1, 1]$, then for any $M > 0$ there exists some $x \in [-1, 1]$ such that

$$|g'(x)| > M.$$

We see that

$$\left| 2x \sin\left(\frac{1}{x^2}\right) + \cos\left(\frac{1}{x^2}\right) \frac{-2}{x}\right| \leq 2|x| + \left|\cos\left(\frac{1}{x^2}\right) \frac{-2}{x}\right|.$$

Hence, the first term is bounded by 2. So we claim that $\cos\left(\frac{1}{x^2}\right) \frac{-2}{x}$ has no finite limit in some $c \in [-1, 1]$.

To prove this claim, suppose for the sake of contradiction that the former converges to 0 when x approaches 0. By sequential formulation, let $(x_n)_{n=1}^{\infty}$ be defined as

$$x_n = \frac{1}{\sqrt{2\pi n}}.$$

This sequence converges to 0. So if g' converged at $c = 0$

$$\lim_{n\to\infty} \cos\left(\frac{1}{x_n^2}\right)\frac{-2}{x_n} = \cos(2\pi n)\left(-2\sqrt{2\pi n}\right) = -2\sqrt{2\pi n}$$

that diverges to $-\infty$, which is a contradiction. We conclude that the whole function is not bounded at $c = 0$.

$\square$

3. Let $f : I \to \mathbb{R}$ be differentiable at $c \in I$. Establish the **Straddle Lemma**: Given $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if $u, v \in I$ satisfy $c - \delta(\varepsilon) < u \le c \le v < c + \delta(\varepsilon)$, then we have $|f(v) - f(u) - (v-u)f'(c)| \le \varepsilon(v-u)$. (Hint: The $\delta(\varepsilon)$ is given by Definition 6.1.1. Subtract and add the term $f(c) - cf'(c)$ on the left side and use the Triangle Inequality.)

4. If $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $c \in \mathbb{R}$, show that

$$f'(c) = \lim_{n\to\infty}\left(n\left\{f\left(c + \frac{1}{n}\right) - f(c)\right\}\right).$$

However, show by example that the existence of the limit of this sequence does not imply the existence of $f'(c)$.

5. Given that the function $h(x) := x^3 + 2x + 1$ for $x \in \mathbb{R}$ has an inverse h^{-1} on $\mathbb{R}$, find the value of $(h^{-1})'(y)$ at the points corresponding to $x = 0,\ 1,\ -1$.

Principal Theorems

Convexity and Concavity

1.

Approximation by polynomial functions

1.