Convergence of sequences sheet I

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Problems on convergence of sequences and fundamental real number properties from the books Terence Tao's "Analysis I", Michael Spivak's "Calculus" and Bartle R.G. "Introduction to Real Analysis" plus some of my own.

1 Real numbers

Cauchy Sequences

1. **(Cauchy sequences are bounded)**. Every Cauchy sequence $(a_n)_{n=1}^{\infty}$ is bounded.
   Solution. Since (a_n) is a Cauchy sequence, we have that for any $\varepsilon > 0$, there exists a natural number $n_\varepsilon \geq 1$ such that for any $n, m \geq n_\varepsilon$ $|a_n - a_m| \leq \varepsilon$. This implies that

   $$-\varepsilon + a_m \leq a_n \leq \varepsilon + a_m.$$

   In particular if $\varepsilon = 1$ for any $n, m \geq n_1$ in particular we have

   $$a_{n_1} - 1 \leq a_n \leq a_{n_1} + 1.$$

   Define $K := \max\{|a_{n_1} - 1|, |a_{n_1} + 1|\}$, then

   $$-K \leq a_{n_1} - 1 \leq a_n \leq a_{n_1} + 1 \leq K.$$

   Hence, for any $n \geq n_1$, $|a_n| \leq K$. If $n_1 = 1$ then we are done, else if $n_1 > 1$ then we know that the subsequence $(a_n)_{n=1}^{n_1-1}$ is bounded by some $M \in \mathbb{Q}$, so suffices letting a bound $K' := \max\{K, M\}$.

   $\square$

2. Show that if $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are equivalent sequences of rationals, then $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence if and only if $(b_n)_{n=1}^{\infty}$ is a Cauchy sequence.
   Solution. Assume that (a_n) is a Cauchy sequence. Then, for any $\varepsilon/2 > 0$, there exists a natural number n_ϵ such that for any $n, m \geq n_\varepsilon$: $|a_n - a_m| \leq$

$\varepsilon/2$. Since both $(a_n), (b_n)$ are equivalent there exists an m_ε such that $|a_n - b_n| \leq \varepsilon/2$ for any $n \geq m_\varepsilon$. Let $M := \max(n_\varepsilon, m_\varepsilon)$. Fixing this M we get that both properties for the sequences hold for any $n, m \geq M$ and we are allowed to do the following derivation:

$$|b_n - b_m| = |(a_n + b_n) - (a_m + b_m)| = |(a_n - b_n) + (a_m - b_m)| \leq \\ |a_n - b_n| + |a_m - b_m| \leq 2\varepsilon/2 = \varepsilon.$$

Which shows that (b_n) is a Cauchy sequence. The converse implication has the same derivation, replacing (a_n) with (b_n).

$\square$

3. Let $\varepsilon > 0$. Show that if $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are eventually ε-close, then $(a_n)_{n=1}^{\infty}$ is bounded if and only if $(b_n)_{n=1}^{\infty}$ is bounded.
Solution.

($\Longrightarrow$) Assume that (a_n) is bounded. Thus,

$$|a_n| \leq M$$

For any $n \in \mathbb{N}$ and some rational number $M \geq 0$. Since (a_n) and (b_n) are equivalent, they are eventually $\varepsilon - close$ for some $\varepsilon > 0$. Then, there must exist some integer $N \geq 1$, such that

$$|a_n - b_n| \leq \varepsilon \text{ for every } n \geq N.$$

This implies that

$$|b_n| = |b_n - a_n + a_n| \leq |b_n - a_n| + |a_n|.$$

Thus, by hypothesis

$$|b_n| \leq \varepsilon + M$$

This does not yet imply that b_n is bounded, since we have not checked what happens for the previous terms before N.
Suppose $N = 1$, then we are done. Else, if $N > 1$ then the subsequence $(b_n)_{n=1}^{N-1}$ is bounded by some $K \in \mathbb{Q}$, this is $|b_n| \leq K$. We conclude that

$$|b_n| \leq \max\{K, \varepsilon + M\}.$$

($\Longleftarrow$) Replacing (a_n) with (b_n) the proof is a mirror of the one before.

$\square$

4. **(Formal limits are well-defined).** Let $x = \lim_{n\to\infty} a_n$, $y = \lim_{n\to\infty} b_n$, and $z = \lim_{n\to\infty} c_n$ be real numbers. Then, with the definition of equality for real numbers, we have $x = x$. Also, if $x = y$, then $y = x$. Finally, if $x = y$ and $y = z$, then $x = z$.
Proof. For the case $x = x$, we have to prove that for the Cauchy sequence (a_n), a_n is eventually $\varepsilon - close$ to itself, hence, take $\varepsilon \geq 0$, take any $N \geq 1$, for any $n \geq N$ we will have:

$$|a_n - a_n| = 0 \leq \varepsilon$$

So it shows that this sequence is equivalent to itself. The second case is also trivial, we get that

$$|a_n - b_n| = |b_n - a_n| \leq \varepsilon$$

Given that $(a_n) \equiv (b_n)$. For the transitivity part, given that for any $\varepsilon/2 \geq 0$, there exist numbers $N, M \in \mathbb{N}$ such that

$$|a_n - b_n| \leq \varepsilon/2 \text{ for any } n \geq N$$

$$|b_n - c_n| \leq \varepsilon/2 \text{ for any } n \geq M$$

taking $K := \max(N, M)$ both properties hold at the same time, then

$$|b_n - c_n| + |a_n - b_n| \leq \varepsilon \text{ for any } n \geq K$$

Thus, by triangle inequality:

$$|b_n - c_n + a_n - b_n| = |a_n - c_n| \leq \varepsilon \text{ for any } n \geq K$$

$\square$

5. **(Multiplication is well-defined).** Let $x = \lim_{n\to\infty} a_n$, $y = \lim_{n\to\infty} b_n$, and $x' = \lim_{n\to\infty} a'_n$ be real numbers. Then xy is also a real number. Furthermore, if $x = x'$, then $xy = x'y$.
Proof. Since (a_n) and (b_n) are Cauchy sequences, for any $\varepsilon > 0$ there must exist an integer $N, M \geq 1$ such that for any $n, m \geq K := \max(N, M)$

$$|a_n - a_m| \leq \varepsilon \text{ and } |b_n - b_m| \leq \varepsilon.$$

So our goal is to show that

$$|b_n a_n - b_m a_m| \leq \varepsilon.$$

We first note that $b_n a_n - b_m a_m = (b_n - b_m)a_n + b_m(a_n - a_m)$, hence

$$|(b_n - b_m)a_n + b_m(a_n - a_m)| \leq |(b_n - b_m)a_n| + |b_m(a_n - a_m)|.$$

By problem 1, (a_n) and (b_n) are bounded, so there must exist a rational number $M \geq 0$ such that $|a_n|, |b_n| \leq M$ for any $n \geq 1$ (where we found M as the maximum between the bounds of both sequences). This means that the former inequality is bounded by

$$M(|b_n - b_m| + |a_n - a_m|) \leq 2M\varepsilon.$$

This finishes up our proof as $2M\varepsilon > 0 \iff \varepsilon > 0$.

For the next part of the problem, let $x' = \lim_{n\to\infty} a'_n$. The goal is to show that given any $\varepsilon > 0$ there exists a natural number N such that for any $n, m \geq N$

$$|b_n a_n - b_n a'_n| \leq \varepsilon,$$

as we want them to be equivalent sequences. Hence, given that $x = x'$ then (a_n) is equivalent to (a'_n) so for any $\varepsilon/K > 0$ there exists some N' such that for any $n, m \geq N'$

$$|a_n - a'_n| \leq \frac{\varepsilon}{K}.$$

Since (b_n) is bounded by $K \in \mathbb{Q}$ then

$$|b_n a_n - b_n a'_n| = |b_n(a_n - a'_n)| \leq |b_n||a_n - a'_n| \leq K\frac{\varepsilon}{K} = \varepsilon.$$

$\square$

6. Let a and b be rational numbers. Show that $a = b$ if and only if $\lim_{n\to\infty} a = \lim_{n\to\infty} b$ (i.e., the Cauchy sequences $a, a, a, a, \ldots$ and $b, b, b, b, \ldots$ are equivalent if and only if $a = b$).
Solution. We claim that for any a, b rational numbers, then $|a - b| \leq \varepsilon$ for any $\varepsilon \geq 0$ if and only if $a = b$. To prove this claim assume for the sake of contradiction that if $|a - b| \leq \varepsilon$ for any $\varepsilon \geq 0$ then $a \neq b$. Then without loss of generality assume $a > b$. Hence $|a - b| = a - b$. Then, if we take $\varepsilon_0 = 0$, $a - b = 0$, which implies that $a = b$. Conversely, if $a = b$, then $|a - b| = |0| = 0$ which is, by hypothesis, true.
Then, let $a = b$. Suppose for the sake of contradiction that $\lim_{n\to\infty} a_n \neq \lim_{n\to\infty} b_n$. So there must exist some $\varepsilon_0 \geq 0$ such that for any $N \geq 1 : |a_n - b_n| > \varepsilon_0$ for some values of $n \geq N$. Fixing this n we get that $|a_n - b_n| = |a - a| = 0 > \varepsilon_0$, a contradiction, since $\varepsilon_0 > 0$.
Conversely, suppose that given two equal real numbers $\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n$ such that $a_n = a, b_n = b$ for any $n \geq 1$, implies that $a \neq b$. Since $\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n$ both sequences (a_n) and (b_n) are equivalent. Hence, for any positive number $\varepsilon \geq 0$ there exists an N such that:

$$|a - b| \leq \varepsilon$$

for every $n \geq N$. Since a and b do not depend in the choice of N, by the last claim, $a = b$.

$\square$

7. Show that $\lim_{n\to\infty} \frac{1}{n} = 0$.
Solution. First we show that this sequence is Cauchy. We have to show that, for any $\varepsilon > 0$ there exists some integer $N \geq 1$ such that for any $n, m \geq N$

$$\left|\frac{1}{n} - \frac{1}{m}\right| \leq \varepsilon.$$

By Archimedean property, we can find

$$\varepsilon/2 > \frac{1}{M}$$

where M is a positive integer, hence

$$M > 2/\varepsilon.$$

So for $n, m \geq M$

$$\frac{1}{m}, \frac{1}{n} < \varepsilon/2.$$

And note that

$$\left|\frac{1}{n} - \frac{1}{m}\right| \leq \left|\frac{1}{n} + \frac{1}{m}\right| \leq 2/M \leq \varepsilon.$$

So this sequence is a Cauchy sequence, and it converges.

Now we are in position to ask whether this sequence converges to a positive or negative real number. And we claim that neither these cases are possible. To prove this claim suffices proving that this sequence is not bounded away from zero.

We firstly note that the sequence $1/n$ cannot be negative. Suppose that there existed some rational number $c > 0$ in which $1/n \geq c$ for any $n \geq 1$. This implies that if this were the case, $n \leq c$. By Archimedean property we can find some $N \geq 1$ in which

$$\frac{1}{n} < Nc$$

So suffices letting $a_{Nn} = 1/Nn$ to show that $a_n < c$. A more rudimentary method for this last part would be noting that $c := p/q$ with p, q positive integers, then one can let $n = q$ so $a_q < c$.

$\square$

8. Let $(a_n)_{n=0}^{\infty}$ be a sequence of rational numbers which is bounded. Let $(b_n)_{n=0}^{\infty}$ be another sequence of rational numbers which is equivalent to $(a_n)_{n=0}^{\infty}$. Show that $(b_n)_{n=0}^{\infty}$ is also bounded.
Solution. If one does not want to use problem 3, one can proceed as follows. Let $K \in \mathbb{R}$ be a bound for (a_n), this means that for any $n \in \mathbb{N}$

$$|a_n| \leq K$$

As (b_n) is equivalent to (a_n), then for any $\varepsilon \geq 0$, there exist some $N \geq 1$ such that for every $n \geq N$:

$$|a_n - b_n| \leq \varepsilon$$

This implies that

$$a_n - \varepsilon \leq b_n \leq \varepsilon + a_n \text{ for any } n \geq N.$$

Let $\varepsilon = K$, then for any $n \geq N_K$

$$b_n \leq K + a_n \leq K + K = 2K.$$

and

$$b_n \geq a_n - K \geq -2K.$$

If $N_K = 1$ then we are done, else if $N_K > 1$ we can find an upper bound M for $(b_n)_{n=1}^{N_K-1}$ and conclude that

$$|b_n| \leq \max\{2K, M\} \text{ for any } n \geq 1.$$

$\square$

9. **(Basic properties of positive reals).** For every real number x, exactly one of the following three statements is true: (a) x is zero; (b) x is positive; (c) x is negative. A real number x is negative if and only if $-x$ is positive. If x and y are positive, then so are $x + y$ and xy.
Solution.

   (a) Suppose for the sake of contradiction that a number can be both zero and positive. Let $x := \lim_{n\to\infty} a_n$, $x > 0$ and $0 := \lim_{n\to\infty} 0_n$.
   This means that (a_n) is equivalent to (0_n), thus, for any $\varepsilon > 0$, there exists an N such that for any $n \geq N$:

   $$|a_n| \leq \varepsilon$$

   By definition of positive real number, we know that there exists some lower bound $c \in \mathbb{Q}$, $c > 0$ such that $a_n \geq c$ for any $n \geq 1$. Thus, let $\varepsilon = c - 1$, this implies that $a_n < c$ for every $n \geq N_c$, which is a contradiction.

$\blacksquare$

(b) Suppose that a real number can be both negative and positive simultaneosly, let $x := \lim_{n\to\infty} a_n$, $x > 0$ and $y := \lim_{n\to\infty} b_n$, $y < 0$ be real numbers. Our assumption implies that (a_n) and (b_n) are equivalent. This means that for any $\varepsilon \geq 0$, there exists some $N \geq 1$ such that for any $n \geq N$,

$$|a_n - b_n| \leq \varepsilon$$

By definition of negative real number, there exists a rational number $-c < 0$ such that $b_n \leq -c$. This implies that $b_n < 0$, furthermore, it implies that $b_n = -|b_n|$. Thus:

$$|a_n - b_n| = |a_n + |b_n|| \leq a_n + |b_n| \leq \varepsilon.$$

This implies that

$$a_n \leq \varepsilon + b_n$$

Let $\varepsilon = c$, it follows that:

$$a_n \leq c + b_n \leq c + (-c) = 0$$

This is absurd, since a_n is a strictly positive rational number.

$\blacksquare$

(c) Now, we are going to prove that if a real number $x \in \mathbb{R}$ is negative, then $-x$ is positive, and viceversa.

For the direct implication, let (a_n) be a Cauchy sequence of rational numbers negatively bounded away from 0, then $x := \lim_{n\to\infty} a_n$ will be a negative real number.

Suppose, for the sake of contradiction, that the sequence $(-a_n)$ is not a Cauchy sequence positively bounded away from 0.

If that were the case, then there must exist some $n_0 \geq 0$ such that $-a_{n_0} \leq 0$. This implies that $-a_{n_0} \leq -c$, where $-c \leq 0$. Hence $a_{n_0} \geq c$. This would imply that (a_n) is not negatively bounded away from zero, which is a contradiction, by our hypothesis.

Now, we would like to verify that $\lim_{n\to\infty} a_n + \lim_{n\to\infty}(-a_n) = 0$, to do this we might prove that their absolute value is the same "at the limit". This is straightforward since $|a_n| = |-a_n|$.

The converse implication is analogous.

$\blacksquare$

(d) We claim that if x, y are both positive real numbers, then $x + y$ is positive. Suppose, for the sake of contradiction, that the sequence $(a + b)_n := (a_n + b_n)$ is not positively bounded away from zero. This would imply that there exists some $n_0 \geq 1$ such that $(a + b)_{n_0} \leq 0$, thus $a_{n_0} + b_{n_0} \leq 0$. This would imply that a_{n_0} or b_{n_0} are negative or zero, which is contradictory to our hypothesis. An analogous reasoning is used to prove the case $x \times y > 0$.

$\square$

10. Prove that given any two real numbers $x < y$, we can find a rational number q such that $x < q < y$.
Solution. We use the results proved in problem 11. Since $y - x > 0$, then by Archimedean property there exists a positive integer N such that

$$N(y - x) > 1$$

hence

$$Ny > Nx + 1.$$

By problem 11,

$$\lfloor Nx \rfloor + 1 > Nx \geq \lfloor Nx \rfloor$$

We see that

$$Ny > \lfloor Nx \rfloor + 1 > Nx$$
$$(\text{since } Ny > Nx + 1 > \lfloor Nx \rfloor + 1)$$

and therefore

$$y > \frac{\lfloor Nx \rfloor + 1}{N} > x$$

$\square$

11. Show that for every real number x there is exactly one integer N such that $N \leq x < N + 1$. (This integer N is called the integer part of x, and is sometimes denoted $N = \lfloor x \rfloor$.)
First Solution. Suppose that there existed two positive integers N, M such that

$$N \leq x < N + 1 \text{ and } M \leq x < M + 1.$$

This would imply that

$$N - M \leq 0 \leq N - M.$$

This implies that $N = M$.

For the existence part, without loss of generality, assume $x > 0$, this implies that x has a decomposition in a positively lowerly bounded Cauchy sequence. Thus, there must exist a rational number $c > 0$ such that $a_n \geq c$ for any integer $n \geq 0$. Also, this sequence is upperly bounded by some rational $K \geq 0$. These two rationals are our first approximation to N:

$$c \leq a_n \leq K.$$

As c, K and a_n are rational numbers, there must exist a natural number N such that

$$0 \leq c \leq a_n \leq K \leq N$$

In fact, suppose $|c - a_n| \leq 1$, then the natural number 1 will satisfy that $0 \leq c \leq a_n \leq 1$, similarly with a_n and K.

Formarly speaking, as (a_n) is a Cauchy sequence, let $\varepsilon \geq 0$, let N_ε be a positive integer; for any $n \geq N_\varepsilon$

$$|a_n - a_m| \leq \varepsilon$$

and

$$|a_n - a_m| \leq |c - K|$$

We determine N by the following sequence of steps:
1) Start with setting $N_0 := K$ and $N_0' := c$, this is, N_0 is a rational upper bound of (a_n) and N_0' is a rational lower bound of (a_n). Also set $\varepsilon = |c - N_0|$. Then, since (a_n) is a Cauchy sequence, there exists $N_\varepsilon \geq 1$ such that for any $n, m \geq N_\varepsilon$

$$|a_n - a_m| \leq \varepsilon = |N_0' - N_0|.$$

2) Lowerly bound N_0 by an integer N_1. Upperly bound N_0' by an integer N_1'.

Set $\varepsilon = |N_1' - N_1|$. Do the same argument as for (1) and the result is

$$|a_n - a_m| \leq |N_1' - N_1|.$$

3) While $|N_i' - N_i| \geq 1$ repeat step (2).

At the end, we will get some $N_k > N_k'$ such that $1 = |N_k - N_k'| \geq |a_n - a_m|$. This implies that $N_k \geq \lim_{n\to\infty} a_n > N_k'$ or $N_k > \lim_{n\to\infty} a_n \geq N_k'$. One can show this last thing by seeing that the sequence (a_{n+N_ε}) is upperly bounded by N_k and lowerly bounded by N_k', this means that

$$N_k \geq a_m > N_k'$$

thus

$$N_k \geq \lim_{n\to\infty} a_{n+N_\varepsilon} > N_k'.$$

Since (a_n) and (a_{n+N_ε}) converge to the same limit, this concludes our proof. Note that since their difference is 1 by construction then $N_k = N'_k + 1$.

Second Solution. For the existence part, let $S := \{n \in \mathbb{Z} : n \leq x\}$ and $S' := \{n \in \mathbb{Z} : n > x\}$. We first claim that these sets are non-empty. Note that we still do not have the least upper bound property, but we can proceed by Archimedean property. Let $\varepsilon = 1$, then there exists a positive integer M such that $M\varepsilon > x$, this integer is just $N + 1$ (if it was N it might occur that it were equal to x), this implies that $S' \neq \emptyset$.

Now, for S, taking some positive integer M that is less than a lower bound of (a_n) suffices; bounding a rational by an integer is always possible.

Thus, S' has minimum and it is $N + 1$, therefore $N \notin S'$. This implies that $N \leq x$, thus $N \in S$. We conclude that $N = \max(S)$. Thus: $N + 1 > x \geq N$.

$\square$

12. Show that for any positive real number $x > 0$ there exists a positive integer N such that $x > \frac{1}{N} > 0$.
First Solution. By the last problem, we can find a minimal integral upper bound M for x such that

$$M > x \geq M - 1.$$

Also, we can find a rational number $q = m/n$, with $0 \leq n + 1 < m$ such that

$$M > m/n \geq x \text{ and } x \geq (m - n)/n.$$

In particular,

$$x > (m - n - 1)/n.$$

Let $N := n$, this results in:

$$x > (m - n - 1)/n \geq (m - n - 1)/n(m - n - 1) = 1/n.$$

Since $n + 1 < m$, then $m - n - 1 > 0$, so we are not dividing by zero.

Second Solution. Since $x > 0$, then $1/x$ is also a real number greater than zero. Using the Archimedean property, there exists some N such that for $\varepsilon = 1$,

$$N\varepsilon > 1/x$$

This implies that

$$1/N < x.$$

$\square$

13. Let x, y be real numbers and let $\varepsilon > 0$ be a positive real. Show that $|x-y| < \varepsilon$ if and only if $y - \varepsilon < x < y + \varepsilon$, and that $|x - y| \leq \varepsilon$ if and only if $y - \varepsilon \leq x \leq y + \varepsilon$.
Solution.

($\implies$) For the direct implication note that $r \leq |r|$ and $-r \leq |r|$ for any $r \in \mathbb{R}$. If $|x - y| < \varepsilon$, then

$$x - y \leq |x - y| < \varepsilon$$

so $x < y + \varepsilon$. Similarly,

$$-(x - y) = y - x \leq |x - y| < \varepsilon$$

so $-x < \varepsilon - y$, thus $x > y - \varepsilon$.
We conclude that

$$y - \varepsilon < x < y + \varepsilon.$$

($\impliedby$) For the converse implication, we have that

$$y - \varepsilon < x < y + \varepsilon$$
$$\implies -\varepsilon < x - y < \varepsilon.$$

Recalling the definition of absolute value:

$$|x - y| := \begin{cases} x - y & \text{if } x \geq y \\ -(x - y) & \text{if } x < y \end{cases}$$

In the first case $|x - y| = x - y < \varepsilon$. In the second case $-(x - y) = y - x > -\varepsilon$.
The second claim can be analogously proved.

$\square$

14. Let x and y be real numbers. Show that $x \leq y + \varepsilon$ for all real numbers $\varepsilon > 0$ if and only if $x \leq y$. Show that $|x - y| \leq \varepsilon$ for all real numbers $\varepsilon > 0$ if and only if $x = y$.
Solution. The second part of the exercise has been already proved, in case of rationals, but it is the exact same proof, see problem 6.

If $x \leq y + \varepsilon$ for any real number $\varepsilon > 0$, suppose for the sake of contradiction, that $x > y$. Let $\varepsilon := (x - y)/2$. By hypothesis:

$$x \leq y + \varepsilon$$

Then

$$x \leq y + x/2 - y/2 = (y + x)/2$$

Since we assumed that $x > y$,

$$(y + x)/2 < 2x/2 = x$$

This implies that $x < x$. A clear absurd.

For the converse it is straightforward, since $x \leq y$, then $x \leq y + \varepsilon$.

$\square$

15. Let $(a_n)_{n=1}^{\infty}$ be a Cauchy sequence of rationals, and let x be a real number. Show that if $a_n \leq x$ for all $n \geq 1$, then $\lim_{n\to\infty} a_n \leq x$. Similarly, show that if $a_n \geq x$ for all $n \geq 1$, then $\lim_{n\to\infty} a_n \geq x$.
First Solution. Let $L := \lim_{n\to\infty} a_n$, suppose, for the sake of contradiction, that $L > x$. This would imply that $L - x > 0$, thus, take $\varepsilon = L - x$.

Since (a_n) converges (it is a Cauchy sequence), there exists a natural number $N \geq 1$ such that for any $n \geq N$:

$$|a_n - L| \leq \varepsilon = L - x$$

Thus

$$x \leq a_n \leq 2L - x$$

This implies that $x \geq a_n$ for any $n \geq N$. By hypothesis we also have $x \leq a_n$ for any $n \geq 1$, so we conclude that $x = a_n$ for any $n \geq N$.

Thus (a_{n+N}) becomes a constant sequence, and converges to x. Since $\lim_{n\to\infty} a_{n+N} = \lim_{n\to\infty} a_n$, this implies that $x = L$, which is a contradiction (We supposed that $L > x$).

The second part is analogous.

$\square$

Least upper bound of a subset of $\mathbb{R}$ / Completeness

1. Let E be a subset of the real numbers $\mathbb{R}$, and suppose that E has a least upper bound M which is a real number, i.e., $M = \sup(E)$. Let $-E$ be the set
$$-E := \{-x : x \in E\}.$$
Show that $-M$ is the greatest lower bound of $-E$, i.e., $-M = \inf(-E)$.
Solution. Suppose, for the sake of contradiction, that $-M$ were not the infimum of $-E$.

Two cases are possible.

(a) The first case is that there exists some element $-x \in -E$ such that $-x < -M$. This implies that $x > M$, which is a contradiction since M is the least upper bound of E.

(b) The second case is that $-M$ is not the greatest lower bound of $-E$. Then there exists a rational number q such that

$$-M < q < -x.$$

by problem 10 of the last section. This implies that

$$M > -q > x.$$

Since $-q \neq x$, then $-q \notin E$, so it is a smaller upper bound than M, which is a contradiction since $M := \inf(E)$.

□

2. Let E be a non-empty subset of $\mathbb{R}$, let $n \geq 1$ be an integer, and let $L < K$ be integers. Suppose that $\frac{K}{n}$ is an upper bound for E, but that $\frac{L}{n}$ is not an upper bound for E. Show that there exists an integer $L < m \leq K$ such that $\frac{m}{n}$ is an upper bound for E, but that $\frac{m-1}{n}$ is not an upper bound for E. (Hint: prove by contradiction, and use induction. It may also help to draw a picture of the situation.)

First Solution. Without loss of generality assume $K > 0$. Since $K/n > 1/n$, we can argue that $1/n \in E$ for some $n \geq 1$. Strictly speaking $1/n$ has two possible options, or it belongs in E, or it does not, or it belongs to the subset

$$F := \{m \in \mathbb{Z} : m/n \notin E, K/n \geq m/n \text{ and } m/n \text{ upper bounds } E\}.$$

This is, F is the set of the upper bounds of E that are lesser than K/n. This set is non-empty since $K/n \in F$. Since we do not yet know whether $L/n \in E$, we might want to see between the rationals from L/n to K/n, thus we see any m/n, choosing out from $K - L$ integers.

By induction in $K - L$, we claim that $F \neq \emptyset$ and there exists some m such that $m/n \in F$ but $(m-1)/n \notin F$. Our base step will be when $K - L = 1$, we can choose $m = K$ then $m/n \in F$, and $m - 1 = L$ which makes $(m-1)/n$ not an element of F.

Now suppose that for $K - L \leq k$, $F \neq \emptyset$, and we can find some m such that $m/n \in F$ but $(m-1)/n \notin F$.

Now for our inductive thesis, let $K - L = k + 1$. The set F is non-empty, and particularly, it has a maximum which is K/n. By well-ordering principle, there exists a minimum m/n of F such that $L < m \leq K$. By construction, m/n is an upper bound, and $(m-1)/n$, since $m/n = \min(F)$ must not be in F.

Second Solution. We use the same sets as the last solution.

For the inductive thesis we can partition E with the next relation, let $0 \leq i < K - L$:

$$E_i := \{x \in E : (L + i + 1)/n \geq x > (L + i)/n\}.$$

And let F_i be defined as:

$$F_i := \{x \in \mathbb{R} : (K - i - 1)/n \leq x < (K - i)/n\}.$$

Now, consider overlapping these partitions, starting from their origins (L and K), their intersection will be zero (informally speaking) for several i's, but there is exactly one in which $E_i \subseteq F_i$. Formally speaking, let

$$R_i := F_i \cap E_i.$$

Take the first i_0 on which $R_i \neq \emptyset$, then $E_{i_0} \subseteq F_{i_0}$, this is by the fact that E_{i_0} takes elements of E, which is a subset of $\mathbb{R}$. This, specifically, means that there exists an upper bound m/n with $m = L + i_0 + 1$ which is then an element of F since this m is less than K.

We now test out if $(m-1)/n$ is not an upper bound for E. Note that

$$\frac{m-1}{n} = \frac{L + i_0}{n}.$$

Since $R_{i_0} \neq \emptyset$, then there exists some real number $x \in E$ such that

$$\frac{L + i_0}{n} < x \leq \frac{L + i_0 + 1}{n}.$$

This implies that $(L + i_0)/n$ is not an upper bound for E.

(Was induction even needed in both solutions? Although it is the induction structure, in both cases it is not needed $-$ I think $-$ to use the hypothesis, in the following one, although, I am going to use it clearly.)

Third Solution. This solution operates in the same tracks as the last one, but it is more algorithmic.

We now consider each partition of the last solution as intervals, and we are going to search with two pointers the first interval in which there exists the upper bound m/n with the property that $(m-1)/n$ is not an upper bound.

Cover a subset (or superset) of E with subintervals of lenght $1/n$ starting from L/n and finishing in K/n. We initialize our right pointer p_i in K/n, and our left pointer q_i in L/n, we see that

$$E \cap \bigcup_{i=L/n}^{K/n} E_i \neq \emptyset.$$

Where

$$E_i = \left[\frac{L+q_k}{n}, \frac{K-p_k}{n}\right]$$

Our inductive base will be when $K - L = 1$. In this case suffices choosing left and right pointers equal zero.

Our inductive hypothesis is that within the $K - L = k$ subintervals, one can find some interval

$$\left[\frac{m-1}{n}, \frac{m}{n}\right] = \left[\frac{L+q_k}{n}, \frac{K-p_k}{n}\right]$$

in which $(K - p_k)/n$ upper bounds E but $(L + q_k)/n$ does not.

We claim that if $K-L = k+1$, then there exists some interval $[(m-1)/n, m/n]$ in which m/n is an upper bound of E, but $(m-1)/n$ is not. By inductive hypothesis, there are two possibilities, either the pointer p_i starts from further right, or the pointer q_i starts from further left with difference $1/n$. Either case, the union

$$E \cap \bigcup_{i=L/n}^{K/n} E_i \neq \emptyset.$$

So, since

$$\bigcup_{i=L/n}^{K/n} E_i \subset \bigcup_{i=L/n}^{K+1/n} E_i,$$

we can find some index q_{n_0} and p_{m_0} such that the interval

$$\left[\frac{L+q_{n_0}}{n}, \frac{K-p_{m_0}}{n}\right]$$

has the property that $\frac{L+q_{n_0}}{n}$ is not an upper bound but $\frac{K-p_{m_0}}{n}$ is.

□

3. Let E be a non-empty subset of $\mathbb{R}$, let $n \geq 1$ be an integer, and let m, m' be integers with the properties that $\frac{m}{n}$ and $\frac{m'}{n}$ are upper bounds for E, but $\frac{m-1}{n}$ and $\frac{m'-1}{n}$ are not upper bounds for E. Show that $m = m'$. (Hint: again, drawing a picture will be helpful.)
   Solution. Suppose for the sake of contradiction that $m < m'$. Then, $m \leq m' - 1$ and
   $$\frac{m}{n} \leq \frac{m'-1}{n}.$$

This implies that $\dfrac{m'-1}{n}$ is an upper bound for E, which is a contradiction. Thus, $m \geq m'$.

If we suppose $m > m'$ then $m-1 \geq m'$ so that in an analogous way as the above case,

$$\frac{m-1}{n} \geq \frac{m'}{n}$$

which is a contradiction since $\dfrac{m-1}{n}$ is not an upper bound. We conclude that $m = m'$.

$\square$

4. Let $q_1, q_2, q_3, \ldots$ be a sequence of rational numbers with the property that $|q_n - q_{n'}| \leq \frac{1}{M}$ whenever $M \geq 1$ is an integer and $n, n' \geq M$. Show that $q_1, q_2, q_3, \ldots$ is a Cauchy sequence. Furthermore, if $S := \lim_{n\to\infty} q_n$, show that $|q_M - S| \leq \frac{1}{M}$ for every $M \geq 1$. (Hint: use Problem 3.)
Solution. Let $(q_n)_{n=1}^{\infty} = q_1, q_2, \ldots$ such as the problem statement. Consider the set $Q_M := \{m/M \in \mathbb{Q} : 1 \geq m/M \geq 1/M\}$, where $0 < m$ is an integer.

We see that $|q_n - q_{n'}|$ is a lower bound of Q_M for every integer $n, n' \geq M$. Let $\varepsilon > 0$ be a rational number, then $\varepsilon = p/q$ for some $p, q \in \mathbb{Z}$ such that $pq > 0$. Then, $|q_n - q_{n'}|$ is a lower bound of Q_q. This means that

$$|q_n - q_{n'}| \leq \frac{1}{q} \leq \frac{p}{q} = \varepsilon.$$

More generally, let $\varepsilon \in \mathbb{R}$ be any real number such that $\varepsilon > 0$, then we can find some $M \in \mathbb{Z}$, $M > 0$ such that $\varepsilon > 1/M$ (by problem 12 of the first section), so we know that $|q_n - q_{n'}|$ is a lower bound of Q_M. This means that

$$|q_n - q_{n'}| \leq \frac{1}{M} \leq \varepsilon.$$

So (q_n) is a Cauchy sequence.
Furthermore, we note that

$$q_{n'} - \frac{1}{M} \leq q_n \leq \frac{1}{M} + q_{n'}.$$

This implies that

$$q_{n'} - \frac{1}{M} \leq S \leq \frac{1}{M} + q_{n'} \text{ (by problem 15);}$$

$$\implies -\frac{1}{M} \leq S - q_{n'} \leq \frac{1}{M}.$$

So we get that

$$|S - q_{n'}| \leq \frac{1}{M}.$$

In particular, since $n' \geq M$ we might let $n' = M$, which ends up our proof.

$\square$

5. Establish an analogue of Proposition 5.4.14 −Problem 10−, in which "rational" is replaced by "irrational".
First Solution. The only irrational number we know yet is $\sqrt{2}$.
We claim that $\sqrt{2}/N$ is irrational for any integer $N \geq 1$.
By induction on N, our base case is when $N = 1$, in this case $\sqrt{2}/N = \sqrt{2}$ which is irrational.
Our inductive hypothesis is that for $N \geq 1$, $\sqrt{2}/N$ is irrational. So now we want to show that for an integer $M = N + 1$, $\sqrt{2}/M$ is also irrational.
For the sake of contradiction, suppose that $\sqrt{2}/M$ is not irrational. If $\sqrt{2}/M$ were rational, then

$$\left(\frac{\sqrt{2}}{M}\right)\left(\frac{M}{N}\right)$$

would also be rational, but this is a contradiction as

$$\left(\frac{\sqrt{2}}{M}\right)\left(\frac{M}{N}\right) = \frac{\sqrt{2}}{N};$$

which by inductive hypothesis is irrational. This concludes the proof of our claim.
Note that $1 < \sqrt{2} < 2$, so we get that

$$\frac{1}{M} < \frac{\sqrt{2}}{M} < \frac{2}{M}.$$

Let $x, y \in \mathbb{R}$ such that $x < y$. We can find a rational number q such that $x < q < y$ by problem 10. We claim that there exists an integer M in which

$$\frac{\sqrt{2}}{M} < y$$

holds. By problem 12 there exists some integer M' such that $y > 1/M'$. Also we have derived that

$$\frac{1}{M'} < \frac{\sqrt{2}}{M'} < \frac{2}{M'},$$

so

$$\frac{\sqrt{2}}{2M'} < \frac{2}{2M'} = \frac{1}{M'}.$$

This means that suffices choosing $M := 2M'$. Furthermore, the following inequalities will hold:

$$x < q < y,$$

$$x < \frac{\sqrt{2}}{M} < y,$$

hence adding them up:

$$2x < q + \frac{\sqrt{2}}{M} < 2y.$$

So we finally get that

$$x < \frac{q}{2} + \frac{\sqrt{2}}{2M} < y.$$

The inner sum cannot be rational, as if it were then both summands would have to be rational, which is not the case.

(Idea of) Solution. We first introduce a way to check out if a real number is rational, this is, let $(a_n)_{n=1}^{\infty}$ be a rational Cauchy sequence, let $(q_n)_{n=1}^{\infty}$ be any constant sequence $q, q, \ldots$ with $q \in \mathbb{Q}$; then (a_n) converges to a rational number q if and only if (a_n) is equivalent to (q_n).

This criterion works well by the last problem, since for any positive integer M

$$|q_n - q_m| = 0 \leq \frac{1}{M}$$

for any $n, m \geq M$, then (q_n) is a Cauchy sequence, and it converges to q. Hence

$$-\frac{1}{M} + q \leq q_n \leq \frac{1}{M} + q.$$

So furthermore, since (a_n) converges to q, for any $\varepsilon > 0$, there exist some $M' \geq 1$ such that for any $n \geq M'$:

$$-\varepsilon + q \leq a_n \leq \varepsilon + q.$$

So let $K := \max(M, M')$ we have that

$$-\varepsilon + \frac{1}{K} \leq a_n - q_n \leq \varepsilon - \frac{1}{K}. \text{ (Subtracting inequalities)}$$

This means that

$$-\varepsilon \leq a_n - q_n \leq \varepsilon \text{ (Since } K \geq 1).$$

So (a_n) is equivalent to (q_n).

Conversely, if (a_n) is equivalent to (q_n), then for any $\varepsilon > 0$, there exists some integer $N \geq 1$ such that for any $n \geq N$:

$$|a_n - q_n| \leq \varepsilon.$$

Then

$$|a_n - q| \leq \varepsilon \text{ (since } q_n = q)$$

So (a_n) converges to q.

We claim that there exists a rational Cauchy sequence that does not converge to a rational number. To prove this claim simply let (a_n) be a sequence in which

$$\lim_{n\to\infty} a_n = \sup\{x \in \mathbb{R} : x \geq 0,\, x^2 < 2\} = \sqrt{2}.$$

This means that the set of rational Cauchy sequences

$$S := \{(a_n)_{n=1}^{\infty} : \lim_{n\to\infty} a_n \notin \mathbb{Q}\}$$

is non-empty.

By the criterion we have already proved, any sequence in S will not be equivalent to any constant rational sequence. Let $(s_n)_{n=1}^{\infty}$ be a sequence in S. One can find some $\varepsilon > 0$ such that for any $N \geq 1$, there exists some $n \geq N$ such that

$$|s_n - q| > \varepsilon.$$

This means that $s_n > \varepsilon + q$.

(Can we finish up this idea? I think an idea such as in the first solution will be eventually needed in spite of the sequence convergence ideas in this one, if not let me know.)

$\square$

6. Show that $\sup\{1 - 1/n : n \in \mathbb{N}\} = 1$.
First Solution. Let $S = \sup\{1 - 1/n : n \in \mathbb{N}\}$. Clearly 1 is an upper bound of the set, and 0 is its minimum.

So suppose, for the sake of contradiction, that 1 were not the least upper bound. Let $s := \sup S$. So, we have that for any $n \in \mathbb{N}$:

$$1 - \frac{1}{n} < s \leq 1.$$

In particular:

$$1 - \frac{1}{n} < s < 1 + \frac{1}{n}$$

For any $n \in \mathbb{N}$. Let $\varepsilon > 0$ be any real number. We know that, there exists some N in which $\varepsilon > 1/N$, so we get that

$$1 - \varepsilon < s < 1 + \varepsilon;$$

thus

$$-\varepsilon < s - 1 < \varepsilon$$

So by problem 14, $s = 1$.

Second Solution. Let $(a_n) := 1$, and $(b_n) := 1/n$ be convergent sequences. We claim that the sequence $(a_n - b_n)_{n=1}^{\infty}$ is increasing and bounded. Clearly $(a_n - b_n)_{n=1}^{\infty}$ is bounded by 1. Then:

$$1 - \frac{1}{n} < 1 - \frac{1}{n+1}.$$

So this proves our claim. Then, by limit laws theorem (see next section) we have that

$$\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n = 1 - 0 = 1.$$

Since this sequence is bounded and increasing, it converges to $\sup\{1-1/n : n \in \mathbb{N}\}$.

$\square$

7. If $S := \{1/n - 1/m : n, m \in \mathbb{N}\}$, find $\inf S$ and $\sup S$.
Solution. We claim that 1 is the supremum of this sequence, and -1 is its infimum. To prove our claim suppose that 1 were an upper bound but not the supremum of S.

Let $s := \sup S$, this implies that for any $n, m \geq 1$

$$\frac{1}{n} - \frac{1}{m} \leq s < 1.$$

Let q be a rational number satisfying $s < q < 1$. This implies that its numerator will be less than its denominator. Let $1 - 1/m \in S$ with $m \geq 1$ any positive integer. Then, by construction

$$q = \frac{a}{b} > 1 - \frac{1}{m}$$

In particular, choose $m := b$ then $1/b < q$, so

$$\frac{a}{b} > \frac{b-1}{b}.$$

So $a > b - 1$, where a, b are positive integers, so $a \geq b$, this implies that $q \geq 1$, which is a contradiction. (We assumed $q < 1$.)

Moreover, let $-S := \{1/m - 1/n : n, m \in \mathbb{N}\}$. This set is the same as S. So by problem 1 of this section we know that $-\sup S = \inf -S$, so $-1 = \inf -S = \inf S$.

$\square$

8. Let $S \subseteq \mathbb{R}$ be nonempty. Prove that if a number $u \in \mathbb{R}$ has the properties:

   (i) For every $n \in \mathbb{N}$, the number $u - \frac{1}{n}$ is not an upper bound of S,

   (ii) For every $n \in \mathbb{N}$, the number $u + \frac{1}{n}$ is an upper bound of S,

   then $u = \sup S$.

   Solution. Let $\varepsilon > 0$ be a positive real number, then one can find a positive integer N in which $\varepsilon > 1/N$. Let $s = \sup S$, which exists since $S \neq \emptyset$. Note that the hypothesis essentially says that

   $$u - \frac{1}{n} \leq s \leq u + \frac{1}{n}.$$

   Since s is the supremum of S. Hence, for any n there must exist some $\varepsilon > 0$ in which $\varepsilon > 1/n$,

   $$-\varepsilon \leq |s - u| \leq \varepsilon.$$

   Which implies that $s = u$.

   $\square$

9. Let S be a nonempty bounded set in $\mathbb{R}$.

(a) Let $a > 0$, and let $aS := \{as : s \in S\}$. Prove that

$$\inf(aS) = a \inf S, \quad \sup(aS) = a \sup S.$$

(b) Let $b < 0$ and let $bS = \{bs : s \in S\}$. Prove that

$$\inf(bS) = b \sup S, \quad \sup(bS) = b \inf S.$$

Solution. This problem provides a more general property to that of problem 1.

(a) First suppose that $a \inf S$ were not the infimum of aS. This implies that there exists some $ax \in aS$, such that $ax \le a \inf S$, or that $a \inf S$ is not a maximal lower bound.

In the first case we see that if $ax \le a \inf S$, then $x \in S$ and $x \le \inf S$, which is absurd.

In the second case, if there existed an infimum $M := \inf(aS)$ such that $M \ge a \inf S$, then $M/a \ge \inf S$, this implies that either $M/a \in S$, or that there exists some $x \in S$ such that $x \le M/a$ (both cases come by the fact that M/a will not be a lower bound for S).

This would imply respectively that, $M \in aS$, or that $ax \le M$, a contradiction since M is the infimum of aS.

In the case of the least upper bounds the reasoning is analogous.

(b) Since $b < 0$, let $a = |b|$, we have that $b = -a$. By part a, the set aS has supremum $a \sup S$, so by problem 1 of this section the set $-(aS)$ has infimum $-\sup aS = -a \sup S$. We conclude that $\inf bS = b \sup S$

The remaining case is analogous as well.

□

10. Let S be a set of nonnegative real numbers that is bounded above and let $T := \{x^2 : x \in S\}$. Prove that if $u = \sup S$, then $u^2 = \sup T$. Give an example that shows the conclusion may be false if the restriction against negative numbers is removed.
Solution. Suppose that u^2 were not $\sup T$, this would imply that $u^2 \le x^2$ for some $x^2 \in T$. Note if that were the case, then $u \le x^2/u$.

So since $u \ge x$ for any $x \in S$,

$$u \le \frac{x^2}{u} \le \frac{x^2}{x} = x.$$

This implies that $u \le x$, so u is not $\sup S$, a contradiction.

For the last part, let $S := \{x \in \mathbb{R} : -1 < x < 0\}$. The supremum of this set is 0, but the supremum of T should be 1, which is not 0^2.

□

11. Let X be a nonempty set and let $f : X \to \mathbb{R}$ have bounded range in $\mathbb{R}$. If $a \in \mathbb{R}$, show that Example 2.4.1(a) implies that

$$\sup\{a + f(x) : x \in X\} = a + \sup\{f(x) : x \in X\}.$$

Show that we also have

$$\inf\{a + f(x) : x \in X\} = a + \inf\{f(x) : x \in X\}.$$

Remark. Example 2.4.1 (a): Let S be a nonempty set that is bounded above, let $a + S := \{a + s : s \in S\}$, then $\sup(a + S) = a + \sup S$.

Solution. Let $f(X) := \{f(x) : x \in X\}$ be defined as the range of X. Since $f(X)$ is nonempty and it is bounded above, it has a supremum $\sup f(X)$. We define $a + f(X)$ as $\{a + f(x) : f(x) \in f(X)\}$. By example 2.4.1 this set has a supremum $a + \sup f(X)$.

Now, we show that $A := \{a + f(x) : f(x) \in f(X)\}$ is the same set as $B := \{a + f(x) : x \in X\}$. By replacement axiom let $a + f(x) \in A$, then $a + f(x)$ satisfies that $f(x) \in f(X)$ for some $f(x)$. This means, by definition of $f(X)$, that $x \in X$.

□

12. Let A and B be bounded nonempty subsets of $\mathbb{R}$, and let $A+B := \{a+b : a \in A, b \in B\}$. Prove that

$$\sup(A + B) = \sup A + \sup B \quad \text{and} \quad \inf(A + B) = \inf A + \inf B.$$

Solution. We firstly show that $\sup A+\sup B \geq \sup(A+B)$, and secondly that $\sup A + \sup B \leq \sup(A + B)$.

For the first part suppose that the inequality were strict, thus $\sup A + \sup B > \sup(A + B)$, then one can find some $x \in \mathbb{R}$ such that

$$\sup A + \sup B > x > \sup(A + B).$$

If this were the case, x would also be upper bound for $A + B$, meaning that for any $a \in A$ and $b \in B$, $x \geq a + b$.

But since $\sup A + \sup B > x$, then $x \leq a_0 + b_0$ for some $a_0 \in A$, $b_0 \in B$, since x would be upperly bounded by $\sup A + \sup B$. So there is no other option for x but to be equal to $a_0 + b_0$ using the last paragraph. Thus $\sup A + \sup B \geq \sup(A + B)$.

For the second part, we argue analogously, and get that $\sup A + \sup B \leq \sup(A + B)$. We conclude that $\sup A + \sup B = \sup(A + B)$.

□

13. Let X be a nonempty set, and let f and g be defined on X and have bounded ranges in $\mathbb{R}$. Show that

$$\sup\{f(x)+g(x) : x \in X\} \leq \sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}$$

and that

$$\inf\{f(x) : x \in X\} + \inf\{g(x) : x \in X\} \leq \inf\{f(x)+g(x) : x \in X\}.$$

Give examples to show that each of these inequalities can be either equalities or strict inequalities.
Solution. Let $N \in \mathbb{R}$, $N \geq 0$ be such that $|f(x)| \leq N$ for any $x \in X$, and let $M \in \mathbb{R}$, $M \geq 0$ be such that $|g(x)| \leq M$ for each $x \in X$. We note that

$$\sup\{f(x)+g(x) : x \in X\} \leq \sup\{N+g(x) : x \in X\}$$

thus

$$\sup\{f(x)+g(x) : x \in X\} \leq N + \sup\{g(x) : x \in X\} \text{ (by problem 11).}$$

In particular since one can choose $N := \sup\{f(x) : x \in X\}$ (but it would not be necessarily $|f(x)|$ but only $f(x)$ for any $x \in X$).

The case for the infima is analogous.

□

14. Let $X = Y := \{x \in \mathbb{R} : 0 < x < 1\}$. Define $h : X \times Y \to \mathbb{R}$ by $h(x,y) := 2x + y$.
(a) For each $x \in X$, find $f(x) := \sup\{h(x,y) : y \in Y\}$; then find $\inf\{f(x) : x \in X\}$.
(b) For each $y \in Y$, find $g(y) := \inf\{h(x,y) : x \in X\}$; then find $\sup\{g(y) : y \in Y\}$. Compare with the result found in part (a).
Solution. (a) We note that $h(x,y)$ is a linear mapping. Fix $x \in (0,1)$, this will make us look at the set

$$\{2x + y : y \in Y\},$$

thus,

$$\sup\{2x+y : y \in Y\} = 2x + \sup\{y : y \in Y\} \text{ (by problem 11).}$$

Since $\sup Y = 1$, the function results in $f(x) = 2x + 1$. Then, $f(x)$ is also linear and $\inf\{f(x) : x \in X\}$ will be when f approaches 0, thus it will result into 1.

(b) Let $y \in (0,1)$, then $\inf\{2x + y : x \in X\} = y + \inf\{2x : x \in X\} = y + 0 = y$. Thus, $g(y) = y$. Then $\sup\{g(y) : y \in Y\}$ will be 1.

$\square$

15. Perform the computations of the preceding exercise for the function $h : X \times Y \to \mathbb{R}$ defined by

$$h(x, y) := \begin{cases} 0 & \text{if } x < y, \\ 1 & \text{if } x \geq y. \end{cases}$$

Solution. (a) Informally speaking, this function describes the same as if we had an lower triangular matrix filled with ones (although clearly, indices are discrete).

We get that $f(x) = \sup\{h(x, y) : y \in Y\}$ will be 1, since for any $x \in X$ we can find $y \in Y$ such that $x \geq y$ (if the interval X were closed this may not occur as we could let $x = 0$). The infimum of f will be 1 as well, since it will never output 0.

(b) For finding $g(y) := \inf\{h(x, y) : x \in X\}$ fix $y \in Y$, as the other case, one can find $x \in X$ such that $x < y$ for any $y \in Y$. Hence $g(y) = 0$. Thus its supremum will also be 0.

$\square$

2 Real number sequences

1. **(Convergent sequences are Cauchy).** Suppose that $(a_n)_{n=m}^{\infty}$ is a convergent sequence of real numbers. Then $(a_n)_{n=m}^{\infty}$ is also a Cauchy sequence.

Proof. Since (a_n) is a convergent sequence of real numbers, then there exists a finite real number $L := \lim_{n\to\infty} a_n$ such that for any $\varepsilon > 0$, there exists some integer $N \geq 1$ in which for any $n \geq N$:

$$|a_n - L| \leq \varepsilon/2;$$

so let $m \geq M$ for some positive integer M, take $K := \max(N, M)$, we get that

$$|a_n - L| + |L - a_m| \leq \varepsilon$$

Then, by triangle inequality:

$$|(a_n - L) + (L - a_m)| = |a_n - a_m| \leq \varepsilon$$

for any $n, m \geq K$.

$\square$

2. **(Limit Laws).** Let $(a_n)_{n=m}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ be convergent sequences of real numbers, and let x, y be the real numbers $x := \lim_{n\to\infty} a_n$ and $y := \lim_{n\to\infty} b_n$.

   (a) The sequence $(a_n + b_n)_{n=m}^{\infty}$ converges to $x + y$; in other words,

   $$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n.$$

   (b) The sequence $(a_n b_n)_{n=m}^{\infty}$ converges to xy; in other words,

   $$\lim_{n\to\infty}(a_n b_n) = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right).$$

   (c) For any real number c, the sequence $(ca_n)_{n=m}^{\infty}$ converges to cx; in other words,

   $$\lim_{n\to\infty}(ca_n) = c\lim_{n\to\infty} a_n.$$

   (d) The sequence $(a_n - b_n)_{n=m}^{\infty}$ converges to $x - y$; in other words,

   $$\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n.$$

   (e) Suppose that $y \neq 0$, and that $b_n \neq 0$ for all $n \geq m$. Then the sequence $(b_n^{-1})_{n=m}^{\infty}$ converges to y^{-1}; in other words,

   $$\lim_{n\to\infty} b_n^{-1} = \left(\lim_{n\to\infty} b_n\right)^{-1}.$$

   (f) Suppose that $y \neq 0$, and that $b_n \neq 0$ for all $n \geq m$. Then the sequence $\left(\frac{a_n}{b_n}\right)_{n=m}^{\infty}$ converges to $\frac{x}{y}$; in other words,

   $$\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}.$$

   (g) The sequence $(\max(a_n, b_n))_{n=m}^{\infty}$ converges to $\max(x, y)$; in other words,

   $$\lim_{n\to\infty}\max(a_n, b_n) = \max\left(\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n\right).$$

   (h) The sequence $(\min(a_n, b_n))_{n=m}^{\infty}$ converges to $\min(x, y)$; in other words,

   $$\lim_{n\to\infty}\min(a_n, b_n) = \min\left(\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n\right).$$

   Solution.

(a) Let $\varepsilon/2 > 0$, then there must exist integers $N, M \geq 1$ in which for any $n \geq N, M$;

$$|a_n - x| \leq \varepsilon/2 \text{ and } |b_n - y| \leq \varepsilon/2.$$

Then, for $n \geq K := \max(N, M)$ we have that

$$\varepsilon \geq |a_n - x| + |b_n - y| \geq |(a_n - x) + (b_n - y)| = |a_n + b_n - (x + y)|.$$

(b) We know that there must exist integers $N, M \geq 1$ in which for any $n \geq N, M$;

$$|a_n - x| \leq \varepsilon \text{ and } |b_n - y| \leq \varepsilon.$$

Particularly, for $n \geq K := \max(N, M)$ both inequalities shown above hold. We have to prove that for any $\varepsilon > 0$, there exists an integer K' such that for any $n \geq K'$:

$$|a_n b_n - xy| \leq \varepsilon.$$

We can note that $a_n b_n - xy = b_n(a_n - x) + x(b_n - y)$, hence for any $n \geq K$;

$$|b_n(a_n - x) + x(b_n - y)| \leq |b_n(a_n - x)| + |x(b_n - y)| \leq \varepsilon(|b_n| + |x|).$$

But we know that b_n is bounded by some $c > 0$, so $|b_n| \leq c$

$$|a_n b_n - xy| \leq \varepsilon(2c) \text{ (since } b_n \leq c \text{ implies } x \leq c).$$

So, in fact we note that $\varepsilon > 0 \iff \varepsilon/2c > 0$, which then implies that the above inequality is bounded by ε. So suffices letting $K' := K$.

(c) Let $c \neq 0$. Since (a_n) converges, for any ε/c there must exist an integer $N \geq 1$ such that

$$|a_n - x| \leq \varepsilon/c.$$

for any $n \geq N$. In particular we note that

$$|ca_n - cx| = |c||(a_n - x)| \leq c\varepsilon/c = \varepsilon.$$

If $c = 0$ then $(ca_n) = 0, 0, \ldots$ so

$$|ca_n - cx| = 0 < \varepsilon.$$

(d) Apply the result of (a) to the sequences (a_n) and $(-b_n)$.

(e) Since we have that $b_n \neq 0$ for any $n \geq m$ and it converges to a nonzero real number we can ask when does it becomes bounded away from zero. This is, we claim that there exists some integer $N \geq m$ in which for any $n \geq N$,

$$|b_n| \geq c \text{ for some real number } c > 0.$$

To prove this claim, assume $y > 0$ (so we are proving, in particular, that b_n is positively bounded away from zero). We first see that for any $\varepsilon > 0$, there exists some $M \geq m$ such that

$$y - \varepsilon \leq b_n \leq \varepsilon + y.$$

So we can choose, say, $\varepsilon := (y-c)/2$, with $0 < c < y$. Hence, $y-(y-c)/2 \leq b_n$, so

$$\frac{y+c}{2} \leq b_n.$$

Thus,

$$\frac{2c}{2} = c \leq b_n \text{ for any } n \geq M.$$

This proves our claim, we can prove analogously that if $y < 0$, the sequence is eventually negatively bounded away from zero.

Now we claim that for any $\varepsilon > 0$, there exists some $N \geq m$ such that for any $n \geq N$:

$$\left| \frac{1}{b_n} - \frac{1}{y} \right| \leq \varepsilon.$$

Then we have to prove that

$$\frac{|y - b_n|}{|b_n y|} \leq \varepsilon.$$

The numerator is simply $|b_n - y|$ that we know that is less than or equal to ε starting at some $N \geq m$. And since the denominator is bounded away from zero by our claim, then clearly this quotient is bounded by ε. Formally speaking, since b_n converges to y, then for any $cy\varepsilon > 0$

$$\frac{|b_n - y|}{b_n y} \leq \frac{|b_n - y|}{cy} \leq (cy\varepsilon)/cy \leq \varepsilon.$$

(f) Using part (e) and part (b) the result follows, as we multiply (a_n) by (b_n^{-1}).

(g) Without loss of generality assume that $x \geq y$. Since (a_n) converges to x and (b_n) converges to y, we can ask when does (a_n) becomes "larger" than (b_n).

This is, we claim that for some $N \geq m$, for any $n \geq N$: $a_n \geq b_n$. To prove this claim, let $\varepsilon := (x - y)/2$, we know that there must exist some $N_\varepsilon, M_\varepsilon \geq m$ such that for any $n \geq N_\varepsilon$

$$|a_n - x| \leq (x-y)/2 \text{ and } |b_n - y| \leq (x-y)/2$$

Hence,

$$b_n \leq (x-y)/2 + y = (x+y)/2 \text{ and } (y-x)/2 + x = (x+y)/2 \leq a_n.$$

So the second implies inequality implies that $a_n \geq b_n$ for any $n \geq K := \max(N_\varepsilon, M_\varepsilon)$. This means that $\lim_{n\to\infty} a_n \geq \lim_{n\to\infty} b_n$, so since starting from K, $\max(a_n, b_n) = a_n$, we conclude that

$$\lim_{n\to\infty} \max(a_n, b_n) = \lim_{n\to\infty} a_{n+K} = x = \max(x,y) \text{ (last thing being by construction).}$$

(h) Analogous proof as item (g), in fact one implies the other, since we can see what happens to the minimum in the same proof.

□

3. Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers, such that $a_{n+1} > a_n$ for each natural number n. Prove that whenever n and m are natural numbers such that $m > n$, then we have $a_m > a_n$. (We refer to these sequences as increasing sequences.)
Solution. This is fairly straightforward, if $m > n$, then there exists a positive integer k such that $m = n + k$, thus $m - k = n$. This means that one can start from a_m and do the following recursive argument

$$a_m > a_{m-1},\ a_{m-1} > a_{m-2}, \ldots,\ a_{m-k+1} > a_{m-k} = a_n.$$

By transitivity, $a_m > a_n$.

□

4. Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers, and let L be a real number. Show that $(a_n)_{n=m}^{\infty}$ converges to L if and only if, given any real $\varepsilon > 0$, one can find an $N \geq m$ such that $|a_n - L| \leq \varepsilon$ for all $n \geq N$.
Is any insight needed here, or both directions are implied by the definition of convergence?

5. Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers, let c be a real number, and let $m' \geq m$ be an integer. Show that $(a_n)_{n=m}^{\infty}$ converges to c if and only if $(a_n)_{n=m'}^{\infty}$ converges to c.
Solution.

($\Longrightarrow$) If $(a_n)_{n=m}^{\infty}$ converges to c, given any $\varepsilon > 0$ one can find some integer $N \geq m$ such that for any $n \geq N$

$$|a_n - c| \leq \varepsilon$$

Then, one can see that for any $n \geq M = \max(N, m')$

$$|a_n - c| \leq \varepsilon.$$

So both sequences converge to c.

$(\impliedby)$ Since $(a_n)_{n=m'}^{\infty}$ converges to c, there exists some $K \geq m'$ such that for any $n \geq K$

$$|a_n - c| \leq \varepsilon.$$

In particular for any $n \geq K \geq m' \geq m$ the above inequality holds. This implies that $(a_n)_{n=m}^{\infty}$ also converges to c.

$\square$

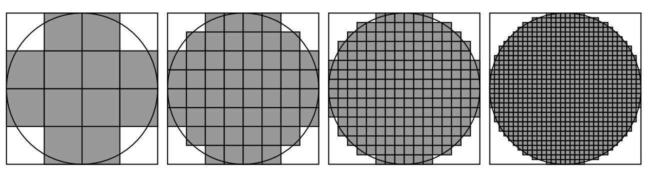
6. Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers, let c be a real number, and let $k \geq 0$ be a non-negative integer. Show that $(a_n)_{n=m}^{\infty}$ converges to c if and only if $(a_{n+k})_{n=m}^{\infty}$ converges to c.
   Solution. First note that if $n \geq m - k$ the sequence $(a_{n+k})_{n=m-k}^{\infty}$ will be simply $(a_n)_{n=m}^{\infty}$, which by hypothesis converges.

   So the sequence $(a_n)_{n=m'}^{\infty}$ where $m' \leq m - k$ will also converge by the last problem, hence: $(a_{n+k})_{n=m'}^{\infty}$ converges, so we also see that $m' \leq m$, hence, by the converse of last problem again $(a_{n+k})_{n=m}^{\infty}$ will converge.

$\square$

3 First application

1. We have the following false visual derivation and we have to prove it wrong, given a 1-sided square, by doing the approximations as it shows we should get that $\pi = 4$.



Proof of falseness.

Definition 1. We will call an $n-$*tessellation* a grid refinement process where each square is divided into four smaller squares of side a_n , doubling the grid resolution in each step.

Definition 2. We will call a *contact point* a point on the unit circle C that is also a vertex of a grid square. The amount of contact points is measured by the sequence (b_n).

We first see that $(a_n)_{n=1}^{\infty}$ converges to zero, since

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{2^n} = 0.$$

We also see that $(b_n)_{n=1}^{\infty}$ does not converge. This is because for each tessellation at least 1 vertex of any grid square is in C, hence

$$b_n \geq \frac{n}{4} \to \infty.$$

Assuming that for any n-tessellation only squares are added, each one adjacent to the former ones we see that since (a_n) converges, then the "taxicab" distance between each adjacent contact point decreases exponentially. So by triangle inequality, their euclidean distance decreases exponentially as well

$$2a_n \geq \sqrt{2}a_n.$$

We define the set

$$X := \{x \in C : x \text{ is a contact point}\},$$

We claim that there exist a point in X that is not ε-adherent for some $\varepsilon > 0$ to some other point of X.

Consider $c \in C$. Suppose that there existed an n-tessellation such that for any $\varepsilon > 0$ given any $q \in X$, the ball of center c and radius ε contains q. I.e. $d(c, q) \leq \varepsilon$ (where $d()$ denotes euclidean distance), this would mean that X is dense.

Let q_0, q_1 be fixed points in X such that $d(q_0, q_1) \leq \sqrt{2}a_n$, these points can be found since for some n-tessellation the euclidean distance between contact points converges to zero.

Furthermore, since C is dense we can find $q_0 < c < q_1$. By our supposition for any $\varepsilon > 0$, $d(q_0, c) \leq \varepsilon$ and $d(c, q_1) \leq \varepsilon$.
Let $3\varepsilon \leq \sqrt{2}a_n$. We have that

$$d(q_0, c) + d(c, q_1) \geq d(q_0, q_1)$$

Therefore,

$$2\varepsilon \geq d(q_0, q_1) \geq \sqrt{2}a_n$$

Hence $3\varepsilon > 2\varepsilon \geq \sqrt{2}a_n$, which is a contradiction.

We conclude that the approximation does not cover the perimeter of C, so saying $\pi = 4$ is false.

$\square$