Linear mappings sheet IV

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Problems in linear algebra: problems from the books of Titu Andreescu "Essential Linear Algebra" and Sheldon Axler "Linear Algebra Done Right". The topics of this week are isomorphisms, change of bases, rank of a matrix and dual spaces.

Isomorphisms and invertibility

1. Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that the following are equivalent:

   (a) T is invertible.

   (b) $Tv_1, \ldots, Tv_n$ is a basis of V for every basis $v_1, \ldots, v_n$ of V.

   (c) $Tv_1, \ldots, Tv_n$ is a basis of V for some basis $v_1, \ldots, v_n$ of V.

   Solution. (a) $\implies$ (b)

2. Suppose V is finite-dimensional, U is a subspace of V, and $S \in \mathcal{L}(U, V)$. Prove that there exists an invertible linear map T from V to itself such that $Tu = Su$ for every $u \in U$ if and only if S is injective.
   Solution. For the direct implication, if T is invertible, then it is injective, The restriction of T to a subset U will be injective, and will behave as expected. The mapping will not be surjective, although this is not an issue.

   For the converse implication, if S is injective, U and $\operatorname{Im} S$ are isomorphic. Hence there exists an isomorphism $S' : V \setminus U \to V \setminus \operatorname{Im} S$. Build T such that

   $$Tv = \begin{cases} Sv & \text{if } v \in U; \\ S'v & \text{if } v \in V \setminus U \end{cases}.$$

   □

   (Idea of) Solution. For the converse implication, if S is injective, we first determine the amount of elements in the intersection

$$U \cap \text{Im } S = \begin{cases} \{0\}; \\ \text{not only zero subspace of } U \text{ or Im } S; \\ \text{a subspace with the same dimension of } U \text{ and Im } S \end{cases}.$$

The first case implies that any basis of U will not be able to span any vector of Im S, the second case implies that there exists a basis of U that spans some vectors of Im S, and the last one implies that a basis of U will also be a basis for Im S. For the third implication there is the identity map, that will do the job of changing any vector of U to its corresponding vector of Im S, as both spaces have same dimension ($\ker S = \{0\}$).

$$u_1 \mapsto Tu_1, \ldots, u_n \mapsto Tu_n.$$

Then, extending this map to V:

$$v_1 \mapsto Tv_1, \ldots, u_1 \mapsto Tu_1, \ldots, u_n \mapsto Tu_n, \ldots, v_m \mapsto Tv_m.$$

Doing the same reasoning as for the direct implication.

For the second case, let $\mathcal{B}$ be a base for U, and let $\mathcal{C}$ be a base for Im S. Consider the subspace $W \subseteq V$, $W = \text{span}(\mathcal{B} \cup \mathcal{C})$, such that $W = U \oplus$ Im S. Let $\varphi_U : W \to U$ be defined as a projection mapping, that is, when valued at some $w \in W$, will output the only $u \in U$ such that for some $x \in$ Im S: $x = u - w$. Define $\varphi_{\text{Im } S} : W \to$ Im T the same way, but interchanging x with u. A map $T : W \to W$ will be

$$Tw = \begin{cases} \varphi_U(w) & \text{if } w \in U; \\ \varphi_{\text{Im } S}(w) & \text{if } w \in \text{Im } S \end{cases}.$$

For $w \in U$, Tw will be itself as $u = x + 0$. Similarly for $x \in$ Im S. Then we can add linearly independent vectors to $\mathcal{B}$ and $\mathcal{C}$.

(Can we finish up this idea? I find it hard to assume both bases will always be linearly independent, besides, the idea turned out too complex.)

3. Suppose that W is finite-dimensional and $S, T \in \mathcal{L}(V, W)$. Prove that $\text{null}\, S = \text{null}\, T$ if and only if there exists an invertible $E \in \mathcal{L}(W)$ such that $S = ET$.
Solution. Let k be the dimension of $\ker T$, let n be the dimension of V and m be the dimension of W. Since $\ker T = \ker S$ then, by the rank-nullity theorem, dim Im T = dim Im S, as the input space is the same for both mappings. Let $\mathcal{B}$ be a basis of V such as

$$\{v_1, \ldots, v_{n-k}, \ldots, v_n\}$$

Where the vectors $v_{n-k}, \ldots, v_n$ are those of a basis of $\ker T$ and $\ker S$. Let $\mathcal{C}$ and $\mathcal{D}$ be bases of W containing the bases of Im T and Im S respectively in a way such as $\{w_1, \dots, w_{n-k}, \dots, w_m\}$, where $w_1, \dots, w_{n-k}$ are vectors from Im T or Im S. Let the matrices $M(T)$ and $M(S)$ be associated to T and S respectively. Consider $Id : W_{\mathcal{C}} \to W_{\mathcal{D}}$, where $Id((w_i)_{\mathcal{C}}) = (w_i)_{\mathcal{D}}$ for vectors w_i of their respective bases; and $1 \leq i \leq n-k$. We have the following diagram

$$\begin{tikzcd} V_{\mathcal{B}} \arrow[r, "T"] \arrow[dr, "S"'] & W_{\mathcal{C}} \arrow[d, leftrightarrow, "Id_{\mathcal{DC}}"] \\ & W_{\mathcal{D}} \end{tikzcd}$$

We see that $E = Id_{\mathcal{DC}}$. So $T = E \circ S$, thus $S = E^{-1} \circ T = Id_{\mathcal{CD}} \circ T$. Reciprocally, if there exists an invertible mapping $E \in \mathcal{L}(W)$ such that $ET = S$, the matrix $M(E)$ will be invertible, then it will be square, and its rank will be $\dim W$. A null column C_j of $M(S)$ (for some basis of W) will be given by

$$C_j = M(E)_{\cdot 1} M(T)_{1j} + \cdots + M(E)_{\cdot m} M(T)_{mj} = O_{\cdot j}$$

Where each column of $M(E)$ is linearly independent (since its rank its m). Thus, the entries of the jth column of $M(T)$ must be zero, thus, the column itself its zero.

$\square$

4. Suppose that V is finite-dimensional and $S, T \in \mathcal{L}(V, W)$. Prove that $\operatorname{range} S = \operatorname{range} T$ if and only if there exists an invertible $E \in \mathcal{L}(V)$ such that $S = TE$.

   (Idea of) Solution. There is, most likely, an analogous reasoning to that of the previous problem, but I will provide a solution using a concept of the book (column rank and column-row factorization) where I got the problem from (Linear Algebra Done Right).

   Let n be the dimension of V, and let m be the dimension of W. Let $M(T)$ and $M(S)$ be the matrices associated to each mapping, the choosen bases do not matter yet.

   As $\operatorname{Im} T = \operatorname{Im} S$, then $\dim \operatorname{Im} T = \dim \operatorname{Im} S$, the column rank r of $M(T)$ is equal to the column rank of $M(S)$. Thus, let $\{C_{\cdot 1}, \dots, C_{\cdot n}\}$ be each column of $M(T)$, and let $\{D_{\cdot 1}, \dots, D_{\cdot n}\}$ be each column of $M(S)$.

   From these columns we can derive linearly independent columns $\{C'_{\cdot 1}, \dots, C'_{\cdot r}\}$ for $M(T)$ and $\{D'_{\cdot 1}, \dots, D'_{\cdot r}\}$ for $M(S)$ (for example, by using the gaussian algorithm in the traspose of $M(T)$ and $M(S)$). So let $M(T')$ be that matrix constructed by merging each column $C'_{\cdot i}$, for $1 \leq i \leq r$:

$$M(T') = \begin{pmatrix} \vdots & \cdots & \vdots \\ C'_{\cdot 1} & \cdots & C'_{\cdot r} \\ \vdots & \cdots & \vdots \end{pmatrix}$$

This matrix has size $m \times r$. Similarly for each column $D'_{\cdot i}$:

$$M(S') = \begin{pmatrix} \vdots & \cdots & \vdots \\ D'_{\cdot 1} & \cdots & D'_{\cdot r} \\ \vdots & \cdots & \vdots \end{pmatrix}.$$

Note that if $m > r$, this would mean that $S', T' : U \to W$ are not surjective.

We claim that both are, although, necessarily injective. To prove this claim first note that the dimension of U is $\dim \operatorname{Im} T = \dim \operatorname{Im} S$, so suppose there were at least one (linearly independent) kernel vector of T in T', thus

$$Tv = 0_V \implies T(v_{\mathcal{B}}) = \begin{pmatrix} 0_{11} \\ \vdots \\ 0_{1m} \end{pmatrix}$$

for some basis of W. It is clearly not linearly independent, so we do not gather as a column for $M(T')$. (Does this argument suffices?) An analogous reasoning is used to prove that S' is injective.

Let $\mathcal{U} = \{u_1, \ldots, u_r\}$ be a basis for U, then, by the previous claim, $T'u_1, \ldots, T'u_r$ will be a basis for $\operatorname{Im} T'$ and for $\operatorname{Im} S'$. As well as $S'u_1, \ldots, S'u_r$: $\mathcal{D}$.

Consider a basis for V extending the basis of U: $\mathcal{A}$, and a basis for W by extending $\mathcal{B}$ or $\mathcal{C}$, we will get matrices for T as

$$M_{\mathcal{A}\mathcal{B}}(T) \text{ and } M_{\mathcal{A}\mathcal{C}}(T)$$

We can build a basis for V in which $M(T')$ is a submatrix of $M(T)$ and even $M(S')$ will be a submatrix of $M(T)$. The way to make them is to take the union with $n - r$ kernel basis vectors with $\mathcal{U}$, instead of linearly extending them, call this basis $\mathcal{A}'$. Matrices will have the following shape:

$$M_{\mathcal{A}'\mathcal{B}}(T) = \begin{pmatrix} M_{\mathcal{U}\mathcal{B}}(T') & O_{n-r} \\ M_{\mathcal{U}\mathcal{B}}(T') & O_{(n-r)(m-r)} \end{pmatrix}$$

We can make this matrix better, as it is injective, T' will have a square submatrix of size r contained in the first quadrant of $M_{\mathcal{A}'\mathcal{B}(T)}$

$$M_{\mathcal{A}'\mathcal{B}}(T) = \begin{pmatrix} M_{\mathcal{U}\mathcal{B}'}(T') & O_{n-r} \\ O_{m-r} & O_{(n-r)(m-r)} \end{pmatrix}$$

Thus:

$$M_{\mathcal{A}'\mathcal{B}}(T) = \begin{pmatrix} M_{\mathcal{U}\mathcal{B}'}(T') \\ O_{m-r} \end{pmatrix} I_{\max n,m} = M(T')I_{\max n,m}$$

Then: $M(S') = M(T')P$. Moreover, by column-row factorization theorem

$$M(T) = M(T')R_T \text{ and } M(S) = M(S')R_S$$

Where R_T and R_S are $r \times n$ matrices. Therefore:

$$M(S) = M(S')R_S = M(T)P$$

(Can we end up this idea? This will anyways end up in a change of basis. If there is a diverse method let me know.)

Solution.

□

5. Suppose V and W are finite-dimensional and $S, T \in \mathcal{L}(V, W)$. Prove that there exist invertible $E_1 \in \mathcal{L}(V)$ and $E_2 \in \mathcal{L}(W)$ such that $S = E_2 T E_1$ if and only if $\dim \operatorname{null} S = \dim \operatorname{null} T$.
   Solution. For the converse, we note that $\ker T$ is isomorphic to $\ker S$. And by rank-nullity theorem, Im T is isomorphic to Im S as well. Moreover $V \setminus \ker T$ is isomorphic to $V \setminus \ker S$. So the mapping $\mathrm{id} : V_{\mathcal{B}} \to V_{\mathcal{C}}$ can be decomposed as id

6. Suppose V is finite-dimensional and $T : V \to W$ is a surjective linear map of V onto W. Prove that there is a subspace U of V such that $T|_U$ is an isomorphism of U onto W.
   Solution. Let $n = \dim V$, $m = \dim W$. Since T is surjective, the dimension of W will be at most that of V. If this is the case take U as the whole V.

   If the dimension of W is strictly less than that of V, by the rank-nullity theorem, there are $n-m$ kernel vectors. Take $\mathcal{B}$, a basis for V containing a basis $\mathcal{K}$ of $\ker T$, this basis can be constructed starting with $\mathcal{K}$ and linearly extending it to $\mathcal{B}$. The subspace U will be constructed as $\operatorname{span}(\mathcal{B} \setminus \mathcal{K})$, and for any vector $u \in U$, $Tu \neq 0_W$. The dimension of U will be $n-(n-m) = m$, meaning that U and W are isomorphic.

   Lastly, recall that dim Im $T = \dim W$, then dim Im $T = \dim U$. Let $T_{\restriction U} : U \to W$ be T restricted to U, since $Tu \neq 0_W$ for any $u \in U$, this mapping is injective. Therefore, by rank-nullity, again,

$$\dim \operatorname{Im} T_{\restriction U} = \dim U = \dim \operatorname{Im} T = \dim W.$$

Thus, $T_{\restriction U}$ is surjective, and since it is injective, will also make an isomorphism between U and W.

$\square$

7. Suppose V and W are finite-dimensional and U is a subspace of V. Let

$$\mathcal{E} = \{T \in \mathcal{L}(V, W) \colon U \subseteq \operatorname{null} T\}.$$

   (a) Show that $\mathcal{E}$ is a subspace of $\mathcal{L}(V, W)$.

   (b) Find a formula for $\dim \mathcal{E}$ in terms of $\dim V$, $\dim W$, and $\dim U$.

Solution. (a) This part is straightforward, take $0_{\mathcal{E}}$ as the null mapping $T = O$. The sum of mappings restricted to some U containing all vectors of their kernel will also have the vectors of U as it kernel. And the same goes for multiplication by a scalar.

(b) Let $n = \dim V$, $m = \dim W$. Since $\mathcal{L}(V, W)$ is isomorphic to $\mathbb{F}^{n,m}$ by the matrix mapping M, and $\mathcal{E}$ is a subspace of it, we will have an isomorphism to some subspace $\mathbb{F}^{p,q}$ with dimension $p \times q$ of $\mathbb{F}^{n,m}$: $M : \mathcal{E} \to \mathbb{F}^{p,q}$ (note it is the same mapping M).

Let $T \in \mathcal{E}$, its associated matrix $M(T)$ will have at least one null column for some basis $\mathcal{B}$ of V (it is not immediately obvious that a null column will appear for any basis of V, but it is certain that there exists, for any mapping $T \in \mathcal{E}$, at least one since $\dim \operatorname{Im} T = \operatorname{rank} M(T) \leq \dim V$). So the dimension of $\mathcal{E}$ will be given from those matrices that have $\dim U$ null columns for some basis $\mathcal{B}$. Therefore: $\dim \mathcal{E} = (\dim(V) - \dim(U)) \dim W$.

$\square$

8. Suppose V is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that

$$ST \text{ is invertible} \iff S \text{ and } T \text{ are invertible.}$$

First Solution. This first solution involves determinants and it is straightforward, I will provide a second solution, as the author supposedly intended a solution without involving them.

Since ST is invertible, $\det(M(S)M(T)) \neq 0$, thus $\det(M(S))\det(M(T)) \neq 0$, this implies that $\det(M(S)) \neq 0$ and $\det(M(T)) \neq 0$. This proves that S and T are invertible.

Conversely, if S and T are invertible, $\det(M(S)) \neq 0$, $\det(M(T)) \neq 0$, thus $\det(M(ST)) = \det(M(S)M(T)) = \det(M(S))\det(M(T)) \neq 0$, this means that ST is invertible.

Second Solution.

9. Suppose V is finite-dimensional and $S, T, U \in \mathcal{L}(V)$ and $STU = I$. Show that T is invertible and that $T^{-1} = US$.

10. Show that the result in Exercise 12 can fail without the hypothesis that V is finite-dimensional.

11. Prove or give a counterexample: If V is a finite-dimensional vector space and $R, S, T \in \mathcal{L}(V)$ are such that RST is surjective, then S is injective.

12. Suppose $T \in \mathcal{L}(V)$ and $v_1, \ldots, v_m$ is a list in V such that $Tv_1, \ldots, Tv_m$ spans V. Prove that $v_1, \ldots, v_m$ spans V.

13. Prove that every linear map from $\mathbb{F}^{n,1}$ to $\mathbb{F}^{m,1}$ is given by a matrix multiplication. In other words, prove that if $T \in \mathcal{L}(\mathbb{F}^{n,1}, \mathbb{F}^{m,1})$, then there exists an $m \times n$ matrix A such that $Tx = Ax$ for every $x \in \mathbb{F}^{n,1}$.

14. Suppose V is finite-dimensional and $S \in \mathcal{L}(V)$. Define $\mathcal{A} \in \mathcal{L}(\mathcal{L}(V))$ by $\mathcal{A}(T) = ST$ for $T \in \mathcal{L}(V)$.

    (a) Show that $\dim \operatorname{null} \mathcal{A} = (\dim V)(\dim \operatorname{null} S)$.

    (b) Show that $\dim \operatorname{range} \mathcal{A} = (\dim V)(\dim \operatorname{range} S)$.

15. Show that V and $\mathcal{L}(\mathbb{F}, V)$ are isomorphic vector spaces.

16. Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T has the same matrix with respect to every basis of V if and only if T is a scalar multiple of the identity operator.

17. Suppose $q \in \mathcal{P}(\mathbb{R})$. Prove that there exists a polynomial $p \in \mathcal{P}(\mathbb{R})$ such that
$$q(x) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$$
for all $x \in \mathbb{R}$.

18. Suppose $T \in \mathcal{L}(V)$ and $v_1, \ldots, v_n$ is a basis of V. Prove that
$$\mathcal{M}(T, (v_1, \ldots, v_n)) \text{ is invertible} \iff T \text{ is invertible.}$$

19. Suppose that $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are bases of V. Let $T \in \mathcal{L}(V)$ be such that $Tv_k = u_k$ for each $k = 1, \ldots, n$. Prove that
$$\mathcal{M}(T, (v_1, \ldots, v_n)) = \mathcal{M}(I, (u_1, \ldots, u_n), (v_1, \ldots, v_n)).$$

20. Suppose A and B are square matrices of the same size and $AB = I$. Prove that $BA = I$.

21. Let V be a vector space over a field $\mathbb{F}$ of dimension n. Let $T \colon V \to V$ be a projection (recall that this is a linear map such that $T \circ T = T$).

    (a) Prove that $V = \ker(T) \oplus \operatorname{Im}(T)$.

(b) Prove that there is a basis of V in which the matrix of T is

$$\begin{pmatrix} I_i & 0 \\ 0 & O_{n-i} \end{pmatrix}$$

for some $i \in \{0, 1, \ldots, n\}$.

Solution. (a) First we prove that $\ker T \cap \text{Im } T = \{0\}$. Suppose that there exists some linearly independent vector v of V such that $Tv = 0$ and $v = Tu$ for some $u \in V$. Then

$$Tv = (T \circ T)u = Tu = 0.$$

Then $u \in \ker T$, and $Tu \in \ker T$ as well. So we conclude that $v = Tu = 0$; a contradiction, since v is linearly independent.

The second step is to prove that there exists an unique decomposition of v as a sum of vectors of $\ker T$ and Im T. For the existence part, since we have already proved that $\ker T \cap$ Im $T = \{0\}$, suffices to notice that the intersection between their bases is null, so the union of them will form a basis for V, by the rank-nullity theorem.

For the uniqueness part, suppose that for some $v \in V$, $v = k + Tu$ and $v = k' + Tu'$, for $k, k' \in \ker T$ and $Tu, Tu' \in \text{Im } T$. Thus

$$k + Tu = k' + Tu'$$

Taking T in both sides (both terms are in V, so it is OK)

$$T(k + Tu) = T(k' + Tu')$$

$$(T \circ T)u = (T \circ T)u'$$

$$Tu = Tu'.$$

So these both vectors of Im T are the same ones. We then conclude, from the first equation that $k = k'$.

(b) This matrix is constructed the same way to that of problem 2 of section 2 of "Kernel, range and matrices sheet", where i depends on the dimension of the image of T.

□

22. Let V be a vector space over $\mathbb{C}$ or $\mathbb{R}$ of dimension n. Let $T\colon V \to V$ be a symmetry (that is, a linear transformation such that $T \circ T = \text{id}$ is the identity map of V).

(a) Prove that $V = \ker(T - \text{id}) \oplus \ker(T + \text{id})$.

(b) Deduce that there exists $i \in [0, n]$ and a basis of V such that the matrix of T with respect to this basis is

$$\begin{pmatrix} I_i & 0 \\ 0 & -I_{n-i} \end{pmatrix}.$$

First Solution. This solution uses eigenspaces, but I do not think the author intended their use yet, so I will provide another solution without involving them.

The following lemma will be handful to prove the desired results.

Lemma 1. *The matrix of a symmetric operator $T : V \to V$, over a finite dimensional $\mathbb{C}-$vector space with some basis $\mathcal{B}$, is a symmetric matrix.*

Proof. Let $\mathcal{B}$ be a basis for V, since $T \circ T = \mathrm{id}$, the equation $M_{\mathcal{B}}(T)M_{\mathcal{B}}(T) = I_n$ will hold. Let a_{ij} be the $ij-$th entry of $M(T)$, hence

$$\sum_{k=1}^{n} a_{ik}a_{kj} = \delta_{ij}$$

so if $i \neq j$,

Second Solution. (a) Let $P := \frac{1}{2}(\mathrm{id} - T)$, and $Q := \frac{1}{2}(\mathrm{id} + T)$. These two mappings are projections from V to V (clearly). To prove this, using the fact that $\mathcal{L}(V)$ is a vector space itself

$$P^2 = \frac{1}{2}(\mathrm{id} - T)\frac{1}{2}(\mathrm{id} - T) = \frac{1}{4}(\mathrm{id} - T)(\mathrm{id} - T)$$

$$= \frac{1}{4}(\mathrm{id} - T)(\mathrm{id} - T) = \frac{1}{4}(\mathrm{id} - T - T + T^2)$$

$$= \frac{1}{4}(\mathrm{id} - T - T + T^2) = \frac{1}{4}(2\,\mathrm{id} - 2T) = P$$

Similar procedure for Q:

$$Q^2 = \frac{1}{4}(\mathrm{id} + T)(\mathrm{id} + T) = \frac{1}{4}(\mathrm{id} + 2T + T^2) = \frac{1}{4}(2\,\mathrm{id} + 2T) = Q.$$

The mapping $P + Q$ is the identity map, and their composition $P \circ Q = Q \circ P$ is the null map. We then claim that for any vector $v \in V$, $v = Pu + Qw$, and furthermore, that this decomposition is unique (meaning that $V = \mathrm{Im}\ P \oplus \mathrm{Im}\ Q$).

To prove this claim, note that the first condition is obvious since $P + Q = \mathrm{id}$, so remains showing that $\mathrm{Im}\ P \cap \mathrm{Im}\ Q = \{0_V\}$. Assume that there exists a vector $v \in V$ belonging to both $\mathrm{Im}\,P$ and $\mathrm{Im}\,Q$, thus,

$$v = P(u), \text{ and } v = Q(w).$$

$$\implies P(u) = Q(w)$$

$$\implies P(u) = (P \circ Q)(w) = 0_V$$

and similarly

$$Q(u) = (P \circ Q)(w) = 0_V$$

So $v = 0_V$. This proves our claim. Using the last problem we also know that $V = \ker P \oplus \operatorname{Im} P$ and $V = \ker Q \oplus \operatorname{Im} Q$. So we get three different expressions for V counting also that of $V = \operatorname{Im} P \oplus \operatorname{Im} Q$. Without loss of generality assume now that $\ker P = \operatorname{Im} Q$, and $\ker Q = \operatorname{Im} P$. This means that $V = \ker Q \oplus \ker P$ as well. Let $v \in \ker P$, thus

$$Pv = \frac{1}{2}(v - Tv) = 0$$

$$\iff 0 = Tv - v$$

so, this shows the equivalence between $\ker P$ and $\ker(T - \mathrm{id})$. Doing the same for Q, we get that:

$$Qv = 0 \iff v + Tv = 0 \iff Tv + v = 0.$$

So we get that $V = \ker P \oplus \ker Q \iff V = \ker(T - \mathrm{id}) \oplus \ker(T + \mathrm{id})$.

(b) As $T \circ T = \mathrm{id}$, we get that $(M_{\mathcal{B}\mathcal{B}}(T))^2 = I_n$ for a base $\mathcal{B}$ of V. Then, $M_{\mathcal{B}\mathcal{B}}(T) = M_{\mathcal{B}\mathcal{B}}(T)^{-1}$, this reduces the threshold of matrices as it only can be diagonal.

Now, we would like to have that a basis for V were the union of the bases of $\ker(T - \mathrm{id})$ and $\ker(T + \mathrm{id})$. Consider the mapping $T - \mathrm{id}$, this mapping will be zero if and only if $Tv = v$, so any vector of the basis of $\ker(T - \mathrm{id})$ will satisfy that $Tv = v$, similarly with $T + \mathrm{id}$, we will get that vectors of the basis of $\ker(T + \mathrm{id})$ are those in which $Tv = -v$.

Thus, let $\mathcal{B} = \{v_1, \ldots, v_i\}$ be a basis for $\ker(T - \mathrm{id})$, and $\mathcal{C} = \{v_{i+1}, \ldots, v_n\}$ be a basis for $\ker(T + \mathrm{id})$. A basis for V will be $\mathcal{D} = \mathcal{B} \cup \mathcal{C}$, and a basis for the arrival V will be $\mathcal{D}$ as well. This builds up the desired matrix.

$\square$

23. Let V be the vector space of polynomials with complex coefficients whose degree does not exceed 3. Let $T : V \to V$ be the map defined by

$$T(P) = P + P'.$$

Prove that T is linear and find the matrix of T with respect to the basis $1, X, X^2, X^3$ of V.

Solution. To prove this mapping is linear, let $c \in \mathbb{R}$ and let P, Q be polynomials with complex coefficients whose degree does not exceed 3:

$$T(P+cQ) = (P+cQ) + (P+cQ)' = P + cQ + P' + cQ' =$$
$$P + P' + cQ + cQ' = T(P) + cT(Q)$$

And note that if $P = c$, constant polynomial, $P' = 0$, in particular with $c = 0$. The matrix for T will be:

$$M(T) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

□

24. (a) Find the matrix with respect to the canonical basis of the map which projects a vector $v \in \mathbb{R}^3$ to the xy-plane.
    (b) Find the matrix with respect to the canonical basis of the map which sends a vector $v \in \mathbb{R}^3$ to its reflection with respect to the xy-plane.
    (c) Let $\theta \in \mathbb{R}$. Find the matrix with respect to the canonical basis of the map which sends a vector $v \in \mathbb{R}^2$ to its rotation through an angle θ, counterclockwise.

25. Let V be a vector space of dimension n over F. A *flag* in V is a family of subspaces
$$V_0 \subset V_1 \subset \cdots \subset V_n$$
such that $\dim V_i = i$ for all $i \in [0, n]$. Let $T : V \to V$ be a linear transformation. Prove that the following statements are equivalent:
    (a) There is a flag $V_0 \subset \cdots \subset V_n$ in V such that $T(V_i) \subset V_i$ for all $i \in [0, n]$.
    (b) There is a basis of V with respect to which the matrix of T is upper-triangular.

Solution. Since $V_0 \subset V_1 \subset \cdots \subset V_n$, we can find a basis for any V_k by extending one from V_{k-1}. Call $\mathcal{B}_k$ a basis for V_k that is (recursively) extended from $\mathcal{B}_{k-1}$.

For the direct implication, let us start with some fixed k. Since $T(V_k) \subset V_k$, there exist at most $k-1$ basis vectors from $\mathcal{B}_k$ that form a basis for $T(V_k)$.

Among these $k-1$ vectors it can occur that each one of these are of $\mathcal{B}_{k-1}$, or that within these, there is the only vector $v_k \in \mathcal{B}_k \setminus \mathcal{B}_{k-1}$, and the other $k-2$ are within $\mathcal{B}_{k-1}$.

For the first case, for any k, we can choose as a basis for $T(V_k)$ exactly $\mathcal{B}_{k-1}$.

Clearly $T(V_{k-1})$ is a subspace of $T(V_k)$ since $\mathcal{B}_{k-1}$ is gotten by linearly extending $\mathcal{B}_{k-2}$. It follows that $T(V_{k-1}) \subset T(V_k)$ for any $k \geq 1$.

Let $v_k \in \mathcal{B}_k \setminus \mathcal{B}_{k-1}$ for any $1 \leq k \leq n$, this vector will not be in any basis of $T(V_j)$ for $1 \leq j < k$ by construction, but will be a basis vector only for V_k. Its image $Tv_k \in T(V_k)$ will result in the following column vector:

$$\begin{pmatrix} a_{1k} \\ \vdots \\ a_{(k-1)k} \\ 0_{kk} \end{pmatrix}$$

As it is spanned by $k-1$ basis vectors from $\mathcal{B}_{k-1}$.

So, gathering the k basis vectors of V the matrix is constructed, note that its principal diagonal is zero, but it is OK since for being upper triangular this does not matter.

For the converse implication let $A \in M_n(\mathbb{F})$ be an upper triangular matrix. We see that the rank m of A is $n-1 \leq m \leq n$. Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be a basis for V.

Consider the subspaces $V_k := \langle v_1, \ldots, v_k \rangle$. By induction on k we will prove that $T(V_k) \subset V_k$. The intuition is that the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{pmatrix}$$

since input and output bases are the same, represents that the mapping of, say v_k spans itself "plus" the $k-1$ earlier ones. We will see that considering subspaces while cumulating basis vectors of V: $\langle v_1 \rangle$, $\langle v_1, v_2 \rangle$, and so on; the dimension of their image will be less than or equal themselves (depending mainly on the zeroes in the diagonal).

When $k = 1$ we have that for $V_1 = \langle v_1 \rangle$, $T(V_1) = \lambda a_{11} v_1 \in \langle v_1 \rangle$.

When $k = 2$ we have that for $V_2 = \langle v_1, v_2 \rangle$, $T(V_2) = \langle T(v_1), T(v_2) \rangle = \lambda a_{11} v_1 + \delta(a_{12} v_1 + a_{22} v_2) = b_1 v_1 + b_2 v_2 \in V_2$.

For the inductive thesis, we will show that for $V_{k+1} := \langle v_1, \ldots, v_k, v_{k+1} \rangle$ it follows that $T(V_{k+1}) \subseteq V_{k+1}$. Since

$$T(V_{k+1}) = \langle T(v_1), \ldots, T(v_k), T(v_{k+1}) \rangle = T(V_k) + \langle T(v_{k+1}) \rangle = \\ T(V_k) + \lambda \sum_{i=1}^{k+1} a_{i(k+1)} v_i,$$

then

$$T(V_{k+1}) = \sum_{i=1}^{k} b_i v_i + \lambda \left(\sum_{i=1}^{k+1} a_{i(k+1)} v_i \right)$$

$$= \sum_{i=1}^{k} b_i v_i + \left(\sum_{i=1}^{k+1} c_i v_i \right) = \sum_{i=1}^{k+1} d_i v_i \in \langle v_1, \dots, v_{k+1} \rangle = V_{k+1}.$$

$\square$

26. Let V be a vector space over a field F, and let $T_1, \dots, T_n : V \to V$ be linear transformations. Prove that

$$\bigcap_{i=1}^{n} \ker(T_i) \subseteq \ker \left(\sum_{i=1}^{n} T_i \right).$$

(Idea of) Solution. Using *Grassmann formula* and rank-nullity theorem we get some interesting inequalities. By arguing by induction on n, for the base case $n = 1$ we get the trivial inclusion $\ker(T_1) \subseteq \ker(T_1)$. Now, suppose that for any $k \geq 1$ the inclusion

$$\bigcap_{i=1}^{k} \ker(T_i) \subseteq \ker \left(\sum_{i=1}^{k} T_i \right).$$

holds. In the right hand side we note that despite having a sum in it, the dimension of V will be invariant anyways, this means that

$$\dim V = \dim \ker T_i + \dim \operatorname{Im} \, T_i = \dim \ker \left(\sum_{i=1}^{k} T_i \right) + \dim \operatorname{Im} \, \left(\sum_{i=1}^{k} T_i \right).$$

Call S the sum $\sum_{i=1}^{k} T_i$. Note that

$$\dim \ker S + \dim \operatorname{Im} \, S = \dim \ker T_i + \dim \operatorname{Im} \, T_i \geq 0$$

Thus:

$$\dim \ker S - \dim \ker T_i = \dim \operatorname{Im} \, T_i - \dim \operatorname{Im} \, S \; (*)$$

Then, using Grassmann formula:

$$\dim(\ker T_i + \ker S) = \dim \ker T_i + \dim \ker S - \dim(\ker T_i \cap \ker S) \geq 0.$$

Which implies that

$$\dim \ker T_i + \dim \ker S \geq \dim(\ker T_i \cap \ker S) \; (**).$$

Summing $(*)$ and $(**)$

$$\dim \ker S \geq \dim \ker T_i + (\dim \operatorname{Im} \, T_i - \dim \operatorname{Im} \, S).$$

Then:

$$\dim \ker S \geq \dim(\ker T_i \cap \ker S) + (\dim \mathrm{Im}\ T_i - \dim \mathrm{Im}\ S).$$

This, although, does still not prove the desired result, but holds for any $k \geq 1$. It could occur that the right hand side were negative. Despite this, by our inductive hypothesis, we can say that $\dim \ker S \geq \dim(\bigcap_{i=1}^{k} \ker T_i)$. For the inductive thesis, when $n = k+1$, we shall prove

$$\bigcap_{i=1}^{k+1} \ker(T_i) \subseteq \ker(S).$$

(This idea is interesting, but I do not think that it can be pushed to get the desired result, although it may be true that the result holds for $n = k+1$, $-$that $\dim \ker S \leq \dim \ker(\bigcap_{i=1}^{k+1} \ker(T_i))-$ this does not tell us anything about the inclusion of the subspaces, we can still have them disjoint except for zero.)

Solution. Let $v \in \displaystyle\bigcap_{i=1}^{n} \ker(T_i)$, we claim that this vector is also in $\ker \left(\displaystyle\sum_{i=1}^{n} T_i \right)$.

To prove this claim first note that any vector $v' \in \ker \left(\displaystyle\sum_{i=1}^{n} T_i \right)$ will satisfy:

$$T_1 v' + \cdots + T_n v' = 0_V.$$

In particular, since v is in each $\ker T_i$,

$$T_1 v + \cdots + T_n v = 0_V, \text{ so } v \in \ker \left(\sum_{i=1}^{n} T_i \right).$$

$\square$

27. Let V be a vector space over a field F, and let $T_1, T_2 : V \to V$ be linear transformations such that

$$T_1 \circ T_2 = T_1 \quad \text{and} \quad T_2 \circ T_1 = T_2.$$

Prove that

$$\ker(T_1) = \ker(T_2).$$

Solution. Let $v \in \ker T_1$, we note that

$T_1 v = 0 \iff T_2(T_1 v) = T_2 v = 0.$ (since any linear mapping applied to 0 is also 0)

This implies that $v \in \ker T_1 \iff v \in \ker T_2$. Similarly, let $u \in \ker T_2$

$$T_2 u = 0 \iff T_1(T_2 v) = T_1 u = 0.$$

So we conclude that $\ker T_1 = \ker T_2$.

$\square$

28. Let V be a vector space over F, and let $T : V \to V$ be a linear transformation such that

$$\ker(T) = \ker(T^2) \quad \text{and} \quad \operatorname{Im}(T) = \operatorname{Im}(T^2).$$

Prove that

$$V = \ker(T) \oplus \operatorname{Im}(T).$$

Solution. We claim that T is necessarily a projection. To prove this, assume that T were not a projection. Take $v \in V, v \neq 0$, then $Tv \neq T^2 v$, let $u \in V$ such that $u = T^2 v$.

As $u \in \operatorname{Im} T^2$, then $u \in \operatorname{Im} T$, this means that $u = Tv'$ for some $v' \in V$, suppose that $v' \neq v$. Then it must hold that $Tv' = T^2 v$, so T is not injective.

Let $k \in \ker T$, such that $k \neq 0$, then $k \in \ker T^2$, thus $Tk = T^2 k = 0$, which is a contradiction.

Then, the result yields by problem 21.

$\square$

29. Let V be a finite dimensional vector space. Let $T : V \to V$ be a linear operator, and let $T^n : V \to V$ denote T applied n times. Prove that there exists an integer N such that

$$V = \ker T^N \oplus \operatorname{Im} T^N.$$

Solution. First note that the kernel of a mapping is stable under T, only possibly increasing its dimension when applying T again. If it continues increasing when applying T, the resulting mapping will be the null mapping for some $N \geq 2$, which is a projection, leading to the result immediately.

Else, if the dimension of both $\operatorname{Im} T$ and $\ker T$ become stable, we claim that we would get, starting from a certain integer j

$$\ker T^j = \ker T^{j+1} \text{ and } \operatorname{Im} T^j = \operatorname{Im} T^{j+1}.$$

Which would imply that T becomes a projection starting from j by the last problem, which also leads to the result immediately. To prove this claim, note that $\ker T$ is actually stable under T, this means that for any $v \in \ker T$

$$v \in \ker T \implies v \in \ker T^2 \implies \cdots \implies v \in \ker T^j$$

in particular, when applying T j times to a basis $\mathcal{K}$ of $\ker T$, every vector of it will be basis vectors of T^{j+1}. Now as we assumed that $\dim \ker T^j = \dim \ker T^{j+1}$, $\mathcal{K}$ forms a basis for $\ker T^j$ and $\ker T^{j+1}$, which means that $\ker T^j = \ker T^{j+1}$. In the case of Im T, we note that $V \setminus \ker T^j = V \setminus \ker T^{j+1} = \cdots = V \setminus \ker T^{j+n}$, hence, $\operatorname{Im} T$ will also become stable starting from j. This proves our claim.

□

Rank of a matrix

1. Let $A, B \in M_3(F)$ be two matrices such that $AB = O_3$. Prove that
$$\min(\operatorname{rank}(A), \operatorname{rank}(B)) \leq 1.$$

2. Let $A \in M_3(\mathbb{C})$ be a matrix such that $A^2 = O_3$.
   (a) Prove that A has rank 0 or 1.
   (b) Deduce the general form of all matrices $A \in M_3(\mathbb{C})$ such that $A^2 = O_3$.

3. Find the rank of the matrix $A = [\cos(i - j)]_{1 \leq i,j \leq n}$.

4. (a) Let V be an n-dimensional vector space over F, and let $T : V \to V$ be a linear transformation. Let T^j be the j-fold iterate of T (so $T^2 = T \circ T$, $T^3 = T \circ T \circ T$, etc.). Prove that:
$$\operatorname{Im}(T^n) = \operatorname{Im}(T^{n+1}).$$
   Hint: Check that if $\operatorname{Im}(T^j) = \operatorname{Im}(T^{j+1})$ for some j, then $\operatorname{Im}(T^k) = \operatorname{Im}(T^{k+1})$ for $k \geq j$.
   (b) Let $A \in M_n(\mathbb{C})$ be a matrix. Prove that A^n and A^{n+1} have the same rank.

5. Let $A \in M_n(F)$ be a matrix of rank 1. Prove that:
$$A^2 = \operatorname{Tr}(A)A.$$

6. Let $A \in M_m(F)$ and $B \in M_n(F)$. Prove that:
$$\operatorname{rank} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \operatorname{rank}(A) + \operatorname{rank}(B).$$

7. Prove that for any matrices $A \in M_{n,m}(F)$ and $B \in M_m(F)$, we have:
$$\operatorname{rank} \begin{bmatrix} I_n & A \\ 0 & B \end{bmatrix} = n + \operatorname{rank}(B).$$

8. Let $n > 2$ and let $A = [a_{ij}] \in M_n(\mathbb{C})$ be a matrix of rank 2. Prove the existence of real numbers x_i, y_i, z_i, t_i for $1 \leq i \leq n$ such that for all $i, j \in \{1, 2, \ldots, n\}$, we have:

$$a_{ij} = x_i y_j + z_i t_j.$$

9. Let $A = [a_{ij}]_{1 \leq i,j \leq n}$ and $B = [b_{ij}]_{1 \leq i,j \leq n}$ be complex matrices such that:

$$a_{ij} = 2ij - b_{ij}$$

for all integers $1 \leq i, j \leq n$. Prove that:

$$\operatorname{rank}(A) = \operatorname{rank}(B).$$

10. Let $A \in M_n(\mathbb{C})$ be a matrix such that $A^2 = A$, i.e., A is the matrix of a projection. Prove that:

$$\operatorname{rank}(A) + \operatorname{rank}(I_n - A) = n.$$

11. Let $n > k$ and let $A_1, \ldots, A_k \in M_n(\mathbb{R})$ be matrices of rank $n - 1$. Prove that $A_1 A_2 \cdots A_k$ is nonzero. *Hint:* Using Sylvester's inequality, prove that:

$$\operatorname{rank}(A_1 \cdots A_j) \geq n - j \quad \text{for } 1 \leq j \leq k.$$

12. Let $A \in M_n(\mathbb{C})$ be a matrix of rank at least $n - 1$. Prove that:

$$\operatorname{rank}(A^k) \geq n - k \quad \text{for } 1 \leq k \leq n.$$

Hint: Use Sylvester's inequality.

13. (a) Prove that for any matrix $A \in M_n(\mathbb{R})$, we have:

$$\operatorname{rank}(A) = \operatorname{rank}({}^\top A A).$$

Hint: If $X \in \mathbb{R}^n$ is a column vector such that ${}^\top A A X = 0$, write ${}^\top X \, {}^\top A A X = 0$ and express the left-hand side as a sum of squares.

(b) Let $A = \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}$. Find the rank of A and $A^\top A$, and conclude that part (a) of the problem is no longer true if $\mathbb{R}$ is replaced with $\mathbb{C}$.

14. Let A be an $m \times n$ matrix with rank r. Prove that there is an $m \times m$ matrix B with rank $m - r$ such that:

$$BA = O_{m,n}.$$

15. (Generalized inverses) Let $A \in M_{m,n}(F)$. A generalized inverse of A is a matrix $X \in M_{n,m}(F)$ such that:

$$AXA = A.$$

(a) If $m = n$ and A is invertible, show that the only generalized inverse of A is A^{-1}.

(b) Show that a generalized inverse of A always exists.

(c) Give an example to show that the generalized inverse need not be unique.

Duality

1.

Product and quotient of a vector space

1. Let V be a finite-dimensional vector space over F, and let $W \subset V$ be a subspace. For a vector $v \in V$, define

$$[v] = \{v + w : w \in W\}.$$

Note that $[v_1] = [v_2]$ if and only if $v_1 - v_2 \in W$. Define the quotient space V/W to be

$$V/W = \{[v] : v \in V\}.$$

Addition and scalar multiplication in V/W are defined as follows:

$$[u] + [v] = [u + v] \quad \text{and} \quad a[v] = [av],$$

where $a \in F$. It is known that these operations are well-defined and that V/W, equipped with this structure, is a vector space.

(a) Show that the map $\pi : V \to V/W$ defined by $\pi(v) = [v]$ is linear with kernel W.

(b) Show that

$$\dim(W) + \dim(V/W) = \dim(V).$$

(c) Suppose $U \subset V$ is any subspace such that $W \oplus U = V$. Show that the restriction $\pi|_U : U \to V/W$ is an isomorphism, i.e., a bijective linear map.

(d) Let $T : V \to U$ be a linear map, let $W \subset \ker(T)$ be a subspace of V, and let $\pi : V \to V/W$ be the projection onto the quotient space. Show that there exists a unique linear map $S : V/W \to U$ such that $T = S \circ \pi$.