Linear mappings and geometry sheet

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Problems on linear algebra: books of Andreescu "Essential Linear Algebra", Sheldon Axler "Linear Algebra Done Right" and Dan Pedoe "Geometry: A comprehensive course". The topics of this week are a continuation of the last week's sheet and some problems in geometry as application for the topics seen yet.

1 Geometry in $\mathbb{R}^2$ and $\mathbb{R}^3$

1. Given the sphere $S : x^2 + y^2 + z^2 - 2x - 4y - 6z = 2$ and the line

$$r : \begin{cases} x = 5 + t \\ y = 2 - t \\ z = 3 + t \end{cases}, \ \text{t} \in \mathbb{R}.$$

a) Find the coordinates of the center C of the sphere S, and the its radius.
b) Determine a parametric and cartesian equation of the plane passing through C and perpendicular to the line r.
c) Find the intersection between the sphere S and the line r.
Solution. a) Finding the general form of the equation of S we have

$$S : x^2 + y^2 + z^2 - 2x - 4y - 6z - 2 = 0 \iff S : (x-1)^2 + (y-2)^2 + (z-3)^2 = 16$$

Hence the center of S is the point $(1, 2, 3)$, and its radius is 4.
b) Let $\overrightarrow{v_\pi} = \overrightarrow{v_r}$, where $\overrightarrow{v_r} = (1, -1, 1)$ is the directional vector of r. Then the cartesian equation of π results in

$$\pi : (x - 1) - (y - 2) + (z - 3) = 0$$

c) Simply substituting the values of $(x, y, z) = (5 + t, 2 - t, 3 + t)$ in S we get:

$$((5 + t) - 1)^2 - ((2 - t) - 2)^2 + ((3 + t) - 3)^2 - 16 = 0$$

$$\implies (4+t)^2 - (-t)^2 + (t)^2 - 16 = 0$$

$$\implies (4+t)^2 - 16 = 8t + t^2 = t(8+3t) = 0$$

Finally, $t = 0$ and $t = 8/3$. So, substituting in the equation of r we have the 3-tuples $(5,2,3), (5+8/3, 2-8/3, 3+8/3)$.

$\square$

2. Given the lines

$$r : \begin{cases} x = t \\ y = -t+1 \\ z = 2 \end{cases} \quad , \text{ t} \in \mathbb{R}.$$

$$s : \begin{cases} 3x - y = 0 \\ 2x + z = 1 \end{cases}$$

a) Find the intersection of r and s and determine if they are orthogonal.
b) Verify that $Q(1,0,2)$ is in r and determine the plane over Q and perpendicular to s.
c) Find the projection Q' of Q over s, and the distance between Q and s.
Solution.

3. Given the point $P(1,1,1)$, the plane $\pi : x + y + 2z - 1 = 0$ and the line

$$r : \begin{cases} x = 1-t \\ y = 2+t \\ z = t \end{cases} \quad , \text{ t} \in \mathbb{R}.$$

a) Find the cartesian equation of the plane σ that contains r and the point P.
b) Find the projection of P over π and the symmetric of P over π.
c) Find an equation for the projection of r over π.
Solution. a) The directional vector of r is $v_r = (-1,1,1)$. Now, assuming that P is in r:

$$r : \begin{cases} 1 = 1-t \implies t = 0 \\ 1 = 2+0 \implies 1 = 2 \\ 1 = t \end{cases}$$

We get an absurd result, then $P \notin r$. Now, let $Q(1,2,0)$, for $t = 0$, be in r. The vector from P to Q is defined as $\overline{PQ} = Q - P = (0,1,-1)$. Now, a normal vector (u_x, u_y, u_z) of σ must be perpendicular to both $\overline{PQ}$ and v_r. We find it by doing the mixed product between both, thus the determinant of the matrix

$$\begin{pmatrix} u_x & u_y & u_z \\ -1 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

Which results in:

$$(-2, -1, -1)$$

Thus, the cartesian equation is:

$$\sigma : -2(x-1) - (y-1) - (z-1) = 2x + y + z - 4 = 0.$$

b) Let r' be the line perpendicular to π that passes through the point P. One directional vector of r': $v_{r'}$ is equal to v_π. Then, for finding the normal vector of π we have

$$\pi : x + y + 2z - 1 = (x-1) + (y-0) + 2(z-0) = 0.$$

Which shows that $(1, 1, 2)$ suffices. So the parametric equation for r' is

$$r' : \begin{cases} x = 1 + t \\ y = 1 + t \\ z = 1 + 2t \end{cases}, t \in \mathbb{R}.$$

Doing the intersection between π and r' we see that:

$$(1+t) + (1+t) + 2(1+2t) - 1 = 6t + 3 = 0.$$

Hence, $t = -1/2$. So the projection of P on π is found by replacing this value on the parametric equation of r (or in the equation of π). Which results in $P' = (\frac{1}{2}, \frac{1}{2}, 0)$. Since P' is the medium point between P and the symmetric point P'', then $(\frac{1}{2}, \frac{1}{2}, 0) = (\frac{a}{2}, \frac{b}{2}, \frac{c}{2})$, where $(a, b, c) = P'' - P = (1 - x_p, 1 - y_p, 1 - z_p)$. Then, by equaling the vectors we get that $P'' = (x_p, y_p, z_p) = (0, 0, -1)$.

c) We firstly show that the intersection $r \cap \pi \neq \emptyset$, by replacing every variable in function of $t \in \mathbb{R}$ of the equation of r to the equation of π

$$r \cap \pi : (1-t) + (2+t) + 2t - 1 = 2t + 2 = 0$$

Hence $t = -1$. Thus, the point R of intersection is $(2, 1, -1)$. Let Q be the same point as the last item $Q = (1, 2, 0) \in r$. Thus, the projection of Q over r is given by the line r'' with directional vector $u = (1, 1, 2)$.

$$r'' : \begin{cases} x = 1 + t \\ y = 2 + t \\ z = 2t \end{cases}$$

The intersection between r'' and π gives out the value $t = -1/3$. Then, replacing it in the equation of r'' results in the point $S = (\frac{2}{3}, \frac{5}{3}, \frac{-2}{3})$. Finally, the projected line is just the line that has the directional vector $\overline{RS} = R - S = (\frac{4}{3}, -\frac{2}{3}, -\frac{1}{3})$ and passes through R

$$r'' = \begin{cases} x = 2 + \frac{4}{3}s \\ y = 1 + -\frac{2}{3}s \\ z = -1 + -\frac{1}{3}s \end{cases}, \ s \in \mathbb{R}.$$

$\square$

4. Given the sphere $S : x^2+y^2+z^2-2y-2z = 0$ and the plane $\pi : 2x+y+1 = 0$:
Show that $S \cap \pi$ is a circumference, and find its radius and center.
Solution.

5. Show that the theorem:
"If C is any point on the line determined by two distinct points A and B, then we may always write

$$C = (1-t)A + tB$$

where the ratio of the real numbers $t : (1-t)$ is equal to the ratio of the lenghts of the segments $\overline{AC} : \overline{CB}$."
Can also be formulated thus:
If C is any point on the line determined by the two distinct points A and B then we may write $C = kA + k'B$, where $\overline{AC} : \overline{BC} = k' : k$, and $k' + k = 1$.
Solution. Since $k' + k = 1$, then $k = 1 - k'$. We get that $C = (1-k')A + k'B$. Conversely, let $t = k'$ and $(1-t) = k$, then $t + (1-t) = k' + k = 1$.

$\square$

6. If D, E are the midpoints of the sides BC, CA of a triangle ABC prove, using vectors, that DE is parallel to BA, and that $\overline{DE} : \overline{BA} = 1/2$.
Solution. Let $v = D - E$ be a vector from E to D. Let $u = B - A$ be a vector from A to B. We claim that $v = t \cdot u$ for some $t \in \mathbb{R}$. Clearly v can be written as $\frac{(B-C)}{2} - \frac{(C-A)}{2}$.

Taking $t = 1/2$ we prove our claim. This implies that the segments $\overline{DE}$ and $\overline{BA}$ are parallel and that the lenght of $\overline{DE}$ is half as that of $\overline{AB}$.

$\square$

7. If D, E and F be the midpoints of the sides BC, CA and AB of triangle ABC, and AP, BP, CP meet the opposite sides in L, M and N, P being any point in the plane, prove that the joins of D, E and F to the midpoints of AL, BM and CN are concurrent.

Solution. Consider barycentric coordinates of the triangle $\triangle ABC$, where $A = (1, 0, 0)$, $B = (0, 1, 0)$, $C = (0, 0, 1)$. We then have a variable point $P = (p, q, r)$ such that $P = pA + qB + rC$ and $p + q + r = 1$. The points D, E, F will be given by

$$D = \frac{B+C}{2} = (0, \frac{1}{2}, \frac{1}{2}),$$
$$E = \frac{C+A}{2} = (\frac{1}{2}, 0, \frac{1}{2}),$$
$$F = \frac{A+B}{2} = (\frac{1}{2}, \frac{1}{2}, 0).$$

Since the points L, M, N pass through AP, BP and CP and are within the segments $\overline{BC}$, $\overline{AC}$, $\overline{AB}$ respectively, we can get a parametric equation for any point $X \in \overline{AP}$ (for example) and then set it to be on the bounds of $\overline{BC}$, thus finding an expression for L.

$$X = (1-t)A + tP,\ t \in \mathbb{R}.$$

We get that X is the tuple $(1 - t + tp, tq, tr)$ and since we bounded X to be in $\overline{BC}$ then

$$(1-t+tp, tq, tr) = (0, tq, tr) \implies 1-t+tp = 0 \implies t(1-p) = 1 \implies t = \frac{1}{1-p}.$$

So, the point L is found when $X = (0, \frac{q}{1-p}, \frac{r}{1-p})$. Now remains to do the same argument for M and N. For M we get the point $Y \in \overline{BP}$ that is also bounded within $\overline{AC}$, then:

$$Y = (1-s)B + sP \implies Y = (sp, 1 - s + sq, sr).$$

Hence:

$$1 - s + sq = 0 \implies s = \frac{1}{1-q}.$$

So, for M we get that $M = (\frac{p}{1-q}, 0, \frac{r}{1-q})$. And lastly we get that $N = (\frac{p}{1-r}, \frac{q}{1-r}, 0)$.

Then, let D', E', F' be the midpoints of $\overline{AL}, \overline{BM}, \overline{CN}$.

$$D' = \frac{A+L}{2} = (\frac{1}{2}, \frac{q}{2(1-p)}, \frac{r}{2(1-p)})$$

$$E' = \frac{B+M}{2} = (\frac{p}{2(1-q)}, \frac{1}{2}, \frac{r}{2(1-q)})$$

$$F' = \frac{C+N}{2} = (\frac{p}{2(1-r)}, \frac{q}{2(1-r)}, \frac{1}{2}).$$

Now, consider the triangle $\triangle DEF$, which has the same baricenter as $\triangle ABC$. We claim that the points D', E', F' are contained in $\triangle DEF$ and furthermore, the cevians DD', EE', FF' are concurrent in $\triangle DEF$. If $D' = E' = F'$ then all these three are the baricenter, and DD', EE', FF' are the medians of the triangle, so we are done. Else, we have to prove that there exist real numbers α, β, γ such that $\alpha + \beta + \gamma = 1$ and $D' = \alpha D + \beta E + \gamma F$, in the case of D'. Hence, we get the system:

$$D' : \begin{cases} x = \alpha \cdot 0 + \beta \cdot \frac{1}{2} + \gamma \frac{1}{2} \\ y = \alpha \cdot \frac{1}{2} + \beta \cdot 0 + \gamma \cdot \frac{1}{2} \\ z = \alpha \cdot \frac{1}{2} + \beta \cdot \frac{1}{2} + \gamma \cdot 0 \end{cases} = \begin{cases} x = \beta \cdot \frac{1}{2} + \gamma \frac{1}{2} \\ y = \alpha \cdot \frac{1}{2} + \gamma \cdot \frac{1}{2} \\ z = \alpha \cdot \frac{1}{2} + \beta \cdot \frac{1}{2} \end{cases} .$$

Replacing values of D':

$$\begin{cases} \frac{1}{2} = \beta \cdot \frac{1}{2} + \gamma \frac{1}{2} \\ \frac{q}{2(1-p)} = \alpha \cdot \frac{1}{2} + \gamma \cdot \frac{1}{2} \\ \frac{r}{2(1-p)} = \alpha \cdot \frac{1}{2} + \beta \cdot \frac{1}{2} \end{cases}$$

From the system we get that: $\beta + \gamma = 1$ thus, $\alpha = 0$. Hence D' is not just in DEF but exactly in the segment $\overline{EF}$. Doing similar computations for the other points we get that $E' \in \overline{DF}$ and $F' \in \overline{DE}$.

(How can I end up using Ceva's theorem here?)

8. The side AB of a triangle ABC is divided internally at P_{AB} in the ratio $k_1 : k_2$, externally at P'_{AB} in the same ratio, the side BC is divided internally at P_{BC} in the ratio $k_2 : k_3$, and externally at P'_{BC} in the same ratio, and the side CA is divided internally at P_{CA} in the ratio $k_3 : k_1$ and externally at P'_{CA} in the same ratio.
a) Show that the lines AP_{BC}, BP_{CA} and and CP_{AB} are concurrent.
b) Show that the line $P_{CA}P_{AB}$ contains the point P'_{BC}.
c) Show that the points $P'_{AB}, P'_{BC}, P'_{CA}$ are collinear.

9. If the point P lies in the plane of triangle ABC, but is distinct from the vertices of the triangle, and the parallelograms $PBLC, PCMA, PANB$ are completed, prove that the segments AL, BM, CN bisect each other.

2 Linear mappings

1. Suppose $w_1, \ldots, w_n$ is a basis of W and V is finite-dimensional. Suppose $T \in L(V, W)$. Prove that there exists a basis $v_1, \ldots, v_m$ of V such that all

the entries in the first row of $M(T)$ (with respect to the bases $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$) are 0 except for possibly a 1 in the first row, first column.
Solution. Let dim range $T = k$. Consider a basis of the range T to be $\{Tv_1, \ldots, Tv_k\}$ for $k \leq n$. Now, given that range T is a subspace of W, for a fixed basis $\{w_1, \ldots, w_k\}$, there must exist a subset $\{w_{i_1}, \ldots, w_{i_k}\}$ that spans range T.

Then, there exists an isomorphism S : span $\{w_{i_1}, \ldots, w_{i_k}\} \to$ range T such that $Sw_{i_1} = Tv_1, \ldots, Sw_{i_k} = Tv_k$. Meaning that $S^{-1}Tv_j = w_{i_j}$. That is, since Tv_j is linearly independent for any $1 \leq j \leq k$ and S is, particularly, injective, the vectors $S^{-1}Tv_j$ are linearly independent and form a basis for range T. Hence, given any basis for range T, one can find k vectors $w_{i_1}, \ldots, w_{i_k}$ in the basis of W, such that for some isomorphism S: $Sw_{i_j} = Tv_j$ for $1 \leq j \leq k$.

Now, for V, as it is not fixed, we can find a subspace $U \subseteq V$ which satisfies that range $T = \{Tu : u \in U\}$ and $\ker T \cap U = \{0\}$. Let $\{u_1, \ldots, u_k\}$ be a set such that $Tu_j = Sw_{i_j} \neq 0$ for $1 \leq j \leq k$. We claim that this set forms a basis for U.

To prove this claim we first note that u_j is not a vector of $\ker T$. Then $\text{span}\{u_1, \ldots, u_k\} \cap \ker T = \{0\}$, and $Tu_j = Sw_{i_j} = Tv_j$ for any $j \leq k$ and for some isomorphism S; thus $\{Tu_1, \ldots, Tu_j\}$ is a basis for range T. (Is this circular?)

Finally, we have that, taking the union of $\{u_1, \ldots, u_k\}$ with a basis of ker T: $\{x_1, \ldots, x_{m-k}\}$ so that the basis of V becomes $\{u_1, \ldots, u_k, x_1, \ldots, x_{m-k}\}$; we have a block matrix

$$M(T) = \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix}.$$

Note that, informally speaking, there are k 1's distributed in A and B, just one in any of the m rows, so by pigeonhole principle, there must be at least a zero in some row, particularly within the first and kth column. Now, depending in what the basis of W is, we might have it with a $S^{-1}Tv_1$ in the beginning, giving a 1 in first row, first column. By the construction of the basis of U we also will have 0's elsewhere in the row.

Remark. Note that the argument in the first paragraph is the same as a change of basis. S could also be denoted by Id: $W_{\mathcal{A}} \to W_{\mathcal{B}}$, where $\mathcal{B}$ is a basis of W that contains the basis of Im T. In the next sheet of linear algebra, I will try to compact the solution of this problem with the tools of the change of bases matrices.

□

2. Suppose A is an m-by-n matrix and B is an n-by-p matrix. Prove that

$$(AB)_{j,\cdot} = A_{j,\cdot}B$$

for each $1 \leq j \leq m$. In other words, show that *row* j of AB equals *row* j of A times B.

3. Suppose $\mathbf{a} = (a_1 \ \cdots \ a_n)$ is a 1-by-n matrix and $\mathbf{B}$ is an n-by-p matrix. Prove that
$$\mathbf{aB} = a_1\mathbf{B}_{1,\cdot} + \cdots + a_n\mathbf{B}_{n,\cdot}.$$
In other words, show that $\mathbf{aB}$ is a linear combination of the rows of $\mathbf{B}$, with the scalars that multiply the rows coming from $\mathbf{a}$.

4. Suppose $\mathbf{A}$ is an m-by-n matrix with $\mathbf{A} \neq 0$. Prove that the rank of $\mathbf{A}$ is 1 if and only if there exist $(c_1, \ldots, c_m) \in \mathbb{F}^m$ and $(d_1, \ldots, d_n) \in \mathbb{F}^n$ such that
$$\mathbf{A}_{j,k} = c_j d_k$$
for every $j = 1, \ldots, m$ and every $k = 1, \ldots, n$.

5. Suppose $T \in \mathcal{L}(V)$, and $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are bases of V. Prove that the following are equivalent:

   (a) T is injective.

   (b) The columns of $\mathcal{M}(T)$ are linearly independent in $\mathbb{F}^{n,1}$.

   (c) The columns of $\mathcal{M}(T)$ span $\mathbb{F}^{n,1}$.

   (d) The rows of $\mathcal{M}(T)$ span $\mathbb{F}^{1,n}$.

   (e) The rows of $\mathcal{M}(T)$ are linearly independent in $\mathbb{F}^{1,n}$.

6. Suppose V and W are finite-dimensional and $T \in L(V, W)$. Prove that $\dim \text{range } T = 1$ if and only if there exist a basis of V and a basis of W such that with respect to these bases, all entries of $M(T)$ equal 1. **Solution.**

7. Suppose V is finite-dimensional, and $\dim V \geq 1$. Prove that the set of noninvertible operators on V is not a subspace of $L(V)$.

8. Suppose V is finite-dimensional, U is a subspace of V, and $S \in L(U, V)$. Prove there exists an invertible operator $T \in L(V)$ such that $Tu = Su$ for every $u \in U$ if and only if S is injective.

9. Suppose $T \in L(U, V)$ and $S \in L(V, W)$ are both invertible linear maps. Prove that $ST \in L(U, W)$ is invertible and that $(ST)^{-1} = T^{-1}S^{-1}$.

10. Given the matrix
$$M = \begin{pmatrix} 1 & -2 & -1 & 0 \\ 1 & 2h & h+1 & h \\ 2 & 0 & h & h \end{pmatrix}$$

a) Determine the rank of M when varying $h \in \mathbb{R}$.
b) Putting $h = 1$, let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be a linear mapping associated to M with respect to the canonical bases. Find the dimension and a basis of both $\ker T$ and range T.

11. Let $a \in \mathbb{R}$ and let $T_a : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear mapping defined as

$$T_a(x, y, z) = (x - z, 2x - ay + az, x + ay + (a + 1)z).$$

a) Determine which values of $a \in \mathbb{R}$ results in $\ker T \neq \{0\}$.
b) Putting $a = 0$, determine the dimension and bases of $\ker T$ and range T.
c) Putting $a = 1$, determine the matrix $M_{FE}(T)$ where E is the canonical basis of $\mathbb{R}^3$ and $F = \{(1, 0, 1), (-1, 0, 3), (1, 2, 0)\}$.

12. If $A = (a_{ij}) \in M_{m_1,n_1}(\mathbb{F})$ and $B \in M_{m_2,n_2}(\mathbb{F})$ are matrices, the **Kronecker product** or **tensor product** of A and B is the matrix $A \otimes B \in M_{m_1 m_2, n_1 n_2}(\mathbb{F})$ defined by

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1,n_1}B \\ a_{21}B & a_{22}B & \cdots & a_{2,n_1}B \\ \vdots & \vdots & \vdots & \vdots \\ a_{m_1,1}B & a_{m_1,2}B & \cdots & a_{m_1,n_1}B \end{pmatrix}.$$

1) Do we always have $A \otimes B = B \otimes A$?
2) Check that $I_m \otimes I_n = I_{mn}$.
3) Prove that if $A_1 \in M_{m_1,n_1}(\mathbb{F})$, $A_2 \in M_{n_1,r_1}(\mathbb{F})$, $B_1 \in M_{m_2,n_2}(\mathbb{F})$, $B_2 \in M_{n_2,r_2}(\mathbb{F})$, then

$$(A_1 \otimes B_1)(A_2 \otimes B_2) = (A_1 A_2) \otimes (B_1 B_2).$$

4) Prove that if $A \in M_m(\mathbb{F})$ and $B \in M_n(\mathbb{F})$ then

$$A \otimes B = (A \otimes I_n)(I_m \otimes B).$$

13. Let $A, B \in M_n(\mathbb{R})$ such that

$$A + B = I_n \text{ and } A^2 + B^2 = O_n.$$

Prove that A and B are invertible and that

$$(A^{-1} + B^{-1})^n = 2^n I_n.$$

for all positive integers n.
Solution. Note that from the given two identities we get another one for free, this is

$$(I_n)^2 = I_n = A^2 + AB + BA + B^2 = (A^2 + B^2) + AB + BA = AB + BA.$$

Hence $I_n = AB + BA$. So the next results hold

$$A + AB + B + BA$$

$$= A(I_n + B) + B(I_n + A)$$

$$= A(I_n + I_n - A) + B(I_n + I_n - B) = 2I_n$$

$$\implies A(2I_n - A) + B(2I_n - B) = 2I_n$$

This identity has interesting symmetry properties, let $X = 2I_n - A$ and $Y = 2I_n - B$.

If $(X, Y) = (I_n, I_n)$ then we would get an absurd: $I_n = 2I_n$, so the matrices $A = B = I_n$ cannot exist.

If $(X, Y) = (A, B)$ we would get that $O_n = I_n$, another absurd result. So, in this case, the equations $2I_n - A = A$ and $2I_n - B = B$ must be false, hence $A, B \neq I_n$ again.

If $(X, Y) = (B, A)$, then we get $I_n = 2I_n$, which is, again, absurd. So, $2I_n - A = B$ cannot hold, meaning that $A \neq 2I_n - B$: $A \neq Y$, similarly we get $B \neq X$.

Now, if $(X, Y) = (Y, Y)$, then we get $2I_n - A = 2I_n - B$, which implies that $A = B$, meaning that $2B(2I_n - B) = 2I_n$, thus meaning that $B^{-1} = (2I_n - B)$ and analogously $A^{-1} = (2I_n - A)$ for $(X, Y) = (X, X)$.

(Is this idea worth keeping forward? I couldn't finish it in this track.)

14. Let $A \in M_n(\mathbb{R})$ be a matrix such that $A^2 = \mu A$, where μ is a real number with $\mu \neq -1$. Prove that

$$(I_n + A)^{-1} = I_n - \frac{1}{\mu + 1}A.$$

Solution. Assuming that the inverse exists, it follows that

$$I_n = (I_n + A)(I_n - \frac{1}{\mu + 1}A)$$

$$I_n = I_n + A - \frac{1}{\mu + 1}A - \frac{1}{\mu + 1}\mu A$$

$$I_n = I_n + -A(-1 + \frac{1}{\mu + 1} + \frac{1}{\mu + 1}\mu)$$

Where, the terms inside the parenthesis become zero. So we get that $I_n = I_n$.

$\square$

15. Suppose that an upper-triangular matrix $A \in M_n(\mathbb{C})$ is invertible. Prove that A^{-1} is also upper-triangular.
Solution. Since A is invertible, $A^{-1}A = I_n$. The i, j entry of this matrix is defined as

$$(A^{-1}A)_{ij} = \sum_{k=1}^{n} b_{ik}a_{kj} = \delta_{ij}$$

Since A is upper triangular, it has this shape:

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ & \ddots & \vdots \\ 0_{ij} & & a_{nn} \end{pmatrix}$$

So the amount of zeroes for the jth column will be $n-j$ starting from the row $r = j+1$. And the (minimum) amount of zeroes for the ith row will be $i-1$ starting at the column $c = 1$ and ending at $c = i-1$.

This data tells us that we can start from a larger index of the summation, since there are certain rows and columns which will not sum anything from some index to another. Now, we will call any i, j entry of A^{-1} as b_{ij}.

Suppose that for some natural number $i > j$ occurred that $b_{ij} \neq 0$ (which would imply that A^{-1} is not upper triangular). In spite of this, since I_n is upper triangular

$$I_{ij} = \sum_{k=1}^{n} b_{ik}a_{kj} = 0.$$

So,

$$\sum_{k=1}^{n} b_{ik}a_{kj} = \sum_{k=1}^{j} b_{ik}a_{kj} + \sum_{k=j+1}^{n} b_{ik}a_{kj} = \sum_{k=1}^{j} b_{ik}a_{kj} = 0.$$

Since for the jth column of A, there will be $n-j$ zeroes, starting from the row $k = j+1$. We see that $a_{kj} = 0$ only for $n \geq k \geq j+1$.

Now, when $k = j$ clearly $b_{ij}a_{jj} = 0$ if and only if, $a_{jj} = 0$ which by the last paragraph results impossible.

Then, suppose that $b_{ij} = 0$ when $i = j$. By doing reasonings in the same fashion, we would get that the diagonal of I_n is composed by 0's, which is absurd again. So, the only way to get an identity matrix is when A^{-1} is upper triangular.

$\square$

16. Let $\theta \in \mathbb{R}$, and let

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

a) Prove that A is orthogonal.
b) Find all values of θ for which A is symmetric.
c) Find all values of θ for which A is skew-symmetric.
Solution. a) The traspose of A will be

$${}^{\top}A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

So the product results in

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

by the identity $\cos^2\theta + \sin^2\theta = 1$.

b) Those values are when $\sin\theta = -\sin\theta$. Thus if $\theta = 2k\pi$ for $k \in \mathbb{Z}$, $\sin\theta = 0$.

c) Those values are when $\cos\theta = 0$, thus when $\theta = \pi/2 + 2k\pi$ for $k \in \mathbb{Z}$.

□

17. Which matrices $A \in M_n(\mathbb{F}_2)$ are the sum of a symmetric matrix and of a skew-symmetric matrix?

Remark. The field $\mathbb{F}_2$ is the binary numbers field, or boolean field with elements 0 or 1.

Solution. Let $B \in M_n(\mathbb{F}_2)$ be symmetric, and let $C \in M_n(\mathbb{F}_2)$ be anti-symmetric. Thus ${}^{\top}B = B$ and ${}^{\top}C = -C$. In the boolean field, the opposite matrix is itself, so anti-symmetry implies symmetry. We then get that from $A = B + C$, ${}^{\top}A = {}^{\top}(B + C) = B - C$. Therefore

$$A + {}^{\top}A = B + C + B - C = B + B = O_n$$

So $A = -{}^{\top}A$, then $A = {}^{\top}A$: A is symmetric.

□

18. Let the following matrices be square with same dimension, prove that
a) The product of two symmetric matrices is a symmetric matrix if and only if the two matrices commute.
b) The product of two antisymmetric matrices is a symmetric matrix if and only if the two matrices commute.

c) The product of a symmetric and a skew-symmetric matrix is a skew-symmetric matrix if and only if the two matrices commute.
Solution.
a) ($\implies$) ${}^{\top}(AB) = AB \implies AB = BA$.
We get that ${}^{\top}(AB) = {}^{\top}B\,{}^{\top}A = BA$ since both B and A are symmetric. So $AB = BA$.
($\impliedby$) $AB = BA \implies {}^{\top}(AB) = AB$.
Similarly, ${}^{\top}(AB) = {}^{\top}B\,{}^{\top}A = BA = AB$.
b)

19. Is it true that if the square of a matrix $A \in M_n(\mathbb{R})$ is symmetric, then A is symmetric?
Solution. No, it is not always the case. A counterexample is the matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}^2.$$

□

20. Consider the map $\varphi : M_3(\mathbb{R}) \to M_3(\mathbb{R})$ defined by

$$\varphi(A) = {}^{\top}A + 2A.$$

Prove that φ is linear.
Solution.

21. Let $A \in M_n(\mathbb{R})$ be a matrix such that $A \cdot {}^{\top}A = O_n$. Prove that $A = O_n$.
Solution. Since $A \cdot {}^{\top}A = O_n$, then we can deduce that $A \cdot {}^{\top}A = -A \cdot {}^{\top}A$. In fact this matrix is anti-symmetric. Since

$${}^{\top}(A \cdot {}^{\top}A) = {}^{\top}(-A \cdot {}^{\top}A) = -(A \cdot {}^{\top}A).$$

Furthermore, this matrix is also symmetric

$${}^{\top}(A \cdot {}^{\top}A) = A \cdot {}^{\top}A.$$

Then, the only matrix that satisfy both options is the null matrix O_n.

□

22. Find the skew-symmetric matrices $A \in M_n(\mathbb{R})$ such that $A^2 = O_n$.
Solution. Note that the diagonal must be formed of zeroes. Then we would like to have this sort of matrix

$$A = \begin{pmatrix} 0 & & -X \\ & \ddots & \\ X & & 0 \end{pmatrix}.$$

Where X can be formed by any entry $a_{ij} \in \mathbb{F}$. But the problem begins as we note that any entry in the diagonal of A is not zero if $a_{ij} \neq 0$.

The entry a_{11} of A^2 will be $0^2 - x_{12}^2 - \cdots - x_{1n}^2$ independently of the arrangement or value of any x_{1i} The only way to get that $A^2 = 0$ is to make $x_{ij} = 0$, for any $x_{ij} \in X$ (the reasoning is iterative; to fix a_{11} we have to make the first row of A null, in consequence, the first column of ${}^{\top}A$ is null as well, we repeat this logic with each a_{ii}).

□

23. Let $A_1, \ldots, A_k \in M_n(\mathbb{R})$ be matrices such that

$$ {}^{\top}A_1 \cdot A_1 + \cdots + {}^{\top}A_k \cdot A_k = O_n $$

Prove that $A_1 = \cdots = A_k = O_n$.
Solution. Let $M = {}^{\top}A_1 \cdot A_1 + \cdots + {}^{\top}A_k \cdot A_k$. Let a_{ij} be any entry of ${}^{\top}A_m$, and b_{ij} be any entry of A_m. The entries of the diagonal of M will be given by

$$ M_{ii} = \sum_{m=1}^{k} \left(\sum_{l=1}^{n} a_{il} b_{li} \right)_m . $$

Since the diagonal is invariant under the matrix traspose: $b_{li} = a_{il}$. Then

$$ \sum_{m=1}^{k} \left(\sum_{l=1}^{n} a_{il} b_{li} \right)_m = \sum_{m=1}^{k} \left(\sum_{l=1}^{n} a_{il}^2 \right)_m . $$

So if we want $\displaystyle\sum_{l=1}^{n} a_{il}^2 = 0$, any a_{il} must be zero. That is, the ith row of ${}^{\top}A_m$ must be zero in order to have its diagonal entry a_{ii} equal to zero. Thus, any column of A_m must be null, so $A_m = O_n$ for $1 \leq m \leq k$.

□

24. Describe all upper-triangular matrices $A \in M_n(\mathbb{R})$ such that

$$ A \cdot {}^{\top}A = {}^{\top}A \cdot A. $$

Solution. Let A be any upper-triangular matrix. Consider the sum of A and ${}^{\top}A$; this new matrix simplifies the computation of the i, j entry of their product, and has interesting symmetry properties. This sum matrix is symmetric and leaves unaltered the two matrices except for the diagonal, which is twice the original, for comodity we subtract a diagonal matrix with entries $b_{ii} = a_{ii}$. Call this new matrix P.

$$P = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{1n} & \cdots & a_{nn} \end{pmatrix}$$

The entry $({}^\top A \cdot A)_{ii}$ is given by $a_{1i}^2 + a_{2i}^2 + \cdots + a_{ii}^2$. Furthermore, the $(A \cdot {}^\top A)_{ii}$ entry will be given by the same sum $a_{1i}^2 + a_{2i}^2 + \cdots + a_{ii}^2$, since ${}^\top P = P = A + {}^\top A = {}^\top A + A$. More generally, the entry $(A \cdot {}^\top A)_{ij}$ will be $a_{1j} \cdot a_{i1} + \cdots + a_{ij} \cdot a_{ii}$. And dually, $({}^\top A \cdot A)_{ij} = a_{j1} \cdot a_{1i} + \cdots + a_{ji} \cdot a_{ii} = (A \cdot {}^\top A)_{ij}$. So we claim that any upper-triangular matrix A will commute with its traspose. By induction on the dimension n of A, if $n = 2$ then

$$A = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}, \; {}^\top A = \begin{pmatrix} a_{11} & 0 \\ a_{12} & a_{22} \end{pmatrix}, \; P = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}.$$

then $(A\,{}^\top A)_{12} = a_{11} \cdot a_{12} = a_{12} \cdot a_{11} = ({}^\top A A)_{12}$. Now, assume that for any dimension $n = k$, $k \in \mathbb{N}$, any upper-triangular matrix A will commute with its traspose. The matrix A' with dimension $k+1$ will be recursively made by adding a non-null vector V with $k+1$ rows at the $k+1$ column to a upper-triangular matrix A with dimension k, and filling the empty bottom row with all zeroes.

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ & \ddots & \vdots \\ 0_{ij} & & a_{nn} \end{pmatrix}, \; A' = \begin{pmatrix} a_{11} & \cdots & a_{1k} & a_{1(k+1)} \\ & \ddots & \vdots & \vdots \\ & & a_{kk} & a_{k(k+1)} \\ 0_{ij} & \cdots & 0_{(k+1)k} & a_{k+1(k+1)} \end{pmatrix} = \begin{pmatrix} A & V \\ 0_{ij} & V \end{pmatrix}.$$

We note, that P is a submatrix of $P' = {}^\top A' + A'$, P, so the product of $A\,{}^\top A$ remains unaltered, and it commutes by hypothesis:

$$P' = \begin{pmatrix} A + {}^\top A & V \\ V & V \end{pmatrix}.$$

Let P'' be

$$P'' = \begin{pmatrix} O_k & V \\ V & V \end{pmatrix}.$$

This matrix is symmetric, and the product between V and itself clearly commutes.

25. Prove that for all matrices $A, B \in M_2(\mathbb{C})$ we have

$$\det(A + B) = \det A + \det B + Tr(A)Tr(B) - Tr(AB).$$

26. Prove that for all matrices $A \in M_2(\mathbb{C})$ we have

$$\det A = \frac{(Tr(A))^2 - Tr(A^2)}{2}.$$

Solution. The solution of this problem boils down to making the calculations which are easy since the size of the matrix is 2.

27. Let $A, B \in M_2(\mathbb{R})$ be commuting matrices. Prove that

$$\det(A^2 + B^2) \geq 0.$$

28. Let $A, B \in M_2(\mathbb{R})$ be such that $AB = BA$ and $\det(A^2 + B^2) = 0$. Prove that $\det A = \det B$.

29. Suppose W is finite-dimensional and $T_1, T_2 \in L(V, W)$. Prove that $\ker T_1 = \ker T_2$ if and only if there exists an invertible operator $S \in L(W)$ such that $T_1 = ST_2$.

30. Suppose V is finite-dimensional and $T_1, T_2 \in L(V, W)$. Prove that range $T_1 =$ range T_2 if and only if there exists an invertible operator $S \in L(V)$ such that $T_1 = T_2 S$.

31. Suppose V and W are finite-dimensional and $T_1, T_2 \in L(V, W)$. Prove that there exist invertible operators $R \in L(V)$ and $S \in L(W)$ such that

$$T_1 = ST_2R$$

if and only if

$$\dim(\ker T_1) = \dim(\ker T_2).$$

32. A magic square is a 3×3 matrix of real numbers where the sum of all diagonals, columns, and rows is equal. Find the dimension of the set of magic squares, as a real vector space under addition.
Solution. We claim that the set

$$M_C := \left\{ M \in M_3(\mathbb{R}) : \sum_{k=1}^{3} M_{ki} \text{ for any } 1 \leq i \leq 3 \right\}$$

is a vector space. First we note that if we sum two "non-zero" matrices $A, B \in M_C$, the result is another matrix $C \in M_C$, with the sum of its columns being the sum of the respective columns of A and B. The null matrix O_3 is in M_C, and works as a null element for the sum. And the scalar multiplication λA is in M_C, with columns being $\lambda-$times the columns of A. This proves our claim.

Suppose that there existed a matrix $B \in M_R$ such that it row sum did not equal the column sum of ${}^{\top}B \in M_C$, in other words

$$\sum_{k=1}^{3} B_{ki} \neq \sum_{k=1}^{3} ({}^{\top}B)_{ik} \text{ for some } 1 \leq i \leq 3.$$

then

$$\sum_{k=1}^{3} B_{ki} \neq \sum_{k=1}^{3} B_{ki} \text{ for some } 1 \leq i \leq 3,$$

which is absurd. This result shows that M_R has the same dimension as M_C, and that a basis for it is the traspose of a basis for M_C.

Then let

$$M_D := \left\{ M \in M_3(\mathbb{R}) : tr(M) = \sum_{k=0}^{3} M_{3-k,k} \right\},$$

this set is a vector space by analogous reasons as the former ones. In spite of this the occupied entries are not each one but just 5, so the dimension of this space will be $\dim M_3(\mathbb{R}) - 5 = 9 - 5 = 4$.

For the columns case, we see that the entries at the last column and last row depend on the choice of the entries $a_{11}, a_{12}, a_{21}, a_{22}$. In other words, the only required basis will be, say the canonical basis for 2×2 square matrices of dimension 4.

We conclude computing the dimension of the intersection between all three spaces.

$$\dim(M_C \cap M_R \cap M_D) = \dim(M_C \cap M_D)$$

33. Prove that two matrices $A, B \in M_n(\mathbb{R})$ are equal if and only if for each $M \in M_n(\mathbb{R})$ it holds that

$$tr(AM) = tr(BM).$$

Solution. For the direct implication it is straightforward. Assume that $A, B \in M_n(\mathbb{R})$ are equal, thus

$$AM = BM, \text{ so } tr(AM) = tr(BM).$$

For the converse implication suppose that for any $M \in M_n(\mathbb{R})$ it held that

$$tr(AM) = tr(BM),$$

then, as the trace operator is linear we get that $tr(AM) - tr(BM) = tr(AM - BM) = 0$.

As the trace operator is null if and only if each entry in the main diagonal is zero, we see that the main diagonal of $AM - BM$ is formed by all zeroes.

Suppose for the sake of contradiction that $A \neq B$. The diagonal entries of AM and BM are given by

$$(AM)_{ii} = \sum_{k=1}^{n} A_{ik}M_{ki},\ (BM)_{ii} = \sum_{l=1}^{n} B_{il}M_{li}.$$

So the overall trace will be

$$tr(AM - BM) = \sum_{i=1}^{n} \left(\sum_{k=1}^{n} A_{ik}M_{ki} - \sum_{l=1}^{n} B_{il}M_{li} \right).$$

Operating the sum we get the following chain of equalities

$$\begin{aligned} tr(AM - BM) &= \sum_{i=1}^{n} \left(\sum_{k=1}^{n} A_{ik}M_{ki} \right) - \sum_{i=1}^{n} \left(\sum_{k=1}^{n} B_{ik}M_{ki} \right) \\ &= \sum_{i=1}^{n} \left(\sum_{k=1}^{n} A_{ik}M_{ki} - \sum_{k=1}^{n} B_{ik}M_{ki} \right) \\ &= \sum_{i=1}^{n} \left(\sum_{k=1}^{n} A_{ik}M_{ki} - B_{ik}M_{ki} \right) \\ &= \sum_{i=1}^{n} \left(\sum_{k=1}^{n} M_{ki}(A_{ik} - B_{ik}) \right). \end{aligned}$$

We see that it the inner sum results into zero when $M_{ki} = 0$ or $A_{ik} = B_{ik}$ for every $1 \leq k \leq n$. The above sum can be further compacted by Fubini's theorem

$$tr(AM - BM) = \sum_{1 \leq i,k \leq n} M_{ki}(A_{ik} - B_{ik}).$$

Since we supposed $A \neq B$, there exists an entry (i_0, k_0) such that $A_{i_0k_0} \neq B_{i_0k_0}$. We construct a matrix M by the formula

$$M_{ij} := \begin{cases} 1 \text{ if } (i,j) = (i_0, k_0) \\ 0 \text{ if not} \end{cases}.$$

Then,

$$\sum_{1 \leq i,k \leq n} M_{ki}(A_{ik} - B_{ik}) = A_{i_0k_0} - B_{i_0k_0} \neq 0.$$

Which contradicts the fact that $tr(AM - BM) = 0$ for any $M \in M_n(\mathbb{R})$.

$\square$

34. Given $A \in M_n(\mathbb{R})$, find an expression for $tr(^\top A \cdot A)$ and prove that $A = 0$ if and only if $tr(^\top A \cdot A) = 0$. Give a counterexample of this fact in the case $A \in M_n(\mathbb{C})$.