Discrete Mathematics Sheet I

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Problems in Discrete Math from the books: "A Path to Combinatorics For Undergraduates" by Titu Andreescu, "102 Combinatorial Problems" by Titu Andreescu, "Graph Theory With Applications" by J. A. Bondy and J. Ullman's "Automata Theory, Languages, and Computation". These weeks problems will be about an introduction to counting techniques, graph theory, and automaton theory.

Counting Techniques

1. Determine the number of functions

$$f : \{1, 2, \ldots, 1999\} \to \{2000, 2001, 2002, 2003\}$$

satisfying the condition that

$$f(1) + f(2) + \cdots + f(1999) \text{ is odd.}$$

2. Find the number of two-digit positive integers that are divisible by both of their digits.

3. For any set S, let $|S|$ denote the number of elements in S, and let $n(S)$ be the number of subsets of S, including the empty set and S itself. If A, B, and C are sets for which

$$n(A) + n(B) + n(C) = n(A \cup B \cup C)$$

and

$$|A| = |B| = 100,$$

then what is the minimum possible value of $|A \cap B \cap C|$?

4. For how many pairs of consecutive integers in $\{1000, 1001, 1002, \ldots, 2000\}$ is no carrying required when the two integers are added?

5. Each of two boxes contains both black and white marbles, and the total number of marbles in the two boxes is 25. One marble is taken out of each box randomly. The probability that both marbles are black is $\frac{27}{50}$. What is the probability that both marbles are white?

6. There are ten girls and four boys in Mr. Fat's combinatorics class. In how many ways can these students sit around a circular table such that no boys are next to each other?

7. Find the number of ordered triples of sets (A, B, C) such that

$$A \cup B \cup C = \{1, 2, \ldots, 2003\} \quad \text{and} \quad A \cap B \cap C = \emptyset.$$

8. Compute the number of sets of three distinct elements that can be chosen from the set $\{2^1, 2^2, 2^3, \ldots, 2^{2000}\}$ such that the three elements form an increasing geometric progression.

9. Let n be an integer greater than four, and let $P_1P_2 \ldots P_n$ be a convex n-sided polygon. Zachary wants to draw $n-3$ diagonals that partition the region enclosed by the polygon into $n-2$ triangular regions and that may intersect only at the vertices of the polygon. In addition, he wants each triangular region to have at least one side that is also a side of the polygon. In how many different ways can Zachary do this?

10. How many five-digit numbers are divisible by three and also contain 6 as one of their digits?

11. Two squares on an 8×8 chessboard are called *touching* if they have at least one common vertex. Determine if it is possible for a king to begin in some square and visit all the squares exactly once in such a way that all moves except the first are made into squares touching an even number of squares already visited.

12. A total of 119 residents live in a building with 120 apartments. We call an apartment *overpopulated* if there are at least 15 people living there. Every day the inhabitants of an overpopulated apartment have a quarrel and each goes off to a different apartment in the building (so they can avoid each other). Is it true that this process will necessarily be completed someday?

Graph theory

1. Show that two simple graphs G and H are isomorphic if and only if there is a bijection $\theta : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $\theta(u)\theta(v) \in E(H)$.

2. Let G be simple. Show that $\varepsilon = \binom{\nu}{2}$ if and only if G is complete.

3. Show that

(a) $\varepsilon(K_{m,n}) = mn$;

(b) if G is simple and bipartite, then $\varepsilon \leq \nu^2/4$.

4. A k-partite graph is one whose vertex set can be partitioned into k subsets so that no edge has both ends in any one subset; a complete k-partite graph is one that is simple and in which each vertex is joined to every vertex that is not in the same subset. The complete m-partite graph on n vertices in which each part has either $[n/m]$ or $\{n/m\}$ vertices is denoted by $T_{m,n}$. Show that

   (a) $\varepsilon(T_{m,n}) = \binom{n-k}{2} + (m-1)\binom{k+1}{2}$, where $k = [n/m]$;

   (b) if G is a complete m-partite graph on n vertices, then $\varepsilon(G) \leq \varepsilon(T_{m,n})$, with equality only if $G \cong T_{m,n}$.

5. The k-cube is the graph whose vertices are the ordered k-tuples of 0's and 1's, two vertices being joined if and only if they differ in exactly one coordinate. (The graph shown in figure 1.4b is just the 3-cube.) Show that the k-cube has 2^k vertices, $k2^{k-1}$ edges and is bipartite.

6. (a) The complement G^e of a simple graph G is the simple graph with vertex set V, two vertices being adjacent in G^e if and only if they are not adjacent in G. Describe the graphs K_n^e and $K_{m,n}^e$.

   (b) A simple graph G is self-complementary if $G \cong G^e$. Show that if G is self-complementary, then $\nu \equiv 0, 1 \pmod 4$.

7. The k-cube is the graph whose vertices are the ordered k-tuples of 0's and 1's, two vertices being joined if and only if they differ in exactly one coordinate. (The graph shown in figure 1.4b is just the 3-cube.) Show that the k-cube has 2^k vertices, $k2^{k-1}$ edges and is bipartite.

8. (a) The *complement* G^* of a simple graph G is the simple graph with vertex set V, two vertices being adjacent in G^* if and only if they are not adjacent in G. Describe the graphs K_n^* and $K_{m,n}^*$.

   (b) A simple graph G is *self-complementary* if $G \cong G^*$. Show that if G is self-complementary, then $\nu \equiv 0, 1 \pmod 4$.

9. An *automorphism* of a graph is an isomorphism of the graph onto itself.

   (a) Show, using exercise 1.2.5, that an automorphism of a simple graph G can be regarded as a permutation on V which preserves adjacency, and that the set of such permutations forms a group $\Gamma(G)$ (the *automorphism group* of G) under the usual operation of composition.

   (b) Find $\Gamma(K_n)$ and $\Gamma(K_{m,n})$.

   (c) Find a nontrivial simple graph whose automorphism group is the identity.

   (d) Show that for any simple graph G, $\Gamma(G) = \Gamma(G^*)$.

(e) Consider the permutation group Λ with elements $(1)(2)(3)$, $(1,2,3)$ and $(1,3,2)$. Show that there is no simple graph G with vertex set $\{1,2,3\}$ such that $\Gamma(G) = \Lambda$.

(f) Find a simple graph G such that $\Gamma(G) = \Lambda$. (Frucht, 1939 has shown that every abstract group is isomorphic to the automorphism group of some graph.)

10. A simple graph G is *vertex-transitive* if, for any two vertices u and v, there is an element g in $\Gamma(G)$ such that $g(u) = v$; G is *edge-transitive* if, for any two edges $\{u_1, v_1\}$ and $\{u_2, v_2\}$, there is an element h in $\Gamma(G)$ such that $h(\{u_1, v_1\}) = \{u_2, v_2\}$. Find

    (a) a graph which is vertex-transitive but not edge-transitive;

    (b) a graph which is edge-transitive but not vertex-transitive.

11. What is the number of edges in a K^n?

12. Let $d \in \mathbb{N}$ and $V := \{0,1\}^d$; thus, V is the set of all 0-1 sequences of length d. The graph on V in which two such sequences form an edge if and only if they differ in exactly one position is called the *d-dimensional cube*. Determine the average degree, number of edges, diameter, girth and circumference of this graph. (Hint for the circumference: induction on d.)

13. Let G be a graph containing a cycle C, and assume that G contains a path of length at least k between two vertices of C. Show that G contains a cycle of length at least $\sqrt{k}$.

14. Is the bound in Proposition 1.3.2 best possible?

15. Let v_0 be a vertex in a graph G, and $D_0 := \{v_0\}$. For $n = 1, 2, \ldots$ inductively define $D_n := N_G(D_0 \cup \ldots \cup D_{n-1})$. Show that $D_n = \{\, v \mid d(v_0, v) = n \,\}$ and $D_{n+1} \subseteq N(D_n) \subseteq D_{n-1} \cup D_{n+1}$ for all $n \in \mathbb{N}$.

16. Show that $\operatorname{rad}(G) \leqslant \operatorname{diam}(G) \leqslant 2\operatorname{rad}(G)$ for every graph G.

17. Prove the weakening of Theorem 1.3.4 obtained by replacing average with minimum degree. Deduce that $|G| \geqslant n_0(d/2, g)$ for every graph G as given in the theorem.

18. Show that graphs of girth at least 5 and order n have a minimum degree of $o(n)$. In other words, show that there is a function $f\colon \mathbb{N} \to \mathbb{N}$ such that $f(n)/n \to 0$ as $n \to \infty$ and $\delta(G) \leqslant f(n)$ for all such graphs G.

19. Show that every connected graph G contains a path or cycle of length at least $\min\{2\delta(G), |G|\}$.

20. Show that a connected graph of diameter k and minimum degree d has at least about $kd/3$ vertices but need not have substantially more.

Finite automata

1. We defined $\hat{\delta}$ by breaking the input string into any string followed by a single symbol (in the inductive part, Equation 2.1). However, we informally think of $\hat{\delta}$ as describing what happens along a path with a certain string of labels, and if so, then it should not matter how we break the input string in the definition of $\hat{\delta}$. Show that in fact, $\hat{\delta}(q, xy) = \hat{\delta}(\hat{\delta}(q, x), y)$ for any state q and strings x and y. *Hint: Perform an induction on* $|y|$.

2. Show that for any state q, string x, and input symbol a, $\hat{\delta}(q, ax) = \hat{\delta}(\hat{\delta}(q, a), x)$. *Hint: Use Exercise 2.2.2.*

3. Give DFA's accepting the following languages over the alphabet $\{0, 1\}$:

    a) The set of all strings ending in 00.
    b) The set of all strings with three consecutive 0's (not necessarily at the end).
    c) The set of strings with 011 as a substring.

4. Give DFA's accepting the following languages over the alphabet $\{0, 1\}$:

    a) The set of all strings such that each block of five consecutive symbols contains at least two 0's.
    b) The set of all strings whose tenth symbol from the right end is a 1.
    c) The set of strings that either begin or end (or both) with 01.
    d) The set of strings such that the number of 0's is divisible by five, and the number of 1's is divisible by 3.

5. Give DFA's accepting the following languages over the alphabet $\{0, 1\}$:

    a) The set of all strings beginning with a 1 that, when interpreted as a binary integer, is a multiple of 5. For example, strings 101, 1010, and 1111 are in the language; 0, 100, and 111 are not.
    b) The set of all strings that, when interpreted *in reverse* as a binary integer, is divisible by 5. Examples of strings in the language are 0, 10011, 1001100, and 0101.

6. Let A be a DFA and q a particular state of A, such that $\delta(q, a) = q$ for all input symbols a. Show by induction on the length of the input that for all input strings w, $\tilde{\delta}(q, w) = q$.

7. Let A be a DFA and a a particular input symbol of A, such that for all states q of A we have $\delta(q, a) = q$.

    a) Show by induction on n that for all $n \geq 0$, $\tilde{\delta}(q, a^n) = q$, where a^n is the string consisting of n a's.
    b) Show that either $\{a\}^* \subseteq L(A)$ or $\{a\}^* \cap L(A) = \emptyset$.

8. Let $A = (Q, \Sigma, \delta, q_0, \{q_f\})$ be a DFA, and suppose that for all a in Σ we have $\delta(q_0, a) = \delta(q_f, a)$.

   a) Show that for all $w \neq \epsilon$ we have $\tilde{\delta}(q_0, w) = \tilde{\delta}(q_f, w)$.

   b) Show that if x is a nonempty string in $L(A)$, then for all $k > 0$, x^k (i.e., x written k times) is also in $L(A)$.

9. Consider the DFA with the following transition table:

	0	1
$\to A$	A	B
$*B$	B	A

   Informally describe the language accepted by this DFA, and prove by induction on the length of an input string that your description is correct. *Hint:* When setting up the inductive hypothesis, it is wise to make a statement about what inputs get you to each state, not just what inputs get you to the accepting state.

10. Repeat Exercise 9 for the following transition table:

	0	1
$\to *A$	B	A
$*B$	C	A
C	C	C

   Informally describe the language accepted by this DFA, and prove by induction on the length of an input string that your description is correct. *Hint:* When setting up the inductive hypothesis, describe what inputs lead to each state (not just the accepting state).