Linear mappings sheet IV

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Problems in linear algebra: problems from the books of Titu Andreescu "Essential Linear Algebra" and Sheldon Axler "Linear Algebra Done Right". The topics of this week are isomorphisms, change of bases, rank of a matrix and dual spaces.

Isomorphisms and invertibility

1. Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that the following are equivalent:

   (a) T is invertible.

   (b) $Tv_1, \ldots, Tv_n$ is a basis of V for every basis $v_1, \ldots, v_n$ of V.

   (c) $Tv_1, \ldots, Tv_n$ is a basis of V for some basis $v_1, \ldots, v_n$ of V.

   Solution. (a) $\implies$ (b)

2. Suppose V is finite-dimensional, U is a subspace of V, and $S \in \mathcal{L}(U, V)$. Prove that there exists an invertible linear map T from V to itself such that $Tu = Su$ for every $u \in U$ if and only if S is injective.
   Solution. For the direct implication, if T is invertible, then it is injective, The restriction of T to a subset U will be injective, and will behave as expected. The mapping will not be surjective, although this is not an issue.

   For the converse implication, if S is injective, U and Im S are isomorphic. Hence there exists an isomorphism $S' : V \setminus U \to V \setminus \text{Im } S$. Build T such that

$$Tv = \begin{cases} Sv & \text{if } v \in U; \\ S'v & \text{if } v \in V \setminus U \end{cases}.$$

$\square$

   (Idea of) Solution. For the converse implication, if S is injective, we first determine the amount of elements in the intersection

$$U \cap \text{Im } S = \begin{cases} \{0\}; \\ \text{not only zero subspace of } U \text{ or Im } S; \\ \text{a subspace with the same dimension of } U \text{ and Im } S \end{cases}.$$

The first case implies that any basis of U will not be able to span any vector of Im S, the second case implies that there exists a basis of U that spans some vectors of Im S, and the last one implies that a basis of U will also be a basis for Im S. For the third implication there is the identity map, that will do the job of changing any vector of U to its corresponding vector of Im S, as both spaces have same dimension ($\ker S = \{0\}$).

$$u_1 \mapsto Tu_1, \ldots, u_n \mapsto Tu_n.$$

Then, extending this map to V:

$$v_1 \mapsto Tv_1, \ldots, u_1 \mapsto Tu_1, \ldots, u_n \mapsto Tu_n, \ldots, v_m \mapsto Tv_m.$$

Doing the same reasoning as for the direct implication.

For the second case, let $\mathcal{B}$ be a basis for U, and let $\mathcal{C}$ be a basis for Im S. Consider the subspace $W \subseteq V$, $W = \text{span}(\mathcal{B} \cup \mathcal{C})$, such that $W = U \oplus$ Im S. Let $\varphi_U : W \to U$ be defined as a projection mapping, that is, when valued at some $w \in W$, will output the only $u \in U$ such that for some $x \in$ Im S: $x = u - w$. Define $\varphi_{\text{Im } S} : W \to \text{Im } T$ the same way, but interchanging x with u. A map $T : W \to W$ will be

$$Tw = \begin{cases} \varphi_U(w) & \text{if } w \in U; \\ \varphi_{\text{Im } S}(w) & \text{if } w \in \text{Im } S \end{cases}.$$

For $w \in U$, Tw will be itself as $u = x + 0$. Similarly for $x \in$ Im S. Then we can add linearly independent vectors to $\mathcal{B}$ and $\mathcal{C}$.

(Can we finish up this idea? I find it hard to assume both bases will always be linearly independent, besides, the idea turned out too complex.)

3. Suppose that W is finite-dimensional and $S, T \in \mathcal{L}(V, W)$. Prove that $\text{null}\, S = \text{null}\, T$ if and only if there exists an invertible $E \in \mathcal{L}(W)$ such that $S = ET$.

   Solution. Let k be the dimension of $\ker T$, let n be the dimension of V and m be the dimension of W. Since $\ker T = \ker S$ then, by the rank-nullity theorem, dim Im T = dim Im S, as the input space is the same for both mappings. Let $\mathcal{B}$ be a basis of V such as

   $$\{v_1, \ldots, v_{n-k}, \ldots, v_n\}$$

   Where the vectors $v_{n-k}, \ldots, v_n$ are those of a basis of $\ker T$ and $\ker S$. Let $\mathcal{C}$ and $\mathcal{D}$ be bases of W containing the bases of Im T and Im S respectively in a way such as $\{w_1, \dots, w_{n-k}, \dots, w_m\}$, where $w_1, \dots, w_{n-k}$ are vectors from Im T or Im S. Let the matrices $M(T)$ and $M(S)$ be associated to T and S respectively. Consider $Id : W_{\mathcal{C}} \to W_{\mathcal{D}}$, where $Id((w_i)_{\mathcal{C}}) = (w_i)_{\mathcal{D}}$ for vectors w_i of their respective bases; and $1 \leq i \leq n-k$. We have the following diagram

$$\begin{tikzcd}
V_{\mathcal{B}} \arrow[r, "T"] \arrow[dr, "S"'] & W_{\mathcal{C}} \arrow[d, leftrightarrow, "Id_{\mathcal{DC}}"] \\
 & W_{\mathcal{D}}
\end{tikzcd}$$

We see that $E = Id_{\mathcal{DC}}$. So $T = E \circ S$, thus $S = E^{-1} \circ T = Id_{\mathcal{CD}} \circ T$. Reciprocally, if there exists an invertible mapping $E \in \mathcal{L}(W)$ such that $ET = S$, the matrix $M(E)$ will be invertible, then it will be square, and its rank will be $\dim W$. A null column C_j of $M(S)$ (for some basis of W) will be given by

$$C_j = M(E)_{\cdot 1} M(T)_{1j} + \cdots + M(E)_{\cdot m} M(T)_{mj} = O_{\cdot j}$$

Where each column of $M(E)$ is linearly independent (since its rank its m). Thus, the entries of the jth column of $M(T)$ must be zero, thus, the column itself its zero.

$\square$

4. Suppose that V is finite-dimensional and $S, T \in \mathcal{L}(V, W)$. Prove that $\operatorname{range} S = \operatorname{range} T$ if and only if there exists an invertible $E \in \mathcal{L}(V)$ such that $S = TE$.

(Idea of) Solution. There is, most likely, an analogous reasoning to that of the previous problem, but I will provide a solution using a concept of the book (column rank and column-row factorization) where I got the problem from (Linear Algebra Done Right).

Let n be the dimension of V, and let m be the dimension of W. Let $M(T)$ and $M(S)$ be the matrices associated to each mapping, the choosen bases do not matter yet.

As $\operatorname{Im} T = \operatorname{Im} S$, then $\dim \operatorname{Im} T = \dim \operatorname{Im} S$, the column rank r of $M(T)$ is equal to the column rank of $M(S)$. Thus, let $\{C_{\cdot 1}, \dots, C_{\cdot n}\}$ be each column of $M(T)$, and let $\{D_{\cdot 1}, \dots, D_{\cdot n}\}$ be each column of $M(S)$.

From these columns we can derive linearly independent columns $\{C'_{\cdot 1}, \dots, C'_{\cdot r}\}$ for $M(T)$ and $\{D'_{\cdot 1}, \dots, D'_{\cdot r}\}$ for $M(S)$ (for example, by using the gaussian algorithm in the traspose of $M(T)$ and $M(S)$). So let $M(T')$ be that matrix constructed by merging each column $C'_{\cdot i}$, for $1 \leq i \leq r$:

$$M(T') = \begin{pmatrix} \vdots & \cdots & \vdots \\ C'_{\cdot 1} & \cdots & C'_{\cdot r} \\ \vdots & \cdots & \vdots \end{pmatrix}$$

This matrix has size $m \times r$. Similarly for each column $D'_{\cdot i}$:

$$M(S') = \begin{pmatrix} \vdots & \cdots & \vdots \\ D'_{\cdot 1} & \cdots & D'_{\cdot r} \\ \vdots & \cdots & \vdots \end{pmatrix}.$$

Note that if $m > r$, this would mean that $S', T' : U \to W$ are not surjective.

We claim that both are, although, necessarily injective. To prove this claim first note that the dimension of U is $\dim \operatorname{Im} T = \dim \operatorname{Im} S$, so suppose there were at least one (linearly independent) kernel vector of T in T', thus

$$Tv = 0_V \implies T(v_{\mathcal{B}}) = \begin{pmatrix} 0_{11} \\ \vdots \\ 0_{1m} \end{pmatrix}$$

for some basis of W. It is clearly not linearly independent, so we do not gather as a column for $M(T')$. (Does this argument suffices?) An analogous reasoning is used to prove that S' is injective.

Let $\mathcal{U} = \{u_1, \ldots, u_r\}$ be a basis for U, then, by the previous claim, $T'u_1, \ldots, T'u_r$ will be a basis for $\operatorname{Im} T'$ and for $\operatorname{Im} S'$. As well as $S'u_1, \ldots, S'u_r$: $\mathcal{D}$.

Consider a basis for V extending the basis of U: $\mathcal{A}$, and a basis for W by extending $\mathcal{B}$ or $\mathcal{C}$, we will get matrices for T as

$$M_{\mathcal{AB}}(T) \text{ and } M_{\mathcal{AC}}(T)$$

We can build a basis for V in which $M(T')$ is a submatrix of $M(T)$ and even $M(S')$ will be a submatrix of $M(T)$. The way to make them is to take the union with $n - r$ kernel basis vectors with $\mathcal{U}$, instead of linearly extending them, call this basis $\mathcal{A}'$. Matrices will have the following shape:

$$M_{\mathcal{A}'\mathcal{B}}(T) = \begin{pmatrix} M_{\mathcal{UB}}(T') & O_{n-r} \\ M_{\mathcal{UB}}(T') & O_{(n-r)(m-r)} \end{pmatrix}$$

We can make this matrix better, as it is injective, T' will have a square submatrix of size r contained in the first quadrant of $M_{\mathcal{A}'\mathcal{B}(T)}$

$$M_{\mathcal{A}'\mathcal{B}}(T) = \begin{pmatrix} M_{\mathcal{U}\mathcal{B}'}(T') & O_{n-r} \\ O_{m-r} & O_{(n-r)(m-r)} \end{pmatrix}$$

Thus:

$$M_{\mathcal{A}'\mathcal{B}}(T) = \begin{pmatrix} M_{\mathcal{U}\mathcal{B}'}(T') \\ O_{m-r} \end{pmatrix} I_{\max n,m} = M(T') I_{\max n,m}$$

Then: $M(S') = M(T')P$. Moreover, by column-row factorization theorem

$$M(T) = M(T')R_T \text{ and } M(S) = M(S')R_S$$

Where R_T and R_S are $r \times n$ matrices. Therefore:

$$M(S) = M(S')R_S = M(T)P$$

(Can we end up this idea? This will anyways end up in a change of basis. If there is a diverse method let me know.)

Solution.

□

5. Suppose V and W are finite-dimensional and $S, T \in \mathcal{L}(V, W)$. Prove that there exist invertible $E_1 \in \mathcal{L}(V)$ and $E_2 \in \mathcal{L}(W)$ such that $S = E_2 T E_1$ if and only if $\dim \operatorname{null} S = \dim \operatorname{null} T$.
   Solution. For the converse, we note that $\ker T$ is isomorphic to $\ker S$. And by rank-nullity theorem, Im T is isomorphic to Im S as well. Moreover $V \setminus \ker T$ is isomorphic to $V \setminus \ker S$. So the mapping $\mathrm{id} : V_{\mathcal{B}} \to V_{\mathcal{C}}$ can be decomposed as id

6. Suppose V is finite-dimensional and $T \colon V \to W$ is a surjective linear map of V onto W. Prove that there is a subspace U of V such that $T|_U$ is an isomorphism of U onto W.
   Solution. Let $n = \dim V$, $m = \dim W$. Since T is surjective, the dimension of W will be at most that of V. If this is the case take U as the whole V.

   If the dimension of W is strictly less than that of V, by the rank-nullity theorem, there are $n-m$ kernel vectors. Take $\mathcal{B}$, a basis for V containing a basis $\mathcal{K}$ of $\ker T$, this basis can be constructed starting with $\mathcal{K}$ and linearly extending it to $\mathcal{B}$. The subspace U will be constructed as $\operatorname{span}(\mathcal{B} \setminus \mathcal{K})$, and for any vector $u \in U$, $Tu \neq 0_W$. The dimension of U will be $n - (n-m) = m$, meaning that U and W are isomorphic.

   Lastly, recall that dim Im $T = \dim W$, then dim Im $T = \dim U$. Let $T_{\restriction U} : U \to W$ be T restricted to U, since $Tu \neq 0_W$ for any $u \in U$, this mapping is injective. Therefore, by rank-nullity, again,

$$\dim \operatorname{Im} T_{\restriction U} = \dim U = \dim \operatorname{Im} T = \dim W.$$

Thus, $T_{\restriction U}$ is surjective, and since it is injective, will also make an isomorphism between U and W.

$\square$

7. Suppose V and W are finite-dimensional and U is a subspace of V. Let

$$\mathcal{E} = \{T \in \mathcal{L}(V, W) \colon U \subseteq \operatorname{null} T\}.$$

   (a) Show that $\mathcal{E}$ is a subspace of $\mathcal{L}(V, W)$.

   (b) Find a formula for $\dim \mathcal{E}$ in terms of $\dim V$, $\dim W$, and $\dim U$.

Solution. (a) This part is straightforward, take $0_{\mathcal{E}}$ as the null mapping $T = O$. The sum of mappings restricted to some U containing all vectors of their kernel will also have the vectors of U as it kernel. And the same goes for multiplication by a scalar.

(b) Let $n = \dim V$, $m = \dim W$. Since $\mathcal{L}(V, W)$ is isomorphic to $\mathbb{F}^{n,m}$ by the matrix mapping M, and $\mathcal{E}$ is a subspace of it, we will have an isomorphism to some subspace $\mathbb{F}^{p,q}$ with dimension $p \times q$ of $\mathbb{F}^{n,m}$: $M : \mathcal{E} \to \mathbb{F}^{p,q}$ (note it is the same mapping M).

Let $T \in \mathcal{E}$, its associated matrix $M(T)$ will have at least one null column for some basis $\mathcal{B}$ of V (it is not immediately obvious that a null column will appear for any basis of V, but it is certain that there exists, for any mapping $T \in \mathcal{E}$, at least one since $\dim \operatorname{Im} T = \operatorname{rank} M(T) \leq \dim V$). So the dimension of $\mathcal{E}$ will be given from those matrices that have $\dim U$ null columns for some basis $\mathcal{B}$. Therefore: $\dim \mathcal{E} = (\dim(V) - \dim(U)) \dim W$.

$\square$

8. Suppose V is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that

$$ST \text{ is invertible} \iff S \text{ and } T \text{ are invertible.}$$

First Solution. This first solution involves determinants and it is straightforward, I will provide a second solution, as the author supposedly intended a solution without involving them.

Since ST is invertible, $\det(M(S)M(T)) \neq 0$, thus $\det(M(S))\det(M(T)) \neq 0$, this implies that $\det(M(S)) \neq 0$ and $\det(M(T)) \neq 0$. This proves that S and T are invertible.

Conversely, if S and T are invertible, $\det(M(S)) \neq 0$, $\det(M(T)) \neq 0$, thus $\det(M(ST)) = \det(M(S)M(T)) = \det(M(S))\det(M(T)) \neq 0$, this means that ST is invertible.

Second Solution.

9. Suppose V is finite-dimensional and $S, T, U \in \mathcal{L}(V)$ and $STU = I$. Show that T is invertible and that $T^{-1} = US$.

10. Show that the result in Exercise 12 can fail without the hypothesis that V is finite-dimensional.

11. Prove or give a counterexample: If V is a finite-dimensional vector space and $R, S, T \in \mathcal{L}(V)$ are such that RST is surjective, then S is injective.

12. Suppose $T \in \mathcal{L}(V)$ and $v_1, \ldots, v_m$ is a list in V such that $Tv_1, \ldots, Tv_m$ spans V. Prove that $v_1, \ldots, v_m$ spans V.

13. Prove that every linear map from $\mathbb{F}^{n,1}$ to $\mathbb{F}^{m,1}$ is given by a matrix multiplication. In other words, prove that if $T \in \mathcal{L}(\mathbb{F}^{n,1}, \mathbb{F}^{m,1})$, then there exists an $m \times n$ matrix A such that $Tx = Ax$ for every $x \in \mathbb{F}^{n,1}$.

14. Suppose V is finite-dimensional and $S \in \mathcal{L}(V)$. Define $\mathcal{A} \in \mathcal{L}(\mathcal{L}(V))$ by $\mathcal{A}(T) = ST$ for $T \in \mathcal{L}(V)$.

    (a) Show that $\dim \operatorname{null} \mathcal{A} = (\dim V)(\dim \operatorname{null} S)$.

    (b) Show that $\dim \operatorname{range} \mathcal{A} = (\dim V)(\dim \operatorname{range} S)$.

15. Show that V and $\mathcal{L}(\mathbb{F}, V)$ are isomorphic vector spaces.

16. Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T has the same matrix with respect to every basis of V if and only if T is a scalar multiple of the identity operator.

17. Suppose $q \in \mathcal{P}(\mathbb{R})$. Prove that there exists a polynomial $p \in \mathcal{P}(\mathbb{R})$ such that

$$q(x) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$$

for all $x \in \mathbb{R}$.

Solution. Let $\mathcal{P}_{\leq n} := \{1, x, x^2, \ldots, x^n\}$ be a basis for any polynomial of degree less than or equal to n. Then

$$q(x) = a_0(1) + a_1(x) + \cdots + a_n(x^n)$$

(in particular we let $a_0 = q(0) = p(3)$) and

$$p(x) = b_0(1) + b_1(x) + \cdots + b_n(x^n) = \sum_{i=0}^{n} b_i x^i$$

for some $a_0, \ldots, a_n$ and $b_0, \ldots, b_n \in \mathbb{R}$. So,

$$2xp'(x) = 2x \sum_{i=1}^{n} ib_i x^{i-1} = 2 \sum_{i=1}^{n} ib_i x^i$$

and

$$(x^2+x)p''(x) = (x^2+x)\sum_{i=2}^{n} i(i-1)b_i x^{i-2} =$$

$$\sum_{i=2}^{n} x^2 i(i-1)b_i x^{i-2} + i(i-1)xb_i x^{i-2} = \sum_{i=2}^{n} i(i-1)b_i x^i + i(i-1)b_i x^{i-1}$$

$$= \sum_{i=2}^{n} i(i-1)b_i x^i + \sum_{i=2}^{n} i(i-1)b_i x^{i-1}.$$

The final expression for $q(x)$ results in

$$q(x) = \left(\sum_{i=2}^{n} i(i-1)b_i x^i + \sum_{i=2}^{n} i(i-1)b_i x^{i-1}\right) + \left(2\sum_{i=1}^{n} ib_i x^i\right) + \sum_{i=0}^{n} b_i 3^i.$$

That by a substitution $i \to i+1$ in the first summand becomes

$$\left(\sum_{i=2}^{n} i(i-1)b_i x^i + \sum_{i=1}^{n-1} (i+1)ib_{i+1} x^i\right) + \left(2\sum_{i=1}^{n} ib_i x^i\right) + \sum_{i=0}^{n} b_i 3^i.$$

Now our goal is to find out if and what identity transformation maps from q in $\mathcal{P}_{\leq n}$ basis to q with a basis that satisfies the above equation. If we find out that it exists, then the problem finishes right there.

Equating coefficients we get:

$$a_0 = \sum_{i=0}^{n} b_i 3^i;$$

$$a_1 = 2b_1 + 2b_2;$$

$$a_k = 2kb_k + k(k-1)b_k + (k+1)kb_{k+1} \text{ for } 2 \leq k < n$$

so

$$a_k = kb_k(k+1) + (k+1)kb_{k+1} \text{ for } 2 \leq k < n$$

and finally

$$a_n = n(n-1)b_n + 2nb_n = (n(n-1)+2n)b_n = n(n+1)b_n.$$

Therefore:

$$s : \begin{cases} a_0 = \sum_{i=0}^{n} b_i 3^i \\ a_k = kb_k(k+1) + (k+1)kb_{k+1} \text{ for } 2 \leq k < n \\ a_n = n(n+1)b_n \end{cases} .$$

In other words, the matrix of the identity mapping becomes:

$$\begin{pmatrix}
p(3) & 0 & 0 & 0 & \cdots & 0 \\
0 & 2b_1 & 0 & 0 & \cdots & 0 \\
0 & 2b_2 & 6b_2 & 0 & \cdots & 0 \\
0 & 0 & 6b_3 & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & kb_k(k+1)+(k+1)kb_{k+1} & 0 \\
0 & 0 & 0 & \cdots & 0 & n(n+1)b_n
\end{pmatrix}.$$

Since the rank of this matrix is n and there are n unknowns b_i, by Rouche-Frobenius theorem there exists an unique solution. Furthermore, each solution is recursively given by:

$$s : \begin{cases} b_n = \dfrac{a_n}{n(n+1)} \\ b_k = \dfrac{a_k - (k+1)kb_{k+1}}{k(k+1)} \\ b_0 = a_0 \end{cases} .$$

□

18. Suppose $T \in \mathcal{L}(V)$ and $v_1, \ldots, v_n$ is a basis of V. Prove that

$$\mathcal{M}(T, (v_1, \ldots, v_n)) \text{ is invertible} \iff T \text{ is invertible.}$$

Solution.

19. Suppose that $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are bases of V. Let $T \in \mathcal{L}(V)$ be such that $Tv_k = u_k$ for each $k = 1, \ldots, n$. Prove that

$$\mathcal{M}(T, (v_1, \ldots, v_n)) = \mathcal{M}(I, (u_1, \ldots, u_n), (v_1, \ldots, v_n)).$$

Solution. This is straightforward, let $\mathcal{U} = (u_1, \ldots, u_n)$, $\mathcal{V} = (v_1, \ldots, v_n)$, and let $\mathrm{id}_{\mathcal{V}\mathcal{U}} : V_{\mathcal{V}} \to V_{\mathcal{U}}$ be the identity mapping. It follows that $\mathrm{id}_{\mathcal{V}\mathcal{U}}(v_k) = u_k$, so $T_{\mathcal{V}}(v_k) = \mathrm{id}_{\mathcal{V}\mathcal{U}}(v_k)$ for $1 \leq k \leq n$. In fact, since u_k is not an specified linear combination of vectors of $\mathcal{V}$ we see that $M(\mathrm{id}_{\mathcal{V}\mathcal{U}}(v_k)) = I_n$.

So we conclude that since $T_{\mathcal{V}}(v_k) = \mathrm{id}_{\mathcal{V}\mathcal{U}}(v_k)$, then $M(T_{\mathcal{V}}(v_k)) = I_n$.

□

20. Suppose A and B are square matrices of the same size and $AB = I$. Prove that $BA = I$.
Solution. Clearly $B = A^{-1}$. (Is any particular insight needed for this one?)

$\square$

21. Let V be a vector space over a field $\mathbb{F}$ of dimension n. Let $T\colon V \to V$ be a projection (recall that this is a linear map such that $T \circ T = T$).

    (a) Prove that $V = \ker(T) \oplus \operatorname{Im}(T)$.

    (b) Prove that there is a basis of V in which the matrix of T is

    $$\begin{pmatrix} I_i & 0 \\ 0 & O_{n-i} \end{pmatrix}$$

    for some $i \in \{0, 1, \ldots, n\}$.

    Solution. (a) First we prove that $\ker T \cap \text{Im } T = \{0\}$. Suppose that there exists some linearly independent vector v of V such that $Tv = 0$ and $v = Tu$ for some $u \in V$. Then

    $$Tv = (T \circ T)u = Tu = 0.$$

    Then $u \in \ker T$, and $Tu \in \ker T$ as well. So we conclude that $v = Tu = 0$; a contradiction, since v is linearly independent.

    The second step is to prove that there exists an unique decomposition of v as a sum of vectors of $\ker T$ and Im T. For the existence part, since we have already proved that $\ker T \cap$ Im $T = \{0\}$, suffices to notice that the intersection between their bases is null, so the union of them will form a basis for V, by the rank-nullity theorem.

    For the uniqueness part, suppose that for some $v \in V$, $v = k + Tu$ and $v = k' + Tu'$, for $k, k' \in \ker T$ and $Tu, Tu' \in$ Im T. Thus

    $$k + Tu = k' + Tu'$$

    Taking T in both sides (both terms are in V, so it is OK)

    $$T(k + Tu) = T(k' + Tu')$$

    $$(T \circ T)u = (T \circ T)u'$$

    $$Tu = Tu'.$$

    So these both vectors of Im T are the same ones. We then conclude, from the first equation that $k = k'$.

    (b) This matrix is constructed the same way to that of problem 2 of section 2 of "Kernel, range and matrices sheet", where i depends on the dimension of the image of T.

$\square$

22. Let V be a vector space over $\mathbb{C}$ or $\mathbb{R}$ of dimension n. Let $T\colon V \to V$ be a symmetry (that is, a linear transformation such that $T \circ T = \mathrm{id}$ is the identity map of V).

    (a) Prove that $V = \ker(T - \mathrm{id}) \oplus \ker(T + \mathrm{id})$.

    (b) Deduce that there exists $i \in [0, n]$ and a basis of V such that the matrix of T with respect to this basis is

$$\begin{pmatrix} I_i & 0 \\ 0 & -I_{n-i} \end{pmatrix}.$$

First Solution. This solution uses eigenspaces, but I do not think the author intended their use yet, so I will provide another solution without involving them.

The following lemma will be handful to prove the desired results.

Lemma 1. *The matrix of a symmetric operator $T : V \to V$ over a finite dimensional $\mathbb{C}-$vector space with some basis $\mathcal{B}$ is a symmetric matrix.*

Proof. Let $\mathcal{B}$ be a basis for V, since $T \circ T = \mathrm{id}$, the equation $M_{\mathcal{B}}(T)M_{\mathcal{B}}(T) = I_n$ will hold. Let a_{ij} be the $ij-$th entry of $M(T)$, then

$$M(T)^2_{ij} = \sum_{k=1}^{n} a_{ik}a_{kj} = \delta_{ij}$$

We will not take into account the diagonal entries, as they do not guarantee anything about the symmetry of M. Although they must be nonzero, since otherwise the matrix would not be invertible.

Let $i \neq j$, then

$$M(T)^2_{ij} = \sum_{k=1}^{n} a_{ik}a_{kj} = 0 \text{ and } M(T)^2_{ji} = \sum_{k=1}^{n} a_{jk}a_{ki} = 0.$$

Therefore

$$\sum_{k=1}^{n} a_{ik}a_{kj} = \sum_{k=1}^{n} a_{jk}a_{ki}$$

we see that symmetry property holds, as $(i, j) \mapsto (j, i)$ implies that the sum equals. Furthermore, if $k = i$

$$\sum_{1 \leq k < i} a_{ik}a_{kj} + a_{ii}a_{ij} + \sum_{i < k \leq n} a_{ik}a_{kj} = \sum_{1 \leq k < i} a_{jk}a_{ki} + a_{ji}a_{ii} + \sum_{i < k \leq n} a_{jk}a_{ki}.$$

So subtracting each side must also be zero

$$\sum_{1\le k<i} a_{ik}a_{kj} + a_{ii}a_{ij} + \sum_{i<k\le n} a_{ik}a_{kj} - \sum_{1\le k<i} a_{jk}a_{ki} - a_{ji}a_{ii} - \sum_{i<k\le n} a_{jk}a_{ki} = 0.$$

If we suppose $a_{ii}a_{ij} \neq a_{ji}a_{ii}$, then $a_{ii}(a_{ij} - a_{ji}) \neq 0$, so $a_{ii} \neq 0$ and $a_{ij} - a_{ji} \neq 0$, so $a_{ij} \neq a_{ji}$. As i and j were chosen arbitrarily we conclude in this paragraph that under the former supposition, for any $i \neq j$: $a_{ij} \neq a_{ji}$. Then it follows that

$$\sum_{k=1}^{n} a_{ik}a_{kj} \neq \sum_{k=1}^{n} a_{jk}a_{ki},$$

as for each term, $a_{ik}a_{kj} \neq a_{ki}a_{jk}$. A contradiction of the equation stated at the beginning.

■

Since the matrix of a symmetry is symmetric, it is diagonalizable, so there exists a set $A = \{\lambda_1, \dots, \lambda_n\}$ of eigenvalues of T. Furthermore, there exists an orthonormal basis $\mathcal{C}$ of V, that we can build by finding each eigenvector in $\ker(T - \lambda_i \mathrm{id})$ for $1 \le i \le n$. Thus $Tv_i = \lambda_i v_i$, so

$$v_i = \lambda_i T v_i \quad \text{by using symmetry property,}$$

but also

$$v_i = \frac{Tv_i}{\lambda_i}.$$

Then

$$\lambda_i T v_i - \frac{Tv_i}{\lambda_i} = Tv_i(\lambda_i - \frac{1}{\lambda_i}) = Tv_i\left(\frac{\lambda_i^2 - 1}{\lambda_i}\right) = 0$$

We conclude seeing that

$$Tv_i(\lambda_i^2 - 1) = Tv_i(\lambda_i + 1)(\lambda_i - 1) = 0.$$

So $\lambda_i = 1$ or $\lambda_i = -1$ for any $1 \le i \le n$, so $m(1) = i$ and $m(-1) = n - i$. Then it follows that $\mathcal{C} = \{v_1, \dots, v_i, -v_{i+1}, \dots, -v_n\}$, so the symmetry matrix is diagonal with each entry being 1 or -1. We also see that $\mathcal{C} = \ker(T - \mathrm{id}) \oplus \ker(T + \mathrm{id})$.

□

Second Solution. (a) Let $P := \frac{1}{2}(\mathrm{id} - T)$, and $Q := \frac{1}{2}(\mathrm{id} + T)$. These two mappings are projections from V to V. To prove this, using the fact that $\mathcal{L}(V)$ is a vector space itself

$$P^2 = \frac{1}{2}(\mathrm{id} - T)\frac{1}{2}(\mathrm{id} - T) = \frac{1}{4}(\mathrm{id} - T)(\mathrm{id} - T)$$

$$= \frac{1}{4}(\mathrm{id} - T)(\mathrm{id} - T) = \frac{1}{4}(\mathrm{id} - T - T + T^2)$$

$$= \frac{1}{4}(\mathrm{id} - T - T + T^2) = \frac{1}{4}(2\,\mathrm{id} - 2T) = P$$

Similar procedure for Q:

$$Q^2 = \frac{1}{4}(\mathrm{id} + T)(\mathrm{id} + T) = \frac{1}{4}(\mathrm{id} + 2T + T^2) = \frac{1}{4}(2\,\mathrm{id} + 2T) = Q.$$

The mapping $P + Q$ is the identity map, and their composition $P \circ Q = Q \circ P$ is the null map. We then claim that for any vector $v \in V$, $v = Pu + Qw$, and furthermore, that this decomposition is unique (meaning that $V = \mathrm{Im}\ P \oplus \mathrm{Im}\ Q$).

To prove this claim, note that the first condition is obvious since $P + Q = \mathrm{id}$, so remains showing that $\mathrm{Im}\ P \cap \mathrm{Im}\ Q = \{0_V\}$. Assume that there exists a vector $v \in V$ belonging to both $\mathrm{Im}\, P$ and $\mathrm{Im}\, Q$, thus,

$$v = P(u), \text{ and } v = Q(w).$$

$$\implies P(u) = Q(w)$$

$$\implies P(u) = (P \circ Q)(w) = 0_V$$

and similarly

$$Q(u) = (P \circ Q)(w) = 0_V$$

So $v = 0_V$. This proves our claim. Using the last problem we also know that $V = \ker P \oplus \mathrm{Im}\, P$ and $V = \ker Q \oplus \mathrm{Im}\, Q$. So we get three different expressions for V counting also that of $V = \mathrm{Im}\, P \oplus \mathrm{Im}\, Q$. Without loss of generality assume now that $\ker P = \mathrm{Im}\, Q$, and $\ker Q = \mathrm{Im}\, P$. This means that $V = \ker Q \oplus \ker P$ as well. Let $v \in \ker P$, thus

$$Pv = \frac{1}{2}(v - Tv) = 0$$

$$\iff 0 = Tv - v$$

so, this shows the equivalence between $\ker P$ and $\ker(T - \mathrm{id})$. Doing the same for Q, we get that:

$$Qv = 0 \iff v + Tv = 0 \iff Tv + v = 0.$$

So we get that $V = \ker P \oplus \ker Q \iff V = \ker(T - \mathrm{id}) \oplus \ker(T + \mathrm{id})$.

(b) As $T \circ T = \mathrm{id}$, we get that $(M_{\mathcal{B}\mathcal{B}}(T))^2 = I_n$ for a basis $\mathcal{B}$ of V. Then, $M_{\mathcal{B}\mathcal{B}}(T) = M_{\mathcal{B}\mathcal{B}}(T)^{-1}$, this reduces the threshold of matrices as it only can be diagonal.

Now, we would like to have that a basis for V were the union of the bases of $\ker(T - \mathrm{id})$ and $\ker(T + \mathrm{id})$. Consider the mapping $T - \mathrm{id}$, this mapping will be zero if and only if $Tv = v$, so any vector of the basis of $\ker(T - \mathrm{id})$ will satisfy that $Tv = v$, similarly with $T + \mathrm{id}$, we will get that vectors of the basis of $\ker(T + \mathrm{id})$ are those in which $Tv = -v$.

Thus, let $\mathcal{B} = \{v_1, \dots, v_i\}$ be a basis for $\ker(T - \mathrm{id})$, and $\mathcal{C} = \{v_{i+1}, \dots, v_n\}$ be a basis for $\ker(T + \mathrm{id})$. A basis for V will be $\mathcal{D} = \mathcal{B} \cup \mathcal{C}$, and a basis for the arrival V will be $\mathcal{D}$ as well. This builds up the desired matrix.

□

23. Let V be the vector space of polynomials with complex coefficients whose degree does not exceed 3. Let $T : V \to V$ be the map defined by

$$T(P) = P + P'.$$

Prove that T is linear and find the matrix of T with respect to the basis $1, X, X^2, X^3$ of V.

Solution. To prove this mapping is linear, let $c \in \mathbb{R}$ and let P, Q be polynomials with complex coefficients whose degree does not exceed 3:

$$\begin{gathered} T(P + cQ) = (P + cQ) + (P + cQ)' = P + cQ + P' + cQ' = \\ P + P' + cQ + cQ' = T(P) + cT(Q) \end{gathered}$$

And note that if $P = c$, constant polynomial, $P' = 0$, in particular with $c = 0$. The matrix for T will be:

$$M(T) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

□

24. (a) Find the matrix with respect to the canonical basis of the map which projects a vector $v \in \mathbb{R}^3$ to the xy-plane.

    (b) Find the matrix with respect to the canonical basis of the map which sends a vector $v \in \mathbb{R}^3$ to its reflection with respect to the xy-plane.

    (c) Let $\theta \in \mathbb{R}$. Find the matrix with respect to the canonical basis of the map which sends a vector $v \in \mathbb{R}^2$ to its rotation through an angle θ, counterclockwise.

25. Let V be a vector space of dimension n over F. A *flag* in V is a family of subspaces

$$V_0 \subset V_1 \subset \cdots \subset V_n$$

such that $\dim V_i = i$ for all $i \in [0, n]$. Let $T : V \to V$ be a linear transformation. Prove that the following statements are equivalent:

   (a) There is a flag $V_0 \subset \cdots \subset V_n$ in V such that $T(V_i) \subset V_i$ for all $i \in [0, n]$.

   (b) There is a basis of V with respect to which the matrix of T is upper-triangular.

Solution. Since $V_0 \subset V_1 \subset \cdots \subset V_n$, we can find a basis for any V_k by extending one from V_{k-1}. Call $\mathcal{B}_k$ a basis for V_k that is (recursively) extended from $\mathcal{B}_{k-1}$.

For the direct implication, let us start with some fixed k. Since $T(V_k) \subset V_k$, there exist at most $k-1$ basis vectors from $\mathcal{B}_k$ that form a basis for $T(V_k)$.

Among these $k-1$ vectors it can occur that each one of these are of $\mathcal{B}_{k-1}$, or that within these, there is the only vector $v_k \in \mathcal{B}_k \setminus \mathcal{B}_{k-1}$, and the other $k-2$ are within $\mathcal{B}_{k-1}$.

For the first case, for any k, we can choose as a basis for $T(V_k)$ exactly $\mathcal{B}_{k-1}$.

Clearly $T(V_{k-1})$ is a subspace of $T(V_k)$ since $\mathcal{B}_{k-1}$ is gotten by linearly extending $\mathcal{B}_{k-2}$. It follows that $T(V_{k-1}) \subset T(V_k)$ for any $k \geq 1$.

Let $v_k \in \mathcal{B}_k \setminus \mathcal{B}_{k-1}$ for any $1 \leq k \leq n$, this vector will not be in any basis of $T(V_j)$ for $1 \leq j < k$ by construction, but will be a basis vector only for V_k. Its image $Tv_k \in T(V_k)$ will result in the following column vector:

$$\begin{pmatrix} a_{1k} \\ \vdots \\ a_{(k-1)k} \\ 0_{kk} \end{pmatrix}$$

As it is spanned by $k-1$ basis vectors from $\mathcal{B}_{k-1}$.

So, gathering the k basis vectors of V the matrix is constructed, note that its principal diagonal is zero, but it is OK since for being upper triangular this does not matter.

For the converse implication let $A \in M_n(\mathbb{F})$ be an upper triangular matrix. We see that the rank m of A is $n-1 \leq m \leq n$. Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be a basis for V.

Consider the subspaces $V_k := \langle v_1, \ldots, v_k \rangle$ (with $V_0 = \langle 0 \rangle$). By induction on k we will prove that $T(V_k) \subset V_k$. The intuition is that the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{pmatrix}$$

since input and output bases are the same, represents that the mapping of, say v_k spans itself "plus" the $k-1$ earlier ones. We will see that considering subspaces while cumulating basis vectors of V: $\langle v_1 \rangle$, $\langle v_1, v_2 \rangle$, and so on; the dimension of their image will be less than or equal themselves (depending mainly on the zeroes in the diagonal).

When $k = 1$ we have that for $V_1 = \langle v_1 \rangle$, $T(V_1) = \lambda a_{11} v_1 \in \langle v_1 \rangle$.

When $k = 2$ we have that for $V_2 = \langle v_1, v_2 \rangle$, $T(V_2) = \langle T(v_1), T(v_2) \rangle = \lambda a_{11} v_1 + \delta(a_{12} v_1 + a_{22} v_2) = b_1 v_1 + b_2 v_2 \in V_2$.

For the inductive thesis, we will show that for $V_{k+1} := \langle v_1, \ldots, v_k, v_{k+1} \rangle$ it follows that $T(V_{k+1}) \subseteq V_{k+1}$. Since

$$T(V_{k+1}) = \langle T(v_1), \ldots, T(v_k), T(v_{k+1}) \rangle = T(V_k) + \langle T(v_{k+1}) \rangle = \\ T(V_k) + \lambda \sum_{i=1}^{k+1} a_{i(k+1)} v_i,$$

then

$$T(V_{k+1}) = \sum_{i=1}^{k} b_i v_i + \lambda \left(\sum_{i=1}^{k+1} a_{i(k+1)} v_i \right)$$

$$= \sum_{i=1}^{k} b_i v_i + \left(\sum_{i=1}^{k+1} c_i v_i \right) = \sum_{i=1}^{k+1} d_i v_i \in \langle v_1, \ldots, v_{k+1} \rangle = V_{k+1}.$$

□

26. Let V be a vector space over a field F, and let $T_1, \ldots, T_n : V \to V$ be linear transformations. Prove that

$$\bigcap_{i=1}^{n} \ker(T_i) \subseteq \ker\left(\sum_{i=1}^{n} T_i \right).$$

(Idea of) Solution. Using *Grassmann formula* and rank-nullity theorem we get some interesting inequalities. By arguing by induction on n, for the base case $n = 1$ we get the trivial inclusion $\ker(T_1) \subseteq \ker(T_1)$. Now, suppose that for any $k \geq 1$ the inclusion

$$\bigcap_{i=1}^{k} \ker(T_i) \subseteq \ker\left(\sum_{i=1}^{k} T_i \right).$$

holds. In the right hand side we note that despite having a sum in it, the dimension of V will be invariant anyways, this means that

$$\dim V = \dim\ker T_i + \dim \mathrm{Im}\ T_i = \dim\ker\left(\sum_{i=1}^{k} T_i\right) + \dim \mathrm{Im}\ \left(\sum_{i=1}^{k} T_i\right).$$

Call S the sum $\sum_{i=1}^{k} T_i$. Note that

$$\dim\ker S + \dim \mathrm{Im}\ S = \dim\ker T_i + \dim \mathrm{Im}\ T_i \geq 0$$

Thus:

$$\dim\ker S - \dim\ker T_i = \dim \mathrm{Im}\ T_i - \dim \mathrm{Im}\ S\ (*)$$

Then, using Grassmann formula:

$$\dim(\ker T_i + \ker S) = \dim\ker T_i + \dim\ker S - \dim(\ker T_i \cap \ker S) \geq 0.$$

Which implies that

$$\dim\ker T_i + \dim\ker S \geq \dim(\ker T_i \cap \ker S)\ (**).$$

Summing $(*)$ and $(**)$

$$\dim\ker S \geq \dim\ker T_i + (\dim \mathrm{Im}\ T_i - \dim \mathrm{Im}\ S).$$

Then:

$$\dim\ker S \geq \dim(\ker T_i \cap \ker S) + (\dim \mathrm{Im}\ T_i - \dim \mathrm{Im}\ S).$$

This, although, does still not prove the desired result, but holds for any $k \geq 1$. It could occur that the right hand side were negative. Despite this, by our inductive hypothesis, we can say that $\dim\ker S \geq \dim(\bigcap_{i=1}^{k} \ker T_i)$. For the inductive thesis, when $n = k+1$, we shall prove

$$\bigcap_{i=1}^{k+1} \ker(T_i) \subseteq \ker(S).$$

(This idea is interesting, but I do not think that it can be pushed to get the desired result, although it may be true that the result holds for $n = k+1$, $-$that $\dim\ker S \leq \dim\ker(\bigcap_{i=1}^{k+1} \ker(T_i))-$ this does not tell us anything about the inclusion of the subspaces, we can still have them disjoint except for zero.)

Solution. Let $v \in \bigcap_{i=1}^{n} \ker(T_i)$, we claim that this vector is also in $\ker\left(\sum_{i=1}^{n} T_i\right)$.

To prove this claim first note that any vector $v' \in \ker\left(\sum_{i=1}^{n} T_i\right)$ will satisfy:

$$T_1 v' + \cdots + T_n v' = 0_V.$$

In particular, since v is in each $\ker T_i$,

$$T_1 v + \cdots + T_n v = 0_V, \text{ so } v \in \ker\left(\sum_{i=1}^{n} T_i\right).$$

$\square$

27. Let V be a vector space over a field F, and let $T_1, T_2 : V \to V$ be linear transformations such that

$$T_1 \circ T_2 = T_1 \quad \text{and} \quad T_2 \circ T_1 = T_2.$$

Prove that

$$\ker(T_1) = \ker(T_2).$$

Solution. Let $v \in \ker T_1$, we note that

$$T_1 v = 0 \iff T_2(T_1 v) = T_2 v = 0. \text{ (since any linear mapping applied to 0 is also 0)}$$

This implies that $v \in \ker T_1 \iff v \in \ker T_2$. Similarly, let $u \in \ker T_2$

$$T_2 u = 0 \iff T_1(T_2 v) = T_1 u = 0.$$

So we conclude that $\ker T_1 = \ker T_2$.

$\square$

28. Let V be a vector space over F, and let $T : V \to V$ be a linear transformation such that

$$\ker(T) = \ker(T^2) \quad \text{and} \quad \operatorname{Im}(T) = \operatorname{Im}(T^2).$$

Prove that

$$V = \ker(T) \oplus \operatorname{Im}(T).$$

Solution. We claim that T is necessarily a projection. To prove this, assume that T were not a projection. Take $v \in V, v \neq 0$, then $Tv \neq T^2 v$, let $u \in V$ such that $u = T^2 v$.

As $u \in \operatorname{Im} T^2$, then $u \in \operatorname{Im} T$, this means that $u = Tv'$ for some $v' \in V$, suppose that $v' \neq v$. Then it must hold that $Tv' = T^2 v$, so T is not injective.

Let $k \in \ker T$, such that $k \neq 0$, then $k \in \ker T^2$, thus $Tk = T^2 k = 0$, which is a contradiction.

Then, the result yields by problem 21.

□

29. Let V be a finite dimensional vector space. Let $T : V \to V$ be a linear operator, and let $T^n : V \to V$ denote T applied n times. Prove that there exists an integer N such that

$$V = \ker T^N \oplus \text{ Im } T^N.$$

Solution. First note that the kernel of a mapping is stable under T, only possibly increasing its dimension when applying T again. If it continues increasing when applying T, the resulting mapping will be the null mapping for some $N \geq 2$, which is a projection, leading to the result immediately.

Else, if the dimension of both Im T and $\ker T$ become stable, we claim that we would get, starting from a certain integer j

$$\ker T^j = \ker T^{j+1} \text{ and } \operatorname{Im} T^j = \operatorname{Im} T^{j+1}.$$

Which would imply that T becomes a projection starting from j by the last problem, which also leads to the result immediately. To prove this claim, note that $\ker T$ is actually stable under T, this means that for any $v \in \ker T$

$$v \in \ker T \implies v \in \ker T^2 \implies \cdots \implies v \in \ker T^j$$

in particular, when applying T j times to a basis $\mathcal{K}$ of $\ker T$, every vector of it will be basis vectors of T^{j+1}. Now as we assumed that $\dim \ker T^j = \dim \ker T^{j+1}$, $\mathcal{K}$ forms a basis for $\ker T^j$ and $\ker T^{j+1}$, which means that $\ker T^j = \ker T^{j+1}$. In the case of Im T, we note that $V \setminus \ker T^j = V \setminus \ker T^{j+1} = \cdots = V \setminus \ker T^{j+n}$, hence, $\operatorname{Im} T$ will also become stable starting from j. This proves our claim.

□

Rank of a matrix

1. Let $A, B \in M_3(F)$ be two matrices such that $AB = O_3$. Prove that

$$\min(\operatorname{rank}(A), \operatorname{rank}(B)) \leq 1.$$

2. Let $A \in M_3(\mathbb{C})$ be a matrix such that $A^2 = O_3$.

   (a) Prove that A has rank 0 or 1.

   (b) Deduce the general form of all matrices $A \in M_3(\mathbb{C})$ such that $A^2 = O_3$.

3. Find the rank of the matrix $A = [\cos(i - j)]_{1 \leq i,j \leq n}$.

4. (a) Let V be an n-dimensional vector space over F, and let $T : V \to V$ be a linear transformation. Let T^j be the j-fold iterate of T (so $T^2 = T \circ T$, $T^3 = T \circ T \circ T$, etc.). Prove that:

$$\mathrm{Im}(T^n) = \mathrm{Im}(T^{n+1}).$$

Hint: Check that if $\mathrm{Im}(T^j) = \mathrm{Im}(T^{j+1})$ for some j, then $\mathrm{Im}(T^k) = \mathrm{Im}(T^{k+1})$ for $k \geq j$.

   (b) Let $A \in M_n(\mathbb{C})$ be a matrix. Prove that A^n and A^{n+1} have the same rank.

5. Let $A \in M_n(F)$ be a matrix of rank 1. Prove that:

$$A^2 = \mathrm{Tr}(A)A.$$

6. Let $A \in M_m(F)$ and $B \in M_n(F)$. Prove that:

$$\mathrm{rank} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \mathrm{rank}(A) + \mathrm{rank}(B).$$

7. Prove that for any matrices $A \in M_{n,m}(F)$ and $B \in M_m(F)$, we have:

$$\mathrm{rank} \begin{bmatrix} I_n & A \\ 0 & B \end{bmatrix} = n + \mathrm{rank}(B).$$

8. Let $n > 2$ and let $A = [a_{ij}] \in M_n(\mathbb{C})$ be a matrix of rank 2. Prove the existence of real numbers x_i, y_i, z_i, t_i for $1 \leq i \leq n$ such that for all $i, j \in \{1, 2, \ldots, n\}$, we have:

$$a_{ij} = x_i y_j + z_i t_j.$$

9. Let $A = [a_{ij}]_{1 \leq i,j \leq n}$ and $B = [b_{ij}]_{1 \leq i,j \leq n}$ be complex matrices such that:

$$a_{ij} = 2ij - b_{ij}$$

for all integers $1 \leq i, j \leq n$. Prove that:

$$\mathrm{rank}(A) = \mathrm{rank}(B).$$

10. Let $A \in M_n(\mathbb{C})$ be a matrix such that $A^2 = A$, i.e., A is the matrix of a projection. Prove that:

$$\mathrm{rank}(A) + \mathrm{rank}(I_n - A) = n.$$

11. Let $n > k$ and let $A_1, \ldots, A_k \in M_n(\mathbb{R})$ be matrices of rank $n - 1$. Prove that $A_1 A_2 \cdots A_k$ is nonzero. *Hint:* Using Sylvester's inequality, prove that:

$$\mathrm{rank}(A_1 \cdots A_j) \geq n - j \quad \text{for } 1 \leq j \leq k.$$

12. Let $A \in M_n(\mathbb{C})$ be a matrix of rank at least $n-1$. Prove that:

$$\operatorname{rank}(A^k) \geq n-k \quad \text{for } 1 \leq k \leq n.$$

Hint: Use Sylvester's inequality.

13. (a) Prove that for any matrix $A \in M_n(\mathbb{R})$, we have:

$$\operatorname{rank}(A) = \operatorname{rank}({}^\top AA).$$

Hint: If $X \in \mathbb{R}^n$ is a column vector such that ${}^\top AAX = 0$, write ${}^\top X\,{}^\top AAX = 0$ and express the left-hand side as a sum of squares.

(b) Let $A = \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}$. Find the rank of A and $A^\top A$, and conclude that part (a) of the problem is no longer true if $\mathbb{R}$ is replaced with $\mathbb{C}$.

14. Let A be an $m \times n$ matrix with rank r. Prove that there is an $m \times m$ matrix B with rank $m-r$ such that:

$$BA = O_{m,n}.$$

15. (Generalized inverses) Let $A \in M_{m,n}(F)$. A generalized inverse of A is a matrix $X \in M_{n,m}(F)$ such that:

$$AXA = A.$$

(a) If $m = n$ and A is invertible, show that the only generalized inverse of A is A^{-1}.

(b) Show that a generalized inverse of A always exists.

(c) Give an example to show that the generalized inverse need not be unique.

Duality

1.

Product and quotient of a vector space

1. Let V be a finite-dimensional vector space over F, and let $W \subset V$ be a subspace. For a vector $v \in V$, define

$$[v] = \{v + w : w \in W\}.$$

Note that $[v_1] = [v_2]$ if and only if $v_1 - v_2 \in W$. Define the quotient space V/W to be

$$V/W = \{[v] : v \in V\}.$$

Addition and scalar multiplication in V/W are defined as follows:

$$[u] + [v] = [u + v] \quad \text{and} \quad a[v] = [av],$$

where $a \in F$. It is known that these operations are well-defined and that V/W, equipped with this structure, is a vector space.

(a) Show that the map $\pi : V \to V/W$ defined by $\pi(v) = [v]$ is linear with kernel W.

(b) Show that
$$\dim(W) + \dim(V/W) = \dim(V).$$

(c) Suppose $U \subset V$ is any subspace such that $W \oplus U = V$. Show that the restriction $\pi|_U : U \to V/W$ is an isomorphism, i.e., a bijective linear map.

(d) Let $T : V \to U$ be a linear map, let $W \subset \ker(T)$ be a subspace of V, and let $\pi : V \to V/W$ be the projection onto the quotient space. Show that there exists a unique linear map $S : V/W \to U$ such that $T = S \circ \pi$.