Linear mappings sheet IV

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Problems in linear algebra: problems from the books of Titu Andreescu "Essential Linear Algebra" and Sheldon Axler "Linear Algebra Done Right". The topics of these weeks are isomorphisms, change of bases and rank of a matrix.

Isomorphisms and invertibility

1. Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that the following are equivalent:

   (a) T is invertible.

   (b) $Tv_1, \ldots, Tv_n$ is a basis of V for every basis $v_1, \ldots, v_n$ of V.

   (c) $Tv_1, \ldots, Tv_n$ is a basis of V for some basis $v_1, \ldots, v_n$ of V.

   Solution.

(a) $\implies$ (b) Since T is injective, it preserves linearly independence when applied on any basis $v_1, \ldots, v_n$ of V, and since T is surjective $\langle Tv_1, \ldots, Tv_n \rangle = V$.

(b) $\implies$ (c) Trivial since "every" implies "some".

(c) $\implies$ (a) Since $Tv_1, \ldots, Tv_n$ is a basis of V for some $v_1, \ldots, v_n$, then T is surjective as $\text{Im}(T) = V$. Since $Tv_1, \ldots, Tv_n$ is linearly independent, let $Tv = 0$

$$Tv = \lambda_1 Tv_1 + \cdots + \lambda_n Tv_n = 0_V \iff \lambda_i = 0 \text{ for } 1 \leq i \leq n.$$

therefore $v = 0_V$. We conclude that T is injective an therefore invertible.

□

2. Suppose V is finite-dimensional, U is a subspace of V, and $S \in \mathcal{L}(U, V)$. Prove that there exists an invertible linear map T from V to itself such that $Tu = Su$ for every $u \in U$ if and only if S is injective.

Solution. For the direct implication, if T is invertible, then it is injective, The restriction of T to a subset U will be injective, and will behave as expected. The mapping will not be surjective, although this is not an issue.

For the converse implication, if S is injective, U and Im S are isomorphic. Hence there exists an isomorphism $S' : V \setminus U \to V \setminus$ Im S. Build T such that

$$Tv = \begin{cases} Sv & \text{if } v \in U; \\ S'v & \text{if } v \in V \setminus U \end{cases}.$$

$\square$

3. Suppose that W is finite-dimensional and $S, T \in \mathcal{L}(V, W)$. Prove that $\operatorname{null} S = \operatorname{null} T$ if and only if there exists an invertible $E \in \mathcal{L}(W)$ such that $S = ET$.
Solution.

($\Longrightarrow$) Let k be the dimension of $\ker T$, let n be the dimension of V and m be the dimension of W. Since $\ker T = \ker S$ then, by the rank-nullity theorem, dim Im T = dim Im S. Let $\mathcal{B}$ be a basis of V such as

$$\{v_1, \ldots, v_{n-k}, \ldots, v_n\}$$

Where the vectors $v_{n-k}, \ldots, v_n$ are those of a basis of $\ker T$ and $\ker S$.

Let $\mathcal{C}$ and $\mathcal{D}$ be bases of W containing the bases of Im T and Im S respectively such as $\{w_1, \ldots, w_{n-k}, \ldots, w_m\}$, where $w_1, \ldots, w_{n-k}$ are vectors from Im T or Im S.

Consider Id : $W_{\mathcal{C}} \to W_{\mathcal{D}}$, where $\mathrm{Id}((w_i)_{\mathcal{C}}) = (w_i)_{\mathcal{D}}$ for vectors w_i of their respective bases, and for $1 \leq i \leq n - k$.

We have the following diagram

$$\begin{tikzcd} V_{\mathcal{B}} \arrow[r, "T"] \arrow[dr, "S"'] & W_{\mathcal{C}} \arrow[d, leftrightarrow, "Id_{\mathcal{D}\mathcal{C}}"] \\ & W_{\mathcal{D}} \end{tikzcd}$$

Consider $E = Id_{\mathcal{D}\mathcal{C}}$. So $T = E \circ S$, thus $S = E^{-1} \circ T = Id_{\mathcal{C}\mathcal{D}} \circ T$.

($\Longleftarrow$) Reciprocally, if there exists an invertible mapping $E \in \mathcal{L}(W)$ such that $ET = S$, the matrix $M(E)$ will be invertible, so its rank will be $\dim W$. A null column C_j of $M(S)$ (for some basis of W) will be given by

$$C_j = M(E)_{\cdot 1} M(T)_{1j} + \cdots + M(E)_{\cdot m} M(T)_{mj} = O_{\cdot j}$$

Where each column of $M(E)$ is linearly independent (since its rank its m). Thus, the entries of the jth column of $M(T)$ must be zero, thus, the column itself is null. We conclude that $M(T)$ must have the corresponding column null, so the kernel of the mapping will be equal.

$\square$

4. Suppose that V is finite-dimensional and $S, T \in \mathcal{L}(V, W)$. Prove that $\operatorname{range} S = \operatorname{range} T$ if and only if there exists an invertible $E \in \mathcal{L}(V)$ such that $S = TE$.
Solution.

($\Longrightarrow$) Let $\dim V = n$, $\dim W = m$ such that $n \geq m$. Let $\dim \operatorname{Im} T = i$. We see that $\dim \ker(T) = n - i$, which is also equal to $\dim \ker(S)$ by rank-nullity.

Since $\operatorname{Im}T = \operatorname{Im}S$, we can choose simply a basis for both their images as $\{w_1, \ldots, w_i\}$ with $u_k \in \operatorname{Im} S$ for $1 \leq k \leq i$. Consider $\{v_1, \ldots, v_{n-i}\} = \ker T$ and $\{u_1, \ldots, u_{n-i}\} = \ker S$. Extend $\ker S$ to a basis of V by taking the union with some vectors $y_n, y_{n-1}, \ldots, y_{n-i+1}$. Thus:

$$V_{\mathcal{B}} = \langle u_1, \ldots, u_{n-i} \rangle + \langle y_n, y_{n-1}, \ldots, y_{n-i+1} \rangle$$

Let $E : V_{\mathcal{B}} \to V$ be applied over the basis of $V_{\mathcal{B}}$, we define E for the kernel vectors of S just by

$$Eu_i = v_i.$$

and for the non-kernel vectors, since $\operatorname{Im}(T) = \operatorname{Im}(S)$ we can find a basis vector $x_i \in V$ such that $T(x_i) = S(y_i)$, we define

$$Ey_i = x_i.$$

To test out our construction let $u_i \in \ker S$, then

$$S(u_i) = (TE)v = T(v_i) = 0_V$$

and for $y_i \notin \ker S$:

$$S(y_i) = (TE)(y_i) = T(x_i) \in \operatorname{Im}(T).$$

($\Longleftarrow$) For the converse implication, let E be an invertible mapping such that $S = TE$. It is not hard to see that $\dim \operatorname{Im}(TE) = \dim \operatorname{Im}(T)$, one can show this by noticing that since E is invertible, the dimension of its image is n, so $\operatorname{Im} E$ is V. So $T : V \to W$ is equivalent to $T : \operatorname{Im}(E) \to W$. Hence $\dim \operatorname{Im}(S) = \dim \operatorname{Im}(T)$. Consider a basis $\mathcal{A} = \{v_1, \ldots, v_n\}$ of V, and a basis $\mathcal{B} = \{Ev_1, \ldots, Ev_n\}$ of $\operatorname{Im}(E) = V$.

Since $S = TE$

$$(TE)v = (TE)(\lambda_1 v_1 + \cdots + \lambda_n v_n) = T(\lambda_1 E v_1 + \cdots + \lambda_n E v_n),$$

we see that $\lambda_1 E v_1 + \cdots + \lambda_n E v_n$ is just v in $\mathcal{B}$ basis. So $S(v_{\mathcal{A}}) = T(v_{\mathcal{B}})$. We conclude that $\mathrm{Im}(S) = \{S(v_{\mathcal{A}}) \in W : v_{\mathcal{A}} \in V\} = \{T(u_{\mathcal{B}}) \in W : u_{\mathcal{B}} = E^{-1} v_B \in V\} = \mathrm{Im}(T)$.

□

5. Suppose V and W are finite-dimensional and $S, T \in \mathcal{L}(V, W)$. Prove that there exist invertible $E_1 \in \mathcal{L}(V)$ and $E_2 \in \mathcal{L}(W)$ such that $S = E_2 T E_1$ if and only if $\dim \operatorname{null} S = \dim \operatorname{null} T$.

   Solution. This problem might be more important among the last four ones, for the next section it would say that the rank does not vary when multiplying at the left and at the right by invertible matrices.

($\implies$) We claim that the dimension of the kernel of $E_2 T E_1$, with E_1, E_2 bijective mappings, is the same as the dimension of the kernel of T.

Since E_1 is surjective $(TE_1)(V) = T(E_1 V) = T(V)$ (as we showed in last problem), so $\dim \mathrm{Im}(T) = \dim \mathrm{Im}(TE_1)$ and $\dim \ker(T) = \dim \ker(TE_1)$.

Then, since E_2 is particularly injective $E_2(Tv) = 0_V$ if and only if $v \in \ker T$. So $\dim \ker(E_2 T) = \dim \ker(T)$. We conclude this paragraph with $\dim \ker(E_2 T E_1) = \dim \ker(T)$.

($\impliedby$) Let $\dim \ker T = k$. We note that $\ker T$ is isomorphic to $\ker S$. And by rank-nullity theorem, Im T is isomorphic to Im S as well.

Let $\mathcal{B}_S := \{u_1, \ldots, u_k, u_{k+1} \ldots u_n\}$ be a basis of V such that $\{u_1, \ldots, u_k\}$ spans $\ker S$ and $\{Su_{k+1}, \ldots, Su_n\}$ spans $\mathrm{Im}(S)$.

Let $\mathcal{B}_T := \{v_1, \ldots, v_k, v_{k+1} \ldots v_n\}$ be a basis of V such that $\{v_1, \ldots, v_k\}$ spans $\ker T$, and $\{Tv_{k+1}, \ldots, Tv_n\}$ spans $\mathrm{Im}(T)$.

We can find an isomorphism $E_1 : V_{\mathcal{B}_S} \to V_{\mathcal{B}_T}$ in which

$$E_1 u_i = v_i \text{ for any } 1 \leq i \leq n.$$

Consider linearly extending $\{Tv_k, \ldots, Tv_n\}$ to a basis $\mathcal{C}_T$ of W

$$\mathcal{C}_T := \{Tv_{k+1}, \ldots, Tv_n, w_1, \ldots, w_{m-n}\}.$$

Similarly let $\mathcal{C}_S := \{Su_{k+1}, \ldots, Su_n, x_1, \ldots, x_{m-n}\}$ be another basis of W. There exists an isomorphism $E_2 : W_{\mathcal{C}_T} \to W_{\mathcal{C}_S}$ defined as

$$E_2 : \begin{cases} E_2 T v_i = S u_i \\ E_2 w_i = x_i \end{cases}.$$

Hence, if $u_j \in \ker S$ with $1 \leq j \leq k$

$$(E_2 T E_1) u_j = (E_2 T) v_j = E_2(0) = 0 = S u_j,$$

if $u_i \notin \ker S$ with $k+1 \leq i \leq n$, then

$$(E_2 T E_1) u_i = (E_2 T) v_i = E_2 T v_i = S u_i.$$

So the identity holds.

Another way to see the things we have been doing so far is to look at the following morphism diagram

$$
\begin{tikzcd}
V_{\mathcal{B}_S} \arrow[r, "E_1"] & V_{\mathcal{B}_T} \arrow[r, "T"] & W_{\mathcal{C}_T} \arrow[r, "E_2"] & W_{\mathcal{C}_S}
\end{tikzcd}
$$

that we can arrange to be a commutative diagram

$$
\begin{tikzcd}
V_{\mathcal{B}_S} \arrow[r, "S"] \arrow[d, "E_1"] & W_{\mathcal{C}_S} \\
V_{\mathcal{B}_T} \arrow[r, "T"] & W_{\mathcal{B}_T} \arrow[u, "E_2"']
\end{tikzcd}
$$

which means that $S = E_2 T E_1$.

$\square$

6. Suppose V is finite-dimensional and $T\colon V \to W$ is a surjective linear map of V onto W. Prove that there is a subspace U of V such that $T|_U$ is an isomorphism of U onto W.

   Solution. Let $n = \dim V$, $m = \dim W$. Since T is surjective, the dimension of W will be at most that of V. If this is the case take U as the whole V.

   If the dimension of W is strictly less than that of V, by the rank-nullity theorem, there are $n-m$ kernel vectors. Take $\mathcal{B}$, a basis for V containing a basis $\mathcal{K}$ of $\ker T$, this basis can be constructed starting with $\mathcal{K}$ and linearly extending it to $\mathcal{B}$. The subspace U will be constructed as $\operatorname{span}(\mathcal{B}\setminus\mathcal{K})$, and for any vector $u \in U$, $Tu \neq 0_W$. The dimension of U will be $n-(n-m) = m$, meaning that U and W are isomorphic.

   Lastly, recall that $\dim \text{ Im } T = \dim W$, then $\dim \text{ Im } T = \dim U$. Let $T_{\upharpoonright U} : U \to W$ be T restricted to U, since $Tu \neq 0_W$ for any $u \in U$, this mapping is injective. Therefore, by rank-nullity, again,

$$\dim \operatorname{Im}\, T_{\upharpoonright U} = \dim U = \dim \operatorname{Im}\, T = \dim W.$$

   Thus, $T_{\upharpoonright U}$ is surjective, and since it is injective, will also make an isomorphism between U and W.

$\square$

7. Suppose V and W are finite-dimensional and U is a subspace of V. Let

$$\mathcal{E} = \{T \in \mathcal{L}(V,W)\colon U \subseteq \operatorname{null} T\}.$$

   (a) Show that $\mathcal{E}$ is a subspace of $\mathcal{L}(V,W)$.

(b) Find a formula for $\dim \mathcal{E}$ in terms of $\dim V$, $\dim W$, and $\dim U$.

Solution. (a) This part is straightforward, take $0_{\mathcal{E}}$ as the null mapping $T = O$. The sum of mappings restricted to some U containing all vectors of their kernel will also have the vectors of U as it kernel. And the same goes for multiplication by a scalar.

(b) Let $n = \dim V$, $m = \dim W$. Since $\mathcal{L}(V, W)$ is isomorphic to $\mathbb{F}^{n,m}$ by the matrix mapping M, and $\mathcal{E}$ is a subspace of it, we will have an isomorphism to some subspace $\mathbb{F}^{p,q}$ with dimension $p \times q$ of $\mathbb{F}^{n,m}$: $M : \mathcal{E} \to \mathbb{F}^{p,q}$ (note it is the same mapping M).

Let $T \in \mathcal{E}$, its associated matrix $M(T)$ will have at least one null column for some basis $\mathcal{B}$ of V (it is not immediately obvious that a null column will appear for any basis of V, but it is certain that there exists, for any mapping $T \in \mathcal{E}$, at least one since dim Im T = rank $M(T) \le \dim V$). So the dimension of $\mathcal{E}$ will be given from those matrices that have $\dim U$ null columns for some basis $\mathcal{B}$. Therefore: $\dim \mathcal{E} = (\dim(V) - \dim(U)) \dim W$.

$\square$

8. Suppose V is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that

$$ST \text{ is invertible} \iff S \text{ and } T \text{ are invertible.}$$

First Solution. This first solution involves determinants and it is straightforward, I will provide a second solution, as the author supposedly intended a solution without involving them.

Since ST is invertible, $\det(M(S)M(T)) \neq 0$, thus $\det(M(S))\det(M(T)) \neq 0$, this implies that $\det(M(S)) \neq 0$ and $\det(M(T)) \neq 0$. This proves that S and T are invertible.

Conversely, if S and T are invertible, $\det(M(S)) \neq 0$, $\det(M(T)) \neq 0$, thus $\det(M(ST)) = \det(M(S)M(T)) = \det(M(S))\det(M(T)) \neq 0$, this means that ST is invertible.

Second Solution.

($\Longrightarrow$) We see that $\text{rank}(ST) \le \text{rank}(T) \le V$, but since ST is invertible it follows that $\text{rank}(T) = \dim V$, so T is surjective and therefore is invertible. Since T is invertible, $(ST)T^{-1}$ is invertible, hence S is invertible.

($\Longleftarrow$) If S and T are invertible, then $\text{Im}(ST) = S(\text{Im}\, T) = S(V) = V$. So ST is surjective and therefore invertible.

$\square$

9. Suppose V is finite-dimensional and $S, T, U \in \mathcal{L}(V)$ and $STU = I$. Show that T is invertible and that $T^{-1} = US$.
Solution.

(a) Since $STU = \mathrm{Id}$, then each one of the three mappings has to be invertible by problem (8), in particular T.

(b) By part (a) all three mappings are invertible. We claim that $TUS = \mathrm{Id}$. To prove this claim we see that since $STU = \mathrm{Id}$, then $TU = S^{-1}$, so

$$STU = SS^{-1} = S^{-1}S = TUS = \mathrm{Id}.$$

We conclude that $T^{-1} = US$.

□

10. Show that the result in Exercise 9 can fail without the hypothesis that V is finite-dimensional.
Solution. Consider the space of real number sequences. Let T be a right-shift, let S be a left-shift and multiplying each element by its term number and let U be each element divided by its term number. For instance for $(a_n)_{n=1}^{\infty}$

$$U(a_n) = a_1, \frac{a_2}{2}, \frac{a_3}{3}, \dots$$
$$TU(a_n) = 0, a_1, \frac{a_2}{2}, \frac{a_3}{3}, \dots$$
$$STU(a_n) = a_1, a_2, a_3, \dots$$

If $T^{-1} = US$, then

$$US(a_n) = \frac{2a_2}{2}, \frac{3a_3}{3}, \frac{4a_4}{4}, \dots = a_2, a_3, a_4, \dots$$

so

$$TUS(a_n) = 0, a_2, a_3, a_4, \dots \neq (a_n).$$

□

11. Prove or give a counterexample: If V is a finite-dimensional vector space and $R, S, T \in \mathcal{L}(V)$ are such that RST is surjective, then S is injective.
Solution. Since RST is surjective then $\mathrm{rank}(RST) = \dim V$. Also, $\mathrm{rank}(RST) \leq \mathrm{rank}(S) \leq \dim V$, hence $\mathrm{rank}(S) = \dim V$. We conclude that S is invertible and therefore injective.

□

12. Suppose $T \in \mathcal{L}(V)$ and $v_1, \dots, v_m$ is a list in V such that $Tv_1, \dots, Tv_m$ spans V. Prove that $v_1, \dots, v_m$ spans V.
Solution. Since $Tv_1, \dots, Tv_m$ spans V, T is surjective, so by rank-nullity T is invertible. Let $w \in V$, then

$$Tw = T(\lambda_1 v_1 + \cdots + \lambda_m v_m)$$

applying T^{-1}

$$w = \lambda_1 v_1 + \cdots + \lambda_m v_m$$

so $v_1, \dots, v_m$ spans V.

$\square$

13. Prove that every linear map from $\mathbb{F}^{n,1}$ to $\mathbb{F}^{m,1}$ is given by a matrix multiplication. In other words, prove that if $T \in \mathcal{L}(\mathbb{F}^{n,1}, \mathbb{F}^{m,1})$, then there exists an $m \times n$ matrix A such that $Tx = Ax$ for every $x \in \mathbb{F}^{n,1}$.

Solution. Let $e_1, \dots, e_n$ be the canonical basis of $\mathbb{F}^{n,1}$ and $d_1, \dots, d_m$ be the canonical basis of $\mathbb{F}^{m,1}$

$$x = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = a_1 e_1 + \cdots + a_n e_n.$$

If $T \in \mathcal{L}(\mathbb{F}^{n,1}, \mathbb{F}^{m,1})$, then T is given by the formula

$$T(x) = T \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = a_1 T(e_1) + \cdots + a_n T(e_n).$$

Where $T(e_i) = \lambda_{1i} d_1 + \cdots + \lambda_{mi} d_m \in \mathbb{F}^{m,1}$. Therefore, $a_i T(e_i) = a_i \lambda_{1i} d_1 + \cdots + a_i \lambda_{mi} d_m$. We see that the j-th entry of $a_i T(e_i)$ is $a_i \lambda_{ji}$, so the j-th entry of $T(x)$ will be

$$T(x)_j = \sum_{k=1}^{n} a_k \lambda_{jk}$$

which matches with the matrix product Ax with

$$A = \begin{pmatrix} \lambda_{11} & \cdots & \lambda_{1n} \\ \vdots & & \vdots \\ \lambda_{m1} & \cdots & \lambda_{mn} \end{pmatrix}$$

$\square$

14. Suppose V is finite-dimensional and $S \in \mathcal{L}(V)$. Define $\mathcal{A} \in \mathcal{L}(\mathcal{L}(V))$ by $\mathcal{A}(T) = ST$ for $T \in \mathcal{L}(V)$.

   (a) Show that $\dim \operatorname{null} \mathcal{A} = (\dim V)(\dim \operatorname{null} S)$.

   (b) Show that $\dim \operatorname{range} \mathcal{A} = (\dim V)(\dim \operatorname{range} S)$.

Solution.

   (a) If $ST = O$ it must happen that $\operatorname{Im} T \subseteq \ker S$, so $\dim \operatorname{Im} T \leq \dim \ker S$. In other words, all mappings $T : V \to \ker S$, whose amount is $(\dim V)(\dim \ker S)$.

   (b) Since $\dim \mathcal{L}(V) = (\dim V)^2$, by rank-nullity and using part (a)

$$\dim \operatorname{Im} \mathcal{A} = (\dim V)^2 - (\dim V)(\dim \ker S)$$

so

$$\dim \operatorname{Im} \mathcal{A} = (\dim V)((\dim V) - (\dim \ker S)) = (\dim V)(\operatorname{Im}(S)).$$

$\square$

15. Show that V and $\mathcal{L}(\mathbb{F}, V)$ are isomorphic vector spaces.
Solution. Notice that $\dim \mathbb{F} = 1$ since for instance 1 is a basis of $\mathbb{F}$. Then, $\dim \mathcal{L}(\mathbb{F}, V) = (\dim \mathbb{F})(\dim V) = \dim V$. Since both spaces have same dimensions we can find an isomorphism $\phi : V \to \mathcal{L}(\mathbb{F}, V)$.

$\square$

16. Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T has the same matrix with respect to every basis of V if and only if T is a scalar multiple of the identity operator.
Solution.

($\Longrightarrow$) Let $A = M(T)$. By hypothesis, let P be any invertible matrix, it follows that
$$P^{-1}AP = A$$
therefore
$$AP = PA.$$
Then necessarily $A = cI_n$ for some $c \in \mathbb{F}$. (Otherwise, it suffices letting P invertible and not symmetric to achieve a contradiction).

($\Longleftarrow$) If $A = cI_n$ for some $c \in \mathbb{F}$ then
$$P^{-1}AP = cP^{-1}P = cI_n.$$
So A remains unchanged under change of bases.

$\square$

17. Suppose $q \in \mathcal{P}(\mathbb{R})$. Prove that there exists a polynomial $p \in \mathcal{P}(\mathbb{R})$ such that
$$q(x) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$$
for all $x \in \mathbb{R}$.
Solution. Let $\mathcal{P}_{\leq n} := \{1, x, x^2, \ldots, x^n\}$ be a basis for any polynomial of degree less than or equal to n. Then
$$q(x) = a_0(1) + a_1(x) + \cdots + a_n(x^n)$$
(in particular we let $a_0 = q(0) = p(3)$) and
$$p(x) = b_0(1) + b_1(x) + \cdots + b_n(x^n) = \sum_{i=0}^{n} b_i x^i$$
for some $a_0, \ldots, a_n$ and $b_0, \ldots, b_n \in \mathbb{R}$. So,

$$2xp'(x) = 2x\sum_{i=1}^{n} ib_i x^{i-1} = 2\sum_{i=1}^{n} ib_i x^i$$

and

$$(x^2+x)p''(x) = (x^2+x)\sum_{i=2}^{n} i(i-1)b_i x^{i-2} =$$

$$\sum_{i=2}^{n} x^2 i(i-1)b_i x^{i-2} + i(i-1)xb_i x^{i-2} = \sum_{i=2}^{n} i(i-1)b_i x^i + i(i-1)b_i x^{i-1}$$

$$= \sum_{i=2}^{n} i(i-1)b_i x^i + \sum_{i=2}^{n} i(i-1)b_i x^{i-1}.$$

The final expression for $q(x)$ results in

$$q(x) = \left(\sum_{i=2}^{n} i(i-1)b_i x^i + \sum_{i=2}^{n} i(i-1)b_i x^{i-1}\right) + \left(2\sum_{i=1}^{n} ib_i x^i\right) + \sum_{i=0}^{n} b_i 3^i.$$

That by a substitution $i \to i+1$ in the first summand becomes

$$\left(\sum_{i=2}^{n} i(i-1)b_i x^i + \sum_{i=1}^{n-1} (i+1)ib_{i+1} x^i\right) + \left(2\sum_{i=1}^{n} ib_i x^i\right) + \sum_{i=0}^{n} b_i 3^i.$$

Now our goal is to find out if and what identity transformation maps from q in $\mathcal{P}_{\leq n}$ basis to q with a basis that satisfies the above equation. If we find out that it exists, then the problem finishes right there.

Equating coefficients we get:

$$a_0 = \sum_{i=0}^{n} b_i 3^i;$$

$$a_1 = 2b_1 + 2b_2;$$

$$a_k = 2kb_k + k(k-1)b_k + (k+1)kb_{k+1} \text{ for } 2 \leq k < n$$

so

$$a_k = kb_k(k+1) + (k+1)kb_{k+1} \text{ for } 2 \leq k < n$$

and finally

$$a_n = n(n-1)b_n + 2nb_n = (n(n-1)+2n)b_n = n(n+1)b_n.$$

Therefore:

$$s : \begin{cases} a_0 = \sum_{i=0}^{n} b_i 3^i \\ a_k = kb_k(k+1) + (k+1)kb_{k+1} \text{ for } 2 \leq k < n \\ a_n = n(n+1)b_n \end{cases} .$$

In other words, the matrix of the identity mapping becomes:

$$\begin{pmatrix} p(3) & 0 & 0 & 0 & \cdots & 0 \\ 0 & 2b_1 & 0 & 0 & \cdots & 0 \\ 0 & 2b_2 & 6b_2 & 0 & \cdots & 0 \\ 0 & 0 & 6b_3 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & kb_k(k+1) + (k+1)kb_{k+1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & n(n+1)b_n \end{pmatrix}.$$

Since the rank of this matrix is n and there are n unknowns b_i, by Rouche-Frobenius theorem there exists an unique solution. Furthermore, each solution is recursively given by:

$$s : \begin{cases} b_n = \dfrac{a_n}{n(n+1)} \\ b_k = \dfrac{a_k - (k+1)kb_{k+1}}{k(k+1)} \\ b_0 = a_0 \end{cases} .$$

□

18. Suppose $T \in \mathcal{L}(V)$ and $v_1, \ldots, v_n$ is a basis of V. Prove that

$$\mathcal{M}(T, (v_1, \ldots, v_n)) \text{ is invertible } \iff T \text{ is invertible.}$$

Solution.

($\implies$) Since $A = M(T)$ is invertible, all of its columns are linearly independent. This implies that any column vector $C \in \mathbb{F}^{n,1}$ is a linear combination of those in A.

Consider an isomorphism $\phi : \mathbb{F}^{n,1} \to V$ defined as $\phi(e_i) = v_i$ for e_i any basis vector of $\mathbb{F}^{n,1}$. Since each $C_i \in A$ is a basis vector of $\mathbb{F}^{n,1}$, then

$$\begin{aligned} \phi(C_i) &= \lambda_1 \phi(e_1) + \cdots + \lambda_n \phi(e_n) \\ &= \lambda_1 v_1 + \cdots + \lambda_n v_n = T(v_1) \end{aligned}$$

is a basis vector of V for $1 \leq i \leq n$. We conclude that $T(v_1), \ldots, T(v_n)$ is a basis of V, so T is surjective and therefore invertible.

($\Longleftarrow$) If T is invertible, then $T(v_1), \ldots, T(v_n)$ is a basis of V, using the inverse mapping of the direct implication part, we see that each one of these correspond to an unique column vector in A that is linearly independent.

□

19. Suppose that $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are bases of V. Let $T \in \mathcal{L}(V)$ be such that $Tv_k = u_k$ for each $k = 1, \ldots, n$. Prove that

$$\mathcal{M}(T, (v_1, \ldots, v_n)) = \mathcal{M}(I, (u_1, \ldots, u_n), (v_1, \ldots, v_n)).$$

Solution. This is straightforward, let $\mathcal{U} = (u_1, \ldots, u_n)$, $\mathcal{V} = (v_1, \ldots, v_n)$, and let $\mathrm{id}_{\mathcal{V}\mathcal{U}} : V_{\mathcal{V}} \to V_{\mathcal{U}}$ be the identity mapping. It follows that $\mathrm{id}_{\mathcal{V}\mathcal{U}}(v_k) = u_k$, so $T_{\mathcal{V}}(v_k) = \mathrm{id}_{\mathcal{V}\mathcal{U}}(v_k)$ for $1 \leq k \leq n$. In fact, since u_k is not an specified linear combination of vectors of $\mathcal{V}$ we see that $M(\mathrm{id}_{\mathcal{V}\mathcal{U}}(v_k)) = I_n$.

So we conclude that since $T_{\mathcal{V}}(v_k) = \mathrm{id}_{\mathcal{V}\mathcal{U}}(v_k)$, then $M(T_{\mathcal{V}}(v_k)) = I_n$.

□

20. Suppose A and B are square matrices of the same size and $AB = I$. Prove that $BA = I$.
Solution. Clearly $B = A^{-1}$. (Is any particular insight needed for this one?)

□

21. Let V be a vector space over a field $\mathbb{F}$ of dimension n. Let $T \colon V \to V$ be a projection (recall that this is a linear map such that $T \circ T = T$).

    (a) Prove that $V = \ker(T) \oplus \mathrm{Im}(T)$.

    (b) Prove that there is a basis of V in which the matrix of T is

$$\begin{pmatrix} I_i & 0 \\ 0 & O_{n-i} \end{pmatrix}$$

    for some $i \in \{0, 1, \ldots, n\}$.

Solution. (a) First we prove that $\ker T \cap \mathrm{Im}\ T = \{0\}$. Suppose that there exists some linearly independent vector v of V such that $Tv = 0$ and $v = Tu$ for some $u \in V$. Then

$$Tv = (T \circ T)u = Tu = 0.$$

Then $u \in \ker T$, and $Tu \in \ker T$ as well. So we conclude that $v = Tu = 0$; a contradiction, since v is linearly independent.

The second step is to prove that there exists an unique decomposition of v as a sum of vectors of $\ker T$ and $\text{Im } T$. For the existence part, since we have already proved that $\ker T \cap \text{ Im } T = \{0\}$, suffices to notice that the intersection between their bases is null, so the union of them will form a basis for V, by the rank-nullity theorem.

For the uniqueness part, suppose that for some $v \in V$, $v = k + Tu$ and $v = k' + Tu'$, for $k, k' \in \ker T$ and $Tu, Tu' \in \text{Im } T$. Thus

$$k + Tu = k' + Tu'$$

Taking T in both sides (both terms are in V, so it is OK)

$$T(k + Tu) = T(k' + Tu')$$

$$(T \circ T)u = (T \circ T)u'$$

$$Tu = Tu'.$$

So these both vectors of $\text{Im } T$ are the same ones. We then conclude, from the first equation that $k = k'$.

(b) This matrix is constructed the same way to that of problem 2 of section 2 of "Kernel, range and matrices sheet", where i depends on the dimension of the image of T.

□

22. Let V be a vector space over $\mathbb{C}$ or $\mathbb{R}$ of dimension n. Let $T\colon V \to V$ be a symmetry (that is, a linear transformation such that $T \circ T = \text{id}$ is the identity map of V).

    (a) Prove that $V = \ker(T - \text{id}) \oplus \ker(T + \text{id})$.

    (b) Deduce that there exists $i \in [0, n]$ and a basis of V such that the matrix of T with respect to this basis is

    $$\begin{pmatrix} I_i & 0 \\ 0 & -I_{n-i} \end{pmatrix}.$$

First Solution. This solution uses eigenspaces, but I do not think the author intended their use yet, so I will provide another solution without involving them.

The following lemma will be handful to prove the desired results.

Lemma 1. *The matrix of a symmetric operator $T : V \to V$ over a finite dimensional $\mathbb{C}-$vector space with some basis $\mathcal{B}$ is a symmetric matrix.*

Proof. Let $\mathcal{B}$ be a basis for V, since $T \circ T = \mathrm{id}$, the equation $M_{\mathcal{B}}(T)M_{\mathcal{B}}(T) = I_n$ will hold. Let a_{ij} be the $ij-$th entry of $M(T)$, then

$$M(T)^2_{ij} = \sum_{k=1}^{n} a_{ik}a_{kj} = \delta_{ij}$$

We will not take into account the diagonal entries, as they do not guarantee anything about the symmetry of M. Although they must be nonzero, since otherwise the matrix would not be invertible.

Let $i \neq j$, then

$$M(T)^2_{ij} = \sum_{k=1}^{n} a_{ik}a_{kj} = 0 \text{ and } M(T)^2_{ji} = \sum_{k=1}^{n} a_{jk}a_{ki} = 0.$$

Therefore

$$\sum_{k=1}^{n} a_{ik}a_{kj} = \sum_{k=1}^{n} a_{jk}a_{ki}$$

we see that symmetry property holds, as $(i, j) \mapsto (j, i)$ implies that the sum equals. Furthermore, if $k = i$

$$\sum_{1 \leq k < i} a_{ik}a_{kj} + a_{ii}a_{ij} + \sum_{i < k \leq n} a_{ik}a_{kj} = \sum_{1 \leq k < i} a_{jk}a_{ki} + a_{ji}a_{ii} + \sum_{i < k \leq n} a_{jk}a_{ki}.$$

So subtracting each side must also be zero

$$\sum_{1 \leq k < i} a_{ik}a_{kj} + a_{ii}a_{ij} + \sum_{i < k \leq n} a_{ik}a_{kj} - \sum_{1 \leq k < i} a_{jk}a_{ki} - a_{ji}a_{ii} - \sum_{i < k \leq n} a_{jk}a_{ki} = 0.$$

If we suppose $a_{ii}a_{ij} \neq a_{ji}a_{ii}$, then $a_{ii}(a_{ij} - a_{ji}) \neq 0$, so $a_{ii} \neq 0$ and $a_{ij} - a_{ji} \neq 0$, so $a_{ij} \neq a_{ji}$. As i and j were chosen arbitrarily we conclude in this paragraph that under the former supposition, for any $i \neq j$: $a_{ij} \neq a_{ji}$.

Then it follows that

$$\sum_{k=1}^{n} a_{ik}a_{kj} \neq \sum_{k=1}^{n} a_{jk}a_{ki},$$

as for each term, $a_{ik}a_{kj} \neq a_{ki}a_{jk}$. A contradiction of the equation stated at the beginning.

■

Since the matrix of a symmetry is symmetric, it is diagonalizable, so there exists a set $A = \{\lambda_1, \ldots, \lambda_n\}$ of eigenvalues of T. Furthermore, there exists an orthonormal basis $\mathcal{C}$ of V, that we can build by finding each eigenvector in $\ker(T - \lambda_i \mathrm{id})$ for $1 \leq i \leq n$. Thus $Tv_i = \lambda_i v_i$, so

$$v_i = \lambda_i T v_i \quad \text{by using symmetry property,}$$

but also

$$v_i = \frac{Tv_i}{\lambda_i}.$$

Then

$$\lambda_i T v_i - \frac{Tv_i}{\lambda_i} = Tv_i(\lambda_i - \frac{1}{\lambda_i}) = Tv_i \left(\frac{\lambda_i^2 - 1}{\lambda_i} \right) = 0.$$

We conclude seeing that

$$Tv_i(\lambda_i^2 - 1) = Tv_i(\lambda_i + 1)(\lambda_i - 1) = 0.$$

So $\lambda_i = 1$ or $\lambda_i = -1$ for any $1 \leq i \leq n$, so $m(1) = i$ and $m(-1) = n - i$. Then it follows that $\mathcal{C} = \{v_1, \ldots, v_i, -v_{i+1}, \ldots, -v_n\}$, so the symmetry matrix is diagonal with each entry being 1 or -1. We also see that $\mathcal{C} = \ker(T - \mathrm{id}) \oplus \ker(T + \mathrm{id})$.

$\square$

Second Solution. (a) Let $P := \frac{1}{2}(\mathrm{id} - T)$, and $Q := \frac{1}{2}(\mathrm{id} + T)$. These two mappings are projections from V to V. To prove this, using the fact that $\mathcal{L}(V)$ is a vector space itself

$$P^2 = \frac{1}{2}(\mathrm{id} - T)\frac{1}{2}(\mathrm{id} - T) = \frac{1}{4}(\mathrm{id} - T)(\mathrm{id} - T)$$

$$= \frac{1}{4}(\mathrm{id} - T)(\mathrm{id} - T) = \frac{1}{4}(\mathrm{id} - T - T + T^2)$$

$$= \frac{1}{4}(\mathrm{id} - T - T + T^2) = \frac{1}{4}(2\,\mathrm{id} - 2T) = P$$

Similar procedure for Q:

$$Q^2 = \frac{1}{4}(\mathrm{id} + T)(\mathrm{id} + T) = \frac{1}{4}(\mathrm{id} + 2T + T^2) = \frac{1}{4}(2\,\mathrm{id} + 2T) = Q.$$

The mapping $P + Q$ is the identity map, and their composition $P \circ Q = Q \circ P$ is the null map. We then claim that for any vector $v \in V$, $v = Pu + Qw$, and furthermore, that this decomposition is unique (meaning that $V = \mathrm{Im}\ P \oplus \mathrm{Im}\ Q$).

To prove this claim, note that the first condition is obvious since $P + Q = \mathrm{id}$, so remains showing that $\mathrm{Im}\ P \cap \mathrm{Im}\ Q = \{0_V\}$. Assume that there exists a vector $v \in V$ belonging to both $\mathrm{Im}\, P$ and $\mathrm{Im}\, Q$, thus,

$$v = P(u), \text{ and } v = Q(w).$$

$$\implies P(u) = Q(w)$$

$$\implies P(u) = (P \circ Q)(w) = 0_V$$

and similarly

$$Q(u) = (P \circ Q)(w) = 0_V$$

So $v = 0_V$. This proves our claim. Using the last problem we also know that $V = \ker P \oplus \operatorname{Im} P$ and $V = \ker Q \oplus \operatorname{Im} Q$. So we get three different expressions for V counting also that of $V = \operatorname{Im} P \oplus \operatorname{Im} Q$. Without loss of generality assume now that $\ker P = \operatorname{Im} Q$, and $\ker Q = \operatorname{Im} P$. This means that $V = \ker Q \oplus \ker P$ as well. Let $v \in \ker P$, thus

$$Pv = \frac{1}{2}(v - Tv) = 0$$

$$\iff 0 = Tv - v$$

so, this shows the equivalence between $\ker P$ and $\ker(T - \mathrm{id})$. Doing the same for Q, we get that:

$$Qv = 0 \iff v + Tv = 0 \iff Tv + v = 0.$$

So we get that $V = \ker P \oplus \ker Q \iff V = \ker(T - \mathrm{id}) \oplus \ker(T + \mathrm{id})$.

(b) As $T \circ T = \mathrm{id}$, we get that $(M_{\mathcal{BB}}(T))^2 = I_n$ for a basis $\mathcal{B}$ of V. Then, $M_{\mathcal{BB}}(T) = M_{\mathcal{BB}}(T)^{-1}$, this reduces the threshold of matrices as it only can be diagonal.

Now, we would like to have that a basis for V were the union of the bases of $\ker(T - \mathrm{id})$ and $\ker(T + \mathrm{id})$. Consider the mapping $T - \mathrm{id}$, this mapping will be zero if and only if $Tv = v$, so any vector of the basis of $\ker(T - \mathrm{id})$ will satisfy that $Tv = v$, similarly with $T + \mathrm{id}$, we will get that vectors of the basis of $\ker(T + \mathrm{id})$ are those in which $Tv = -v$.

Thus, let $\mathcal{B} = \{v_1, \ldots, v_i\}$ be a basis for $\ker(T - \mathrm{id})$, and $\mathcal{C} = \{v_{i+1}, \ldots, v_n\}$ be a basis for $\ker(T + \mathrm{id})$. A basis for V will be $\mathcal{D} = \mathcal{B} \cup \mathcal{C}$, and a basis for the arrival V will be $\mathcal{D}$ as well. This builds up the desired matrix.

□

23. Let V be the vector space of polynomials with complex coefficients whose degree does not exceed 3. Let $T : V \to V$ be the map defined by

$$T(P) = P + P'.$$

Prove that T is linear and find the matrix of T with respect to the basis $1, X, X^2, X^3$ of V.

Solution. To prove this mapping is linear, let $c \in \mathbb{R}$ and let P, Q be polynomials with complex coefficients whose degree does not exceed 3:

$$T(P + cQ) = (P + cQ) + (P + cQ)' = P + cQ + P' + cQ' =$$
$$P + P' + cQ + cQ' = T(P) + cT(Q)$$

And note that if $P = c$, constant polynomial, $P' = 0$, in particular with $c = 0$. The matrix for T will be:

$$M(T) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

□

24. (a) Find the matrix with respect to the canonical basis of the map which projects a vector $v \in \mathbb{R}^3$ to the xy-plane.

    (b) Find the matrix with respect to the canonical basis of the map which sends a vector $v \in \mathbb{R}^3$ to its reflection with respect to the xy-plane.

    (c) Let $\theta \in \mathbb{R}$. Find the matrix with respect to the canonical basis of the map which sends a vector $v \in \mathbb{R}^2$ to its rotation through an angle θ, counterclockwise.

Solution.

(a) The mapping P must be such that if $v = (a, b, c)$ then $Pv = (a, b, 0)$. Therefore, $P(e_1) = e_1$, $P(e_2) = e_2$ and $P(e_3) = (0, 0, 0)$. Hence,

$$M(P) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(b) The mapping S must be such that if $v = (a, b, c)$ then $Pv = (a, b, -c)$. Therefore

$$M(S) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

(c) Let $v \in \mathbb{R}^3$. Consider the unitary vector u

$$u = \frac{v}{|v|}.$$

Note that $u = (\cos(\alpha), \sin(\alpha))$ for some α since

$$|u| = 1 = \sqrt{\cos(\alpha)^2 + \sin(\alpha)^2}$$

for any $0 \leq \alpha \leq 2\pi$. We expect $R_\theta(u)$ to preserve the modulus of u and we assume that $R_\theta e_1$ is linearly independent on $R_\theta e_2$ in a way that rotating two orthogonal vectors preserves their orthogonality. Hence, letting $R_\theta e_1 = r_1$ and $R_\theta e_2 = r_2$ we get

$$R_\theta u = R_\theta(e_1 \cos(\alpha)) + R_\theta(e_2 \sin(\alpha))$$

$$= \cos(\alpha) r_1 + \sin(\alpha) r_2$$

We claim that $r_1 = (\cos\theta, \sin\theta)$ and that $r_2 = (-\sin\theta, \cos\theta)$. Since these vectors have modulus equal to 1 and are orthonormal, the only thing that remains checking is that $|R_\theta u| = 1$. The matrix of R_θ will be

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

and multiplying by u we get that

$$R_\theta u = \begin{pmatrix} \cos\theta\cos\alpha - \sin\theta\sin\alpha \\ \sin\theta\cos\alpha + \cos\theta\sin\alpha \end{pmatrix} = \begin{pmatrix} \cos(\alpha+\theta) \\ \sin(\alpha+\theta) \end{pmatrix}.$$

It follows that $|R_\theta u| = 1$. Hence, $R_\theta v = |v| R_\theta u$.

$\square$

25. Let V be a vector space of dimension n over F. A *flag* in V is a family of subspaces

$$V_0 \subset V_1 \subset \cdots \subset V_n$$

such that $\dim V_i = i$ for all $i \in [0, n]$. Let $T : V \to V$ be a linear transformation. Prove that the following statements are equivalent:

(a) There is a flag $V_0 \subset \cdots \subset V_n$ in V such that $T(V_i) \subset V_i$ for all $i \in [0, n]$.

(b) There is a basis of V with respect to which the matrix of T is upper-triangular.

Solution. Since $V_0 \subset V_1 \subset \cdots \subset V_n$, we can find a basis for any V_k by extending one from V_{k-1}. Call $\mathcal{B}_k$ a basis for V_k that is (recursively) extended from $\mathcal{B}_{k-1}$.

($\implies$) For the direct implication, let us start with some fixed k. Since $T(V_k) \subset V_k$, there exist at most $k-1$ basis vectors from $\mathcal{B}_k$ that form a basis for $T(V_k)$.

Among these $k-1$ vectors it can occur that each one of these are of $\mathcal{B}_{k-1}$, or that within these, there is the only vector $v_k \in \mathcal{B}_k \setminus \mathcal{B}_{k-1}$, and the other $k-2$ are within $\mathcal{B}_{k-1}$.

For the first case, for any k, we can choose as a basis for $T(V_k)$ exactly $\mathcal{B}_{k-1}$.

Clearly $T(V_{k-1})$ is a subspace of $T(V_k)$ since $\mathcal{B}_{k-1}$ is gotten by linearly extending $\mathcal{B}_{k-2}$. It follows that $T(V_{k-1}) \subset T(V_k)$ for any $k \geq 1$.

Let $v_k \in \mathcal{B}_k \setminus \mathcal{B}_{k-1}$ for any $1 \leq k \leq n$, this vector will not be in any basis of $T(V_j)$ for $1 \leq j < k$ by construction, but will be a basis vector only for V_k. Its image $Tv_k \in T(V_k)$ will result in the following column vector:

$$\begin{pmatrix} a_{1k} \\ \vdots \\ a_{(k-1)k} \\ 0_{kk} \end{pmatrix}$$

As it is spanned by $k-1$ basis vectors from $\mathcal{B}_{k-1}$.

So, gathering the k basis vectors of V the matrix is constructed, note that its principal diagonal is zero, but it is OK since for being upper triangular this does not matter.

($\Longleftarrow$) For the converse implication let $A \in M_n(\mathbb{F})$ be an upper triangular matrix. We see that the rank m of A is $n-1 \leq m \leq n$. Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be a basis for V.

Consider the subspaces $V_k := \langle v_1, \ldots, v_k \rangle$ (with $V_0 = \langle 0 \rangle$). By induction on k we will prove that $T(V_k) \subset V_k$. The intuition is that the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{pmatrix}$$

since input and output bases are the same, represents that the mapping of, say v_k spans itself "plus" the $k-1$ earlier ones. We will see that considering subspaces while cumulating basis vectors of V: $\langle v_1 \rangle$, $\langle v_1, v_2 \rangle$, and so on; the dimension of their image will be less than or equal themselves (depending mainly on the zeroes in the diagonal).

Basis If $k = 1$ we have that for $V_1 = \langle v_1 \rangle$, $T(V_1) = T(\langle v_1 \rangle) = \langle Tv_1 \rangle$. Consider a vector in $T(V_1)$, we see that it has the form $\lambda a_{11} v_1$ for some $\lambda \in \mathbb{F}$, where a_{11} is the respective entry in A. This inmediatly means that $Tv_1 \in V_1$ by definition of span.

If $k = 2$ we have that for $V_2 = \langle v_1, v_2 \rangle$, $T(V_2) = \langle T(v_1), T(v_2) \rangle = \lambda a_{11} v_1 + \delta(a_{12} v_1 + a_{22} v_2)$ for any $\lambda, \delta \in \mathbb{F}$. Then let $b_1 = \lambda a_{11} + \delta a_{12}$ and let $b_2 = \delta a_{22}$. We see that $T(V_2) = b_1 v_1 + b_2 v_2$ when varying $b_1, b_2 \in \mathbb{F}$. Thus $T(V_2) \subseteq \langle v_1, v_2 \rangle = V_2$.

Hypothesis Suppose that if $V_k = \langle v_1, \ldots, v_k \rangle$, then $T(V_k) \subseteq V_k$. We can write

$$T(V_k) = \left\{ \sum_{i=1}^{k} b_i v_i : b_i \in \mathbb{F}, v_i \in V_k \right\}.$$

Thesis For the inductive thesis, we will show that for $V_{k+1} := \langle v_1, \ldots, v_k, v_{k+1} \rangle$ it follows that $T(V_{k+1}) \subseteq V_{k+1}$. We see that

$$T(V_{k+1}) = \langle T(v_1), \dots, T(v_k), T(v_{k+1}) \rangle = T(V_k) + \langle T(v_{k+1}) \rangle$$

then by hypothesis any vector $v \in T(V_k)$ is of the form

$$Tv = \sum_{i=1}^{k} b_i v_i \text{ for some } b_i \in \mathbb{F}$$

so in general we see that under varying $\lambda, b_i \in \mathbb{F}$ (we do not vary a_{ij} since it is given by the matrix)

$$\begin{aligned} T(V_{k+1}) &= \sum_{i=1}^{k} b_i v_i + \lambda \left(\sum_{i=1}^{k+1} a_{i(k+1)} v_i \right) \\ &= \sum_{i=1}^{k} b_i v_i + \left(\sum_{i=1}^{k+1} c_i v_i \right) = \sum_{i=1}^{k+1} d_i v_i. \end{aligned}$$

when varying $d_i \in \mathbb{F}$. And this is by definition $\langle v_1, \dots, v_{k+1} \rangle = V_{k+1}$.

$\square$

26. Let V be a vector space over a field F, and let $T_1, \dots, T_n : V \to V$ be linear transformations. Prove that

$$\bigcap_{i=1}^{n} \ker(T_i) \subseteq \ker \left(\sum_{i=1}^{n} T_i \right).$$

Solution. Let $v \in \bigcap_{i=1}^{n} \ker(T_i)$, we claim that this vector is also in $\ker \left(\sum_{i=1}^{n} T_i \right)$. To prove this claim first note that any vector $v' \in \ker \left(\sum_{i=1}^{n} T_i \right)$ will satisfy:

$$T_1 v' + \dots + T_n v' = 0_V.$$

So seeing v is in each $\ker T_i$ derives into

$$T_1 v + \dots + T_n v = 0_V, \text{ so } v \in \ker \left(\sum_{i=1}^{n} T_i \right).$$

$\square$

27. Let V be a vector space over a field F, and let $T_1, T_2 : V \to V$ be linear transformations such that

$$T_1 \circ T_2 = T_1 \quad \text{and} \quad T_2 \circ T_1 = T_2.$$

Prove that

$$\ker(T_1) = \ker(T_2).$$

Solution. Let $v \in \ker T_1$, we note that

$$T_1 v = 0 \implies (T_2 \circ T_1)(v) = T_2(0) = 0.$$
(since any linear mapping applied to 0 is also 0)

This implies that $\ker(T_1) \subseteq \ker(T_2)$. And symmetrically

$$T_2 v = 0 \implies (T_1 \circ T_2)(v) = T_1(0) = 0.$$

This implies that $\ker(T_2) \subseteq \ker(T_1)$. We conclude that $\ker T_1 = \ker T_2$.

$\square$

28. Let V be a vector space over F, and let $T : V \to V$ be a linear transformation such that

$$\ker(T) = \ker(T^2) \quad \text{and} \quad \mathrm{Im}(T) = \mathrm{Im}(T^2).$$

Prove that
$$V = \ker(T) \oplus \mathrm{Im}(T).$$

Solution. We claim that T is necessarily a projection. To prove this, assume that T were not a projection. Take $v \in V, v \neq 0$ so that $Tv \neq T^2 v$. Then we can let $u \in V$ such that $u = T^2 v$.

Since $u \in \mathrm{Im}\, T^2$, then $u \in \mathrm{Im}\, T$, thus $u = Tv'$ for some $v' \in V$. Suppose $v' \neq v$. Then it must hold that $Tv' = T^2 v$ by construction, so T is not injective.

Since T is not injective, its kernel is not null, so we can find some $k \in \ker T$, such that $k \neq 0$. We see that $k \in \ker T^2$, thus $Tk = T^2 k = 0$. This is a contradiction on the first supposition as we found $k \in V$ such that $Tk = T^2 k$.

Then, the result yields by problem 21.

$\square$

29. Let V be a finite dimensional vector space. Let $T : V \to V$ be a linear operator, and let $T^n : V \to V$ denote T applied n times. Prove that there exists an integer N such that

$$V = \ker T^N \oplus \mathrm{Im}\ T^N.$$

Solution. First note that the kernel of a mapping is stable under T, only possibly increasing its dimension when applying T again. If it continues increasing when applying T, the resulting mapping will be the null mapping for some $N \geq 2$, which is a projection, leading to the result immediately.

Else, if the dimension of both Im T and $\ker T$ become stable, we claim that we would get, starting from a certain integer j

$$\ker T^j = \ker T^{j+1} \text{ and } \operatorname{Im} T^j = \operatorname{Im} T^{j+1}.$$

Which would imply that T becomes a projection starting from j by the last problem, which also leads to the result immediately.

To prove this claim, note that $\ker T$ is always stable under T, this means that for any $v \in \ker T$

$$v \in \ker T \implies v \in \ker T^2 \implies \cdots \implies v \in \ker T^j.$$

In particular, when applying T j times to a basis $\mathcal{K}$ of $\ker T$, every vector of it will be basis vectors of the kernel of T^{j+1}.

Since we assumed that $\dim \ker T^j = \dim \ker T^{j+1}$, $\mathcal{K}$ forms a basis for $\ker T^j$ and $\ker T^{j+1}$, which means that $\ker T^j = \ker T^{j+1}$ as well. In the case of Im T, we note that $V \setminus \ker T^j = V \setminus \ker T^{j+1} = \cdots = V \setminus \ker T^{j+n}$, hence, $\operatorname{Im} T$ will also become stable starting from j. This proves our claim.

$\square$

Rank of a matrix

1. Let $T_1, T_2 : V \to W$ be linear transformations. Prove that

$$|\operatorname{rank}(T_1) - \operatorname{rank}(T_2)| \leq \operatorname{rank}(T_1 + T_2) \leq \operatorname{rank}(T_1) + \operatorname{rank}(T_2).$$

Solution. Since

$$\operatorname{Im}(T_1) + \operatorname{Im}(T_2) := \{T_1 u + T_2 v : T_1 u, T_2 v \in V\}$$

then for any $v \in V$, $(T_1 + T_2)v = T_1 v + T_2 v \in \operatorname{Im}(T_1) + \operatorname{Im}(T_2)$. Therefore $\operatorname{Im}(T_1 + T_2) \subseteq \operatorname{Im}(T_1) + \operatorname{Im}(T_2)$, we see that $\dim \operatorname{Im}(T_1 + T_2) \leq \dim(\operatorname{Im}(T_1) + \operatorname{Im}(T_2))$. Using Grassmann's identity we get

$$\dim(\operatorname{Im}(T_1) + \operatorname{Im}(T_2)) \leq \dim \operatorname{Im}(T_1) + \dim \operatorname{Im}(T_2).$$

So we conclude that $\dim \operatorname{Im}(T_1 + T_2) \leq \dim \operatorname{Im}(T_1) + \dim \operatorname{Im}(T_2)$.

On the other hand, assume without loss of generality $\operatorname{rank}(T_1) \geq \operatorname{rank}(T_2)$. Let $T_1 = (T_1 + T_2) + (-T_2)$ so that $\operatorname{rank}(T_1) = \operatorname{rank}((T_1 + T_2) + (-T_2))$. By the last result we get that

$$\dim \operatorname{Im}((T_1 + T_2) + (-T_2)) \leq \dim \operatorname{Im}(T_1 + T_2) + \dim \operatorname{Im}(-T_2)$$

but since $\dim \operatorname{Im}(-T_2) = \dim \operatorname{Im}(T_2)$, and replacing the left hand side by T_1

$$\dim \operatorname{Im}(T_1) \leq \dim \operatorname{Im}(T_1 + T_2) + \dim \operatorname{Im}(T_2).$$

We conclude that

$$\dim \operatorname{Im}(T_1) - \dim \operatorname{Im}(T_2) \leq \dim \operatorname{Im}(T_1 + T_2).$$

If we assume $\operatorname{rank}(T_1) \leq \operatorname{rank}(T_2)$ we get an analogous result. We conclude that for both cases

$$|\operatorname{rank}(T_1) - \operatorname{rank}(T_2)| \leq \operatorname{rank}(T_1 + T_2).$$

$\square$

2. **(Sylvester's Inequality).** Prove that for all $A, B \in M_n(F)$ we have

$$\operatorname{rank}(AB) \geq \operatorname{rank}(A) + \operatorname{rank}(B) - n.$$

Solution. An equivalent inequality (via Rank-Nullity) is

$$\operatorname{rank}(AB) \geq \operatorname{rank}(A) + \operatorname{rank}(B) - \operatorname{rank}(A) - \operatorname{null}(A)$$

hence

$$\operatorname{rank}(AB) \geq \operatorname{rank}(B) - \operatorname{null}(A) \geq 0.$$

Consider the image space associated to each matrix columns mappings, say $T, S : V \to V$ associated to A and B respectively with $\dim V = n$. Let $\dim \operatorname{Im}(T) = t$ and $\dim \operatorname{Im}(S) = s$. Assume without loss of generality $r \leq s$.

Let a basis of V be $\mathcal{C} = \{u_1, \ldots, u_n\}$ such that $U = \{u_1, \ldots, u_s\}$ and $K = \{u_{s+1}, \ldots, u_n\}$ where $S(U)$ spans $\operatorname{Im}(S)$ and K spans $\ker(S)$.

Consider the mapping $T_{\restriction \operatorname{Im}(S)} : \operatorname{Im}(S) \to V$ that we get applying T on $S(U)$ this mapping has the same image as $(T \circ S)$. Since S is not necessarily surjective then $\operatorname{Im}(T_{\restriction \operatorname{Im}(S)}) \subseteq \operatorname{Im}(T)$.

By rank-nullity theorem we see that

$$\dim \operatorname{Im}(S) = \dim \operatorname{Im}(T_{\restriction \operatorname{Im}(S)}) + \dim \ker(T_{\restriction \operatorname{Im}(S)})$$

so

$$\dim \operatorname{Im}(T \circ S) = \dim \operatorname{Im}(S) - \dim \ker(T_{\restriction \operatorname{Im}(S)}).$$

So finally, we see that $T_{\restriction \operatorname{Im}(S)}(v) = 0 \iff v \in \ker(T)$, this is as we see that none of the vectors of $\operatorname{Im}(S)$ are zero, in other words $\ker(T_{\restriction \operatorname{Im}(S)}) = \operatorname{Im}(S) \cap \ker(T)$, so $\ker(T_{\restriction \operatorname{Im}(S)}) \subseteq \ker(T)$. Thus

$$\dim \operatorname{Im}(T \circ S) \geq \dim \operatorname{Im}(S) - \dim \ker(T).$$

$\square$

3. Let $A, B \in M_3(F)$ be two matrices such that $AB = O_3$. Prove that

$$\min(\operatorname{rank}(A), \operatorname{rank}(B)) \leq 1.$$

Solution. Suppose $\operatorname{rank}(A) = \min(\operatorname{rank}(A), \operatorname{rank}(B)) = r \geq 2$. Since $AB = O_3$, its rank is zero. By Sylvester's inequality

$$\operatorname{rank}(A) + \operatorname{rank}(B) \leq 3.$$

We also get that $\operatorname{rank}(B) \geq 2$. We conclude that

$$\operatorname{rank}(A) + \operatorname{rank}(B) \geq 4 > 3 \geq \operatorname{rank}(A) + \operatorname{rank}(B).$$

So $\operatorname{rank}(A) + \operatorname{rank}(B) > \operatorname{rank}(A) + \operatorname{rank}(B)$, which is absurd.

□

4. Let $A \in M_3(\mathbb{C})$ be a matrix such that $A^2 = O_3$.

   (a) Prove that A has rank 0 or 1.

   (b) Deduce the general form of all matrices $A \in M_3(\mathbb{C})$ such that $A^2 = O_3$.

Solution. (a) By Sylvester's inequality:

$$2 \operatorname{rank}(A) \leq 3,$$

so $\operatorname{rank}(A) \leq \frac{3}{2}$, we see that necessarily $\operatorname{rank}(A) \leq 1$.

(b) An example of these kind of matrices is

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a & 0 & 0 \end{pmatrix}.$$

In general the only restriction that these have is that we cannot put the pivot in the center.

□

5. Find the rank of the matrix $A = [\cos(i - j)]_{1 \leq i,j \leq n}$.

Remark. Useful identity: $\cos(a - b) = \cos(a)\cos(b) + \sin(a)\sin(b)$.

Solution. Let $r = \operatorname{rank}(A)$ The entry a_{11} of this matrix is just 1, as well as the entry a_{nn}. In general any diagonal entry a_{ii} of this matrix is 1.

Given that cos is a even function, this matrix is symmetric. Consider the following submatrix A_k for any $n \geq k \geq 1$

$$\begin{pmatrix} 1 & a_{12} & \cdots & a_{1k} \\ a_{12} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{k-1k} \\ a_{1k-1} & \cdots & a_{kk-1} & 1 \end{pmatrix}.$$

The entries of the sub-diagonal $a_{12} \cdots a_{kk}$ are the same since $\cos(1-2) = \cos((k-1)-k)$. In fact, for each sub-diagonal d_p we see that their respective value is $\cos(1-p)$. So the matrix is better represented by

$$A_k = \begin{pmatrix} 1 & a_{12} & \cdots & a_{1k} \\ a_{12} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{12} \\ a_{1k} & \cdots & a_{12} & 1 \end{pmatrix}.$$

When $k = 2$,

$$\begin{pmatrix} 1 & \cos(1) \\ \cos(1) & 1 \end{pmatrix}.$$

that has determinant $r_2 = 1 - \cos^2(1) = \sin^2(1) > 0$. So the matrix has rank greater or equal than 2. When $k = 3$,

$$\begin{pmatrix} 1 & \cos(1) & \cos(2) \\ \cos(1) & 1 & \cos(1) \\ \cos(2) & \cos(1) & 1 \end{pmatrix}.$$

(How can I finish this problem? It is almost impossible to compute this determinant by hand, and calculators may have errors, my guess is that its rank is 2)

$\square$

6. (a) Let V be an n-dimensional vector space over F, and let $T : V \to V$ be a linear transformation. Let T^j be the j-fold iterate of T (so $T^2 = T \circ T$, $T^3 = T \circ T \circ T$, etc.). Prove that:

$$\mathrm{Im}(T^n) = \mathrm{Im}(T^{n+1}).$$

Hint: Check that if $\mathrm{Im}(T^j) = \mathrm{Im}(T^{j+1})$ for some j, then $\mathrm{Im}(T^k) = \mathrm{Im}(T^{k+1})$ for $k \geq j$.

(b) Let $A \in M_n(\mathbb{C})$ be a matrix. Prove that A^n and A^{n+1} have the same rank.

Solution.

(a) By problem (29) of last section we know that there exists some $j \geq 1$ such that for any $k \geq j$

$$V = \ker(T^k) \oplus \mathrm{Im}(T^k)$$

but we cannot tell if j is greater than or fewer than n, which is the dimension of V. By induction over n we claim that $j = n$.

Basis If $n = 1$ then a basis of V is, say v_1, so $\mathrm{Im}(T)$ can be spanned via $T(v_1)$. Hence, a generic mapping on v_1 is

$$T(v_1) = \lambda v_1.$$

so $\mathrm{Im}(T) = \langle v_1 \rangle$. We conclude that since

$$T^2(v_1) = \lambda^2 v_1$$

then $\mathrm{Im}(T^2) = \langle v_1 \rangle$ as well.

Hypothesis Suppose that for any dimension $n = \dim V$ with $n = 1, \ldots, i$ and a mapping $T : V \to V$ it sufficed to let $j = i$ in order for the identity

$$\mathrm{Im}(T^k) = \mathrm{Im}(T^{k+1})$$

to hold for any $k \geq j$.

Thesis We claim that for $n = \dim V$ with $n = i + 1$ it suffices to let $j = i + 1$ in order to have

$$\mathrm{Im}(T^{i+1}) = \mathrm{Im}(T^{i+2})$$

for any $k \geq j$ and for any mapping $T : V \to V$.
Suppose T were invertible, then $\mathrm{Im}(T^i) = V = \mathrm{Im}(T^{i+1})$.
If T were not invertible, then $\mathrm{Im}(T) \subset V$. We can define $S : \mathrm{Im}(T) \to \mathrm{Im}(T)$, where $S := T_{\restriction \mathrm{Im}T}$.
By hypothesis, since $\dim \mathrm{Im}(T) \leq i$, then for any $k \geq i$ (in particular for $i + 1$) we have

$$\mathrm{Im}(S^k) = \mathrm{Im}(S^{k+1})$$

we conclude seeing that $\mathrm{Im}(S) = \mathrm{Im}(T)$, so by the formula above

$$\mathrm{Im}(T^{i+1}) = \mathrm{Im}(T^{i+2}).$$

(b) Since $A = M(T)$, then $A^k = M(T^k)$, so the result follows from part (a) noticing that since each image is equal then their dimension is also equal.

$\square$

7. Let $A \in M_n(F)$ be a matrix of rank 1. Prove that:

$$A^2 = \mathrm{Tr}(A)A.$$

Solution. Since $\text{rank}(A) = 1$, there exist only one linearly independent column, say

$$C_1 = \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix}.$$

This means that the j-th column C_j of A will be a scalar multiple of C_1, so that

$$A = \begin{pmatrix} \vdots & & \vdots & & \vdots \\ C_1 & \cdots & \lambda_j C_1 & \cdots & \lambda_n C_1 \\ \vdots & & \vdots & & \vdots \end{pmatrix}.$$

$\text{Tr}(A)$ is then given by

$$\sum_{k=1}^{n} \lambda_k a_{k1} = \lambda_1 a_{11} + \cdots + \lambda_n a_{n1}.$$

Hence, let $b_{ij} \in \text{Tr}(A)A$,

$$b_{ij} = a_{ij} \sum_{k=1}^{n} \lambda_k a_{k1} = \lambda_j a_{i1} \sum_{k=1}^{n} \lambda_k a_{k1} = \sum_{k=1}^{n} (\lambda_k a_{k1}) \lambda_j a_{i1}.$$

The c_{ij} entry of A^2 will be

$$c_{ij} = \sum_{k=1}^{n} a_{ik} a_{kj} = \sum_{k=1}^{n} (\lambda_k a_{i1})(\lambda_j a_{k1}) = \sum_{k=1}^{n} (\lambda_k a_{k1}) \lambda_j a_{k1}.$$

We conclude that $b_{ij} = c_{ij}$ for any $1 \leq i, j \leq n$.

□

8. Let $A \in M_m(F)$ and $B \in M_n(F)$. Prove that:

$$\text{rank} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \text{rank}(A) + \text{rank}(B).$$

Solution. It is straightforward, since the matrix is constructed that way, we get no possible overlapping between rows nor columns. Therefore, we can find the rank of each matrix independently by row or column elimination (Gauss' algorithm).

□

9. Prove that for any matrices $A \in M_{n,m}(F)$ and $B \in M_m(F)$, we have:

$$\text{rank} \begin{bmatrix} I_n & A \\ 0 & B \end{bmatrix} = n + \text{rank}(B).$$

Solution. Consider the block matrix $(I_n|A)$, its rank will be n since $\text{rank}(I_n) = n$ and $\text{rank}(I_n|A) \leq \min(n, n+m) = n$. In other words, A has only linearly dependent columns. We can perform column eliminations using I_n to A and not affect columns of B because of the 0 block.

Finally the block matrix $(0|B)$ will have rank of at most m. Since B occupies columns from $n+1$ to $n+m$ its rank will add up to that of $(I_n|A)$. So we conclude that

$$\text{rank}\begin{bmatrix} I_n & A \\ 0 & B \end{bmatrix} = n + \text{rank}(B).$$

$\square$

10. Let $n > 2$ and let $A = [a_{ij}] \in M_n(\mathbb{R})$ be a matrix of rank 2. Prove the existence of real numbers x_i, y_i, z_i, t_i for $1 \leq i \leq n$ such that for all $i, j \in \{1, 2, \ldots, n\}$, we have:

$$a_{ij} = x_i y_j + z_i t_j.$$

Solution. Since A has rank 2 we can find two invertible matrices $P, Q \in GL_n(\mathbb{C})$ such that

$$A = P\Delta Q, \text{ where } \Delta = \begin{pmatrix} I_2 & O \\ O & O \end{pmatrix} \text{ with } O \text{ being the null (rectangular) matrix of the required size.}$$

We claim that $P \times \Delta$ is a block matrix of the form

$$(P\Delta) = \begin{pmatrix} P_2 & O \\ P_2 & O \end{pmatrix}$$
with P_2 being the first and second column of P.

To prove this claim let $1 \leq i, j \leq n$. The i, j-th entry of $P\Delta$ is given by

$$\sum_{k=1}^{n} p_{ik}\gamma_{kj}.$$

where

$$\gamma_{ij} = \begin{cases} \delta_{ij} & \text{if } 1 \leq (i \text{ and } j) \leq 2 \\ 0 & \text{if not} \end{cases} \text{ gives the entries of } \Delta.$$

We see that

$$\sum_{k=1}^{n} p_{ik}\gamma_{kj} = \begin{cases} \sum_{k=1}^{n} p_{ik}\delta_{kj} & \text{if } 1 \leq (k \text{ and } j) \leq 2 \\ 0 & \text{if not} \end{cases}.$$

Thus

$$\sum_{k=1}^{n} p_{ik}\gamma_{kj} = \sum_{k=1}^{2} p_{ik}\delta_{kj} = p_{i1}\delta_{1j} + p_{i2}\delta_{2j} \text{ if } j \leq 2$$

$$= p_{i1} \text{ if } j = 1 \text{ and } p_{i2} \text{ if } j = 2.$$

Iterating over $1 \leq i \leq n$ we conclude that the resulting matrix has two non-null columns at $j = 1$ and $j = 2$.

Therefore the i, j-th entry of $(P\Delta)Q$, where q_{ij} denotes the i, j-th entry of Q, will be

$$\sum_{k=1}^{n} (P\Delta)_{ik} q_{kj} = \sum_{1 \leq k \leq 2} p_{ik} q_{kj} \text{ since for any } k > 2 \ p_{ik} = 0.$$

Hence, the product results in just $p_{i1}q_{1j} + p_{i2}q_{2j}$, the result follows.

□

11. Let $A = [a_{ij}]_{1 \leq i,j \leq n}$ and $B = [b_{ij}]_{1 \leq i,j \leq n}$ be complex matrices such that:

$$a_{ij} = 2ij - b_{ij}$$

for all integers $1 \leq i, j \leq n$. Prove that:

$$\text{rank}(A) = \text{rank}(B).$$

Remark. I have made a typo in the problem statement, it is supposed to be $a_{ij} = 2^{i-j} b_{ij}$. The problem seems unsolvable, but we can bound $\text{rank}(A) \leq \text{rank}(B) + 1$.

Solution. An equivalent formulation of the problem is to prove that $\text{rank}(A) - \text{rank}(B) = 0$. We see that

$$a_{ij} + b_{ij} = 2ij.$$

Consider the auxiliar matrix $C = (c_{ij})_{1 \leq i,j \leq n}$ with $c_{ij} = ij$. Clearly this matrix is symmetric as $ij = ji$.

We claim that the rank of this matrix is 1. By induction on the dimension k of the matrix

Basis Let $k = 2$, then

$$C = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}.$$

The second row is two times the first row, so we can eliminate it using Gauss' algorithm.

Hypothesis Suppose that for any $k \geq 2$ it held that the matrix (c_{ij}) with $c_{ij} := ij$ has rank equals to 1.

Thesis We claim that for $k+1$ the matrix $(c_{ij})_{1\leq i,j\leq k+1}$ with $c_{ij} = ij$ has rank 1. By hypothesis the submatrix $C' = (c_{ij})_{1\leq i,j\leq k}$ has rank 1. The matrix C has this shape

$$\begin{pmatrix} 1 & 2 & \cdots & k & k+1 \\ 2 & 4 & \cdots & 2k & 2(k+1) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ k & 2k & \cdots & k^2 & k(k+1) \\ k+1 & 2(k+1) & \cdots & k(k+1) & (k+1)^2 \end{pmatrix}$$

where by hypothesis there exists a series of steps of Gauss' algorithm such that we can end up with this matrix

$$\begin{pmatrix} 1 & 2 & \cdots & k & k+1 \\ 0 & 0 & \cdots & 0 & a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & a_k \\ k+1 & 2(k+1) & \cdots & k(k+1) & (k+1)^2 \end{pmatrix}.$$

We see that the $(k+1)$th row is $(k+1)$ times the first row, in other words the matrix becomes

$$\begin{pmatrix} 1 & 2 & \cdots & k & k+1 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

which has rank 1. Since C has rank 1, $2C$ has rank 1 as well.

Since C is symmetric, A and B must be symmetric, otherwise if $a_{ij} \neq a_{ji}$ then $c_{ij} = a_{ij} + b_{ij} \neq a_{ji} + b_{ji} = c_{ji}$ which is false. By problem (1) of this section,

$$|\operatorname{rank}(A) - \operatorname{rank}(B)| \leq \operatorname{rank}(A+B) = 1$$

Without loss of generality assume that $\operatorname{rank}(A) > \operatorname{rank}(B)$. We know that $\operatorname{rank}(A) \leq \operatorname{rank}(B) + 1$. Hence

$$\operatorname{rank}(B) + 1 \geq \operatorname{rank}(A) > \operatorname{rank}(B).$$

From the second inequality we infer $\operatorname{rank}(A) \geq \operatorname{rank}(B) + 1$ and from the first we infer that $\operatorname{rank}(B) + 1 \geq \operatorname{rank}(A)$, we conclude that $\operatorname{rank}(A) = \operatorname{rank}(B) + 1$.

Hence, $\operatorname{rank}(A) - \operatorname{rank}(B) = 1$.

$\blacksquare$

12. Let $A = (a_{ij})_{1\le i,j\le n}$, $B = (b_{ij})_{1\le i,j\le n}$ be complex matrices such that

$$a_{ij} = 2^{i-j} \cdot b_{ij}$$

for all integers $1 \le i, j \le n$. Prove that rank $A =$ rank B.
Solution. Let $\operatorname{rank}(B) = r$.

We first notice that for any $1 \le i \le n$, $a_{ii} = b_{ii}$ since $a_{ii} = 2^0 b_{ii} = b_{ii}$.

Since $\operatorname{rank}(B) = r$ there exist r linearly independent columns $C_1, \ldots, C_r$ in B. By induction on r, we claim that $\operatorname{rank}(A) = r$.

Basis If $r = 1$, then the column

$$C_j = \begin{pmatrix} b_{1j} \\ \vdots \\ b_{nj} \end{pmatrix}$$

is linearly independent. Hence,

$$\left\langle \begin{pmatrix} b_{11} \\ \vdots \\ b_{n1} \end{pmatrix}, \cdots, \begin{pmatrix} b_{1j} \\ \vdots \\ b_{nj} \end{pmatrix}, \cdots, \begin{pmatrix} b_{1n} \\ \vdots \\ b_{nn} \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} b_{1j} \\ \vdots \\ b_{nj} \end{pmatrix} \right\rangle.$$

This means that any column is a scalar multiple of C_j. So we see that

$$B = \begin{pmatrix} \lambda_1 b_{1j} & \cdots & b_{1j} & \cdots & \lambda_n b_{1j} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ \lambda_1 b_{nj} & \cdots & b_{nj} & \cdots & \lambda_n b_{nj} \end{pmatrix}$$

Hence, the i-th column of A with $i \neq j$ will be

$$\begin{pmatrix} 2^{1-i} b_{1i} \\ \vdots \\ 2^{n-i} b_{ni} \end{pmatrix} = \begin{pmatrix} 2^{1-i} \lambda_1 b_{1j} \\ \vdots \\ 2^{n-i} \lambda_1 b_{nj} \end{pmatrix} = \begin{pmatrix} \alpha_1 b_{1j} \\ \vdots \\ \alpha_n b_{nj} \end{pmatrix}$$

More generally

$$A = \begin{pmatrix} c_{11} b_{1j} & \cdots & c_{1n} b_{1j} \\ \vdots & \ddots & \vdots \\ c_{n1} b_{nj} & \cdots & c_{nn} b_{nj} \end{pmatrix}$$

with $c_{ij} = 2^{i-j}\lambda_i$. Consider dividing the i-th row by $\lambda_i b_{ij}$ (assuming $b_{ij} \neq 0$, but if it were 0 then the i-th row would be null so thats OK anyways), then we get the matrix

$$\begin{pmatrix} 1 & \cdots & 2^{1-n} \\ \vdots & \ddots & \vdots \\ 2^{n-1} & \cdots & 1 \end{pmatrix}$$

We claim that this matrix does not have a minor of rank 2, consider the submatrix

$$\begin{pmatrix} 2^{i-j} & 2^{i-(j+1)} \\ 2^{(i+1)-j} & 2^{(i+1)-(j+1)} \end{pmatrix} = \begin{pmatrix} 2^{i-j} & 2^{i-j-1} \\ 2^{i+1-j} & 2^{i-j} \end{pmatrix}$$

this matrix has determinant $2^{2(i-j)} - 2^{i-j+i-j} = 2^{2(i-j)} - 2^{2i-2j} = 2^{2(i-j)} - 2^{2(i-j)} = 0$. Hence its rank is 1.

Hypothesis Suppose that for any $1 \leq r < n$ the rank of A is r.

Thesis We have to prove that if $r = n$, then the rank of A is n. Since the rank of B is n, each column $C_1, \dots, C_n$ of B is linearly independent. By hypothesis, we can find $D_1, \dots, D_{n-1}$ linearly independent columns of A. We want to prove that there exists a column D_n such that

$$\sum_{j=1}^{n} \lambda_j D_j = 0_V \iff \lambda_j = 0 \text{ for any } 1 \leq j \leq n.$$

Let

$$D_n = \begin{pmatrix} 2^{1-n} b_{1n} \\ 2^{2-n} b_{2n} \\ \vdots \\ b_{nn} \end{pmatrix}$$

Then

$$\sum_{j=1}^{n} \lambda_j D_j = 0_V$$

if and only if for each row, it follows that

$$\sum_{j=1}^{n} \lambda_j 2^{i-j} b_{ij} = 0_V.$$

Hence suppose that there existed a non-null λ_k for some $1 \leq k \leq n$, then

$$\begin{aligned} &\lambda_1 2^{i-1} b_{i1} + \cdots + \lambda_k 2^{i-k} b_{ik} + \cdots + \lambda_k 2^{i-n} b_{in} = 0_V \\ &\implies \frac{\lambda_1}{-\lambda_k 2^{i-k}} 2^{i-1} b_{i1} + \cdots + \frac{\lambda_k}{-\lambda_k 2^{i-k}} 2^{i-n} b_{in} = b_{ik} \\ &\implies \frac{\lambda_1}{-\lambda_k} \frac{2^{i-1}}{2^{i-k}} b_{i1} + \cdots + \frac{\lambda_k}{-\lambda_k} \frac{2^{i-n}}{2^{i-k}} b_{in} = b_{ik} \\ &\implies \alpha_1 \frac{2^{i-1}}{2^{i-k}} b_{i1} + \cdots + \alpha_n \frac{2^{i-n}}{2^{i-k}} b_{in} = b_{ik} \\ &\implies \alpha_1 2^{k-1} b_{i1} + \cdots + \alpha_n 2^{k-n} b_{in} = b_{ik}. \end{aligned}$$

Therefore, for any column k of B, we see that it is a linear combination of the remaining $n-1$ columns since i does not vary in the left hand side coefficients. A contradiction of the hypothesis.

□

13. Let $A \in M_n(\mathbb{C})$ be a matrix such that $A^2 = A$, i.e., A is the matrix of a projection. Prove that:

$$\operatorname{rank}(A) + \operatorname{rank}(I_n - A) = n.$$

Solution. Let $T : V \to V$ be the projection associated to A. It suffices to prove $\operatorname{rank}(A) = n - \operatorname{rank}(I_n - A)$ or in other words $\dim \operatorname{Im}(T) = n - \dim \operatorname{Im}(\mathrm{id} - T)$.

By rank-nullity we infer that the problem statement is equivalent to proving that

$$\dim \operatorname{Im}(T) = \dim \ker(\mathrm{id} - T).$$

We claim that $\operatorname{Im}(T) = \ker(\mathrm{id} - T)$. To prove this claim, we let $Tv \in \operatorname{Im}(T)$, then

$$Tv \in \ker(\mathrm{id} - T) \iff (\mathrm{id} - T)Tv = 0_V$$

But using projection property

$$Tv - T(Tv) = Tv - Tv = 0_V$$

hence $Tv \in \ker(\mathrm{id} - T)$ so $\operatorname{Im}(T) \subseteq \ker(\mathrm{id} - T)$.

Then, we let $v \in \ker(\mathrm{id} - T)$, we see that

$$(\mathrm{id} - T)v = 0_V \text{ if } v - Tv = 0_V$$

hence $v = Tv$ so $v \in \operatorname{Im}(T)$ and $\ker(\mathrm{id} - T) \subseteq \operatorname{Im}(T)$.

We conclude that $\operatorname{Im}(T) = \ker(\mathrm{id} - T)$.

□

14. Let $n > k$ and let $A_1, \dots, A_k \in M_n(\mathbb{R})$ be matrices of rank $n - 1$. Prove that $A_1 A_2 \cdots A_k$ is nonzero. *Hint:* Using Sylvester's inequality, prove that:

$$\operatorname{rank}(A_1 \cdots A_j) \geq n - j \quad \text{for } 1 \leq j \leq k.$$

Solution.

Claim 1. *Let $A_1, \dots, A_k \in M_n(\mathbb{R})$ be matrices of rank $n - 1$. The inequality*

$$rank(A_1 \cdots A_j) \geq n - j$$

holds for any $1 \leq j \leq k$.

Proof. By induction on j,

Basis If $j = 1$ then $\text{rank}(A_1) = n - 1$ by hypothesis.
If $j = 2$ then by Sylvester's inequality

$$\text{rank}(A_1 A_2) \geq \text{rank}(A_1) + \text{rank}(A_2) - n$$

therefore

$$\text{rank}(A_1 A_2) \geq (n-1) + (n-1) - n = n - 2.$$

Hypothesis Suppose that for $j < n$ matrices $A_1, \dots, A_j$ whose rank is $n - 1$ it followed that

$$\text{rank}(A_1 \cdots A_j) \geq n - j.$$

Thesis Consider $j + 1$ matrices $A_1, \dots, A_j$ whose rank is $n - 1$. We have to prove that

$$\text{rank}(A_1 \cdots A_j A_{j+1}) \geq n - (j+1).$$

By hypothesis, the matrix $B = A_1, \dots, A_j$ has rank $n - j$. Therefore by Sylvester's inequality

$$\text{rank}(BA_{j+1}) \geq \text{rank}(B) + \text{rank}(A_{j+1}) - n$$

$$\geq (n-j) + (n-1) - n = n - (j+1)$$

By our claim we see that $\text{rank}(A_1 A_2 \cdots A_k) \geq n - k$. This means that there exist $n - k > 0$ linearly independent columns. Since these are linearly independent they cannot be null columns.

□

15. Let $A \in M_n(\mathbb{C})$ be a matrix of rank at least $n - 1$. Prove that:

$$\text{rank}(A^k) \geq n - k \quad \text{for } 1 \leq k \leq n.$$

Hint: Use Sylvester's inequality.
Solution. By induction on k

Basis If $k = 1$ then

$$\text{rank}(A) \geq n - 1$$

by hypothesis.
If $k = 2$ then

$$\text{rank}(A^2) \geq 2\,\text{rank}(A) - n \geq (2n-2) - n = n - 2.$$

Hypothesis Suppose that for any $1 \leq k < n$ for a matrix A with rank of at least $n - 1$ it held that

$$\text{rank}(A^k) \geq n - k.$$

Thesis For $k+1 \leq n$ we have to prove that

$$\operatorname{rank}(A^{k+1}) \geq n-(k+1).$$

By hypothesis the matrix $B = A^k$ has rank of at least $n-k$. Therefore by Sylvester's inequality,

$$\operatorname{rank}(BA) \geq (n-k)+(n-1)-n = n-(k+1).$$

We conclude that $\operatorname{rank}(A^k) \geq n-k$ for any $1 \leq k \leq n$.

$\square$

16. (a) Prove that for any matrix $A \in M_n(\mathbb{R})$, we have:

$$\operatorname{rank}(A) = \operatorname{rank}({}^{\top}AA).$$

Hint: If $X \in \mathbb{R}^n$ is a column vector such that ${}^{\top}AAX = 0$, write ${}^{\top}X\,{}^{\top}AAX = 0$ and express the left-hand side as a sum of squares.

(b) Let $A = \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}$. Find the rank of A and $A^{\top}A$, and conclude that part (a) of the problem is no longer true if $\mathbb{R}$ is replaced with $\mathbb{C}$.

Solution.

(a) The following lemma was proven as an example problem in the reference book (Andreescu's, problem 1.52).

Lemma 2. *Let $X \in F^n$ be a vector with coordinates $x_1, \ldots, x_n$, considered as a matrix in $M_{n,1}(F)$. For any matrix $A \in M_n(F)$ we have*

$${}^{t}X({}^{t}A \cdot A)X = \sum_{i=1}^{n} (a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n)^2.$$

$\blacksquare$

Let

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

be a column vector with $x_i \in \mathbb{R}$. We can find the nullity of ${}^{\top}AA$ solving the linear system $({}^{\top}AA)X = 0$. Since $X \in \operatorname{null}({}^{\top}AA)$ we see that ${}^{t}X({}^{t}A \cdot A)X = 0$. Therefore using lemma (2) on A as the problem statement we have

$$\sum_{i=1}^{n} (a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n)^2 = 0.$$

We see that each summand has to be 0 due to the square, hence we end up with a system of n equations and n unknowns

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = 0 \\ \vdots \\ a_{i1}x_1 + \cdots + a_{in}x_n = 0 \\ \vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n = 0 \end{cases}$$

Hence this system is the same as $AX = 0$. In other words, the solutions of $({}^\top AA)X = 0$ are the same as $AX = 0$. So the dimension of the set solutions for both of them (i.e. their nullity in this case) is the same, so their rank must be equal.

(b)

$$A^T A = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Hence its rank is zero.

$$A = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \longrightarrow_{R_2 = R_2 - iR_1} \begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix}$$

Hence its rank is 1.

□

17. Let A be an $m \times n$ matrix with rank r. Prove that there is an $m \times m$ matrix B with rank $m - r$ such that:

$$BA = O_{m,n}.$$

Solution. Let $A = PJ_rQ$ with

$$J_r = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \in M_{m,n}(\mathbb{F}),$$

and $P \in GL_m(\mathbb{F})$ and $Q \in GL_n(\mathbb{F})$. Let $B = K_{m-r}P^{-1}$ with

$$K_{m-r} = \begin{pmatrix} 0 & 0 \\ 0 & I_{m-r} \end{pmatrix} \in M_{m,m}.$$

Therefore $AB = K_{m-r}(P^{-1}P)J_rQ = K_{m-r}J_rQ$. We conclude that $K_{m-r}J_r$ is a matrix of size $m \times n$ and it is null. So the product with Q will also result in a $m \times n$ null matrix.

□

18. (Generalized inverses) Let $A \in M_{m,n}(F)$. A generalized inverse of A is a matrix $X \in M_{n,m}(F)$ such that:

$$AXA = A.$$

(a) If $m = n$ and A is invertible, show that the only generalized inverse of A is A^{-1}.

(b) Show that a generalized inverse of A always exists.

(c) Give an example to show that the generalized inverse need not be unique.

Solution.

(a) Suppose we had $X \neq A^{-1}$ with $X \in M_n(F)$ such that

$$AXA = AA^{-1}A$$

then

$$XA = A^{-1}A.$$

Therefore,

$$XA - A^{-1}A = O_n$$
$$(X - A^{-1})A = O_n$$

So necessarily $X - A^{-1} = O_n$, otherwise A should have rank zero but it is not possible since it is invertible. We conclude that $X = A^{-1}$.

(b) Let $A = PJ_rQ$ with

$$J_r = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \in M_{m,n}(F),$$

$P \in GL_m(F)$ and $Q \in GL_n(F)$. Let $X = Q^{-1}K_rP^{-1}$ with $K_r \in M_{n,m}(F)$ defined as

$$\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \in M_{n,m}(F)$$

Hence, $XA = Q^{-1}K_rP^{-1}PJ_rQ = Q^{-1}K_rJ_rQ$. Therefore, $AXA = PJ_rQ(Q^{-1}K_rJ_rQ) = P(J_rK_rJ_r)Q$. In other words, suffices $J_rK_rJ_r = J_r$. We see that

$$K_rJ_r = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \in M_n(F).$$

So multiplying it again by J_r the result follows.

(c) Consider the 1×2 matrix

$$A = \begin{pmatrix} 1 & 2 \end{pmatrix}$$

If we want to find a 2×1 matrix

$$B = \begin{pmatrix} a & b \end{pmatrix}$$

such that $ABA = A$, then we see that

$$BA = \begin{pmatrix} a & 2a \\ b & 2b \end{pmatrix}$$

so $ABA = A$ becomes a system

$$\begin{cases} a + 2b = 1 \\ 2a + 4b = 2 \end{cases}$$

where the second equation is twice the first. We see that for this example in particular suffices $B = (1 - 2b, b)^t$ for any $b \in F$.

$\square$