Normalization Theory sheet I

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Exercises in normalization theory from J. Ullman's "Database Systems - The Complete Book". This week's topics are closures of functional dependencies sets, projections of FD's sets, minimal bases, BCNF and 3NF.

Rules about Functional Dependencies

1. **3.2.1:** Consider a relation with schema $R(A, B, C, D)$ and FD's

$$AB \to C, \quad C \to D, \quad D \to A.$$

   (a) What are all the nontrivial FD's that follow from the given FD's? You should restrict yourself to FD's with single attributes on the right side.

   (b) What are all the keys of R?

   (c) What are all the superkeys for R that are not keys?

2. **3.2.2:** Repeat Exercise 3.2.1 for the following schemas and sets of FD's:

   i) $S(A, B, C, D)$ with FD's $A \to B$, $B \to C$, and $B \to D$.

   ii) $T(A, B, C, D)$ with FD's $AB \to C$, $BC \to D$, $CD \to A$, and $AD \to B$.

   iii) $U(A, B, C, D)$ with FD's $A \to B$, $B \to C$, $C \to D$, and $D \to A$.

   Solution.

   i) The

3. **3.2.4:** Show that each of the following are *not* valid rules about FD's by giving example relations that satisfy the given FD's (following the "if") but not the FD that allegedly follows (after the "then").

   (a) If $A \to B$ then $B \to A$.

   (b) If $AB \to C$ and $A \to C$, then $B \to C$.

   (c) If $AB \to C$, then $A \to C$ or $B \to C$.

4. **3.2.5** Show that if a relation has no attribute that is functionally determined by all the other attributes, then the relation has no nontrivial FD's at all.
   Solution. Let R be a relation schema. By hypothesis, for any set of attributes $X \in R$, $X^+ = X$. So if a nontrivial FD such as $X \longrightarrow Y$ existed, it would imply that $Y \nsubseteq X$, so $X^+ = XY$, a contradiction.

   $\square$

5. **3.2.6** Let X and Y be sets of attributes. Show that if $X \subseteq Y$, then $X^+ \subseteq Y^+$, where the closures are taken with respect to the same set of FD's.
   Solution. Since $X \subseteq Y$ and $Y \subseteq Y^+$, then necessarily $X \subseteq Y^+$. Since $X \subseteq X^+$, we see that each element of $X^+ \cap X$ is in Y^+. With this in mind, we have shown the inclusion in the trivial DF's case.

   Now for the case of nontrivial dependencies $x \in X^+ \setminus X$ we have to show that $x \in Y^+$. Since x is nontrivial then there exists some $z \in X$ such that $x \nsubseteq z$ and

   $$z \longrightarrow x.$$

   Since $z \in Y$ as well, we conclude that $x \in Y^+$.

   $\square$

6. **3.2.7** Prove that $(X^+)^+ = X^+$.
   Solution. Since it is obvious that $X^+ \subseteq (X^+)^+$ (the closure of a FD set always contains itself), remains showing that $(X^+)^+ \subseteq X^+$.

   Let $x \in (X^+)^+$. Then there exists an $y \in X^+$ such that

   $$y \longrightarrow x.$$

   So there must also exist a $z \in X$ such that

   $$z \longrightarrow y,$$

   hence, by transitivity $z \longrightarrow x$. We conclude that $x \in X^+$.

   $\square$

7. **3.2.8** We say a set of attributes X is *closed* (with respect to a given set of FD's) if $X^+ = X$. Consider a relation with schema $R(A, B, C, D)$ and an unknown set of FD's. If we are told which sets of attributes are closed, we can discover the FD's. What are the FD's if:

   (a) All sets of the four attributes are closed.

(b) The only closed sets are $\varnothing$ and $\{A, B, C, D\}$.

(c) The closed sets are $\varnothing$, $\{A, B\}$, and $\{A, B, C, D\}$.

Solution.

(a) By exercise 3.2.5 R has no non-trivial dependencies. So $\{A, B, C, D\}^+ = \{A, B, C, D\}$.

(b) We claim that there does exist at least one superkey of the relation. First we note that the unordered pairs of attributes is

$$\binom{4}{2} > 4,$$

so there must exist at least two pairs X, Y such that $X \cap Y \neq \varnothing$ (by pigeonhole principle). In other words, since the closure of any attribute is at least two (they are not closed) we see that

$$X^+ = XZ \text{ and } Y^+ = YZ$$

or

$$X^+ = XZ \text{ and } Z^+ = ZY$$

So in the first case

$$X \longrightarrow Z \text{ and } Y \longrightarrow Z$$

then

$$XY \longrightarrow Z.$$

In fact, since neither Z is closed, $Z \longrightarrow T$, hence by transitivity $XY \longrightarrow T$, we conclude that $(XY)^+ = XYZT = ABCD$.

In the second case, we get that X is a superkey by transitivity.

So the FD's will look like $\{AB \longrightarrow C, C \longrightarrow D\}$ or $\{A \longrightarrow B, B \longrightarrow C, C \longrightarrow D, D \longrightarrow A\}$ with respect to both cases.

(c) Consider the sub relations $R_1 = \pi_{A,B}(R)$ and $R_2 = \pi_{C,D}(R)$. For R_1, since $A^+ \neq A$ and $B^+ \neq B$ but $(AB)^+ = AB$ it will suffice letting $A \longrightarrow B$ and $B \longrightarrow A$. In fact, this is the only way it can be done; as if we let $A \longrightarrow X$ with $X \subseteq R_2$ then

$$AB \longrightarrow A \longrightarrow X, \text{ so } (AB)^+ = ABX \neq AB.$$

For R_2 we have that $C^+ \neq C$ and $D^+ \neq D$ and that $(CD)^+ \neq CD$, since these are not closed. We then see that CD is a superkey by seeing that since

$$CD \longrightarrow C,\ CD \longrightarrow D,\ C \longrightarrow X,\ D \longrightarrow Y \text{ (with } X \text{ and } Y \text{ not necessarily different),}$$

then

$$CD \longrightarrow XY.$$

And if $XY = X$, then since $(CD)^+ = CDX$, but since CDX is not closed, by transitivity $(CD)^+ = CDXZ = ABCD$. Then, the FD's for R_2 will be of the form $\{C \longrightarrow D, D \longrightarrow A\}$, $\{C \longrightarrow A, D \longrightarrow B\}$ or $\{C \longrightarrow A, D \longrightarrow A, CD \longrightarrow B\}$.

□

8. **Exercise 3.2.11** Show that if an FD F follows from some given FD's, then we can prove F from the given FD's using Armstrong's axioms (defined in the box "A Complete Set of Inference Rules" in Section 3.2.7). *Hint:* Examine Algorithm 3.7 and show how each step of that algorithm can be mimicked by inferring some FD's by Armstrong's axioms.
Solution. Let $R = \{A_1, \ldots, A_n\}$ be a relation. Let $\{B_1, \ldots, B_m\}$ be a subset of R. Suppose F had several attributes $\{C_1, \ldots, C_k\}$ on the right hand side, Algorithm 3.7 says F holds even when splitting it into several one-attributed FD's, in other words

$$B_1, \ldots, B_m \longrightarrow F \iff$$

$$\begin{cases} B_1, \ldots, B_m \longrightarrow C_1 \\ \vdots \\ B_1, \ldots, B_m \longrightarrow C_k \end{cases}.$$

There are now some cases we are ought to study; first suppose that some C_i were an element of $B_1, \ldots, B_m$ for some $m \leq n$, then

$$\{C_i\} \subseteq \{B_1, \ldots, B_m\} \implies B_1, \ldots, B_m \longrightarrow C_i$$

More generally if we had a larger subset of F, say $\{C_1, \ldots, C_q\}$ with $q \leq k$ and each element of it satisfying the above relation, then by union rule we can say that

$$\{C_1, \ldots, C_q\} \subseteq \{B_1, \ldots, B_m\} \implies B_1, \ldots, B_m \longrightarrow C_1, \ldots, C_q,$$

verifying reflexivity.

Let $\{B_1, \ldots, B_m\} \longrightarrow C_i$ such that $C_i \longrightarrow C_j$ for some i, j. Then we see that

$$C_i \in \{B_1, \ldots, B_m\}^+$$

so

$$B_1, \ldots, B_m, C_i \longrightarrow C_j.$$

Meaning that $C_j \in \{B_1, \ldots, B_m, C_i\}^+ = \{B_1, \ldots, B_m\}^+$. In other words

$$B_1, \ldots, B_m \longrightarrow C_i \text{ and } C_i \longrightarrow C_j \implies B_1, \ldots, B_m \longrightarrow C_j,$$

verifying transitivity.

Lastly, since $B_1, \ldots, B_m \longrightarrow F$, necessarily for any other subset X of R, we have

$$B_1, \ldots, B_m, X \longrightarrow B_1, \ldots, B_m \longrightarrow F \text{ (by reflexivity)}$$

and

$$B_1, \ldots, B_m, X \longrightarrow X. \text{ (by reflexivity)}$$

So by union rule

$$B_1, \ldots, B_m X \longrightarrow FX,$$

verifying augmentation.

□

Design Theory for Relational Databases

1. **3.3.1:** For each of the following relation schemas and sets of FD's:

   (a) $R(A, B, C, D)$ with FD's $AB \to C$, $C \to D$, and $D \to A$.

   (b) $R(A, B, C, D)$ with FD's $B \to C$ and $B \to D$.

   (c) $R(A, B, C, D)$ with FD's $AB \to C$, $BC \to D$, $CD \to A$, and $AD \to B$.

   (d) $R(A, B, C, D)$ with FD's $A \to B$, $B \to C$, $C \to D$, and $D \to A$.

   (e) $R(A, B, C, D, E)$ with FD's $AB \to C$, $DE \to C$, and $B \to D$.

   (f) $R(A, B, C, D, E)$ with FD's $AB \to C$, $C \to D$, $D \to B$, and $D \to E$.

   do the following:

   i) Indicate all the BCNF violations. Do not forget to consider FD's that are not in the given set, but follow from them. However, it is not necessary to give violations that have more than one attribute on the right side.

ii) Decompose the relations, as necessary, into collections of relations that are in BCNF.

Solution.

(a) Doing the closure of C: $C^+ = CDA \neq R$. So we split R in

$$R = R_1 \bowtie R_2,$$
$$\text{where } R_1 = CDA \text{ and}$$
$$R_2 = C \cup (R \setminus C^+) = C \cup (ABCD \setminus CDA) = CB.$$

We see that R_2 is in BCNF as it only have 2 attributes, and $R_1 = CDA$ has a projected FD's set $S_1 := \{C \to D, D \to A\}$. So $D^+ = DA \neq R_1$, we decompose it again

$$R_1 = R_3 \bowtie R_4$$

$R_3 = DA$ and $R_4 = D \cup (R_1 \setminus DA) = DC$. All three of them have two attributes so they are in BCNF.

We conclude that the BCNF decomposition of R is

$$R_2(C, B), R_3(D, A), R_4(D, C).$$

(b) Doing the closure of B we see that $B^+ = BCD \neq R$, so $B \to C$ violates BCNF. We decompose R into $R_1 = B^+ = BCD$ and $R_2 = B \cup (R \setminus B^+) = B \cup (ABCD \setminus BCD) = AB$. Their respective set of FD's $S_1 = \{B \to C, B \to D\}$ and $S_2 = \varnothing$ (where $\varnothing$ denotes the absence of nontrivial FD's).

We conclude that R_1 is in BCNF since $B^+ = BCD = R_1$ and R_2 is also in BCNF since it holds only trivial FD's, so the decomposition is

$$R_1(B, C, D), R_2(A, B).$$

(c) We see that $(AB)^+ = ABCD$, $(BC)^+ = BCDA$, $(CD)^+ = CDAB$, $(AD)^+ = ADBC$, so it already is in BCNF.

(d) We see that $A^+ = ABCD$, $B^+ = BCDA$, $C^+ = CDAB$ and $D^+ = DABC$, so it already is in BCNF.

(e) We see that $(AB)^+ = ABCD \neq ABCDE$, so we decompose R into $R_1 = ABCD$ and $R_2 = (AB) \cup (R \setminus (AB)^+) = (AB) \cup (E) = ABE$. Their respective set of FD's are $S_1 = \{AB \to C, B \to D\}$ and $S_2 = \varnothing$.

R_1 is not in BCNF since $B^+ = BD \neq ABCD$. We decompose R_1 into $R_3 = BD$ and $R_4 = B \cup (ABCD \setminus BD) = ABC$. R_3 is in BCNF, and R_4 has a FD's set $\{AB \to C\}$, so it is in BCNF. We conclude that the BCNF decomposition is

$$R_2(A, B, E), R_3(B, D), R_4(A, B, C).$$

(f) We see that $C \to D$ violates BCNF since $C^+ = CDBE \neq ABCDE$. We decompose R into $R_1 = BCDE$ and $R_2 = C \cup (ABCDE \setminus BCDE) = AC$. R_2 is in BCNF as it has only two attributes. R_1 has FD's set $\{C \to D, D \to B, D \to E\}$, suffices taking $D^+ = DBE$ to see that it is not yet in BCNF. We decompose R_1 into $R_3 = DBE$ and $R_4 = D \cup (BCDE \setminus DBE) = D \cup (C) = CD$. We see that the FD's set of R_3 is $\{D \to B, D \to E\}$ so it is in BCNF. We conclude that the BCNF decomposition is

$$R_2(A, C), R_3(D, B, E), R_4(C, D).$$

□

2. **3.3.2:** Consider a relation R whose schema is the set of attributes $\{A, B, C, D\}$ with FD's $A \to B$ and $A \to C$. Either is a BCNF violation, because the only key for R is $\{A, D\}$. Suppose we begin by decomposing R according to $A \to B$. Do we ultimately get the same result as if we first expand the BCNF violation to $A \to BC$? Why or why not?
Solution.

   (a) We start with a FD's set $S = \{A \to B, A \to C\}$; decomposing R with $A \to B$ we get $R_1 = A^+ = ABC$ and $R_2 = A \cup (R \setminus ABC) = AD$. With FD's set $S_1 = \{A \to B, A \to C\}$ and $S_2 = \varnothing$.

   (b) We decompose with $A \to BC$, $A^+ = ABC$; so $R_1 = ABC$ and $R_2 = AD$. With FD's set $S_1 = \{A \to BC\}$ and $S_2 = \varnothing$.

   We conclude that the decompositions yield the same subrelations, but the FD's set for the second case is smaller (but equivalent under splitting).

□

3. **3.3.3:** Let R be as in Exercise 3.3.2, but let the FD's be $A \to B$ and $B \to C$. Again compare decomposing using $A \to B$ first against decomposing by $A \to BC$ first.
Solution.

   (a) Decomposing with $A \to B$ and seeing $A^+ = ABC$ gives us subrelations $R_1 = ABC$ and $R_2 = AD$. R_2 is in BCNF. The FD's set for R_1 is $S_1 = \{A \to B, B \to C\}$, so $B^+ = BC \neq ABC$. We decompose R_1 in

   $$R_3 = BC \text{ and } R_4 = B \cup (R_1 \setminus BC) = AB.$$

   We conclude that the BCNF decomposition for this case is

   $$R_2(A, D), R_3(B, C), R_4(A, B).$$

   (b) Decomposing with $A \to BC$ we get $R_1 = ABC$ and $R_2 = AD$ with $S_1 = \{A \to BC\}$

4. **3.3.4:** Suppose we have a relation schema $R(A, B, C)$ with FD $A \to B$. Suppose also that we decide to decompose this schema into $S(A, B)$ and $T(B, C)$. Give an example of an instance of relation R whose projection onto S and T and subsequent rejoining does not yield the same relation instance. That is, $\pi_{A,B}(R) \bowtie \pi_{B,C}(R) \neq R$.
   Solution.

5. Let $\mathcal{F}$ and $\mathcal{F}_{min}$ be two equivalent sets of FD's for a relation R, where

   (a) $\mathcal{F}$ contains FD's with arbitrary right-hand sides (e.g., $X \to Y$ where $|Y| \geq 1$),

   (b) $\mathcal{F}_{min}$ is a minimal basis.

   Prove or disprove the following claim:

   Claim:
   Decomposing R into BCNF using $\mathcal{F}_{min}$ results in a number of subrelations that is *less than or equal to* the number of subrelations obtained using $\mathcal{F}$ instead.
   Solution.

6. Prove that the BCNF decomposition algorithm ends after finitely many steps. Furthermore prove that each output subrelation is in BCNF.
   Solution.

Decomposition: The Good, Bad, and Ugly