

Finite and infinite series sheet

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*Problems on series and products over finite and infinite sets: books of Terence Tao's "Analysis I", Donald E. Knuth's "The Art of Computer Programming".
As well as problems of the Putnam Mathematical Competition and the International Math Olympiad.*

Convergence of series

1. Let $\sum_{n=m}^{\infty} a_n$ be a formal series of real numbers. Then $\sum_{n=m}^{\infty} a_n$ converges if and only if, for every real number $\varepsilon > 0$, there exists an integer $N \geq m$ such that

$$\left| \sum_{n=p}^q a_n \right| \leq \varepsilon \quad \text{for all } p, q \geq N.$$

Solution.

- (\implies) If $\sum_{n=m}^{\infty} a_n$ converges then, by definition, the partial sum sequence

$$(S_N)_{N=m}^{\infty} = \left(\sum_{n=m}^N a_n \right)_{N=m}^{\infty}$$

converges to L . Therefore, for any $\varepsilon > 0$ there exists some $M \geq m$ such that for any $p-1, q \geq M$

$$|S_q - S_{p-1}| \leq \varepsilon.$$

Without loss of generality assume $p-1 \leq q$, it follows that

$$\left| \sum_{n=m}^q a_n - \sum_{n=m}^{p-1} a_n \right| = \left| \sum_{n=p}^q a_n \right| \leq \varepsilon.$$

- (\impliedby) Doing the reverse process of the direct implication the result follows.

□

2. **(Zero test).** Let $\sum_{n=m}^{\infty} a_n$ be a convergent series of real numbers. Then we must have $\lim_{n \rightarrow \infty} a_n = 0$. To put this another way, if $\lim_{n \rightarrow \infty} a_n$ is non-zero or divergent, then the series $\sum_{n=m}^{\infty} a_n$ is divergent.
Solution. If $\sum_{n=m}^{\infty} a_n$ converges then by last problem, for any $\varepsilon > 0$, there exists some $M \geq m$ such that for any $p, q \geq M$

$$\left| \sum_{n=p}^q a_n \right| \leq \varepsilon. \quad (1)$$

Suppose for the sake of contradiction that $\lim_{n \rightarrow \infty} a_n = L > 0$ so that for any $\varepsilon > 0$ there exists some M' such that for any $n \geq M'$

$$|a_n - L| < \varepsilon$$

so that in particular $a_n \geq L - \varepsilon$.

Therefore, for any $p, q \geq K := \max\{M, M'\}$

$$\left| \sum_{n=p}^q a_n \right| \geq \left| \sum_{n=p}^q L - \varepsilon \right| = (q - p + 1) |L - \varepsilon|.$$

Hence, let $\varepsilon = L/2$, we see that if the above inequality were true

$$\left| \sum_{n=p}^q a_n \right| \geq \frac{(q - p + 1)L}{2}.$$

Then let $q = p + 1$, it follows that

$$\left| \sum_{n=p}^q a_n \right| \geq L > \varepsilon.$$

A contradiction of inequality (1).

□

3. **(Absolute convergence test).** Let $\sum_{n=m}^{\infty} a_n$ be a formal series of real numbers. If this series is absolutely convergent, then it is also conditionally convergent. Furthermore, in this case we have the triangle inequality

$$\left| \sum_{n=m}^{\infty} a_n \right| \leq \sum_{n=m}^{\infty} |a_n|.$$

Solution. If $\sum_{n=m}^{\infty} |a_n|$ converges to L , then for any $\varepsilon > 0$ there exists some $M \geq m$ such that for any $p, q \geq M$

$$\left| \sum_{n=p}^q |a_n| \right| \leq \varepsilon.$$

Hence,

$$\sum_{n=p}^q |a_n| \leq \varepsilon.$$

By triangle inequality over finite series, for any $p, q \geq M$ we have

$$\left| \sum_{n=p}^q a_n \right| \leq \sum_{n=p}^q |a_n|.$$

We conclude that $\sum_{n=p}^q a_n$ converges. Trivially, we also note that again, by triangle inequality over finite series, each partial sum $S_N = \sum_{n=m}^N a_n$ is smaller than or equal to $T_N = \sum_{n=m}^N |a_n|$, meaning that by comparison principle $\lim_{N \rightarrow \infty} S_N \leq \lim_{N \rightarrow \infty} T_N$.

□

4. (Series laws).

- (a) If $\sum_{n=m}^{\infty} a_n$ is a series of real numbers converging to x , and $\sum_{n=m}^{\infty} b_n$ is a series of real numbers converging to y , then $\sum_{n=m}^{\infty} (a_n + b_n)$ is also a convergent series, and converges to $x + y$.

In particular, we have

$$\sum_{n=m}^{\infty} (a_n + b_n) = \sum_{n=m}^{\infty} a_n + \sum_{n=m}^{\infty} b_n.$$

- (b) If $\sum_{n=m}^{\infty} a_n$ is a series of real numbers converging to x , and c is a real number, then $\sum_{n=m}^{\infty} (ca_n)$ is also a convergent series, and converges to cx .

In particular, we have

$$\sum_{n=m}^{\infty} (ca_n) = c \sum_{n=m}^{\infty} a_n.$$

- (c) Let $\sum_{n=m}^{\infty} a_n$ be a series of real numbers, and let $k \geq 0$ be an integer. If one of the two series $\sum_{n=m}^{\infty} a_n$ and $\sum_{n=m+k}^{\infty} a_n$ are convergent, then the other one is also, and we have the identity

$$\sum_{n=m}^{\infty} a_n = \sum_{n=m}^{m+k-1} a_n + \sum_{n=m+k}^{\infty} a_n.$$

- (d) Let $\sum_{n=m}^{\infty} a_n$ be a series of real numbers converging to x , and let k be an integer. Then $\sum_{n=m+k}^{\infty} a_n$ also converges to x .

5. (Comparison test.)

6. (Geometric series.)
7. (Cauchy criterion.)
8. (Root test.)
9. (Ratio test.)
10. Let $(a_n)_{n \geq 1}$ be a sequence of positive real numbers such that the series $\sum_{n=1}^{\infty} a_n$ converges. Show that the series

$$\sum_{n=1}^{\infty} a_n^{\frac{n}{n+1}}$$

Also converges.

Solution. Consider the partial sum sequence $(S_N)_{N=m}^{\infty}$ such that $S_n = \sum_{n=m}^N a_n^{\frac{n}{n+1}}$. We see that

$$b_n = \frac{n}{n+1} = \frac{(n+1) - 1}{n+1} = 1 - \frac{1}{n+1}$$

so that $\lim_{n \rightarrow \infty} b_n = 1$. By zero test, since $\sum_{n=1}^{\infty} a_n$ converges, necessarily $\lim_{n \rightarrow \infty} a_n = 0$. We therefore can study the behaviour of the sequence

$$\lim_{n \rightarrow \infty} a_n^{b_n}.$$

That results into

$$\lim_{n \rightarrow \infty} \frac{a_n}{a_n^{\frac{1}{n+1}}}.$$

Note that we cannot use limit laws theorem since we do not know whether $a_n^{\frac{1}{n+1}}$ might converge. Suppose that it converged to $c \in \mathbb{R}$, then

$$\lim_{n \rightarrow \infty} a_n^{b_n} = \frac{1}{c} \lim_{n \rightarrow \infty} a_n = 0.$$

Then one thing we shall also verify is that it is bounded above by for example $\frac{1}{n(n+1)}$ thus by comparison principle the requested series converge.

Series expansions