Finite and infinite series sheet

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Problems on series and products over finite and infinite sets: books of Terence Tao's "Analysis I", Donald E. Knuth's "The Art of Computer Programming". As well as problems of the Putnam Mathematical Competition and the International Math Olympiad.

Convergence of series

1. Let $\sum_{n=m}^{\infty} a_n$ be a formal series of real numbers. Then $\sum_{n=m}^{\infty} a_n$ converges if and only if, for every real number $\varepsilon > 0$, there exists an integer $N \geq m$ such that

$$\left| \sum_{n=p}^{q} a_n \right| \leq \varepsilon \quad \text{for all } p, q \geq N.$$

Solution.

($\implies$) If $\sum_{n=m}^{\infty} a_n$ converges then, by definition, the partial sum sequence

$$(S_N)_{N=m}^{\infty} = \left(\sum_{n=m}^{N} a_n \right)_{N=m}^{\infty}$$

converges to L. Therefore, for any $\varepsilon > 0$ there exists some $M \geq m$ such that for any $p-1, q \geq M$

$$|S_q - S_{p-1}| \leq \varepsilon.$$

Without loss of generality assume $p - 1 \leq q$, it follows that

$$\left| \sum_{n=m}^{q} a_n - \sum_{n=m}^{p-1} a_n \right| = \left| \sum_{n=p}^{q} a_n \right| \leq \varepsilon.$$

($\impliedby$) Doing the reverse process of the direct implication the result follows.

$\square$

2. **(Zero test)**. Let $\sum_{n=m}^{\infty} a_n$ be a convergent series of real numbers. Then we must have $\lim_{n\to\infty} a_n = 0$. To put this another way, if $\lim_{n\to\infty} a_n$ is non-zero or divergent, then the series $\sum_{n=m}^{\infty} a_n$ is divergent.
Solution. If $\sum_{n=m}^{\infty} a_n$ converges then by last problem, for any $\varepsilon > 0$, there exists some $M \geq m$ such that for any $p, q \geq M$

$$\left|\sum_{n=p}^{q} a_n\right| \leq \varepsilon. \tag{1}$$

Suppose for the sake of contradiction that $\lim_{n\to\infty} a_n = L > 0$ so that for any $\varepsilon > 0$ there exists some M' such that for any $n \geq M'$

$$|a_n - L| < \varepsilon$$

so that in particular $a_n \geq L - \varepsilon$.
Therefore, for any $p, q \geq K := \max\{M, M'\}$

$$\left|\sum_{n=p}^{q} a_n\right| \geq \left|\sum_{n=p}^{q} L - \varepsilon\right| = (q - p + 1)\left|L - \varepsilon\right|.$$

Hence, let $\varepsilon = L/2$, we see that if the above inquality were true

$$\left|\sum_{n=p}^{q} a_n\right| \geq \frac{(q-p+1)L}{2}.$$

Then let $q = p + 1$, it follows that

$$\left|\sum_{n=p}^{q} a_n\right| \geq L > \varepsilon.$$

A contradiction of inequality (1).

□

3. **(Absolute convergence test)**. Let $\sum_{n=m}^{\infty} a_n$ be a formal series of real numbers. If this series is absolutely convergent, then it is also conditionally convergent. Furthermore, in this case we have the triangle inequality

$$\left|\sum_{n=m}^{\infty} a_n\right| \leq \sum_{n=m}^{\infty} |a_n|.$$

Solution. If $\sum_{n=m}^{\infty} |a_n|$ converges to L, then for any $\varepsilon > 0$ there exists some $M \geq m$ such that for any $p, q \geq M$

$$\left|\sum_{n=p}^{q} |a_n|\right| \leq \varepsilon.$$

Hence,

$$\sum_{n=p}^{q} |a_n| \leq \varepsilon.$$

By triangle inequality over finite series, for any $p, q \geq M$ we have

$$\left| \sum_{n=p}^{q} a_n \right| \leq \sum_{n=p}^{q} |a_n|.$$

We conclude that $\sum_{n=p}^{q} a_n$ converges. Trivially, we also note that again, by triangle inequality over finite series, each partial sum $S_N = \sum_{n=m}^{N} a_n$ is smaller than or equal to $T_N = \sum_{n=m}^{N} |a_n|$, meaning that by comparison principle $\lim_{N\to\infty} S_N \leq \lim_{N\to\infty} T_N$.

$\square$

4. (**Series laws**).

   (a) If $\sum_{n=m}^{\infty} a_n$ is a series of real numbers converging to x, and $\sum_{n=m}^{\infty} b_n$ is a series of real numbers converging to y, then $\sum_{n=m}^{\infty}(a_n + b_n)$ is also a convergent series, and converges to $x + y$.
   In particular, we have

   $$\sum_{n=m}^{\infty} (a_n + b_n) = \sum_{n=m}^{\infty} a_n + \sum_{n=m}^{\infty} b_n.$$

   (b) If $\sum_{n=m}^{\infty} a_n$ is a series of real numbers converging to x, and c is a real number, then $\sum_{n=m}^{\infty}(ca_n)$ is also a convergent series, and converges to cx.
   In particular, we have

   $$\sum_{n=m}^{\infty} (ca_n) = c \sum_{n=m}^{\infty} a_n.$$

   (c) Let $\sum_{n=m}^{\infty} a_n$ be a series of real numbers, and let $k \geq 0$ be an integer. If one of the two series $\sum_{n=m}^{\infty} a_n$ and $\sum_{n=m+k}^{\infty} a_n$ are convergent, then the other one is also, and we have the identity

   $$\sum_{n=m}^{\infty} a_n = \sum_{n=m}^{m+k-1} a_n + \sum_{n=m+k}^{\infty} a_n.$$

   (d) Let $\sum_{n=m}^{\infty} a_n$ be a series of real numbers converging to x, and let k be an integer. Then $\sum_{n=m+k}^{\infty} a_{n-k}$ also converges to x.

5. **(Comparison test.)**

6. **(Geometric series.)**
If $|x| > 1$ then $\lim_{n\to\infty} x^n = \infty$, so $\sum_{n=1}^{\infty} x^n$ does not converge.
If $|x| < 1$ then the sequence converges to 0, so the sum could converge. In particular we know that $\sum_{n=0}^{k} x^n = \frac{1-x^{k+1}}{1-x}$. It follows that
$$\sum_{n=0}^{\infty} x^n = \lim_{k\to\infty} = \frac{1-x^{k+1}}{1-x} = \frac{1}{1-x}$$
.

7. **(Cauchy criterion.)**

8. **(Root test.)**

9. **(Ratio test.)**

10. Let $(a_n)_{n\geq 1}$ be a sequence of positive real numbers such that the series $\sum_{n=1}^{\infty} a_n$ converges. Show that the series
$$\sum_{n=1}^{\infty} a_n^{\frac{n}{n+1}}$$
Also converges.
Solution. Consider the partial sum sequence $(S_N)_{N=m}^{\infty}$ such that $S_n = \sum_{n=m}^{N} a_n^{\frac{n}{n+1}}$. We see that
$$b_n = \frac{n}{n+1} = \frac{(n+1)-1}{n+1} = 1 - \frac{1}{n+1}$$
so that $\lim_{n\to\infty} b_n = 1$. By zero test, since $\sum_{n=1}^{\infty} a_n$ converges, necessarily $\lim_{n\to\infty} a_n = 0$. We therefore can study the behaviour of the sequence
$$\lim_{n\to\infty} a_n^{b_n}.$$
That results into
$$\lim_{n\to\infty} \frac{a_n}{a_n^{\frac{1}{n+1}}}.$$
Note that we cannot use limit laws theorem since we do not know whether $a_n^{\frac{1}{n+1}}$ might converge. Suppose that it converged to $c \in \mathbb{R}$, then
$$\lim_{n\to\infty} a_n^{b_n} = \frac{1}{c} \lim_{n\to\infty} a_n = 0.$$
Then one thing we shall also verify is that it is bounded above by for example $\frac{1}{n(n+1)}$ thus by comparison principle the requested series converge.

Series expansions