Convergence Sheet II

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Problems on convergence of sequences, functions and continuity. The problems are from the books of M. Spivak "Calculus", T. Tao "Analysis I", R. G. Bartle "Introduction to Real Analysis" and Titu Andreescu "Putnam and Beyond".

Suprema and Infima of sequences

1. Let E be a subset of $\mathbb{R}^*$. Then the following statements are true.

   (a) For every $x \in E$ we have $x \leq \sup(E)$ and $x \geq \inf(E)$.

   (b) Suppose that $M \in \mathbb{R}^*$ is an upper bound for E, i.e., $x \leq M$ for all $x \in E$. Then we have $\sup(E) \leq M$.

   (c) Suppose that $M \in \mathbb{R}^*$ is a lower bound for E, i.e., $x \geq M$ for all $x \in E$. Then we have $\inf(E) \geq M$.

   Solution.

2. **(Least upper bound property)**. Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers, and let x be the extended real number

   $$x := \sup(a_n)_{n=m}^{\infty}.$$

   Then we have $a_n \leq x$ for all $n \geq m$. Also, whenever $M \in \mathbb{R}^*$ is an upper bound for a_n (i.e., $a_n \leq M$ for all $n \geq m$), we have $x \leq M$.

   Finally, for every extended real number y for which $y < x$, there exists at least one $n \geq m$ for which

   $$y < a_n \leq x.$$

   Solution. The first two propositions follow directly from the definitions of suprema and infima of sequences and sets of extended real numbers.

   For the final proposition, if $x = +\infty$, and $y \in \mathbb{R}$ then given that $a_n \in \mathbb{R}$, (a_n) is divergent. Therefore, for any K there exists some $n \geq m$ such that $a_n \geq K$. In particular there exists some $n_0 \geq m$ such that $a_n \geq K_0 > y$.

   If $x \in \mathbb{R}$ then $y \in \mathbb{R}$ for any $y < x$. Suppose for the sake of contradiction that there existed at least one $y_0 < x$ such that for any $n \geq m$, $a_n \leq y_0$.

Then immediately we get a contradiction, as y_0 is an upper bound smaller than x.

Note that $x, y = \infty$ but $y < x$ would mean that $\infty < \infty$ which is false.

$\square$

3. (**Monotone bounded sequences converge**). Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers which has some finite upper bound $M \in \mathbb{R}$, and which is also increasing (i.e., $a_{n+1} \geq a_n$ for all $n \geq m$). Then $(a_n)_{n=m}^{\infty}$ is convergent, and in fact

$$\lim_{n\to\infty} a_n = \sup(a_n)_{n=m}^{\infty} \leq M.$$

Solution. Let ε_n be the real number $\sup(a_n) - a_n$. We see that (ε_n) is a decreasing positive sequence, has supremum $\sup(a_n) - a_1$ and infimum $\sup(a_n) - \sup(a_n) = 0$.

Intuitively we see that the change of the first n terms condition the growing of the following $m \geq n$ terms: for instance, suppose $a_2 - a_1 = \frac{M-a_1}{2}$, then for any $n, m \geq 2$, $|a_n - a_m| \leq \frac{M-a_1}{2}$. Finally, note that $|a_n - a_m| = a_{n+k} - a_n$ for some $k \in \mathbb{Z}$.

Formally speaking, we state the following claim:

(1) For any $n, m \geq 1$, we have that $|a_n - a_m| \leq \varepsilon_{\min\{n,m\}}$.

To prove this claim, we have that $|a_n - a_m| = a_{n+k} - a_n \leq \sup(a_n) - a_n = \varepsilon_n$.

Then, let $\varepsilon > 0$ be any real number, we claim that there exists some $N \geq 1$ such that for any $n \geq N$

$$|a_n - \sup(a_n)| \leq \varepsilon.$$

Suppose we had bounded $|a_n - a_m|$ by ε_k with $k = \min\{n, m\}$, due to the increasing behaviour of (a_n), $|a_n - a_{m+1}|$ will be bounded also by $\varepsilon_{k+1} \leq \varepsilon_k$ since

$$|a_n - a_m| = |a_{m+k} - a_m| \leq \sup(a_n) - a_m \implies |a_{m+k} - a_{m+1}| \leq \sup(a_n) - a_{m+1}.$$

In other words for any ε_k suffices letting $N := k$ in order to $|a_n - a_m| \leq \varepsilon_k$ for any $n, m \geq N$. Then, since $\inf(\varepsilon_n) = 0$, by problem (2), for any $\varepsilon > 0$ we can find at least one k such that $\varepsilon > \varepsilon_k \geq 0$, hence this shows that

$$|a_n - a_m| \leq \varepsilon$$

for a given ε and for every $n, m \geq N$, so the sequence converges by Cauchy criterion. Then, by construction

$$|\sup(a_n) - a_n| = \varepsilon_n < \varepsilon.$$

□

4. Explain why Proposition 6.3.10 fails when $x > 1$. In fact, show that the sequence $(x^n)_{n=1}^{\infty}$ diverges when $x > 1$. (Hint: prove by contradiction and use the identity $(1/x)^n x^n = 1$ and the limit laws in Theorem 6.1.19.) Compare this with the argument in Example 1.2.3; can you now explain the flaws in the reasoning in that example?

   Remark. **Proposition 6.3.10.** *Let* $0 < x < 1$. *Then we have* $\lim_{n\to\infty} x^n = 0$.

   Solution. Let $x > 1$. Suppose for the sake of contradiction that $\lim_{n\to\infty} x^n$ converged to $L \in \mathbb{R}$. Consider the sequence $\left(\frac{1}{x}\right)^n$, this sequence converges to 0 by squeeze theorem with $\frac{1}{n}$. Therefore, by limit laws (since both sequences supposedly converge) we should get that

$$\lim_{n\to\infty} \left(\frac{1}{x}\right)^n \cdot x^n = \lim_{n\to\infty} \left(\frac{1}{x}\right)^n \lim_{n\to\infty} x^n = 0 \cdot L = 0.$$

   In spite of this, the limit should be equal to

$$\lim_{n\to\infty} \left(\frac{1}{x^n}\right) \cdot x^n = \lim_{n\to\infty} \frac{x^n}{x^n} = 1.$$

   We conclude that $0 = 1$ which is false. We conclude that x^n does not converge if $x > 1$, note that in this case the test of falseness we used was not the epsilon definition but the limit laws theorem.

   Regarding *Example* 1.2.3, that essentially says that for any $x \in \mathbb{R}$

$$L = \lim_{n\to\infty} x^n = \lim_{m+1\to\infty} x^{m+1} = x \lim_{m+1\to\infty} x^{m+1}$$

   so that by limit laws

$$x \lim_{m+1\to\infty} x^m = xL,$$

   concluding that

$$xL = L$$

   or that $L = 0$ or $x = 1$. the two main flaws are: first, assuming that the sequence converges to a finite limit L for any x (that we have shown it does not) and the second (related to the first), that we can apply limit laws in possibly divergent sequences. The condition of the limit laws theorem to hold is the sequences to converge in first place.

□

5. **(Limits are limit points.)** Let $(a_n)_{n=m}^{\infty}$ be a sequence which converges to a real number c. Then c is a limit point of $(a_n)_{n=m}^{\infty}$, and in fact it is the only limit point of $(a_n)_{n=m}^{\infty}$.

Subsets of the real line

Limits and continuity