

# Linear mappings sheet IV

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*Problems in linear algebra: problems from the books of Titu Andreescu “Essential Linear Algebra” and Sheldon Axler “Linear Algebra Done Right”. The topics of this week are isomorphisms, change of bases, rank of a matrix and dual spaces.*

## Isomorphisms and invertibility

1. Suppose  $V$  is finite-dimensional and  $T \in \mathcal{L}(V)$ . Prove that the following are equivalent:
  - (a)  $T$  is invertible.
  - (b)  $Tv_1, \dots, Tv_n$  is a basis of  $V$  for every basis  $v_1, \dots, v_n$  of  $V$ .
  - (c)  $Tv_1, \dots, Tv_n$  is a basis of  $V$  for some basis  $v_1, \dots, v_n$  of  $V$ .

**Solution.** (a)  $\implies$  (b)

2. Suppose  $V$  is finite-dimensional,  $U$  is a subspace of  $V$ , and  $S \in \mathcal{L}(U, V)$ . Prove that there exists an invertible linear map  $T$  from  $V$  to itself such that  $Tu = Su$  for every  $u \in U$  if and only if  $S$  is injective.

**Solution.** For the direct implication, if  $T$  is invertible, then it is injective, The restriction of  $T$  to a subset  $U$  will be injective, and will behave as expected. The mapping will not be surjective, although this is not an issue.

For the converse implication, if  $S$  is injective,  $U$  and  $\text{Im } S$  are isomorphic. Hence there exists an isomorphism  $S' : U \rightarrow \text{Im } S$ . Build  $T$  such that

$$Tv = \begin{cases} Sv & \text{if } v \in U; \\ S'v & \text{if } v \in V \setminus U. \end{cases}$$

□

**(Idea of) Solution.** For the converse implication, if  $S$  is injective, we first determine the amount of elements in the intersection

$$U \cap \text{Im } S = \begin{cases} \{0\}; \\ \text{not only zero subspace of } U \text{ or } \text{Im } S; \\ \text{a subspace with the same dimension of } U \text{ and } \text{Im } S \end{cases}.$$

The first case implies that any basis of  $U$  will not be able to span any vector of  $\text{Im } S$ , the second case implies that there exists a basis of  $U$  that spans some vectors of  $\text{Im } S$ , and the last one implies that a basis of  $U$  will also be a basis for  $\text{Im } S$ . For the third implication there is the identity map, that will do the job of changing any vector of  $U$  to its corresponding vector of  $\text{Im } S$ , as both spaces have same dimension ( $\ker S = \{0\}$ ).

$$u_1 \mapsto Tu_1, \dots, u_n \mapsto Tu_n.$$

Then, extending this map to  $V$ :

$$v_1 \mapsto Tv_1, \dots, u_1 \mapsto Tu_1, \dots, u_n \mapsto Tu_n, \dots, v_m \mapsto Tv_m.$$

Doing the same reasoning as for the direct implication.

For the second case, let  $\mathcal{B}$  be a basis for  $U$ , and let  $\mathcal{C}$  be a basis for  $\text{Im } S$ . Consider the subspace  $W \subseteq V$ ,  $W = \text{span}(\mathcal{B} \cup \mathcal{C})$ , such that  $W = U \oplus \text{Im } S$ . Let  $\varphi_U : W \rightarrow U$  be defined as a projection mapping, that is, when valued at some  $w \in W$ , will output the only  $u \in U$  such that for some  $x \in \text{Im } S$ :  $x = u - w$ . Define  $\varphi_{\text{Im } S} : W \rightarrow \text{Im } S$  the same way, but interchanging  $x$  with  $u$ . A map  $T : W \rightarrow W$  will be

$$Tw = \begin{cases} \varphi_U(w) & \text{if } w \in U; \\ \varphi_{\text{Im } S}(w) & \text{if } w \in \text{Im } S. \end{cases}$$

For  $w \in U$ ,  $Tw$  will be itself as  $u = x + 0$ . Similarly for  $x \in \text{Im } S$ . Then we can add linearly independent vectors to  $\mathcal{B}$  and  $\mathcal{C}$ .

(Can we finish up this idea? I find it hard to assume both bases will always be linearly independent, besides, the idea turned out too complex.)

3. Suppose that  $W$  is finite-dimensional and  $S, T \in \mathcal{L}(V, W)$ . Prove that  $\text{null } S = \text{null } T$  if and only if there exists an invertible  $E \in \mathcal{L}(W)$  such that  $S = ET$ .

**Solution.** Let  $k$  be the dimension of  $\ker T$ , let  $n$  be the dimension of  $V$  and  $m$  be the dimension of  $W$ . Since  $\ker T = \ker S$  then, by the rank-nullity theorem,  $\dim \text{Im } T = \dim \text{Im } S$ , as the input space is the same for both mappings. Let  $\mathcal{B}$  be a basis of  $V$  such as

$$\{v_1, \dots, v_{n-k}, \dots, v_n\}$$

Where the vectors  $v_{n-k}, \dots, v_n$  are those of a basis of  $\ker T$  and  $\ker S$ . Let  $\mathcal{C}$  and  $\mathcal{D}$  be bases of  $W$  containing the bases of  $\text{Im } T$  and  $\text{Im } S$  respectively

in a way such as  $\{w_1, \dots, w_{n-k}, \dots, w_m\}$ , where  $w_1, \dots, w_{n-k}$  are vectors from  $\text{Im } T$  or  $\text{Im } S$ . Let the matrices  $M(T)$  and  $M(S)$  be associated to  $T$  and  $S$  respectively. Consider  $Id : W_C \rightarrow W_D$ , where  $Id((w_i)_C) = (w_i)_D$  for vectors  $w_i$  of their respective bases; and  $1 \leq i \leq n - k$ . We have the following diagram

$$\begin{array}{ccc} V_B & \xrightarrow{T} & W_C \\ & \searrow S & \uparrow Id_{DC} \\ & & W_D \end{array}$$

We see that  $E = Id_{DC}$ . So  $T = E \circ S$ , thus  $S = E^{-1} \circ T = Id_{CD} \circ T$ . Reciprocally, if there exists an invertible mapping  $E \in \mathcal{L}(W)$  such that  $ET = S$ , the matrix  $M(E)$  will be invertible, then it will be square, and its rank will be  $\dim W$ . A null column  $C_j$  of  $M(S)$  (for some basis of  $W$ ) will be given by

$$C_j = M(E)_{\cdot 1} M(T)_{1j} + \dots + M(E)_{\cdot m} M(T)_{mj} = O_{\cdot j}$$

Where each column of  $M(E)$  is linearly independent (since its rank is  $m$ ). Thus, the entries of the  $j$ th column of  $M(T)$  must be zero, thus, the column itself is zero.

□

4. Suppose that  $V$  is finite-dimensional and  $S, T \in \mathcal{L}(V, W)$ . Prove that  $\text{range } S = \text{range } T$  if and only if there exists an invertible  $E \in \mathcal{L}(V)$  such that  $S = TE$ .

**(Idea of) Solution.** There is, most likely, an analogous reasoning to that of the previous problem, but I will provide a solution using a concept of the book (column rank and column-row factorization) where I got the problem from (Linear Algebra Done Right).

Let  $n$  be the dimension of  $V$ , and let  $m$  be the dimension of  $W$ . Let  $M(T)$  and  $M(S)$  be the matrices associated to each mapping, the chosen bases do not matter yet.

As  $\text{Im } T = \text{Im } S$ , then  $\dim \text{Im } T = \dim \text{Im } S$ , the column rank  $r$  of  $M(T)$  is equal to the column rank of  $M(S)$ . Thus, let  $\{C_{\cdot 1}, \dots, C_{\cdot n}\}$  be each column of  $M(T)$ , and let  $\{D_{\cdot 1}, \dots, D_{\cdot n}\}$  be each column of  $M(S)$ .

From these columns we can derive linearly independent columns  $\{C'_{\cdot 1}, \dots, C'_{\cdot r}\}$  for  $M(T)$  and  $\{D'_{\cdot 1}, \dots, D'_{\cdot r}\}$  for  $M(S)$  (for example, by using the gaussian algorithm in the transpose of  $M(T)$  and  $M(S)$ ). So let  $M(T')$  be that matrix constructed by merging each column  $C'_{\cdot i}$ , for  $1 \leq i \leq r$ :

$$M(T') = \begin{pmatrix} \vdots & \cdots & \vdots \\ C'_{\cdot 1} & \cdots & C'_{\cdot r} \\ \vdots & \cdots & \vdots \end{pmatrix}$$

This matrix has size  $m \times r$ . Similarly for each column  $D'_{\cdot i}$ :

$$M(S') = \begin{pmatrix} \vdots & \cdots & \vdots \\ D'_{\cdot 1} & \cdots & D'_{\cdot r} \\ \vdots & \cdots & \vdots \end{pmatrix}.$$

Note that if  $m > r$ , this would mean that  $S', T' : U \rightarrow W$  are not surjective.

We claim that both are, although, necessarily injective. To prove this claim first note that the dimension of  $U$  is  $\dim \text{Im } T = \dim \text{Im } S$ , so suppose there were at least one (linearly independent) kernel vector of  $T$  in  $T'$ , thus

$$Tv = 0_V \implies T(v_{\mathcal{B}}) = \begin{pmatrix} 0_{11} \\ \vdots \\ 0_{1m} \end{pmatrix}$$

for some basis of  $W$ . It is clearly not linearly independent, so we do not gather as a column for  $M(T')$ . (Does this argument suffices?) An analogous reasoning is used to prove that  $S'$  is injective.

Let  $\mathcal{U} = \{u_1, \dots, u_r\}$  be a basis for  $U$ , then, by the previous claim,  $T'u_1, \dots, T'u_r$  will be a basis for  $\text{Im } T'$  and for  $\text{Im } S'$ . As well as  $S'u_1, \dots, S'u_r$ :  $\mathcal{D}$ .

Consider a basis for  $V$  extending the basis of  $U$ :  $\mathcal{A}$ , and a basis for  $W$  by extending  $\mathcal{B}$  or  $\mathcal{C}$ , we will get matrices for  $T$  as

$$M_{\mathcal{A}\mathcal{B}}(T) \text{ and } M_{\mathcal{A}\mathcal{C}}(T)$$

We can build a basis for  $V$  in which  $M(T')$  is a submatrix of  $M(T)$  and even  $M(S')$  will be a submatrix of  $M(T)$ . The way to make them is to take the union with  $n - r$  kernel basis vectors with  $\mathcal{U}$ , instead of linearly extending them, call this basis  $\mathcal{A}'$ . Matrices will have the following shape:

$$M_{\mathcal{A}'\mathcal{B}}(T) = \begin{pmatrix} M_{\mathcal{U}\mathcal{B}}(T') & O_{n-r} \\ M_{\mathcal{U}\mathcal{B}}(T') & O_{(n-r)(m-r)} \end{pmatrix}$$

We can make this matrix better, as it is injective,  $T'$  will have a square submatrix of size  $r$  contained in the first quadrant of  $M_{\mathcal{A}'\mathcal{B}}(T)$

$$M_{\mathcal{A}'\mathcal{B}}(T) = \begin{pmatrix} M_{\mathcal{U}\mathcal{B}'}(T') & O_{n-r} \\ O_{m-r} & O_{(n-r)(m-r)} \end{pmatrix}$$

Thus:

$$M_{\mathcal{A}'\mathcal{B}}(T) = \begin{pmatrix} M_{\mathcal{U}\mathcal{B}'}(T') \\ O_{m-r} \end{pmatrix} I_{\max n, m} = M(T') I_{\max n, m}$$

Then:  $M(S') = M(T')P$ . Moreover, by column-row factorization theorem

$$M(T) = M(T')R_T \text{ and } M(S) = M(S')R_S$$

Where  $R_T$  and  $R_S$  are  $r \times n$  matrices. Therefore:

$$M(S) = M(S')R_S = M(T)P$$

(Can we end up this idea? This will anyways end up in a change of basis. If there is a diverse method let me know.)

**Solution.**

□

5. Suppose  $V$  and  $W$  are finite-dimensional and  $S, T \in \mathcal{L}(V, W)$ . Prove that there exist invertible  $E_1 \in \mathcal{L}(V)$  and  $E_2 \in \mathcal{L}(W)$  such that  $S = E_2 T E_1$  if and only if  $\dim \text{null } S = \dim \text{null } T$ .

**Solution.** For the converse, we note that  $\ker T$  is isomorphic to  $\ker S$ . And by rank-nullity theorem,  $\text{Im } T$  is isomorphic to  $\text{Im } S$  as well. Moreover  $V \setminus \ker T$  is isomorphic to  $V \setminus \ker S$ . So the mapping  $\text{id} : V_{\mathcal{B}} \rightarrow V_{\mathcal{C}}$  can be decomposed as  $\text{id}$

6. Suppose  $V$  is finite-dimensional and  $T : V \rightarrow W$  is a surjective linear map of  $V$  onto  $W$ . Prove that there is a subspace  $U$  of  $V$  such that  $T|_U$  is an isomorphism of  $U$  onto  $W$ .

**Solution.** Let  $n = \dim V$ ,  $m = \dim W$ . Since  $T$  is surjective, the dimension of  $W$  will be at most that of  $V$ . If this is the case take  $U$  as the whole  $V$ .

If the dimension of  $W$  is strictly less than that of  $V$ , by the rank-nullity theorem, there are  $n - m$  kernel vectors. Take  $\mathcal{B}$ , a basis for  $V$  containing a basis  $\mathcal{K}$  of  $\ker T$ , this basis can be constructed starting with  $\mathcal{K}$  and linearly extending it to  $\mathcal{B}$ . The subspace  $U$  will be constructed as  $\text{span}(\mathcal{B} \setminus \mathcal{K})$ , and for any vector  $u \in U$ ,  $Tu \neq 0_W$ . The dimension of  $U$  will be  $n - (n - m) = m$ , meaning that  $U$  and  $W$  are isomorphic.

Lastly, recall that  $\dim \text{Im } T = \dim W$ , then  $\dim \text{Im } T = \dim U$ . Let  $T|_U : U \rightarrow W$  be  $T$  restricted to  $U$ , since  $Tu \neq 0_W$  for any  $u \in U$ , this mapping is injective. Therefore, by rank-nullity, again,

$$\dim \operatorname{Im} T|_U = \dim U = \dim \operatorname{Im} T = \dim W.$$

Thus,  $T|_U$  is surjective, and since it is injective, will also make an isomorphism between  $U$  and  $W$ .

□

7. Suppose  $V$  and  $W$  are finite-dimensional and  $U$  is a subspace of  $V$ . Let

$$\mathcal{E} = \{T \in \mathcal{L}(V, W) : U \subseteq \operatorname{null} T\}.$$

- (a) Show that  $\mathcal{E}$  is a subspace of  $\mathcal{L}(V, W)$ .
- (b) Find a formula for  $\dim \mathcal{E}$  in terms of  $\dim V$ ,  $\dim W$ , and  $\dim U$ .

**Solution.** (a) This part is straightforward, take  $0_{\mathcal{E}}$  as the null mapping  $T = O$ . The sum of mappings restricted to some  $U$  containing all vectors of their kernel will also have the vectors of  $U$  as its kernel. And the same goes for multiplication by a scalar.

(b) Let  $n = \dim V$ ,  $m = \dim W$ . Since  $\mathcal{L}(V, W)$  is isomorphic to  $\mathbb{F}^{n,m}$  by the matrix mapping  $M$ , and  $\mathcal{E}$  is a subspace of it, we will have an isomorphism to some subspace  $\mathbb{F}^{p,q}$  with dimension  $p \times q$  of  $\mathbb{F}^{n,m}$ :  $M : \mathcal{E} \rightarrow \mathbb{F}^{p,q}$  (note it is the same mapping  $M$ ).

Let  $T \in \mathcal{E}$ , its associated matrix  $M(T)$  will have at least one null column for some basis  $\mathcal{B}$  of  $V$  (it is not immediately obvious that a null column will appear for any basis of  $V$ , but it is certain that there exists, for any mapping  $T \in \mathcal{E}$ , at least one since  $\dim \operatorname{Im} T = \operatorname{rank} M(T) \leq \dim U$ ). So the dimension of  $\mathcal{E}$  will be given from those matrices that have  $\dim U$  null columns for some basis  $\mathcal{B}$ . Therefore:  $\dim \mathcal{E} = (\dim(V) - \dim(U)) \dim W$ .

□

8. Suppose  $V$  is finite-dimensional and  $S, T \in \mathcal{L}(V)$ . Prove that

$$ST \text{ is invertible} \iff S \text{ and } T \text{ are invertible.}$$

**First Solution.** This first solution involves determinants and it is straightforward, I will provide a second solution, as the author supposedly intended a solution without involving them.

Since  $ST$  is invertible,  $\det(M(S)M(T)) \neq 0$ , thus  $\det(M(S)) \det(M(T)) \neq 0$ , this implies that  $\det(M(S)) \neq 0$  and  $\det(M(T)) \neq 0$ . This proves that  $S$  and  $T$  are invertible.

Conversely, if  $S$  and  $T$  are invertible,  $\det(M(S)) \neq 0$ ,  $\det(M(T)) \neq 0$ , thus  $\det(M(ST)) = \det(M(S)M(T)) = \det(M(S)) \det(M(T)) \neq 0$ , this means that  $ST$  is invertible.

**Second Solution.**

9. Suppose  $V$  is finite-dimensional and  $S, T, U \in \mathcal{L}(V)$  and  $STU = I$ . Show that  $T$  is invertible and that  $T^{-1} = US$ .
10. Show that the result in Exercise 12 can fail without the hypothesis that  $V$  is finite-dimensional.
11. Prove or give a counterexample: If  $V$  is a finite-dimensional vector space and  $R, S, T \in \mathcal{L}(V)$  are such that  $RST$  is surjective, then  $S$  is injective.
12. Suppose  $T \in \mathcal{L}(V)$  and  $v_1, \dots, v_m$  is a list in  $V$  such that  $Tv_1, \dots, Tv_m$  spans  $V$ . Prove that  $v_1, \dots, v_m$  spans  $V$ .
13. Prove that every linear map from  $\mathbb{F}^{n,1}$  to  $\mathbb{F}^{m,1}$  is given by a matrix multiplication. In other words, prove that if  $T \in \mathcal{L}(\mathbb{F}^{n,1}, \mathbb{F}^{m,1})$ , then there exists an  $m \times n$  matrix  $A$  such that  $Tx = Ax$  for every  $x \in \mathbb{F}^{n,1}$ .
14. Suppose  $V$  is finite-dimensional and  $S \in \mathcal{L}(V)$ . Define  $\mathcal{A} \in \mathcal{L}(\mathcal{L}(V))$  by  $\mathcal{A}(T) = ST$  for  $T \in \mathcal{L}(V)$ .
  - (a) Show that  $\dim \text{null } \mathcal{A} = (\dim V)(\dim \text{null } S)$ .
  - (b) Show that  $\dim \text{range } \mathcal{A} = (\dim V)(\dim \text{range } S)$ .
15. Show that  $V$  and  $\mathcal{L}(\mathbb{F}, V)$  are isomorphic vector spaces.
16. Suppose  $V$  is finite-dimensional and  $T \in \mathcal{L}(V)$ . Prove that  $T$  has the same matrix with respect to every basis of  $V$  if and only if  $T$  is a scalar multiple of the identity operator.
17. Suppose  $q \in \mathcal{P}(\mathbb{R})$ . Prove that there exists a polynomial  $p \in \mathcal{P}(\mathbb{R})$  such that
 
$$q(x) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$$
 for all  $x \in \mathbb{R}$ .
18. Suppose  $T \in \mathcal{L}(V)$  and  $v_1, \dots, v_n$  is a basis of  $V$ . Prove that
 
$$\mathcal{M}(T, (v_1, \dots, v_n)) \text{ is invertible} \iff T \text{ is invertible.}$$
19. Suppose that  $u_1, \dots, u_n$  and  $v_1, \dots, v_n$  are bases of  $V$ . Let  $T \in \mathcal{L}(V)$  be such that  $Tv_k = u_k$  for each  $k = 1, \dots, n$ . Prove that
 
$$\mathcal{M}(T, (v_1, \dots, v_n)) = \mathcal{M}(I, (u_1, \dots, u_n), (v_1, \dots, v_n)).$$
20. Suppose  $A$  and  $B$  are square matrices of the same size and  $AB = I$ . Prove that  $BA = I$ .
21. Let  $V$  be a vector space over a field  $\mathbb{F}$  of dimension  $n$ . Let  $T: V \rightarrow V$  be a projection (recall that this is a linear map such that  $T \circ T = T$ ).
  - (a) Prove that  $V = \ker(T) \oplus \text{Im}(T)$ .

(b) Prove that there is a basis of  $V$  in which the matrix of  $T$  is

$$\begin{pmatrix} I_i & 0 \\ 0 & O_{n-i} \end{pmatrix}$$

for some  $i \in \{0, 1, \dots, n\}$ .

**Solution.** (a) First we prove that  $\ker T \cap \operatorname{Im} T = \{0\}$ . Suppose that there exists some linearly independent vector  $v$  of  $V$  such that  $Tv = 0$  and  $v = Tu$  for some  $u \in V$ . Then

$$Tv = (T \circ T)u = Tu = 0.$$

Then  $u \in \ker T$ , and  $Tu \in \ker T$  as well. So we conclude that  $v = Tu = 0$ ; a contradiction, since  $v$  is linearly independent.

The second step is to prove that there exists a unique decomposition of  $v$  as a sum of vectors of  $\ker T$  and  $\operatorname{Im} T$ . For the existence part, since we have already proved that  $\ker T \cap \operatorname{Im} T = \{0\}$ , suffices to notice that the intersection between their bases is null, so the union of them will form a basis for  $V$ , by the rank-nullity theorem.

For the uniqueness part, suppose that for some  $v \in V$ ,  $v = k + Tu$  and  $v = k' + Tu'$ , for  $k, k' \in \ker T$  and  $Tu, Tu' \in \operatorname{Im} T$ . Thus

$$k + Tu = k' + Tu'$$

Taking  $T$  in both sides (both terms are in  $V$ , so it is OK)

$$T(k + Tu) = T(k' + Tu')$$

$$(T \circ T)u = (T \circ T)u'$$

$$Tu = Tu'.$$

So these both vectors of  $\operatorname{Im} T$  are the same ones. We then conclude, from the first equation that  $k = k'$ .

(b) This matrix is constructed the same way to that of problem 2 of section 2 of “Kernel, range and matrices sheet”, where  $i$  depends on the dimension of the image of  $T$ .

□

22. Let  $V$  be a vector space over  $\mathbb{C}$  or  $\mathbb{R}$  of dimension  $n$ . Let  $T: V \rightarrow V$  be a symmetry (that is, a linear transformation such that  $T \circ T = \operatorname{id}$  is the identity map of  $V$ ).

(a) Prove that  $V = \ker(T - \operatorname{id}) \oplus \ker(T + \operatorname{id})$ .



- (b) Deduce that there exists  $i \in [0, n]$  and a basis of  $V$  such that the matrix of  $T$  with respect to this basis is

$$\begin{pmatrix} I_i & 0 \\ 0 & -I_{n-i} \end{pmatrix}.$$

**First Solution.** This solution uses eigenspaces, but I do not think the author intended their use yet, so I will provide another solution without involving them.

The following lemma will be helpful to prove the desired results.

**Lemma 1.** *The matrix of a symmetric operator  $T : V \rightarrow V$  over a finite dimensional  $\mathbb{C}$ -vector space with some basis  $\mathcal{B}$  is a symmetric matrix.*

**Proof.** Let  $\mathcal{B}$  be a basis for  $V$ , since  $T \circ T = \text{id}$ , the equation  $M_{\mathcal{B}}(T)M_{\mathcal{B}}(T) = I_n$  will hold. Let  $a_{ij}$  be the  $ij$ -th entry of  $M(T)$ , then

$$M(T)_{ij}^2 = \sum_{k=1}^n a_{ik}a_{kj} = \delta_{ij}$$

We will not take into account the diagonal entries, as they do not guarantee anything about the symmetry of  $M$ . Although they must be nonzero, since otherwise the matrix would not be invertible.

Let  $i \neq j$ , then

$$M(T)_{ij}^2 = \sum_{k=1}^n a_{ik}a_{kj} = 0 \text{ and } M(T)_{ji}^2 = \sum_{k=1}^n a_{jk}a_{ki} = 0.$$

Therefore

$$\sum_{k=1}^n a_{ik}a_{kj} = \sum_{k=1}^n a_{jk}a_{ki}$$

we see that symmetry property holds, as  $(i, j) \mapsto (j, i)$  implies that the sum equals. Furthermore, if  $k = i$

$$\sum_{1 \leq k < i} a_{ik}a_{kj} + a_{ii}a_{ij} + \sum_{i < k \leq n} a_{ik}a_{kj} = \sum_{1 \leq k < i} a_{jk}a_{ki} + a_{ji}a_{ii} + \sum_{i < k \leq n} a_{jk}a_{ki}.$$

So subtracting each side must also be zero

$$\sum_{1 \leq k < i} a_{ik}a_{kj} + a_{ii}a_{ij} + \sum_{i < k \leq n} a_{ik}a_{kj} - \sum_{1 \leq k < i} a_{jk}a_{ki} - a_{ji}a_{ii} - \sum_{i < k \leq n} a_{jk}a_{ki} = 0.$$

If we suppose  $a_{ii}a_{ij} \neq a_{ji}a_{ii}$ , then  $a_{ii}(a_{ij} - a_{ji}) \neq 0$ , so  $a_{ii} \neq 0$  and  $a_{ij} - a_{ji} \neq 0$ , so  $a_{ij} \neq a_{ji}$ . As  $i$  and  $j$  were chosen arbitrarily we conclude in this paragraph that under the former supposition, for any  $i \neq j$ :  $a_{ij} \neq a_{ji}$ . Then it follows that

$$\sum_{k=1}^n a_{ik}a_{kj} \neq \sum_{k=1}^n a_{jk}a_{ki},$$

as each for each term,  $a_{ik}a_{kj} \neq a_{ki}a_{jk}$ . A contradiction of the equation stated at the beginning. ■

Since the matrix of a symmetry is symmetric, it is diagonalizable, so there exists a set  $A = \{\lambda_1, \dots, \lambda_n\}$  of eigenvalues of  $T$ . Furthermore, there exists an orthonormal basis  $\mathcal{C}$  of  $V$ , that we can build by finding each eigenvector in  $\ker(T - \lambda_i \text{id})$  for  $1 \leq i \leq n$ . Thus  $Tv_i = \lambda_i v_i$ , so

$$v_i = \lambda_i T v_i \quad \text{by using symmetry property,}$$

but also

$$v_i = \frac{T v_i}{\lambda_i}.$$

Then

$$\lambda_i T v_i - \frac{T v_i}{\lambda_i} = T v_i \left( \lambda_i - \frac{1}{\lambda_i} \right) = T v_i \left( \frac{\lambda_i^2 - 1}{\lambda_i} \right) = 0$$

We conclude seeing that

$$T v_i (\lambda_i^2 - 1) = T v_i (\lambda_i + 1)(\lambda_i - 1) = 0.$$

So  $\lambda_i = 1$  or  $\lambda_i = -1$  for any  $1 \leq i \leq n$ , so  $m(1) = i$  and  $m(-1) = n - i$ . Then it follows that  $\mathcal{C} = \{v_1, \dots, v_i, -v_{i+1}, \dots, -v_n\}$ , so the symmetry matrix is diagonal with each entry being 1 or  $-1$ . We also see that  $\mathcal{C} = \ker(T - \text{id}) \oplus \ker(T + \text{id})$ . □

**Second Solution.** (a) Let  $P := \frac{1}{2}(\text{id} - T)$ , and  $Q := \frac{1}{2}(\text{id} + T)$ . These two mappings are projections from  $V$  to  $V$ . To prove this, using the fact that  $\mathcal{L}(V)$  is a vector space itself

$$\begin{aligned} P^2 &= \frac{1}{2}(\text{id} - T) \frac{1}{2}(\text{id} - T) = \frac{1}{4}(\text{id} - T)(\text{id} - T) \\ &= \frac{1}{4}(\text{id} - T)(\text{id} - T) = \frac{1}{4}(\text{id} - T - T + T^2) \end{aligned}$$

$$= \frac{1}{4}(\text{id} - T - T + T^2) = \frac{1}{4}(2\text{id} - 2T) = P$$

Similar procedure for  $Q$ :

$$Q^2 = \frac{1}{4}(\text{id} + T)(\text{id} + T) = \frac{1}{4}(\text{id} + 2T + T^2) = \frac{1}{4}(2\text{id} + 2T) = Q.$$

The mapping  $P + Q$  is the identity map, and their composition  $P \circ Q = Q \circ P$  is the null map. We then claim that for any vector  $v \in V$ ,  $v = Pu + Qw$ , and furthermore, that this decomposition is unique (meaning that  $V = \text{Im } P \oplus \text{Im } Q$ ).

To prove this claim, note that the first condition is obvious since  $P + Q = \text{id}$ , so remains showing that  $\text{Im } P \cap \text{Im } Q = \{0_V\}$ . Assume that there exists a vector  $v \in V$  belonging to both  $\text{Im } P$  and  $\text{Im } Q$ , thus,

$$v = P(u), \text{ and } v = Q(w).$$

$$\implies P(u) = Q(w)$$

$$\implies P(u) = (P \circ Q)(w) = 0_V$$

and similarly

$$Q(u) = (P \circ Q)(w) = 0_V$$

So  $v = 0_V$ . This proves our claim. Using the last problem we also know that  $V = \ker P \oplus \text{Im } P$  and  $V = \ker Q \oplus \text{Im } Q$ . So we get three different expressions for  $V$  counting also that of  $V = \text{Im } P \oplus \text{Im } Q$ . Without loss of generality assume now that  $\ker P = \text{Im } Q$ , and  $\ker Q = \text{Im } P$ . This means that  $V = \ker Q \oplus \ker P$  as well. Let  $v \in \ker P$ , thus

$$Pv = \frac{1}{2}(v - Tv) = 0$$

$$\iff 0 = Tv - v$$

so, this shows the equivalence between  $\ker P$  and  $\ker(T - \text{id})$ . Doing the same for  $Q$ , we get that:

$$Qv = 0 \iff v + Tv = 0 \iff Tv + v = 0.$$

So we get that  $V = \ker P \oplus \ker Q \iff V = \ker(T - \text{id}) \oplus \ker(T + \text{id})$ .

(b) As  $T \circ T = \text{id}$ , we get that  $(M_{\mathcal{B}\mathcal{B}}(T))^2 = I_n$  for a basis  $\mathcal{B}$  of  $V$ . Then,  $M_{\mathcal{B}\mathcal{B}}(T) = M_{\mathcal{B}\mathcal{B}}(T)^{-1}$ , this reduces the threshold of matrices as it only can be diagonal.

Now, we would like to have that a basis for  $V$  were the union of the bases of  $\ker(T - \text{id})$  and  $\ker(T + \text{id})$ . Consider the mapping  $T - \text{id}$ , this mapping will be zero if and only if  $Tv = v$ , so any vector of the basis of  $\ker(T - \text{id})$  will satisfy that  $Tv = v$ , similarly with  $T + \text{id}$ , we will get that vectors of the basis of  $\ker(T + \text{id})$  are those in which  $Tv = -v$ .

Thus, let  $\mathcal{B} = \{v_1, \dots, v_i\}$  be a basis for  $\ker(T - \text{id})$ , and  $\mathcal{C} = \{v_{i+1}, \dots, v_n\}$  be a basis for  $\ker(T + \text{id})$ . A basis for  $V$  will be  $\mathcal{D} = \mathcal{B} \cup \mathcal{C}$ , and a basis for the arrival  $V$  will be  $\mathcal{D}$  as well. This builds up the desired matrix.

□

23. Let  $V$  be the vector space of polynomials with complex coefficients whose degree does not exceed 3. Let  $T : V \rightarrow V$  be the map defined by

$$T(P) = P + P'.$$

Prove that  $T$  is linear and find the matrix of  $T$  with respect to the basis  $1, X, X^2, X^3$  of  $V$ .

**Solution.** To prove this mapping is linear, let  $c \in \mathbb{R}$  and let  $P, Q$  be polynomials with complex coefficients whose degree does not exceed 3:

$$\begin{aligned} T(P + cQ) &= (P + cQ) + (P + cQ)' = P + cQ + P' + cQ' = \\ &P + P' + cQ + cQ' = T(P) + cT(Q) \end{aligned}$$

And note that if  $P = c$ , constant polynomial,  $P' = 0$ , in particular with  $c = 0$ . The matrix for  $T$  will be:

$$M(T) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

□

24. (a) Find the matrix with respect to the canonical basis of the map which projects a vector  $v \in \mathbb{R}^3$  to the  $xy$ -plane.  
 (b) Find the matrix with respect to the canonical basis of the map which sends a vector  $v \in \mathbb{R}^3$  to its reflection with respect to the  $xy$ -plane.  
 (c) Let  $\theta \in \mathbb{R}$ . Find the matrix with respect to the canonical basis of the map which sends a vector  $v \in \mathbb{R}^2$  to its rotation through an angle  $\theta$ , counterclockwise.
25. Let  $V$  be a vector space of dimension  $n$  over  $F$ . A *flag* in  $V$  is a family of subspaces

$$V_0 \subset V_1 \subset \dots \subset V_n$$

such that  $\dim V_i = i$  for all  $i \in [0, n]$ . Let  $T : V \rightarrow V$  be a linear transformation. Prove that the following statements are equivalent:

- (a) There is a flag  $V_0 \subset \cdots \subset V_n$  in  $V$  such that  $T(V_i) \subset V_i$  for all  $i \in [0, n]$ .
- (b) There is a basis of  $V$  with respect to which the matrix of  $T$  is upper-triangular.

**Solution.** Since  $V_0 \subset V_1 \subset \cdots \subset V_n$ , we can find a basis for any  $V_k$  by extending one from  $V_{k-1}$ . Call  $\mathcal{B}_k$  a basis for  $V_k$  that is (recursively) extended from  $\mathcal{B}_{k-1}$ .

For the direct implication, let us start with some fixed  $k$ . Since  $T(V_k) \subset V_k$ , there exist at most  $k - 1$  basis vectors from  $\mathcal{B}_k$  that form a basis for  $T(V_k)$ .

Among these  $k - 1$  vectors it can occur that each one of these are of  $\mathcal{B}_{k-1}$ , or that within these, there is the only vector  $v_k \in \mathcal{B}_k \setminus \mathcal{B}_{k-1}$ , and the other  $k - 2$  are within  $\mathcal{B}_{k-1}$ .

For the first case, for any  $k$ , we can choose as a basis for  $T(V_k)$  exactly  $\mathcal{B}_{k-1}$ .

Clearly  $T(V_{k-1})$  is a subspace of  $T(V_k)$  since  $\mathcal{B}_{k-1}$  is gotten by linearly extending  $\mathcal{B}_{k-2}$ . It follows that  $T(V_{k-1}) \subset T(V_k)$  for any  $k \geq 1$ .

Let  $v_k \in \mathcal{B}_k \setminus \mathcal{B}_{k-1}$  for any  $1 \leq k \leq n$ , this vector will not be in any basis of  $T(V_j)$  for  $1 \leq j < k$  by construction, but will be a basis vector only for  $V_k$ . Its image  $Tv_k \in T(V_k)$  will result in the following column vector:

$$\begin{pmatrix} a_{1k} \\ \vdots \\ a_{(k-1)k} \\ 0_{kk} \end{pmatrix}$$

As it is spanned by  $k - 1$  basis vectors from  $\mathcal{B}_{k-1}$ .

So, gathering the  $k$  basis vectors of  $V$  the matrix is constructed, note that its principal diagonal is zero, but it is OK since for being upper triangular this does not matter.

For the converse implication let  $A \in M_n(\mathbb{F})$  be an upper triangular matrix. We see that the rank  $m$  of  $A$  is  $n - 1 \leq m \leq n$ . Let  $\mathcal{B} = \{v_1, \dots, v_n\}$  be a basis for  $V$ .

Consider the subspaces  $V_k := \langle v_1, \dots, v_k \rangle$  (with  $V_0 = \langle 0 \rangle$ ). By induction on  $k$  we will prove that  $T(V_k) \subset V_k$ . The intuition is that the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{pmatrix}$$

since input and output bases are the same, represents that the mapping of, say  $v_k$  spans itself “plus” the  $k-1$  earlier ones. We will see that considering subspaces while cumulating basis vectors of  $V$ :  $\langle v_1 \rangle$ ,  $\langle v_1, v_2 \rangle$ , and so on; the dimension of their image will be less than or equal themselves (depending mainly on the zeroes in the diagonal).

When  $k = 1$  we have that for  $V_1 = \langle v_1 \rangle$ ,  $T(V_1) = \lambda a_{11}v_1 \in \langle v_1 \rangle$ .

When  $k = 2$  we have that for  $V_2 = \langle v_1, v_2 \rangle$ ,  $T(V_2) = \langle T(v_1), T(v_2) \rangle = \lambda a_{11}v_1 + \delta(a_{12}v_1 + a_{22}v_2) = b_1v_1 + b_2v_2 \in V_2$ .

For the inductive thesis, we will show that for  $V_{k+1} := \langle v_1, \dots, v_k, v_{k+1} \rangle$  it follows that  $T(V_{k+1}) \subseteq V_{k+1}$ . Since

$$\begin{aligned} T(V_{k+1}) &= \langle T(v_1), \dots, T(v_k), T(v_{k+1}) \rangle = T(V_k) + \langle T(v_{k+1}) \rangle = \\ &= T(V_k) + \lambda \sum_{i=1}^{k+1} a_{i(k+1)}v_i, \end{aligned}$$

then

$$\begin{aligned} T(V_{k+1}) &= \sum_{i=1}^k b_i v_i + \lambda \left( \sum_{i=1}^{k+1} a_{i(k+1)} v_i \right) \\ &= \sum_{i=1}^k b_i v_i + \left( \sum_{i=1}^{k+1} c_i v_i \right) = \sum_{i=1}^{k+1} d_i v_i \in \langle v_1, \dots, v_{k+1} \rangle = V_{k+1}. \end{aligned}$$

□

26. Let  $V$  be a vector space over a field  $F$ , and let  $T_1, \dots, T_n : V \rightarrow V$  be linear transformations. Prove that

$$\bigcap_{i=1}^n \ker(T_i) \subseteq \ker \left( \sum_{i=1}^n T_i \right).$$

**(Idea of) Solution.** Using *Grassmann formula* and rank-nullity theorem we get some interesting inequalities. By arguing by induction on  $n$ , for the base case  $n = 1$  we get the trivial inclusion  $\ker(T_1) \subseteq \ker(T_1)$ . Now, suppose that for any  $k \geq 1$  the inclusion

$$\bigcap_{i=1}^k \ker(T_i) \subseteq \ker \left( \sum_{i=1}^k T_i \right).$$

holds. In the right hand side we note that despite having a sum in it, the dimension of  $V$  will be invariant anyways, this means that

$$\dim V = \dim \ker T_i + \dim \operatorname{Im} T_i = \dim \ker \left( \sum_{i=1}^k T_i \right) + \dim \operatorname{Im} \left( \sum_{i=1}^k T_i \right).$$

Call  $S$  the sum  $\sum_{i=1}^k T_i$ . Note that

$$\dim \ker S + \dim \operatorname{Im} S = \dim \ker T_i + \dim \operatorname{Im} T_i \geq 0$$

Thus:

$$\dim \ker S - \dim \ker T_i = \dim \operatorname{Im} T_i - \dim \operatorname{Im} S \quad (*)$$

Then, using Grassmann formula:

$$\dim(\ker T_i + \ker S) = \dim \ker T_i + \dim \ker S - \dim(\ker T_i \cap \ker S) \geq 0.$$

Which implies that

$$\dim \ker T_i + \dim \ker S \geq \dim(\ker T_i \cap \ker S) \quad (**).$$

Summing  $(*)$  and  $(**)$

$$\dim \ker S \geq \dim \ker T_i + (\dim \operatorname{Im} T_i - \dim \operatorname{Im} S).$$

Then:

$$\dim \ker S \geq \dim(\ker T_i \cap \ker S) + (\dim \operatorname{Im} T_i - \dim \operatorname{Im} S).$$

This, although, does still not prove the desired result, but holds for any  $k \geq 1$ . It could occur that the right hand side were negative. Despite this, by our inductive hypothesis, we can say that  $\dim \ker S \geq \dim(\bigcap_{i=1}^k \ker T_i)$ . For the inductive thesis, when  $n = k + 1$ , we shall prove

$$\bigcap_{i=1}^{k+1} \ker(T_i) \subseteq \ker(S).$$

(This idea is interesting, but I do not think that it can be pushed to get the desired result, although it may be true that the result holds for  $n = k + 1$ , –that  $\dim \ker S \leq \dim \ker(\bigcap_{i=1}^{k+1} \ker(T_i))$ – this does not tell us anything about the inclusion of the subspaces, we can still have them disjoint except for zero.)

**Solution.** Let  $v \in \bigcap_{i=1}^n \ker(T_i)$ , we claim that this vector is also in  $\ker\left(\sum_{i=1}^n T_i\right)$ .

To prove this claim first note that any vector  $v' \in \ker\left(\sum_{i=1}^n T_i\right)$  will satisfy:

$$T_1 v' + \cdots + T_n v' = 0_V.$$

In particular, since  $v$  is in each  $\ker T_i$ ,

$$T_1v + \cdots + T_nv = 0_V, \text{ so } v \in \ker \left( \sum_{i=1}^n T_i \right).$$

□

27. Let  $V$  be a vector space over a field  $F$ , and let  $T_1, T_2 : V \rightarrow V$  be linear transformations such that

$$T_1 \circ T_2 = T_1 \quad \text{and} \quad T_2 \circ T_1 = T_2.$$

Prove that

$$\ker(T_1) = \ker(T_2).$$

**Solution.** Let  $v \in \ker T_1$ , we note that

$$T_1v = 0 \iff T_2(T_1v) = T_2v = 0. \text{ (since any linear mapping applied to 0 is also 0)}$$

This implies that  $v \in \ker T_1 \iff v \in \ker T_2$ . Similarly, let  $u \in \ker T_2$

$$T_2u = 0 \iff T_1(T_2v) = T_1u = 0.$$

So we conclude that  $\ker T_1 = \ker T_2$ .

□

28. Let  $V$  be a vector space over  $F$ , and let  $T : V \rightarrow V$  be a linear transformation such that

$$\ker(T) = \ker(T^2) \quad \text{and} \quad \text{Im}(T) = \text{Im}(T^2).$$

Prove that

$$V = \ker(T) \oplus \text{Im}(T).$$

**Solution.** We claim that  $T$  is necessarily a projection. To prove this, assume that  $T$  were not a projection. Take  $v \in V, v \neq 0$ , then  $Tv \neq T^2v$ , let  $u \in V$  such that  $u = T^2v$ .

As  $u \in \text{Im } T^2$ , then  $u \in \text{Im } T$ , this means that  $u = Tv'$  for some  $v' \in V$ , suppose that  $v' \neq v$ . Then it must hold that  $Tv' = T^2v$ , so  $T$  is not injective.

Let  $k \in \ker T$ , such that  $k \neq 0$ , then  $k \in \ker T^2$ , thus  $Tk = T^2k = 0$ , which is a contradiction.

Then, the result yields by problem 21.

□



29. Let  $V$  be a finite dimensional vector space. Let  $T : V \rightarrow V$  be a linear operator, and let  $T^n : V \rightarrow V$  denote  $T$  applied  $n$  times. Prove that there exists an integer  $N$  such that

$$V = \ker T^N \oplus \operatorname{Im} T^N.$$

**Solution.** First note that the kernel of a mapping is stable under  $T$ , only possibly increasing its dimension when applying  $T$  again. If it continues increasing when applying  $T$ , the resulting mapping will be the null mapping for some  $N \geq 2$ , which is a projection, leading to the result immediately.

Else, if the dimension of both  $\operatorname{Im} T$  and  $\ker T$  become stable, we claim that we would get, starting from a certain integer  $j$

$$\ker T^j = \ker T^{j+1} \text{ and } \operatorname{Im} T^j = \operatorname{Im} T^{j+1}.$$

Which would imply that  $T$  becomes a projection starting from  $j$  by the last problem, which also leads to the result immediately. To prove this claim, note that  $\ker T$  is actually stable under  $T$ , this means that for any  $v \in \ker T$

$$v \in \ker T \implies v \in \ker T^2 \implies \dots \implies v \in \ker T^j$$

in particular, when applying  $T$   $j$  times to a basis  $\mathcal{K}$  of  $\ker T$ , every vector of it will be basis vectors of  $T^{j+1}$ . Now as we assumed that  $\dim \ker T^j = \dim \ker T^{j+1}$ ,  $\mathcal{K}$  forms a basis for  $\ker T^j$  and  $\ker T^{j+1}$ , which means that  $\ker T^j = \ker T^{j+1}$ . In the case of  $\operatorname{Im} T$ , we note that  $V \setminus \ker T^j = V \setminus \ker T^{j+1} = \dots = V \setminus \ker T^{j+n}$ , hence,  $\operatorname{Im} T$  will also become stable starting from  $j$ . This proves our claim.

□

## Rank of a matrix

1. Let  $A, B \in M_3(F)$  be two matrices such that  $AB = O_3$ . Prove that

$$\min(\operatorname{rank}(A), \operatorname{rank}(B)) \leq 1.$$

2. Let  $A \in M_3(\mathbb{C})$  be a matrix such that  $A^2 = O_3$ .

- (a) Prove that  $A$  has rank 0 or 1.
- (b) Deduce the general form of all matrices  $A \in M_3(\mathbb{C})$  such that  $A^2 = O_3$ .

3. Find the rank of the matrix  $A = [\cos(i - j)]_{1 \leq i, j \leq n}$ .

4. (a) Let  $V$  be an  $n$ -dimensional vector space over  $F$ , and let  $T : V \rightarrow V$  be a linear transformation. Let  $T^j$  be the  $j$ -fold iterate of  $T$  (so  $T^2 = T \circ T$ ,  $T^3 = T \circ T \circ T$ , etc.). Prove that:

$$\text{Im}(T^n) = \text{Im}(T^{n+1}).$$

*Hint:* Check that if  $\text{Im}(T^j) = \text{Im}(T^{j+1})$  for some  $j$ , then  $\text{Im}(T^k) = \text{Im}(T^{k+1})$  for  $k \geq j$ .

- (b) Let  $A \in M_n(\mathbb{C})$  be a matrix. Prove that  $A^n$  and  $A^{n+1}$  have the same rank.

5. Let  $A \in M_n(F)$  be a matrix of rank 1. Prove that:

$$A^2 = \text{Tr}(A)A.$$

6. Let  $A \in M_m(F)$  and  $B \in M_n(F)$ . Prove that:

$$\text{rank} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \text{rank}(A) + \text{rank}(B).$$

7. Prove that for any matrices  $A \in M_{n,m}(F)$  and  $B \in M_m(F)$ , we have:

$$\text{rank} \begin{bmatrix} I_n & A \\ 0 & B \end{bmatrix} = n + \text{rank}(B).$$

8. Let  $n > 2$  and let  $A = [a_{ij}] \in M_n(\mathbb{C})$  be a matrix of rank 2. Prove the existence of real numbers  $x_i, y_i, z_i, t_i$  for  $1 \leq i \leq n$  such that for all  $i, j \in \{1, 2, \dots, n\}$ , we have:

$$a_{ij} = x_i y_j + z_i t_j.$$

9. Let  $A = [a_{ij}]_{1 \leq i, j \leq n}$  and  $B = [b_{ij}]_{1 \leq i, j \leq n}$  be complex matrices such that:

$$a_{ij} = 2ij - b_{ij}$$

for all integers  $1 \leq i, j \leq n$ . Prove that:

$$\text{rank}(A) = \text{rank}(B).$$

10. Let  $A \in M_n(\mathbb{C})$  be a matrix such that  $A^2 = A$ , i.e.,  $A$  is the matrix of a projection. Prove that:

$$\text{rank}(A) + \text{rank}(I_n - A) = n.$$

11. Let  $n > k$  and let  $A_1, \dots, A_k \in M_n(\mathbb{R})$  be matrices of rank  $n - 1$ . Prove that  $A_1 A_2 \cdots A_k$  is nonzero. *Hint:* Using Sylvester's inequality, prove that:

$$\text{rank}(A_1 \cdots A_j) \geq n - j \quad \text{for } 1 \leq j \leq k.$$

12. Let  $A \in M_n(\mathbb{C})$  be a matrix of rank at least  $n - 1$ . Prove that:

$$\text{rank}(A^k) \geq n - k \quad \text{for } 1 \leq k \leq n.$$

*Hint:* Use Sylvester's inequality.

13. (a) Prove that for any matrix  $A \in M_n(\mathbb{R})$ , we have:

$$\text{rank}(A) = \text{rank}({}^TAA).$$

*Hint:* If  $X \in \mathbb{R}^n$  is a column vector such that  ${}^TAA X = 0$ , write  ${}^T X {}^T AA X = 0$  and express the left-hand side as a sum of squares.

- (b) Let  $A = \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}$ . Find the rank of  $A$  and  $A^T A$ , and conclude that part (a) of the problem is no longer true if  $\mathbb{R}$  is replaced with  $\mathbb{C}$ .

14. Let  $A$  be an  $m \times n$  matrix with rank  $r$ . Prove that there is an  $m \times m$  matrix  $B$  with rank  $m - r$  such that:

$$BA = O_{m,n}.$$

15. (Generalized inverses) Let  $A \in M_{m,n}(F)$ . A generalized inverse of  $A$  is a matrix  $X \in M_{n,m}(F)$  such that:

$$AXA = A.$$

- (a) If  $m = n$  and  $A$  is invertible, show that the only generalized inverse of  $A$  is  $A^{-1}$ .  
 (b) Show that a generalized inverse of  $A$  always exists.  
 (c) Give an example to show that the generalized inverse need not be unique.

## Duality

- 1.

## Product and quotient of a vector space

1. Let  $V$  be a finite-dimensional vector space over  $F$ , and let  $W \subset V$  be a subspace. For a vector  $v \in V$ , define

$$[v] = \{v + w : w \in W\}.$$

Note that  $[v_1] = [v_2]$  if and only if  $v_1 - v_2 \in W$ . Define the quotient space  $V/W$  to be

$$V/W = \{[v] : v \in V\}.$$

Addition and scalar multiplication in  $V/W$  are defined as follows:

$$[u] + [v] = [u + v] \quad \text{and} \quad a[v] = [av],$$

where  $a \in F$ . It is known that these operations are well-defined and that  $V/W$ , equipped with this structure, is a vector space.

(a) Show that the map  $\pi : V \rightarrow V/W$  defined by  $\pi(v) = [v]$  is linear with kernel  $W$ .

(b) Show that

$$\dim(W) + \dim(V/W) = \dim(V).$$

(c) Suppose  $U \subset V$  is any subspace such that  $W \oplus U = V$ . Show that the restriction  $\pi|_U : U \rightarrow V/W$  is an isomorphism, i.e., a bijective linear map.

(d) Let  $T : V \rightarrow U$  be a linear map, let  $W \subset \ker(T)$  be a subspace of  $V$ , and let  $\pi : V \rightarrow V/W$  be the projection onto the quotient space. Show that there exists a unique linear map  $S : V/W \rightarrow U$  such that  $T = S \circ \pi$ .