Given:

$$P = \{p_1, \ldots, p_n\}, K$$

$$K, p_i \in \mathbb{N}^+$$

Does there exist an M such that

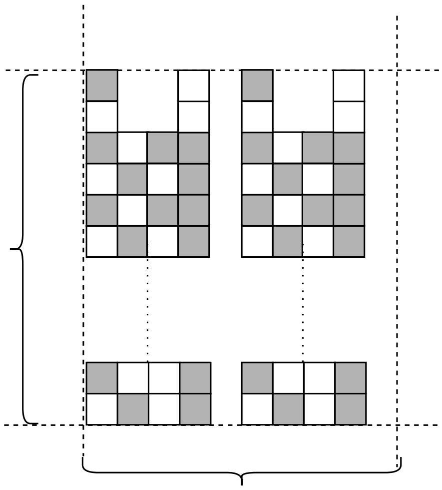
$$M = \{q_1, \ldots, q_m\} \subseteq P$$

$$\sum_{i=1}^{m} q_i = K$$

Initial gameboard:

$2\left(\sum P + K\right)$

rows



10 columns

Consists of two identical "wells"

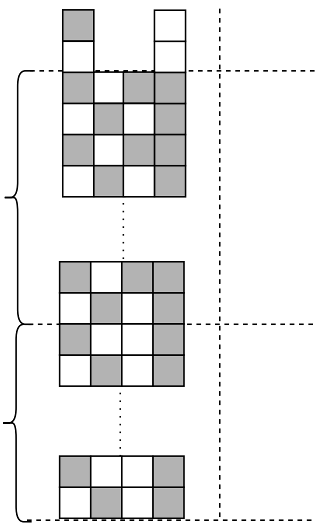
The well:

Checkered patttern in c3

$2\sum P$

White cells in a straight line in c3

$2(K-1)$



5 columns and

$2\left(K + \sum P\right)$

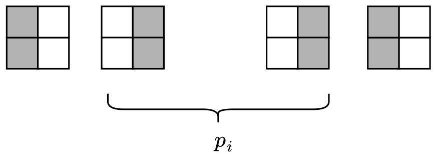
rows per well

Transformation:

Phase 1

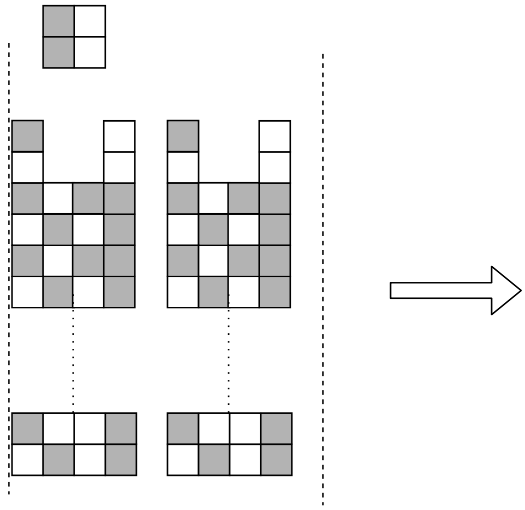
$\forall\, p_i$

create the following block sequence:

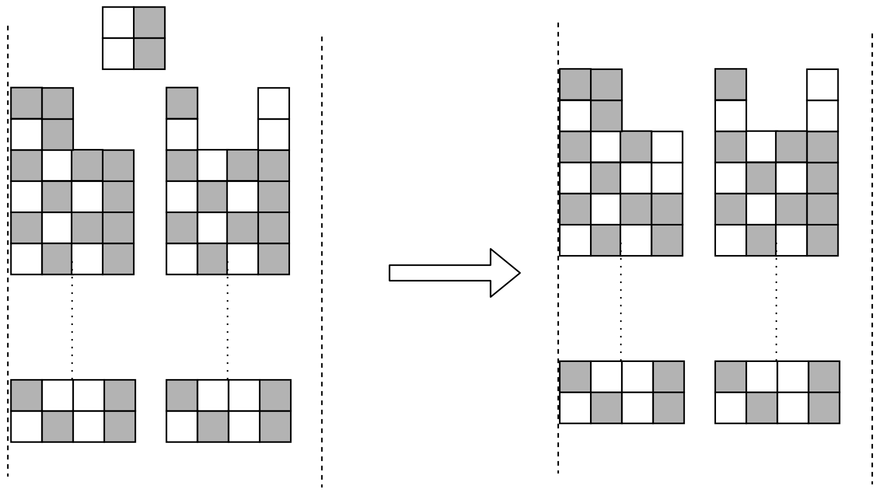


The first block in this sequence "opens" a well:

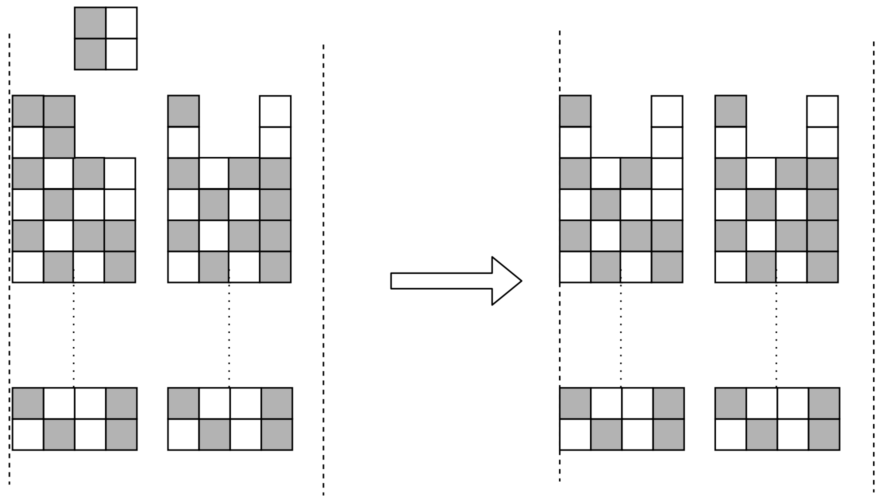
For now, assume that the player wants to place the blocks in such a way that every block fixation will clear cells on the gameboad

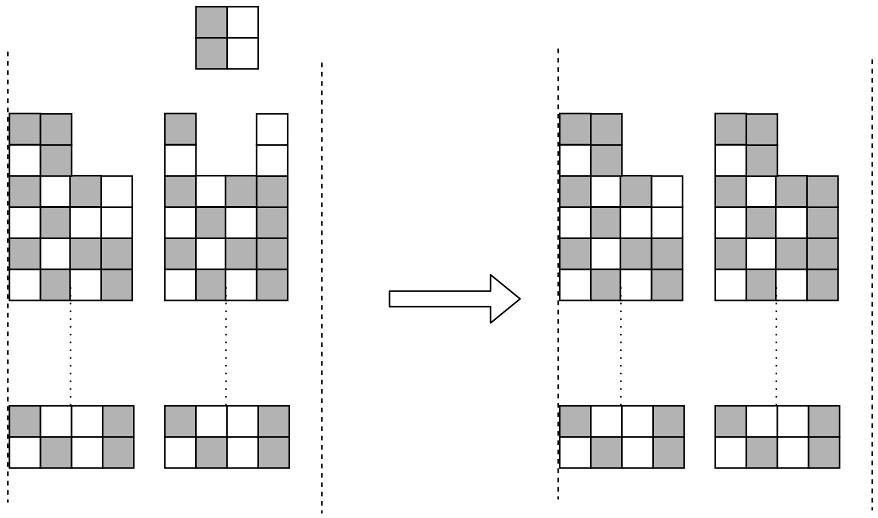


The blocks in the middle of the sequence must all be placed in the open well at the far right, or else the the wells will be permanently blocked for the rest of the game. Further into proof we will see that this implies a non-optimal trajectory.



The last block of the
sequence either closes
the open well, or opens
the closed well



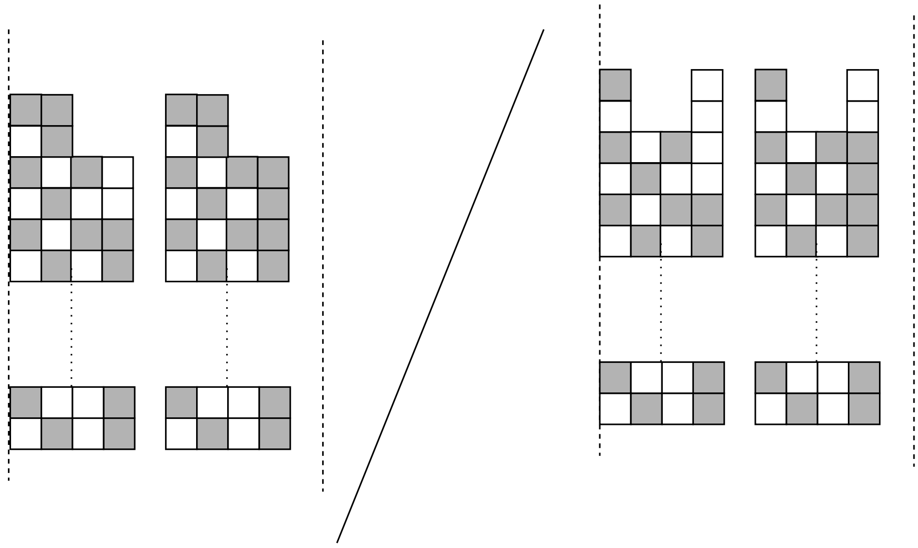


Lemma: a well is closed permanently in this phase
if and only if a block has been place which did not
mark any cells on fixation **TODO: prove**

Assuming the player has placed the blocks in the way described, the
following holds when the sequence corresponding to p_i begins:

1. Either both of the wells are closed, or both of the wells are open. Neither of them is in a permanently blocked state.

2. Columns 1, 3 and 5 are unchanged from the initial gameboard. Column 2 is unchanged except from the top two rows (which may be black or empty).



3. Let (this does not change the semantics of the given subset sum instance)

$$p_0 = 0$$

.
Then in total

$$2(i-1) + \sum_{j=0}^{i-1} p_j$$

block has been placed

assuming every block clears exactly four cells on fixation (**TODO: prove)**

$$8(i-1) + 4\sum_{j=0}^{i-1} p_j$$

cells has been cleared

4. There exists

$$M_1, M_2 \subseteq \{p_0, \ldots, p_{i-1}\}, M_1 \cap M_2 = \varnothing, M_1 \cup M_2 = \{p_0, \ldots, p_{i-1}\}$$

.
För well q it holds that the all of the rows

$$\left[2\Big(K - 1 + \sum P - \sum M_q\Big) + 1,\ 2\Big(K - 1 + \sum P\Big)\right]$$

have white cells in c4 (B_q + 3)

The rows

$$\left[1, 2\Big(K - 1 + \sum P - \sum M_q\Big)\right]$$

have black cells in c4 (B_q + 3)

When every block corresponding to the elements of the given set has been placed

$$i = n + 1$$

.

1. Either both of the wells are closed, or both of the wells are open. Neither of them is in a permanently blocked state.

2. Columns 1, 3 and 5 are unchanged from the initial gameboard. Column 2 is unchanged except from the top two rows (which may be black or empty).

2. In total

$$2n + \sum P$$

blocks have been placed and

$$8n + 4\sum P$$

cells have been cleared

3. There exists

$$M_1, M_2 \subseteq P, M_1 \cap M_2 = \varnothing, M_1 \cup M_2 = P$$

.

For well q it holds that all of the rows

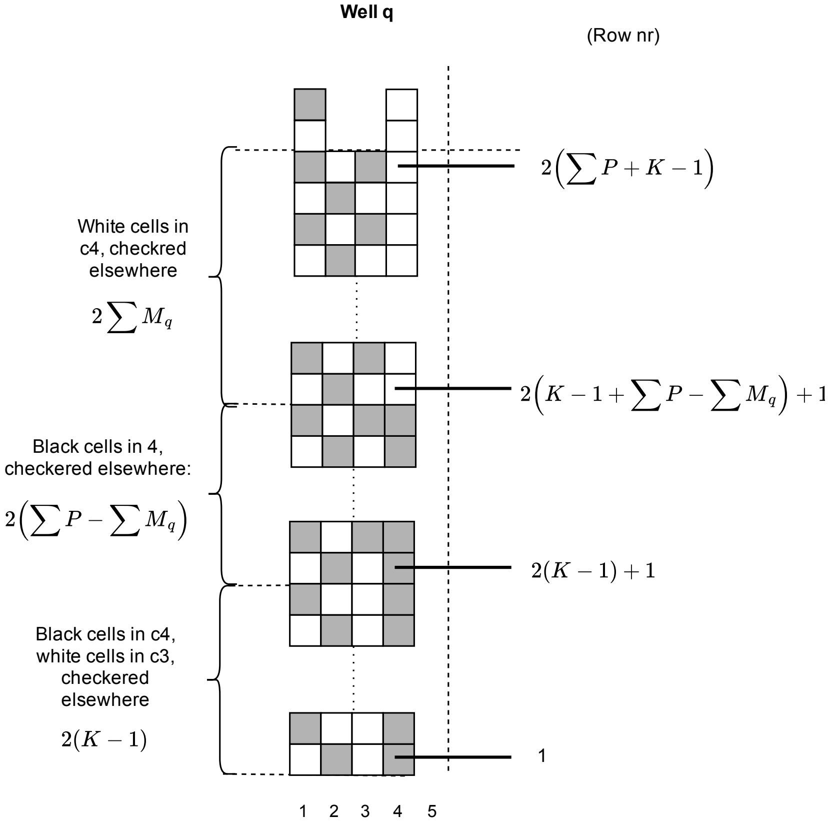
$$\left[2\left(K - 1 + \sum P - \sum M_q\right) + 1,\ 2\left(K - 1 + \sum P\right)\right]$$

have white cells in c4 (B_q + 3)

All of the rows

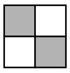
have black cells in i c4 (B_q + 3)

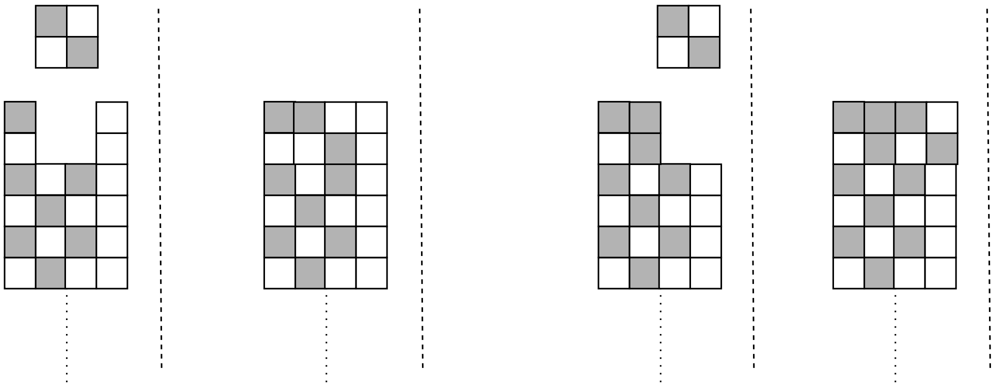
Well q



Phase 2

Both of the wells are not permanently blocked. If one or both of them are, according to lemma a block has been placed which did not mark any cells.

A block of type  forces the permanent closure of one well

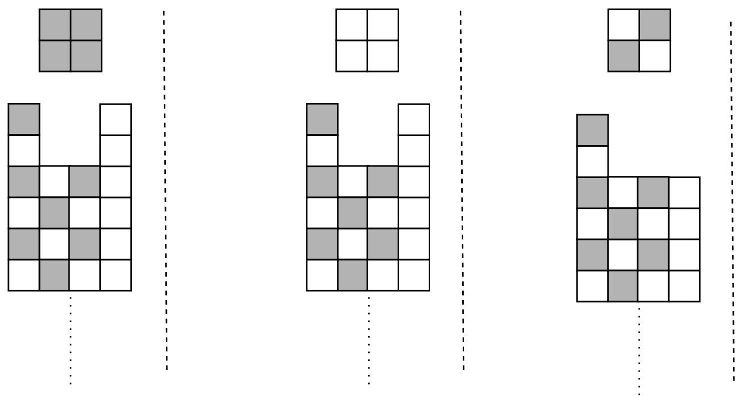


Remark: If one or more of the wells are permanently blocked before this phase, this phase will trigger a certain game over, making it impossible to clear more cells. Therefore it is required to only place block which instantly clears cells in order to clear the optimal amount of cells for this sequence.

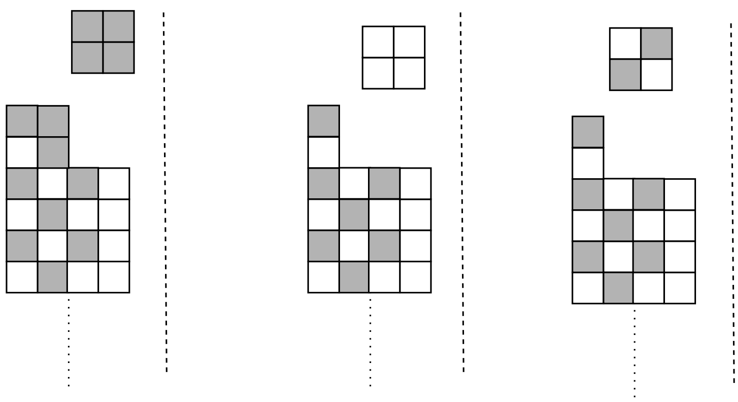
Phase 3

one full-black and one full-white block forces the topmost two rows in c2-c4 to be cleared. A checkered block must then be placed in c2-c3 (otherwise the well becomes blocked)

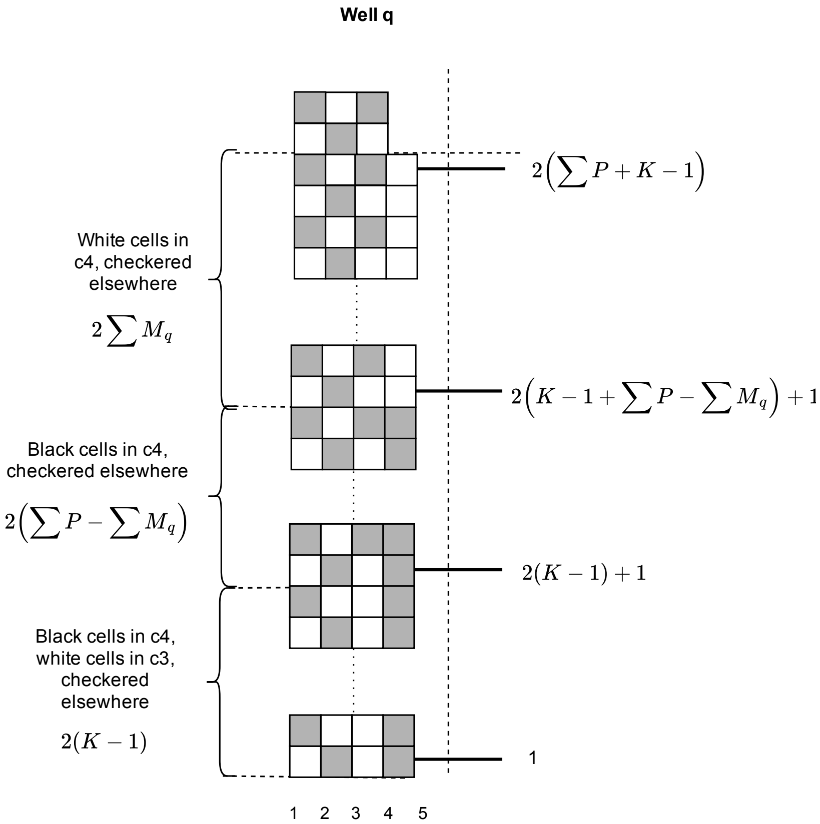
Case 1:



Case 2:



Phase 4



Now only one well is open, therefore a block can only be placed in one possible possition, otherwise a game over is triggered

Remark: Black cells in c4

$$2\left(\sum P - \sum M_q\right) + 2(K-1)$$

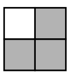
$$\sum M_q = K \implies$$

$$2\left(\sum P - K\right) + 2(K-1) = 2\left(\sum P - 1\right)$$

Idea:

Place

$\sum P - 1$



blocks

If

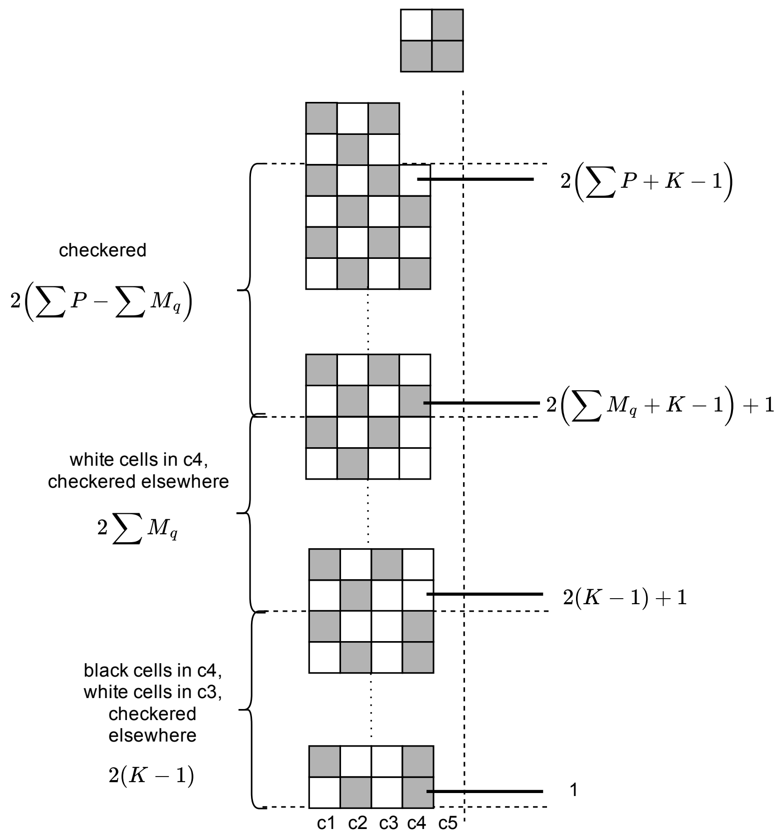
$$\sum M_q = K$$

all possible black cells in (in c4) will be cleared

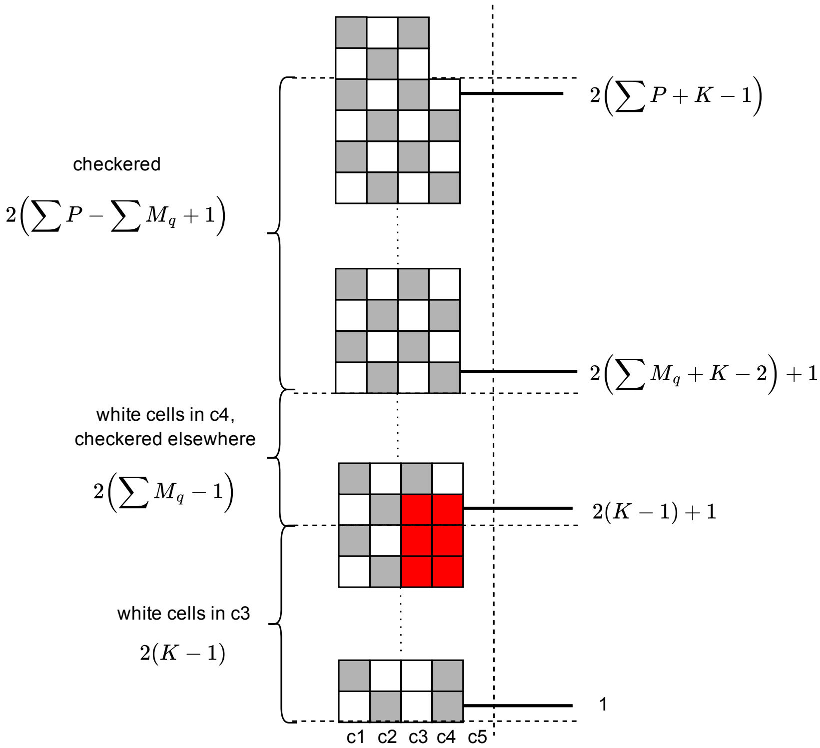
When block

$$\sum P - \sum M_q + 1$$

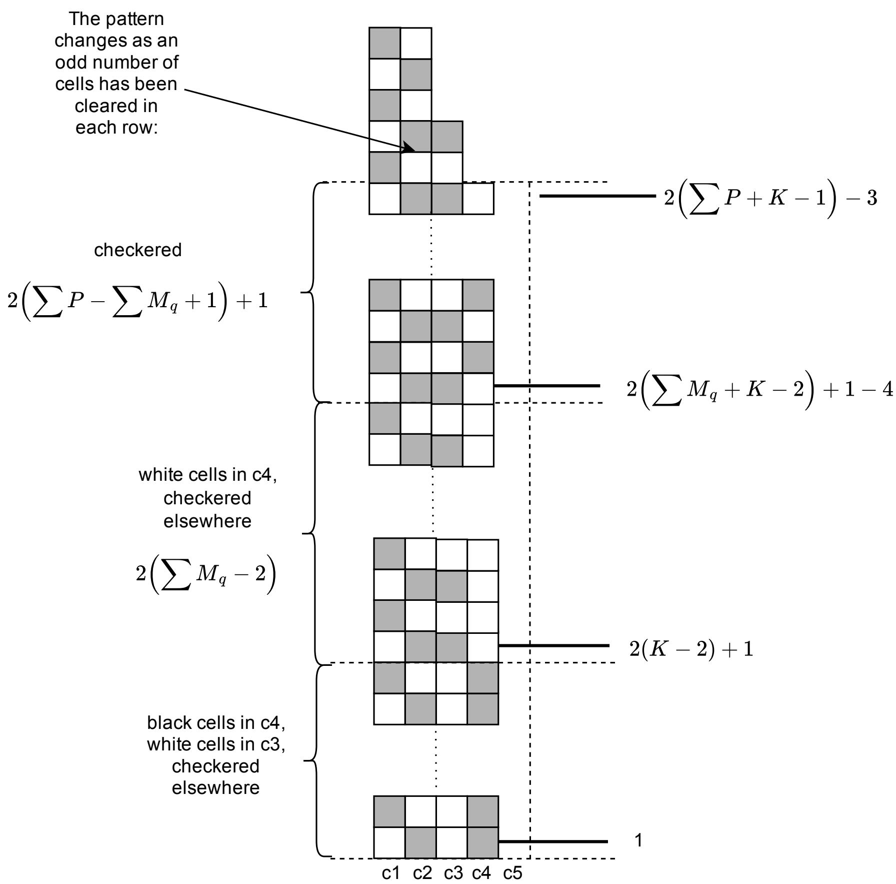
is about to be placed:



The well is now about to collapse:



checkered
$2\left(\sum P - \sum M_q + 1\right)$
white cells in c4, checkered elsewhere
$2\left(\sum M_q - 1\right)$
white cells in c3
$2(K-1)$
$2\left(\sum P + K - 1\right)$
$2\left(\sum M_q + K - 2\right) + 1$
$2(K-1)+1$
1
c1 c2 c3 c4 c5



The pattern changes as an odd number of cells has been cleared in each row:
checkered
$2\left(\sum P - \sum M_q + 1\right) + 1$
white cells in c4, checkered elsewhere
$2\left(\sum M_q - 2\right)$
black cells in c4, white cells in c3, checkered elsewhere
$2\left(\sum P + K - 1\right) - 3$
$2\left(\sum M_q + K - 2\right) + 1 - 4$
$2(K-2)+1$
1
c1 c2 c3 c4 c5

Phase 5

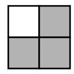
Lemma: The only way to clear cells is to place blocks in c4-c5

Sketch of proof (swedish): c1-c2 kan inte clearas (randing terräng). Läggs blocken i c3-c4 clearas ingen terräng eftersom c3 byggs på med samma terräng. c4 fylls på med två svarta som aldrig kommer att möta två andra svarta att ta ut.

Continue placing the remaining

$$\sum M_q - 2$$

blocks



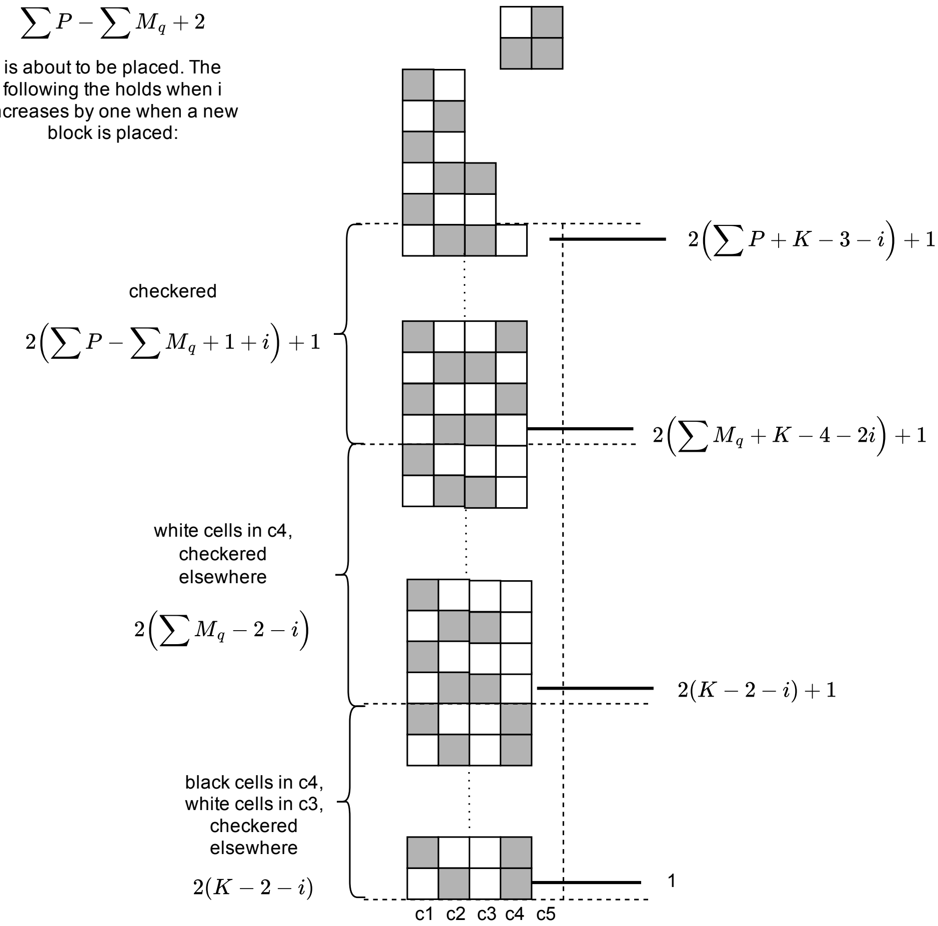
let

$$i = 0$$

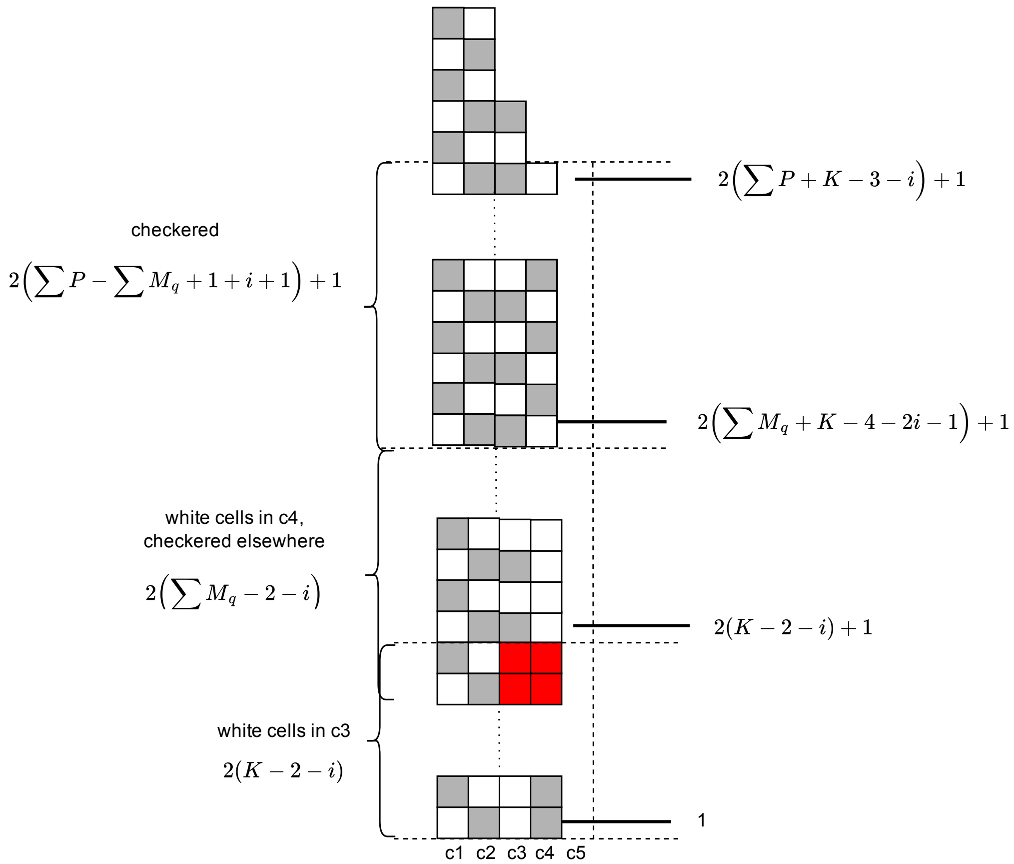
when block

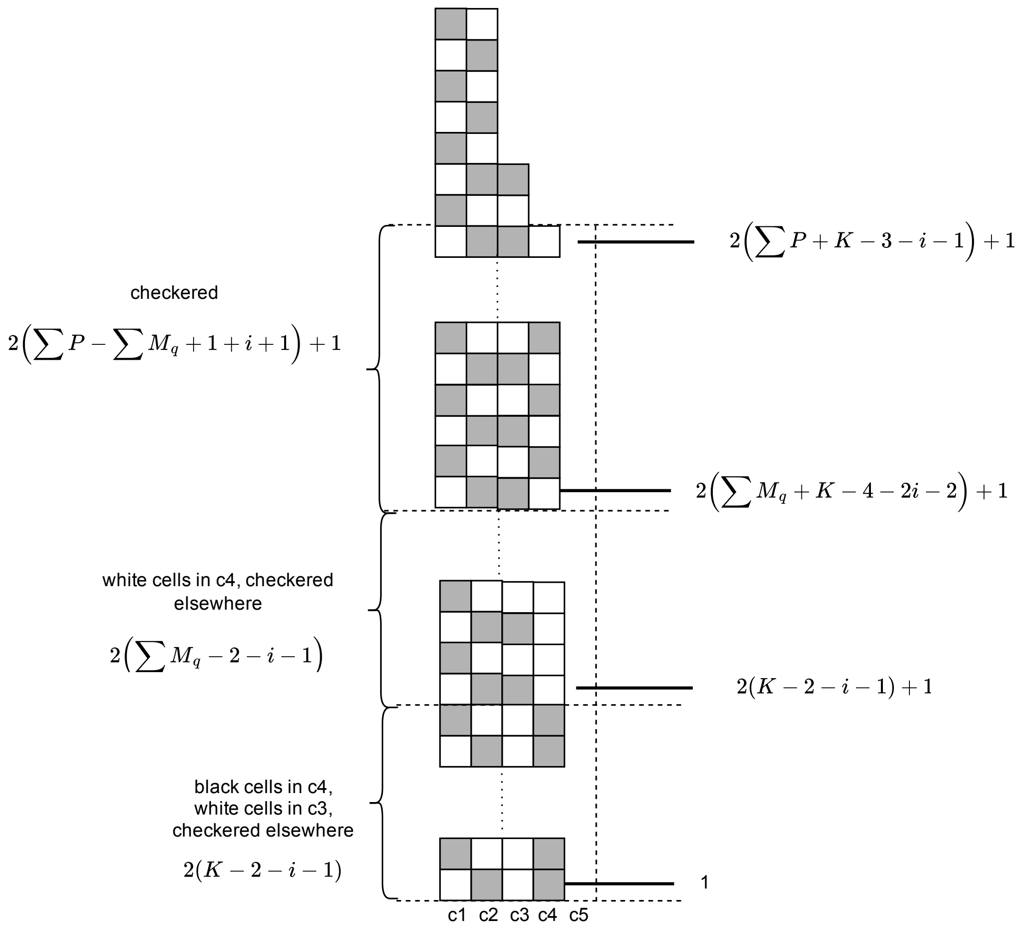
$$\sum P - \sum M_q + 2$$

is about to be placed. The following the holds when i increases by one when a new block is placed:



Remark: Only cell blocks of height two will be cleared in each row, therefore the pattern doesn't change during this phase





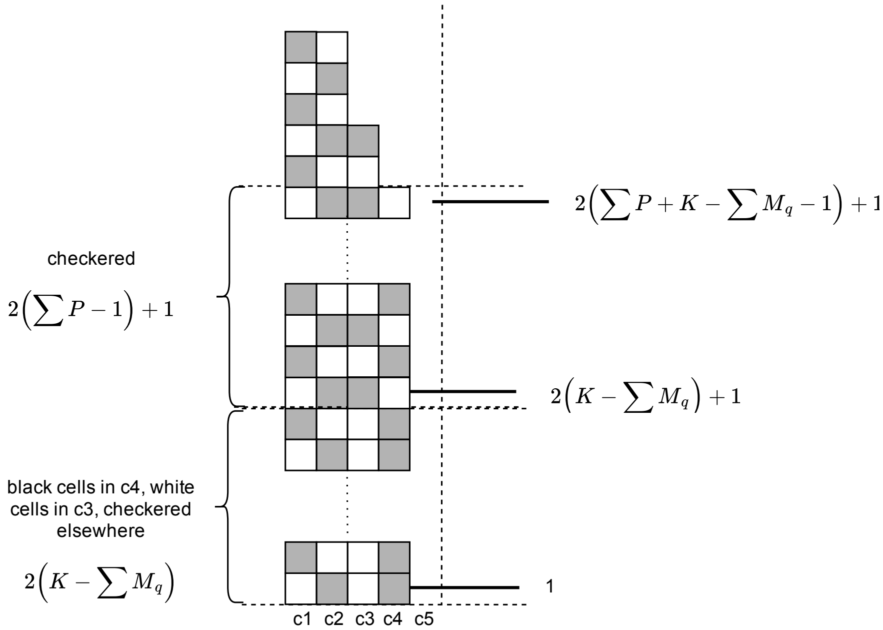
let

$$i = i + 1$$

the invariant now holds for the next iteration

When

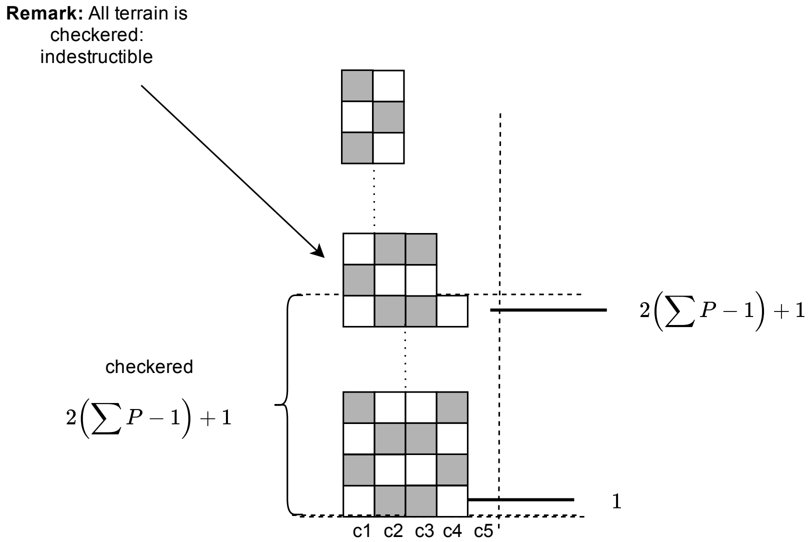
$i = \sum M_q - 2$



$K = \sum M_q \implies$

Every possible black cell in c4 is cleared, every possible white cell in c3 is cleared

:



$$K < \sum M_q \implies$$

c4 will contain too many white cells, ie too few black cells. These cannot be cleared by any placed block.

$$K > \sum M_q \implies$$

c3 will contain too many white cells, these are not cleared by the corresponding white cells in c4

$$K = \sum M_q \implies$$

All possible white cells in c3 will be cleared by corresponding white cells in c4. All possible black cells in c4 will be cleared by placed blocks. As many cells as possible is cleared, using the given block sequence.

Conclusion: There exists an optimal trajectory (placement of blocks) if and only if there exist an M such that

$$M = \{q_1, \ldots, q_m\} \subseteq P$$

$$\sum_{i=1}^{m} q_i = K$$

Given:

$$P = \{p_1, \ldots, p_n\}, K$$

$$K, p_i \in \mathbb{N}^+$$