Week 02, problems.

Linear Congruences, System of congruences, Chinese Remainder Theorem.

1. a) What can the remainder of an integer be when divided by 273 if 57 times that integer gives a remainder of 99 when divided by 273?
   b) We know that 12 times an integer x gives a remainder that is 3 more than the number x itself when divided by 86. What is the remainder of x divided by 86?
   c) Determine all the possible positive integers n that give the same remainder when $52n+3$ and $n+7$ are divided by 85.

2. For what integer is it true that its remainders are 2 and 3 when divided by 7 and 9 respectively.

3. Determine all positive integers which give a remainder of $1, 2$ and 4 when divided by $3, 7$ and 11 respectively.

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4. (MT+'19) Determine all the integers between 1 and 1000 for which $n+10$ divided by 36, and $n-10$ divided by 38 both give a remainder of 1.

5. a) (MT+'20) We multiply an integer n by 17 and the result gives the remainder 23 when it is divided by 65. What can the remainder be when we divide n by 130?
   b) (MT'17) For an integer n, its product with 115 gives a remainder that is 110 more than of n itself when divided by 344. What is the remainder when we divide n by 344?
   c) (MT'14) For the integer n, the last two digits of $43n-1$ and $2n+2$ are the same. What are these two digits?

6. Determine all positive integers x for which the following congruences hold:
   $3x \equiv 2 \pmod 5$, $\quad 4x \equiv 3 \pmod 7$, $\quad x \equiv 2 \pmod{11}$.

7. (MT'24) Determine all the integers between 1 and 2024 for which the last two digits are 11 when expressed in base 6 and in base 8.

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8. (a) A millipede wants to count its legs. It knows that every millipede has at most 344 legs. If it counts its legs by 13's then 3 are left out, and if by 17's then 10 are left out. How many legs does the millipede have?
   (b) Another millipede wants to use this method as well. If it counts its legs by 16's then 5 are left out, and if by 20's then 15. Show that it made a mistake.
   (c) The king of millipedes learns about the method also. If it counts its legs by 6's then 5 are left out, by 7's then 6 are left out and if by 8's then 7. How many legs does it have?

9. a) (MT'20) Determine the numbers between 1 and 111 which, when multiplied by 1111, give the remainder 11 if we divide them by 2020.
   b) (MT+'11) What can the remainder of an integer n be when divided by 202 if $53n-1$ is divisible by 202?
   c) (MT+'13) We know that for some integer n, when we divide $37n+9$ and $n+10$ by 235, we get the same remainder. What can this common remainder be?

10. (MT'23) Determine all the integers n between 1 and 2023 for which both the fraction $\frac{n-2}{21}$ and the fraction $\frac{n-5}{166}$ has an integer value.

11. (MT'13) The remainder of the integer n when divided by 82 is 3. What can the remainder of n be when divided by 182?

12. (MT+'12) How many integers n are there between 1 and 1000 for which there is an integer m such that the equation $37n+218m=10$ holds?

13. (MT+'23) On the third planet of the Great Vogon System, every month is 29 days long since ancient times, and the remaining 22 days at the end of the year are spent by the residents' excessive drinking of the famous pálinka. However, an in-depth analysis by the National Bank recently showed that the 22-day break at the end of the year has an unfavorable effect on the performance of the economy. That is why the 32-day month is introduced, so the number of days omitted from months is reduced to 5. How many days does the year consist of on the third planet, if we know that the year on the fourth planet (which of course is longer than that of the third) is 1000 days long.

14. (A classical problem of Bhaskara, al-Haitham, Fibonacci, from Kraft) When the eggs in a basket are taken out 2 at a time, there is one egg left in the basket. When they are taken out $3, 4, 5$ and 6 at a time, there are $2, 3, 4$ and 5 eggs, respectively, left in the basket. If 7 at a time are taken, then none are left. What is the least number of eggs in the basket?

Final Answers

1. a) $x \equiv 64, 155, 246 \pmod{273}$, b) $x \equiv 55 \pmod{86}$, c) no solutions
2. numbers of the form $63 \cdot k + 30$
3. numbers of the form $231 \cdot k + 37$
4. numbers of the form $684 \cdot k + 315$, so 315 and 999.
5. a) 9 or 74, b) 7 or 179, c) 68 (note these are digits of $2n + 2$)
6. numbers of the form $385 \cdot k + 244$
7. there are no such numbers
8. a) 146, b) $(a, m)|b$, c) 167 or 335
9. a) no solutions, b) 61, c) 121
10. 1997
11. All odd values are possible
12. $124 + 218k$
13. 805
14. 119, this is a nice puzzle, you could either do it the long way, solve all the congruences with methods learned so far, OR you could look at the remainders and exclaim EUREKA!