Week 03, problems.

Euler-Fermat Theorem, Polynomial time algorithms.

1. (MT+'23) What is the remainder when 499^{4201} is divided by 539?

2. (MT'18, MT++'18) The two codes written below in C take a positive integer a (written in decimal system) as input. The first computes the square of a, while the second computes the sum of the digits of a. Assume that the computer uses the "normal" basic operations (addition, subtraction, multiplication, division,...). Determine whether the algorithms are polynomial or not. (`floor(a/10.0)` gives the lower integer part of $\frac{a}{10}$.)

```
x = a; y = 0;
while (x > 0) {
   x = x-1;
   y = y+a;
}
printf(''Result: %d'', y);
```

```
x = 0; y = 0;
while (a > 0) {
   x = floor(a/10.0);
   y =y+a-10*x;
   a = x;
}
printf(''Result: %d'', y);
```

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3. a) Determine the last two digits of 303^{404}.
   b) (MT++'21) Determine the remainder we get when we divide 701^{701701} by 99.

4. The input of the imaginary algorithm A is the positive integer m written in the decimal system. Decide whether the statements below are true or false.
   a) If A stops after at most $5m^2$ steps for each input m then A is a polynomial algorithm.
   b) If A stops after at most $100 \cdot (\log_3 m)^5$ steps for each input m then A is a polynomial algorithm.
   c) If A makes at least m steps for each m then A is not a polynomial algorithm for sure.d) If there is an m for which A makes at least m steps then A is not a polynomial algorithm for sure.
   e) If A makes at least m steps for each even m then A is not a polynomial algorithm for sure.

5. (MT'05) Let A be the arithmetic progression whose first term is 32, and whose difference is 51. (So the first few terms of A are $32, 83, 134, ...$) Determine the remainder we get if we divide the product of the first 32 terms of A by 51.

6. (MT+'19) How many positive integers are there which are not greater than 504 and have a multiple which gives 1 as a remainder when divided by 504?

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7. The two codes written below in C take a positive integer a (written in decimal system) as input. The first calculates $\lfloor \sqrt{a} \rfloor$while the second calculates $\lfloor \log_2 a \rfloor$. Assume that the computer uses the "normal" basic operations (addition, subtraction, multiplication, division,...). Determine whether the algorithms are polynomial or not.

```
x = 0; y = 0;
while (y <= a) {
   x = x+1;
   y = x * x;
}
printf(''Result: %d'', x-1);
```

```
x = 0; y = 1;
while (y <= a) {
   x = x+1;
   y = 2 * y;
}
printf(''Result: %d'', x-1);
```

8. Determine the remainder for the following:
   a) (MT++'20) 7^{3234} divided by 80.

b) (MT'21) $2021^{2021} - 2021^{101}$ divided by 600.
c) 39^{1200} divided by 26.

9. (MT'07) Let $n = 200705111601$. Determine the last three digits of n^n.
10. (MT'14) Determine the remainder we get if we divide $46^{47^{48}}$ by 25.
11. * (MT+'23) Show that there is no integer x for which $x^6 \equiv 2 \pmod{201}$ holds.
12. * (MT+'24) Let a, b be positive integers that are co-prime. What is the remainder of $a^{\varphi(b)} + b^{\varphi(a)}$ when divided by ab?

Final Answers

1. 499
2. no,yes
3. 81,8
4. false, true, true, false, true
5. 1
6. 144
7. no,yes
8. a) 49, b) 0, c) 13
9. 601
10. 46
11.
12.