CSC165 Assignment 1

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1 Question

1.1 a)

$\forall k \in \mathbb{Z} : k \geq 1 \Rightarrow \frac{1}{(k+2)^2} < \frac{1}{k} - \frac{1}{k+1} < \frac{1}{(k+0)^2}$

1.2 b)

$\forall k \in \mathbb{Z} : k \geq 1 \Rightarrow \frac{1}{(k+2)^2} < \frac{1}{k+1} - \frac{1}{k+2} < \frac{1}{(k-1)^2}$

1.3 c)

$\forall k \in \mathbb{Z} : k \geq 1 \Rightarrow (k-2)^2 < k^2 + k < (k+2)^2$

1.4 b)

Prove: $\forall x, y, z \in \mathbb{Z}, x \nmid yz \Rightarrow x \nmid y \wedge x \nmid z$
  $\exists x, yz \in \mathbb{Z}, x|y \vee x|z \Rightarrow x|yz$ # Take the contrapositive
  Assume x, y, z $\in \mathbb{Z}$ # Generic integers
    Assume $x|y$
    Then $y = x * k$ # By the definition of division, let k $\in \mathbb{Z}$
    Then $z * y = x * k * z$ # Multiply both sides by z
    Then $z * y = x(k * z)$ # Manipulation laws
    Then $z * y = x * (s)$ # By definition of division, s = (k * z)
    Therefore $x|yz$
  $\exists x, yz \in \mathbb{Z}, x|y \vee x|z \Rightarrow x|yz$ # Reintroduce existential quantifier
Therefore $\forall x, y, z \in \mathbb{Z}, x \nmid yz \Rightarrow x \nmid y \wedge x \nmid z$ # Reintroduce original statement, proved true by contrapositive

2 Question

P : Prime Numbers

Prove: $\forall x, y, z \in P, x^2 + y^2 \neq z^2$

$\exists x, y, z \in P, x^2 + y^2 = z^2$ # Proof by contradiction, so take the negation

Assume $x, y \in P$ # So x and y are generic prime numbers

So $x^2 = z^2 - y^2$ # Through algebra

So $x^2 = (z - y)(z + y)$ # Through factoring

So $x * x = (z - y)(z + y)$ # Expanding

Therefore $x * x \neq (z - y)(z + y)$ # For these to be equal, by the laws of unique prime decomposition, x would need to be equal to (z - y) and also (z + y), which is not possible, and therefore raises a contradiction.

Therefore $x^2 \neq z^2 - y^2$ # simplify

Therefore $x^2 + y^2 \neq z^2$ # Reintroduce original statement

$\forall x, y, z \in P, x^2 + y^2 \neq z^2$ # Reintroduce universal quantifier, proved by contradiction

3 Question

3.1 a)

$Def_1 : \forall y \in \mathbb{R}, \forall y \in \mathbb{Z}, (y = \lfloor x \rfloor) \Leftrightarrow (y \leq x \wedge (\forall z \in \mathbb{Z}, (z \leq x) \Rightarrow (z \leq y)))$

Prove $S_1 : \forall n \in \mathbb{Z}, \forall y \in \mathbb{R}, (0 \leq y) \wedge (y<1) \Rightarrow (\lfloor n + y \rfloor = n)$

Assume $n \in \mathbb{Z}, y \in \mathbb{R}$ # n is a generic integer and y is a generic real number

Assume $(0 \leq y) \wedge (y<1)$ # Assume antecedent

Let x = n + y # x from Def_1 equal to n + y in floor definition in S_1

Let y = n # y from Def_1 equal to n in floor definition in S_1

So $(n = \lfloor n + y \rfloor) \Leftrightarrow (n \leq n + y \wedge (\forall z \in \mathbb{Z}, (z \leq n + y) \Rightarrow (z \leq n)))$

Substitute into Def_1

Therefore $n \leq n + y$ # This shows that $y \in \mathbb{R}+$, case proved from Def_1

Assume $n, z \in \mathbb{Z}$ # Generic integers

Then $(z \leq n + y) \Rightarrow (z \leq n)$ # Using Def_1

Then $(z>n) \Rightarrow (z>n + y)$ # Using the contrapositive

Assume $(z>n)$ # Assume antecedent

So $(z \geq n + 1)$ # By the definition of integers

$(y<1)$ # By definition

Therefore $(z \geq n + y)$

Therefore $(z \leq n + y) \Rightarrow (z \leq n)$ # Case proved from Def_1

Therefore $(n \leq n + y \wedge (\forall z \in \mathbb{Z}, (z \leq n + y) \Rightarrow (z \leq n)))$ # Cases proved from Def_1 for S_1

Therefore $(\lfloor n + y \rfloor = n)$ # S_1 consequent proved

Therefore $(0 \leq y) \wedge (y<1) \Rightarrow (\lfloor n + y \rfloor = n)$ # Introduce antecedent

Therefore $\forall n \in \mathbb{Z}, \forall y \in \mathbb{R}, (0 \leq y) \wedge (y<1) \Rightarrow (\lfloor n+y \rfloor = n)$ # Reintroduce universal quantifiers, S_1 proved using Def_1

3.2 b)

$S_2 : \forall x \in \mathbb{R}, \exists y \in \mathbb{R}, (0 \leq y) \wedge (y<1) \wedge (x = \lfloor x \rfloor + y)$

Assume $x \in \mathbb{R}$ # Generic real

Let $y = x - \lfloor x \rfloor$ # Construct y through algebra and $y \in \mathbb{R}$

Then $(\lfloor x \rfloor \leq x \wedge (\forall z \in \mathbb{Z}, (z \leq x) \Rightarrow (z \leq \lfloor x \rfloor)))$ # Substitution in Def_1

Then $(\lfloor x \rfloor \leq x) \wedge (z \leq x)$ # By floor definition

So $x - \lfloor x \rfloor<1$ # Nature of integers

Therefore $y < 1$ # Because $y = x - \lfloor x \rfloor$

Assume $x \geq \lfloor x \rfloor$ # Assumed by Def_1

Then $x - \lfloor x \rfloor \geq 0$ # Definition of subtraction

Therefore $y \geq 0$

Therefore $(0 \leq y) \wedge (y<1)$ # First two cases true by Def_1

Therefore $\exists y \in \mathbb{R}, (0 \leq y) \wedge (y<1)$ # Introduce existential quantifier

Therefore $\exists y \in \mathbb{R}, (0 \leq y) \wedge (y<1) \wedge (x = \lfloor x \rfloor + y)$ # Introduce last case

Therefore $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, (0 \leq y) \wedge (y<1) \wedge (x = \lfloor x \rfloor + y)$ # Reintroduce universal quantifier

3.3 C)

Prove: $\forall x \in \mathbb{R}, \forall n \in \mathbb{Z}, (\lfloor x+n \rfloor = \lfloor x \rfloor + n)$

Assume $\exists y \in \mathbb{R}, (x = \lfloor x \rfloor + y), (0 \leq y) \wedge (y<1)$ # As proved in S_2

So $n_0 = \lfloor \lfloor x \rfloor + y + n \rfloor$ # By substituting into S_1

Then $\lfloor x+n \rfloor = \lfloor (\lfloor x \rfloor + y) + n \rfloor$ # By substituting into S_1

Therefore $n_0 = \lfloor \lfloor x \rfloor + y + n \rfloor \in \mathbb{Z}$

Then $\lfloor x \rfloor \in \mathbb{Z}$ # By Def_1 for x

Then $n \in \mathbb{Z}$ # By $\mathbb{N} \subset \mathbb{Z}$

Then $\lfloor x \rfloor + n \in \mathbb{Z}$

Then $\lfloor x+n \rfloor = n \in \mathbb{Z}$ # By the definition of S_2

Therefore $\lfloor x+n \rfloor = \lfloor x \rfloor + n$ # As both statements $\in \mathbb{Z}$

$\forall x \in \mathbb{R}, \forall n \in \mathbb{Z}, (\lfloor x+n \rfloor = \lfloor x \rfloor + n)$ # Reintroduce universal quantifiers