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PROBLEM SET 6 (T-445-GRTH)

You need to collect **65** points to get a full score **but** you cannot get more than **X** points (in total) from a problem section with annotation **max X**.

Please make sure to:
1. Write your name/email(s) on your work (replace my name above).
2. Write your answers in `\begin{solution} ... \end{solution}` blocks.
3. Write clear and concise proofs: points may be deducted for vagueness.

1. Intersection, Chordal, Outerplanar Graphs (max 55)

Definition. A planar graph G is *outerplanar* if G has an embedding in the plane in such a way that every vertex is on the boundary of the infinite face.

1 (5 points) Prove that every connected unit interval graph has a Hamiltonian path.

Solution. Take a unit interval representation of the graph and order the intervals from the lowest beginning to the highest one. Begin the path at the vertex of the first interval and connect it to the second one and so one. This is possible because it is a connected graph and the intervals all have the same length, so there exists no interval, that is a proper subset of an other interval. □

2 (8 points) Prove that every outer planar graph on n vertices has at most $2n - 3$ edges.

Solution. We prove the claim by induction. For n=1,2 or 3, it is trivial.
The assumption holds for an arbitrary n.
Take an outer planar graph on n+1 vertices. We delete a vertex with degree less or equal to 2. The remaining graph is still an outer planar graph and has at most $2n - 3$ edges by induction hypothesis. So the graph on $n + 1$ vertices has at most $2(n + 1) - 3$ which shows the assumption.
It remains to show, that there exists a vertex of degree less or equal to two. Either you have a leaf. Than it is true. If not, the outer edges form a cycle. If there are no inner edges, it is true. If there is an inner edge, it can not connect two adjacent vertices. There need to be one in between, with degree 2. So the assumption holds. □

3 (10 points) Let G be a graph in which any two simple cycles have at most one vertex in common.
- **a** Let B be a block in G. What can you say about the structure of B?
- **b** Prove that G is an outer planar graph.

Solution. a) B is at most 2-connected.
b) Let G be a graph in which any two simple cycles have at most one vertex in common. Then let H be a subgraph of G such that H is homeomorphic to K_4 or $K_{2,3}$. For any two C_3 subgraphs of H, c_1 and c_2, $|c_1 \cup c_2| = 2$, which

contradicts the definition of G so H cannot be a subgraph of G. Therefore by Theorem 7.59 in the book, if neither K_4 or $K_{2,3}$ is homeomorphic to any subgraph of G, then the graph G is outer planar.

$\square$

4 (7 points) Prove that every unit coin graph on $n > 3$ vertices has at most $3n - 7$ edges.

Solution. A vertex can have at most 6 neighbours by geometrical properties. We take a representation with unit coins and order it from left to right. Take the leftmost vertex. It only has neighbours to his right and therefor maximal three. By induction, the claim holds if it is true for a base case. Take 4 vertices. They can maximal have 5 edges. So the assumption holds. $\square$

5 (8 points) Prove that for any unit disc graph G, $D(G) \leq 3(\omega(G) - 1)$.

Solution. Let G be a unit disk graph. Then for each vertex $v \in V(G)$, the radius of v is 1. Then if you take an outer vertex u then all it's neighbours will be within an angle less than 180 degrees. Therefore we can have a vertex cover with three sections, each with a radius less than 1. and an angle of 60 degrees or less. Then the vertices in these sections combined make a complete graph so the degree of u is equal to $3(w(G) - 1)$. Therefore the degeneracy would be $D(G) \leq 3(\omega(G) - 1)$. $\square$

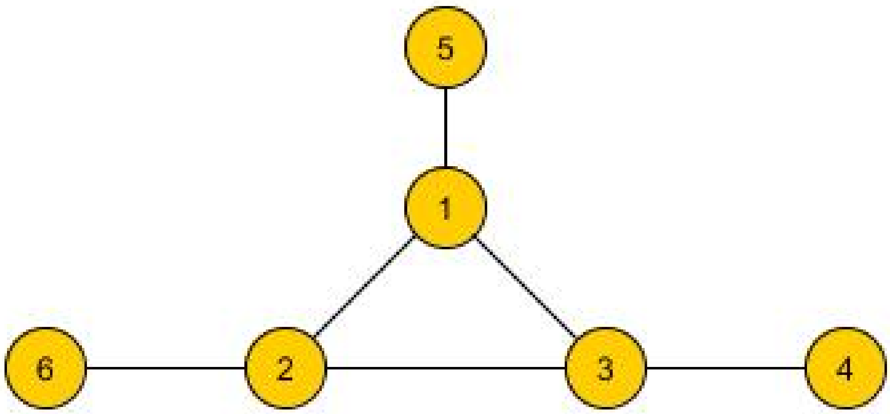
6 (20 points) A graph G is a *split graph* if $V(G)$ can be partitioned into subsets X, Y, such that $G[X]$ is a clique and $G[Y]$ is a null graph.

a Prove that split graphs are chordal.

b Prove that the complement of a split graph is a split graph.

c Find a split graph that is not an interval graph.

Solution. a) Let G' be any induced subgraph. If it contains only vertices from the clique, it is obviously triangulated. If it does not contain only vertices from the clique, it is not an induced cycle, because one vertex is isolated.
b) The complement of a complete graph is a null graph and the other way around. So our partition is the same as for the original graph.
c) An example is displayed in the following graphic



It is not an interval graph, because the vertices 1,2 and 3 need to have something in common, but each of them needs to have an end, that is not the same as the others. This is only possible for two of them. $\square$

7 (20 points) A graph G is a *threshold graph* if there is a non-negative number B and a non-negative number a_v for each vertex $v \in V(G)$, such that for any subset $U \subseteq V(G)$, U is an independent set *if and only if* $\sum_{v \in U} a_v \leq B$. Use this definition of threshold graphs to prove:

a K_n is a threshold graph.

b Adding an isolated vertex to a threshold graph gives a threshold graph.

c Adding a dominating vertex (a vertex that is connected to every other vertex) to a threshold graph gives a threshold graph.

d Every threshold graph is a split graph.

Note: *Every* threshold graph can be built by repeatedly doing the two operations above (no proof required).

Solution. a) Let $a_v = B = 1$ for all v. The only independent sets in K_n are sets with one vertex, so this shows, that K_n is a threshold graph.
b)Let v be the isolated vertex. Define $a_v = 0$. Now, this vertex itself is isolated, and if it is part of an isolated subset, it does not change the sum, and so it stays a threshold graph.
c) Case 1: All values could be chosen as 0, so all n vertices before are isolated. For each of these vertices set $a_v = 1$. For the new vertex set $a_v = n + 1$ and set $B = n + 2$. Therefor it is threshold.
Case 2: All other cases: Set the $a_v = B$ for the new vertex. So itself is isolated but adding any other vertex rises the value.
d) By the hint, we know that they are build up by the operations of b) and c). So we can start with one vertex and create the threshold graph. Every dominating vertex is put to the set, that is a clique and every isolated vertex to the other set. Our first vertex could be put in either of the sets. So, every threshold graph is a split graph. $\square$

2. Powers of Graphs

8 (5 points) Let $\{[a_1, b_1], \dots, [a_n, b_n]\}$ be an interval representation of a graph G. Show that $\{[a_1, b_1'], \dots, [a_n, b_n']\}$, where

$$b_i' = \max_j \{b_j : [a_j, b_j] \cap [a_i, b_i] \neq \emptyset\},$$

is an interval representation of the graph G^2.

Solution. We show that for an arbitrary vertex, all other vertices that have maximum distance of two in the original graph are adjacent with the new representation. Therefor, we show that they intersect the nwe interval.
Wlog we can look at the vertex, represented by the interval $[a_1, b_1]$. The old neighbours are included easily because $b_i' \geq b_i$. So take a vertex with distance two to this vertex. Either the interval is completely on the left or the right of $[a_1, b_1]$. If it is on the right, there exists a vertex representing interval, that intersects both of them. Therefor b_1' is high enough, so this two intervals intersect. For an interval completely on the left, the statement is the same, just with the changing b_i' of the other interval. In total it is a representation of G^2. □

9 (10 points) Let T be a tree. Show that $\chi(T^2) = \Delta(T) + 1$.

3. Perfect Graphs

10 (8 points) Prove that the complement of an odd cycle C_{2k+1} with $k > 1$ is not a perfect graph.

Solution. The complement of C_5 is again C_5. C_5 is not perfect, because the chromatic number is 3 but the clique number is 2. Take an odd cycle as in the task. Number the edges once around in a circle. Take the complement a graph. It has a subgraph, that is a 5cycle. For example, We can take the cycle along the vertices 1,3,5,2,4,1. □

11 (10 points) Let G, H be two perfect graphs whose intersection is a complete graph. Prove that $G \cup H$ is perfect.

Solution. Take an arbitrary subset S of vertices of $G \cup H$. If it contains only vertices from one of these graphs, it is perfect. Assume, that it contains vertices from both graphs. Wlog the subset is connected (unconnected things don't change the chromatic or the clique numbers). Take the two subsets $S_1 = S \cap G$ and $S_2 = S \cap H$. For both subsets, the chromatic and the clique number are the same. Because the vertices in common form a complete graph, they all have different colours. So the we take the higher clique number and the higher chromatic number. This are the numbers of our subset, so the graph is perfect. □

12 (9 points) Show that perfection is closed neither under edge deletion nor (simple) contractions.

Solution. Take C_6. It is perfect but contraction by any edge results in C_5, that is not perfect.
Take C_5 and connect two adjacent vertices to a new vertex. This graph is

perfect, but deleting one of the two edges, that are not in C_5 result in a not perfect graph. $\square$

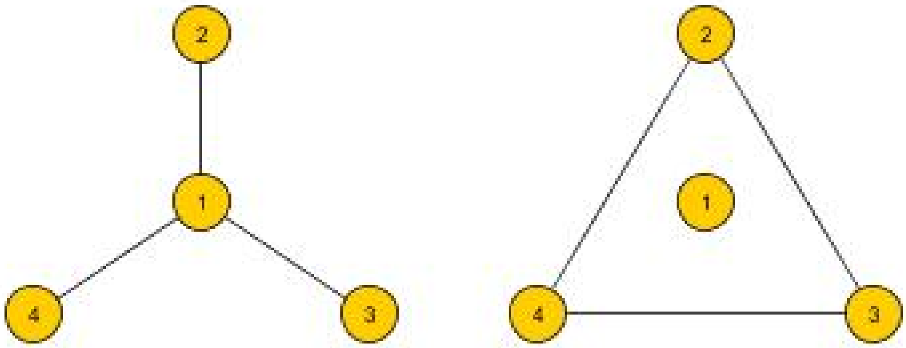
4. Claw-Free Graphs

Definition. A *claw* graph is a star with three leaves, i.e. has four vertices, three of them adjacent to the fourth one. A graph is *claw-free* if it doesn't contain a claw as an *induced* subgraph.

Definition. The line graph $L(G)$ of a graph G is such that each vertex of $L(G)$ represents an edge of G, and two vertices of $L(G)$ are adjacent if and only if their corresponding edges are incident in G (share a vertex). For instance, the line graph of a star is a complete graph.

13 (5 points) Prove that the complement of a triangle-free graph is claw-free.

Solution. Assume there is a claw in the complement. The complement of the complement is the original graph again. So we look at the complement of a claw. It is a triangle, as shown in the following picture. Therefor the original graph needs to have a triangle if the complement has a claw.



$\square$

14 (5 points) Prove that for any graph G, the line graph $L(G)$ is claw-free.

Solution. If there is claw in L(G), the center vertex of this claw has 3 neighbours. An edge is ending in maximum 2 vertices, so in G two of the adjacent edges need to share a vertex. This would mean, they are connected as well, which is a contradiction to the definition of a claw. Therefor, L(G) is claw-free $\square$

15 (7 points) Prove that for any claw-free graph G, $\frac{\Delta(G)}{2} \leq \chi(G) \leq \Delta(G)+1$.

Solution. We have already shown the upper bound for $\chi(G)$by greedy colouring in the lecture. If $\Delta(G) \leq 2$, the claim is trivial, because you need at least one colour. So assume $\Delta(G) > 2$. Let$v$ be a vertex with maximum degree and let $N(v)$ be the neighbourhood of this vertex. The induced graph $[N(v)]$ can maximum contain two cliques, because otherwise you could choose one vertex out of three cliques, that create a claw. Let the bigger or equal clique be S. We can conclude

$$\chi(G) \geq \chi(S \cup \{v\}) = |S| \geq \frac{\Delta(G)}{2}.$$

$\square$

16 (6 points) Prove that every claw-free interval graph is a unit interval graph (i.e. the class of claw-free interval graphs is precisely the class of unit interval graphs).

Solution. If there is a claw, you need three other intervals, that intersect one interval, but not each other. This is obviously not possible for intervals with unit length.
If there is no claw, there are maximum two sets of other intervals, that are not allowed to intersect. This is possible by increasing the length to the other direction. This holds for every vertex.
In total both sets are equivalent. $\square$