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PROBLEM SET 2 (T-445-GRTH)

You need to collect **65** points to get a full score **but** you cannot get more than **X** points (in total) from a problem section with annotation **max X**.

Please make sure to:
1. Write your name/email(s) on your work (replace my name above).
2. Write your answers in `\begin{solution} ... \end{solution}` blocks given after each problem. Turn in a single LaTeX-generated pdf.
3. Write clear and concise proofs: points may be deducted for vagueness.

1. More on Trees (max 40)

1 (5 points) Show that every tree T has at least $\Delta(T)$ leaves, where $\Delta(T)$ denotes the maximum degree of T.

Solution. □

2 (15 points) State necessary and sufficient conditions on an ordered n-tuple of positive integers $(d_1, \dots, d_n)$ with $d_1 \leq d_2 \leq \cdots \leq d_n$ in order that there be a tree T on vertices $u_1, \dots, u_n$ with $\deg_T(u_i) = d_i$ for each $i \in \{1, \dots, n\}$.

Solution. □

3 (10 points) Let $\Delta \geq 3$, and let $d_\Delta(n)$ be the maximum number of nodes of degree Δ that a tree on n vertices may have. Use induction to show that:

$$d_\Delta(n) \leq \left\lfloor \frac{n-2}{\Delta - 1} \right\rfloor .$$

Solution. □

4 **a** (10 points) Show that if a tree T has a longest path of even length, then the mid-vertex of one such longest path is the mid-vertex of every longest path in T. Hint: First show that two such paths can't be vert.-disjoint.

b (5 points) Prove that the common vertex is a center of T.

Solution. □

5 (5 points) A full ternary tree is an ordered rooted tree where each vertex, except the leaves, has exactly 3 children. Hence, all of the internal vertices have degree four, except the root which has degree 3. Prove the *Full Ternary Tree Theorem* which states that a regular ternary tree has $n = 3k + 1$ vertices, k of them internal and $2k + 1$ of them leaves.

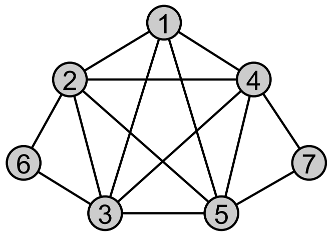
Solution. □

2. Spanning Trees (**max 15**)

6 (7 points) Describe a procedure for finding a spanning tree in a graph. Prove that it indeed finds a spanning tree in every connected graph. Apply it to the graph from the following exercise.

Solution. □

7 (10 points) How many different spanning trees does the following graph contain?



Solution. □

3. Eulerian Graphs (**max 25**)

8 Consider the 3×3 chessboard, and let Q denote the 9 squares of the board. Let $H_{3,3} = (Q, E)$ denote the simple graph on vertex set Q so that $(q_1, q_2) \in E$ if and only if a rook at square q_1 can reach q_2 in a single move (a rook can move horizontally and vertically arbitrary distance).

a (5 points) Run *Fleury's algorithm* for computing an Eulerian circuit on $H_{3,3}$. The algorithm on a simple graph $G = (V, E)$ is as follows:

> Pick a vertex v_1 arbitrarily. Having picked $v_1, \ldots, v_k$, set $G_k = G - \{e_{1,2}, e_{2,3}, \ldots, e_{k-2,k-1}, e_{k-1,k}\}$, where $e_{i,j}$ denotes an edge connecting v_i to v_j. If there is a non-bridge (in G_k) edge connecting v_k to a vertex u then let $v_{k+1} = u$. Otherwise, let v_{k+1} be any neighbor of v_k in G_k. If the degree of v_k in G_k is 0 then terminate. Repeat.

b (5 points) Consider the general $n \times m$ chessboard, for $n, m \geq 1$, and similarly the graph $H_{n,m}$ so that two vertices (squares) are adjacent if and only if a rook can get from one to the other in one move. For which values n, m does the graph $H_{m,n}$ contain a Euler circuit?

c (10 points) Prove that Fleury's algorithm always finds an Eulerian circuit if there is one.

Solution. □

9 (10 points) For an integer k, let G be a connected graph that contains $2k$ vertices of odd degree. Show that there exist k edge-disjoint subgraphs $G_1, \ldots, G_k$ such that

- $E(G) = E(G_1) \cup E(G_2) \cup \cdots \cup E(G_k)$,
- each G_i has an Eulerian trail.

Hint: Add k edges to G so that it becomes Eulerian.

Solution. □

10 (5 points) Prove that a balanced weakly connected graph is strongly connected.

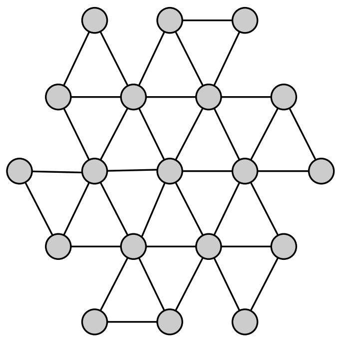
Solution. □

4. Hamiltonian Graphs (max 20)

11 (10 points) Let m and n be positive integers. Consider the *grid graph* $G_{m,n} = (V, E)$ with vertex set $V = \{1, \dots, m\} \times \{1, \dots, n\}$, where vertices $u = (x, y) \in V$ and $v = (x', y') \in V$ are adjacent if and only if $|x - x'| + |y - y'| = 1$. Find necessary and sufficient conditions that m and n must satisfy in order for the graph $G_{m,n}$ to be Hamiltonian.

Solution. □

12 (10 points) Prove that the following graph is not Hamiltonian.



Solution. □

5. Connectivity (max 30)

13 (5 points) Show that a vertex u in a graph G is a cut-vertex if, and only if, there are vertices $v, w \in V(G) \setminus \{u\}$ such that every path between v and w contains the vertex u.

Solution. □

14 (10 points) Show that if G is a graph on n vertices and m edges, we have:

$$\kappa(G) \leq \kappa'(G) \leq \left\lfloor \frac{2m}{n} \right\rfloor.$$

Solution. □

15 (10 points) Let G be a connected graph. Show that if $C \subseteq E(G)$ has an even number of edges in common with every edge cut of G, then C is an edge-disjoint union of cycles.

Solution. □

16 (5 points) For a tree T on n vertices and with maximum degree Δ, what is the minimum number of vertices in its block-cutpoint graph $BC(T)$?

Solution. $\square$