# CSC165 Assignment 1

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## 1 Question

### 1.1 a)

$\forall k \in \mathbb{Z} : k \geq 1 \Rightarrow \frac{1}{(k+1)^2} < \frac{1}{k} - \frac{1}{k+1} < \frac{1}{(k+0)^2}$

### 1.2 b)

$\forall k \in \mathbb{Z} : k \geq 1 \Rightarrow \frac{1}{(k+2)^2} < \frac{1}{k+1} - \frac{1}{k+2} < \frac{1}{(k+0)^2}$

### 1.3 c)

$\forall k \in \mathbb{Z} : k \geq 1 \Rightarrow (k-2)^2 < k^2 + k < (k+2)^2$

### 1.4 d)

Prove A: $\forall k \in \mathbb{Z} : k \geq 1 \Rightarrow \frac{1}{(k+1)^2} < \frac{1}{k} - \frac{1}{k+1} < \frac{1}{(k+0)^2}$

Assume $1 \leq k \in \mathbb{Z}$

Let $k \in \mathbb{Z}$

Then $k \leq k + k$

$\leq k^2$

$\leq k^2 + k$

$< k^2 + 2k$ # $1 \leq k \in \mathbb{Z}$ then 2k will always be larger than k

Then $k^2 + k < k^2 + 2k$

Then $\frac{1}{k^2+k} > \frac{1}{k^2+2k}$ # The larger denominator is the smaller number.

Then $\frac{1}{k} > \frac{1}{k^2+k} > \frac{1}{k^2+2k}$

Then $\frac{1}{k} > \frac{1}{k^2+k} > \frac{1}{k^2+2k+1}$

Then $\frac{1}{(k+0)^2} > \frac{1}{k} - \frac{1}{k+1} > \frac{1}{(k+1)^2}$

Prove C: $\forall k \in \mathbb{Z} : k \geq 1 \Rightarrow (k-2)^2 < k^2 + k < (k+2)^2$

Assume $1 \leq k \in \mathbb{Z}$

Let $k \in \mathbb{Z}$

Then $k \leq k + k$

$\leq k^2$

$\leq k^2 + k$

$> k^2 - k$ # $1 \leq k \in \mathbb{Z}$ k is always positive

Then $k^2 - k < k^2 + k < k^2 + 2k$

Then $k^2 - 4k < k^2 + k < k^2 + 4k$

Then $k^2 - 4k + 4 < k^2 + k < k^2 + 4k + 4$ # $k^2 < k^2 + k$ since $1 \leq k \in \mathbb{Z}$

Then $(k-2)^2 < k^2 + k < (k+2)^2$ # Algebra

**Weaker A**

Disproof: $\forall k \in \mathbb{Z} : k \geq -2 \Rightarrow \frac{1}{(k+1)^2} < \frac{1}{k} - \frac{1}{k+1} < \frac{1}{(k+0)^2}$

Assume $-2 \leq k \in \mathbb{Z}$

Let $k = -2$ # For 0, and -1 the middle rational functions of the inequality becomes undefined.

Then $\frac{1}{(-2+1)^2} < \frac{1}{-2} - \frac{1}{-2+1} < \frac{1}{(-2+0)^2}$

$= \frac{1}{1} \not< \frac{1}{2} \not< \frac{1}{4}$

**Weaker C**

Disproof: $\forall k \in \mathbb{Z} : k \geq 0 \Rightarrow (k-2)^2 < k^2 + k < (k+2)^2$

Assume $0 \leq k \in \mathbb{Z}$

Let $k = 0$

Then $(0-2)^2 < 0^2 + k < (0+2)^2$

$= 4 \not< 0 < 4$

**Stronger A**

Prove: $\forall k \in \mathbb{Z} : k \geq 2 \Rightarrow \frac{1}{(k+1)^2} < \frac{1}{k} - \frac{1}{k+1} < \frac{1}{(k+0)^2}$

Assume $2 \leq k \in \mathbb{Z} \wedge (k \geq 1 \Rightarrow \frac{1}{(k+1)^2} < \frac{1}{k} - \frac{1}{k+1} < \frac{1}{(k+0)^2})$

Then $k \geq 2 \geq 1$

Then $\frac{1}{(2+1)^2} < \frac{1}{2} - \frac{1}{2+1} < \frac{1}{(2+0)^2}$

Then $\frac{1}{9} < \frac{1}{6} < \frac{1}{4}$

Therefore $(k \geq 1 \Rightarrow \frac{1}{(k+1)^2} < \frac{1}{k} - \frac{1}{k+1} < \frac{1}{(k+0)^2}) \Rightarrow (k \geq 2 \Rightarrow \frac{1}{(k+1)^2} < \frac{1}{k} - \frac{1}{k+1} < \frac{1}{(k+0)^2})$

**Stronger C**

Prove: $\forall k \in \mathbb{Z} : k \geq 2 \Rightarrow (k-2)^2 < k^2 + k < (k+2)^2$

Assume $2 \leq k \in \mathbb{Z} \wedge (k \geq 1 \Rightarrow (k-2)^2 < k^2 + k < (k+2)^2)$

Then $k \geq 2 \geq 1$

Then $(2-2)^2 < 2^2 + 2 < (2+2)^2$

Then $0 < 6 < 16$

Therefore $(k \geq 1 \Rightarrow (k-2)^2 < k^2 + k < (k+2)^2) \Rightarrow (k \geq 2 \Rightarrow (k-2)^2 < k^2 + k < (k+2)^2)$

### 1.5 e)

Prove B: $\forall k \in \mathbb{Z} : k \geq 1 \Rightarrow \frac{1}{(k+2)^2} < \frac{1}{k+1} - \frac{1}{k+2} < \frac{1}{(k+0)^2}$

Assume $(k \geq 1 \land \frac{1}{(k+1)^2} < \frac{1}{k} - \frac{1}{k+1} < \frac{1}{(k+0)^2})$

Let $k \in \mathbb{Z}$

Then $\frac{1}{(k+1)^2} < \frac{1}{k} - \frac{1}{k+1}$

Then $\frac{1}{(k+1)^2} < \frac{1}{k} - \frac{1}{k+2}$

Then $\frac{1}{(k+1)^2} \not< \frac{1}{k+1} - \frac{1}{k+2}$ # when k = 1

Then $\frac{1}{(k+2)^2} < \frac{1}{k+1} - \frac{1}{k+2}$

Then $\frac{1}{(k+2)^2} < \frac{1}{k+1} - \frac{1}{k+2} < \frac{1}{(k+0)^2}$

## 2 Question

### 2.1 a)

Prove $\forall x \in \mathbb{R} : [Q(x) \Rightarrow Q(x+1)]$

Assume $Q(x)$

Let $x \in \mathbb{R}$

Let $p, q \in \mathbb{Z}$

Let $m \in \mathbb{Q}$

Then $x = \frac{p}{q}$ # Definition of Rational Number

Then $1 = \frac{1}{1} = \frac{p}{q} \in \mathbb{Q}$

Then $x \land 1 \in \mathbb{Q}$

Then $(x+1) = m \in (Q)$ # The properties of addition of Rational numbers

Therefore $(x+1) \in (Q)$

### 2.2 b)

Prove $\forall x \in \mathbb{R} : [\neg Q(x) \Rightarrow \neg Q(x+1)]$

Proof by Contradiction

Assume $\neg Q(x) \land Q(x+1)$

Let $x \in \mathbb{R}$

Let $p, q \in \mathbb{Z}$

Let $m \in \mathbb{Q}$

Then $x + 1 = \frac{p}{q}$

Then $x = \frac{p}{q} - 1$

Then $x = \frac{p}{q} - \frac{1}{1}$

Then $x \in \mathbb{Q}$ # The properties of subtraction of Rational numbers

Therefore $Q(x) \land \neg Q(x)$ # Contradiction

## 2.3 c)

Prove: $\forall x, y \in \mathbb{R} : [[\neg Q(x) \wedge \neg Q(y)] \Rightarrow \neg Q(xy)]$

Disproof: $\exists x, y \in \mathbb{R} : [[\neg Q(x) \wedge \neg Q(y)] \Rightarrow Q(xy)]$

Assume $\neg Q(x) \wedge \neg Q(y)$

Let $x = \sqrt{2} \in \mathbb{R}$

Let $y = \sqrt{2} \in \mathbb{R}$

Then $xy = \sqrt{2} \cdot \sqrt{2} = 2$

Then $2 = \frac{2}{1} \in \mathbb{Q}$

Therefore $\neg Q(x) \wedge \neg Q(y) \Rightarrow Q(xy)$

## 2.4 d)

Prove: $\forall x, y \in \mathbb{R} : [[\neg Q(xy)] \Rightarrow Q(x) \wedge \neg Q(y)]$

Assume $[Q(x) \wedge \neg Q(y)] \wedge Q(xy)$

Let $x \in \mathbb{R}$

Let $y \in \neg\mathbb{Q}$

Let $p, q, s, t \in \mathbb{Z}$

Then $x = \frac{p}{q}$

Then $xy = \frac{s}{t}$ # Properties of Rational numbers

Then $y \cdot \frac{p}{q} = \frac{s}{t}$ # Subsitution

Then $y = \frac{sq}{tp} \in \mathbb{Q}$ # Algebra

Then $Q(y)$ # Which is a contradiction

## 2.5 d) Converse

Prove: $\forall x, y \in \mathbb{R} : [[Q(x) \wedge \neg Q(y)] \Rightarrow \neg Q(xy)]$

Disproof: $\exists x, y \in \mathbb{R} : [[Q(x) \wedge \neg Q(y)] \Rightarrow Q(xy)]$

Assume $[Q(x) \wedge \neg Q(y)]$

Let $x = 0, y = \sqrt{2}$

Then $xy = 0 \cdot \sqrt{2} = 0$

Then $0 = \frac{0}{0} \wedge 0 \in \mathbb{Z}$

Then $0 \in \mathbb{Q}$

Therefore $Q(xy)$