Strjál stærðfræði 2

Lokapróf á vorönn 2017 - Formúlublað

Ef 2x2 fylkið $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ hefur ákveðu $\det A = a \cdot d - b \cdot c \neq 0$

þá er fylkið andhverfanlegt og andhverfan er $A^{-1} = \frac{1}{\det A} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

LÍKINDAREIKNINGUR:

$P(n,r) = \frac{n!}{(n-r)!}$ $\quad C(n,r) = \binom{n}{r} = \frac{n!}{r! \cdot (n-r)!}$ $\quad \binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$

THEOREM 1 The number of r-permutations of a set of n objects with repetition allowed is n^r.

THEOREM 2 There are $C(n+r-1, r)$ r-combinations from a set with n elements when repetition of elements is allowed.

THEOREM 3 The number of different permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2, ..., and n_k indistinguishable objects of type k, is

$$\frac{n!}{n_1! n_2! \cdots n_k!}.$$

THEOREM 4 The number of ways to distribute n distinguishable objects into k distinguishable boxes so that n_i objects are placed into box i, $i = 1, 2, \ldots, k$, equals

$$\frac{n!}{n_1! n_2! \cdots n_k!}.$$

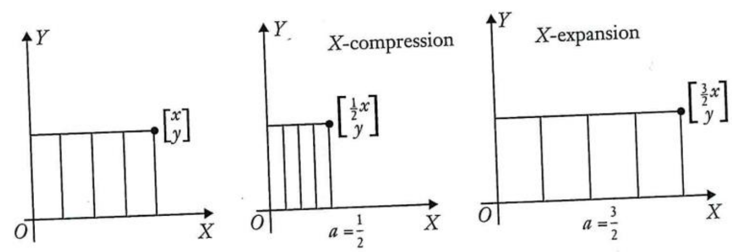
$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

$$b(k; n, p) = C(n,k) \cdot p^k \cdot q^{n-k}$$

X-stríkkun ("X-expansion") og X-þjöppun ("X-compression"):

Example 3

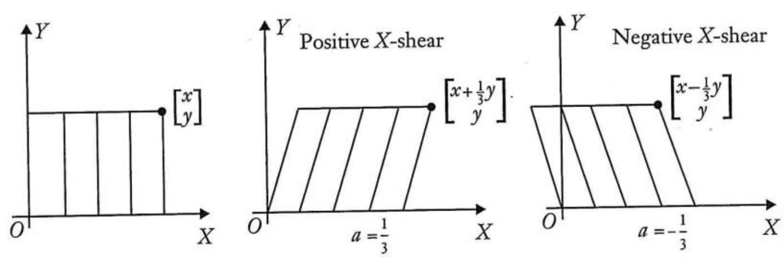
If $a > 0$, the matrix transformation $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax \\ y \end{bmatrix}$ induced by the matrix $A = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$ is called an ***X*-expansion** of $\mathbb{R}^2$ if $a > 1$, and an ***X*-compression** if $0 < a < 1$. The reason for the name is clear in the diagram below. Similarly, if $b > 0$ the matrix $\begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix}$ gives rise to ***Y*-expansions** and ***Y*-compressions**.



X-skekking ("X-shear"):

Example 4

If a is a number, the matrix transformation $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + ay \\ y \end{bmatrix}$ induced by the matrix $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ is called an ***X*-shear** of $\mathbb{R}^2$ (**positive** if $a > 0$ and **negative** if $a < 0$). Its effect is illustrated below when $a = \frac{1}{3}$ and $a = -\frac{1}{3}$.



Snúningur rangsælis um upphafspunkt ("Rotation counterclockwise about origin"):

Example 5

If θ is any angle, let R_θ denote the transformation that rotates $\mathbb{R}^2$ counterclockwise about the origin through the angle θ. Then R_θ is the matrix transformation induced by the matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

Speglun ("reflection") um beina línu gegnum upphafspunkt:

Example 10

Let L denote the line through the origin in $\mathbb{R}^2$ that makes an angle θ with the positive X axis. If $Q : \mathbb{R}^2 \to \mathbb{R}^2$ is reflection in L, show that Q is linear with matrix $\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$.

Fylki fyrir línulega vörpun ("The matrix of a linear transformation"):

Theorem 2

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a transformation.

1. T is linear if and only if it is a matrix transformation.
2. If T is linear, then T is induced by a unique matrix A, given in terms of its columns by

$$A = [T(E_1)\ T(E_2)\ \cdots\ T(E_n)]$$

where $\{E_1, E_2, \ldots, E_n\}$ is the standard basis of $\mathbb{R}^n$.

Theorem 1

Let $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ be vectors. Then:

1. $\vec{v} = \vec{w}$ as vectors if and only if $x = x_1$, $y = y_1$ and $z = z_1$.
2. $\|\vec{v}\| = \sqrt{x^2 + y^2 + z^2}$.
3. $\vec{v} = \vec{0}$ if and only if $\|\vec{v}\| = 0$.
4. $\|a\vec{v}\| = |a|\|\vec{v}\|$ for any scalar a.

Stikajöfnur beinnar línu ("Parametric equations of a line"):

Parametric Equations of a Line

The line through $P_0(x_0, y_0, z_0)$ with direction vector $\vec{d} = [a\ b\ c]^T \neq \vec{0}$ is given by

$$\begin{aligned} x &= x_0 + ta \\ y &= y_0 + tb \quad t \text{ any scalar} \\ z &= z_0 + tc \end{aligned}$$

In other words, the point $P(x, y, z)$ is on this line if and only if a real number t exists such that $x = x_0 + ta$, $y = y_0 + tb$, and $z = z_0 + tc$.

Innfeldi vektora ("Dot product of vectors"):

Theorem 2

Let $\vec{v}$ and $\vec{w}$ be nonzero vectors. If θ is the angle between $\vec{v}$ and $\vec{w}$, then

$$\vec{v} \cdot \vec{w} = \|\vec{v}\|\|\vec{w}\| \cos\theta.$$

Ofanvarp vektors ("Projection of a vector"):

Theorem 4

Let $\vec{u}$ and $\vec{d} \neq \vec{0}$ be vectors.

1. The projection $\vec{u}_1$ of $\vec{u}$ on $\vec{d}$ is given by $\mathrm{proj}_{\vec{d}}\vec{u} = \frac{\vec{u} \cdot \vec{d}}{\|\vec{d}\|^2}\vec{d}$.
2. The vector $\vec{u} - \mathrm{proj}_{\vec{d}}\vec{u}$ is orthogonal to $\vec{d}$.

Jafna fyrir plan ("Equation of a plane"):

Scalar Equation of a Plane

The plane through $P_0(x_0, y_0, z_0)$ with normal $\vec{n} = [a\ b\ c]^T \neq \vec{0}$ is given by

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

In other words, the point $P(x, y, z)$ is on this plane if and only if x, y, and z satisfy this equation.

Krossfeldi vektora ("cross product"):

Given vectors $\vec{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$, define the **cross product** $\vec{v} \times \vec{w}$ by

$$\vec{v}_1 \times \vec{v}_2 = \begin{bmatrix} y_1 z_2 - z_1 y_2 \\ -(x_1 z_2 - z_1 x_2) \\ x_1 y_2 - y_1 x_2 \end{bmatrix}.$$

Einsleit hnit ("Homogeneous coordinates"):

The idea is to represent a point $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ as a 3×1 column $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$, called the homogeneous coordinates of $\vec{v}$. Then translation by $\vec{w} = \begin{bmatrix} p \\ q \end{bmatrix}$ can be achieved by multiplying by a 3×3 matrix:

$$\begin{bmatrix} 1 & 0 & p \\ 0 & 1 & q \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + p \\ y + q \\ 1 \end{bmatrix} = \begin{bmatrix} T_{\vec{w}}(\vec{v}) \\ 1 \end{bmatrix}$$

Tafla um hornaföll

x	$\sin x$	$\cos x$
0	0	1
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1	0
$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$
π	0	-1
$\frac{3\pi}{2}$	-1	0
2π	0	1

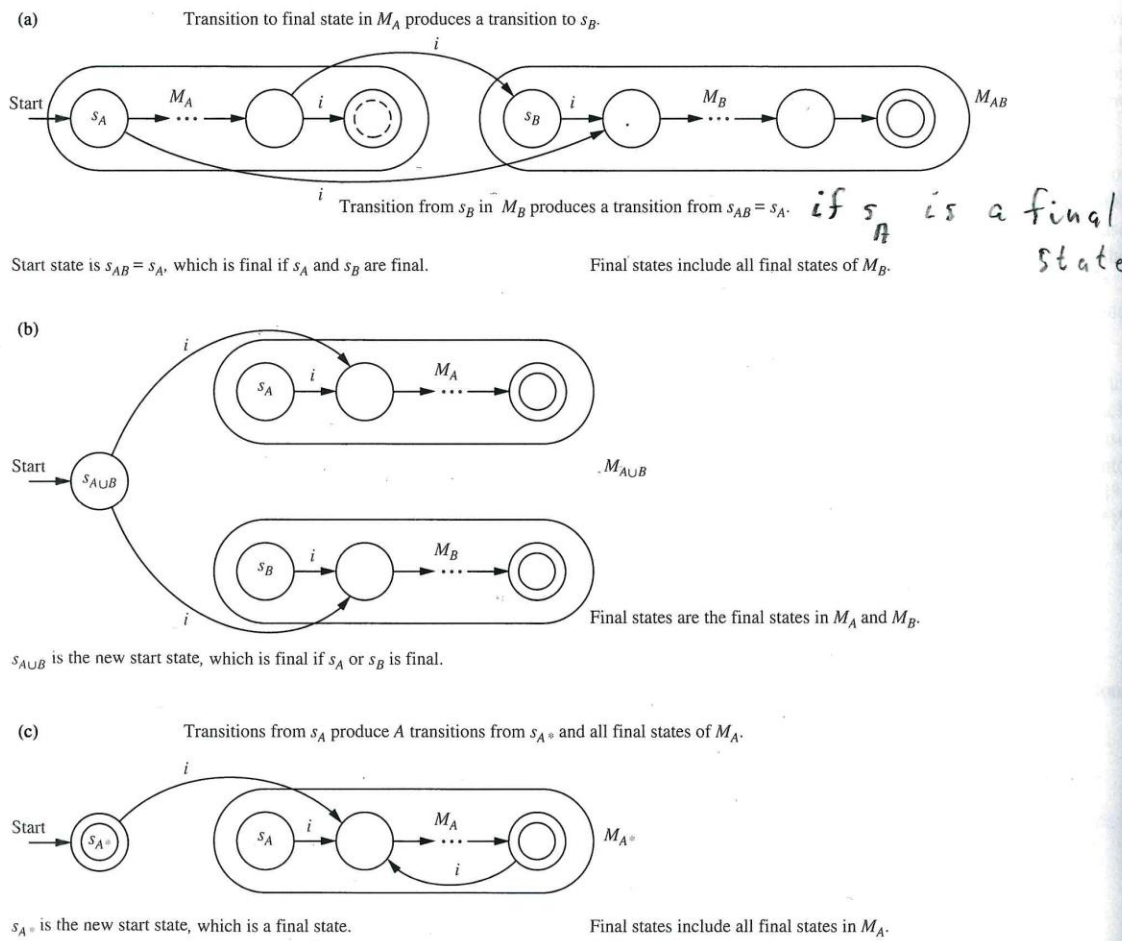


FIGURE 2 **Building Automata to Recognize Concatenations, Unions, and Kleene Closures.**