SEBASTIAN PLENZ (sebastianp16@ru.is) AND AXEL STEINGRIMSSON (axels16@ru.is)

PROBLEM SET 5 (T-445-GRTH)

You need to collect **60** points to get a full score **but** you cannot get more than **X** points (in total) from a problem section with annotation **max X**.

Please make sure to:
1. Write your name/email(s) on your work (replace my name above).
2. Write your answers in `\begin{solution} ... \end{solution}` blocks.
3. Write clear and concise proofs: points may be deducted for vagueness.

1. INDEPENDENT SETS (max 30)

1 (10 points) Find a recursive formula for counting the number of *maximal* independent sets in P_n ($n \geq 1$). Use it to derive a formula counting the number of maximal independent sets in C_n.

Solution. We order the vertices from one end to the other. Take v_1. From every set, create two new sets by adding the vertices v_{i+2} or v_{i+3} to the set of v_i was the last vertex. Redo this step as often as it is possible(both new vertices still exist). If only v_{i+2}exists, add it to the set. Do the same again but starting with v_2. Count all sets. This is the number of maximal independent sets in P_n.
For C_n we can do the same, but if we start with v_1, it is not allowed to have v_n in the same set. So the first and the second part create the same amount of independent sets. Just do the algorithm with starting at v_2 and double the number. □

2 (5 points) Show that:

a For every simple graph G, $\alpha(G) \geq \frac{|V(G)|}{D(G)+1}$.

b For every planar graph G, $\alpha(G) \geq \frac{|V(G)|}{4}$.

Solution.
a) For a simple graph G, if for each vertex v_i in G, you remove v_i and its adjacent neighbours, and continue removing each vertex in this fashion until there are no vertices left, then $\{v_1, v_2, ..., v_i\}$ is an independent set. Because each vertex can have at most $D(G)$ neighbours, we can remove at most $D(G) + 1$ vertices. Therefore there can be at most $\frac{V(G)}{D(G)+1}$ removals, and therefore the independent set must have at least this many vertices.
Therefore for every simple graph G, $\alpha(G) \geq \frac{|V(G)|}{D(G)+1}$.
b) From the lecture we know $\alpha(G) \geq \frac{|V(G)|}{\chi(G)}$. The 4-color-theorem says every planar graph has chromatic number ≤ 4. In total

$$\alpha(G) \geq \frac{|V(G)|}{\chi(G)} \geq \frac{|V(G)|}{4}$$

. □

3 (7 points) Show that:

a For every simple graph G, $\alpha(G) \le |V(G)| - \delta(G)$.

b For every simple triangle-free graph G, $\alpha(G) \ge \Delta(G)$.

Solution. a) Take an arbitrary maximal independent set and a vertex v from this set. All neighbors of this vertex can not be in the maximal independent set and this are at least $\delta(G)$. So $\alpha(G) \le |V(G)| - \delta(G)$. b) Take a vertex v with degree $\Delta(G)$. Take all adjacent vertices of v. This is an indepent set, because the graph is triangle-free and has size $\Delta(G)$. So the claim holds. □

4 (10 points) Show that every simple triangle-free graph has an independent set of size of at least $\lfloor \sqrt{n} \rfloor$.

Solution. For a simple triangle-free graph G we know that if there exists a vertex u with a degree of greater than $\lfloor \sqrt{n} \rfloor$, then there exists an independent set with each neighbour adjacent to u, because the neighbours cannot be adjacent to each other because if they were it would create a triangle. Otherwise all vertices would have a degree of less than $\lfloor \sqrt{n} \rfloor$, The we know that if there exists a vertex v with a degree of less than $\lfloor \sqrt{n} \rfloor$, then there must exist a maximal independent set greater than $\lfloor \sqrt{n} \rfloor$.
Therefore based on the upper and lower bounds being greater than $\lfloor \sqrt{n} \rfloor$, every simple triangle-free graph has an independent set of size of at least $\lfloor \sqrt{n} \rfloor$. □

2. Bipartite Matching (max 26)

5 (5 points) Let $G = (X \cup Y, E)$ be a simple bipartite graph. Suppose that there is a $k \in \mathbb{N}$ such that $d(x) \ge k \ge d(y)$ for all $x \in X, y \in Y$. Show that G has a matching saturating X (covering all of X).

6 (6 points) Let $G = (X \cup Y, E)$ be a simple bipartite graph. Let A be the set of vertices in G of maximum degree. Show that G has a matching saturating $A \cap X$.

7 (6 points) Let $G = (X \cup Y, E)$ be a simple bipartite graph such that $d(x) \ge 1$ for all $x \in X$ and $d(x) \ge d(y)$ for all $\{x, y\} \in E$ ($x \in X, y \in Y$). Show that G has a matching saturating X.

8 (10 points) Let $G = (X \cup Y, E)$ be a simple bipartite graph containing a perfect matching. Prove that there is a vertex $x \in X$, such that for every incident edge $\{x, y\}$, there is a perfect matching that contains $\{x, y\}$.

Solution. Let $G = (X \cup Y, E)$ be a simple bipartite graph containing a perfect matching. A matching is only perfect if it has $\lfloor \frac{n}{2} \rfloor$ edges, which means G has an even amount of vertices and that $|X| = |Y|$ so the perfect match M in G must have $\frac{n}{2}$ edges. Then if M is perfect and G is not disconnected we know that for each match in M there must be an edge not in M that connects each match. This implies that there exists an alternating path P in G, which means for each edge in M that connects u, v, there is are two corresponding edges in P/M that connect both it's vertices. Therefore the edges in P/M provide a perfect match not equal to M. □

9 (6 points) Let X, Y be disjoint independent sets in a simple graph G, such that $|X| = \alpha(G)$. Prove that $G[X \cup Y]$ has a matching of size $|Y|$.

Solution. For every vertex in Y, we can find a vertex in X and connect them, such that it is a matching. This is possible because otherwise, you would find a subset $S \subseteq Y$ and the set of adjacent vertices $S' \subseteq Y$ with $|S| > |S'|$. But if this holds, we can replace S' by S in X and still have an independent subset. This subset would have more vertices than $\alpha(G)$ and therefor is a contradiction. Hence we can find a matching of size $|Y|$. $\square$

3. More Matching (max 50)

10 (6 points) Let G be a simple graph with an even number of vertices such that $d(v) \geq \frac{|V(G)|}{2}$ for every vertex $v \in V(G)$. Show that G has a perfect matching.

Solution. If G is a simple graph with an even number of vertices, then arbitrarily splitting G into two sets $G = S1 \cup S2$ such that $|S1| = |S2|$. Then we know from the definition that each vertex $v \in V(G)$, $d(v) \geq \frac{|V(G)|}{2}$, so we know that each set contains $\frac{|V(G)|}{2}$ and each vertex must have a degree of at least $\frac{|V(G)|}{2}$. Therefore a vertex can be connected to at most $\frac{|V(G)|}{2} - 1$ vertices in it's own set, and therefore every vertex must be connected to a vertex in the opposite set. Therefore every pair of vertices have at least one edge connecting them, implying that a perfect match must exist between $S1$ and $S2$ and therefore in G as well. $\square$

11 (9 points) Let t be a tree on n vertices with l leaves. Show that t has a matching of size at least $\lceil \frac{n-l}{2} \rceil$.

12 (10 points) For every $k \geq 1$, show that every simple k-regular graph has a matching of size at least $\frac{n}{4-\frac{2}{k}}$.
Hint: show that this bound holds for every maximal matching.

13 (10 points) Show that a tree has either one perfect matching or none. For a graph G, denote by $o(G)$ the number of components of G of odd cardinality. Prove that, for a tree T, T has a perfect matching, if and only if, $o(T - v) = 1$ for all vertices v.

14 (15 points) Prove that every simple bridgeless cubic graph (3-regular graph) has a perfect matching. Furthermore, show that there is a cubic graph that has a bridge and does not have a perfect matching.
Hint: Use Tutte's theorem.

15 (13 points) Let G be a graph and M be a maximal matching in G.

a Show that if there is no M-augmenting path of length three then $|M| \geq \frac{2}{3} \cdot opt$, where *opt* is the size of a maximum matching.

b Suppose now that for a given $k > 1$, there is no M-augmenting path of length $2k+1$ or shorter. Prove a better bound on $|M|$ than the one above, and show that your bound is tight by providing an example.