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PROBLEM SET 6 (T-445-GRTH)

You need to collect **65** points to get a full score **but** you cannot get more than **X** points (in total) from a problem section with annotation **max X**.

Please make sure to:
1. Write your name/email(s) on your work (replace my name above).
2. Write your answers in \begin{solution} ... \end{solution} blocks.
3. Write clear and concise proofs: points may be deducted for vagueness.

1. INTERSECTION, CHORDAL, OUTERPLANAR GRAPHS (**max 55**)

Definition. A planar graph G is *outerplanar* if G has an embedding in the plane in such a way that every vertex is on the boundary of the infinite face.

1 (5 points) Prove that every connected unit interval graph has a Hamiltonian path.

2 (8 points) Prove that every outerplanar graph on n vertices has at most $2n-3$ edges.

3 (10 points) Let G be a graph in which any two simple cycles have at most one vertex in common.
- **a** Let B be a block in G. What can you say about the structure of B?
- **b** Prove that G is an outerplanar graph.

4 (7 points) Prove that every unit coin graph on $n > 3$ vertices has at most $3n-7$ edges.

5 (8 points) Prove that for any unit disc graph G, $D(G) \leq 3(\omega(G)-1)$.

6 (20 points) A graph G is a *split graph* if $V(G)$ can be partitioned into subsets X, Y, such that $G[X]$ is a clique and $G[Y]$ is a null graph.
- **a** Prove that split graphs are chordal.
- **b** Prove that the complement of a split graph is a split graph.
- **c** Find a split graph that is not an interval graph.

7 (20 points) A graph G is a *threshold graph* if there is a non-negative number B and a non-negative number a_v for each vertex $v \in V(G)$, such that for any subset $U \subseteq V(G)$, U is an independent set *if and only if* $\sum_{v \in U} a_v \leq B$. Use this definition of threshold graphs to prove:
- **a** K_n is a threshold graph.
- **b** Adding an isolated vertex to a threshold graph gives a threshold graph.
- **c** Adding a dominating vertex (a vertex that is connected to every other vertex) to a threshold graph gives a threshold graph.
- **d** Every threshold graph is a split graph.

Note: *Every* threshold graph can be built by repeatedly doing the two operations above (no proof required).

2. Powers of Graphs

8 (5 points) Let $\{[a_1, b_1], \dots, [a_n, b_n]\}$ be an interval representation of a graph G. Show that $\{[a_1, b'_1], \dots, [a_n, b'_n]\}$, where

$$b'_i = \max_j \{b_j : [a_j, b_j] \cap [a_i, b_i] \neq \emptyset\},$$

is an interval representation of the graph G^2.

9 (10 points) Let T be a tree. Show that $\chi(T^2) = \Delta(T) + 1$.

3. Perfect Graphs

10 (8 points) Prove that the complement of an odd cycle C_{2k+1} with $k > 1$ is not a perfect graph.

11 (10 points) Let G, H be two perfect graphs whose intersection is a complete graph. Prove that $G \cup H$ is perfect.

12 (9 points) Show that perfection is closed neither under edge deletion nor (simple) contractions.

4. Claw-Free Graphs

Definition. A *claw* graph is a star with three leaves, i.e. has four vertices, three of them adjacent to the fourth one. A graph is *claw-free* if it doesn't contain a claw as an *induced* subgraph.

Definition. The line graph $L(G)$ of a graph G is such that each vertex of $L(G)$ represents an edge of G, and two vertices of $L(G)$ are adjacent if and only if their corresponding edges are incident in G (share a vertex). For instance, the line graph of a star is a complete graph.

13 (5 points) Prove that the complement of a triangle-free graph is claw-free.

14 (5 points) Prove that for any graph G, the line graph $L(G)$ is claw-free.

15 (7 points) Prove that for any claw-free graph G, $\frac{\Delta(G)}{2} \leq \chi(G) \leq \Delta(G)+1$.

16 (6 points) Prove that every claw-free interval graph is a unit interval graph (i.e. the class of claw-free interval graphs is precisely the class of unit interval graphs).