Notations

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Contents

Abstract

This is a rough notes mainly created for being a cheat sheet for the notations in Moeglin and Waldspurger, hopefully as well as containing some reading notes (an ϵ difference from copying the book, hopefully). The idea is to understand them in simple cases hopefully: number field, no or double covering, SL_2, GL_2, GL_3, Sp_4, etc... The use of mathbf, that is, finite covering of some (sub)group, is messy, in the sense that it should not matter, that is, be treated as not considering covering.

0 Notations

I.1.1:

- k global field
- $\mathbb{A}$ ring of adeles of k
- v finite place of k
- ϑ_v ring of integers
- $\mathbb{A}_f$ ring of finite adeles
- $\mathbb{A}_\infty$ product over archimedean places
- G connected reductive algebraic group defined over k
- $i'_G : G \hookrightarrow GL_n$ embedding defined over k, with closed image
- $i_G : G \hookrightarrow GL_{2n}$ is
$$i_G(g) = \begin{pmatrix} i'_G(g) & 0 \\ 0 & {}^t i'_G(g)^{-1} \end{pmatrix}$$
- K maximal compact subgroup of $G(\mathbb{A})$
- $\mathbf{G}$ (resp. other bold letters) a topological group which is a finite central covering of $G(\mathbb{A})$ (roughly, resp. other groups)
- U unipotent

I.1.4:

- M Levi subgroup
- P_0 minimal parabolic subgroup
- M_0 standard Levi
- Z_G centre of G
- $\mathrm{Rat}(M)$ the group of rational characters of M (homomorphisms as algebraic groups of M into the multiplicative group $\mathbb{G}_m$)

- $\mathfrak{a}_M^* = \mathrm{Rat}(M) \bigotimes_{\mathbb{Z}} \mathbb{C}$
- $\mathfrak{a}_M = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Rat}(M), \mathbb{C})$
- $\mathfrak{a}_{M,\mathbb{Q}}^* = \mathrm{Rat}(M) \otimes_{\mathbb{Q}} \mathbb{R}$
- $\mathfrak{a}_M = \mathfrak{a}_{M,\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$
- $\mathrm{Re}\mathfrak{a}_M^* = \mathrm{Rat}(M) \otimes_{\mathbb{Z}} \mathbb{R}$
- $\mathrm{Re}\mathfrak{a}_M = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Rat}(M), \mathbb{R})$
- $M^1 = \bigcap_{\chi \in \mathrm{Rat}(M)} \ker |\chi|$
- T_M maximal split torus of Z_M
- X_M the group of continuous homomorphisms of $M(\mathbb{A})$ into $\mathbb{C}^*$ which are trivial on M^1
- $\log_M$ natural map of $M(\mathbb{A})$ into $\mathfrak{a}_M$ defined by

$$\forall m \in M(\mathbb{A}), \forall \chi \in \mathrm{Rat}(M), \langle \chi, \log_M(m) \rangle = \log(m^{|\chi|})$$

- $\kappa : \mathfrak{a}_M^* \to X_M$ surjective morphism of groups
- $\mathrm{Re}X_M = \kappa(\mathrm{Re}\mathfrak{a}_M^*)$
- $\mathrm{Im}X_M = \kappa(i\mathrm{Re}\mathfrak{a}_M^*)$
- X_M^G the subgroup of X_M consisting of the characters of M/M^1 trivial on Z_G
- $m_P : \mathbf{G} \to \mathbf{M}^1 \backslash \mathbf{M}$ map defined by

$$m_P(g) = \mathbf{M}^1 m \text{ if } g = umk, u \in U(\mathbb{A}), m \in \mathbf{M}, k \in \mathbf{K}$$

I.1.5:

- $Z_G^1 = Z_G(\mathbb{A}) \cap G^1$

I.1.6:

- T_0 maximal split torus of the centre of M_0
- $R(T_0, G)$ the set of roots of G relative to T_0
- res_0 the restriction of M_0 to T_0
- $\check{\beta}$ coroot
- $\langle \chi, \check{\beta} \rangle := \langle \mathrm{res}_0 \chi, \check{\beta} \rangle$
- $R^+(T_0, C)$ a set of positive roots

- $\triangle_0$ a subset of simple roots
- $\triangle_0^M = \triangle_0 \cap R(T_0, M)$
- T_M maximal split torus in the centre of M
- $\triangle_M$ the subset of $R(T_M, G)$ which consists of the non-trivial restrictions of elements of $\triangle_0$
- $\mathrm{Re}X_{M_0}^M$ the set consisting of those of $\mathrm{Re}X_{M_0}$ trivial on the centre of $M(\mathbb{A})$
- $\mathrm{Re}(\mathfrak{a}_M^{M'})^*$ the real vector subspace of $\mathrm{Re}\mathfrak{a}_M^*$ generated by $R(T_M, M')$

I.1.7:

- W the Weyl group of G, where $W = \mathrm{Norm}_{G(k)} T_0(k)/\mathrm{Cent}_{G(k)} T_0(k)$
- w representative of W in $G(k)$
- W_M the Weyl group of M
- $W(M)$ the set of elements $w \in W$ of minimal length in their class wW_M, such that wMw^{-1} is again a standard Levi

I.1.11:

- $\check{\alpha} = n^{-1}\check{\alpha}_0$ where for unique indivisible root $\alpha_0 \in R^+(T_M, G)$ and unique $n \in \mathbb{Z}$ such that $\alpha = n\alpha_0$, if $\check{\alpha}_0$ is defined
- α^* *see the corresponding section*

I.1.13:

- ρ_0 half-sum of positive roots of T_0
- ρ_P the half-sum of positive roots of M in Lie U

I.2.1:

- A_{M_0} the unique connected subgroup of T_0 which projects onto $\mathbb{R}_+^{*R}$
- $A_M = A_{M_0} \cap Z_M$
- $\mathbf{M}^c = \mathbf{M}^1$ in number fields
- $S = \{pak : p \in \omega, a \in A_{M_0}(t_0), k \in \mathbf{K}\}$ where ω is a compact subset of P_0, t_0 an element of $\mathbf{M}_0$
- $S^P = \omega A_{M_0} \mathbf{K}$ if $P = P_0$

I.2.2:

- $\|g\|$ height function:

$$\|g\| = \prod_v \sup\{|g_{r,s}|_v : r, s = 1, \cdots, 2n\}.$$

I.2.3:

- δ the right translation action
- $\mathscr{U}$ the enveloping algebra of the Lie algebra of $\mathbf{G}_\infty$

I.2.5:

- $L^2(G(k)\backslash\mathbf{G})_\xi$ the square integrable functions on $G(k)\backslash\mathbf{G}$ with central character ξ,

I.2.6:

- $\phi_P(g) = \int_{U(k)\backslash U(\mathbb{A})}\phi(ug)du$ measurable, locally L^1 function on $U(\mathbb{A})\backslash\mathbf{G}$

I.2.7:

- $\hat{\triangle}^M$ the basis of $(\mathfrak{a}^M_{M_0})^*$ dual to the basis of coroots $\{\check{\alpha};\alpha\in\triangle^M\}$ of $\mathfrak{a}^M_{M_0}$
- $\mathbf{M}_0(P,t)$ the set of elements $m\in\mathbf{M}_0$ such that

$$m^\alpha > t^\alpha \text{ for all } \alpha\in\triangle_M,$$

$$m^\omega \leq t^\omega \text{ for all } \omega\in\hat{\triangle}^M$$

I.2.9:

- $A^G_M = \{a\in A_M : \log_G a = 0\}$
- $s\phi$ the alternating sum $s\phi(g) = \sum_{P=MU:P_0\subset P\subset G}(-1)^{rg(G)-rg(M)}\phi_P(g)$

I.2.13:

- $\hat{\tau}_P$ the characteristic function of the sum of $\mathrm{Re}\mathfrak{a}_G$ and of the interior of the cone of $\mathrm{Re}\mathfrak{a}^G_M$ generated by the elements of $\triangle_M$.
- $\wedge^T$ the truncation operator, see 1.2.16

I.2.17:

- $\mathfrak{z}$ the Bernstein centre of $\mathbf{G}_{v_0}$
- $\phi_k : M(k)\backslash\mathbf{M}\to\mathbb{C}$ by

$$\phi_k(m) = m^{-\rho_P}\phi(mk)$$

  where $\phi : U(\mathbb{A})M(k)\backslash\mathbf{G}\to\mathbb{C}$ and $k\in\mathbf{K}$
- $\langle\phi_0,\phi\rangle = \int_{Z_M U(\mathbb{A})M(k)\backslash G}\overline{\phi_0}(g)\phi(g)dg$ where $\phi_0\in A_0(U(\mathbb{A})M(k)\backslash\mathbf{G})_{\xi^*}$

I.2.18:

- $\phi^{cusp}\in A_0(U(\mathbb{A})M(k)\backslash\mathbf{G})_\xi$ such that

$$\langle\phi_0,\phi^{cusp}\rangle = \langle\phi_0,\phi\rangle$$

  for all $\phi_0\in A_0(U(\mathbb{A})M(k)\backslash\mathbf{G})_\xi$

- $A_0(\mu,\sigma)_\xi$ the eigensubspace of elements of $A_0(U(\mathbb{A})M(k)\backslash\mathbf{G})_\xi$ for $\mathfrak{z}$ with eigenvalue μ, on which $\mathbf{K}$ acts via σ

I.3.2:

- $A(U(\mathbb{A})M(k)\backslash\mathbf{G})_{Z,\xi} = \bigoplus_\eta A(U(\mathbb{A})M(k)\backslash\mathbf{G})_\eta$ over $\eta \in \mathrm{Hom}(Z_M,\mathbb{C}^*)$ such that $\eta|_{Z_G} = \xi$
- $\Pi_0(M)_\xi$ the set of isomorphism classes of irreducible representations of $\mathbf{M}$ occurring as submodules of $A_0(M(k)\backslash\mathbf{M})_\xi$
- $A_0(M(k)\backslash\mathbf{M})_\pi$ the isotypic component of type π of $A_0(M(k)\backslash\mathbf{M})_\xi$ where $\pi \in \Pi_0(\mathbf{M})_\xi$
- χ_π the character of irreducible representation π of $\mathbf{M}$
- $\mathrm{Re}\pi = \mathrm{Re}\chi_\pi$
- $\mathrm{Im}\pi = \pi \otimes (-\mathrm{Re}\pi)$
- $-\pi$ the contragredient representation of π
- $-\overline{\pi}$ the conjugate of the contragredient representation of π

I.3.3 and I.3.5:

- $D(M,\phi)$ see 1.3.7
- $\Pi_0(\mathbf{M},\phi)$ see 1.3.7

I.3.4:

- $C_0(U(\mathbb{A})M(k)\mathbf{G})$ see 1.3.10

I.3.5:

- ϕ_P^{cusp} the unique cuspidal form that is weakly equivalent to some automorphic component, see 1.3.14

I.4.2:

- $A(d,R,Y)$ set of some functions, see 1.4.2

I.4.3:

- $s(\varphi,r) = \sup\{\|g\|^{-r}|\varphi(g)|; g \in \mathbf{G}\}$ see 1.4.4

I.4.4.:

- $A((V_P,\Gamma_P,N_P)_{P_0\subset P\subset G})$ set of some automorphic forms, see 1.4.6

1 Hypotheses, automorphic forms, constant terms

1.1 Hypotheses and general notation

We require the group K to have the properties

1. $G(\mathbb{A}) = P_0(\mathbb{A})K$;
2. for every $P = MU$ standard parabolic of G, we have $P(\mathbb{A}) \cap K = (M(\mathbb{A}) \cap K)(U(\mathbb{A}) \cap K)$ and $M(\mathbb{A}) \cap K$ is still a compact maximal subgroup of $M(\mathbb{A})$.

From the second property, when choosing K we also fix a choice of maximal compact subgroup of $M(\mathbb{A})$ for every standard Levi.

Example 1.1.1. If we consider $\mathbf{G} = \mathrm{GL}_n$, and the standard Levi to be just diagonal entries, then the rational characters $\mathrm{Rat}(M)$ are roughly rational functions in the diagonal entries, for example, the determinant map. Then $\mathfrak{a}_M^*$ is simply giving this group the structure as a complex vector space and its dual $\mathfrak{a}_M$ can be viewed as, for example, the evaluation of the rational function on certain diagonal entries. When M is bigger than just diagonal, for example, contains a 2×2 block, then $\mathfrak{a}_M$ can be larger than diagonal.

Characters, root systems.
Now each $\chi \in \mathrm{Rat}(M)$ defines an algebraic character for every place v of k from $M(k_v)$ into k_v^*. Take each $m \in M(\mathbb{A})$, we can write $m = (m_v)$ and then we would be able to compute $m_v^{\chi_v}$ which denotes the value of χ_v at the point m_v, then we evaluate this using the absolute value of this place then taking product, where we write

$$m^{|\chi|} := \prod_v |m_v^{\chi_v}|_v,$$

and then the group M^1 which is the intersection of the kernel of $|\chi|$ over all rational character χ can be viewed as a generalization of $\mathbb{Q} \hookrightarrow \mathbb{A}$ with χ being the Hecke characters in the GL_1 case.

Then the definition of X_M being the group of continuous homomorphisms of $M(\mathbb{A})$ into $\mathbb{C}^*$ trivial on M^1 can be viewed as a generalization of the Hecke character.

Remark 1.1.2. The group $\mathrm{Rat}(M)$ is really a free $\mathbb{Z}-$module since M is connected. Hence we can pick a basis $\chi_1, \cdots, \chi_R$ of this module and denote by V the image of $\mathbb{A}^*$ under the absolute value map. Hence in the number field case, V simply is $\mathbb{R}_+^*$.

Lemma 1.1.3. *Take connected algebraic group G and its maximal torus A, the character groups $X(G)$ then can be restricted to $X(A)$, where the map is injective and its image is a finite index subgroup of $X(A)$.*

Proof. Consider the derived group G_D of G, which would then be a semisimple group and hence has no non-trivial character, i.e., $X(G_D)$ is trivial. Now consider the map

$$f : (G_D \times A)/N \longrightarrow G,$$

$$(x, y) \longmapsto xy^{-1}$$

where $N := \{(x, x) | x \in G_D \cap A\}$, which is indeed an isomorphism. Hence $X(G) \simeq X(G_D \times A/N)$ and note that N is finite. Therefore, $X(G)$ is of finite index inside $X(G_D \times A) \simeq X(G_D) \times X(A)$. Now $X(G_D)$ is trivial hence $X(G)$ is of finite index inside $X(A)$ and the map is injective. $\square$

Proposition 1.1.4. *Define a map* $j : M(\mathbb{A}) \to V^R = (R_+^*)^R$ *in the number field case by*

$$j : M(\mathbb{A}) \longrightarrow V^R,$$

$$m \longmapsto (m^{|\chi_1|}, \cdots, m^{|\chi_R|}).$$

The kernel of this map is $\mathbf{M}^1$ *and its image is all of* V^R. *Moreover, this map defines a topological group isomorphism from* $M(\mathbb{A})/M^1$ *onto* $j(M(\mathbb{A}))$.

Proof. Recall that $\mathbf{M}^1 = \bigcap_{i=1}^R \ker |\chi_i|$ by definition, then $\ker(j) \subset \mathbf{M}^1$. On the other hand, any element in $\mathbf{M}^1$ will be mapped to $(1, \cdots, 1)$ and hence $\ker(j) = \mathbf{M}^1$.

For the image, let us just consider the number field case. First suppose $M(\mathbb{A})$ is itself a split torus, i.e., we have $M(\mathbb{A}) \simeq \mathbb{G}_m^R$. Then j really is a map from $\mathbb{A}^{*R}$ to its value group under absolute value map for each coordinate and hence is surjective.

If on the other hand $M(\mathbb{A})$ is not a split torus itself, we can simply look at such inside it, i.e., $T_M \subset Z_M \subset M$. Now consider the restriction $\chi_i|_{T_M}$ which gives an character on T_M but we can see that all such χ_i do not really generate $\mathrm{Rat}(T_M)$. However, such restricted characters generate a subgroup of $\mathrm{Rat}(T_M)$ and such a subgroup is of finite index by previous lemma. As a topological group, there is no finite index subgroup of R_+^* other than the whole group, thus the images of $\chi_i|_{T_M}$ would indeed generate all of V^R. We then have the diagram

$$\begin{tikzcd} M(\mathbb{A})/M^1 \arrow[rr, Rightarrow, "j"] \arrow[dr, "f"'] & & V^R = (\mathbb{R}_+^*)^R \arrow[dl, "p"] \\ & \mathbb{C}^* & \end{tikzcd}$$

commutes, meaning that any character on $M(\mathbb{A})/M^1$ factors through p, which from topology results we know that must have the form $p(a_1, \cdots, a_R) = a_1^{s_1} \cdots a_R^{s_R}$ for complex numbers s_i. $\square$

This result makes us conclude the following theorem.

Theorem 1.1.5. *The group* X_M *has the property that if* $\lambda \in X_M$, *there exists* $\chi_1, \cdots, \chi_R \in$ *Rat*(M) *and* $s_1, \cdots, s_R \in \mathbb{C}$ *such that for all* $m \in M(\mathbb{A})$, *we have*

$$m^\lambda = (m^{|\chi_1|})^{s_1} \cdots (m^{|\chi_R|})^{s_R}.$$

In the case of number fields, it tells us that we have the surjective homomorphism

$$\kappa : \mathfrak{a}_M^* \longrightarrow X_M$$

which is indeed a bijection.

Example 1.1.6. We can think about the GL_1 case, where X_M is really Hecke characters and $\mathfrak{a}_M^*$ by definition are complexified rational characters on $\mathbb{A}^*$. Roughly speaking, this is the result that Hecke characters on $\mathbb{Q}$ has the form $|\det|^s$ without the unitary part (trivial on M^1).

Definition 1.1.7. With the notations as above, we can define a natural **logarithm** map $\log_M$ for $\chi \in \mathrm{Rat}(M)$ and $m \in M$ we have $\log_M$:

$$M(\mathbb{A}) \longrightarrow \mathfrak{a}_M,$$

$$m \longmapsto (\chi \mapsto \langle \chi, \log_M(m) \rangle = \log(m^{|\chi|})),$$

with kernel M^1 and image real values. In number field case, $\log_M(\mathbf{M}/\mathbf{M}^1) \simeq \mathrm{Re}\mathfrak{a}_M$.

Recall that we write X_M^G for the subgroup of X_M consisting of the characters of M/M^1 trivial on Z_G, the centre of G and the map m_P which takes g to $M^1 m$ if $g = umk$ for the UMK decomposition. Consider any larger Levi subgroup M' that contains M, it suggests that we would have the diagram

$$\begin{tikzcd}
 & M \arrow[rr, "\log_M"] \arrow[dr, two heads] \arrow[dd, hook] & & \mathrm{Re}\mathfrak{a}_M \arrow[dd, hook, "f^*"] & \mathrm{Re}\mathfrak{a}_M^* \simeq \mathrm{Re}X_M \\
 & & M^1\backslash M \arrow[ur, "\tilde{\log}_M"'] & & \\
\mathbf{G} \arrow[urr, "m_P"] \arrow[drr, two heads, "m_{P'}"'] & M' \arrow[rr, "\log_{M'}"] \arrow[dr, two heads] & & \mathrm{Re}\mathfrak{a}_{M'} & \mathrm{Re}\mathfrak{a}_{M'}^* \simeq \mathrm{Re}X_{M'} \arrow[uu, hook, "f\text{,inclusion}"'] \\
 & & (M')^1\backslash M' \arrow[ur, "\tilde{\log}_{M'}"'] & &
\end{tikzcd}$$

commutes.

For a set of roots given by T_0, maximal split torus of the minimal Levi M_0, denoted by $R(T_0, G)$, we have a pairing $\langle , \rangle$ between the subgroup of rational characters of a split torus and subgroup of its one-parameter subgroups. For every coroot $\check{\beta}$ in this root system, for every $\chi \in \mathrm{Rat}(M_0)$ where M_0 minimal Levi, we have the pairing induced by restriction map from M_0 to T_0. Extending this pairing linearly, in number fields case, we define thus a pairing for all $\lambda \in X_{M_0}$ and $\check{\beta}$ coroot. This tells us the following.

Proposition 1.1.8. *Every coroot $\check{\beta}$ can be identified with an element in $\mathrm{Re}\mathfrak{a}_{M_0}$. In number fields, for any $\lambda \in X_{M_0}$, i.e., continuous characters on M_0 that are trivial on M_0^1, we can define $\langle \lambda, \check{\beta} \rangle$.*

We obtain an isomorphism

$$\mathrm{Re}X_{M_0} \simeq \mathrm{Re}\mathfrak{a}_{M_0}^* = \mathrm{Rat}(M_0) \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathrm{Rat}(T_0) \otimes_{\mathbb{Z}} \mathbb{R} \hookleftarrow \{\text{roots}\}$$

and thus we can simply view roots as elements in $\mathrm{Re}X_{M_0}$ and by duality, view coroots as elements in $\mathrm{Re}\mathfrak{a}_{M_0}$, the linear forms on $\mathrm{Re}X_{M_0}$. They are indeed rational elements in the latter ones.

Now we can think about the root system given by T_0 inside M, denoted by $R(T_0, M)$, and in this way, we can define its simple roots to be the simple roots that lies in it, that is, $\triangle_0^M = \triangle_0 \cap R(T_0, M)$. However, we write $R(T_M, G)$ where T_M is the maximal split torus in the center of M which is **not** really a root system of G. Indeed, $R(T_M, G)$ can be identified with a subset of $\mathrm{Re}X_M$ generating it.

Remark 1.1.9. Consider M_0 minimal Levi, we have the restriction map $|_{M_0}$ from any Levi M to M_0. By duality, this induces an embedding of $\mathrm{Re}\mathfrak{a}_M^*$ in $\mathrm{Re}\mathfrak{a}_{M_0}^*$. The map projecting from $\mathrm{Re}\mathfrak{a}_{M_0}^*$ onto $\mathrm{Re}\mathfrak{a}_M^*$ would induce then a restriction map from $R(T_0, G)$ to $R(T_M, G) \cap \{0\}$ as we identified them above.

Denote by $\mathrm{Re}(\mathfrak{a}_{M_0}^M)^*$ the real vector subspace in $\mathrm{Re}\mathfrak{a}_{M_0}^*$ generated by elements of $R(T_0, M)$ (again, we identified root systems with them). We have

$$\mathrm{Re}\mathfrak{a}_{M_0}^* \simeq \mathrm{Re}\mathfrak{a}_M^* \oplus \mathrm{Re}(\mathfrak{a}_{M_0}^M)^*,$$

where $\mathrm{Re}(\mathfrak{a}_{M_0}^M)^* \simeq \mathrm{Re}X_{M_0}^M$, those $\mathrm{Re}X_{M_0}$ that are trivial on the centre of $M(\mathbb{A})$. With similar notation fashion, in number field case, we also have that by restriction, the map $\mathfrak{a}_M^* \to X_M$ gives an isomorphism

$$(\mathfrak{a}_M^{M'})^* \simeq X_M^{M'}.$$

Finally we arrive at the isomomorphism

$$\mathrm{Re}X_M \simeq \mathrm{Re}X_{M'} \oplus \mathrm{Re}X_M^{M'},$$

and by working dually, we also have the decomposition

$$\mathfrak{a}_{M_0} = \mathfrak{a}_{M_0}^M \oplus \mathfrak{a}_M$$

where the first term on the right hand side is the complexification of the real vector subspace of $\mathrm{Re}\mathfrak{a}_{M_0}$ generated by the coroots associated with $R(T_0, M)$.

Weyl elements

Recall the definition W the Weyl group of G, which is the quotient

$$W = \mathrm{Norm}_{G(k)}T_0(k)/\mathrm{Cent}_{G(k)}T_0(k).$$

Every $w \in W$ can be represented by some element of $G(k)$ and since we are able to embed $G(k)$ into GL it is natural to have the **Bruhat decomposition**

$$G(k) = \bigcup_{w \in W} P_0(k) w P_0(k).$$

[TBD]

Scalar product, measures

Recall the property of root system and since we identified them with our main objects, we have that on the space $\mathrm{Re}\mathfrak{a}_{M_0}^*$, we can choose some positive definite scalar product which is invariant under W. This can be extended to any $\mathrm{Re}\mathfrak{a}_M^* \simeq \mathrm{Re}X_M$ for any M standard Levi by the restriction map from M to M_0. By extension of scalars this also gives a symmetric bilinear form $(,)$ on $\mathfrak{a}_M^* \times \mathfrak{a}_M^*$ and by Hermitian extension $\langle , \rangle$ on

$\mathfrak{a}_M^*$, and by duality on $\mathfrak{a}_M$. Denote $\|\cdot\|$ to be the norms induced by these Hermitian products.

We use the κ map above and define $\mathrm{Im}X_M := \kappa(i\mathrm{Re}\mathfrak{a}_M^*)$. In the case of number field, we obtain a positive definite Hermitian product on X_M. The real vector spaces $\mathrm{Re}\mathfrak{a}_{M_0}$ and $\mathrm{Re}\mathfrak{a}_{M_0}^*$ are hence also equipped with Haar measures where the measure of a length 1 cube should be 1.

Here we want to have the measure of $U(k)\backslash U(\mathbb{A})$ to be 1, with $U(k)$ having counting measure, which induces a Haar measure of $U(\mathbb{A})$, which we call du. As $\mathbf{K}$ is (cover of) compact, we can simply normalize it to give a Haar measure dk such that $\int_{\mathbf{K}} dk = 1$. Further we simply fix a measure on $\mathbf{M}_0$. Then take ρ_0 the half-sum of positive roots of T_0 and f some continuous function with compact support (so that the integral later will be well-defined without worrying about M) on $\mathbf{G}$, we can indeed define a Haar measure dg on $\mathbf{G}$ such that

$$\int_{\mathbf{G}} f(g)dg = \int_{U_0(\mathbb{A})\times\mathbf{M}_0\times\mathbf{K}} f(umk)m^{-2\rho_0}dkdmdu$$

holds. The same method can be generalized to define a measure on any standard Levi $\mathbf{M}$ by using the already defined dg such that

$$\int_{G} f(g)dg = \int_{U(\mathbb{A})\times\mathbf{M}\times\mathbf{K}} f(umk)m^{-2\rho_P}dkdmdu.$$

In addition, the map $\lambda \mapsto \kappa(i\lambda)$ from $\mathrm{Re}\mathfrak{a}_M^G$ to $\mathrm{Im}X_M^G$ induces a Haar measure on $\mathrm{Im}X_M^G$.

Example 1.1.10. In the case of SL_2, for example, all the definitions for Haar measure are the familiar ones, and we note that the half sum of positive roots is in this case the modular character $\delta^{1/2}$ where $P = B$ the Borel subgroup. Now the Levi M is the torus $\mathrm{diag}(y^{1/2}, y^{-1/2})$ corresponding to the Lie algebra trace zero, and the root is simply has weight 2 so the power of m here is simply 2 again and m parameterized by y which would give us the classical Poincare metric $dxdy/y^2$.

1.2 Automorphic forms: growth, constant terms

Torus and height

In the number field case, we fix an embedding $\mathbb{R}_+^* \hookrightarrow \mathbb{A}^*$ such that in Archimedean places it is the same value and 1 on non-Archimedean places. Therefore, we are able to compose $\mathbb{R}_+^* \hookrightarrow \mathbb{A}^* \hookrightarrow T_0(\mathbb{A})$ indeed identifying $\mathbb{R}_+^{*R}$ with a split subgroup of $T_0(\mathbb{A})$ as it is simply connected. Now we define A_{M_0} to be the **unique** connected subgroup of T_0 which projects onto V^R (V again is $\mathbb{R}_+^*$). Taking

$$A_{M_0} = \bigcap_{w\in W} wA_{M_0}w^{-1},$$

we can safely say that A_{M_0} is invariant under W.

Generally, if we want A_M for standard Levi M, which is defined to be $A_{M_0} \cap Z_M$, we then define a map $\log^M$ such that

$$\begin{tikzcd}
M_0 \arrow[r, "\log : m \mapsto (\log(m^{|\chi|}))"] \arrow[dr, "\log^M"'] & \mathfrak{a}_{M_0} \arrow[d, "\mathrm{pr}"] \\
& \mathfrak{a}^M_{M_0}
\end{tikzcd}$$

commutes. Then we can show the following.

Lemma 1.2.1. *Let M_0 be minimal Levi and M some standard Levi contains M_0. Let U be the unipotent subgroup that generates M with T_0. Therefore, T_0 acts on U via roots $\alpha \in R(T_0, M)$ identified with characters. If for $a \in A_{M_0}$ we have that $a^\alpha = 1$ the trivial map, then a commutes with M, that is, $a \in Z_M$.*

Proof. Root space decomposition. Make use of the identity $t_0 n(b_0)^\alpha t_0^{-1} = n(t_0^{2\alpha} b_0)$? □

Proposition 1.2.2. *Indeed, we have $A_M = \{a \in A_{M_0} : \log^M a = 0\}$. That is, we have*

$$\begin{tikzcd}
A_{M_0} \arrow[rr, "\log^M"] \arrow[dr, two heads] & & \mathfrak{a}^M_{M_0} \\
& A_{M_0}/A_M \arrow[ur] &
\end{tikzcd}$$

commutes.

Proof. One direction is clear, that is, for any $a \in A_M = A_{M_0} \cap Z_M$, we have a commutes with all elements in M and thus the roots acts as zero map which is $a^\alpha = 1$ when composed with exponential map. Then under the log map we would obtain zero in $\mathfrak{a}_{M_0}$ and thus $\mathfrak{a}^M_{M_0}$.

On the other hand, take $a \in A_{M_0}$ such that $\log^M a = 0$. If we can show $a \in Z_M$ then we are done by definition. By our choice of a, it commutes with M_0 and to show that it commutes with M we need to check the part is not generated by M_0, that is, the part that is generated by some unipotent subgroup U where the torus T_0 acts via roots α. By the previous lemma, we only need to show that $a^\alpha = 1$. In number fields, recall the definition of A_{M_0} so characters on $a \in A_{M_0}$ only take real values. That is, $a^{|\alpha|}$ is the same as a^α and since $\log^M(a) = 0$, by definition, it means that a is either in the kernel of the map $m \mapsto \log(m^{|\chi|})$ or under log the image of a is in the kernel of pr from $\mathfrak{a}_{M_0}$ to $\mathfrak{a}^M_{M_0}$. If it is the latter case, then it means that $\log_{M_0}(a) \in \mathfrak{a}_M$, a contradiction. Therefore it must be in the kernel of $\log_{M_0}$ which means that by definition $a^{|\alpha|} = 1$. Hence $a \in A_M$. □

If we define $A^G_M = \{a \in A_M : \log_G a = 0\}$, then at least in number field case, we have decomposition

$$A_M = A_G \oplus A^G_M.$$

The set $A_M \backslash M / M^1$ is finite, in number field cases, moreover, we can choose $\mathbf{M}^1$ such that

$$\mathbf{M} = A_{\mathbf{M}} \mathbf{M}^1.$$

We define the "matrix norm", **height** as following. First we have an embedding from the reductive groups to matrices ((GL_n really). Then let us write the image of g under the embedding in coordinates, i.e., $i(g) = (g_{r,s})_{r,s=1,\cdots 2n}$. For each place v, we can evaluate its "local height" by $\|g\|_v = \sup\{|g_{r,s}|_v : r, s = 1, \cdots, 2n\}$. Then taking product over all places we obtain the **height**

$$\|g\| = \prod_v \sup\{|g_{r,s}|_v : r, s = 1, \cdots, 2n\}.$$

Theorem 1.2.3. *For the height function we defined, there are many properties worth noting:*

1. *there is some $c > 0$ such that $\|g\| > c$ for all $g \in \mathbf{G}$, that is, the height of a reductive group is bounded below,*
2. *there is some $c > 0$ such that $\|g_1 g_2\| < c\|g_1\|\|g_2\|$ for every $g_1, g_2 \in G$, that is, the norm of product is "bounded" by the product of the norm,*
3. *there is some $c, r > 0$ such that $\|g^{-1}\| < c\|g\|^r$ for all $g \in \mathbf{G}$,*
4. *the height is related to the norm on $\mathfrak{a}_{M_0}$, that is, there is some $c, c'' > 0$ and $c' \in \mathbb{R}$ such that for all $a \in A_{M_0}$ we have*

   $$c\|\log_{M_0}\| \leq \log\|a\| + c' \leq c''\|\log_{M_0} a\|,$$

5. *let $\lambda \in Re(X_{M_0})$, there is some $c, r > 0$ such that*

   $$m_{P_0}(g)^\lambda \leq c\|g\|^r$$

   for all $g \in \mathbf{G}$,
6. *there is $\lambda \in ReX_{M_0}$ and $c > 0$ such that*

   $$\|g\| \leq c m_{P_0}(g)^\lambda$$

   for all $g \in \mathbf{G}^c \cap S$, which with (5) shows that $\|.\|$ is somehow "controlled by" the evaluation of some characters on m_{P_0}.
7. *There is $c > 0$ such that for all $\gamma \in G(k), g \in S$, we have*

   $$\|g\| \leq c\|\gamma g\|;$$

8. *there exists $c, t, t' > 0$ such that for all $a \in A_G$ and $g \in \mathbf{G}^c$, we have*

   $$c\|a\|^{t'}\|g\|^t \leq \|ag\|.$$

We omit the proof for now.

Moderate growth

Consider space of functions V stable under right translations by **G**, in this book this action is denoted by δ. We know we can therefore define the action of the algebra of functions of **G** on V as well as the algebra of differential operators, which will still be denoted by δ as they are "induced" by the translation action.

Example 1.2.4. We can consider for example SL_n and the algebra of function are just functions defined on it and the differential operators are really the universal enveloping algebra from its Lie algebra.

Definition 1.2.5. Let $\phi : G \to \mathbb{C}$ be a function, then we say ϕ has **moderate growth** if there is $c, r \in \mathbb{R}$ such that for all $g \in \mathbf{G}$, we have

$$|\phi(g)| \leq c\|g\|^r.$$

Moreover, in the number field case, we denote G_∞ to be the product of Archimedean places, and assume ϕ is C^∞, we say that ϕ and its derivatives (with respect to $\mathscr{U}$) have **uniformly moderate growth** if

$$\exists r \in \mathbb{R}, \forall X \in \mathscr{U}, \exists c_X \in \mathbb{R}, \forall g \in \mathbf{G}$$

we have

$$|\delta(X)\phi(g)| \leq c_X\|g\|^r.$$

Proposition 1.2.6. *In number field cases, consider functions that are $U(\mathbb{A})M(k)$ invariant, that is, functions on $U(\mathbb{A})M(k)\backslash\mathbf{G} \to \mathbb{C}$, then ϕ has moderate growth if and only if ϕ and its derivatives have uniformly moderate growth. This can be explicitly written as: there is some $c, r \in \mathbb{R}$ and $\lambda \in ReX_{M_0}$ such that for all $a \in A_M, k \in A_M, m \in M^1 \cap S^P$ we have*

$$|\phi(amk)| \leq c\|a\|^r m_{P_0}(m)^\lambda.$$

Proposition 1.2.7. *There is $r > 0$ and for every compact subset $C \subset \mathbf{G}$, there is $c > 0$ such that*

1. *for all $x, y \in \mathbf{G}$, we have*
$$\sum_{\gamma\in\mathbf{G}(k)} 1_C(x^{-1}\gamma y) \leq c\|x\|^r;$$
2. *for all $x, y \in \mathbf{G}^c$, we have*
$$\int_{G(k)Z_G} 1_C(x^{-1}\gamma y)d\gamma \leq c\|x\|^r.$$
3. *There exists $r' > 0$ and for every compact C we have $c > 0$ such that for all $x \in \mathbf{G}$, $y \in S$, we have*
$$\sum_{\gamma\in G(k)} 1_C(x^{-1}\gamma y) \leq c\|x\|^{r'}\|y\|^{-r};$$
4. *for all $x \in \mathbf{G}^c, y \in S \cap \mathbf{G}^c$ we have*
$$\int_{G(k)Z_G} 1_C(x^{-1}\gamma y)d\gamma \leq c\|x\|^{r'}\|y\|^{-r}.$$

Those are used to prove several properties regarding the functions we care about here. For instance, let $f : \mathbf{G} \to \mathbb{C}$ be a smooth function with compact support, then if we have $\phi : G(k)\backslash\mathbf{G} \to \mathbb{C}$ be a function with moderate growth, then $\delta(f)\phi$ (the action of f on ϕ induced by right translation) has moderate growth and in number field case, its derivatives and itself have uniformly moderate growth. It might also be good to list the similar property for L^2 functions as well.

Proposition 1.2.8 (For square-integrable functions.). *Let ξ be a unitary character of Z_G, trivial on $Z_G(k) \cap Z_{\mathbf{G}}$, let $\phi \in L^2(G(k)\backslash\mathbf{G})_\xi$. Then $\delta(f)\phi$ satisfies the same property as above.*

In this chapter functions will be mainly with moderate growth as some results can give results for L^2 functions as well. For instance,

Proposition 1.2.9. *For all $\epsilon > 0$, there is a function $f : \mathbf{G} \to \mathbb{C}$, smooth with compact support, K—biinvariant, such that*

$$\|\delta(f)\phi - \phi\| < \epsilon.$$

The function $\delta(f)\phi$ is smooth K—biinvariant and has moderate growth.

Definition 1.2.10. Let $P = MU$ be a standard parabolic of G, ϕ measurable and locally L^1 on $U(k)\backslash\mathbf{G}$. Define a measurable locally L^1 function ϕ_P, the **constant term of ϕ with respect to** P on $U(\mathbb{A})\backslash\mathbf{G}$ by

$$\phi_P(g) = \int_{U(k)\backslash U(\mathbb{A})} \phi(ug)du.$$

If ϕ left invariant under $G(k)$, has moderate growth and smooth, so is ϕ_P.

The constant terms play an important role in this theory. First of all, we deal with the approximation.

Approximation of a function by its constant term

Here we assume k to be a number field only. Let us denote rg(G) and rg(M) for the semi-simple ranks of G and M, for example, rg(G) $= |\triangle_0|$.

As the proof can be technical, let us write the result we want to have here first.

Theorem 1.2.11. *Suppose k number field, K'_f compact open subgroup of $\mathbf{G}_f$. Let $r > 0, \lambda \in ReX_{M_0}, X \in \mathscr{U}$ and $c > 0$. Then there exist two finite subsets $\{X_i : i = 1, \cdots, N\} \subset \mathscr{U}$ and $\{c_i : i = 1, \cdots, N\} \subset \mathbb{R}^*_+$ such that the following property is satisfied:*

Let ϕ be a smooth function on $G(k)\backslash\mathbf{G}$, right K'_f invariant. Suppose that for all $i \in \{1, \cdots, N\}$ and all $g \in \mathbf{G}$, we have the inequality

$$|\delta(X_i)\phi(g)| \leq c_i\|g\|^r.$$

Furthermore, define a function $s\phi$ on S by

$$s\phi(g) = \sum_{P=MU:P_0 \subset P \subset G} (-1)^{rg(g)-rg(M)}\phi_P(g).$$

Then for all $a \in A_G, g \in G^1 \cap S$, we have the inequality

$$|\delta(X)s\phi(ag)| \leq c\|a\|^r m_{p_0}(g)^\lambda.$$

Roughly speaking, this theorem tells us that there are some derivatives of ϕ if they are bounded above, and if we look at the "alternating sum" over all the constant terms of ϕ with respect to all parabolic subgroups containing the minimal standard one, this sum's derivative under the enveloping algebra is also "bounded" in terms of A_G and its Levi part, per se.

By omitting the proof and even the lemma needed for the proof, we can proceed to the next definitions.

Definition 1.2.12. Let $\phi : S \to \mathbb{C}$, and recall that $S = \{pak, p \in \omega, a \in A_{M_0}(t_0), k \in \mathbf{K}\}$ where ω compact subset of P_0. We say ϕ is **rapidly decreasing** if there exists $r > 0$ and if for all $\lambda \in \mathrm{Re}X_{M_0}$, there exists $c > 0$ such that for all $a \in A_G$, $g \in \mathbf{G}^1 \cap S$, we have

$$|\phi(ag)| \leq c\|a\|^r m_{P_0}(g)^\lambda.$$

Further, if we look at functions on a different (indeed larger) domain, that is, $\phi : G(k)\backslash\mathbf{G} \to \mathbb{C}$, we say that ϕ is **rapidly decreasing** if the restriction of ϕ to S is rapidly decreasing.

This allows us to restate Theorem 1.2.11 as follows:

Theorem 1.2.13. *Take k number field. Let $\phi : G(k)\backslash\mathbf{G} \to \mathbb{C}$ be a smooth $\mathbf{K}$–finite function. Define $s\phi$ on S by*

$$s\phi(g) = \sum_{P=MU:P_0\subset P\subset G} (-1)^{rg(G)-rg(M)}\phi_P(g).$$

If ϕ and its derivatives have uniformly moderate growth. Then $s\phi$ is rapidly decreasing.

Arthur's Truncation

This part is note even proved in the book itself. It refers the proof to the papers "A trace formula for reductive groups I: terms associated to classes in $G(\mathbb{Q})$" and "A trace formula for reductive groups. II: applications of a truncation operator" by Arthur.

Let us first set the notations. The important new notation here is $\hat{\tau}_P$, which is defined to be the characteristic function of the sum of $\mathrm{Re}\mathfrak{a}_G$ and of the interior of the cone of $\mathrm{Re}\mathfrak{a}_M^G$ generated by the elements of $\triangle_M$. Furthermore, *fix* some $T \in \mathrm{Re}\mathfrak{a}_{M_0}$ where T is supposed to be sufficiently positive, that is, for all $\alpha \in \triangle_0$, we have the pairing $\langle\alpha, T\rangle$ will be large enough. Denote by also T_M the projection of T onto $\mathrm{Re}\mathfrak{a}_M$ (do not confuse with T_M subtorus of M). We can now state our first lemma.

Example 1.2.14. For instance, if we consider the case $\mathbf{G} = GL_2$ and its parabolic to be the Borel $P = B$. Then the $\triangle_M$ would be $\mathbb{R}^2$ while $\mathfrak{a}_G$ is $x_1 = x_2$. The interior of the cone corresponds to the part $x_1 > x_2$. Then $\hat{\tau}_P = 1_{\mathrm{Re}\mathfrak{a}_G+(x_1>x_2)} = 1_{x_1\geq x_2}$. This function is really characteristic function on some cone area in the Weyl chamber. What T does is to shift the center from the origin to T.

Lemma 1.2.15. *Let $r > 0, c > 0$, there exist $r' > 0, c' > 0$ such that the following property is satisfied: Let $\phi : P(k)\backslash\mathbf{G} \to \mathbb{C}$ be a function. Suppose that*

$$|\phi(g)| \leq c\|g\|^r$$

for all $g \in \mathbf{G}$. Let $g \in \mathbf{G}$. Then

$$\sum_{\gamma\in P(k)\backslash G(k)} |\phi(\gamma g)|\hat{\tau}_P(\log_M(m_P(\gamma g)) - T_M) \leq c'\|g\|^{r'}.$$

In particular, the sum if finite.

Proof. Arthur's first paper mentioned above, lemma 5.1 and corollary 5.2. □

This allows us to define the *truncation operator*.

Definition 1.2.16. Let $\phi : G(k)\backslash \mathbf{G} \to \mathbb{C}$ be a locally L^1 function. The **truncation operator** $\wedge^T$ acts on ϕ by

$$\wedge^T\phi(g) = \sum_{P=MU:P_0\subset P\subset G} ((-1)^{\mathrm{rg}(G)-\mathrm{rg}(M)} \times \sum_{\gamma\in P(k)\backslash G(k)} \hat{\tau}_P(\log_M(m_P(\gamma g)) - T_M)\phi_P(\gamma g)).$$

Remark 1.2.17. If ϕ is a cusp form, then the truncation operator acts as identity, that is, $\wedge^T\phi = \phi$.

Let us list several results regarding such defined operator. Note that if ϕ is just locally L^1, then the truncated function is only defined almost everywhere. If we further suppose that ϕ locally bounded, and if ϕ continuous, then $\wedge^T\phi(g)$ is defined for all g and $\wedge^T$ is *locally bounded*. Then if ϕ moderate growth, so does its truncation.

Proposition 1.2.18. *Let $\phi : G(k)\backslash \mathbf{G} \to \mathbb{C}$ be a locally L^1 function. Then we have the equality*

$$\wedge^T \wedge^T \phi(g) = \wedge^T\phi(g)$$

for almost all g. (If ϕ is also locally bounded then the above is true for all g.)

Proposition 1.2.19. *Let ξ be a unitary character of Z_G, ϕ_1 and ϕ_2 two locally L^1 functions on $G(k)\backslash \mathbf{G}$ of 'central character' ξ. Suppose ϕ_1 has moderate growth and ϕ_2 is rapidly decreasing (recall we assume k number field only), then*

$$\int_{Z_G G(k)\backslash \mathbf{G}} \overline{\wedge^T\phi_1}(g)\phi_2(g)dg = \int_{Z_G G(k)\backslash \mathbf{G}} \overline{\phi_1}(g) \wedge^T \phi_2(g)dg.$$

Automorphic forms

We finally can define the *automorphic forms* now!

Definition 1.2.20. Let $P = MU$ be a standard parabolic subgroup of G and $\phi : U(\mathbb{A})M(k)\backslash \mathbf{G} \to \mathbb{C}$ be a function. We say that ϕ is **automorphic** if it satisfies the following:

1. ϕ has moderate growth;
2. ϕ is smooth;
3. ϕ is $\mathbf{K}-$finite;
4. ϕ is $\mathfrak{z}-$finite.

Somehow we are familiar with functions on $M(k)\backslash \mathbf{M}$, and we can move the domain as follows. For $\phi : U(\mathbb{A})M(k)\backslash \mathbf{G} \to \mathbb{C}$ and $k \in \mathbf{K}$, define a function on $M(k)\backslash \mathbf{M}$ by

$$\phi_k(m) = m^{-\rho_P}\phi(mk),$$

and recall that ρ_P is the half-sum of positive roots of M in $\mathrm{Lie}(U)$. That is, given ϕ and each element in the maximal compact subgroup we can associate an automorphic form on $M(k)\backslash \mathbf{M}$.

This correspondence is really induced by the map $P' \mapsto P' \cap M$. Then the automorphic forms defined on $M(k)\backslash \mathbf{M}$ is really in correspondence with the forms defined on $U(\mathbb{A})M(k)\backslash \mathbf{G}$. An important identification is then:

Remark 1.2.21. By definition above, function ϕ is automorphic if and only if it is smooth, $\mathbf{K}-$finite and for all $j \in \mathbf{K}$ we have ϕ_j is automorphic over $M(k)\backslash \mathbf{M}$. The difficulty is the proof with respect to (4).

Let ξ be a character of Z_M, that is, a continuous homomorphism from Z_M to $\mathbb{C}^*$, the so-called "**central character**". We denote by $A(U(\mathbb{A})M(k)\backslash \mathbf{G})_\xi$ the subspace of elements $\phi \in A(U(\mathbb{A})M(k)\backslash \mathbf{G})$ satisfying that for all $g \in \mathbf{G}, z \in Z_M$, we have

$$\phi(zg) = z^{\rho_P}\xi(z)\phi(g).$$

In other words, the functions such that the value of z factors thru a character and of course, with consideration of the modular character. We also can assume $\xi|_{Z_M(k) \cap Z_M} = 1$. Otherwise by multiplying elements in the latter set, we would obtain $\phi = 0$, which is not interesting to us. Note further that we can consider the action of G on these two spaces, however, in number field cases, the condition (3), that is $\mathbf{K}-$finite condition, is not stable under the action. Instead, we can consider the action of

$$\text{Lie}(\mathbf{G}_\infty) \times \mathbf{K}_\infty \times \mathbf{G}_f$$

on these two spaces.

Further, we have the following results passing from $M(k)\backslash \mathbf{M}$ to $U(\mathbb{A})M(k)\backslash \mathbf{G}$:

Proposition 1.2.22. *Let $\{P_i = M_i U_i : i = 1, \cdots n\}$ collection of parabolic subgroups of G and for all i, we define V_i to be a finite dimensional subspace of $A(U_i(\mathbb{A})M_i(k)\backslash \mathbf{G})$. Then there is a function $f : \mathbf{G} \to \mathbb{C}$ of compact support, smooth and $\mathbf{K}-$biinvariant, such that for all $i = 1, \cdots, n$ and all $\phi \in V_i$, we have*

$$\delta(f)\phi = \phi.$$

Furthermore, in the number field case, for all P and ϕ, the function itself and its derivatives have moderate growth.

The above proposition mainly tells us that the automorphic forms we defined can be "averaged" by values on compact sets (consider the function f to be the characteristic function on K, for example). Moreover, its derivatives are also well-behaved.

Cuspidal forms

Recall our elementary ideas about cuspidal forms are functions that vanish at the cusps. In the classical case, that is, SL_2, we have only one proper parabolic subgroup P that is the upper triangular matrices. In general, we need to be more careful about this.

Definition 1.2.23 (Cuspidal forms)**.** Let $P = MU$ be a standard parabolic subgroup of G, and ϕ be an automorphic form on $U(\mathbb{A})M(k)\backslash \mathbf{G}$. That is, for one standard parabolic subgroup P, we consider automorphic forms that are defined with respect to this particular parabolic subgroup. Then we consider all parabolic subgroups P' that contain this minimal parabolic P_0 and properly contained in the parabolic P we consider. Then we say ϕ is **cuspidal** if for all P' (correspondingly U') we have

$$\phi_{P'}(g) = \int_{U'(k)\backslash U'(\mathbb{A})} \phi(ug)du = 0,$$

where we recall definition 1.2.10 for the meaning of the notation, that is, constant terms of ϕ with respect to P'. We use subscript 0 to denote cuspidal forms, that is, $A_0(U(\mathbb{A})M(k)\backslash\mathbf{G})$ and $A_0(U(\mathbb{A})M(k)\backslash\mathbf{G})_\xi$ for the obvious meanings. Further, in number field cases, ϕ is "rapidly decreasing" if it is cuspidal.

For any character μ of the Bernstein center $\mathfrak{z}$ and any irreducible representation σ of $\mathbf{K}$, we define $A_0(\mu,\sigma)_\xi$ to be the eigenspace of elements in $A_0(U(\mathbb{A})M(k)\backslash\mathbf{G})_\xi$ where $\mathfrak{z}$ acts with eigenvalue μ and $\mathbf{K}$ acts via σ. Note that they must be compatible to be nontrivial.

Definition 1.2.24 (Inner product)**.** For any $\phi \in A(U(\mathbb{A})M(k)\backslash\mathbf{G})_\xi$ an automorphic form with central character ξ, and $\phi_0 \in A_0(U(\mathbb{A})M(k)\backslash\mathbf{G})_{\xi^*}$, where $\xi^* = \overline{\xi}^{-1}$, we can define an inner product between them as

$$\langle \phi_0, \phi \rangle = \int_{Z_M U(\mathbb{A})M(k)\backslash\mathbf{G}} \overline{\phi}_0(g)\phi(g)dg.$$

Remark 1.2.25. We require at least one of them to be cuspidal, similar to the classical case, where there is no natural inner product on modular forms but if one is cusp form the definition would work. Moreover, if ξ is unitary character, we have $\xi^* = \overline{\xi}^{-1}$, and indeed this gives an inner product on the space $A_0(U(\mathbb{A})M(k)\backslash\mathbf{G})_\xi$ and thus this space is a pre-Hilbertian space. If we take $P = G$ the whole group, this gives $L^2_0(G(k)\backslash\mathbf{G})_\xi$.

Proposition 1.2.26. *There is a unique cuspidal form attached to each automorphic form worth mentioning. Consider* $\phi \in A(U(\mathbb{A})M(k)\backslash\mathbf{G})_\xi$*, there is a unique cuspidal form* ϕ^{cusp} *in* $A_0(U(\mathbb{A})M(k)\backslash\mathbf{G})_\xi$ *such that*

$$\langle \phi_0, \phi^{cusp} \rangle = \langle \phi_0, \phi \rangle$$

for all $\phi_0 \in A_0(U(\mathbb{A})M(k)\backslash\mathbf{G})$*.*

Remark 1.2.27. The idea is that, given any automorphic form with some central character, its inner product with cuspidal forms in the same space is no difference from some unique cuspidal form in this space.

Moreover, if we consider the automorphic space with central character ξ, where the Lie algebra acts via character μ and maximal compact subgroup has representation σ, we have that $A_0(\mu,\sigma)_\xi$ is finite dimensional, with $A_0(\mu^*,\sigma)_{\xi^*}$ being its dual and finite dimensional as well, where μ^* is defined to be the character from μ via the inverse map $g \mapsto g^{-1}$ composed with complex conjugation.

Now let us define $\phi^{\mathrm{cusp}}(\mu,\sigma) \in A_0(\mu,\sigma)_\xi$ that satisfies the above equation for $\phi_0 \in A_0(\mu^*,\sigma)_{\xi^*}$. Importantly, ϕ being $\mathbf{K}-$finite and $\mathfrak{z}-$finite implies that $\phi^{\mathrm{cusp}}(\mu,\sigma) = 0$ for almost all (μ,σ). Then let

$$\phi^{\mathrm{cusp}} = \sum_{\mu,\sigma} \phi^{\mathrm{cusp}}(\mu,\sigma).$$

1.3 Cuspidal components

A_M-finite functions, decomposition of an automorphic form

Let us look at the characters on A_M first, i.e., we are looking at the continuous homomorphism of A_M to $\mathbb{C}^*$, denoted by $\mathrm{Hom}(A_M, \mathbb{C}^*)$. We are familiar with the properties of X_M, the characters on M that is trivial on M^1. Then restriction of M to A_M gives a map from X_M to $\mathrm{Hom}(A_M, \mathbb{C}^*)$. If we call the kernel of this map ker $= X_M(A_M)$, we have the diagram

$$\begin{tikzcd} X_M \arrow[rr] \arrow[dr, two heads] & & \mathrm{Hom}(A_M, \mathbb{C}^*) \\ & X_M/X_M(A_M) \arrow[ur, hook] & \end{tikzcd}$$

commutes. The identity element in $\mathrm{Hom}(A_M, \mathbb{C}^*)$ are characters trivial on A_M and its preimage are thus characters on M that are trivial on A_M. Now in the number field case, recall that we have the identity $\mathbf{M} = A_M\mathbf{M}^1$. Furthermore, as elements in X_M are already trivial on $\mathbf{M}^1$, such characters are just trivial on the whole M, that is, $X_M(A_M) = \{0\}$. Now in number field cases, we can identify **the enveloping algebra of the Lie algebra of the real group** A_M as

$$\mathfrak{z}(A_M) \simeq \mathrm{Sym}_{\mathbb{R}}(\mathfrak{a}_M) \otimes_{\mathbb{R}} \mathbb{C}.$$

Therefore, we identify $\mathfrak{z}(A_M)$ with the symmetric polynomial algebra over the complex space $\mathfrak{a}_M^*$, and the characters on A_M.

It is worth mentioning that in function field cases, by the map $z \mapsto \hat{z}$ (*Fourier-Mellin transform* $\hat{z}$) given by

$$\hat{z}(\lambda) = \int_{A_M} z(a)a^{-\lambda}da,$$

the enveloping algebra can be identified with $X_M/X_M(A_M)$.

Proposition 1.3.1. *Every smooth and $\mathfrak{z}(A_M)-$finite function on A_M is a linear combination of functions of the form*

$$a \mapsto a^\lambda Q(\log_M a),$$

where $Q \in \mathfrak{q}_M$, a basis of the space $\mathbb{C}[Re\mathfrak{a}_M]$ of polynomials over $Re\mathfrak{a}_M$, consisting of homogeneous elements, and $\lambda \in \Lambda$ some finite set in $X_M/X_M(A_M)$, with some further conditions for degree of Q.

The degree is restricted as follows. Consider this finite set $\Lambda \subset X_M/X_M(A_M)$ and a family $N = (N_\lambda)_{\lambda\in\Lambda}$ of nonnegative integers. We denote $\mathfrak{z}(A_M, \Lambda, N)$ the space of $z \in \mathfrak{z}(A_M)$ such that for all $\lambda \in \Lambda$, we have that its transformation $\hat{z}$ vanishes to the order at least N_λ in N (recall that $\hat{z}$ is a function on λ the characters). Note that this is an ideal of finite codimension of $\mathfrak{z}(A_M)$. Now we require $\deg(Q) < N_\lambda$. This identification will be useful for the later decomposition.

Convention 1.3.2. Let $\mathrm{Hom}(Z_M, \mathbb{C}^*)$ be the set of characters of Z_M. Moreover, we can denote by $A(U(\mathbb{A})M(k)\backslash\mathbf{G})_Z$ the summation such that

$$A(U(\mathbb{A})M(k)\backslash\mathbf{G})_Z := \sum_{\xi\in\mathrm{Hom}(Z_M,\mathbb{C}^*)} A(U(\mathbb{A})M(k)\backslash\mathbf{G})_\xi.$$

That is, this is all the automorphic forms with central characters coming from the central characters of Z_M.

Let $Q \in \mathbb{C}[\mathrm{Re}\mathfrak{a}_M]$ and $\psi \in A(U(\mathbb{A})M(k)\backslash\mathbf{G})_Z$, then

$$g \longmapsto Q(\log_M m_P(g))\psi(g)$$

is a function on $\mathbf{G}$. When elements $um \in U(\mathbb{A})M(k)$ acts on the left, by the definition of the function m_P, we see that it is left invariant. The function so-defined is an automorphic form on $U(\mathbb{A})M(k)\backslash\mathbf{G}$.

Now consider any element $\phi \in A(U(\mathbb{A})M(k)\backslash\mathbf{G})$, by definition and the arguments above, we have that ϕ is $\mathfrak{z}^M-$finite. Then we have some finite set $\Lambda \subset X_M$, associated with some integer set N such that ϕ is killed by $\mathfrak{z}(A_M : \Lambda, N)$. The functions

$$a \longmapsto f_{\lambda,Q,g}(a) := (am_P(g))^\lambda Q(\log_M(am_P(g)))$$

where $\mathfrak{q}_M$ a basis of $\mathbb{C}[\mathrm{Re}\mathfrak{a}_M]$, on A_M form a basis of smooth functions killed by $\mathfrak{z}(A_M : \Lambda, N)$. This gives us a result from differential equations that for ϕ on $\mathbf{G}$ we ought to have that we would have elements $\psi_{\lambda,Q} \in \mathbb{C}$ such that

$$\phi(ag) = \sum_{\lambda\in\Lambda} \sum_{Q\in\mathfrak{q}_M;\deg(Q)<N_\lambda} f_{\lambda,Q,g}(a)\psi_{\lambda,Q}(g).$$

By the uniqueness and existence of ψ we have the following claim.

Proposition 1.3.3. *The $\psi_{\lambda,Q}$ defined as above is automorphic.*

Proof. We deduce this result by reducing degrees. First suppose $\deg(Q) = N_\lambda - 1$, that is, the degree of polynomial Q is one less than the order that $\hat{z}$ vanishes on λ. Therefore, we can take some $z \in \mathfrak{z}(A_M)$ such that for all $a \in A_M$, $g \in \mathbf{G}$, we have

$$(z.f_{\lambda,Q,g})(a) = z.((am_P(g))^\lambda Q(\log_M(am_P(g)))) = (am_P(g))^\lambda$$

and also

$$(z.f_{\lambda',Q',g})(a) = 0$$

for other pairs (λ', Q'). This is because we argued before that z can be viewed as general characters on A_M, then characters can separate the basis of function on A_M. Now consider the function ϕ_g on A_M instead of G but defined to be

$$\phi_g(a) = \phi(ag).$$

Then we look at

$$(z.\phi_g)(a) = z.\phi(ag) = z\sum\sum f_{\lambda,Q,g}(a)\psi_{\lambda,Q}(g) = (am_P(g))^\lambda\psi_{\lambda,Q}(g).$$

Therefore, we have $\psi_{\lambda,Q}(g) = m_P(g)^{-\lambda}(z\phi)(g)$. Clearly m_P is automorphic by definition, and $z\phi$ is also automorphic, hence $\psi_{\lambda,Q}$ is automorphic as well.

If $\deg(Q) = N_\lambda - 2$, we replace $\phi(g)$ by

$$\phi(g) - \sum_{\lambda\in\Lambda} \sum_{Q\in\mathfrak{q}:\deg(Q)=N_\lambda-1} m_P(g)^\lambda Q(\log_M m_P(g))\psi_{\lambda,Q},$$

which allows us to show that $\psi_{\lambda,Q}$ is automorphic. □

Furthermore, $\psi_{\lambda,Q}$ is automorphic and left Z_M−finite. Each such ψ is invariant under A_M, and we know Z_M/A_M is compact which indeed is a compact abelian group. This means that we can take Fourier inversion on it of ψ which means we would have the unique decomposition

$$\psi_{\lambda,Q} = \sum_{\xi} \psi_{\lambda,Q,\xi}$$

where ξ goes over a subfinite set of $\mathrm{Hom}(Z_M, \mathbb{C}^*)$ which is indeed a finite sum. Also, $\psi_{\lambda,Q,\xi}$ satisfies

$$\psi_{\lambda,Q,\xi}(zg) = \xi(z)\psi_{\lambda,Q,\xi}(g)$$

for all $z \in Z_M$, $g \in \mathbf{G}$. Then taking $a = 1$ and plug in the finite sum we obtain

$$\phi(g) = \sum_{\lambda,Q,\xi} m_P(g)^\lambda Q(\log_M m_P(g))\psi_{\lambda,Q,\xi}(g),$$

where (λ, Q, ξ) runs over a certain finite set. Now let us define ξ^λ to be the "central character" that is modified

$$\xi^\lambda(z) = z^{\lambda-\rho_P}\xi(z)$$

and $\psi'_{\lambda,Q,\xi}$ by

$$\psi'_{\lambda,Q,\xi} = m_P(g)^\lambda \psi_{\lambda,Q,\xi} \in A(U(\mathbb{A})M(k)\backslash\mathbf{G})_{\xi^\lambda}.$$

Then

$$\phi(g) = \sum_{\lambda,Q,\xi} Q(\log_M m_P(g))\psi'_{\lambda,Q,\xi}(g).$$

This gives us the surjectivity of the following theorem. The injectivity comes from all the uniqueness of the above constructions.

Theorem 1.3.4. *We have an isomorphism*

$$\mathbb{C}[\mathit{Re}\,\mathfrak{a}_M] \otimes A(U(\mathbb{A})M(k)\backslash\mathbf{G})_Z \xrightarrow{\simeq} A(U(\mathbb{A})M(k)\backslash\mathbf{G}).$$

Furthermore, set $A(U(\mathbb{A})M(k)\backslash\mathbf{G})_{Z,\xi} = \bigoplus_\eta A(U(\mathbb{A})M(k)\backslash\mathbf{G})_\eta$*, where the sum runs over* $\eta \in \mathrm{Hom}(Z_M, \mathbb{C}^*)$ *such that* $\eta|_{Z_G} = \xi$*, then we also have an isomorphism*

$$\mathbb{C}[\mathit{Re}\,\mathfrak{a}_M^G] \otimes A(U(\mathbb{A})M(k)\backslash\mathbf{G})_{Z,\xi} \simeq A(U(\mathbb{A})M(k)\backslash\mathbf{G})_\xi.$$

Moreover, for cuspidal forms we have

$$\mathbb{C}[\mathit{Re}\,\mathfrak{a}_M] \otimes A_0(U(\mathbb{A})M(k)\backslash\mathbf{G})_Z \simeq A_0(U(\mathbb{A})M(k)\backslash\mathbf{G}),$$

which we will discuss now.

Cuspidal automorphic forms

Convention 1.3.5. For a central character ξ of Z_M, let us make the following notations:

- let $\Pi_0(M)_\xi$ denote the set of isomorphism classes of irreducible representations of M occurring as submodules of $A_0(M(k)\backslash\mathbf{M})_\xi$. That is, the "cuspidal automorphic representations" of $M(k)\backslash\mathbf{M}$ with central character ξ.

- Moreover, let $\Pi_0(\mathbf{M})$ denote the collection $\bigcup_{\xi \in \mathrm{Hom}(Z_M, \mathbb{C}^*)} \Pi_0(\mathbf{M})_\xi$.
- Further, for $\pi \in \Pi_0(\mathbf{M})_\xi$, we denote $A_0(M(k)\backslash\mathbf{M})_\pi$ the isotypic component of type π of $A_0(M(k)\backslash\mathbf{M})_\xi$.
- Moreover, denote by $A_0(U(\mathbb{A})M(k)\backslash\mathbf{G})_\pi$ the subspace of $\phi \in A_0(U(\mathbb{A})M(k)\backslash\mathbf{G})_\xi$ such that $\phi_k(m) = m^{-\rho_P}\phi(mk) \in A_0(M(k)\backslash\mathbf{M})_\pi$ for all $k \in \mathbf{K}$. That is, it is the functions on $U(\mathbb{A})M(k)\backslash\mathbf{G}$ such that when lift to a function on $M(k)\backslash\mathbf{M}$ it is in the isotypic components. This makes it easier to state the following decompositions.

Proposition 1.3.6. *We have the decomposition*

$$A_0(M(k)\backslash\mathbf{M})_\xi = \bigoplus_{\pi \in \Pi_0(M)_\xi} A_0(U(\mathbb{A})M(k)\backslash\mathbf{M})_\pi$$

and correspondingly

$$A_0(U(\mathbb{A})M(k)\backslash\mathbf{G})_\xi = \bigoplus_{\pi \in \Pi_0(M)_\xi} A_0(U(\mathbb{A})M(k)\backslash\mathbf{G})_\pi.$$

Furthermore, we have the collection

$$A_0(U(\mathbb{A})M(k)\backslash\mathbf{G})_Z = \bigoplus_{\pi \in \Pi_0(\mathbf{M})} A_0(U(\mathbb{A})M(k)\backslash\mathbf{G})_\pi.$$

This allows us to decompose the space of cuspidal forms. Recall the important Theorem 1.3.4, which gives an isomorphism, meaning that for any $\phi \in A_0(U(\mathbb{A})M(k)\backslash\mathbf{G})$, we have some element in $\mathbb{C}[\mathrm{Re}\mathfrak{a}_M]$ and some element in $A_0(U(\mathbb{A})M(k)\backslash\mathbf{G})_Z$ where the latter can be decomposed into $A_0(U(\mathbb{A})M(k)\backslash\mathbf{G})_\pi$, which is indexed by the cuspidal representations of $\mathbf{M}$.

That is, given ϕ and Levi $\mathbf{M}$, we ought to be able to compose it with respect to the data in $\mathbb{C}[\mathrm{Re}\mathfrak{a}_M]$, $\Pi_0(\mathbf{M})$ and $A_0(U(\mathbb{A})M(k)\backslash\mathbf{G})_\pi \subset A_0(U(\mathbb{A})M(k)\backslash\mathbf{G})$. Given such ideas, we are to define the important notion in this section.

Definition 1.3.7. For all $\phi \in A_0(U(\mathbb{A})M(k)\backslash\mathbf{G})$, we define a finite set $D(\mathbf{M}, \phi)$, containing $3-$tuples, in the product set $\mathbb{C}[\mathrm{Re}\mathfrak{a}_M] \times \Pi_0(\mathbf{M}) \times A_0(U(\mathbb{A})M(k)\backslash\mathbf{G})$, namely **a set of cuspidal data for** ϕ with elements (Q, π, ψ) such that

- if $(Q, \pi, \psi) \in D(M, \phi)$, then $\psi \in A_0(U(\mathbb{A})M(k)\backslash\mathbf{G})_\pi$ and
- for all $g \in \mathbf{G}$ we have

$$\phi(g) = \sum_{(Q,\pi,\psi) \in D(M,\phi)} Q(\log_M m_P(g))\psi(g).$$

Moreover, if we suppose $Q \in q_M$, the basis of $\mathbb{C}[\mathrm{Re}\mathfrak{a}_M]$, and $\psi \neq 0$, then the set of cuspidal data $D(M, \phi)$ is unique. Then consider the set of π such that there exist some Q, ψ satisfying $(Q, \pi, \psi) \in D(M, \phi)$, denoted by $\Pi_0(\mathbf{M}, \phi)$, the **projection of** $D(M, \phi)$ **over** $\Pi_0(\mathbf{M})$. We call this projection the **cuspidal support of** ϕ. Note that the cuspidal support so-defined is not a 3-tuple, but sets of representations π.

Remark 1.3.8. The cuspidal support of ϕ defined above relies on the choice of $Q \in \mathfrak{q}_M$, which means it needs a choice of basis. It can also be basis-free defined, as we can define it to be the set of $\pi \in \Pi_0(\mathbf{M})$ such that we have

$$0 \neq \sum Q \otimes \psi \in \mathbb{C}[\mathrm{Re}\mathfrak{a}_M] \otimes_{\mathbb{C}} A_0(U(\mathbb{A})M(k)\backslash\mathbf{G})$$

where the summation is over all the (Q, ψ) such that $(Q, \pi, \psi) \in D(M, \phi)$. The two definitions are equivalent, the idea is just to make it clear that cuspidal support of a cuspidal form is the set of cuspidal representations π such that ϕ can be decomposed into the 3-tuple data restricted to the representation π, i.e., one has some $\mathrm{Re}\mathfrak{a}_M$ coming from the Levi and some automorphic form isotypic to π.

Now we start to discuss terms with representation theories. Therefore we shall set some notations. For π irreducible representation of $\mathbf{M}$, we denote by χ_π the central character of π, indeed a character of Z_M. If π is automorphic or cuspidal, we have χ_π is trivial on $Z_M(k)$, then we can associate such a χ with an element $\mathrm{Re}\chi$ in $\mathrm{Re}X_M$. Indeed, we identify $|\chi|$ with an element of $\mathrm{Re}X_M$.

Convention 1.3.9. Now let us define

$$\mathrm{Re}\pi = \mathrm{Re}\chi_\pi$$

and moreover

$$\mathrm{Im}\pi = \pi \otimes (-\mathrm{Re}\pi).$$

If π is cuspidal, the representation $\mathrm{Im}\pi$ is then unitary. Moreover, we denote

$$-\pi, \text{ the contragredient representation of } \pi$$

and

$$-\overline{\pi}, \text{ the conjugate of the contragredient.}$$

For cuspidal form ϕ, the set of characters χ_π such that π is the cuspidal support of ϕ is called **the set of cuspidal exponents of** ϕ. That is, it is the set $\{\chi_\pi : \pi \in \Pi_0(\mathbf{M}, \phi)\}$.

Cuspidal components and automorphic functions

Convention 1.3.10. Take $P = MU$ standard parabolic subgroup of G. Denote by

$$C_0(U(\mathbb{A})M(k)\backslash\mathbf{G})$$

the space of functions on $U(\mathbb{A})M(k)\backslash\mathbf{G}$ that are linear combinations of functions of the form

$$g \mapsto b(m_P(g))\phi(g),$$

where $\phi \in A_0(U(\mathbb{A})M(k)\backslash\mathbf{G})$ and b is a smooth function with compact support on $\mathbf{M}^1\backslash\mathbf{M}$. That is, the product of a compact support smooth function on the Levi and a cuspidal form.

Definition 1.3.11. Now let ϕ be a measurable function on $G(k)\backslash\mathbf{G}$ having moderate growth, then we define **the cuspidal component of** ϕ along P: the linear form on $C_0(U(\mathbb{A})M(k)\backslash\mathbf{G})$ given by

$$\psi \mapsto \int_{U(\mathbb{A})M(k)\backslash\mathbf{G}} \overline{\phi}_P(g)\psi(g)dg.$$

Note that by this definition, the cuspidal component of ϕ relies on the choice of parabolic subgroup P.

We then have the following important theorem.

Theorem 1.3.12. *Let $\phi : G(k)\backslash\mathbf{G} \to \mathbb{C}$ be a measurable function of moderate growth. Suppose that for every standard parabolic subgroup P of G the cuspidal component of ϕ along P is zero. Then $\phi = 0$ almost everywhere.*

Remark 1.3.13. This theorem also tells us a nonzero (in almost everywhere sense) measurable function with moderate growth, together with some parabolic subgroup, would give a nonzero linear functional (distribution) on the space $C_0(U(\mathbb{A})M(k)\backslash\mathbf{G})$.

Furthermore, this result has analogue to functions ϕ with central character ξ and to the space $C_0(U(\mathbb{A})M(k)\backslash\mathbf{G})_{\xi^*}$ where ξ^* is defined before. It can also be applied to the L^2 cases.

Definition 1.3.14. Furthermore, let ϕ and P be as above. We say the cuspidal component is **automorphic** if there exists an automorphic form ϕ'_P on $U(\mathbb{A})M(k)\backslash\mathbf{G}$ such that the cuspidal component of ϕ along P is equal to the linear form on the space space by

$$\psi \longmapsto \int_{U(\mathbb{A})M(k)\backslash\mathbf{G}} \overline{\phi'_P}(g)\psi(g)dg.$$

That is, in the functional ("weak") sense, it is equivalent to some automorphic form on P. Furthermore, if there is such an automorphic form, we can choose it to be cuspidal as follows.

If we have

$$\phi'_P(g) = \sum_{Q,\xi} Q(\log_M m_p(g))\psi_{Q,\xi}(g)$$

where (Q, ξ) runs through a certain subfinite set of $\mathbb{C}[\mathrm{Re}\mathfrak{a}_M] \times \mathrm{Hom}(Z_M, \mathbb{C}^*)$ and $\psi_{Q,\xi} \in A(U(\mathbb{A})M(k)\backslash\mathbf{G})_\xi$, which is possible as we argued before the decomposition theorem. Now recall that in terms of the inner product, we have the choice of cuspidal forms by Proposition 1.2.26 which allows us to replace ϕ'_P by replacing $\psi_{Q,\xi}$ with $\psi^{cusp}_{Q,\xi}$. This is a unique cuspidal form corresponding to ϕ'_P which is associated with ϕ_P and hence we denote it by ϕ^{cusp}_P.

It is important to not get confused with ϕ^{cusp}_P just defined and $(\phi^{cusp})_P$ the automorphic form averaging over U.

Theorem 1.3.15. *Let ϕ measurable function of moderate growth on $G(k)\backslash\mathbf{G}$, then*

ϕ is automorphic

$$\Longleftrightarrow$$

for every P of G, the cuspidal component of ϕ along P is automorphic.

This tells us after defining the cuspidal component ϕ_P^{cusp} of the constant term of ϕ along P, when P runs through the standard parabolic subgroups, the components determine ϕ.

Convention 1.3.16. Now let us go back to the notation of cuspidal data and cuspidal support. Now given an automorphic form on $G(k)\backslash\mathbf{G}$, we can associate an automorphic form on $U(\mathbb{A})M(k)\backslash\mathbf{G}$ which can be chosen to be cuspidal, and this cuspidal form ϕ_P^{cusp} can give cuspidal data (resp. cuspidal support), which by abuse of notation, we can also call as the cuspidal data (resp. cuspidal support) of ϕ. Therefore we denote $D(M,\phi) = D(M,\phi_P^{cusp})$ and $\Pi_0(\mathbf{M},\phi) = \Pi_0(\mathbf{M},\phi_P^{cusp})$. Here as the cuspidal support is also indexed by P, we call $\Pi_0(\mathbf{M},\phi)$ the **cuspidal support of ϕ along** P and we define the **cuspidal support of** ϕ to be

$$\bigcup_M \Pi_0(\mathbf{M},\phi)$$

over all standard Levi M.

1.4 Upper bounds as functions of the constant term

This section is fundamental yet technical, for now, we will just try to state the results instead of going over the proof.

Upper bounds

We have the following theorem which tells us running through all P standard parabolic subgroups, the cuspidal support of ϕ determines the upper bounds of the function ϕ.

Theorem 1.4.1. *Let ϕ be an automorphic form on $G(k)\backslash\mathbf{G}$, recall that for every standard P of G we have the corresponding cuspidal form ϕ_P^{cusp} that is "weakly equivalent with respect to P" with ϕ, which gives the cuspidal data*

$$D(M,\phi) \subset \mathbb{C}[Re\mathfrak{a}_M] \times \Pi_0(M) \times A_0(U(\mathbb{A})M(k)\backslash\mathbf{G}).$$

The elements in the cuspidal data are the 3-tuples (Q,π,ψ), which gives us the following control: there exists some $c > 0$ such that for all $g \in S (= \{pak : p \in \omega, a \in A_{M_0}(t_0), k \in \mathbf{K}\})$ we have the ***upper bound***

$$|\phi(g)| \leq c \sum_P \sum_{(Q,\pi,\psi)\in D(M,\phi)} m_{P_0}(g)^{Re\pi+\rho_P}(1+\|\log_M m_P(g)\|)^{\deg(Q)},$$

where $\deg(Q)$ is the total degree of Q. Furthermore, for all $\mu \in ReX_{M_0}$, there is $c_\mu > 0$ such that for all $g \in S$, we have the upper bound

$$|\phi(g)| \leq c_\mu \sum_P \sum_{(Q,\pi,\psi)\in D(M,\phi)} m_{P_0}(g)^{Re\pi+\mu^M+\rho_P}(1+\|\log_M m_P(g)\|)^{\deg(Q)}$$

where μ^M is the projection of $\mu \in ReX_{M_0}$ onto $ReX_{M_0}^M$.

We then make use of the following polynomial functions.

Convention 1.4.2. Then, take n, d natural numbers and R positive real number, consider the set $Y = \mathbb{R}^n$ or $\mathbb{Z}^n$, denote by $A(d, R, Y)$ the set of complex-valued functions on Y of the form

$$y \longmapsto \sum_{i=1}^{N} P_i(y) e^{y\lambda_i}$$

where we have

1. for all i,, we have P_i a polynomial on Y with complex coefficients;
2. the sum $\sum_{i=1}^{N}(1 + \deg(P_i)) \leq d$ bounded;
3. for all i, we have $\lambda_i \in \mathbb{C}^n$ and $\|\lambda_i\| < R$ bounded.

Then it is easier to write $y = (y_1, \cdots, y_n)$ and $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{C}^n$, then

$$y\lambda = \sum_{j=1}^{n} y_j \lambda_j$$

and

$$\|\lambda\| = (\sum_{j=1}^{n} |\lambda_j|^2)^{1/2}.$$

For $r \in \mathbb{R}$ and $r > R$, $f \in A(d, R, Y)$, the function

$$y \mapsto |f(y)| e^{-r\|y\|}$$

is bounded on Y. We also write

$$s(f, r) = \sup\{|f(y)| e^{-r\|y\|}; y \in Y\}.$$

Then we have the following theorem.

Theorem 1.4.3. *Let n, d, r, Y as before. There is a subfinite set $Y_0 \subset Y$ and for all $r \in \mathbb{R}, r \geq 2^{(n-1)/2}(R+1)$, there exists $c(r) > 0$ such that for all $f \in A(d, R, Y)$, we have the inequality*

$$s(f, r) \leq c(r) \sup\{|f(y)|; y \in Y_0\}.$$

Uniform upper bounds

Convention 1.4.4. For φ on $\mathbf{G}$ having moderate growth and $r > 0$, we set

$$s(\varphi, r) = \sup\{\|g\|^{-r} |\varphi(g)|; g \in \mathbf{G}\}.$$

It is finite if r is large enough. Let $P = MU$ be standard, V finite dimensional subspace of $A_0(U(\mathbb{A})M(k)\backslash\mathbf{G})$, and Γ compact subset of X_M, and $N \geq 1$ integer, denote by $A(V, \Gamma, N)$ the set of functions on $\mathbf{G}$ of the form

$$g \longmapsto \sum_{i=1}^{N} m_P(g)^{\lambda_i} \varphi_i(g)$$

where for all i we have $\varphi \in V$ and $\lambda_i \in \Gamma$. Then there is a real number $R > 0$ such that for all $\varphi \in A(V, \Gamma, N)$ such that $s(\varphi, R)$ finite.

Theorem 1.4.5. *Let* P, V, Γ, N, R *be as before. There is* $R' \geq R$ *and a subfinite set* $C \subset \mathbf{G}$ *such that for all* $r \geq R'$, *there is* $c(r) > 0$ *such that for all* $\varphi \in A(V, \Gamma, N)$, *we have the inequality*

$$s(\varphi, r) \leq c(r) \sup\{|\varphi(g)| : g \in C\}.$$

We also have the other direction:

$$\sup\{|\varphi(g) : g \in C|\} \leq c'(r) s(\varphi, r)$$

where $c'(r) = \sup\{\|g\|^r : g \in C\}$.

Sequences of automorphic forms

Convention 1.4.6. For every $P = MU$, consider finite dimensional space V_P of the space of cuspidal automorphic forms on $U(\mathbb{A})M(k)\backslash\mathbf{G}$, and a compact subset $\Gamma_P \subset X_M$ and an integer N_P. Then we define

$$A((V_P, \Gamma_P, N_P)_{P_0 \subset P \subset G}) := \{\phi : G(k)\backslash\mathbf{G} \to \mathbb{C} | \text{automorphic satisfies the following:}$$

$$\text{for all } P \text{, there is a family } (\lambda_{P,i}, \varphi_{P,i})_{i=1,\cdots,N_P} \text{ of elements of } \Gamma_P \times V_P \text{ such that}$$

$$\text{we have the equality } \phi_P^{\text{cusp}}(g) = \sum_{i=1}^{N_P} m_P(g)^{\lambda_{P,i}} \varphi_{P,i}(g) \text{ for all } g \in \mathbf{G}\}.$$

This is *not* a vector space. There is some $R > 0$ such that $s(\phi, R)$ is finite for all $\phi \in A((V_P, \Gamma_P, N_P)_{P_0 \subset P \subset G})$.

Theorem 1.4.7. *For every* P *of* G, *let notations be as above, then we have*

1. *there is a smooth, right and left* $\mathbf{K}$*–finite function* $f : \mathbf{G} \to \mathbb{C}$ *with compact support and a map*

$$A((V_P, \Gamma_P, N_P)_{P_0 \subset P \subset G}) \longrightarrow A(G(k)\backslash\mathbf{G})$$
$$\phi \longmapsto \phi',$$

   and for all $r > 0$, *there is* $c_1(r), c_2(r) > 0$ *such that for all* $\phi \in A((V_P, \Gamma_P, N_P)_{P_0 \subset P \subset G})$, *we have*

   (a) $\phi = \delta(f)\phi'$,

   (b) $c_1(r)s(\phi', r) \leq s(\phi, r) \leq c_2(r)s(\phi', r)$.

2. *there is* $R > 0$ *and for all* $r \geq R$, *there is* $c(r) > 0$ *such that for all* $\phi \in A((V_P, \Gamma_P, N_P)_{P_0 \subset P \subset G})$, *we have*

$$s(\phi, r) \leq c(r) \sup\{s(\phi_P^{cusp}, r) : P_0 \subset P \subset G\}.$$

3. *let* $(\phi_n)_{n \in N}$ *be a sequence of elements of* $A((V_P, \Gamma_P, N_P)_{P_0 \subset P \subset G})$ *and for all* P, *let* $\varphi_P \in V_P$. *Suppose that for all* P, *the sequence* $(\phi_{n,P}^{cusp})_{n \in N}$ *converges uniformly to* φ_P *on every compact subset of* $\mathbf{G}$. *Then there exists a unique automorphic form* ϕ *on* $G(k)\backslash\mathbf{G}$ *such that the sequence* $(\phi_n)_{n \in N}$ *converges uniformly to* ϕ *on all compact sets. For all* P, *we have the equality*

$$\phi_P^{cusp} = \varphi_P.$$

Fréchet spaces and holomorphic functions

Definition 1.4.8. If H is a vector space over $\mathbb{C}$ with a topology defined by a countable family of semi-norms, for which it is separated and complete, then we call H a **Fréchet space**.

- Let U be an open set of $\mathbb{C}$ and $f : U \to H$ a function, we say f is **holomorphic** if for all $z \in U$, the function
$$z' \longmapsto \frac{f(z') - f(z)}{z' - z}$$
has a limit in H when z' tends to z. It is equivalent to ask that $l \circ f$ be holomorphic for every continuous linear form $l : H \to \mathbb{C}$.

- If f is n variable complex function, then we say f is **holomorphic** if it is holomorphic with respect to each variable $z_1, \cdots, z_N$ of $\mathbb{C}^N$.

- Moreover, if it is only locally so, that is, if X is a complex analytic manifold, and $f : X \to H$ a function, we say f is **holomorphic** if for every local chart
$$\mathbb{C}^N \supset U \overset{i}{\hookrightarrow} X,$$
the function $f \circ i$ is holomorphic.

- If however, f is only defined almost everywhere on X, then f is **meromorphic** if for every $x \in X$ and every sufficiently small neighborhood V of x, there exist two holomorphic functions
$$f_1 : V \to H,$$
$$0 \neq d : V \to \mathbb{C}$$
such that $d(y)f(y) = f_1(y)$ at every point $y \in V$ where $f(y)$ is defined.

Remark 1.4.9. The classical theorems:

- expansion into a convergent series of a holomorphic function in a polydisk,
- Cauchy's formula,
- if X connected, then the uniqueness of the meromorphic continuation of a function defined on a non-empty open set,

etc., still holds to this situation.

Convention 1.4.10. Let Y be locally compact topological space which is countable at infinity, equipped with some measure. Let $L^2_{\mathrm{loc}}(Y)$ be the space of complex valued functions on Y which are locally square integrable modulo the relation of equal almost everywhere.

Those setup gives us the following discussion. For every compact subset $C \subset Y$, we can define a semi-norm

$$\varphi \longmapsto \|\varphi|_C\|^2 = \int_C |\varphi(y)|^2 dy,$$

that is, square integral over the compact set C. Since we assumed countable, the space $L^2_{\text{loc}}(Y)$ equipped with this semi-norm gives a Fréchet space. For any compact C,the space $L^2(C)$ takes the L^2 norm induced topology. Then it is also a Fréchet space. Now consider the restriction map from Y on C gives

$$L^2_{\text{loc}}(Y) \longmapsto L^2(C)$$
$$\varphi \longmapsto \varphi|_C,$$

which is continuous.

Proposition 1.4.11. *Now let us consider U an open set in $\mathbb{C}^N$, and a function on it $f : U \to L^2_{loc}(Y)$. Then f is holomorphic in this multi-variable complex function sense if and only if, for all C compact sets in Y, we have the function*

$$U \longrightarrow L^2(C)$$
$$z \longmapsto f(z)|_C$$

is holomorphic. Equivalently, it means for all $\varphi \in L^2(C)$, the function

$$U \longrightarrow \mathbb{C}$$
$$z \longmapsto \int_C f(z)(y)\overline{\varphi(y)}dy$$

is holomorphic.

Automorphic forms and holomorphicity
Recall that in the definition of $A((V_P, \Gamma_P, N_P)_{P_0 \subset P \subset G})$, we deal with sequence of automorphic forms. The idea now is to try to parameterize the automorphic forms on a complex parameter.

Take $P = MU$ standard of G, V_P finite dimensional subspsce of the space of cuspidal forms on $U(\mathbb{A})M(k)\backslash\mathbf{G}$, and Γ_P a compact subset of X_M, the characters, and N_P some integer. Let n be another integer, and an open connected subset $D \subset \mathbb{C}^n$ and $f : D \to \mathbb{C}$ a holomorphic function which is not always zero. Set

$$D' = \{z \in D; f(z) \neq 0\}$$

the support of f, let

$$\phi : D' \longrightarrow L^2_{\text{loc}}(\mathbf{G}),$$
$$z \longmapsto \phi_z$$

be a function, that is, it takes complex number z and output some L^2 function. Also, for all P, define

$$\psi_P : D' \longrightarrow L^2_{\text{loc}}(\mathbf{G})$$
$$z \longmapsto \psi_{P,z} = \phi^{\text{cusp}}_{z,P}.$$

Now we have the following.

Theorem 1.4.12. *With notations as above, suppose*

1. *for all* $z \in D'$, *we have* ϕ_z *is an element in* $A((V_P, \Gamma_P, N_P)_{P_0 \subset P \subset G})$, *that is, in the sequence of automorphic forms.*

2. *also that for all* P, *we have* ψ_P *is holomorphic on* D'.

Then the function ϕ *can be continued to a holomorphic function on* D *(originally defined on* D'*) in* $L^2_{loc}(\mathbf{G})$ *if and only if for all* P, *the function* ψ_P *satisfies the same property.*

Furthermore, suppose the two conditions hold, we have such continuations and we still denote by ϕ *and* ψ_P *the continued functions. They then have the following properties:*

1. *for all* $z \in D$, *we have* ϕ_z *automorphic,*

2. *for all* $z \in D$ *and all* P, *we have* $\phi^{cusp}_{z,P} = \psi_{P,z}$

3. *the function*

$$\mathbf{G} \times D \longrightarrow \mathbb{C}$$
$$(g, z) \longmapsto \phi_z(g)$$

   is smooth,

4. *if* $g \in \mathbf{G}$, *the function* $z \mapsto \phi_z(g)$ *is holomorphic on* D *as function of z.*

Exponents of L^2 automorphic forms

We want to state the following theorem which tells us that, if ϕ admits a *unitary* central character, then the characters of its cuspidal supports actually determines its square integrability modulo Z_G.

Theorem 1.4.13. *Let* ξ *be unitary character of* Z_G, $\phi \in A(G(k)\backslash\mathbf{G})$ *automorphic. For every* $P = MU$ *of* G, *let* $\Pi_0(M, \phi)$ *the cuspidal support of* ϕ *along* P. *Then, we have that* ϕ *is square integrable if and only if for all* $P = MU$, *and all* $\pi \in \Pi_0(M, \phi)$, *the character Re*π *can be written in the form*

$$Re\pi = \sum_{\alpha \in \triangle_M} x_\alpha \alpha,$$

where coefficients $x_\alpha \in \mathbb{R}$ *and negative.*

Remark 1.4.14. If ξ unitary, then in any case for all $\pi \in \Pi_0(M, \phi)$ we have $\mathrm{Re}\pi \in \mathrm{Re}X^G_M$.

Remark 1.4.15. For $\mathrm{Re}\mathfrak{a}^G_{M_0}$, take a basis $\{\mathfrak{m}_{\check{\alpha}} : \alpha \in \triangle_0\}$ of it, which is the dual of the basis $\triangle_0$ of $\mathrm{Re}(\mathfrak{a}^G_{M_0})^*$. For $\alpha \in \triangle_M$, take the unique simple root $\beta \in \triangle_0$ that projects onto α, and thus induces the definition of $\mathfrak{m}_{\check{\alpha}} = \mathfrak{m}_{\check{\beta}}$. Then $\{\mathfrak{a}_{\check{\alpha}} : \alpha \in \triangle_M\}$ is the basis of $\mathrm{Re}\mathfrak{a}^G_M$ which is the dual of the basis $\triangle_M$ of $\mathrm{Re}(\mathfrak{a}^G_M)^*$. Now we identify $\mathrm{Re}X^G_M$ with $\mathrm{Re}(\mathfrak{a}^G_M)^*$. The theorem above can be then stated as

$$\langle \mathfrak{m}_{\check{\alpha}}, \mathrm{Re}\pi \rangle < 0$$

for all $\alpha \in \triangle_M$.

2 Decomposition According to Cuspidal Data

2.1 Definitions

Irreducible subrepresentations of automorphic forms and complex analytic structure

References

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