Exercises

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Contents

Disclaimer: This is my exercises on the book "Automorphic Forms and Representations" by D.Bump. The "solutions" I have here contains many faulty, hand-waving and non-sense proofs as well as tons of typos. All such things are my own faults.

Chapter 1

Modular Forms (46/68 Problems)

1.1 Dirichlet L-Functions (9/10)

Exercise 1.1.1. Let χ be a primitive character modulo N. Show that $\tau(\overline{\chi}) = \chi(-1)\overline{\tau(\chi)}$.

Proof. From definition, we have $\tau(\overline{\chi}) = \sum_{n \bmod N} \overline{\chi(n)} e^{2\pi i n/N}$. Also

$$\overline{\tau(\chi)} = \sum_{n \bmod N} \overline{\chi(n)} e^{-2\pi i n/N} = \sum_{-n \bmod N} \overline{\chi(-n)} e^{2\pi i n/N}.$$

Then

$$\chi(-1)\overline{\tau(\chi)} = \sum_{-n \bmod N} \chi(-1)\overline{\chi(-n)} e^{2\pi i n/N} = \sum_{-n \bmod N} \overline{\chi(n)} e^{2\pi i n/N}$$

and note that sending n to $-n$ does not change the congruence class. Hence $\tau(\overline{\chi}) = \chi(-1)\overline{\tau(\chi)}$.

An alternative way is to define another more general sum $c_\chi(n) = \sum_{a=1}^{N} \chi(a) e^{2\pi i a n/N}$ and note that we care only when $(a, N) = 1$ and in this case, sending a to an by a multiple of n does not change the congruence class. Then we can build the equation $\chi(n) c_\chi(n) = \tau(\chi)$ and the result is clear. □

Exercise 1.1.2 (Dirichlet). (a). Show that the identity

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = -\log(1 - x),$$

valid if $|x| < 1$, remains true if $|x| = 1$ and $x \neq 1$, in which case the series is conditionally convergent.

(b). Let χ be a nontrivial primitive character modulo N. Assume $N > 1$, so that χ is nontrivial. Use

$$\chi(n) = \frac{\chi(-1)\tau(\chi)}{N} \sum_{m \bmod N} \overline{\chi(m)} e^{2\pi i n m/N} \tag{1.1}$$

to prove that

$$L(1, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n} = -\frac{\chi(-1)\tau(\chi)}{N} \sum_{m \bmod N} \overline{\chi(m)} \log(1 - e^{2\pi i m/N}). \tag{1.2}$$

From this, deduce that

$$L(1, \chi) = \begin{cases} -\frac{\tau(\chi)}{N} \sum_{m \bmod N} \overline{\chi(m)} \log|1 - e^{2\pi i m/N}| \text{ if } \chi(-1) = 1; \\ \frac{i\pi\tau(\chi)}{N^2} \sum_{m=1}^{N} \overline{\chi(m)} m \text{ if } \chi(-1) = -1. \end{cases}$$

Proof. We prove as follows.

(a). For the case $|x| < 1$, it is in the radius of convergence so it is just an identity from Taylor expansion of $\log(1-x) = \sum_{n=1}^{\infty} -\frac{x^n}{n}$. Then we look at the case when $|x| = 1$ and $x \neq 1$. For this circle of convergence where $x \neq 1$, we use the Dirichlet's test, where we choose one sequence $a_n = \frac{1}{n}$ and another sequence $b_n = x^n$. Then it is clear the monotonicity and limit of a_n satisfies the condition. For the partial sum of b_n, note that it forms a geometric series, then

$$|\sum_{n=1}^{N} b_n| = |\sum_{n=1}^{N} x^n| = |\frac{x(1-x^n)}{1-x}|$$

where $|x(1-x^n)| < 2$ since x is on the unit circle except for 1. Hence the value is bounded by $\frac{2}{|1-x|}$ where the denominator is a constant as we fix x. Hence by the test the sequence converges. However it is clearly not absolutely convergent (harmonic series), hence it is conditionally convergent.

(b). By definition, we have

$$L(1,\chi) = \sum_{n=1}^{\infty} \chi(n)n^{-1} = \sum_{n=1}^{\infty} \frac{\frac{1}{N}\chi(-1)\tau(\chi)\sum_{m \mod N} \overline{\chi(m)}e^{2\pi inm/N}}{n}$$

where by Eq(1.1) we obtain

$$= \frac{1}{N}\chi(-1)\tau(\chi)\sum_{n=1}^{\infty}\sum_{m \mod N} \frac{\overline{\chi(m)}}{n}e^{2\pi inm/N}$$

where we note that the summation inside is a finite sum, hence it is safe to switch the order of summation. Hence we get

$$L(1,\chi) = \frac{1}{N}\chi(-1)\tau(\chi)\sum_{m \mod N}\overline{\chi(m)}\sum_{n=1}^{\infty}\frac{1}{n}e^{2\pi inm/N}$$

then since the summation to the right satisfies the condition of the identity in (a), we have that

$$L(1,\chi) = \frac{1}{N}\chi(-1)\tau(\chi)\sum_{m \mod N}\overline{\chi(m)}(-\log(1-e^{2\pi im/N}))$$

$$= -\frac{1}{N}\chi(-1)\tau(\chi)\sum_{m \mod N}\overline{\chi(m)}\log(1-e^{2\pi im/N}).$$

Hence we obtained Eq(1.2).

Hence, if $\chi(-1) = 1$, then note that this means $\chi(-m) = \chi(N-m) = \chi(-1)\chi(m) = \chi(m)$. This means that we can pair up m and $N-m$ which runs over all such pairs (in the case that N is even, this situation is still valid since the number $N/2$ contributes nothing). Then with the identity $\chi(-m) = \chi(m)$, this gives

$$\begin{aligned} L(1,\chi) &= -\frac{1}{N}\tau(\chi) \sum_{m \mod N} \overline{\chi(m)}(\log(1-e^{2\pi i m/N}) + \log(1-e^{-2\pi i m/N})) \\ &= -\frac{1}{N}\tau(\chi) \sum_{m \mod N} \overline{\chi(m)} \log|1-e^{2\pi i m/N}|. \end{aligned}$$

For the case $\chi(-1) = -1$, similarly, for the pairs we get

$$\begin{aligned} &\overline{\chi(m)}\log(1-e^{2\pi i m/N}) - \overline{\chi(m)}\log(1-e^{-2\pi i m/N}) \\ &= \overline{\chi(m)} \log \frac{1-e^{2\pi m/N}}{1-e^{-2\pi i m/N}}. \end{aligned}$$

Then divide the result by 2, we have

$$L(1,\chi) = \frac{1}{N}\tau(\chi)\sum_{m=1}^{N} \overline{\chi(m)} \log e^{2\pi i m/N} = \frac{i\pi\tau(\chi)}{N^2}\sum_{m=1}^{N}\overline{\chi(m)}m$$

as wanted. □

Recall that for the character χ modulo N to be quadratic means that $\chi(n) = \pm 1$ for all $(n, N) = 1$, but that χ is not identically one.

Exercise 1.1.3. Let p be an odd prime.

(a). Prove that there is a unique quadratic character χ modulo p.

(b). Prove that the number of solutions to $x^2 \equiv a \mod p$ equals $1 + \chi(a)$.

(c). Show that

$$\tau(\chi) = \sum_{n=0}^{p-1} e^{2\pi i n^2/p}.$$

Proof. We prove as follows.

(a). Since p is an odd prime, in the congruence class it admits a primitive t. Then it is clear that $\chi(t)$ determines all values uniquely of the elements in $\mod N$. Then if $\chi(t) = 1$, the character is identically 1 by viewing every element as a power of the primitive. Hence the only quadratic character χ modulo p is the one assigns the primitive t value $\chi(t) = -1$.

(b). Since $(\mathbb{Z}/p\mathbb{Z})^\times$ forms a cyclic group with generator primitive t, each congruence class corresponds to one power of the generator t. Hence if $\chi(a) = 1$, from previous discussion, we have that a must be even power $2k$ of t, hence $x = \pm t^k$ are two solutions. This is unique since if there is another $t^{2k'}$ then p would divide some power of t which is not possible.

For the case that $\chi(a) = -1$, it is then odd power $2k+1$ of t, which is impossible because if $x = t^m$, then p divides $t^{2k+1} - t^{2m}$. Hence in this case, there is no solution.

Therefore, we have the number of solution equals $1 + \chi(a)$.

(c). Recall the definition

$$\tau(\chi) = \sum_{n \mod p} \chi(n) e^{2\pi i n/p}.$$

Now we look at the expression

$$\sum_{n \mod p} (1 + \chi(n)) e^{2\pi i n/p} = \sum_{n \mod p} e^{2\pi i n/p} + \sum_{n \mod p} \chi(n) e^{2\pi i n/p}$$

where the first term on the left hand side is 0 since it is the summation over all p roots of unity and the second term is the Gauss sum $\tau(\chi)$. The left hand side is supported when n is a square in modulo p. However, note that when changing from n to a such that $a^2 \equiv n \mod p$, when running over all a, each term is added twice, hence the result would need to be divided by 2. Therefore both sides give us

$$\tau(\chi) = \sum_{a=0}^{p-1} e^{2\pi i a^2/p}.$$

Hence the result. $\square$

Exercise 1.1.4. Let $\tau = x + iy$, where $x, y \in \mathbb{R}$ and $y > 0$. Let k be an integer greater than or equal to 2. Define

$$f(u) = (u - \tau)^{-k}.$$

Use the residue theorem to show that

$$\hat{f}(v) = \begin{cases} 2\pi i Res(e^{2\pi i u v}(u-\tau)^{-k})|_{u=\tau} \text{ if } v > 0; \\ 0 \text{ if } v \leq 0. \end{cases}$$

Hence

$$\hat{f}(v) = \begin{cases} \frac{(2\pi i)^k}{(k-1)!} v^{k-1} e^{2\pi i v\tau} \text{ if } v > 0; \\ 0 \text{ if } v \leq 0. \end{cases}$$

Conclude that

$$\sum_{n=-\infty}^{\infty} (n-\tau)^{-k} = \frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} n^{k-1} e^{2\pi i n\tau}.$$

Proof. Take the semicircle centered at the origin with radius R such that the semicircle lies in the upper half plane for the case $v > 0$. Then take the path γ going from $-R$ to R and then go from the semicircle to $-R$ again. Then the Fourier transform gives

$$\hat{f}(v) = \int_{-\infty}^{\infty} f(u) e^{2\pi i v u} du$$

and we consider

$$I = \int_{\gamma} f(u) e^{2\pi i v u} du = \int_{\gamma} \frac{e^{2\pi i v u}}{(u-\tau)^k} du = (\int_{-R}^{R} + \int_{\gamma'}) \frac{e^{2\pi i v u}}{(u-\tau)^k} du.$$

The integral over γ' the half circle is zero because the term $e^{2\pi i v u}(u-\tau)^{-k}$ is bounded by $1/u^k$ which means that the integral goes to zero when R goes to infinity. Further, the residue of this expression exists only possible at τ. Hence we have the identity that when $v > 0$ that

$$\hat{f}(v) = 2\pi i Res(e^{2\pi i u v}(u-\tau)^{-k})|_{u=\tau}$$

and for the same reason it is 0 for $v \leq 0$. The second expression of the question follows directly by considering the residue with the expansion of $e^{2\pi i u v}$. Hence by Poisson summation formula the last identity directly follows. □

Exercise 1.1.5 (Quadratic Fields)**.** If K is a quadratic extension of $\mathbb{Q}$, let $\mathfrak{o}_K$ denote the ring of integers in K. Then $\mathfrak{o}_K \simeq \mathbb{Z} \oplus \mathbb{Z}$ as an Abelian group. Let α, β be a $\mathbb{Z}-$basis of $\mathfrak{o}_K$. The discriminant of K is by definition

$$D_k = \begin{vmatrix} \alpha & \beta \\ \alpha' & \beta' \end{vmatrix}^2,$$

where $x \mapsto x'$ denotes conjugation, that is, the nontrivial Galois automorphism of K over $\mathbb{Q}$. Show that this definition is independent of the choice of basis and that $D_K \in \mathbb{Z}$.

Proof. By change of basis, what we do is to apply an invertible linear transformation to the matrix under the square. Now such invertible matrix has all coefficients in $\mathbb{Z}$ since we need to take $\mathbb{Z}-$basis to $\mathbb{Z}-$basis. Such invertible matrix can have determinant only 1 or -1, that is, the only two elements in $\mathbb{Z}^{\times}$. Since D_K takes square, the value is invariant under base change.

Since the discriminant is invariant under change of basis, we can always change the basis to $1, \omega$ which under Galois automorphism has $1, \overline{\omega}$. The discriminant is then given as $\overline{\omega} - \omega$ which is in $\mathbb{Q}$ by Galois theory. Further all elements are algebraic elements, which in $\mathbb{Q}$ are $\mathbb{Z}$, hence D_K is integers. □

Exercise 1.1.6 (Fundamental Discriminants)**.** Part (c) of this exercise assumes the quadratic reciprocity law, and part (d) assumes the definition of a discriminant of a quadratic field.

(a). Prove that if q is a prime power, then there exists a primitive quadratic character modulo q if and only if q equals $4, 8$, or is an odd prime; in each of these cases there is precisely one primitive quadratic character, except that if $q = 8$, there are two, one satisfying $\chi(-1) = 1$ and one satisfying $\chi(-1) = -1$.

(b). Show that if $(m, n) = 1$, then

$$(\mathbb{Z}/mn\mathbb{Z})^{\times} \simeq (\mathbb{Z}/m\mathbb{Z})^{\times} \times (\mathbb{Z}/n\mathbb{Z})^{\times},$$

and deduce that there exists a primitive quadratic Dirichlet character modulo a positive integer d if and only if d is the product of relatively prime factors, each of which is an odd prime, or else equals 4 or 8.

(c). Show that if D is an intger, positive or negative, and if there exists a quadratic character χ that is primitive modulo $|D|$ such that the sign of D is equal to $\chi(-1)$, then

$$\chi(n) = (\frac{D}{n})$$

for odd positive integers n.

The integers D satisfying the condition of (c) are called *fundamental discriminants*. They are in one-to-one correspondence with the primitive quadratic characters.

The restriction to odd n in (c) is undesirable; it is sometimes removed by employing Kronecker's modification of the Jacobi symbol, in which $(\frac{a}{b})$ is sometimes defined even when b is even. Using the Kronecker symbol, one may say that the unique quadratic character modulo $|D|$, where D is a fundamental discriminant, is $n \mapsto (\frac{D}{n})$. This has some disadvantages; for example, (i) above is no longer true for the Kronecker symbol. (Shimura (1973) has proposed yet another modification of the quadratic symbol.) We will avoid using the Kronecker symbol. If D is a fundamental discriminant, we will denote the unique primitive quadratic character modulo $|D|$ such that $\chi(-1)$ has the same sign as D by χ_D.

(d). Prove if K is a quadratic extension of $\mathbb{Q}$, there exists a unique fundamental discriminant D such that $K = \mathbb{Q}(\sqrt{D})$; thus the fundamental discriminants are in one-to-one correspondence with quadratic fields.

(e). Prove that if D is a fundamental discriminant, then $D \equiv 0$ or $1 \mod 4$, and that the ring of integers in $K = \mathbb{Q}(\sqrt{D})$ is $\mathbb{Z} \oplus \mathbb{Z}\tau$ where

$$\tau = \begin{cases} \frac{1}{2}\sqrt{D} \text{ if } D \equiv 0 \pmod 4; \\ \frac{1}{2}(\sqrt{D}+1) \text{ if } D \equiv 1 \pmod 4. \end{cases}$$

Conclude that D is the dicriminant of K. Hence the fundamental discriminants are precisely the discriminants of quadratic fields.

(f). Let D be a fundamental discriminant, p a prime, and $K = \mathbb{Q}(\sqrt{D})$. Show that

$$\begin{cases} p \text{ splits in } K \text{ if and only if } \chi_D(p) = 1; \\ p \text{ remains prime in } K \text{ if and only if } \chi_D(p) = -1; \\ p \text{ ramifies in } K \text{ if and only if } \chi_D(p) = 0. \end{cases}$$

Proof. We prove as follows.

(a). We use the fact that the multiplicative group of $\mathbb{Z}/p^n\mathbb{Z}$ is cyclic when p is odd prime. Hence for the case χ is primitive modulo q where $q = p^n$, it has to assign -1 to the generator. Hence such a character is unique if not trivial. However, from previous exercise we have that χ_1 modulo p is unique if quadratic and it clearly induces a nontrivial quadratic character modulo q. This contradicts the uniqueness of the character. Hence $n = 1$, that is, $q = p$ odd prime.

By the classification of multiplicative structure of $(\mathbb{Z}/n\mathbb{Z})^\times$, we still have $2, 4$ are cyclic and not cyclic for higher order of 2. For 2 by definition of quadratic character there is no such exists. For 4 there exists only one as the multiplicative group is C_2. For $q = 8$, if it is imprimitive, it can only be induced from χ' the quadratic character with conductor 4. Therefore, since the structure we know is $C_2 \times C_2$, if we assign $\chi(3) = 1$, then $\chi(7) = \chi(-1)$ and $\chi(5) = \chi(-3) = \chi(7)\chi(3) = \chi(-1)$. Hence to make the character nontrivial, we give $\chi(-1) = -1$ which builds a primitive quadratic character with $\chi(-1) = -1$. Now if $\chi(3) = -1$, note that the imprimitive character with conductor 8 would have value $\chi(5) = \chi'(5) = \chi'(1) = 1$ and $\chi(7) = \chi'(7) = \chi'(3) = -1$. We need our character to be different from such assignments. Now if $\chi(5) = 1$, then $\chi(3 \times 5) = \chi(3)\chi(5) = -1 = \chi(7)$. Hence it cannot be such a choice. For $\chi(5) = -1$, similarly $\chi(7) = 1$ and then it is nontrivial and not induced. Hence it gives a primitive character that is quadratic and $\chi(-1) = \chi(7) = 1$. Therefore we get the result needed.

Now for higher power of 2, since it has the structure $C_2 \times C_{2^{k-2}}$ where $k > 3$, it is determined by its assignment on the generator of the second coordinate. Each assignment will give a character that can be induced from the corresponding character on $C_2 \times C_2 \simeq (\mathbb{Z}/8\mathbb{Z})^\times$. Hence we exhausted all possibilities when q is a prime power.

Hence we constructed the primitive quadratic character for the cases when $q = 4, 8$ and odd prime, and we proved that there cannot be any for other cases. Hence we showed if and only if.

(b). The identity

$$(\mathbb{Z}/mn\mathbb{Z})^\times \simeq (\mathbb{Z}/m\mathbb{Z})^\times \times (\mathbb{Z}/n\mathbb{Z})^\times$$

for $(m, n) = 1$ just follows from the Chinese Reminder Theorem. From this identity, we have that for any $d = p_1^{k_1} \cdots p_n^{k_n}$, the multiplicative group can be decomposed to $(\mathbb{Z}/p_1^{k_1}\mathbb{Z}) \times \cdots \times (\mathbb{Z}/p_n^{k_n}\mathbb{Z})$. If there is a primitive quadratic character modulo such d, it must be primitive on each component since otherwise it can be induced by the smaller character. Hence by (a) each $p_i^{k_i}$ must be the cases $4, 8$ or odd prime.

On the other hand, if each factor is $4, 8$ or odd prime, from (a) they admits at least one primitive character. The compose of all such primitive characters gives a primitive character on the product. This is because if the compose is not primitive, there is some smaller character which would be imprimitive for one of the factors, a contradiction.

(c). From (b), we have that $|D|$ is a product of some of $4, 8$ and distinct odd prime. Now D has three possibilities: 1. D has only odd prime factor; 2. D has an even factor 4; 3. D has an even factor 8.

For D has only odd prime factor, by (b), we can factor $\chi = \chi_{p_1} \cdots \chi_{p_k}$ for the factorization of D. Now note that we showed χ_{p_i} is unique if primitive and Legendre symbol clearly is a primitive quadratic character, hence it is the only possible character. That is, $\chi(-1) = (\frac{-1}{p_1}) \cdots (\frac{-1}{p_k}) = (\frac{-1}{D})$. Now if D is positive, by definition $\chi(-1)$ is also positive, which means $D \equiv 1 \mod 4$. Now $\chi(n) = (\frac{n}{D})$. and since $D \equiv 1 \mod 4$, the inverse symbol is identical, hence $\chi(n) = (\frac{D}{n})$ as needed. Now if D is negative, similarly and we then have $-D \equiv -1 \mod 4$, and then if $n \equiv -1 \mod 4$, then $\chi(n) = (\frac{n}{-D}) = -(\frac{-D}{n}) = -(\frac{-1}{n})(\frac{D}{n}) = (\frac{D}{n})$. If $n \equiv 1 \mod 4$, then $\chi(n) = (\frac{n}{-D}) = (\frac{-D}{n}) = (\frac{-1}{n})(\frac{D}{n}) = (\frac{D}{n})$. Hence we showed the case for D having only odd prime factor.

For the case that D has an even factor 4 and suppose D is positive, we have $D = 4p_1 \cdot p_k$. Then if $\chi(-1) = 1 = \operatorname{sgn}(D)$, then $\chi(-1) = \chi_4(-1)(\frac{-1}{p_1 \cdots p_k}) = -(\frac{-1}{p_1 \cdots p_k})$ by previous results. Now $\chi(n) = \chi_4(n)(\frac{n}{p_1 \cdots p_k})$. Similarly, if $n \equiv 1 \mod 4, \chi_4(n) = 1$, then $\chi(n) = (\frac{n}{p_1 \cdots p_k}) = (\frac{p_1 \cdots p_k}{n})$. Further, $(\frac{D}{n}) = (\frac{p_1 \cdots p_k}{n})(\frac{4}{n}) = (\frac{p_1 \cdots p_k}{n}) = \chi(n)$. If $n \equiv 3 \mod 4, \chi_4(n) = -1$, then $\chi(n) = -(\frac{n}{p_1 \cdots p_k})$. Then $(\frac{D}{n}) = (\frac{p_1 \cdots p_k}{n}) = -(\frac{n}{p_1 \cdots p_k}) = \chi(n)$. The case that D is negative follows the same procedure as above and in the case D negative for the D has no even factor case.

For the case D has factor 8, it follows similarly as the case D having no even factor and having one even factor 4. There are two cases for primitive quadratic character with conductor 8, both cases are to be checked. Then we have the identity $\chi(n) = (\frac{D}{n})$.

(d). Now any quadratic extension $K = \mathbb{Q}(\alpha)$ is given by joining a root of a monic polynomial x^2+bx+c. Now a root of such a polynomial has the form $-\frac{b}{2}+\frac{\sqrt{M}}{2}$ where

$M = b^2 - 4c$, and can be written as $-\frac{b}{2} + \sqrt{M'}$ where $M' = \frac{b^2}{4} - c$. This means we have $\mathbb{Q}(\alpha) = \mathbb{Q}(-\frac{b}{2} + \sqrt{M'})$ where $b \in \mathbb{Q}$ hence is isomorphic to $\mathbb{Q}(\sqrt{M'})$. Now M' can have arbitrary prime factors, hence say $M' = p_1^{r_1} \cdots p_k^{r_k}$, we take all such p_i that its power has absolute value greater than 1, and take their even power, hence we would obtain $\sqrt{M'} = p_{i_1}^{i_1} \cdots p_{i_t}^{i_t} \sqrt{p_{m_1} \cdots p_{m_j}}$ where the primes under the square root are all power one. Here we denote the part outside square root as P and the part under square root as D, we have $\sqrt{M'} = P\sqrt{D}$ and $K = \mathbb{Q}(\sqrt{M'}) = \mathbb{Q}(P\sqrt{D}) = \mathbb{Q}(\sqrt{D})$ and D satisfies the condition for the fundamental discriminant. Hence the result.

(e). From the proof of part (c), we have that $D \equiv 1 \mod 4$ or $-D \equiv 1 \mod 4$ and note that in both cases $D \equiv 1 \mod 4$, where D is the product of odd primes. For the case that D has factor $4, 8$, it has $D \equiv 0 \mod 4$. Hence $D \equiv 0, 1 \mod 4$ when D is a fundamental discriminant. If $D \equiv 0 \mod 4$, then $\sqrt{D} = 2\sqrt{\prod p_i}$ where p_i can also be even, then from the knowledge of quadratic extension we know that $\tau = \sqrt{\prod p_i}$ hence $\tau = \frac{1}{2}\sqrt{D}$. For the case $D \equiv 1 \mod 4$, it follows from the similar argument for quadratic ring of integers. Hence D is the discriminant, and that it can be viewed as the discriminants of quadratic fields.

(f). From the result of the prime ramification and the quadratic reciprocity law, this is just standard argument for the ramification behavior and Jacobi symbol.

Hence the result. □

Exercise 1.1.7. Explain how to modify the proof of Theorem 1.1.1 to handle the case where $N = 1$, so that $\chi = 1$ and $L(s, \chi)$ is the Riemann zeta function.

Proof. Standard proof for the functional equation of the theta function. □

Exercise 1.1.8 (Riemann (1892))**.** Riemann gave two proofs of the functional equation of ζ, each important in its own way. The proof based on taking the Mellin transform of a theta function as in the proof of Theorem 1.1 is Riemann's second proof. (It was extended to L-functions by Hecke (1918) and (1920)). This exercise, based on Riemann's first proof of the functional equation, leads to a determination of the values of $\zeta(s)$ at the negative odd integers or equivalently, at the positive even integers. Riemann's Paper is discussed at length in Edwards (1974). For the extension to L-functions, see Chapter 4 of Washington (1982).

(a). The *Hankel Contour C* begins and ends at ∞, circling the origin counter-clockwise. Prove that if $re(s)$ is large

$$\int_C (-x)^{s-1} e^{-x} dx = -2i \sin(\pi s) \int_0^\infty t^{s-1} e^{-t} dt$$
$$= -2i \sin(\pi s) \Gamma(s).$$

In this integration, we define $(-x)^{s-1}$ to be $e^{(s-1)\log(-x)}$, where we choose the branch of log that is real when $(-x)$ is real and positive. In view of the well-known identity

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)},$$

this may be rewritten

$$\frac{1}{\Gamma(1-s)} = \frac{i}{2\pi} \int_C (-x)^{s-1} e^{-x} dx.$$

Although we proved this only for $re(s)$ large, observe that the integral is convergent for all s, so by analytic continuation, this formula is valid for all s.

(b). Use the geometric series identity

$$\frac{1}{e^x - 1} = \sum_{n=1}^\infty e^{-nx},$$

valid if $re(x) > 0$, and adapt the calculation of (a) to show that

$$\zeta(s) = -\frac{\Gamma(1-s)}{2\pi i} \int_C \frac{(-x)^{s-1}}{e^x - 1} dx.$$

This formula is valid for all s.

The *Bernoulli numbers* are defined by the identity

$$\frac{t}{e^t - 1} = \sum_{n=1}^\infty B_n \frac{t^n}{n!}.$$

We have $B_0 = 1, B_1 = -1/2, B_2 = 1/6, B_4 = -1/30$. It is not hard to see that $B_n = 0$ if n is odd and greater than 1; if n is even, it is clear that B_n is rational, and it will follow from (d) below that the sign of B_n is $-(-1)^{n/2}$.

(c). Use the functional equation to show that ζ vanishes at the negative even integers. Use the residue theorem to show that if n is a positive even integer, then $\zeta(1-n) = -B_n/n$.

(d). Use the functional equation to deduce that if n is a positive even integer

$$\zeta(n) = -\frac{2^{n-1}\pi^n(-1)^{n/2}B_n}{n!}.$$

Proof. (a). The contour consists of two line segments and a small circle around the origin. By standard argument of complex integral the circle contributes zero and for the two line segment, with value $a+bi$ and $a-bi$ where b goes to zero, by taking branch cut differed by 2π we obtain the identity

$$\int_C (-x)^{s-1}e^{-x}dx = -2i\sin(\pi s)\int_0^\infty t^{s-1}e^{-t}dt.$$

(b). Consider the integral for $re(s) > 0$ that

$$\int_0^\infty \frac{(-x)^{s-1}}{e^x-1} = \sum_{n=1}^\infty \int_0^\infty (-x)^{s-1}e^{-nx}dx$$

and take $x \mapsto x/n$ we obtain

$$= \sum_{n=1}^\infty \frac{1}{n^s}\int_0^\infty (-x)^{s-1}e^{-x}dx = \zeta(s)\Gamma(s).$$

Now we are to integral over the Hankel contour C that as in (a) taking branch cut we obtain

$$\int_C \frac{(-x)^{s-1}}{e^x-1}dx = (e^{\pi i(s-1)} - e^{-\pi i(s-1)})\zeta(s)\Gamma(s),$$

with the identity $\Gamma(s) = \frac{\pi}{\sin(\pi s)\Gamma(1-s)}$ we have

$$\zeta(s) = -\frac{\Gamma(1-s)}{2\pi i}\int_C \frac{(-x)^{s-1}}{e^x-1}dx.$$

Note that it converges for all s.

(c). For even integer $-2n$, we have

$$\zeta(-2n) = -\frac{\Gamma(1-2n)}{1}Res_{x=0}\frac{(-x)^{-2n-1}}{e^x-1}.$$

Now we have the Bernoulli numbers that

$$\frac{t^{-2n-1}}{e^t-1} = \sum_{k=0}^\infty B_k\frac{t^{k-2n-2}}{k!},$$

where the t^{-1} term will be given at odd k where B_k is zero. Hence the residue is zero. Hence $\zeta(-2n) = 0$.

With similar argument, we have

$$\zeta(1-2k) = -\frac{\Gamma(2k)}{2\pi i}\int_C \frac{(-x)^{-2k}}{e^x - 1}dx = -\Gamma(2k)Res_{x=0}\frac{(-x)^{-2k}}{e^x - 1}$$
$$= -\Gamma(2k)B_{2k}\frac{1}{2k!} = -\frac{B_{2k}}{2k}$$

as wanted.

(d). Recall the functional equation

$$\xi(s) = \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$$

hence

$$\xi(n) = \pi^{-n/2}\Gamma(\frac{n}{2})\zeta(n) = \pi^{-(1-n)/2}\Gamma(\frac{1-n}{2})\zeta(1-n)$$
$$= \pi^{(n-1)/2}\Gamma(\frac{1-n}{2})(-\frac{B_n}{n}).$$

Then

$$\zeta(n) = -\pi^{n-1/2}\Gamma(\frac{1-n}{2})\Gamma^{-1}(\frac{n}{2})\frac{B_n}{n}.$$

By the identity that

$$\Gamma(\frac{1}{2} - \frac{n}{2}) = \frac{(-4)^n n!}{(2n)!}\sqrt{\pi},$$

plug in we obtain the desired identity.

Hence the result. □

Exercise 1.1.9. We return to the setting of Exercise 1.2(b). Assume that χ is quadratic, so its conductor $N = |D|$, where D is a fundamental discriminant. Then χ is the quadratic character attached to the quadratic extension $K = \mathbb{Q}(\sqrt{D})$. We recall the factorization of the Dedekind zeta function $\zeta_K(s) = \zeta(s)L(s,\chi)$ (see Lang (1970, Theorem XII.1, p.230)). Thus $L(1,\chi)$ is the residue at $s = 1$ of ζ_K, which is computed classically as in Lang (1970, Theorem XIII.2, p.259). Suppose that $\chi(-1) = -1$ so that K is imaginary quadratic. Then

$$L(1,\chi) = \frac{2\pi h}{w\sqrt{|D|}},$$

where D is the discriminant of K, h is its class number, and w is the number of roots of unity in K (two unless $D = -4$ or -3.) Thus by Exercise 1.2(b),

$$h = i\tau(\chi)w|D|^{-3/2}2^{-1}\sum_{m=1}^{D}\chi(m)m.$$

But $\tau(\chi) = i\sqrt{D}$. (See Washington (1982, Corollary 4.6, p.35) for the evaluation of quadratic Gauss sums. Also, compare Eq.(9,15) in section 1.9.) We obtain *Dirichlet's class number formula*

$$h = -\frac{w}{2|D|}\sum_{m=1}^{D}\chi(m)m. \tag{1.3}$$

Proof. TBD. □

Exercise 1.1.10. (a). Let χ be a primitive character modulo N. Prove, using the functional equation, that $L(s,\chi)$ has a simple zero at $s = 0$ if $\chi(-1) = 1$ and is nonzero at $s = 0$ if $\chi(-1) = -1$.

(b). Stark (1971), (1975), (1976), (1980) has conjectured that if $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL(n,\mathbb{C})$ is a Galois representation such that the Artin L-function $L(s,\rho)$ has a zero of order r at $s = 0$, the leading coefficient in its Taylor expansion is essentially an $r \times r$ "Stark regulator" of units in some number field. The simplest open cases of the conjecture are when $r = 1$. Artin's reciprocity law allows us to consider χ to be a Galois character, namely, it gives a character of $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q}) \simeq (\mathbb{Z}/N\mathbb{Z})^\times, \zeta = e^{2\pi i/N}$, where $a \in (\mathbb{Z}/N\mathbb{Z})^\times$ corresponds to $\sigma_a \in \mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q}), \sigma_a(\zeta) = \zeta^a$. With this identification, reinterpret (a) to show that $r = 1$ if $\chi(-1) = 1$, and $r = 0$ if $\chi(-1) = -1$.

(c). Assume that $\chi(-1) = 1$. In this case Exercise 1.2(b) verifies the Stark conjecture because it shows that

$$L(1,\chi) = -\frac{\tau(\chi)}{N}\sum_{m \mod N}\overline{\chi(m)}\log|\epsilon_m|, \tag{1.4}$$

where $\epsilon_m = (1 - e^{2\pi i m/N})/(1 - e^{2\pi i/N})$. Note that if m and N are coprime, ϵ_m is a unit in $\mathbb{Z}[\zeta]$.

Proof. We prove as follows.

(a). If $\chi(-1) = 1$, then $\epsilon = 0$, and we have

$$\Lambda(s, \chi) = \pi^{-s/2}\Gamma(\frac{s}{2})L(s, \chi)$$

and hence

$$\pi^{-s/2}\Gamma(\frac{s}{2})L(s, \chi) = \tau(\chi)N^{-s}\pi^{(s-1)/2}\Gamma(\frac{1-s}{2})L(1-s, \overline{\chi}).$$

Since $\Gamma(0)$ is a simple pole, and the right hand side is analytic at $s = 0$ and nonzero, we have that $L(0, \chi)$ is a simple zero.

If $\chi(-1) = -1$, then $\epsilon = 1$, and we have

$$\Lambda(s, \chi) = \pi^{-(s+1)/2}\Gamma(\frac{s+1}{2})L(s, \chi)$$

and hence

$$\pi^{-(s+1)/2}\Gamma(\frac{s+1}{2})L(s, \chi) = -\tau(\chi)N^{-s}\pi^{-(2-s)/2}\Gamma(\frac{2-s}{2})L(1-s, \overline{\chi}).$$

For $s = 0$ we have

$$\pi^{-1/2}\Gamma(\frac{1}{2})L(0, \chi) = -\tau(\chi)N^{0}\pi^{-1}\Gamma(1)L(1, \overline{\chi})$$

where no terms are zero hence $L(0, \chi)$ is nonzero.

(b). By this setting, we are looking at a primitive character χ with conductor N, hence we can apply (a). This gives that the order of zero is 1 when $\chi(-1) = 1$ and is 0 when $\chi(-1) = -1$.

(c). Follows from the identities. TBD.

Hence the result. □

1.2 The Modular Group (10/11)

Exercise 1.2.1. Prove Eq.(2.2):

$$\mathrm{Im}(g(z)) = |cz + d|^{-2}y,$$

for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}), z = x + iy \in \mathscr{H}.$

Proof. Compute

$$g(z) = \frac{az+b}{cz+d} = \frac{(az+b)(c\overline{z}+d)}{|cz+d|^2} = \frac{ac|z|^2+bd+adz+bc\overline{z}}{|cz+d|^2}$$
$$= \frac{ac|z|^2+bd+(ad+bc)x+(ad-bc)yi}{|cz+d|^2}$$

where $ad-bc=1$ hence the identity. □

Exercise 1.2.2. Let Γ be a discontinuous subgroup of $\mathrm{SL}(2,\mathbb{R})$, and let Γ' be a subgroup. Let F be a fundamental domain for Γ, and let $\gamma_1,\cdots,\gamma_n$ be a set of coset representatives for $\Gamma'\backslash\Gamma$; that is, $\Gamma = \bigcup \Gamma'\gamma_i$ disjointly. Prove that $\bigcup \gamma_i(F)$ is a fundamental domain for Γ'.

Proof. Since $\gamma_1,\cdots,\gamma_n$ are coset representatives, by definition, we have that any element $\gamma\in\Gamma$ has the form $\gamma=\gamma'\gamma_i$ for some $g'\in\Gamma'$ and $i\in[1,n]$. Now any element c in $\mathscr{H}$ can be translated by $\gamma(c')=c$ such that $c'\in F$ and $\gamma\in\Gamma$. Hence $\gamma'\gamma_i(c')=\gamma'(\gamma_i(c'))=c$ where $\gamma_i(c')\in\gamma_i(F)$. Hence any element can be mapped by elements in Γ' from element in $\gamma_i(F)$ for some i. The other conditions for fundamental domains follows with similar arguments. □

Exercise 1.2.3. (a). Prove that a fundamental domain for $\Gamma(2)$ consists of $x+iy$ such that $-1/2<x<3/2, |z+1/2|>1/2, |z-1/2|>1/2$ and $|z-3/2|>1/2$ (cf. Figure 3).

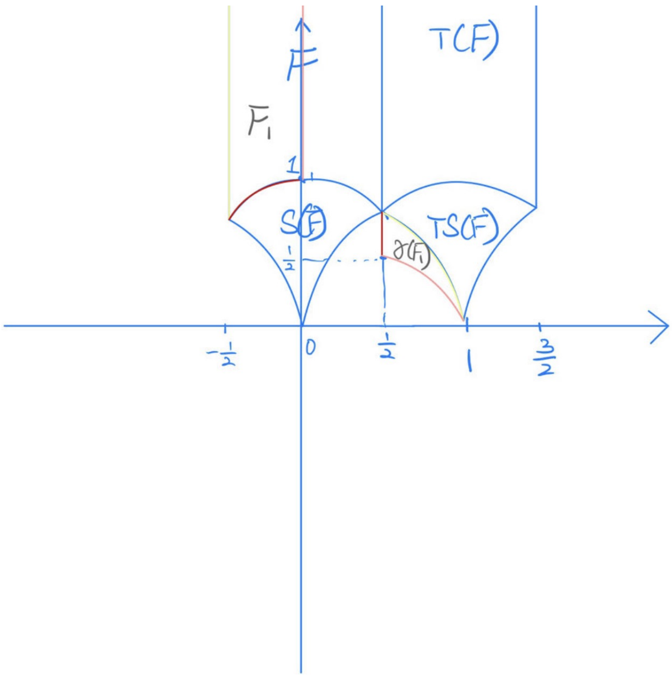
(b). Use the method of Proposition 1.2.3 to prove that $\Gamma(2)$ is generated by $\begin{pmatrix}1 & 2\\ & 1\end{pmatrix}$ and $\begin{pmatrix}1 & \\ 2 & 1\end{pmatrix}$.

Proof. (a). To apply the result from 1.2.2, we first need to look at the action $\Gamma(2)\backslash\Gamma(1)\simeq \mathrm{SL}(2,\mathbb{Z}/2\mathbb{Z})$. Hence we first need to determine the elements in $\mathrm{SL}(2,\mathbb{Z}/2\mathbb{Z})$. They are

$$\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}1&1\\0&1\end{pmatrix},\begin{pmatrix}1&0\\1&1\end{pmatrix},\begin{pmatrix}1&1\\1&0\end{pmatrix},\begin{pmatrix}0&1\\1&0\end{pmatrix},\begin{pmatrix}0&1\\1&1\end{pmatrix}.$$

We label them as $\gamma_1,\cdots,\gamma_6$. Now the actions of them except for γ_3 and γ_6 are clear, where they give the regions (in the sense of Figure 1 in the book) that are

$F, S(F), T(F), T(S(F))$. Hence we are left with γ_3 and γ_6 acting on F. We refer to the picture



.

We need to take care of $\gamma_3(F)$ and $\gamma_6(F)$. First we break F into F_1 and the other part F_2 as in the figure. Then we want to show that $\gamma_3(F_1)$ is the $\gamma(F_1)$ in the figure. Now the mapping of the yellow line $x = -1/2$ is mapped to the arc on $TS(F)$ and the red arc is mapped to the straight line on $x = 1/2$ are clear and easy computation. We now need to determine the mapping of the purple line, the $y-$axis. Now the points on this line has the form $z = yi$ where $y > 1$. The mapping gives

$$\frac{z}{z+1} = \frac{yi}{1+yi} = \frac{y^2}{1+y^2} + \frac{y}{1+y^2}i.$$

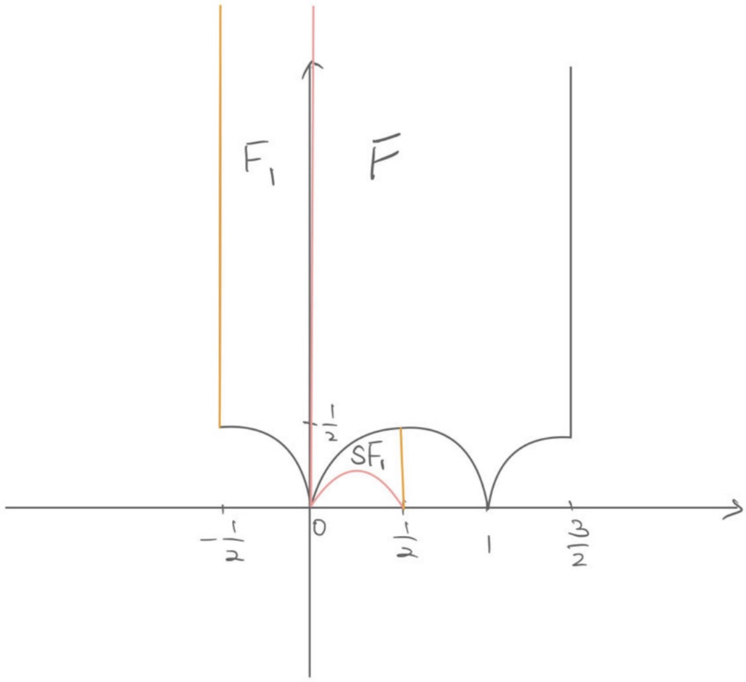
Now note that

$$(\frac{y^2}{1+y^2} - \frac{1}{2})^2 + (\frac{y}{1+y^2})^2 = \frac{y^4/4 - y^2/2 + 1/4 + y^2}{(1+y^2)^2} = \frac{1}{4}.$$

Hence the x coordinate and the y coordinate defines a circle with center $1/2$ and radius $1/2$. Hence it is the purple arc in the figure and hence they together enclose the region $\gamma(F_1)$. By symmetry and computation, $F_2, T(F_1), T(F_2)$ are mapped by γ_3, γ_6 to same shape of regions at different places. Since we can move them freely by action, rearranging them, we obtain the figure 3 in the book as we want.

(b). Denote the two matrix by A and S respectively and denote the fundamental domain we obtained in (a) as F'. The argument will follow immediately if we can show that

$A(F'), A^{-1}(F'), S(F')$ are all the adjacent regions to F'. The action of A and A^{-1} is clear that it horizontally move F' by 2, hence we need to look at the action of S. By similar argument as in (a), we separate the fundamental domain F by four parts and denote the left most one by F_1. By same argument we obtain as the figure below



.

The region of F_1 acted by S is the region above the half circle with center $1/4$ and radius $1/4$. By symmetry, the four parts of F give four such regions which together covers all the region below F. Hence we can apply the same argument as in Proposition 1.2.3 that for any element γ in $\Gamma(2)$ we can find some sequence $\gamma_1, \cdots, \gamma_n$ such that $\gamma_1 = I$ and $\gamma_n = \gamma$ and $\gamma_i \in \Gamma(2)$ and each image of F under γ_i, γ_{i+1} are adjacent to each other. The existence of such property is because F is a fundamental domain of $\Gamma(2)$. Then $\gamma_i^{-1}\gamma_{i+1}(F)$ is adjacent to F, which means it is equal to one of $A(F), A^{-1}(F), S(F)$. Hence $\prod_{i=1}^n \gamma_i^{-1}\gamma_{i+1} = \gamma_1^{-1}\gamma_n = \gamma = \prod B_i$ where B_i is one of A, S and their inverses. Hence $\gamma \in \langle A, S\rangle$, as needed.

Hence the result. $\square$

Exercise 1.2.4. Prove that the stabilizer of an elliptic point is cyclic.

Proof. Denote the elliptic point by a. We first apply the Cayley transform to take a to the zero point and take the upper half plane to the unit circle by

$$z \mapsto \frac{z-a}{z-\overline{a}}.$$

We apply the inverse of this map after applying the action of the stabilizer, that is, we conjugate the action of the stabilizer by this transformation. This means we are now looking at the composition of Mobius maps that take unit circle to unit circle. As a fact of complex analysis it has the form $f(z) = e^{i\theta}\frac{z-t}{1-\overline{t}z}$. We need this map taking zero to zero as zero corresponds to the fixed point. Hence $f(0) = e^{i\theta}(-t) = 0$ which means $t = 0$. Therefore $f(z) = e^{i\theta}z$. Then the map f acts as turning the unit circle by degree θ every time, and we have that such map can only have this form. Therefore, any stabilizer of zero is cyclic with generator f_θ. Hence conjugating by the inverse of the Cayley transform, we obtain that the stabilizer is cyclic.

Another way to show this is that the stabilizer group is SO(2), or say SU(1), hence any discrete subgroup that is also a stabilizer is discrete subgroup of SO(2), which is cyclic. $\square$

Exercise 1.2.5. (a). Let $\mathrm{SL}(2,\mathbb{C})$ act on $\mathbb{P}^1(\mathbb{C})$ by linear functional transformations as in Eq(2.1):

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \to g(z) = \frac{az+b}{cz+d}.$$

Prove that the subgroup that maps the unit disk $\mathcal{D}$ onto itself is

$$\mathrm{SU}(1,1) = \{\begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} : |a|^2 - |b|^2 = 1\}.$$

(b). Prove that the group $\mathrm{SU}(1,1)$ is conjugate to $\mathrm{SL}(2,\mathbb{R})$ in $\mathrm{SL}(2,\mathbb{C})$.

(c). Prove that the subgroup of $\mathrm{SU}(1,1)$ fixing $0 \in \mathcal{D}$ is the group of rotation

$$\begin{pmatrix} e^{i\theta/2} & \\ & e^{-i\theta/2} \end{pmatrix}.$$

Proof. (a). From previous discussion, we know that any such map has the form $f(z) = e^{i\theta}\frac{z-t}{1-\overline{t}z}$. We can view this as

$$f(z) = \frac{e^{i\theta/2}z - e^{i\theta/2}t}{e^{-i\theta/2} - e^{-i\theta/2}\overline{t}z}.$$

If we take $a = e^{i\theta/2}$ and $b = -e^{i\theta/2}t$, we obtain

$$f(z) = \frac{az+b}{\bar{a}+\bar{b}z}.$$

This means that such map corresponds to $\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$ as matrix. Further, since we are looking subgroups in SL(2), we also have $|a|^2 - |b|^2 = 1$, hence it is exactly SU(1, 1).

(b). One way to see this is to consider the Cayley transformation as before. The Cayley transformation takes the upper half plane to the unit circle, and then elements in SU(1, 1) act as automorphism on unit circle, and then the inverse of the Cayley transformation pulls it back. Together they act as $\mathrm{GL}(2, \mathbb{R})$. For determinant, the Cayley transformation has determinant A and its inverse has $1/A$, hence the determinant is 1. Hence it is conjugate to $\mathrm{SL}(2, \mathbb{R})$.

(c). From previous discussion, we already obtained the form of SU(1, 1). Hence we want to show that $b = 0$, which is true if $t = 0$. Note that t represents the point of the fixed point of the map, hence if we want the map to fix zero, then $t = 0$. Hence the group of rotation is obtained.

Hence the result. □

Exercise 1.2.6. (a). *Bruhat decomposition*: Prove that if B is the Borel subgroup of $\mathrm{SL}(2, \mathbb{R})$ consisting of upper triangular matrices and $S = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$, then

$$\mathrm{SL}(2, \mathbb{R}) = B \cup BSB,$$

and the union is disjoint. Thus $\mathrm{SL}(2, \mathbb{R})$ is generated by matrices of the following types:

$$\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}, \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}, \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}.$$

(Making use of the identity (IV.1.16) below, it is possible to dispense with the diagonal matrices here.)

(b). Show that the measure $|y|^{-2}dxdy$ is invariant under the action of $\mathrm{SL}(2, \mathbb{R})$ by checking that it is invariant under generators in part (a).

(c). Show that the volume of $\Gamma(1)\backslash\mathcal{H}$ is finite with respect to this invariant measure.

Proof. We prove as follows.

(a). Since B is a subgroup of $\mathrm{SL}(2,\mathbb{R})$, and S is a subgroup as well, then the right hand side is by definition contained in $\mathrm{SL}(2,\mathbb{R})$. Hence we only need to show another direction.

For any element $g \in \mathrm{SL}(2,\mathbb{R})$ with the traditional way of a, b, c, d with $ad - bc = 1$. There are two possibilities. One is that $c = 0$, then $g = \begin{pmatrix} a & b \\ & d \end{pmatrix}$ which is in B exactly. Note also that here $ad = 1$, i.e., $d = 1/a$.

Another possibility is that $c \neq 0$, then we consider $t_1 = \begin{pmatrix} c^{-1} & a \\ & c \end{pmatrix} \in B$ and $t_2 \begin{pmatrix} 1 & \frac{d}{c} \\ & 1 \end{pmatrix} \in B$. We can compute that

$$t_1 S t_2 = \begin{pmatrix} c^{-1} & a \\ & c \end{pmatrix} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \begin{pmatrix} 1 & \frac{d}{c} \\ & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = g.$$

Hence $\mathrm{SL}(2,\mathbb{R}) \subseteq B \cup BSB$ thus they are identical. For disjoint, note that B contains all $c = 0$, hence it is sufficient to show that BSB cannot have $c = 0$. We compute for arbitrary

$$\begin{pmatrix} x & y \\ & x^{-1} \end{pmatrix} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \begin{pmatrix} t & m \\ & t^{-1} \end{pmatrix} = \begin{pmatrix} ty & ym - \frac{x}{t} \\ \frac{t}{x} & \frac{m}{x} \end{pmatrix}.$$

If we want $c = 0$, that is, $\frac{t}{x} = 0$, it can only be $t = 0$, which would make the third term on the left hand side having element t^{-1} not defined. Hence BSB do not contain matrix with $c = 0$. Thus B and BSB are disjoint.

Therefore, for the first case, g can be generated by $\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}$ and $\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}$. For the second case, the generator is $\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}$ and $\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}$ and $\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$. Hence $\mathrm{SL}(2,\mathbb{R})$ is generated by these three types of matrices.

(b). Take $z = x + yi$. We check by the generators by the order as in the question. The first one gives $a^2z = a^2x + a^2yi$ hence $|y|^{-2}$ gives a^{-4} and $dxdy$ gives a^2 thus they cancel out. Similarly, for the second one, $z = z + b$ where b is real number hence $|y|^{-2}$ is invariant as well as $dxdy$ hence invariant. For the third one, $z = -\frac{1}{z}$. Hence y is mapped to $y/|x^2 + y^2|$ thus it gives $|x^2 + y^2|^2$. Then $dxdy$ gives $|x^2 + y^2|^{-2}$ as well hence they cancel out. Thus the measure $|y|^{-2}dxdy$ is invariant.

(c). Since the measure above is invariant under the group action, it means we can compute the volume of the quotient space by its fundamental domain. Note that the rectangle that covers the fundamental domain has greater value hence we have

$$\mathrm{vol}(F) < \int_{\frac{\sqrt{3}}{2}}^{\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{|y|^2} dxdy = -\frac{1}{|y|}\Big|_{\frac{\sqrt{3}}{2}}^{\infty} = \frac{2}{\sqrt{3}} < \infty.$$

Hence finite volume with respect to this measure.

Hence the result. □

Exercise 1.2.7. Let $\pm I \neq \gamma \in \mathrm{SL}(2, \mathbb{R})$ acting on the Riemann sphere $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$.

(a). If $|\mathrm{tr}(\gamma)| < 2$, show that γ has two fixed points in $\mathbb{P}^1(\mathbb{C})$: one in $\mathscr{H}$ and its complex conjugate. Such an element is called *elliptic*.

(b). If $|\mathrm{tr}(\gamma)| > 2$, show that γ has two fixed points in $\mathbb{P}^1(\mathbb{R})$ and no other fixed points in $\mathbb{P}^1(\mathbb{C})$. Such an element is called *hyperbolic*.

(c). If $|\mathrm{tr}(\gamma)| = 2$, show that γ has a single fixed point in $\mathbb{P}^1(\mathbb{R})$ and no other fixed points in $\mathbb{P}^1(\mathbb{C})$. Such an element is called *parabolic*. If $\mathrm{tr}(\gamma) = 2$, then both eigenvalues of γ are one, in which case the matrix γ is called *unipotent*. If γ is parabolic, then either γ or $-\gamma$ is unipotent.

Proof. We prove as follows.

(a). Let us take the fixed point t, then we have $\frac{at+b}{ct+d} = t$ hence $ct^2 + (d-a)t - b = 0$ which has solution

$$t = \frac{(a-d) \pm \sqrt{(d-a)^2 + 4bc}}{2c}.$$

The value inside the square root is $V = (d-a)^2 + 4bc = d^2 + a^2 - 2ad + 4bc$ where since $ad - bc = 1$ it is also $d^2 + a^2 - 2ad + 4(ad-1) = a^2 + d^2 + 2ad - 4 = (a+d)^2 - 4$.

If $|\mathrm{tr}(\gamma)| < 2$, that is, $|a + d| < 2$, then clearly $V < 0$ hence it has two solutions one of which is in the upper half plane and the other one its complex conjugate.

(b). By similar argument, when trace is greater than 2, it has two real valued solutions. Hence two fixed points in $\mathbb{P}^1(\mathbb{R})$.

(c). By similar argument, there is one fixed point when trace is 2. Further, since the eigenvalue is one, we can take its Jordan canonical form that gives $\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$.

Now there is another way for this question. Consider the matrix over complex field, that is, consider $\mathrm{SL}(2, \mathbb{C})$. Since it is algebraically closed, we have its Jordan canonical form $A = \begin{pmatrix} a & 1 \\ & a \end{pmatrix}$ or $B = \begin{pmatrix} b_1 & \\ & b_2 \end{pmatrix}$. Note that for case A we need to have $a^2 = 1$ and for the case B we need to have $b_1 b_2 = 1$. Hence for A, we have $a = \pm 1$ and hence the trace can only be ± 2. In particular, a is real hence this gives a case for $\mathrm{SL}(2, \mathbb{R})$. For case B, we have $b_1 b_2 = 1$ and if b_2 is real, we have $b_1 + 1/b_1$ which by basic inequality it is greater than or equal to 2. On the other hand, if it is complex, then further by the Jordan canonical form, where the two eigenvalues are complex conjugate to each other, we further have $b_1 + \overline{b_1} = 2Re(b_1)$ where $b_1 \overline{b_1} = 1$ hence $Re(b_1) < 1$ as $Im(b_1) \neq 0$ hence $2Re(b_1) < 2$. Now note that two cases corresponds to the value of trace, in other words, if trace is greater than or equal to 2, it must be A or real B. If trace is less than 2, it corresponds to case B with conjugate complex eigenvalues. This also gives explanation of its eigenvalues and hence eigenvectors. Hence the relation regarding fixing points follows. □

Exercise 1.2.8. (a). Let $\Gamma \subset \mathrm{SL}(2, \mathbb{R})$ be a discontinuous group, and let $\pm I \neq \gamma \in \Gamma$. Show that γ is elliptic if and only if it has finite order. In this case, we call the fixed point of γ in $\mathscr{H}$ an *elliptic fixed point* for Γ.

(b). Show that there are only two orbits of elliptic fixed points for $\mathrm{SL}(2, \mathbb{Z})$, represented by i and $e^{2\pi i/3}$, respectively. [Prove this by examining the fundamental domain.]

Proof. We prove as follows.

(a). The second method of the previous question would be helpful here. We have that γ elliptic hence trace is less than 2 which gives the Jordan canonical form $B = \begin{pmatrix} b & \\ & \bar{b} \end{pmatrix}$ where b is a complex number with norm 1. Now since it has only one fixed point in $\mathcal{H}$, we can actually conjugate the Cayley transform g to take the fixed point to 0 on the unit circle. This means that we obtain from Exercise 1.2.5 that $B \subset g\mathrm{SU}(1)g^{-1}$. In other words, we have $B \subseteq \Gamma \cap g\mathrm{SU}(1)g^{-1}$ where Γ is discontinuous. This implies that B is finite as Γ discrete and $g\mathrm{SU}(1)g^{-1}$ compact by topology fact.

Another way to see this is that we have $B = \begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix}$ by Exercise 1.2.5, and this is rotation of the unit circle. Further we require this action to be acting discontinuous. This means that for two compact sets we can have only finitely many element actions to make them intersect. Now if the value of θ is rational, it is finite as we can find a constant c which would gives $c\theta = 2k\pi$ for some k. For the case θ is irrational, since we have the fact that taking powers of irrational numbers can be approximation to a given number and the value inside an interval is infinite. This means that θ can not be irrational as if so, Γ will not act discontinuously. Hence θ is rational and thus is finite order. (A general fact that every finite subgroup of $\mathbb{Q}/\mathbb{Z}$ is cyclic).

(b). Since we are looking at elliptic fixed points, we first look at the elements that fix them, which by (a) have trace with absolute value less than 2. Since it is over integers, we are looking at trace ± 1 and 0. Now from above discussion we know it must have the form of B. Now for the case trace is 0, this implies $Re(b_1) = 0$ and recall that they are rotations of the unit circle hence it corresponds to $e^{\pi i/2} = \pm i$. For the case trace is ± 1, where $Re(b_1) = \pm 1/2$ hence for the same reason $b_1 = \pm e^{\pi i/3}$ and $b_1 = \pm e^{2\pi i/3}$. On the fundamental domain, they have representatives i and $e^{2\pi i/3}$. (A fact that roots of 1 in a quadratic field have order dividing 4 or 6).

Hence the result. □

Exercise 1.2.9. Let $\Gamma \subset \mathrm{SL}(2, \mathbb{R})$ be a discontinuous group such that the quotient $\Gamma\backslash\mathcal{H}$ has finite volume (cf.Exercise 2.6(c)). Show that $\Gamma\backslash\mathcal{H}$ is compact if and only if Γ contains no parabolic elements.

Proof. For one direction it is clear. If Γ contains no parabolic elements, it means that it has no cusp. Hence $\mathcal{H}$ is exactly $\mathcal{H}^*$ as defined in the book. Hence $\Gamma\backslash\mathcal{H} = \Gamma\backslash\mathcal{H}^*$ which is compact after giving coordinate chart near each point by the discussion in the book.

Hence we want to show the other direction, that is, if $\Gamma\backslash\mathcal{H}$ is compact, then Γ contains no parabolic elements. In fact, the cusps of $\Gamma\backslash\mathcal{H}$ are in bijection with the conjugacy classes of the maximal parabolic subgroups of Γ. Hence compact means no cusps meaning no parabolic elements. $\square$

Exercise 1.2.10. Let Γ be a congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$. Prove that if $a \in \mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$, then there exists a parabolic element $\gamma \in \Gamma$ such that $\gamma(a) = a$ if and only if $a \in \mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$.

Proof. If $\gamma(a) = a$, recall from previous discussion that when parabolic then trace with absolute value 2 then the value inside the square root is 0 hence the value of a is in the base field, that is, in the fraction field of $\mathbb{Z}$ which is $\mathbb{Q}$. For the other direction, we are dealing with the same quadratic root with value inside the square root 0, hence we can simply take $(a-d)/2c = p/q = a$ where a, d, c are all integers. Hence the existence is guaranteed. $\square$

Exercise 1.2.11. Let $\gamma \in \Gamma(1)$ be a hyperbolic element. Show that there exists a real quadratic field $K = \mathbb{Q}(\sqrt{D})$ with $D > 0$ such that the fixed points and eigenvalues of γ lie in K. Show that the eigenvalues of γ are conjugate pair of unites of norm one in K. *Make the following assumption about K: assume that the ring generated by the units of norm one in K is the full ring of integers.* This may or may not be true. Let ϵ, ϵ' be the eigenvalues of γ. If α is a fractional ideal of K, then α is a free $\mathbb{Z}-$module of rank 2; let $\{a_1, a_2\}$ be a basis. Then there exists an element $\gamma \in \mathrm{SL}(2,\mathbb{Z})$ such that $\epsilon(a_1, a_2) = (a_1, a_2)\gamma$. Show that the $\mathrm{GL}(2,\mathbb{Z})-$conjugacy class of γ depends only on the ideal class of α, and that the $\mathrm{GL}(2,\mathbb{Z})-$conjugacy classes of hyperbolic elements with eigenvalues ϵ and ϵ' are thus in bijection with the ideal classes of K. For a fuller discussion of the hyperbolic conjugacy classes in $\mathrm{SL}(2,\mathbb{Z})$, see the references in Volume I of Terras (1985, p.273).

Proof. Since we are looking at integer coefficients, the eigenvalue is the root of the corresponding characteristic polynomial with degree 2 and integer coefficients, the value is rational numbers with a quadratic root. Since γ is hyperbolic, from the discussion of Exercise 1.2.7, we have that the two eigenvalues are real and inverse to each other. Hence the square root is real, i.e., $D > 0$. This D also gives the solution of the fixed point calculation, which coincides with the value inside the square root part of the solutions of the characteristic polynomial, hence both lie in $\mathbb{Q}(\sqrt{D})$.

By the calculation of the characteristic polynomial we obtain

$$\lambda^2 - (a+d)\lambda + 1 = 0$$

with roots $\lambda = \frac{(a+d)\pm\sqrt{(a+d)^2-4}}{2} = \frac{t\pm\sqrt{t^2-4}}{2} = \pm\beta$. Then by the definition of the norm of number fields, we obtain $N(\beta) = \frac{t^2}{4} - \frac{t^2}{4} + 1 = 1$ and same for $N(-\beta)$. They are also complex conjugate to each other.

TBD. □

1.3 Modular Forms For SL$(2,\mathbb{Z})$ (17/19)

Exercise 1.3.1. Verify that the Eisenstein series Eq.(3.3)

$$E_k(z) = \frac{1}{2} \sum_{m,n\in\mathbb{Z};(m,n)\neq(0,0)} (mz+n)^{-k}$$

is absolutely convergent if $k \geq 4$.

Proof. For absolutely convergent, we are looking at $F_k(z) = \sum \frac{1}{|mz+n|^k}$. We can view this as counting over all lattices which is bounded by the double integral $\int\int \frac{dxdy}{(x^2+y^2)^{k/2}}$ and compute by polar coordinates. □

Exercise 1.3.2. This exercise outlines a proof of Jacobi's triple product formula (3.7):

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{(6n+1)^2/24} = q^{1/24} \prod_{n=1}^{\infty} (1-q^n).$$

Let z and w be complex parameters such that $z \in \mathscr{H}$. Let $\Lambda \subset \mathbb{C}$ be the lattice $\{2mz + n | m, n \in \mathbb{Z}\}$. We will also let $q = e^{2\pi i z}$ and $x = e^{2\pi i w}$.

(a). An *elliptic function* with respect to the lattice Λ, is meant a meromorphic function f such that $f(u+\lambda) = f(u)$ for $\lambda \in \Lambda$. Use the maximum modulus principle to show that if f is an elliptic function that has no poles, then f is constant.

(b). Define

$$v(z,w) = \sum_{n=-\infty}^{\infty} q^{n^2} x^n,$$

and let

$$P(z,w) = \prod_{n=1}^{\infty} (1+q^{2n-1}x)(1+q^{2n-1}x^{-1}).$$

Prove that

$$v(z, w+2z) = (qx)^{-1} v(z,w)$$

and that

$$P(z, w+2z) = (qx)^{-1} P(z,w).$$

Hence for fixed z, $f(w) = v(z,w)/P(z,w)$ is an elliptic function.

(c). Prove for fixed z that if $P(z,w) = 0$ then either $w = \frac{1}{2} + z + \lambda$ or else $w = \frac{1}{2} - z + \lambda$ for some $\lambda \in \Lambda$. Show that these values of w are also zeros of $v(z,w)$, and conclude that $f(w)$ has no poles, and hence by (a) is constant. This shows that

$$v(z,w) = \phi(q) P(z,w)$$

where $\phi(q)$ is independent of w.

(d). The Jacobi triple-sum formula will follow if we know that

$$\phi(q) = \prod_{n=1}^{\infty} (1-q^{2n}).$$

To this end, show that

$$v(4z, 1/2) = v(z, 1/4)$$

whereas

$$P(4z,1/2)P(z,1/4) = \prod_{n=1}^{\infty} (1-q^{4n-2})(1-q^{8n-4}).$$

Then

$$\phi(q) = \frac{P(4z,1/2)}{P(z,1/4)} \phi(q^4).$$

Now show that $\phi(q) \to 1$ as $q \to 0$, and thus evaluate $\phi(q)$.

Proof. We prove as follows.

(a). Since it is elliptic function, we can look at its values only on one closed period parallelogram where it is holomorphic so it is bounded on this region and thus the whole complex plane. The holomorphic function that is bounded on the whole complex plane is constant.

In fact, we can prove a stronger result: a nonconstant elliptic function has at least two poles by argument principal.

(b). Since the sum is over all integers, we can change n to $n-1$ as needed, hence we see

$$v(z,w+2z)=\sum e^{2\pi i(zn^2+wn+2zn)}=\sum e^{2\pi i(z(n-1)^2+w(n-1)+2z(n-1))}$$
$$=\sum e^{2\pi i(zn^2+wn-z-w)}=(qx)^{-1}v(z,w).$$

Note that $w\mapsto w+2z$ is the same as $x\mapsto xq^2$ and we see

$$P(z,w+2z)=\prod_{n=1}^{\infty}(1+q^{2n+1}x)(1+q^{2n-3}x^{-1})$$
$$=(1+q^{-1}x^{-1})(1+q^3x)(1+qx^{-1})(1+q^5x)\cdots=(qx)^{-1}(1+qx)(1+q^3x)\cdots$$

where the terms after $(qx)^{-1}$ are exactly $\prod_{n=1}^{\infty}(1+q^{2n-1}x)(1+q^{2n-1}x^{-1})=P(z,w)$. Hence

$$P(z,w+2z)=(qx)^{-1}P(z,w)$$

and thus

$$f(w+2z)=\frac{v(z,w+2z)}{P(z,w+2z)}=\frac{(qx)^{-1}v(z,w)}{(qx)^{-1}P(z,w)}=f(w)$$

and thus it is an elliptic function.

(c). If $P(z,w)=0$ as an infinite product, at least one of its terms is zero, for the addition case, which means that we have $e^{2\pi i(2zn-z+w)}=-1$. Then we have $\pi+2k\pi=2\pi(2zn-z+w)$ hence $w=\frac{1}{2}+(k-2zn)+z$ where $k-2zn$ has fixed z and arbitrary k hence can be viewed as an element in Λ. Thus $w=\frac{1}{2}+\lambda+z$ or $w=\frac{1}{2}+\lambda-z$ for some $\lambda\in\Lambda$. Now for $v(z,w)$ we see

$$v(z,w)=\sum_{n=-\infty}^{\infty}e^{2\pi i(zn^2+wn)}=\sum e^{2\pi i(zn^2+\frac{1}{2}n+zn+\lambda n)}$$
$$=\sum e^{2\pi i(zn^2+\frac{1}{2}n+zn+(k+2zm)n)}=\sum e^{2\pi i(kn+\frac{1}{2}n+zn+zn^2)}=\sum e^{2\pi i((zn^2+zn+2mnz)+(kn+\frac{1}{2}n))}$$

$$= \sum e^{2\pi iz(n^2+n+2mn)} e^{2\pi i(\frac{1}{2}n+kn)}.$$

Now let us first assume that the summation is absolutely convergent, that is, we can change the order of summation. Now the part $t(n) = e^{2\pi i(kn+\frac{1}{2}n)}$ takes value 1 and -1 when n is even and odd respectively. Observe that the part $m(n) = e^{2\pi iz(n^2+n+2mn)}$ is identical under reflection along $\frac{-(2m+1)}{2} = -m - \frac{1}{2}$ where m is integer. Hence $m(n) = m(-2m-1-n)$ and when n is even $-2m-1-n$ is odd and vice versa. Hence we pair the summation as $\sum_{n=-m}^{\infty}(t(n)m(n) + t(-2m-1-n)m(-2m-1-n)) = \sum(t(n)m(n) + t(-2m-1-n)m(n)) = \sum(t(n)m(n) - t(n)m(n)) = 0$. Hence such w is also zero for $v(z,w)$ if it is absolutely convergent (Fubini's Theorem). Now we need to prove the convergence. For convergence, note that $z \in \mathscr{H}$, hence $|q| < 1$ and if $w = \frac{1}{2} + z + \lambda$ we can see that the absolute value of each summand is $|q|^{n(n+1+2m)}$ which we can see the summation is bounded by means of integral over the real line or separate to three parts and each one's boundness can be shown.

Further, by opening the infinite product, we see that $P(z,w)$ is a summation with term $q^{n^2}x^n$ since x is linear growth and the degree of q is going by arithmetic growth hence q^{n^2}. Thus the degree of zeroes should be the same for $v(z,w)$ and $P(z,w)$ and hence $f(w)$ has no pole. Hence $f(w)$ is constant. Therefore $v(z,w) = f(w)P(z,w)$ where $f(w)$ is constant w.r.t w but not necessarily to z, i.e., q. Hence $v(z,w) = \phi(q)P(z,w)$ where ϕ is independent of w.

(d). We have

$$v(4z,\frac{1}{2}) = \sum_{n=-\infty}^{\infty} e^{2\pi in^z} e^{\pi in} = \sum_{n=-\infty}^{\infty} e^{2\pi i(2n)^2 z} e^{\frac{1}{2}\pi i(2n)}$$

and

$$v(z,\frac{1}{4}) = \sum_{n=-\infty}^{\infty} e^{2\pi izn^2} e^{\frac{1}{2}\pi in}.$$

For the latter one, we note that when n is odd, summand $t(n) + t(-n) = 0$ by symmetry, and the change of summation is allowed by the convergence we showed above. Hence we are looking only at even terms, that is, $n = 2k$ for k integers. Hence $v(z,\frac{1}{4}) = \sum_{k=-\infty}^{\infty} e^{2\pi iz(2k)^2} e^{\frac{1}{2}\pi i(2k)}$ which is identical to the first identity. Hence we have

$$v(4z,1/2) = v(z,1/4).$$

For $P(4z,1/2)$ and $P(z,1/4)$, we see

$$P(4z,1/2)=\prod_{n=1}^{\infty}(1+e^{8\pi iz(2n-1)+2\pi i(1/2)})(1+e^{8\pi iz(2n-1)+2\pi i(-1/2)})$$

$$=\prod_{n=1}^{\infty}(1-e^{8\pi iz(2n-1)})(1-e^{8\pi iz(2n-1)})=\prod_{n=1}^{\infty}(1-e^{8\pi iz(2n-1)})^2$$

as $e^{\pm\pi i}=-1$. Similarly

$$P(z,1/4)=\prod_{n=1}^{\infty}(1+ie^{2\pi iz(2n-1)})(1-ie^{2\pi iz(2n-1)})=\prod_{n=1}^{\infty}(1+e^{4\pi iz(2n-1)}).$$

Now term by term we see

$$P(4z,1/2)/P(z,1/4)=\prod_{n=1}^{\infty}(1-e^{4\pi iz(2n-1)})(1-e^{8\pi iz(2n-1)})=\prod_{n=1}^{\infty}(1-q^{4n-2})(1-q^{8n-4}).$$

Since we showed that $\phi(q)$ is independent of w, let us take

$$\phi(q)=\frac{v(z,w)}{P(z,w)}=\frac{v(z,1/4)}{P(z,1/4)}$$

and

$$\phi(q^4)=\frac{v(4z,w)}{P(4z,w)}=\frac{v(4z,1/2)}{4z,1/2}.$$

Then

$$\frac{\phi(q)}{\phi(q^4)}=\frac{v(z,1/4)P(4z,1/2)}{v(4z,1/2)P(z,1/4)}.$$

Hence with above discussion we have $\phi(q)=\frac{P(4z,1/2)}{P(z,1/4)}\phi(q^4)$. When $q\to 0$, we see that $v(z,w)\to 1$ as only $n=0$ term is not going to zero and $P(z,w)\to 1$ as each term is going to 1 in the infinite product. Hence we have $\phi(q)\to 1$ as $q\to 0$. With this limit, we can see

$$\phi(q)=\frac{P(4z,1/2)}{P(z,1/4)}\phi(q^4)=\frac{P(4z,1/2)}{P(z,1/4)}\frac{P(16z,1/2)}{P(4z,1/4)}\phi(q^{16})$$

$$=\frac{P(4z,1/2)}{P(z,1/4)}\frac{P(16z,1/2)}{P(4z,1/4)}\cdots\phi(q^{4t})$$

$$=(\prod_{n=1}^{\infty}(1-q^{4n-2})(1-q^{8n-4}))(\prod(1-q^{16n-8})(1-q^{32n-16})\cdots)\phi(q^{4t})$$

where $t\to\infty$. Note that the product of the infinity products would result in exactly $\prod_{n=1}^{\infty}(1-q^{2n})$. Hence $\phi(q)=(\prod_{n=1}^{\infty}(1-q^{2n}))\phi(q^{\infty})=\prod_{n=1}^{\infty}(1-q^{2n})$.

Hence the Jacobi triple-sum formula follows. □

Exercise 1.3.3. Show that if $\rho = e^{2\pi i/3} \in \mathscr{H}$, and if $3 \nmid k$, then $f(\rho) = 0$ for any modular form of weight k.

Proof. We see for the γ in the hint, $\gamma(\rho) = \rho$ by direct computation. Then for any modular form, we have

$$f(\rho) = f(\gamma\rho) = (-\rho)^k f(\rho)$$

and hence $f(\rho)(1 - (-\rho)^k) = 0$. Now if $(-\rho)^k = 1$, meaning that $-\rho$ is $k-$th root of unity, which is only possible when k is some multiple of 3. Therefore it is $f(\rho) = 0$. □

Exercise 1.3.4. Show that G_4 and G_6 are algebraically independent.

Proof. Being algebraically independent means that there is some polynomial P with two variables that $P(G_4, G_6) = 0$. Since $G_k = cE_k$ for fixed k, we can deduce the setting to E_k. That is, we want to show that there is no nontrivial polynomial such that $P(E_4, E_6) = 0$. Denote the polynomial's degree by m, which we assume to be the minimal degree of such polynomials.

First we show that P, if exists, must be homogeneous. Let us say that $P = f+g$ where f is the homogeneous part and g is the non-homogeneous part. Here by "homogeneous" we mean that E_4 contribute degree 4 and E_6 contribute degree 6, that is, the homogeneous degree is $4a + 6b$, say is d. With this definition, we see that f is then a modular form of weight d since it is product of modular form and the weight follows from the homogeneous definition. That is, we have $f|[\alpha]_d = f$. Applying $|[\alpha]_d$ to P at the zeroes we obtain

$$P = f + g = 0 = 0|[\alpha]_d = f|[\alpha]_d + g|[\alpha]_d = f + g|[\alpha]_d$$

and thus $g = g|[\alpha]_d$. This is impossible since g is not homogeneous and not in $M_d(\Gamma)$. Hence P must be homogeneous. Then P is indeed a modular form of weight d. We hence have P the form

$$P(x, y) = \sum_{4a+6b=d} c_{ab} x^a y^b.$$

We use the important identity that $E_4(\rho) = 0$ and $E_6(i) = 0$ where $\rho = e^{2\pi i/3}$ as defined in Exercise 1.3.3.

Now we write the homogeneous polynomial P as

$$P(x, y) = \sum_{4a+6b=d} c_{ab} x^a y^b = Ax^{\frac{d}{4}} + By^{\frac{d}{6}} + \sum_{4a+6b=d, a,b\neq 0} c_{ab} x^a y^b$$

$$= Ax^{\frac{d}{4}} + By^{\frac{d}{6}} + xyG(x,y)$$

for some G. Now

$$P(E_4,E_6) = A(E_4)^{\frac{d}{4}} + B(E_6)^{\frac{d}{6}} + E_4E_6G(E_4,E_6).$$

If A or B is not zero, it means that for the point ρ or i, we have for instance, A not zero and at ρ that

$$0 = P(E_4,E_6) = 0 + B(E_6)^{\frac{d}{6}}(\rho) + 0.$$

However ρ is not a zero for E_6 hence $B = 0$. Then at i, we again obtain $A(E_4)^{\frac{d}{4}}(i)+0+0 = 0$ and hence $A = 0$. A contradiction. Similarly if $B \neq 0$ we would obtain the same result that A, B must both be zero. This means that

$$P(x,y) = xyG(x,y)$$

for some G. Now since E_4, E_6 are not identically zero, we have that $P(E_4,E_6) = 0$ means that $G(E_4,E_6) = 0$ and hence it is also one such polynomial that satisfies the required condition, but it has strictly less degree than P, which contradicts the minimality of P. Hence a contradiction. Therefore, no such polynomial exists. Hence G_4, G_6 are algebraically independent. $\square$

Exercise 1.3.5. Prove Proposition 1.3.6 in the case where f is not necessarily cuspidal.

Proof. The proof in the book already showed the case when f is cuspidal. Hence we are to show the case when f is not. If f is not cuspidal, we have that $f = a_0 + g$ where g is some cuspidal form. Recall for form of weight k, we have the relation

$$f(iy) = (-1)^{k/2}y^{-k}f(i/y)$$

and as $y \to \infty$ we have $f(iy) = a_0 + g(iy) \to a_0$. As $y \to 0$, by the above relation we have $f(iy) \to a_0$ also. Now the integral

$$\int_0^\infty f(iy)y^s\frac{dy}{y}$$

has convergence problem when it reaches 0 and ∞. For the case of 0, we see $\int_0^c f(iy)y^s\frac{dy}{y} = \frac{y^s}{s}|_0^c = \frac{c^s}{s}$ which when $c \to 0$ is zero except only when $s = 0$ and it is clear that it gives a simple pole. For the case of ∞, similarly, by changing y to $1/y$, we are looking at

$\int_0^M (-1)^{k/2} y^{-k} y^s f(\frac{i}{y})\frac{dy}{y} = (-1)^{k/2}\frac{M^{s-k}}{s-k}$ which also approaches to zero only except for $s=k$, where it gives a simple pole. Hence when f is not cuspidal, it has two simple poles at $s=0$ and $s=k$. □

Exercise 1.3.6. Show that the inner product Eq.(3.13):

$$\langle f, g\rangle = \int_{\Gamma(1)\backslash\mathcal{H}} f(z)\overline{g(z)}y^k \frac{dxdy}{y^2}$$

is defined if only one of f and g is cuspidal and the other is an arbitrary modular form. Prove that the Eisenstein series E_k is orthogonal to the cusp forms (cf. Exercise 1.6.4).

Proof. Assume one of f and g is cuspidal and the other is modular. For the region away from the cusps, it is clearly convergent. The case for both being cuspidal is shown in the book and for the case of only one of them being cuspidal, we can say that without loss of generality f is cuspidal and hence we have $f = a_1 q + a_2 q^2 + \cdots$ and $g = b_0 + b_1 q + \cdots$ Fourier expansions and their product would be $a_1 b_0 q + O(q^2)$ where q decreases very rapidly hence $a_1 b_0 q y^{k-2}$ is still convergent hence it is defined. Note that we can show also that if both are not cuspidal then $\int c y^{k-2} dxdy$ will not be necessarily convergent near the cusps and hence may not be defined.

For the second part, we make use of Hecke operators and hence it will be proved later in 1.6. □

Exercise 1.3.7. (a). Let M be a compact Riemann surface, and let $f : M \to \mathbb{C}$ be a meromorphic function. Assume that f has only one pole, at $m \in M$, which is simple. Extend f to a mapping $M \to \mathbb{P}^1(\mathbb{C})$ by $f(m) = \infty$. Prove that f is an isomorphism of Riemann surfaces.

(b). Define a function $j : \mathcal{H} \to \mathbb{C}$ by $j(z) = G_4^3/\triangle$. Show that j is an automorphic function for SL$(2,\mathbb{Z})$ with a Fourier expansion

$$j(z) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + 864299970q^3 + \cdots .$$

Prove that $j(i) = 1728$ and $j(e^{2\pi i/3}) = 0$. Use part (a) to conclude that j is a bijection of the compactified space

$$\mathrm{SL}(2,\mathbb{Z})\backslash\mathcal{H} \cup \{\infty\} \simeq \mathbb{P}^1(\mathbb{C}).$$

Proof. We prove as follows.

(a). An isomorphism of Riemann surfaces is a map that is bijective holomorphic with holomorphic inverse. We first state a result that a meromorphic function on a compact Riemann surface has same number of poles and zeroes counting multiplicities. Hence by taking $f \mapsto f - c$ we lift the poles and zeroes to any fixed point c, we have that such a function takes each value exactly n times for some n for every value. Since f has only one pole, the value ∞ is taken only once thus every value is taken exactly once and therefore this meromorphic function is bijective. It becomes holomorphic if we extend it to a map to $\mathbb{P}^1(\mathbb{C})$ and thus we can take its inverse which is also a function from $\mathbb{P}^1(\mathbb{C})$ which is compact and satisfies the previous argument. Hence the extended f is holomorphic and bijective with holomorphic inverse. Therefore it is an isomorphism of Riemann surfaces.

(b). By the graded property and previous results, we have that $G_4^3 \in M_{12}(\Gamma)$ and $\triangle \in M_{12}(\Gamma)$ with $\triangle \neq 0$ for any point in $\mathscr{H}$. Hence $j = G_4^3/\triangle$ is an automorphic function on $\Gamma = \mathrm{SL}(2,\mathbb{Z})$. Recall

$$\begin{aligned}\triangle = \frac{1}{1728}(G_4^3 - G_6^2) &= q - 24q^2 + 252q_6 - 1472q^4 + \cdots \\ &= q\prod_{n=1}^{\infty}(1-q^n)^{24},\end{aligned}$$

and

$$G_4 = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n.$$

The coefficients follow directly from the definition and computation, for example, take $G_4^3 = 1 + 720q + O(q^2)$ and $\triangle = q(1 - 24q + O(q^2))$ we obtain the terms $\frac{1}{q} + 744 + O(q)$. To obtain higher order terms, we simply take more accurate terms of G_4^3 and $\triangle$ and do the same computation. We want to prove something more: the coefficients of $j(z)$ are all integers. We write $G_4 = 1+240X$ and $G_6 = 1-504Y$ and $\triangle = \frac{13*240X+2*504Y}{1728} + P(X,Y)$ where P is a polynomial with integer coefficients. We have that $d^5 \equiv d^3 \pmod{12}$ or say $\frac{5}{12}(\sigma_3(n) - \sigma_5(n)) \in \mathbb{Z}$. This would imply that $\frac{720X+1008Y}{1728}$ lies in $q + q^2\mathbb{Z}$.

By computation, we have

$$j(i) = \frac{G_4^3(i)}{G_4^3 - G_6^2} = 1728\frac{1}{1-0} = 1728, j(\rho) = j(e^{2\pi i/3}) = 1728\frac{0}{0-1} = 0.$$

Further, since the Fourier coefficients show that $j(z)$ has only one simple pole at the "cusp" which is infinity, we can extend its domain from $\mathrm{SL}(2,\mathbb{Z})\backslash\mathscr{H}$ to $\mathrm{SL}(2,\mathbb{Z})\backslash\mathscr{H}\cup\{\infty\}$. Also the value is $j(\infty)=\infty$ hence by (a) it induces an isomorphism

$$\mathrm{SL}(2,\mathbb{Z})\backslash\mathscr{H}\cup\{\infty\}\simeq\mathbb{P}^1(\mathbb{C}).$$

Hence the result. □

Exercise 1.3.8. Apply this (Hurwitz genus formula (Lang 1982), etc) now in the case of the canonical map $f:\Gamma(N)\backslash\mathscr{H}^*\to\mathrm{SL}(2,\mathbb{Z})\backslash\mathscr{H}^*$, with $N\geq 2$. Show that the degree n of f is 6 if $N=2$, and $\frac{1}{2}N^3\prod_{p|N}(1-p^{-2})$ if $N>2$. Show that the points of $\mathrm{SL}(2,\mathbb{Z})\backslash\mathscr{H}^*$ that ramify are $i, e^{2\pi i/3}$, and ∞. Show that there are $n/2$ points in the fiber over i, each with ramification index $2, n/3$ points in the fiber over $e^{2\pi i/3}$, each with ramification index 3, and n/N points in the fiber over ∞, each with ramification index N.

Hence show that when $N=2$ or 3, $\Gamma(N)\backslash\mathscr{H}^*$ has genus zero. Confirm this when $N=2$ by examining the fundamental domain in Exercise 1.2.3.

(Note: The fact that all the points in the fiber over the three points that ramify all have the same ramification index is due to the fact that $\Gamma(N)$ is a normal subgroup of $\mathrm{SL}(2,\mathbb{Z})$. This phenomenon does not occur for subgroups that are not normal.)

Proof. To obtain the degree of the covering $\Gamma(N)\backslash\mathscr{H}^*\to\mathrm{SL}(2,\mathbb{Z})\backslash\mathscr{H}^*$, we are first to obtain the index of $\Gamma(N)$ in $\Gamma=\mathrm{SL}(2,\mathbb{Z})$. Now recall that $\Gamma(N)$ is the kernel of $\Gamma\to\mathrm{SL}(2,\mathbb{Z}/N\mathbb{Z})$ which is surjective, hence by the first isomorphism theorem we have

$$\Gamma/\Gamma(N)\simeq\mathrm{SL}(2,\mathbb{Z}/N\mathbb{Z}).$$

Hence the index of $\Gamma(N)$ in Γ is the order of $\mathrm{SL}(2,\mathbb{Z}/N\mathbb{Z})$, which we are to determine.

To determine its order, first we have that if $N=\prod p_i^{r_i}$, then $\mathbb{Z}/N\mathbb{Z}\simeq\mathbb{Z}/\prod p_i^{r_i}\mathbb{Z}$. Hence $\mathrm{GL}(2,\mathbb{Z}/N\mathbb{Z})\simeq\prod\mathrm{GL}(2,\mathbb{Z},p_i^{r_i}\mathbb{Z})$ as we can build isomorphism from $\mathrm{Mat}(n,\mathbb{Z}/N\mathbb{Z})$ to $\mathrm{Mat}(n,\mathbb{Z}/\prod p_i^{r_i}\mathbb{Z})$. Therefore we want to know the order of $\mathrm{GL}(2,\mathbb{Z}/N\mathbb{Z})$ and if we can determine its relation with $\mathrm{SL}(2,\mathbb{Z}/N\mathbb{Z})$ we can obtain the desired result. To obtain the order of $\mathrm{GL}(2,\mathbb{Z}/N\mathbb{Z})$ we want the order of $\mathrm{GL}(2,\mathbb{Z}/p^r\mathbb{Z})$ and the above identity can give us the result.

Now we compute the order of $\mathrm{GL}(2,\mathbb{F}_q)$ as follows. First we have that the order of $\mathrm{GL}(2,\mathbb{F}_p)$ for prime p is taking any elements that are not both zero on the first row which gives p^2-1 and the second row should be linearly independent with the first row, which gives $p(p-1)$, hence we have $(p^2-1)p(p-1)$ as its order. Then we check the canonical map

$$\mathrm{GL}(2,\mathbb{F}_q) \to \mathrm{GL}(2,\mathbb{F}_p)$$

where $q=p^r$. This map is surjective clearly and if we can determine its kernel we can then determine the order of the codomain. Here we need to note that the kernel of this map are all the matrices of the form $I+p\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with all $a,b,c,d \in \mathbb{Z}/p^{r-1}\mathbb{Z}$. Then it can be seen that the kernel has order $(p^{r-1})^4$ and thus we have $|\mathrm{GL}(2,\mathbb{F}_q)| = p^{4r-4}(p^2-1)(p^2-p)$ where $q=p^r$.

Further, for the order of SL, we need to look at the determinant map $\mathrm{GL}(2,\mathbb{F}_q) \to (\mathbb{F}_q)^\times \simeq (\mathbb{Z}/q\mathbb{Z})^\times$. The group SL is the kernel of this map, hence we have

$$\mathrm{GL}(2,\mathbb{F}_q) \simeq (\mathbb{Z}/q\mathbb{Z})^\times \times \mathrm{SL}(2,\mathbb{F}_q).$$

Hence $|\mathrm{SL}(2,\mathbb{F}_q)| = p^{4r-4}(p^2-1)(p^2-p)/\phi(p^r) = p^{4r-4}(p^2-1)(p^2-p)/((p-1)p^{r-1})$.

Therefore, we obtain that the index is

$$\prod_i p_i^{4r_i-4}(p_i^2-1)(p_i^2-p_i)/((p_i-1)p_i^{r-1}) = \prod_i p_i^{3r_i}(1-p_i^{-2})$$
$$= N^3\prod_i(1-p_i^{-2}) = N^3\prod_{p|N}(1-p^{-2}).$$

Now we determined the index of $\Gamma(N)$ in Γ. Recall that for the orbit space $\Gamma(N)\backslash\mathscr{H}^*$, we have that $\pm I$ both act trivially, hence the degree n should be half of the index. Note that this implies that $\Gamma'(N)$ has index $n/2$ if $\Gamma(N)$ has index n when $N \geq 3$ since $-I$ is not in $\Gamma(N)$. It does not hold when $N=2$ as $-I$ is in $\Gamma(2)$, which means that both space has half degree and hence the degree is same as the index.

Putting the above together we have that when $N=2$, we have $n = 2^3(1-\frac{1}{4}) = 8$. When $N \geq 3$, we have $n = \frac{1}{2}N^3\prod_{p|N}(1-p^{-2})$.

If a point of $\mathrm{SL}(2,\mathbb{Z})\backslash\mathscr{H}^*$ ramifies, we first see the multiplication property. Since $\Gamma(N)$ is a normal subgroup of Γ, the quotient forms a group. We see the diagram

$$\begin{tikzcd}
P \in \mathscr{H}^* \arrow[r, "p"] \arrow[d, "p'"] & Q \in \Gamma\backslash\mathscr{H}^* \\
Q' \in \Gamma(N)\backslash\mathscr{H}^* \arrow[ur, "f"] &
\end{tikzcd}$$

commutes. Hence the ramification index has the identity $e(P|Q) = e(P|Q')e(Q'|Q)$. Therefore, if f has degree greater than 1 for any point, this point must be a fixed point on $\mathscr{H}$ by Γ. Therefore, the only points that ramify would be the two representatives of the fixed points, which are $i, e^{2\pi i/3} = \rho, \infty$. Then we also need to show that those three points do ramify. That is, we now have $e(P|Q) > 1$ and we want to show $e(Q'|Q) > 1$. Recall that ρ, i are elliptic points and ∞ is cusp, hence we deal with them separately. For the elliptic points, if we can show that $\Gamma(N)$ has no elliptic points, we have $e(P|Q') = 1$ and thus $e(Q'|Q) > 1$. Recall also that the elliptic points corresponds to matrices with trace absolute value less than 2. For the elliptic point i, the corresponding matrices are $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and for the elliptic point ρ the corresponding matrices are $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ and the matrices it generates as well as their conjugates. However, note that with at least one entry on the diagonal is zero, those elements are not in the congruence subgroup $\Gamma(N)$ for any $N > 1$. Therefore, $\Gamma(N)$ contains no elliptic fixed point, and hence $e(P|Q') = 1$ and thus $e(Q'|Q) > 1$. For the cusp ∞, since any $\Gamma(N)$ must contain $\begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix}$, the cusp is then a ramified point. Hence we conclude that i, ρ, ∞ are the ramified points.

By the commutative diagram above we have that $e(P|Q) = e(P|Q')e(Q'|Q)$. For i, we have that $e(i|Q) = 2$, and hence $e(i|Q') = 2$ since it is not elliptic as we showed above. Hence there must be $n/2$ points in the fiber $f^{-1}(i)$ as its ramification index is 2 and the degree of f is n. For the same reason, there are $n/3$ points in the fiber $f^{-1}(\rho)$ with ramification index 3. For the cusp ∞, note that the matrix with top right corner N is the generator of its stabilizer, which means that its order is N and hence by the same reason as the elliptic points, we have for ∞ the fiber $f^{-1}(\infty)$ has n/N points each with ramification index N.

We have $\Gamma\backslash\mathscr{H}$ has genus zero, hence the formula can be written as

$$(2g-2) = -2n + \sum (e(P_i|Q_i) - 1),$$

$$g = -n + 1 + \sum \frac{e-1}{2}.$$

Hence to determine each value n we want to first determine the summation. For the points corresponding to i, we have that there are $n/2$ such points, and each with ramification index 2, hence the summation is summing over $1 \cdot n/2 = n/2$. Similarly, the points corresponding to ρ would have $n/3$ such points and the summation is $n/3$ times $(3-1) = 2$ thus $2n/3$. Similarly, the case for the cusps would be $n - n/N$. Therefore, if we put in $N = 2, n = 6$ we have

$$g_2 = -6 + 1 + 6/4 + 6/3 + 6/2 - 6/4 = 0.$$

For $N = 3$, we compute $n = 12$ and thus

$$g_3 = -12 + 1 + 12/4 + 12/3 + 12/2 - 12/6 = 0.$$

Therefore, when $N = 2$ or $N = 3$, we have $\Gamma(N)\backslash\mathscr{H}^*$ has genus zero. Recall from Exercise 1.2.3 that the fundamental domain of $\Gamma(2)\backslash\mathscr{H}^*$ can be realized as a region with genus zero and three cusps.

Note that the above arguments rely on the group structure (hence the map is a group homomorphism) of the orbit space, thus it holds rely on the fact that $\Gamma(N)$ is a normal subgroup of Γ. □

Exercise 1.3.9. Show that $\Gamma_0(11)\backslash\mathscr{H}$ has genus one.

Proof. It is possible to develop a similar formula as in the question above for a general $\Gamma_0(N)$, but here we are not going to do so and do direct computation. First note that if $c = 0$, then there should be no elliptic fixed points. Hence in $\Gamma_0(N)$ we only look at the cusps. Now similar to the argument for 1.3.8, that is, consider the natural map p which takes $\Gamma_0(N)/\Gamma(N)$ to the groups of matrices of the form $\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$ which is of order $N\phi(N)$ and hence we would have

$$[\Gamma(1) : \Gamma_0(N)] = N \prod_{p|N} (1 + p^{-1}).$$

In this case it would then be $N + 1$. Now note that by definition $\{\pm I\}$ is indeed inside $\Gamma_0(N)$, which means we treat its order and degree same as the case of $\Gamma(2)$. It can be computed that

$$g = 1 + 2n/6 + n/4 - n/2 - v_\infty/2$$

where v_∞ is the number of inequivalent cusps. Here we have $n = N + 1$ hence $g = 1 + 12/12 - 2/2 = 1$. This means that $\Gamma_0(11)\backslash\mathscr{H}^*$ has genus one. □

Exercise 1.3.10 (Picard's Theorem)**.** Prove that if ϕ is an entire function on $\mathbb{C}$ such there are two complex numbers a and b such that $a, b \notin \phi(\mathbb{C})$, then ϕ is constant.

Proof. We just follow the hint. By Exercise 1.3.8, we concluded that $\Gamma(2)$ has three cusps and $\Gamma(2)\backslash\mathscr{H}^*$ has genus zero. Genus zero means that it would be regarded as the Riemann sphere, which implies that genus zero with three cusps can be regarded as the Riemann sphere minus three points. Hence the map f in the question that has two points not in its value can be regarded as taking values in $\Gamma(2)\backslash\mathscr{H}$. Now recall that the Cayley transform can take $\mathscr{H}$ to the unit disk and hence by the basic results of complex analysis, we have that f is bounded and entire. Hence by Liouville's theorem, it is constant. □

Exercise 1.3.11. Prove that there is bijection between equivalence classes of coverings of U and conjugacy classes of subgroups of the fundamental group $\pi_1(U)$, which associates with the subgroup $\Gamma \subset \pi_1(U)$ the covering map $\Gamma\backslash\widetilde{U} \to U$ induced by projection $\widetilde{p_U} : \widetilde{U} \to U$. Show that $\pi_1(\Gamma\backslash\widetilde{U}) \simeq \Gamma$.

Proof. We follow the hint here. If we have a covering map $p : V \to U$, then by the lifting property, we have that there is a relation $\widetilde{p_U} = p \circ \widetilde{p_V}$ where $\widetilde{p_U}$ is the covering $\widetilde{p_U} : \widetilde{U} \to U$ and $\widetilde{p_V}$ is the covering $\widetilde{p_V} : \widetilde{V} \to V$. Since $\widetilde{V}$ is a universal covering of V, and V is a covering space of U, we have that $\widetilde{V}$ is a universal covering of U and also $\widetilde{U}$ is a universal covering of U. Hence there is map $f_1 : \widetilde{U} \to \widetilde{V}$ and $f_2 : \widetilde{V} \to \widetilde{U}$. As we see that $f_1 f_2(\widetilde{V})$ is a lift of $\widetilde{V}$, and also Id is a lift, with the uniqueness of path lifting property, we have that $f_1 f_2 \simeq \mathrm{Id}$ and $f_2 f_1$ as well. Hence $\widetilde{U}$ and $\widetilde{V}$ are isomorphic as the maps are isomorphisms. Then we can identify $\widetilde{U}$ with $\widetilde{V}$. For the correspondence, assume the base points $u_0 \in U$ and $v_0 \in V$ are that $u_0 = p(v_0)$. Then the fundamental group $\pi_1(U) = \widetilde{p_U}^{-1}(u_0) = \widetilde{p_U}^{-1}(p(v_0))$ contains $\pi_1(V) = \widetilde{p_V}^{-1}(v_0)$ by definition. Conversely, if we have a subgroup Γ of $\pi_1(U)$, we can define a covering space of U as the quotient space $\Gamma\backslash\widetilde{U}$ where the action of Γ is inherited from the natural action of $\pi_1(U)$ on $\widetilde{U}$. That is, the covering map is $p' : \Gamma\backslash\widetilde{U} \to U$ which is induced by $\widetilde{p_U} : \widetilde{U} \to U$. These two constructions are inverse of each other. Hence they are in bijection. Hence $\pi_1(\Gamma\backslash\widetilde{U}) \simeq \Gamma$. □

Exercise 1.3.12. Let $p : V \to U$ and $p' : V' \to U$ be covering maps, and let $\Gamma, \Gamma' \subset \pi_1(U)$ be the subgroups associated with these covering maps by Exercise 1.3.11. Show that p dominates p' if and only if Γ is conjugate in $\pi_1(U)$ to a subgroup of Γ'.

Proof. If p dominates p', we then have

$$\begin{tikzcd} V \arrow[dd, "p"'] \arrow[dr, "q"] & \\ & V' \arrow[dl, "p'"] \\ U & \end{tikzcd}$$

commutes. Then since we can see that V is also a covering space for V', we see that the elements of Γ should be the composition of element of Γ' and some other ones. That is, Γ can be viewed as a subgroup of Γ' when conjugate some element induced by q. Conversely, when we have such an inclusion relation, by the previous exercise, the covering space they induce are quotient space and they would have the quotient relation. Hence it can be seen that p dominates p'. □

Exercise 1.3.13. Conversely, show that if $p : V \to U$ is a covering, and if there exists a group G of automorphisms of V that commutes with p such that G is transitive on the fiber $p^{-1}(u_0)$, then the covering p is regular, and if Γ is the subgroup of $\pi_1(U)$ associated with p by Exercise 1.3.11, then $G \simeq \pi_1(U)/\Gamma$.

Proof. Choose $\beta \in G$, we are to show that $\beta\Gamma\beta^{-1} = \Gamma$. We first have that $\beta\Gamma\beta^{-1} = \pi_1(U, \delta_\beta(y))$ where δ_β is the monodromy around β. Since the deck transformation acts transitively, we can always find some $g \in G$ such that $g(y) = \delta_\beta(y)$. Also for deck transformations, we have $p \circ g = p$ hence we would have the identity. □

Exercise 1.3.14. Show that every covering is dominated by a regular covering.

Proof. The universal covering is clearly regular and it dominates every covering hence the result. A stronger result would be that if we identify the covering space with the corresponding groups, we can always take its abelianization and the corresponding covering space then dominates the covering and is regular. □

Exercise 1.3.15. Let X and Y be compact Riemann surfaces and let $f : X \to Y$ be a holomorphic mapping. Let $P_1, \cdots, P_r \in Y$ be the points that ramify. Let $U = Y - \{P_1, \cdots, P_r\}$, and let $V = f^{-1}(U)$. Then the restriction of f to V is a finite covering of U. Conversely, show that if $f' : V' \to U$ is any finite covering of U, then V' may be identified with an open subset of a compact Riemann surface X', and f' may be extended to a holomorphic mapping $X' \to Y$.

Proof. We follow the hints. If $f' : V' \to U$ is any finite covering of U, what we need is to compactify the space V'. Hence locally we look at the fibers $f'^{-1}(P)$ when P is some exceptional points. The question here is to compactify the "cusps" that is analogous to the case of $\Gamma(1)\backslash\mathscr{H}$ compactifying to $\Gamma(1)\backslash\mathscr{H}^*$. Here what we do is to look at the map $q(z) = \exp(2\pi iz)$. Consider the Cayley transform f that takes the point P_i to zero and if it has ramification index $e(P|Q) = n$, then we compose the map $q(z) = \exp(2\pi iz/n)$ such that $p \circ f$ defines a complex structure locally near P_i. □

Exercise 1.3.16. In the setting of Exercise 1.3.15, the holomorphic mapping $f : X \to Y$ induces an inclusion $F_Y \to F_X$ of the fields of meromorphic functions. Show that the field degree $[F_X : F_Y]$ equals the degree of the cover $V \to U$ and that the cover is regular if and only if F_X/F_Y is a Galois extension, in which case the group $\pi_1(\Gamma\backslash\widetilde{U}) \simeq \Gamma$ of Exercise 1.3.11 is isomorphic to the Galois group $\mathrm{Gal}(F_X/F_Y)$.

Proof. Since the mapping is holomorphic, that is, it has no poles, applying the Hurwitz genus formula we obtain that the degree of the map is the same as the difference of the degrees of the two surfaces. TBD. □

Exercise 1.3.17. Let $Y = \mathbb{P}^1(\mathbb{C})$, let y_0, y_1 and y_∞ be three distinct points of Y, and let $U = Y - \{y_0, y_1, y_\infty\}$. Prove that there exists a regular cover of degree six of U, which can be extended to a holomorphic mapping $f : X \to Y$ of compact Riemann surfaces, and such that $f^{-1}(y_0)$ and $f^{-1}(y_1)$ each consist of three points, with ramification index two, and $f^{-1}(y_\infty)$ consists of two points, each with ramification index three. Use the genus formula (3.19) to show that X has genus zero. Let $p : Z \to Y$ be any holomorphic map from another Riemann surface to Y. Assume that only y_0, y_1 and y_∞ ramify, and that the ramification index of any point in the fiber over y_0 or y_1 is either 1 or 2 and that the ramification index of any point in the fiber over y_∞ is either 1 or 3. Prove that there exists a holomorphic mapping $q : X \to Z$ such that $f = p \circ q$.

Proof. We follow the hints. For the first part, we want to prove the existence of a regular cover with degree 6 of U and it can be extended with some other properties. From topology results, we have that $\pi_1(U)$ is a free group with two generators γ_0 and γ_1 where γ_i is a loop issuing out of the base point and circling the point y_i once counterclockwise before returning to the base point. Then consider the smallest normal subgroup Γ of $\pi_1(U)$ containing γ_0^2, γ_1^2 and $(\gamma_0\gamma_1)^3$ and note that $V = \Gamma\backslash\widetilde{U}$ is isomorphic to S_3, by the previous discussion of its relation with covering space. Then from the results of Exercise 1.3.15, the cover $V \to U$ can be extended to a ramified cover and hence it would have order $|S_3| = 6$. Hence, since $n = 6 = e \cdot f$, we have that for fibers with two points, it has ramification index three and for fibers with three points, it has ramification index two. The second part follows from Exercise 1.3.12. □

Exercise 1.3.18. Prove that there exists an automorphic function z on $\Gamma(2)\backslash\mathscr{H}$ that satisfies the polynomial

$$z^3 - zj - 16j = 0.$$

Proof. We follow the hints. First we identify $\mathrm{SL}(2,\mathbb{Z})\backslash\mathscr{H}$ with $\mathbb{P}^1(\mathbb{C})$ by treating the map j as in Exercise 1.3.7(b), that is, it induces an isomorphism

$$\mathrm{SL}(2,\mathbb{Z})\backslash\mathscr{H} \cup \{\infty\} \simeq \mathbb{P}^1(\mathbb{C}).$$

Then with the results from Exercise 1.3.17, we can take $y_0 = 0, y_1 = 1728, y_\infty = \infty$. If $X = \Gamma(2)\backslash\mathscr{H}$, consider the projection

$$f : \Gamma(2)\backslash\mathscr{H} \longrightarrow \mathrm{SL}(2,\mathbb{Z})\backslash\mathscr{H}$$

has the ramification as described for the map $X \to \mathbb{P}^1(\mathbb{C})$ in Exercise 1.3.17. Combining those two results, we have that we can identify the covering X with $\Gamma(2)\backslash\mathscr{H}$. Now if $j_0 \in \mathbb{C}$, the equation

$$z^3 - zj_0 - 16j_0 = 0$$

has no multiple roots by applying the formula for cubic equations unless $j_0 = 0$ or $j_0 = 1728$. Let us now define a threefold covering from Z to $\mathrm{SL}(2,\mathbb{Z})\backslash\mathscr{H}$ by taking

$$Z = \{(z_0, \tau) \in \mathbb{C} \times \mathrm{SL}(2,\mathbb{Z})\backslash | z_0^3 - z_0 j(\tau) - 16j(\tau) = 0\},$$

with the covering map $p : Z \to \mathrm{SL}(2, \mathbb{Z})\backslash\mathscr{H}$ being the projection on the second component, that is, $(z_0, \tau) \mapsto \tau$. Then note that with such definitions, y_0, y_1, y_∞ are distinct points and the cover is of degree 6 and p is holomorphic, the conditions for the question above is satisfied. Hence we have that there exists a holomorphic map $q : X \to Z$ such that $f = p \circ q$. Then composing q with the projection on the first component gives the required mapping. Hence it shows that there is such an automorphic function z as required. □

Exercise 1.3.19. Prove that there exists an automorphic function on $\Gamma(3)\backslash\mathscr{H}$ whose cube equals j.

Proof. TBD. □

1.4 Hecke Operators (8/14)

Exercise 1.4.1. (a). Assume that H is finite dimensional. Let S be a family of normal operators on H such that if $T_1, T_2 \in S$, then T_1 and T_2 commute. Prove that H has a basis of simultaneous eigenvectors for the operators in S.

(b). Suppose that H is infinite dimensional, with an orthonormal basis $x_n (n \in \mathbb{Z})$. Let T_m be the "shift operator" $T_m x_n = x_{m+n}$. Show that this is a commutative family of normal operators, yet there are no simultaneous eigenvectors in H.

Proof. (a). We assume the ground field K is algebraically closed. Since algebraically closed, any element T in S would have an eigenvector e_1 with eigenvalue λ_1. Now consider $V_1 = (K \cdot e_1)^\perp$. Since T is Hermitian, for $v \in V_1$, we have $\langle Tv, v^\perp \rangle = \langle v, Tv^\perp \rangle = \langle v, \lambda_1 v^\perp \rangle = 0$. Hence $Tv \in V_1$, that is, V_1 is invariant under T. Therefore, T has another eigenvector e_2 with eigenvalue λ_2. Then take $V_2 = (K \cdot e_1 + K \cdot e_2)^\perp$ we can do induction on such process, which gives us a basis of V that are eigenvectors for T. This also gives us that T is diagonalizable.

Hence from the above, we have that $V = \bigoplus V_{\lambda_i}$ where V_{λ_i} are the eigenspaces for each eigenvalue λ_i of the operator T_1. Now we have $T_1 V_{\lambda_1} = \lambda_1 V_{\lambda_1}$, the commuting property gives us $T_1 T_2 v = T_2 T_1 v = \lambda T_2 v$ for $v \in V_{\lambda_1}$. That is, T_1 acts on $T_2 v$ as scalars, which means that $T_2 v$ is in V_{λ_1}. Hence the eigenspace is invariant under

T_2. This tells us there are sub-eigenspaces and eigenvalues for T_2 in V_{λ_1}. Further, each V_{λ_i} can be decomposed as a sum of eigenspaces of T_2. As the space is finite dimensional, we can continue this process until we arrive at a decomposition of $V = \bigoplus V_i$ such that every operator T_j acts as a scalar on each summand. Hence we can choose a basis for each summand V_i and the union of those bases would give us a basis of simultaneous eigenvectors.

(b). The commutativity is clear by definition that $T_iT_jx_n = T_ix_{n+j} = x_{i+n+j} = T_jT_ix_n$. For normality, we see that $\langle T_ix_n, x_m\rangle = \langle x_{i+n}, x_m\rangle = \delta_{i+n,m} = \langle x_n, T_i^*x_m\rangle = \langle x_n, x_{m-i}\rangle$. If $\langle x_n, x\rangle = \langle x_n, y\rangle$ for any n, then we have $x = y$. Hence $T_i^*x_m = x_{m-i}$. Thus TT^* clearly commutes with T^*T when both are defined since they are identity. It has no simultaneous eigenvectors because after ordering the basis in some ways, the operator, say T_1, has no eigenvector (it satisfies no nonzero polynomials).

Hence the result. □

Exercise 1.4.2. Verify the statement that was made in connection with Eq.(4.4):

$$f|T_\alpha|T_\beta = \sum f|\alpha_i\beta_j = \sum_{\sigma\in\Gamma(1)\backslash\mathrm{GL}(2,\mathbb{Q})^+/\Gamma(1)} m(\alpha,\beta;\sigma)f|T_\sigma.$$

That is, that $m(\alpha,\beta;\sigma)$, defined as in the text, depends only on the double coset $\Gamma(1)\sigma\Gamma(1)$.

Proof. Recall that $m(\alpha,\beta;\sigma)$ is the cardinality of the set of indices (i,j) such that $\sigma\in\Gamma(1)\alpha_i\beta_j$. Now we have $\Gamma(1)\alpha\Gamma(1) = \bigsqcup\Gamma(1)\alpha_i$ and $\Gamma(1)\beta\Gamma(1) = \bigsqcup\Gamma(1)\beta_j$, hence

$$\begin{aligned}\Gamma(1)\alpha\Gamma(1)\cdot\Gamma(1)\beta\Gamma(1) &= \Gamma(1)\alpha\Gamma(1)\beta\Gamma(1)\\ &= \bigcup_j \Gamma(1)\alpha\Gamma(1)\beta_j\\ &= \bigcup_{i,j}\Gamma(1)\alpha_i\beta_j.\end{aligned}$$

Now if $\sigma\in\Gamma(1)\alpha_1\beta_a$ and $\sigma\in\Gamma(1)\alpha_2\beta_b$, then $\sigma\in\Gamma(1)\alpha\beta\Gamma(1)$ and same for β, we have $\sigma = u_1\alpha_1\beta_1v_1 = u_2\alpha_2\beta_2v_2$ where $u_1,u_2,v_1,v_2\in\Gamma(1)$. Then $\Gamma(1)\sigma\Gamma(1)\ni u_1^{-1}\sigma v_1^{-1} = \alpha_1\beta_1$ and $\Gamma(1)\sigma\Gamma(1)\ni u_2^{-1}\sigma v_2^{-1} = \alpha_2\beta_2$. Hence the cardinality is also the cardinality of (i,j) where (i,j) is the indices of the coset decomposition of $\Gamma(1)\alpha_i$ and $\Gamma(1)\beta_j$ such that $\alpha_i\beta_j$ is in $\Gamma(1)\sigma\Gamma(1)$. Hence the value of $m(\alpha,\beta;\sigma)$ is determined by the double coset $\Gamma(1)\sigma\Gamma(1)$. □

Exercise 1.4.3. Verify Eq.(4.6):

$$\langle f|\alpha, g\rangle = \langle f, g|\alpha^{-1}\rangle.$$

Proof. Recall the fact of two by two matrices that for $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ we have $S^* = \begin{pmatrix} D & -B \\ -C & A \end{pmatrix}$ that $SS^* = \mathbf{1}$. Hence for $\alpha = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, by assuming the determinant to be one, we can see

$$\langle f|\alpha, g\rangle = \int_F (Cz+D)^{-k} f(\alpha z)\overline{g(z)} y^k y^{-2} dxdy$$

and if we change $z \mapsto \alpha^{-1}z$ we would have

$$\langle f|\alpha, g\rangle = \int_{F'} (C\alpha^{-1}z+D)^{-k} f(z)\overline{g(\alpha^{-1}z)} y'^k y'^{-2} dxdy$$

$$= \int_{F'} |-Cz+A|^{-k}\overline{(-Cz+A)}^{-k} f(z)\overline{g(\alpha^{-1}z)} y'^{k-2} dxdy = \langle f, g|\alpha^{-1}\rangle$$

by the previous identity. Now note that the determinant would be canceled out if it is not one in the above computation, hence we veriefied Eq.(4.6). □

Exercise 1.4.4. (a). Prove that if d_1, d_2 are positive integers, $d_2|d_1$, then $\Gamma(1)\begin{pmatrix} d_1 & \\ & d_2 \end{pmatrix}\Gamma(1)$ is the set of all matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_2(\mathbb{Z})$ such that $ad - bc = d_1 d_2$ and such that the greatest common divisor of a, b, c, d is d_2. (Note that this statement remains true if d_1 and d_2 are positive rational numbers such that d_1/d_2 is an integer if the "greatest common divisor of a, b, c and d" is interpreted to be the positive rational number that generates the same fractional ideal as is generated by a, b, c, and d.)

(b). Prove that if d_1, d_2 are positive integers, $d_2|d_1$, then

$$\Gamma(1)\begin{pmatrix} d_1 & \\ & d_2 \end{pmatrix}\Gamma(1) = \bigsqcup_{a,d>0,ad=d_1d_2,b \pmod d),\gcd(a,b,d)=d_2} \Gamma(1)\begin{pmatrix} a & b \\ & d \end{pmatrix}.$$

(c). Prove Proposition 1.4.4.

Proof. (a). For one direction, we can show by direct computation of

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} km & \\ & m \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

where $ad - bc = eh - fg = 1$. It gives us

$$\begin{pmatrix} akme + bmg & ckmf + bmh \\ ckme + dmg & ckmf + dmh \end{pmatrix}$$

where its determinant is $m^k he(ad - bc) + m^2 gfk(cb - ad) = m^2 k(he - fg) = m^2 k = d_1 d_2$. It is also clear that the greatest common divisor is $m = d_2$. On the other hand, since to first make the matrices upper triangular by invertible elementary matrix operations, it is the same as multiplication by $\mathrm{GL}(2, \mathbb{Z})$ from the left and the column operations are in correspondence with multiplication from the right. The determinant is the multiplication of the determinant of each, hence it can be regarded as $\mathrm{SL}(2, \mathbb{Z}) \begin{pmatrix} d_1 & \\ & d_2 \end{pmatrix} \mathrm{SL}(2, \mathbb{Z})$. Therefore, they are identical.

(b). This is by similar argument that to make it upper triangular it is the same as left multiplication by $\mathrm{SL}(2, \mathbb{Z})$. Hence we can obtain such a decomposition also.

(c). Take $d_1 d_2 = n$, then it follows from the above.

Hence the result. □

Exercise 1.4.5. Prove that the canonical map $\Gamma(1) \to \mathrm{SL}(2, \mathbb{Z}/N\mathbb{Z})$ is surjective, and hence determine the index of $\Gamma(N)$ in $\Gamma(1)$.

Proof. It is already proved in the proof of Exercise 1.3.8. □

Let N be a positive integer. The subgroups $\Gamma_0(N)$ and $\Gamma_1(N)$ are particularly important congruence subgroups of $\mathrm{SL}(2, \mathbb{Z})$. $\Gamma_0(N)$ is the group of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $N|c$; $\Gamma_1(N)$ is the subgroup of $\Gamma_0(N)$ defined by the further conditions $a \equiv d \equiv 1 \pmod N$. Let χ be a Dirichlet character modulo N. We do not require χ to be primitive! We will assume that the weight k is positive and satisfies $\chi(-1) = (-1)^k$. Let $M_k(\Gamma_1(N))$

and $S_k(\Gamma_1(N))$ be defined as in the text, and let $M_k(\Gamma_0(N),\chi)$ and $S_k(\Gamma_1(N))$ be the subspaces of $S_k(\Gamma_1(N))$ defined by the condition that

$$f|\gamma = \chi(d)f \text{ for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).$$

Exercise 1.4.6. Prove that

$$S_k(\Gamma_1(N)) = \bigoplus_{\chi} S_k(\Gamma_0(N),\chi),$$

where the summation is over all Dirichlet characters χ modulo N. Moreover, show that the summands are orthogonal with respect to the Petersson inner product.

Proof. This can be viewed as a corollary of Exercise 1.4.1. Let us consider the operator $\langle d\rangle$ which acts on vectors in $S_k(\Gamma_1(N))$ as

$$\langle d\rangle f = f|_\gamma, \text{ where } \gamma = \begin{pmatrix} a & b \\ c & d' \equiv d \pmod N \end{pmatrix} \in \Gamma_0(N).$$

It is important to check that this operator is well defined. If we have two different choice of γ and γ' with same right bottom corner, as they are elements in $\mathrm{SL}(2,\mathbb{Z})$, the determinant parts are identical. Further, since they are both in $\Gamma_0(N)$, meaning that c and c' both are some multiple of N. Now we have that

$$\begin{pmatrix} a & b \\ & d \end{pmatrix} = \begin{pmatrix} 1 & \\ & d \end{pmatrix}\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}\begin{pmatrix} a & \\ & 1 \end{pmatrix}$$

hence by the fact that f is $\Gamma_1(N)$ invariant and a is determined by d we have that the action is determined by d and thus this operator is well-defined. Now by definition of this operator they commute with each other, and they are clearly with finite order (by the fact that d is determined mod N, from Exercise 1.4.1, we have that they are diagonalizable and they have simultaneous eigenspaces. Those spaces by definition are with eigenvalues $\chi(d)$, and hence they span the whole space and they form a basis for $S_k(\Gamma_1(N))$. Or we can check the orthgonality by Petersson inner product such that

$$\langle f,g\rangle = \int_{\Gamma(1)\backslash H} f(z)\overline{g(z)}y^k\frac{dxdy}{y^2}$$

where the definition means that it is invariant under the action of $\Gamma(1)$. Moreover, this inner product is self-adjoint, and thus if we have two different f, g which are eigenvectors as discussed before corresponding to different eigenvalues λ_1, λ_2, then we have

$$\langle f, \langle d\rangle g\rangle = \langle f, \chi(d)g\rangle = \chi(d)\langle f, g\rangle = \lambda_1\langle f, g\rangle = \langle\langle d\rangle f, g\rangle = \lambda_2\langle f, g\rangle$$

which holds only when $\langle f, g\rangle = 0$. Hence they are orthogonal with respect to the Petersson inner product. □

Exercise 1.4.7. Prove that if f is a modular form for a congruence group, there exists $\alpha \in \mathrm{GL}(2, \mathbb{Q})^+$ such that $f|\alpha \in M_k(\Gamma_1(N))$ for some N.

Proof. If f is a modular form for a congruence group which we denote by $\Gamma(N) = \{\begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \pmod{N}\}$, then consider the element $\alpha = \begin{pmatrix} N & \\ & 1 \end{pmatrix}$. If $f \in M_k(\Gamma(N))$, indeed, the map $f \mapsto f|h$ actually induces an isomorphism between $M_k(\Gamma(N)) \simeq M_k(\Gamma'(N))$ where $\Gamma'(N) = \alpha^{-1}\Gamma(N)\alpha = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}) : a, d \equiv 1 \pmod{N}, c \equiv 0 \pmod{N^2}\}$ which is inside $\Gamma_0(N^2)$. Now we check the isomorphism. Take $\sigma \in \Gamma'(N)$ such that

$$\sigma = \begin{pmatrix} 1 + a'N & b' \\ c'N^2 & 1 + d'N \end{pmatrix}$$

and now

$$f|h(\sigma z) = (\det h)^{k/2} f(h\sigma z)$$

where

$$h\sigma = \begin{pmatrix} N + a'N^2 & b'N \\ c'N^2 & 1 + d'N \end{pmatrix} = \begin{pmatrix} 1 + a'N & b'N \\ c'N & 1 + d'N \end{pmatrix}\begin{pmatrix} N & \\ & 1 \end{pmatrix}$$

where the first term is in $\Gamma(N)$ and hence we have

$$f|h(\sigma z) = (\det h)^{k/2}(c'Nz + 1 + d'N) f(hz) = (c'Nz + 1 + d'N) f|h(z)$$

which implies that $f|h$ is a modular form for $\Gamma'(N)$. Moreover, any element in $\Gamma_1(N^2)$ is in $\Gamma'(N)$ and hence a modular form with respect to $\Gamma'(N)$ is also a modular form with respect to $\Gamma'(N^2)$, which is exactly the space we need. □

Exercise 1.4.8. Prove that a complete set of coset representatives for $\Gamma_0(N)\backslash G_0(N)/\Gamma_0(N)$ consists of the matrices

$$\begin{pmatrix} d_1 & \\ & d_2 \end{pmatrix},$$

where d_2 and d_1 are positive elements of $\mathbb{Z}_\Sigma^\times$ and d_1/d_2 is a positive rational integer prime to N.

Proof. If $\alpha \in G_0(N)$, that is, $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with positive determinant such that $c \in N\mathbb{Z}_\Sigma$, then by Proposition 1.4.2 and Proposition 1.4.3, we can write $\alpha = \gamma_1\delta\gamma_2$ where $\gamma_1, \gamma_2 \in \Gamma(1)$ and δ is the diagonal matrix with d_1/d_2 positive integer. We let $d = \frac{d_1}{d_2}$, then from Exercise 1.4.4, the greatest common divisor of a, b, c, d can be interpreted to be that the positive rational number that generates the same fractional ideal as the ideal generated by a, b, c, d is the rational number d_2. Say $(d_2) = (\frac{d_{2a}}{d_{2b}})$, then $d_1 = \frac{dd_{2a}}{d_{2b}}$. Now we want to adjust γ_2 on the right from the left by multiplying an element in $\Gamma_0(d)$, then the resulting left bottom corner would be $dme + tg$ where since $(d, N) = 1$, by the Chinese Reminder Theorem, we have that we have the choice of making it to a multiple of N. We can have a corresponding inverse on the left and by passing it to the left of δ it would be a right multiplication for γ_1, which will have the same effect on γ_1, making it an element of $\Gamma_0(N)$. Hence we have such coset representatives. □

Exercise 1.4.9. Prove that if $d_2|d_1$, and d_1, d_2 are integers prime to N, then there are the same number of right cosets in

$$\Gamma_0(N)\backslash\Gamma_0(N)\alpha\Gamma_0(N), \alpha = \begin{pmatrix} d_1 & \\ & d_2 \end{pmatrix}$$

as in $\Gamma(1)\backslash\Gamma(1)\alpha\Gamma(1)$ and that we may use the same representatives in both cases.

Proof. From all the previous results, we know that we are actually looking at $\Gamma_0(N)\backslash G_0(N)$ and we hence have

$$\begin{pmatrix} d_1 & \\ & d_2 \end{pmatrix} \begin{pmatrix} a & b \\ cN & e \end{pmatrix}$$

which would give $\begin{pmatrix} ad_2d & bd_2d \\ cd_2N & ed_2 \end{pmatrix}$ which we note that can be bijective to the results of Exercise 1.4.4 where the representatives have requirement that $a, d > 0, ad = d_1d_2, b \pmod{d}$ and $\gcd(a, b, d) = d_2$. Hence we can use the same representatives. □

1.5 Twisting (0/2)

1.6 The Rankin-Selberg Method (2/5)

Exercise 1.6.1. Suppose that $re(s) > 1$. Prove that the series

$$E(z,s) = \pi^{-s}\Gamma(s)\frac{1}{2}\sum_{m,n}\frac{y^s}{|mz+n|^{2s}}$$

is absolutely convergent and that $E(z,s)$ is automorphic, that is, prove

$$E(\gamma(z),s) = E(z,s)$$

for $\gamma \in \mathrm{SL}(2,\mathbb{Z})$.

Proof. The absolutely convergence is determined by the behavior of

$$\sum_{m,n\in\mathbb{Z},(m,n)\neq(0,0)}\frac{1}{|mz+n|^{2s}}$$

since the rest factors are $\pi^{-s}\Gamma(s)\frac{y^s}{2}$ which are fixed and does not affect the convergence. The behavior of the series is proved the same way as Exercise 1.3.1 by the integral test.

For the automorphicity, we check that

$$E(\gamma(z),s) = \pi^{-s}\Gamma(s)\sum_{m,n}\frac{\frac{y^s}{|cz+d|^{2s}}}{|m\gamma(z)+n|^{2s}} = \pi^{-s}\Gamma(s)\sum_{m,n}\frac{y^s}{|(ma+nc)z+bm+dn|^{2s}}$$

where similarly $(ma+nc)$ and $bm+dn$ is a permutation of $\mathbb{Z}\times\mathbb{Z}$ but does not change the series itself. Hence it is same as $E(z,s)$. □

Exercise 1.6.2. (a). Prove the Poisson summation formula for $\mathbb{R}^n$: Let

$$\langle x,y\rangle = x_1y_1 + \cdots x_ny_n$$

be the usual inner product; define the Fourier transform

$$\hat{f}(x) = \int_{\mathbb{R}^n} f(y)e^{2\pi i\langle x,y\rangle}dy.$$

Suppose that f is smooth and of rapid decay; that is, $f(x) = O((1+|x|)^{-M})$ for all $M > 0$. Prove that

$$\sum_{\xi\in\mathbb{Z}^n} f(\xi) = \sum_{\xi\in\mathbb{Z}^n}\hat{f}(\xi).$$

(b). If $z = x + iy \in H$ and $t > 0$, let

$$\Theta(t) = \sum_{(m,n)\in\mathbb{Z}^2} e^{-\pi|mz+n|^2 t/y}.$$

Use the Poisson summation formula to show that $\Theta(t) = t^{-1}\Theta(t^{-1})$.

(c). Prove that

$$E(z,s) = \frac{1}{2}\int_0^\infty (\Theta(t)-1)t^s \frac{dt}{t}.$$

Deduce the analytic continuation and functional equation of $E(z,s)$ from this identity and from (b).

Proof. We prove as below.

(a). From definition, we denote $x \in \mathbb{R}^n, m \in \mathbb{Z}^n$ and we define $F(x) = \sum_{m\in\mathbb{Z}^n} f(x+m)$ and then

$$\hat{F}_k = \int_{[0,1]^n} \sum_{m\in\mathbb{Z}^n} f(x+m)e^{-2\pi ikx}dx$$
$$= \sum_{m\in\mathbb{Z}^n}\int_{[0,1]^n} f(x+m)e^{-2\pi ikx}dx = \sum_{m\in\mathbb{Z}^n}\int_{[m,m+1]^n} f(x)e^{-2\pi ikx}dx$$
$$= \int_{\mathbb{R}^n} f(x)e^{-2\pi ikx}dx = \hat{f(k)},$$

where the steps are justified by the nice property of f. Therefore we have that $F(x) = \sum_{k\in\mathbb{Z}^n} \hat{f(k)}e^{ikx}$. Now let $x = 0$ we obtain

$$F(0) = \sum_{k\in\mathbb{Z}^n} \hat{f(k)} = \sum_{m\in\mathbb{Z}^n} f(m).$$

We can also show this using the "trace formula" way. Consider $K_f(x,y) \in L^2([0,1]^d)$ where

$$K_f(x,y) = \sum_{\xi\in\mathbb{Z}^d} f(x+\xi-y),$$

and it is well-defined by the nice property of f. Then consider the integral operator T_f defined by

$$T_f(\phi)(y) = \int_{[0,1]^d} K_f(x,y)\phi(x)dx = \int_{[0,1]^d} \sum_{\xi\in\mathbb{Z}^d} f(x+\xi-y).$$

This operator T_f is Hilbert-Schmidt and the trace is along the diagonal, i.e., if we denote the volume of $[0,1]^d$ by C we have

$$\mathrm{tr}(T_f) = \int_{[0,1]^d} K_f(x,x)dx = \int_{[0,1]^d} dx \sum_{\xi\in\mathbb{Z}^d} f(\xi) = C\sum_{\xi\in\mathbb{Z}} f(\xi).$$

Now let us look at the spectral side of the trace.

Consider the family $\{e_n\}, n \in \mathbb{Z}^d$, where $e_n(x) = e^{2\pi i \langle n,x\rangle}$. This is a Hilbert basis for $L^2([0,1]^d)$, and then using Fourier expansion we have

$$K_f(x,y) = \sum_{n\in\mathbb{Z}^d} \hat{f(n)} e_n(x)\overline{e_n(y)}$$

and if we fix y and expand in x we have the $n-$Fourier coefficient

$$\int_{[0,1]^d} \sum_{\xi\in\mathbb{Z}^d} f(x+\xi-y)\overline{e_n(x)}dx = \int_{\mathbb{R}^d} f(x-y)\overline{e_n(x)}dx = \int_{\mathbb{R}^d} f(x)\overline{e_n(x+y)}dx$$
$$= \hat{f(n)}\overline{e_n(y)}.$$

Then $\mathrm{tr}T_f = \int_{[0,1]^d} \sum_{n\in\mathbb{Z}^d} \hat{f(n)}1 = C\sum_{n\in\mathbb{Z}^d} \hat{f(n)}$.

(b). Here we are to view Θ as a function on the lattices (m,n), that is, we have $\Theta(t) = \sum_{\gamma\in\mathbb{Z}^2} f(\gamma)$ where

$$f(\gamma) = f((m,n)) = e^{-\pi|mz+n|^2t/y} = e^{-\pi t\frac{(mx+n)^2+(ym)^2}{y}},$$

which means we need to deal with the Fourier transform of this function. We can view $x_1 = (mx+n/\sqrt{y})$ and $x_2 = (ym/\sqrt{y})$ as lattices in $\mathbb{R}^2$, then to do the Fourier transform, we pick x_1 and x_2 as the basis, we are then computing the Fourier transform of $e^{-\pi t(x_1^2+x_2^2)}$ and correspondingly we have the well-known result that the Fourier transform of such functions is

$$\hat{f(y)} = te^{-\pi(y_1^2+y_2^2)/t}$$

and hence we have

$$\Theta(t) = \sum_{x_1,x_2} e^{-\pi t(x_1^2+x_2^2)} = t^{-1}\sum_{y_1,y_2} e^{-\pi t(y_1^2+y_2^2)} = t^{-1}\Theta(t^{-1}).$$

(c). We see the right hand side

$$\frac{1}{2}\int_0^\infty (\Theta(t)-1)t^s\frac{dt}{t} = \frac{1}{2}\int_0^\infty \sum_{(m,n)\neq(0,0)} e^{-\pi|mz+n|^2t/y}t^s\frac{dt}{t}$$
$$= \frac{1}{2}\sum_{(m,n)\neq(0,0)}\int_0^\infty e^{-t}(\frac{y}{\pi|mz+n|^2})^s t^{s-1}dt = \frac{1}{2}\sum_{(m,n)\neq(0,0)} \Gamma(s)\pi^{-s}\frac{y^s}{|mz+n|^{2s}}$$
$$= E(z,s)$$

by definition. Therefore, from the functional equation above, we have that

$$\begin{aligned}
2E(z,s) &= \int_0^\infty (\Theta(t)-1)t^s\frac{dt}{t} + \int_0^1 (\Theta(t)-1)t^s\frac{dt}{t} \\
&= \int_0^\infty (\Theta(t)-1)t^s\frac{dt}{t} + \int_1^\infty (\Theta(\frac{1}{t})-1)t^{-s}\frac{dt}{t} \\
= I(s)+\int_1^\infty (t\Theta(t)-1)t^{-s}\frac{dt}{t} &= I(s)+\int_1^\infty (\Theta(t)-1)t^{1-s}\frac{dt}{t}+\int_1^\infty t^{1-s}\frac{dt}{t}-\int_1^\infty t^{-s}\frac{dt}{t} \\
&= I(s)+I(1-s)-\frac{1}{1-s}-\frac{1}{s} = 2E(z,1-s).
\end{aligned}$$

Note that the final identity we just arrived showed that it is meromorphic over $\mathbb{C}$ except for simple poles at $s=0$ and $s=1$, hence we obtain its continuation.

□

1.7 Hecke Characters and Hilbert Modular Forms (0/6)

1.8 Artin L-Functions and Langlands Functoriality (0/2)

1.9 Maass Forms (0/5)

1.10 Base Change (0/4)

Chapter 2

Automorphic Form and Representations of $\mathrm{GL}_2(2, \mathbb{R})$ (24/35 Problems)

2.1 Maass Forms and the Spectral Problem (9/9)

Exercise 2.1.1. Prove that $R_k \circ \triangle_k = \triangle_{k+2} \circ R_k$ and $L_k \circ \triangle_k = \triangle_{k-2} \circ L_k$.

Proof. We compute directly for the first identity and the second identity follows with the same fashion. Given any f we would have

$$R_k \circ \triangle_k(f)$$
$$= -iy^3\frac{\partial^3 f}{\partial x^3} - iy^3\frac{\partial^3 f}{\partial y^2 \partial x} - ky^2\frac{\partial^2 f}{\partial x^2} + y(-2y\frac{\partial^2 f}{\partial x^2} - y^2\frac{\partial^3 f}{\partial x^2 \partial y} - 2y\frac{\partial^2 f}{\partial y^2} - y^2\frac{\partial^2 f}{\partial y^2} + ik\frac{\partial f}{\partial x} + iky\frac{\partial^2 f}{\partial x \partial y}),$$
$$-\frac{k}{2}y^2\frac{\partial^2 f}{\partial x^2} - \frac{k}{2}y^2\frac{\partial^2 f}{\partial y^2} + \frac{k^2 iy}{2}\frac{\partial f}{\partial x}$$

and on the other hand

$$\triangle_{k+2} \circ R_k(f) = \triangle_{k+2}(iy\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + \frac{k}{2}f)$$
$$= -iy^3\frac{\partial^3 f}{\partial x^3} - y^3\frac{\partial^3 f}{\partial x^2 \partial y} - \frac{ky^2}{2}\frac{\partial^2 f}{\partial x^2} - y^2(\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial x \partial y} + y\frac{\partial^3 f}{\partial x \partial y^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial y^2})$$
$$+y\frac{\partial^3 f}{\partial y^3} + \frac{k}{2}\frac{\partial^2 f}{\partial y^2} + i(k+2)y(iy\frac{\partial^2 f}{\partial x^2} + y\frac{\partial^2 f}{\partial x \partial y} + \frac{k}{2}\frac{\partial y}{\partial x})$$

which agree with each other by comparing terms. The second identity can also follow from such direct computation. □

Exercise 2.1.2. Verify the formulas $d_L(b) = \frac{dxdydu}{y^2u}, d_R(b) = \frac{dxdydu}{yu}$ for left and right Haar measures.

Proof. For any given $b = \begin{pmatrix} u & \\ & u \end{pmatrix}\begin{pmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2} \end{pmatrix}$ we see that it can be written as

$$b = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} uy^{1/2} & \\ & uy^{-1/2} \end{pmatrix}$$

and then the two identities become a simple corollary of Exercise 4.2.1. □

Exercise 2.1.3 (Iwasawa Decomposition)**.** Prove that each element of $G = \mathrm{GL}(2,\mathbb{R})^+$ has a unique representation (Equation

$$g = \begin{pmatrix} u & \\ & u \end{pmatrix}\begin{pmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2} \end{pmatrix} k_\theta, k_\theta = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

).

Proof. The action of $g \in \mathrm{GL}(2, \mathbb{R})^+$ is transitive on the upper half plane and we know that the stabilizer of the point i is the group $Z(\mathbb{R})\mathrm{SO}_2(\mathbb{R})$ where i corresponds to $x + yi$ that it gets sent to. Therefore, we can write G in the unique representation. □

Exercise 2.1.4. Check that the right regular representation, defined by

$$(\rho(g)f)(x) = f(xg), g, x \in G$$

satisfies $\rho(gg')f = \rho(g)\rho(g')f$.

Proof. We check

$$(\rho(gg')f)(x) = f(xgg')$$

and for

$$\rho(g)\rho(g')f(x) = \rho(g)(\rho(g')f(x))$$

let us denote $\rho(g')f = h$ and we have $\rho(g)h(x) = h(xg)$. Now

$$h(xg) = \rho(g')f(xg) = f(xgg') = (\rho(gg')f)(x)$$

Hence the result. □

Exercise 2.1.5. Let H be a Hilbert space, let

$$K = \mathrm{SO}(2) = \{\begin{pmatrix} a & b \\ -b & a \end{pmatrix} |a, b \in \mathbb{R}, a^2 + b^2 = 1\},$$

and let $\rho : K \to \mathrm{End}(H)$ be a unitary representation of K. For $k \in \mathbb{Z}$, let

$$H_k = \{v \in H|\rho(\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix})v = e^{ik\theta}v\}.$$

Prove that the spaces H_k are orthogonal, and that

$$H = \bigoplus H_k$$

as a *Hilbert space direct sum*. This means that every vector $v \in H$ has a unique representation as the sum of a series $v = \sum_{k=-\infty}^{\infty} v_k$, with $v_k \in H_k$ (series convergent in H).

Proof. We know that ρ is a unitary representation of K, hence the operator $\rho(g)$ is a unitary operator in H. Now consider $v_1 \in H_{k_1}$ and $v_2 \in H_{k_2}$, we ought to have $\langle \rho(g)v_1, \rho(g)v_2\rangle = \langle v_1, v_2\rangle$ while the left hand side is also $\langle e^{ik_1\theta}v_1, e^{ik_2\theta}v_2\rangle = e^{i(k_1-k_2)\theta}\langle v_1, v_2\rangle$ meaning that we have $e^{i(k_1-k_2)\theta} = 1$ or the inner product vanishes. Since θ is arbitrary, it only holds if $k_1 - k_2 = 0$ which is not the case by out hypothesis. Hence $\langle v_1, v_2\rangle = 0$. Therefore, they are orthogonal.

Now for the spanning part, let us suppose that there exists some element $v \in H$ such that it is orthogonal to the span of all of H_k. Consider the space $\mathbb{R}/2\pi$ and let U be the neighborhood of its identity. Then by the Stone-Weierstrass Theorem we would have that there is some polynomial in the form of trigonometric (as our space is $\mathbb{R}/2\pi$) such that

$$F(\theta) = \sum_{n=-N}^{n=N} a_n e^{in\theta}$$

such that $\int_0^{2\pi} F(\theta)d\theta = 2\pi$ where $|F(\theta)| < \epsilon$ for $\epsilon > 0$ when away from U. Then consider

$$v' = \frac{1}{2\pi}\int_0^{2\pi} F(\theta)\rho(k_\theta)v d\theta$$

we have

$$|v' - v| = \frac{1}{2\pi}|\int_0^{2\pi} F(\theta)\rho(k_\theta)vd\theta - \int_0^{2\pi} F(\theta)vd\theta|$$

$$= \frac{1}{2\pi}|\int_0^{2\pi} F(\theta)(\rho(k_\theta)v - v)d\theta| = \frac{1}{2\pi}|\int_0^{2\pi}\sum_{n=-N}^{n=N} a_n e^{in\theta}(e^{ik\theta}v - v)d\theta|$$

$$< \epsilon * (2\pi - d(U)) + 1 * d(U)$$

which is then can be taken arbitrarily small. Hence v' can be arbitrarily close to v. However, note that v' is a finite linear combination of elements of H_k and thus it should be orthogonal to v. This is a contradiction to the fact that it can be arbitrarily close to v (by taking inner product $\langle v, v'\rangle$). Therefore $v = 0$ and hence $H = \bigoplus H_k$. □

Exercise 2.1.6. Verify the assertions in the proof of Proposition 2.1.8.

Proof. The assertions are regarding identities so we check them here. If f satisfies

$$\chi(\gamma)f(z) = (\frac{c\overline{z}+d}{|cz+d|})^k f(\frac{az+b}{cz+d}), \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,$$

then

$$F(\gamma gu) = \sigma_k f(\gamma gu) = (f|_k(\gamma gu))(i) = (\det \gamma u)^{k/2}(cz+d)^{-k}(u)^{-k} f|_k(g)(\frac{az+b}{cz+d})$$

$$= (\det \gamma)^{k/2}\chi(\gamma) f|_k(g)(i) = \chi(\gamma)F(g),$$

and thus the second identity

$$F(gk_\theta) = e^{ik\theta}F(g)$$

follows directly from the identity $e^{ik\theta} = \cos(k\theta) + i\sin(k\theta)$. Conversely, let us define

$$f(z) = F(\begin{pmatrix} y & x \\ & 1 \end{pmatrix}),$$

the first identity follows with the same fashion as above but reversed. □

Exercise 2.1.7 (Holomorphic modular forms as Maass forms)**.** Because the raising and lowering operators shift the weight by two, it might seem that by applying them repeatedly we can shift any element of $C^\infty(\Gamma\backslash H, \chi, k)$ into $C^\infty(\Gamma\backslash H, \chi, \epsilon)$ where $\epsilon = 0$ or 1 and reduce the study of Maass forms to weights zero or one. This is almost true; however, there is an exception to this.

(a). Show that if f is a modular form of weight $k > 0$, then $y^{k/2}f(z)$ is a Maass form of weight k (by our definition), with the eigenvalue $\lambda = \frac{k}{2}(1-\frac{k}{2})$ of $\triangle_k$, but that it is annihilated by the lowering operator L_k.

(b). Show that this is the only case where a Maass form with weight can be annihilated by a lowering operator. Describe the Maass forms that are annihilated by raising operators.

Proof. We prove as below.

(a). First we show that it is annihilated by L_k as we can view it as a function regarding z and its conjugate and hence

$$L_k(y^{k/2}f) = (-(z-\overline{z})\frac{\partial}{\partial \overline{z}} - \frac{k}{2})y^{k/2}f$$

$$= -(z-\overline{z})\frac{\partial y^{k/2} f}{\partial \overline{z}} - \frac{ky^{k/2}f}{2} = -(z-\overline{z})\frac{\partial y^{k/2}}{\partial \overline{z}}f - \frac{ky^{k/2}f}{2}$$

$$= \frac{k}{2}(z-\overline{z})^{k/2}y^{k/2}(\frac{1}{2i})^{k/2}f - \frac{ky^{k/2}f}{2} = 0.$$

Hence it is annihilated by L_k. Now we apply the identity that

$$\triangle_k = -R_{k-2}L_k + \frac{k}{2}(1-\frac{k}{2})$$

we would have

$$\triangle_k(f) = -R_{k-2}L_k(f) + \frac{k}{2}(1-\frac{k}{2})(f) = 0 + \frac{k}{2}(1-\frac{k}{2})f$$

which implies that f is one eigenvector of the Laplacian and has eigenvalue $\frac{k}{2}(1-\frac{k}{2})$.

(b). The first assertion follows from the last identity we used. Similarly, for the ones that are annihilated by R_k, they should have eigenvalue $-\frac{k}{2}(1+\frac{k}{2})$.

Hence the result. □

Exercise 2.1.8 (Positivity of the Laplacian). (a). Prove that $\triangle_0$ is a *positive operator* on $C^\infty(\Gamma\backslash H, \chi) = C^\infty(\Gamma\backslash H, \chi, 0)$. That is, we have $\langle \triangle f, f\rangle \geq 0$, with equality only if f is equal to a constant function. Conclude that the eigenvalues of f are all nonnegative real numbers.

(b). Prove that if λ is an eigenvalue of $\triangle_1$, then $\lambda \geq \frac{1}{4}$.

(c). Prove that if $k \geq 0$, and if λ is an eigenvalue of $\triangle_k$, then either $\lambda = \frac{l}{2}(1-\frac{l}{2})$, where l is an integer such that $1 \leq l \leq k, l \equiv k \pmod 2$; or else $\lambda \geq 0$, and in fact $\lambda \geq \frac{1}{4}$ if k is odd.

Proof. We prove as below.

(a). Recall the inner product

$$\langle \triangle_0 f, f\rangle = \int_{\Gamma\backslash H} (\triangle_0 f)\overline{f}\frac{dxdy}{y^2}.$$

Recall also the Stoke's Theorem

$$\int_F \triangle f\overline{g}d\mu dz = -\int_F \triangledown f\overline{\triangledown f}dxdy + \int_{\partial F} h$$

and thus

$$\langle \triangle_0 f, f\rangle = \int_F \triangledown f\overline{\triangledown f}dxdy$$

as the boundary condition is zero by Lemma 2.1.2 on the book. Hence clearly the inner product is nonnegative. It is zero only when the integrand is zero and it means that the function is constant. Therefore, since we have

$$\langle \triangle f, f\rangle = \lambda\langle f, f\rangle \geq 0,$$

the value of λ has to be nonnegative and real. We can also use the identity that $\triangle_0 = -R_{-2}L_0$ and then

$$\langle \triangle_0 f, f\rangle = \langle -R_{-2}L_0 f, f\rangle = -\langle L_0 f, -L_0 f\rangle = \langle L_0 f, L_0 f\rangle$$

which is positive definite and use the definition of L_0.

(b). We use similar arguments that

$$\lambda_1\langle f, f\rangle = \langle \triangle_1 f, f\rangle = \langle -R_{-1}L_1 f + \frac{1}{4}f, f\rangle = \langle L_1 f, L_1 f\rangle + \frac{1}{4}\langle f, f\rangle$$

hence

$$(\lambda_1 - \frac{1}{4})\langle f, f\rangle = \langle L_1 f, L_1 f\rangle \geq 0$$

and thus $\lambda_1 \geq \frac{1}{4}$.

(c). By applying the previous identities we used as well as the identity from Exercise 2.1.1, we would obtain the eigenvalue such that

$$\lambda = \frac{\langle L_k f, L_k f\rangle}{\langle f, f\rangle} + \frac{k}{2}(1 - \frac{k}{2}).$$

Further, we note that by those, we have $\triangle_{k-2}L_k f = \lambda L_k f$, that is, $L_k f$ is an eigenfunction of the operator $\triangle_{k-2}$. TBD.

Hence the result. □

Exercise 2.1.9. Let H be a Hilbert space, let G be a group, and let $\pi : G \to \mathrm{End}(V)$ be a group action such that $\pi(g) : H \to H$ is unitary for all $g \in G$. Suppose that for all $v \in H$ the map $g \mapsto \pi(g)v$ is continuous. Prove that π is a representation.

Proof. The algebraic definition is clear. We need to check the continuity. That is, we want to show that the map

$$G \times H \to H,$$

$$(g, v) \mapsto \pi(g)v$$

is continuous. Pick any open neighborhood of $\pi(g_1)v_1$ and if we have v_2 near v_1 and g_2 near g_1 where their distance is denoted by ϵ and σ, then we consider

$$d = |\pi(g_1)v_1 - \pi(g_2)v_2| \leq |\pi(g_1)v_1 - \pi(g_1)v_2| + |\pi(g_1)v_2 - \pi(g_2)v_2|$$

$$\leq |\pi(g_1)||v_1 - v_2| + |\pi(g_1) - \pi(g_2)||v_2| \leq \epsilon + \sigma|v_2|$$

where since σ can be taken sufficiently small we have that d can be arbitrarily small. Therefore, there is a small neighborhood of (g_1, v_1) such that the image can be arbitrarily small. Hence it is a continuous map. Therefore it is a unitary representation. □

2.2 Basic Lie Theory (6/6)

Exercise 2.2.1 (The adjoint representations of G and $\mathfrak{g}$)**.** Let $G = \mathrm{GL}(n, \mathbb{R})$, and let $\mathfrak{g}$ be its Lie algebra. There exists an n^2-dimensional representation of G on the vector space $\mathfrak{g}$, called the *adjoint representation*, defined by

$$\mathrm{Ad}(g)(X) = gXg^{-1},$$

where the multiplications on the right-hand side are ordinary matrix multiplication. As we have shown, every finite-dimensional representation π of G has a differential $d\pi$. The differential of Ad is a representation of $\mathfrak{g}$ denoted $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$. Prove the formula

$$(\mathrm{ad}X)(Y) = [X, Y].$$

Proof. This is direct calculation as

$$\mathrm{ad}(X)(Y) = \frac{d}{dt}\mathrm{Ad}(\exp(tX))(Y) = \frac{d}{dt}\mathrm{Ad}(e^{tX})(Y)$$

$$= \frac{d}{dt}(1 + tX + g_1(tX))Y(1 - tX + g_2(tX))$$

$$= \frac{d}{dt}(Y + tXY)(1 - tX) = \frac{d}{dt}(Y - tYX + tXY - t^2XYX + \cdots)$$

$$= (-YX + XY - 2tXYX + g(tX))$$

where $g(t, X)$ forms all mean a higher term with respect to tX. Then the last expression is simply $-YX + XY$ when $t = 0$, and thus is exactly $[X, Y]$. Therefore, we have $(\mathrm{ad}X)(Y) = [X, Y]$. □

Exercise 2.2.2. Let $G = \mathrm{GL}(2,\mathbb{R})$. For each of the following Lie subgroups H of G, prove that the Lie algebra $\mathfrak{h}$ is as described.

(a). $H = \mathrm{SL}(2,\mathbb{R}); \mathfrak{h} = \{X \in \mathfrak{gl}(2,\mathbb{R}) | \mathrm{tr}(X) = 0\}$.

(b). $H = \{\begin{pmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2} \end{pmatrix} | x, y \in \mathbb{R}, y > 0\}$; $\mathfrak{h}$ is the linear span of $\hat{H}$ and $\hat{R}$.

(c). $H = \{\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} | x, y \in \mathbb{R}, y > 0\}$; $\mathfrak{h}$ is the linear span of $\hat{R}$.

(d). $H = \{\begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix} | x, y \in \mathbb{R}, y > 0\}$; $\mathfrak{h}$ is the linear span of $\hat{H}$.

(e). $H = \mathrm{SO}(2)$; $\mathfrak{h}$ is the linear span of W.

Proof. We prove as below.

(a). We have the identity that $\det(e^X) = e^{\mathrm{tr}X} = 1$ and thus it holds if and only if $\mathrm{tr}X = 0$.

(b). It follows from (c) and (d).

(c). Consider the curve $\gamma(t) = \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix}$ and its derivative

$$\gamma'(0) = \gamma'(t = 0) = \begin{pmatrix} & 1 \\ & \end{pmatrix}$$

and thus equals $\hat{R}$.

(d). Consider the curve $\gamma(t) = \begin{pmatrix} y^{t/2} & \\ & y^{-t/2} \end{pmatrix}$ and

$$\gamma'(t) = \begin{pmatrix} \frac{1}{2}\ln(y)y^{t/2} & \\ & -\frac{1}{2}\ln(y)y^{-t/2} \end{pmatrix}$$

and

$$\gamma'(0) = \begin{pmatrix} \frac{1}{2}\ln(y) & \\ & -\frac{1}{2}\ln(y) \end{pmatrix}$$

which is a span of $\hat{H}$.

(e). We use the identification of the curve and the elements in SO(2) is of the form

$$\gamma(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \gamma'(t) = \begin{pmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{pmatrix}$$

and

$$\gamma'(0) = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$$

which is the linear span of W.

Hence the result. □

Exercise 2.2.3. Finish the proof of Proposition 2.2.5 by verifying the Laplace-Beltrami operator.

Proof. We can prove it using the identities that are already proved. Now we have

$$\triangle = -(H^2 + 2RL + 2LR)/4$$

and

$$\begin{aligned} -4\triangle = H^2 + 4(y^2\frac{\partial^2}{\partial x^2} + iy^2\frac{\partial^2}{\partial x\partial y} - \frac{y}{2}\frac{\partial^2}{\partial x\partial\theta} + y^2\frac{\partial^2}{\partial y^2} - iy^2\frac{\partial^2}{\partial x\partial y} - \frac{y}{2i}\frac{\partial^2}{\partial y\partial\theta} - \frac{y}{2}\frac{\partial^2}{\partial\theta\partial x} \\ + \frac{y}{2i}\frac{\partial^2}{\partial\theta\partial y} + \frac{1}{4}\frac{\partial^2}{\partial\theta^2}) \\ = 4(y^2\frac{\partial^2}{\partial x^2} - y\frac{\partial^2}{\partial x\partial\theta} + y^2\frac{\partial^2}{\partial y^2}) \end{aligned}$$

which verifies the identity. □

Exercise 2.2.4. (a). Prove that if $X, Y \in \mathfrak{gl}(n, \mathbb{R})$, and if $[X, Y] = 0$, then

$$\exp(X + Y) = \exp(X)\exp(Y).$$

(b). Prove that if $X \in \mathfrak{gl}(n, \mathbb{R})$, then $t \mapsto \exp(tX)$ is a homomorphism $\mathbb{R} \to \mathrm{GL}(n, \mathbb{R})$. Such a homomorphism is called a *one-parameter subgroup*.

(c). Prove that any continuous homomorphism $\mathbb{R} \mapsto \mathrm{GL}(n, \mathbb{R})$ is of the form $t \mapsto \exp(tX)$ for some $X \in \mathfrak{gl}(n, \mathbb{R})$.

Proof. We prove as below.

(a). The condition $[X, Y] = 0$ implies that $XY = YX$. Hence we have

$$\exp(X+Y) = I + (X+Y) + \frac{1}{2}(X+Y)^2 + \frac{1}{6}(X+Y)^3 + \cdots$$

and

$$\exp(X)\exp(Y) = (I + X + \frac{1}{2}X^2 + \frac{1}{6}X^3 + \cdots)(I + Y + \frac{1}{2}Y^2 + \cdots)$$

$$= I + (X+Y) + XY + \frac{1}{2}X^2 + \frac{1}{2}Y^2 + \cdots = I + (X+Y) + \frac{1}{2}(X+Y)^2 + \cdots,$$

hence they are identical.

(b). The identity in $\mathbb{R}$ is zero which gets sent to $\exp(0) = I$ the identity. Moreover, for $a, b \in \mathbb{R}$, we have

$$\exp((a+b)X) = I + ((a+b)X) + \frac{1}{2}((a+b)X)^2 + \cdots$$

$$= \exp(aX)\exp(bX)$$

by previous assertion. Hence it is a homomorphism.

(c). By the homomorphicity, we have that $f : t \mapsto \exp(tX)$ gives $1 \mapsto \exp(X)$ and hence $n \mapsto \exp(nX)$ meaning that $f(1)$ determines the value on all integers. Since $\mathbb{R}$ is divisible and torsion-free, it hence determines the values on $\mathbb{Q}$. By continuity, it determines the value on all $\mathbb{R}$.

Hence the result. □

Exercise 2.2.5. The adjoint representation defined in Exercise 2.2.1 extends to an action $\mathrm{Ad} : G \to \mathrm{End}(U(\mathfrak{g}))$. Prove that if D is in the center of $U(\mathfrak{g})$ and $g \in G$ then $\mathrm{Ad}(g)D = D$.

Proof. This follows from the important identity that $\mathrm{Ad}(\exp(X)) = \exp(\mathrm{ad}_X)$ for $X \in \mathfrak{g}$. Then $\mathrm{ad}_X(D) = [X, D] = 0$ by previous results and hence $\mathrm{Ad}(g)(D) = \mathrm{Ad}(\exp(X))(D) = \exp(\mathrm{ad}_X)(D) = (I + \mathrm{ad}_X + \frac{1}{2}\mathrm{ad}_X^2 + \cdots)D = D$. Hence the identity.

For the first identity, consider $M(t) = \mathrm{Ad}(\exp(tX))$ and by previous exercise we have that it is uniquely determined. □

Exercise 2.2.6. (a). Generalize Exercise 2.2.2(e) by proving that the Lie algebra l of $K = \mathrm{SO}(n)$ consists of the skew-symmetric matrices in $\mathrm{Mat}(n, \mathbb{R})$.

(b). Prove that the exponential map $\exp : l \to K$ is surjective.

Proof. We prove as below.

(a). Consider the curve $\gamma(t) = \exp(tX)$ for the Lie algebra elements X. Then

$$\gamma(t) = I + tX + \frac{1}{2}(tX)^2 + \cdots$$

and by the condition of $\mathrm{SO}(n)$ if we look at only first order term (higher ones will vanish when $t = 0$), we would have

$$I = (I + tX)(I + tX)^T = (I + tX)(I + (tX)^T) = I + (tX)^T + tX + (tX)(tX)^T$$
$$= I + tX^T + tX + t^2$$

and hence $0 = tX^T + tX + t^2X^2 + \cdots$. Then if we take the derivative at $t = 0$ we would have $X^T + X = 0$ which means $X^T = -X$, the definition of skew-symmetric matrices.

(b). For the case $n = 2$, we have that K is the 1-PGD we discussed above. Then the image $\exp(tW)$ is $I + tW + \frac{1}{2}t^2W^2 + \cdots$ where it means that any element in K can be obtained by this map from the Lie algebra as the terms with odd powers of W corresponds to the right top and left bottom corners of K and the even powers of such corresponds to the diagonal.

For the general case, that is, when $n \geq 3$. Note that $A^TA = I$ means that $A^TAA^T = A^T$ and such that $AA^T = I = A^TA$. Further, since the base field is $\mathbb{R}$, it means that $\mathrm{SO}(n)$ here is normal. Normal matrices has spectral theorem that they are similar to diagonal matrices

$$\begin{pmatrix} e^{i\theta_1} & \cdots & 0 \\ \vdots & e^{i\theta_k} & \vdots \\ 0 & \cdots & e^{i\theta_m} \end{pmatrix}$$

Now if n is even, then each $e^{i\theta}$ corresponds to an element in K, meaning that we can also write it as a two-by-two block form. If n is odd, then the only one dimensional matrix here is the identity, that is, all can be written as two-by-two blocks except for one I_1. Now from the case for $n = 2$ we showed that the map is surjective, then we just need to pick all the corresponding preimages to be mapped to the element in $\mathrm{SO}(n)$, where the preimage would be diagonal blocks with one or no identity and each block is indeed a skew-symmetric matrix.

Hence the result. $\square$

2.3 Discreteness of the Spectrum (1/2)

Exercise 2.3.1. Let $\Gamma \subset \mathrm{SL}(2, \mathbb{R})$ be a discontinuous group. Let $z, \xi \in H$. Prove that there exists a constant C such that for all $0 < r < 1$, the number of $\gamma \in \Gamma$ such that (the image γz is in the circle?)

$$\frac{|z - \xi|}{|z - \bar{\xi}|} < r$$

is less than $Cr^2/(1 - r^2)$.

Proof. For any given such r, the region described is a circle with radius r and centered at a point determined by ξ. Now for given z, we can find a small neighborhood B_0 near the image of z under the Cayley transform is completely contained in the fundamental domain. No such image of γz would be intersecting with this ball as Γ is a discontinuous group. Then if $C_\xi(\gamma z)$ is contained in B_r for some γ, then since both are circles, by elementary geometry, we note that at least half of the circle is contained in B_r, which means that if the nonzero volume of the image of B_0 is V_0, at least $V_0/2$ is in B_r exclusively. Pick the least such values, namely V', and by the formula that the ball B_r in non-Euclidean metric is $4\pi r^2/(1-r)^2$ and thus it should contain at most $4\pi M r^2/(V'(1-r)^2)$ number of such points, where M denotes the effect on the volume of the map C_ξ and hence we have the desired result. $\square$

Exercise 2.3.2. Complete the proof of Theorem 2.3.5 by extending the theory of the Green's functions in Proposition 2.3.4 and 2.3.5 to the case of Maass forms of weight k.

Proof. TBD. $\square$

2.4 Basic Representation Theory (6/6)

Exercise 2.4.1. Let K be a compact subgroup of $\mathrm{GL}(n, \mathbb{C})$, and let (π, H) be a unitary representation, so that Theorem 2.4.2 applies.

(a). Let H_1 and H_2 be two irreducible invariant subspaces of H such that the representations of K on H_1 and H_2 are nonisomorphic. Prove that H_1 and H_2 are orthogonal.

(b). Let σ be an isomorphism class of irreducible representations of K. Let $H(\sigma)$ be the sum of all irreducible invariant subspaces of H that are isomorphic to σ. Prove that $H(\sigma)$ is an invariant subspace of H, and that

$$H = \bigoplus_{\sigma} H(\sigma),$$

where the sum runs over the isomorphism classes of irreducible representations of K. (Hilbert space direct sum.) $H(\sigma)$ is called the $\sigma-$isotypic part of H.

(c). In any decomposition of H into a direct sum of irreducible representations, prove that the (possibly infinite) number of terms isomorphic to σ is $\dim(H(\sigma))/\dim(\sigma)$; in particular, the number of terms is independent of the particular direct sum decomposition.

Proof. We prove as below.

(a). Let us consider the intertwining similar to the operator constructed in Proposition 2.4.3 but modified a bit, that is, given f a matrix coefficient $f_{x,y}(g) = \langle \pi(g)x, y \rangle$, we consider

$$L : H_1 \longrightarrow H_2,$$
$$v_1 \longmapsto \int_K \langle \pi(k)x, y \rangle \pi(k^{-1}) v_2 dk,$$

where $v_2 \in H_2$. This is an intertwining operator, where all the definition check is similar to the ones in the proof of Proposition 2.4.3. Since H_1 and H_2 are irreducible, by Schur's lemma, this operator is either an isomorphism or zero. Since they are non-isomorphic, L is a zero operator. Then consider the inner product

$$\langle v_1, L(v_1) \rangle = 0, \text{ for } v_1 \in H_1.$$

By taking $y = v_2$, it is simply

$$0 = \int_K \overline{\langle \pi(k)x, y\rangle}\langle x, \pi(k^{-1})v_2\rangle dk = \int_K \langle \pi(k)x, y\rangle^2 dk$$

which means that $\langle \pi(k)x, y\rangle = 0$ for any $y \in H_2$ and note that our choice is arbitrary for $x \in H_1$. Therefore, it means that the inner product of any elements $x \in H_1, y \in H_2$ is zero, meaning that $H_1 \perp H_2$.

(b). From Peter-Weyl, we have that H decomposed into a direct sum of irreducible unitary representations. Any summand of such decomposition gives an irreducible representation of K which is in $H(\sigma)$. Hence we need to show the other direction which also follows since $H(\sigma)$ is a subspace of H.

(c). This is a corollary of the previous two results. One side is clear, so we need to show that the number of such terms can not be smaller than $\dim(H(\sigma))/\dim(\sigma)$. If it is smaller, meaning that there is some subspace in $H(\sigma)$ not isomorphic to σ, but by (a), they must be orthogonal, which means there can be another summand which is nonisomorphic to σ and is in $H(\sigma)$, which is a contradiction.

Hence the result. □

Exercise 2.4.2. Let $G = \mathrm{GL}(n, \mathbb{R})$, and let $X \in \mathfrak{g} = \mathfrak{gl}(n, \mathbb{R})$.

(a). Let $f \in C^\infty(G)$. Prove that

$$f(ge^X) = \sum_{n=0}^{\infty} \frac{1}{n!}(dX^n f)(g),$$

where $dX : C^\infty(G) \to C^\infty(G)$ is defined by Eq

$$(dXf)(g) = \frac{d}{dt} f(g(I + tX))|_{t=0}.$$

(b). Let (π, H) be a representation of G on a Hilbert space, and let the representation of $\mathfrak{g}$ on the space H^∞ of smooth vectors be defined by Eq

$$\pi(X)f := \lim_{t\to 0} \frac{1}{t}(\pi(\exp(tX))f - f).$$

Prove that for $f \in H^\infty$, we have

$$\pi(e^X)f = \sum_{k=0}^{\infty} \frac{1}{n!} X^n f.$$

Proof. We prove as below.

(a). Recall Lemma 2.2.2 that if $\phi \in C^\infty(G \times \mathbb{R})$ satisfies $\frac{\partial}{\partial t}\phi(g,t) = dX\phi(g,t)$ and the boundary condition for $t = 0$ is 0. Then $\phi(g,t) = 0$ for all $t = 0$. Hence let us consider

$$\phi(g,t) = f(ge^{tX}) - \sum_{n=0}^{\infty} \frac{t^n}{n!}(dX^n f)(g),$$

where the identity follows by definition. Put $t = 0$, we see that $\phi(g,0) = f(g) - f(g) = 0$ since the summation has only $n = 0$ term non-vanishing. Therefore, by Lemma 2.2.2, we have that $\phi(g,t) = 0$ for any $t = 0$. Therefore, we have the desired identity by taking $t = 1$.

(b). If we can show that for $\phi \in H$, the inner product

$$\langle \pi(e^{tX})f - \sum_{n=0}^{\infty} \frac{1}{n!} t^n X^n f, \phi \rangle = 0$$

holds for all $t \in \mathbb{R}$, then by taking $t = 1$ we would obtain the desired identity. Now from all the identities we have and recall that

$$\langle \pi(g)f, \phi\rangle = (Lf)(g),$$

$$((L \circ X)f)(g) = ((dX \circ L)f)(g),$$

we now consider the first term of the inner product that is

$$\langle \pi(e^{tX})f, \phi\rangle = Lf(e^{tX}) = L(\sum_{n=0}^{\infty} \frac{1}{n!}(dX^n f)(g)) = \langle \sum_{n=0}^{\infty} \frac{1}{n!}(dt^n X^n f), \phi\rangle$$

and the second term is

$$-\langle \sum_{n=0}^{\infty} \frac{1}{n!} t^n X^n f, \phi\rangle = -\sum_{n=0}^{\infty} \frac{1}{n!} t^n dX^n \langle f, \phi\rangle$$

which would kill the first term. Therefore the desired inner product is zero.

Hence the result. □

Exercise 2.4.3 (Gelfand Pairs)**.** We say (G,K) is a Gelfand pair if every irreducible unitary representation H of G has at most one linearly independent $K-$fixed vector. Assume that G and K are both finite. Let R be the algebra of K biivariant functions $\phi : G \to \mathbb{C}$, with multiplication given by convolution

$$(\phi_1 * \phi_2)(g) = \sum_{h \in G} \phi_1(gh)\phi_2(h^{-1}).$$

Prove that (G, K) is a Gelfand pair if and only if the convolution algebra R is commutative.

Proof. If (G, K) is a Gelfand pair, then by definition there is at most one dimensional $K-$fixed vector in any irreducible unitary representation H of G. Therefore, let us consider the representation space such that the vectors are functions on G and the action is right translation. This holds here because we are working with finite groups, hence the functions are just group algebras. Therefore, we build an isomorphism between R and H. Now $K-$fixed vectors in R are functions in $\mathbb{C}[G/K]$. If there is only one such vector, the algebra is clearly commutative. Conversely, if this algebra is commutative, then the preimage of this isomorphism is also commutative. Namely, the representation $H|_K$ is commutative. Commutative representation is one-dimensional, hence the result. □

Exercise 2.4.4. Let $G = \mathrm{GL}(n, \mathbb{R})^+$ and $K = \mathrm{SO}(n, \mathbb{R})$. Prove that in the definition of a $(\mathfrak{g}, K)-$module, condition (iii) is automatically satisfied.

Proof. We first use the idea as in Exercise 2.4.2. Let $X \in \mathfrak{g}, Y \in \mathfrak{k}, v \in V$ and let Λ be any linear functional on $\mathfrak{g}$. Define a function ϕ such that

$$K \times \mathbb{R} \to \mathbb{C},$$

$$(g, t) \mapsto \Lambda(\pi(ge^{tY})X\pi(e^{-tY}g^{-1})f - \pi(g)(\mathrm{Ad}(e^{tY})X)\pi(g^{-1})f).$$

Our plan is to show that Lemma 2.2.2 can apply to this function and hence we would obtain the identity

$$\pi(ge^{tY})X\pi(e^{-tY}g^{-1})f = \pi(g)(\mathrm{Ad}(e^{tY})(X))\pi(g^{-1})f$$

and the right hand side is

$$\pi(g)\pi(e^{tY}Xe^{-tY})\pi(g^{-1})f = \pi(ge^{tY})X\pi(e^{-tY}g^{-1})f = \pi(\mathrm{Ad}(ge^{tY})X)f.$$

Further, recall from Exercise 2.2.6(b) that the exponential map from $\mathfrak{k}$ to K in this example is surjective, meaning that for any $k \in K$, we have some $Y \in \mathfrak{k}$ in its preimage under the exponential map such that the identity follows. Therefore, we checked the condition (iii).

Now we only need to apply Lemma 2.2.2 to the function we defined. It satisfies the condition for this lemma because the functional Λ is linear and dX acts as exponential map with derivative at $t = 0$, which is the same definition as $\frac{\partial}{\partial t}\phi(g,t)$. Therefore, we put in $t = 0$, obtaining

$$\phi(g,0) = \Lambda(\pi(g)X\pi(g^{-1})f - \pi(g)(X)\pi(g^{-1})f) = 0,$$

and thus the lemma can apply. Therefore, it is zero for any $t \in \mathbb{R}$. Hence the rest follows from the discussion above. □

Exercise 2.4.5 (Induction from a subgroup of finite index)**.** Let G be a reductive Lie group, K a maximal compact subgroup, and $\mathfrak{g}$ its Lie algebra. Let G_0 be a subgroup of finite index in G, and let $K_0 = G_0 \cap K$. Suppose that every connected component of G contains a unique connected component of K. Show that this implies that $[G : G_0] = [K : K_0]$. Let V_0 be a $(\mathfrak{g}, K_0)-$module. We will construct a $(\mathfrak{g}, K)-$module that is "induced" from V_0. We will denote the actions of both $\mathfrak{g}$ and K_0 on V_0 as π_0 to avoid confusion. Thus we will write $\pi_0(X)v$ instead of Xv for $X \in \mathfrak{g}, v \in V_0$.

The space V consists of all functions $F : K \to V_0$ that satisfy $F(k_0k) = \pi_0(k_0)F(k)$ when $k_0 \in K_0, k \in K$. The action π of K on V is given by

$$(\pi(c)F)(k) = F(kc),$$

and the action π of $\mathfrak{g}$ is given by

$$(\pi(X)F)(k) = \pi_0(\mathrm{Ad}(k)X)F(k).$$

Prove that with these actions of K and $\mathfrak{g}, V$ is a $(\mathfrak{g}, K)-$module.

Proof. The first part follows from that since G is a reductive Lie group and each of its connected component contains a unique connected component of K, if the representative of the cosets G/G_0 is $\{1, r_1, \cdots, r_n\}$, we are then looking at $1G_0 \cap K$. Inside K, there are also this many representatives, hence the result. TBD.

For the second part, we are to check the three conditions for $(\mathfrak{g}, K)-$modules. For condition (i), we need to check that it is $K-$finite. Now we know that $F(k_0k) = \pi_0(k_0)F(k)$, so the value of F is determined by its representative of the cosets K/K_0, which from our first assertion is finite. Therefore, when we consider the action $\pi(K)F(g) = F(gK)$ we can decompose gk into K_0K for $g \in K$ and the finiteness implies the finiteness of the span. For condition (ii), we are to check the derivative is defined. That is, we need to check

$$\pi(X)F = \frac{d}{dt}\pi(\exp(tX))F|_{t=0}$$

is defined. This is clear as we have

$$\pi(X)F(k) = \pi_0(\mathrm{Ad}(k)X)F(k) = \frac{d}{dt}\pi_0(\exp(tkXk^{-1}))F(k)|_{t=0}$$

$$= \frac{d}{dt}F(k(I + k^{-1}Xkt + \cdots))|_{t=0} = \frac{d}{dt}F(k + tXk + \cdots)|_{t=0} = \frac{d}{dt}F(k(\exp(tX)))$$

$$= \frac{d}{dt}\pi(\exp(tX))F(k)|_{t=0},$$

which is the desired property. For condition (iii), we are to check that for $X \in \mathfrak{g}$ and $g \in K$, we have

$$\pi(g)\pi(X)\pi(g^{-1})F(k) = \pi(g)\pi(X)F(kg^{-1}) = \pi(g)\pi_0(\mathrm{Ad}(kg^{-1})X)F(kg^{-1})$$

$$= \frac{d}{dt}F(k\exp(kg^{-1}Xgk^{-1})) = \frac{d}{dt}F(k(I + tgXg^{-1} + \cdots)) = \pi(\mathrm{Ad}(g)X)F(k)$$

which verifies the third condition.

Hence, the actions of K and $\mathfrak{g}$ make V is $(\mathfrak{g}, K)-$module. $\square$

Exercise 2.4.6 (Schur's Lemma for $(\mathfrak{g}, K)-$Modules)**.** Let $\mathfrak{g}$ and K be as in the definition of a $(\mathfrak{g}, K)-$module. Let V and W be irreducible admissible $(\mathfrak{g}, K)-$modules.

(a). Prove that the space of linear maps $\lambda : V \to W$ that commute with the actions of $\mathfrak{g}$ and K is at most one dimensional.

(b). Denoting the $(\mathfrak{g}, K)-$actions of $\mathfrak{g}$ and K on V and on W by π_V and π_W, respectively, show that the space of bilinear maps $\langle , \rangle : V \times W \to \mathbb{C}$ such that

$$\langle v, w\rangle = \langle \pi_V(k)v, \pi_W(k)w\rangle, k \in K$$

and

$$\langle \pi_V(X)v, w\rangle = -\langle v, \pi_W(X)w\rangle$$

is at most one dimensional.

(c). Recall that a pairing $V \times W \to \mathbb{C}$ is *sesquilinear* if it is linear in the first variable and conjugate linear in the second. Prove that the dimension of the space of sesquilinear maps $\langle , \rangle : V \times W \to \mathbb{C}$ which satisfy the above two identities is at most one dimensional.

Proof. We prove as below.

(a). If such a linear map λ exists, it is an intertwining operator. Therefore, since V, W are irreducible, the map must be zero or isomorphism. Now if it is an isomorphism, by Peter-Weyl, we know that they must be in the same $V(\sigma)$ isotypic summand. Therefore, $V(\sigma)$ is invariant under the action of λ and thus there is some eigenvector w in $V(\sigma)$ for λ. Now for the actions of $\mathfrak{g}, K$ we have $\lambda(\pi(g)w) = \pi(g)\lambda(w) = c\pi(g)w$ and thus $\pi(g)w \in \langle w \rangle$. So does the action of K. Therefore, λ acts on this space as scalar. Then by irreducibility, this eigenspace must be the whole space and hence such a λ acts as a scalar meaning the space of such linear maps is at most one dimensional.

(b). If such an inner product exists, we note that it actually induces an intertwining operator between V and W. Namely, we consider

$$\lambda : V \longrightarrow W,$$

$$v \longmapsto w_v,$$

where w_v is chosen as one element in W such that $\langle v, w_v \rangle = 1$. We can check its property, for instance, the action of K would be $\lambda(\pi_V(k)v) = w_{\pi_V(k)v}$ where we should have $\langle \pi_V(k)v, w_{\pi_V(k)v} \rangle = 1$, where we also have $\langle v, w_v \rangle = 1 = \langle \pi_V(k)v, \pi_W(k)w_v \rangle$. Therefore, we have $\lambda(\pi_V(k)v) = w_{\pi_V(k)v} = \pi_W(k)w_v = \pi_W(k)\lambda(v)$. Same argument checks also λ commutes with the action of $\mathfrak{g}$, hence it is an intertwining operator. Since we can have at most one dimensional of such intertwining operator, we can only have at most one dimension of such inner product.

(c). Similar argument as the bilinear case but lift to tensor product.

TBD. $\square$

2.5 Irreducible $(\mathfrak{g}, K)-$Modules for $\mathrm{GL}(2, \mathbb{R})$ (2/8)

Exercise 2.5.1. Let (π, H) be an irreducible admissible representation of $\mathrm{GL}(2, \mathbb{R})^+$. Let t be a complex number. Let (π', H) be the representation on the same space with

$$\pi'(g)v = \det(g)^t \pi(g)v.$$

This representation is sometimes denoted $\pi \otimes \det^t$. Prove that every irreducible admissible representation has a unique representation in the form $\pi \otimes \det^t$, where

$$\pi \begin{pmatrix} u & \\ & u \end{pmatrix} v = v.$$

Prove that this condition is equivalent to the assumption that the eigenvalue μ of Z in π is equal to zero.

Proof. Since the center $\begin{pmatrix} u & \\ & u \end{pmatrix}$ induces an intertwining operator, which is also one-dimensional, we have that it factors through the determinant map, which is a result from topology that it must has the form $|\cdot|^t$ for some $t \in \mathbb{C}$. Hence every irreducible admissible representation has a unique representation in such a form. Further, recall that the eigenvalue of Z has that

$$\mu = u\frac{\partial}{\partial u},$$

hence if $\mu = 0$ then $\partial/\partial u = 0$ which is equivalent to saying that v is invariant under Z. On the other hand, if it is invariant under Z, the partial derivative is hence zero and so is the eigenvalue. □

Exercise 2.5.2. Describe the classification theorem for irreducible admissible $(\mathfrak{g}, K)-$modules and for infinitesemal equivalence classes of admissible representations of $\mathrm{SL}(2, \mathbb{R})$.

Proof. We have the theory for $\mathrm{GL}(2, \mathbb{R})^+$ which is only different from $\mathrm{SL}(2, \mathbb{R})$ by the center $\begin{pmatrix} u & \\ & u \end{pmatrix}$. Then $\mu = 0 = s_1 + s_2$ and $s = s_1 + \frac{1}{2}$ then $\lambda = (s_1 + \frac{1}{2})(\frac{1}{2} - s_1)$. Hence the module classification follows directly from the one for $\mathrm{GL}(2, \mathbb{R})^+$. With this difference, we have that if λ is not of the form $(1 - k/2)k/2$ then we have principal series representations $P(\lambda, \epsilon)$. If it is of such a form and finite-dimensional, then it is the nonunitary representation talked in the book. If it is of such a form and infinite dimensional, then we have $D^{\pm}(k)$, the discrete series and limit of discrete series respectively. □

Exercise 2.5.3. If V is a $(\mathfrak{g}, K)-$module for $\mathrm{GL}(2,\mathbb{R})$, it is automatically also a $(\mathfrak{g}, K)-$module for $\mathrm{GL}(2,\mathbb{R})^+$. Let $\pi : O(2) \to \mathrm{End}(V)$ denote the given representation of K. Let

$$\mu = \pi \begin{pmatrix} -1 & \\ & 1 \end{pmatrix} \in \mathrm{End}(V).$$

(a). Prove that the $(\mathfrak{g}, O(2))-$module structure is uniquely determined by specifying the $(\mathfrak{g}, SO(2))-$module structure, together with the action of μ, which must satisfy

$$\mu^2 = 1, \mu H \mu = -H, \mu R \mu = L, \mu L \mu = R, \mu Z \mu = Z.$$

(b). Prove, conversely, that given a $(\mathfrak{g}, SO(2))-$module V together with an endomorphism μ satisfying this equation, V may be given a unique $(\mathfrak{g}, O(2))-$module structure such that the first equation is true.

Proof. We prove as below.

(a).

□

2.6 Unitaricity and Intertwining Integrals (0/4)

2.7 Representations and the Spectral Problem

2.8 Whittaker Models

2.9 A Theorem of Harish-Chandra

Chapter 3

Automorphic Representations (5/31 Problems)

3.1 Tate's Thesis (3/9)

Exercise 3.1.1. (a). The "no small subgroups" argument. Let G be a topological group having a basis of neighborhoods of the identity consisting of open and compact subgroups. Prove that the kernel of any cotinuous homomorphism of $G \to \mathrm{GL}(m, \mathbb{C})$ contains an open subgroup.

(b). Let F be a non-Archimedean local field. Let $\mathfrak{o}$ be the ring of integers in F and $\mathfrak{p}$ its maximal ideal. Let ψ be a nontrivial additive character of F. Prove that there exists a unique integer m such that ψ is trivial on $\mathfrak{p}^{-m}$, but not on $\mathfrak{p}^{-m-1}$. We call $\mathfrak{p}^{-m}$ the conductor of ψ.

(c). Let k be a finite field, ψ a nontrivial additive character of k, and let ψ_1 be another character. Show that there exists a unique element $a \in k$ such that $\psi_1 = \psi(ax)$ for all $x \in k$.

(d). Let ψ_1 be another additive character of the non-Archimedean local field F. Prove that there exists a unique element $a \in F$ such that $\psi_1(x) = \psi(ax)$ for all $x \in F$.

(e). Define $\psi_a(x) = \psi(ax)$, for $a \in F$. Prove that $a \mapsto \psi_a$ is a topological isomorphism of F with its dual group.

(f). Prove that $F = \mathbb{R}$ or $F = \mathbb{C}$ then F may also be identified with its dual group: let $\psi(x) = e^{2\pi ix}$ for $x \in \mathbb{R}$ or $\psi(x) = e^{4\pi i re(x)}$ for $x \in \mathbb{C}$. Show that $a \mapsto \psi_a$ is a topological isomorphism of F with its dual group.

Proof. We prove as follows.

(a). Consider N as a small neighborhood near the identity such that it contains no nontrivial subgroups. Then the preimage $N' = F^{-1}(N)$ under the continuous homomorphism F would be consisted of open neighborhood basis $G_1, \cdots, G_n, \cdots$ of the identity. If all of them do not map to the identity, we then pick a smaller neighborhood U than N such that it is smaller than any $F(G_i)$. Then the preiamge $F^{-1}(U)$ would not contain any of the neighborhood basis and thus there must be some G_i such that $F(G_i) = 1$. Hence the kernel contains an open subgroup.

(b). We recall that a non-Archimedean ring of integer is compact if and only of it is locally compact. Since it is discrete, it must have finite value and thus trivial on some integer m. It is hence clearly unique by additive property.

(c). It is a classical result that additive characters on finite field with characteristic p has the form $\psi(x) = e^{2\pi ix/p}$ which can be proved by simple ODE arguments. Then the desired property is clear by counting the number of elements in k and the fact that different x induces different such functions.

(d). From previous results, we without loss of generality assume that ψ_1 has conductor $\mathfrak{p}^{-m_1}$ for some integer m_1. We prove by induction. For $N = m_1$, consider $a_N = 0$, then for $x \in \mathfrak{p}^{-N}$ we have $\psi_1(x) = \psi(a_N x) = 0$. If it is true for some greater number N_1, we have $\psi_1(x) = \psi(a_{N_1} x)$. Then for larger N, consider $a_N = a_{N-1} + c_N$ for some $c_N \in \mathfrak{p}^{N-1-m_1}$. Then $\psi(a_N x) = \psi(a_{N-1}x + c_N x) = \psi_1(x)$ by choosing proper c_N. This choice is valid due to the result from (c) as on this part it works as a finite field. Then we take the inverse limit

$$a = \varprojlim a_N$$

which gives us the unique choice of desired a.

(e). Since for this case the topology is discrete, the topological isomorphism just follows from the group isomorphism, which is clear by the uniqueness we obtained above.

(f). TBD (Standard argument, c.f. Poonen's notes on Tate's thesis).

Hence the result. □

Exercise 3.1.2. Let F be an algebraic number field and A its adele ring. Suppose that for each place v of F, we select an additive character ψ_v of F_v such that the conductor of ψ_v is $\mathfrak{o}_v$ for almost all v. Then we may define a character $\psi = \prod_v \psi_v$ of A by $\psi(a) = \prod_v \psi_v(a_v)$ for each adele $a = (a_v) \in A$. The aim of this exercise is to show directly how to choose the ψ_v such that ψ is trivial on F.

(a). Assume that F is $\mathbb{Q}$. Define characters $\psi_v : F_v \to \mathbb{C}^\times$ as follows. If $v = \infty$, let $\psi_\infty : \mathbb{Q}_\infty = \mathbb{R} \to \mathbb{C}$ be defined by $\psi_\infty(x) = e^{2\pi ix}$. If $v = p$ is a finite prime, we have $\mathbb{Q}_p = \mathbb{Z}[1/p] + \mathbb{Z}$, and so we may write any $x \in \mathbb{Q}_p$ as $y + z$, where $y \in \mathbb{Z}[1/p]$ and $z \in \mathbb{Z}_p$. Then define $\psi_v(x) = e^{-2\pi iy}$. Prove that $\psi = \prod_v \psi_v$ is trivial on $\mathbb{Q}$.

(b). If F is a number field, reduce to the case $F = \mathbb{Q}$ by composition with the trace map.

Proof. We prove as follows.

(a). For any element $x = p_1^{r_1} \cdots p_k^{r_k} \in \mathbb{Q}$, we embed it into A and it is clear that on the places that are not one of the p_i the character ψ_v is trivial. For place p_i, we have that $\psi_{p_i}(x) = e^{-2\pi i p_i^{r_i}}$ if $r_i \leq 0$ and we denote $i := j$ or trivial if $r_i > 0$. Then

$$\psi(x) = \prod_j \psi_j(x) = e^{2\pi i(x - p_1^{r_1} - \cdots - p_k^{r_k})} = e^{2\pi i(y)}$$

where y is an integer by this argument. Hence $\psi(x) = 1$.

(b). By composition the previous construction with the trace map, that is, $\psi' = \psi \circ \mathrm{Tr}$, we only need to take care of the places that the element $x \in F$ would have non-trivial value by ψ_v.

Hence the result follows. □

Exercise 3.1.3. Let F be an algebraic number field and A be its adele ring. Let ψ be an additive character of A that is trivial on F.

(a). Prove that every character of A has the form $\psi_a(x) := \psi(ax)$ for some $a \in A$.

(b). Prove that $a \mapsto \psi_a$ is an isomorphism of A with its dual group.

(c). Prove that the character ψ_a is trivial on the discrete, cocompact subgroup F of A if and only if $a \in F$.

(d). Prove that the dual of the discrete group F is the compact group A/F.

Proof. We prove as follows.

(a). It is a basic result for abstract restricted product that the characters on A lies in the space $\prod'(\hat{F}_v, \widehat{F_v/O_v})$. Then by local results we obtained in Exercise 3.1.1, we have the result that $\psi_a(x) = \psi(ax)$.

(b). We build this isomorphism

$$\prod{}'(F_v, O_v) \longrightarrow \prod{}'(\hat{F_v}, F_v\hat{/}O_v),$$

$$(a_v) \longmapsto (\psi_{a_v}).$$

That is, we use the identification from (a) to build this isomorphism locally and then compose them globally. This map locally takes O_v to $F_v\hat{/}O_v$ as for all but finitely many places the conductor of ψ_v is p_v^0. Hence it gives an isomorphism.

(c). If ψ_a is trivial on F of A, then by the above identification, we have that $a \in F$ and vice versa.

(d). As F is discrete and A/F is compact, we have that $A\hat{/}F = F'$ is discrete. As $F-$vector spaces, F' is a subspace of $\hat{A}$ which is isomorphic to A. Hence we can look at F'/F which would be a discrete subspace of A/F which is compact, hence F'/F is finite. However, F is infinite, then F'/F can only be finite if it is zero as a vector space, that is, $F'/F = 0$ and $F' = F$.

Hence the results. □

Exercise 3.1.4. In 1.7, we defined the notion of a Hecke character over a totally real field F. Generalize Proposition 3.1.2 to show that there is a bijection between primitive Hecke characters of F and characters of $A^\times/F^\times$, so that if χ_0 is a Hecke character in the sense of Section 1.7, and if χ is the corresponding character of $A^\times/F^\times$, then for nonramified places v, $\chi_v(\varpi) = \chi_0(\mathfrak{p}_v)$.

Proof. TBD. □

Exercise 3.1.5. (a). Let F be an algebraic number field and A its adele ring. Prove that every quasicharacter of $A^\times/F^\times$ has the form $\chi(x)|x|^s$, where χ is unitary (i.e., a Hecke character) and $s \in \mathbb{R}$.

(b). Let F be a non-Archimedean local field. Prove that every quasicharacter of $F^\times$ has the form $\chi(x)|x|^s$, where χ is a unitary character of $F^\times$.

Proof. TBD. □

3.2 Classical Automorphic Forms and Representations (0/6)

Exercise 3.2.1. (a). With notations as in this section, show that $C(\Gamma\backslash G, \chi, \omega)$ is dense in $L^2(\Gamma\backslash G, \chi, \omega)$.

(b). Prove that the action ρ on $L^2(\Gamma\backslash G, \chi, \omega)$ is continuous.

Proof. TBD. □

3.3 Automorphic Representations of $\mathrm{GL}(n)$ (0/7)

3.4 The Tensor Product Theorem (2/6)

Exercise 3.4.1. Let K be a compact group, and let (π, V) be an irreducible representation. Let $\langle , \rangle$ be an invariant inner product on V.

(a). We define a linear map $\phi : V \otimes V \to \mathrm{End}(V)$ by $\phi(x \otimes y)(z) = \langle z, y\rangle x$. Prove that ϕ is an isomorphism, and that

$$\phi(x \otimes y) \circ \phi(z \otimes w) = \langle z, y\rangle \phi(x \otimes w).$$

(b). Define a map $\Phi : V \otimes V \to C(K)$ by

$$\Phi(x \otimes y)(g) = \dim(V)\langle x, \pi(g)y\rangle.$$

Prove that

$$\Phi(x \otimes y) * \Phi(z \otimes w) = \langle z, y\rangle \Phi(x \otimes w).$$

Conclude that $\phi(x \otimes y) \to \Phi(x \otimes y)$ is a ring homomorphism $\mathrm{End}(V) \to C(V)$.

(c). Prove that $\mathcal{H}_K$ is the algebraic direct sum of the images of the maps defined in (b), as (π, V) ranges through the isomorphism classes of irreducible representations of K.

Proof. We prove as follows.

(a). Recall the basic definition of inner products. Now we check identity $1 \otimes 1 \in V \otimes V$ and $\phi(1 \otimes 1)(z) = \langle z, 1\rangle 1 = z\langle 1, 1\rangle = cz$ for some c scalar where if the ground field is complex then $c = 1$ and hence it shows that identity is mapped to the identity endomorphism. For the kernel of this map, to get zero map, we have $\phi(x \otimes y)(z) = 0 = \langle z, y\rangle x$ for any z which means x or y is zero which is a zero element in $V \otimes V$. On the other hand, any element $f \in \mathrm{End}(V)$ that takes $f(1) = c$ can be mapped from $c \otimes 1$ thus bijective. Therefore it is an isomorphism. For the composition, we check

$$\phi(x \otimes y) \circ \phi(z \otimes w)(t) = \phi(x \otimes y)(\langle t, w\rangle z) = \langle t, w\rangle \phi(x \otimes y)(z)$$

$$= \langle t, w\rangle\langle z, y\rangle x = \langle z, y\rangle\langle t, w\rangle x = \langle z, y\rangle \phi(x \otimes w)(t).$$

(b). By definition, we can check

$$\Phi(x \otimes y) * \Phi(z \otimes w)(t) = \int_G \Phi(x \otimes y)(tg)\Phi(z \otimes w)(g)dg$$

$$= \int_G (\dim^2 V)\langle x, \pi(tg)y\rangle\langle z, \pi(g)w\rangle dg,$$

where by Schur's orthogonality relations (c.f. Knapp 1986, Section I.5), we have

$$= \dim^2 V \frac{1}{\dim V}\langle x, \pi(t)w\rangle\langle z, y\rangle = \langle z, y\rangle\Phi(x \otimes w)(t),$$

as this is an invariant inner product on compact group. Hence as the ring multiplication on $\mathrm{End}(V)$ is by composition and on $C(V)$ is by convolution we have that $\phi(x \otimes y) \mapsto \Phi(x \otimes y)$ is a ring homomorphism.

(c). Direct application of Knapp 1986, Theorem 1.12 (Peter-Weyl Theorem).

Hence the result. □

Exercise 3.4.2. Complete the proof of Proposition 3.4.3.

Proof. First we check that it is a representation. Consider

$$\pi(\phi * \psi)v = \int_K (\phi * \psi)(k)\pi(k)vdk$$

$$= \int_K (\int_K \phi(g)\psi(kg^{-1})dg)\pi(k)vdk = \int_K \int_K \psi(k)\phi(g)dg\pi(kg)vdk$$

$$= \int_K \left(\int_K \psi(k)\pi(k)v dk \right) \phi(g)\pi(g)dg$$
$$= \int_K \phi(g)\pi(g)\pi(\psi)v dg$$
$$= \pi(\phi)\pi(\psi)v.$$

Hence we see that it is a representation. On the other hand, if we are given a smooth representation π of $\mathscr{H}_K$, then by the isomorphism $\mathscr{H}_K \simeq \bigoplus_{i=1}^{\infty} \mathrm{End}_{\mathbb{C}}(V_i)$ we get a representation π' of some endomorphism ϕ on V_i. Recall that as V_i are irreps, ϕ must lie in some $\mathrm{End}_{\mathbb{C}}(V_i)$, and this corresponding representation affords such a smooth representation. □

Exercise 3.4.3. Complete the proof of Proposition 3.4.5.

Proof. □

3.5 Whittaker Models and Automorphic Forms

3.6 Adelization of Classical Automorphic Forms

3.7 Eisenstein Series and Intertwining Integrals (0/2)

Exercise 3.7.1. Discuss the possible poles of $E^\sharp(g, f)$.

Proof. As in the discussion for $E^*(g, f)$, the poles should be determined by the $L-$factors. □

3.8 The Rankin-Selberg Method

3.9 The Global Langlands Conjectures (0/1)

3.10 The Triple Convolution

Chapter 4

Representations of $\mathrm{GL}(2)$ over a $P-$adic Field (42/53 Problems)

4.1 GL(2) **Over a Finite Field (20/20)**

Exercise 4.1.1 (Frobenius Reciprocity)**.** (a). Verify that the correspondence described in the text following Eq(1.3) is an isomorphism.

(b). Let (π_1, V_1) and (π_2, V_2) be representations of the finite group G, and let χ_1, χ_2 be their respective characters. Define their inner product as usual by

$$\langle \chi_1, \chi_2 \rangle_G = \frac{1}{|G|} \sum_{x \in G} \chi_1(x) \overline{\chi_2(x)}.$$

(Here $|G|$ is the cardinality of G). Prove that

$$\langle \chi_1, \chi_2 \rangle_G = \dim \mathrm{Hom}_G(\pi_1, \pi_2).$$

(c). Now let $H \subset G$, let (π, V) be a representation of H, and let (τ, U) be a representation of G. Let χ and σ be their respective characters, let χ^G be the character of the induced representation (π^G, V^G) of G, and let σ_H be the restriction of σ to H. Prove that

$$\langle \chi^G, \sigma \rangle_G = \langle \chi, \sigma_H \rangle_H.$$

Proof. We prove as follows.

(a). From the text, the correspondence is given by

$$f : \mathrm{Hom}_G(U, V^G) \longrightarrow \mathrm{Hom}_H(U_H, V),$$

$$\phi \longmapsto \phi'$$

where $\phi'(u) = \phi(u)(1)$. Conversely we have

$$h : \mathrm{Hom}_H(U_H, V) \longrightarrow \mathrm{Hom}_G(U, V^G),$$

$$\phi' \longmapsto \phi$$

such that $\phi(u)(g) = \phi'(gu)$. We are to check

$$f(r\phi_1 + s\phi_2)(u) = (r\phi_1 + s\phi_2)'(u) = (r\phi_1 + s\phi_2)(u)(1) = (rf(\phi_1) + sf(\phi_2))(u)$$

for homomorphism which is clear. Then we need to check that it is an isomorphism. With the given inverse, we check the composition

$$(h \circ f)(\phi)(u)(g) = h(f(\phi)(u))(g) = h(\phi'(u))(g) = \phi(u)(1)(g) = \phi(u)(1g) = \phi(u)(g)$$

which implies that $h \circ f = \mathrm{Id}$ on left hand side. Similar for the right hand side.

(b). If we have a non-irreducible finite dimensional representation V, then $V = V_1 \oplus V_2$ and it is sufficient to show the identity for irreducible then because of the linearity w.r.t. to direct sum of spaces on both sides, it can imply the case with non-irreducible representations. Hence we assume π_1, π_2 are irreducible.

Hence for two irreducible representations V and V', by Schur's Lemma, if they are isomorphic, then $\dim(\mathrm{Hom}(V, V')) = 1 = \langle \chi, \chi' \rangle_G$ by definition. If they are non-isomorphic, then dimension is 0 and non-isomorphic characters are orthogonal (results from finite group representation theory). Hence the identity is verified for irreducible representations.

To generalize, if we have $V = V_1 \oplus V_2$. The right hand side has the same argument and the left hand side is also linear by the inner product property. Hence the identity follows.

(c). From (b) we obtain $\langle \chi^G, \sigma \rangle_G = \dim(\mathrm{Hom}_G(\pi^G, \sigma))$, and $\langle \chi, \sigma_H \rangle_H = \dim \mathrm{Hom}_H(\pi, \sigma_H)$. Hence we need the dimension to be equal. Indeed, the two spaces are isomorphic from (a), hence they have same dimension. Hence the inner product identity verifies.

Hence the theorem. $\square$

Exercise 4.1.2 (Mackey's Theorem)**.** Let the notation be as in Proposition 4.1.1. Let $x_1, \cdots, x_r$ be a set of double coset representatives for $H_2 \backslash G / H_1$. Let $S_i = H_2 \cap x_i H_1 x_i^{-1}$. Define two representations $(\pi_{1,i}, V_1)$ and $(\pi_{2,i}, V_2)$ of S_i on the vector spaces V_1 and V_2 by $\pi_{1,i}(s) = \pi_1(x_i^{-1} s x_i)$ and $\pi_{2,i}(s) = \pi_2(s)$.

(a). Prove that if $\triangle : G \to \mathrm{Hom}_{\mathbb{C}}(V_1, V_2)$ satisfies Eq.(1.3), then $\triangle(x_i) : V_1 \to V_2$ is an intertwining operator from $\pi_{1,i}$ to $\pi_{2,i}$.

(b). Prove that

$$\mathrm{Hom}_G(V_1^G, V_2^G) \simeq \bigoplus_{i=1}^{r} \mathrm{Hom}_{S_i}(\pi_{1,i}, \pi_{2,i}).$$

Proof. We prove as follows.

(a). The condition "satisfies Eq.(1.3)." should be "Eq.(1.4)". If so, then first for any $s \in S_i$, we have that $s = h_2 = x_i h_1 x_i^{-1}$ for some $h_2 \in H_2$ and $h_1 \in H_1$. This also means we have $h_2 x_i = x_i h_1$ for the corresponding x_i and h_i. Then take any s, we have $\triangle(x_i) \circ \pi_{1,i}(s)(v_1) = \triangle(x_i) \circ \pi_1(x_i^{-1} s x_i)(v_1) = \triangle(s x_i)(v_1) = \triangle(h_2 x_i)(v_1) = \pi_2(h_2) \circ \triangle(x_i)(v_1) = \pi_2(s) \circ \triangle(x_i)(v_1)$. Hence by definition it is an intertwining operator in $\mathrm{Hom}_{S_i}(\pi_{1,i}, \pi_{2,i})$.

(b). To build the isomorphism between $\mathrm{Hom}_G(V_1^G, V_2^G) \simeq \bigoplus_{i=1}^r \mathrm{Hom}_{S_i}(\pi_{1,i}, \pi_{2,i})$, we want to show that for any L in the left hand side space, it has a correspondence intertwining operator in one of the summands of the right hand side space. From (a), it is enough to show that any L gets mapped to some $\triangle(x_i)$. Let us first verify Eq(1.5):

$$[G:H_1]L(f_{a^{-1},v})(1) = \frac{|G|}{|H_1|} \triangle * f_{a^{-1},v}(1) = \frac{1}{|H_1|} \sum_{g \in H_1 a^{-1}} \triangle(g^{-1})(f_{a^{-1}v}(g))$$

$$= \frac{1}{|H_1|} \sum_{g \in H_1 a^{-1}} \triangle(a)(v) = \triangle(a)(v).$$

Since a has a representative x_i in the double coset, we have that $[G:H_1]L(f_{x_i^{-1},v})(1) = \triangle(x_i)(v)$ where v on the left hand side is arbitrarily chosen from V_1, hence it is an element in $\mathrm{Hom}_{S_i}(\pi_{1,i}, \pi_{2,i})$. Also the composition of these two maps are identity which is clear. Then we now build isomorphism between $\mathrm{Hom}_G(V_1^G, V_2^G)$ to $\mathcal{D}$.

Then we look at the space $\mathcal{D}$ and the space $\mathrm{Hom}_{S_i}(\pi_{1,i}, \pi_{2,i})$. Indeed, the correspondence between these two spaces are given by evaluation at x_i, which means we are left to show that the map $\triangle$ depends only on representatives. This is clear since $\triangle(h_2 x_i h_1) = \pi_2(h_2)\triangle(x_i)\pi_1(h_1)$. Hence this tells us the element $\triangle \in \mathcal{D}$ is determined by its value on each x_i. Conversely, the composition of $\triangle(x_i)$ over all representative x_i gives an element in $\mathcal{D}$. Hence we build the isomorphism between $\mathrm{Hom}_{S_i}(\pi_{1,i}, \pi_{2,i})$ and $\mathcal{D}$.

Thus we have $\mathrm{Hom}_G(V_1^G, V_2^G) \simeq \bigoplus_{i=1}^r \mathrm{Hom}_{S_i}(\pi_{1,i}, \pi_{2,i})$. as desired.

Hence the result. □

Exercise 4.1.3. Let F be any field. Prove that in $\mathrm{GL}(n, F)$, every matrix is conjugate to its transpose.

Proof. Let us first state the Corollary in Lang's book: Let A, B be $n \times n$ matrices over a field k and let k' be the extension field of k. Assume that there is an invertible matrix

C' in k' such that $B = C'AC'^{-1}$. Then there is an invertible matrix C in k such that $B = CAC^{-1}$.

Now denote the algebraic closure of F by $\overline{F}$, arbitrary matrix in $\mathrm{GL}(n, F)$ by M, its transpose by M^t. Suppose this condition holds for $\overline{F}$, that is, we have some matrix $C' \in \overline{F}$ such that $M = C'M^tC'^{-1}$, then from the Corollary we have that there is some matrix C in F such that $M = CM^tC^{-1}$. Then it is sufficient to show that the statement holds when $F = \overline{F}$. That is, in $\mathrm{GL}(n, \overline{F})$ and some $C \in \mathrm{GL}(n, \overline{F})$, every matrix M has that $M = CM^tC^{-1}$. Note that the Jordan canonical form is invariant by transpose and its existence is ensured by the algebraic closed field, that is, we have J being the Jordan canonical form of M and M^t. This means that we have invertible matrices A, B such that $J = AMA^{-1}$ and $J = BM^tB^{-1}$. Hence $AMA^{-1} = BM^tB^{-1}$ thus $M = A^{-1}BM^tB^{-1}A$ where $A^{-1}B$ is the inverse of $B^{-1}A$. Thus we have M is conjugate to its transpose over algebraic closed field and by previous discussion it is true over any field F. □

The Finite Stone-Von Neumann Theorem. Let H be a finite *two-step nilpotent group*, also known as a *Heisenberg group*. This means that if Z is the center of H, we assume that $\overline{H} = H/Z$ is Abelian. If A is any subgroup of H containing Z, we will denote $\overline{A} = A/Z \subseteq \overline{H}$; and if $x \in H$, we will denote by $\overline{x}$ its image in $\overline{H}$. Because $\overline{H}$ is Abelian, $\overline{A}$ is normal in $\overline{H}$, and consequently, A is normal in H. Let χ_0 be a character of Z. If $\overline{x}, \overline{y} \in \overline{H}$, then $xyx^{-1}y^{-1} \in Z$ does not depend on the representatives $x, y \in H$ for $\overline{x}, \overline{y} \in \overline{H}$. Thus we may define

$$\langle \overline{x}, \overline{y} \rangle = \chi_0(xyx^{-1}y^{-1}).$$

Exercise 4.1.4. Show that the pairing

$$(\overline{x}, \overline{y}) \mapsto \langle \overline{x}, \overline{y} \rangle$$

is bilinear and skew symmetric; that is, prove that

$$\langle \overline{x_1}\,\overline{x_2}, \overline{y} \rangle = \langle \overline{x_1}, \overline{y} \rangle \langle \overline{x_2}, \overline{y} \rangle, \tag{4.1}$$

$$\langle \overline{x}, \overline{y_1}\,\overline{y_2} \rangle = \langle \overline{x}, \overline{y_1} \rangle \langle \overline{x}, \overline{y_2} \rangle, \tag{4.2}$$

$$\langle \overline{x}, \overline{x} \rangle = 1, \tag{4.3}$$

and

$$\langle \overline{x}, \overline{y} \rangle = \langle \overline{y}, \overline{x} \rangle^{-1}. \tag{4.4}$$

Let G be a finite Abelian group, and let G^* be its character group. If $g \in G, g^* \in G^*$, we will denote $\langle g, g^* \rangle = g^*(g)$ to emphasize the symmetry between the two. It is well known that G^* is a finite Abelian group isomorphic to G, though not canonically. However, there is a canonical isomorphism $G \simeq G^{**}$, and we identify these two groups. If $A \subseteq G$, we denote

$$A^\perp = \{g^* \in G^* | \langle a, g^* \rangle = 1 \text{ for all } a \in A\}.$$

Thus $A^\perp$ consists of all characters of G that vanish on A, and we may identify $A^\perp = (G/A)^*$. We have

$$A^{\perp\perp} = A. \tag{4.5}$$

If A and B are subgroups of G, then clearly

$$(AB)^\perp = A^\perp \cap B^\perp$$

and using Eq.(4.5), this implies that

$$(A \cap B)^\perp = A^\perp B^\perp. \tag{4.6}$$

Returning to the two-step nilpotent group H, we say that χ_0 is *generic* if the pairing $\langle , \rangle$ is nondegenerate, that is, if every character of $\overline{H}$ has the form

$$\overline{x} \mapsto \langle \overline{x}, \overline{y} \rangle$$

for a unique $\overline{y} \in \overline{H}$. In this case, we may use the pairing to identify $\overline{H}$ with its dual group. With the above notation, each subgroup $\overline{A}$ of $\overline{H}$ (corresponding to a subgroup A of H containing Z) gives rise to another subgroup $\overline{A}^\perp$.

We will assume in the remaining exercises that χ_0 is generic. Making this assumption, we say that a subgroup $\overline{A} \subseteq \overline{H}$ is *isotropic* if $\langle \overline{x}, \overline{y} \rangle = 1$ for all $\overline{x}, \overline{y} \in \overline{A}$. We say that $\overline{A}$ is *polarizing* if $\overline{A} = \overline{A}^\perp$; that is, $\langle \overline{x}, \overline{y} \rangle = 1$ for all $\overline{y} \in \overline{A}$ if and only if $\overline{x} \in \overline{A}$. Thus a polarizing subgroup is automatically isotropic. (If A is the preimage of $\overline{A}$ in H, we also say that A is *isotropic* or *polarizing* if $\overline{A}$ has the same corresponding property.)

Proof. For Eq(4.1): Consider

$$\langle \overline{x_1}\,\overline{x_2}, \overline{y}\rangle = \chi_0(x_1x_2y(x_1x_2)^{-1}y^{-1}) = \chi_0(x_1x_2yx_2^{-1}x_1^{-1}y^{-1}).$$

On the other hand

$$\langle \overline{x_1}, \overline{y}\rangle\langle \overline{x_2}, \overline{y}\rangle = \chi_0(x_1yx_1^{-1}y^{-1})\chi_0(x_2yx_2^{-1}y^{-1}) = \chi_0(x_1yx_1^{-1}y^{-1}x_2yx_2^{-1}y^{-1})$$

and now note that since $\overline{x_1}, \overline{x_2}, \overline{y} \in \overline{H}$, we have that $y^{-1}x_2yx_2^{-1} \in Z$, hence

$$\chi_0(x_1yx_1^{-1}y^{-1})\chi_0(x_2yx_2^{-1}y^{-1}) = \chi_0(x_1yx_1^{-1}y^{-1}x_2yx_2^{-1}y^{-1}) = \chi_0(x_1yy^{-1}x_2yx_2^{-1}x_1^{-1}y^{-1})$$

$$= \chi_0(x_1x_2yx_2^{-1}x_1^{-1}y^{-1}) = \langle \overline{x_1}\,\overline{x_2}, \overline{y}\rangle.$$

For Eq(4.2): Similarly consider

$$\langle \overline{x}, \overline{y_1}\,\overline{y_2}\rangle = \chi_0(xy_1y_2x^{-1}y_2^{-1}y_1^{-1}).$$

On the other hand since χ has abelian image, we see

$$\langle \overline{x}, \overline{y_1}\rangle\langle \overline{x}, \overline{y_2}\rangle = \langle \overline{x}, \overline{y_2}\rangle\langle \overline{x}, \overline{y_1}\rangle = \chi_0(xy_1y_2x^{-1}y_2^{-1}xx^{-1}y_1^{-1}) = \langle \overline{x}, \overline{y_1}\,\overline{y_2}\rangle.$$

For Eq(4.3), consider

$$\langle \overline{x}, \overline{x}\rangle = \chi_0(xxx^{-1}x^{-1}) = 1.$$

For Eq(4.4), consider

$$\langle \overline{x}, \overline{y}\rangle\langle \overline{y}, \overline{x}\rangle = \chi_0(xyx^{=-1}y^{-1}yxy^{-1}x^{-1}) = 1.$$

Hence we showed the bilinearity and skew symmetric. □

Exercise 4.1.5. Prove that a maximal isotropic subgroup is polarizing. Thus any isotropic subgroup may be embedded in a polarizing subgroup. In particular, because the trivial subgroup is isotropic, polarizing subgroups exist.

Proof. We follow the spirit of the hint. Take $\overline{A}$ as our arbitrary maximal isotropic subgroup, and we want to show that it is polarizing. To do this, we are to show that $\overline{A} \subseteq \overline{A}^{\perp}$ and vice versa.

Take $\overline{x} \in \overline{A}^{\perp}$, we want to show that $\overline{x} \in \overline{A}$. Now consider the group $\overline{B} = \langle \overline{x}\rangle$. For the group $\overline{A}\,\overline{B}$, the smallest subgroup contains both $\overline{A}$ and $\overline{B}$, if it is isotropic, there are two possibilities. If it is the whole $\overline{H}$, then it means that any two elements has trivial inner product. Then $\langle \overline{x}, \overline{x} \rangle = 1$ meaning $\overline{x} \in \overline{A}^{\perp\perp} = A$. If it is not the whole $\overline{H}$, by the maximality of $\overline{A}$, we have that $\overline{A}\,\overline{B} \subseteq A$ hence it is indeed $\overline{A}$ and thus $\overline{x} \in \overline{A}$. Hence it is sufficient to show that $\overline{A}\,\overline{B}$ is isotropic.

Any element in $\overline{A}\,\overline{B}$ has the form ab for $a \in \overline{A}$ and $b \in \overline{B}$. Then consider any $ab, a'b'$, take the inner product $\langle ab, a'b' \rangle = \langle a, a'b' \rangle \langle b, a'b' \rangle = \langle a, a' \rangle \langle a, b' \rangle \langle b, a' \rangle \langle b, b' \rangle$ by the previous exercise. Since $\overline{A}$ is isotropic, the first term is trivial. Since b, b' are both in $\langle \overline{x} \rangle$, by (4.3), we have the last term is also trivial. Then we have $\langle a, \overline{x} \rangle^n \langle \overline{x}, a' \rangle^m$ left, while since $\overline{x} \in \overline{A}^{\perp}$ they are all trivial. Hence $\overline{A}\,\overline{B}$ by definition is isotropic. Hence by previous argument, $\overline{A}$ is polarizing. □

Exercise 4.1.6. Let $\overline{A}$ be an isotropic subgroup of $\overline{H}$, and let A be its preimage in H. Then χ_0 may be extended to a character of A.

Proof. First recall that χ_0 is a character of Z, and take $Z_0 = \ker(\chi_0)$. Now $\overline{A}$ is isotropic means that $\langle \overline{x}, \overline{y} \rangle = 1$ for any $\overline{x}, \overline{y} \in \overline{A}$, that is, $\chi_0(xyx^{-1}y^{-1}) = 1$ for any $x, y \in A$. Hence $Z_0 = \ker(\chi_0) = xyx^{-1}y^{-1}$ for any $x, y \in A$. Quotient by such a relation is exactly the operation of abelianization of a group. Hence A/Z_0 is the abelianization of A thus is Abelian. Then we have the fact as in the hint that if two finite Abelian groups are under inclusion, then the character on the smaller group can be extended to a character of the larger one. In our setting, the character χ_0 on Z induces a character on Z/Z_0, which is a finite Abelian subgroup of A/Z_0 which we just proved is also Abelian. Hence the character can be extended to A/Z_0. This character can be extended to A by trivial on Z_0. □

Exercise 4.1.7. Let H be a two-step nilpotent group, and let χ_0 be a generic character of its center Z. Let $\overline{A}$ and $\overline{B}$ be polarizing subgroups of $\overline{H}$, and let A and B be their preimages in H. Let χ_A and χ_B be characters of A and B extending χ_0, and let π_A and π_B be the representations of H induced from these characters of A and B.

(a). Show that A, B and BA are normal subgroups of H and that there exists a unique (double) coset $xBA = BxA \in B\backslash H/A$ such that (choosing a representative x of the coset) the characers $s \mapsto \chi_A(x^{-1}sx)$ and $s \mapsto \chi_B(s)$ coincide on $A \cap B$.

(b). Prove that

$$\dim \operatorname{Hom}_H(\pi_A, \pi_B) = 1.$$

Proof. We prove first (a) and then (b).

(a). The normality of the groups are clear. For example for A, since $\overline{A}$ is normal, meaning we have $hah^{-1}N \in A/N$ hence $hah^{-1}N = an'$. Further, since we have $n' \in N \subset A$ we have $hah^{-1} \in A$ and thus A is normal. The normality of the other two follows similar fashion.

Then for the double coset, choose one representative x_r in $B\backslash H/A$, if the two characters coincide on $A \cap B$, it means we have $\chi_B(s) = \chi_A(x^{-1}sx)$ for $s \in A \cap B$, that

(b). Recall from Exercise 4.1.2 that $\operatorname{Hom}_H(\pi_A, \pi_B) = \bigoplus_{i=1}^r \operatorname{Hom}_{S_i}(\pi_{1,i}, \pi_2, i)$, and since from (a) we have only one such coset with representative x, it means the left hand side is equal to $\dim \operatorname{Hom}_{A\cap B}(\chi_A, \chi_B)$ where the two character coincide on this coset. Hence by Schur's Lemma is 1. Hence $\dim \operatorname{Hom}_H(\pi_A, \pi_B) = 1$.

Thus the proof. □

Exercise 4.1.8 (The Stone-Von Neumann Theorem)**.** Let H be a finite two-step nilpotent group, and let χ_0 be a generic character of its center Z. Then there exists a unique isomorphism class of irreducible representations of H with central character χ_0. Such a representation may be constructed as follows: Let A be any polarizing subgroup of H, and let χ_A be any extension of χ_0 to A. Then the representation π of H induced from this character of A is of this class.

Proof. TBD. □

Projective representations and covering groups: Let G be a group, and let A be an Abelian group. By a *central extension* of G by A we mean a group $\tilde{G}$ together with a short exact sequence

$$1 \longrightarrow A \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1$$

such that the image of A is contained in the center of $\tilde{G}$. We say that two extensions $\tilde{G}_1$ and $\tilde{G}_2$ of G by A are *equivalent* if there exists an isomorphism $\tilde{G}_1 \to \tilde{G}_2$ making the following diagram commute:

$$\begin{tikzcd}
1 \arrow[r] & A \arrow[r] \arrow[d, Rightarrow, no head] & \tilde{G}_1 \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, Rightarrow, no head] & 1 \\
1 \arrow[r] & A \arrow[r] & \tilde{G}_2 \arrow[r] & G \arrow[r] & 1
\end{tikzcd}.$$

We recall the definition of the cohomology group $H^2(G, A)$ (where G acts trivially on A). Let $Z^2(G, A)$ be the multiplicative group of all maps ("2-cocycles") $\sigma : G \times G \to A$ satisfying the *cocycle relation*

$$\sigma(g_1, g_2)\sigma(g_1g_2, g_3) = \sigma(g_1, g_2g_2)\sigma(g_2, g_2). \tag{4.7}$$

If $\phi : G \to A$ is any map, then

$$\sigma(g_1, g_2) = \phi(g_1)\phi(g_2)\phi(g_1g_2)^{-1}$$

satisfies Eq.(4.7); Let $B^2(G, A)$ be the subgroup of $Z^2(G, A)$ of elements of this type ("coboundaries"). Then $H^2(G, A) = Z^2(G, A)/B^2(G, A)$. If $\sigma \in Z^2(G, A)$, we denote its class in $H^2(G, A)$ by $[\sigma]$.

Given a two-cocycle $\sigma \in Z^2(G, A)$, we construct a central extension $\tilde{G}_\sigma$ as follows. As a set, $\tilde{G}_\sigma$ is the Cartesian product $G \times A$. The group law is given by

$$(g, a)(g', a') = (gg', aa'\sigma(g, g')). \tag{4.8}$$

The cocycle condition (Eq.4.7) implies that this multiplication satisfies the associative law. The homomorphisms $A \to \tilde{G}_\sigma$ and $\tilde{G}_\sigma \to G$ are given by $a \mapsto (1, a)$ and $(g, a) \mapsto g$.

Exercise 4.1.9. Show that the extension $\tilde{G}_\sigma$ depends only on the cohomology class of σ in $H^2(G, A)$ and, moreover, that any central extension $\tilde{G}$ is equivalent to some $\tilde{G}_\sigma$ for a unique class $[\sigma] \in H^2(G, A)$. Hence there is a bijection between the classes of central extensions of G by A and the elements of $H^2(G, A)$.

Proof. To show depending only on the cohomology class, first we see that as a set, $\tilde{G}_\sigma$ is $G \times A$ which does not change regarding σ. If we have σ_1, σ_2 such that they are both in $[\sigma]$ with different coset, that is, $\sigma_1(g_1, g_2) = \sigma(g_1, g_2)\phi_1(g_1)\phi_1(g_2)\phi_1(g_1g_2)^{-1}$

and $\sigma_2(g_1, g_2) = \sigma(g_1, g_2)\phi_2(g_1)\phi_2(g_2)\phi_2(g_1g_2)^{-1}$. Clearly we can build an isomorphism between them by for example multiply the σ_1 by $\phi_1(g_1g_2)\phi_1(g_2)^{-1}\phi_1(g_1)^{-1}$ and similar to σ_1. Hence they are independent of the cosets.

Since any $\tilde{G}$ has the exact sequence

$$1 \longrightarrow A \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1,$$

it means that we have the isomorphism $G \simeq \tilde{G}/A$. Take any map $\phi : G \to A$ corresponding to a given σ and compose it with the injection ι from A to $\tilde{G}$. Then $\tilde{\phi} = \iota\phi : G \to \tilde{G}$. Then define σ to be $\sigma(g_1, g_2) = \tilde{\phi}(g_1)\tilde{\phi}(g_2)\tilde{\phi}(g_1g_2)$ which satisfies the cocycle relation and hence implies that $\tilde{G}$ and some certain $\tilde{G_\sigma}$ correspond. For the inverse isomorphism, it is similar. This correpondence is unique because under quotient of ϕ, it depends only on the map ι hence is unique. Thus any central extension $\tilde{G}$ can be identified with $\tilde{G_\sigma}$ for some σ and thus it is in bijection with σ. $\square$

By a *projective representation* of a group G, we mean a homomorphism of G into $PGL(n, \mathbb{C})$ for some n, where we recall that $PGL(n, \mathbb{C})$ is $\mathrm{GL}(n, \mathbb{C})$ module its center. Choosing a representative $\rho(g)$ of the image of $g \in G$ in $\mathrm{GL}(n, \mathbb{C})$ under this homomorphism, we find that

$$\rho(gg') = \sigma(g, g')^{-1}\rho(g)\rho(g')$$

for some complex number σ. It follows from the associative law in G that σ satisfies the cocycle relation (Eq.4.7.) and hence determines an extension $\tilde{G} = \tilde{G_\sigma}$. One checks then that $\tilde{\rho} : \tilde{G} \to GL(n, \mathbb{C})$ defined by

$$\tilde{\rho}(g, a) = a\rho(g), g \in G, a \in \mathbb{C}$$

is a representation. Thus, a *projective representation* of G gives rise to a cohomology class in $H^2(G, \mathbb{C})$ and to a true representation of a central extension of G by $\mathbb{C}^\times$.

The Weil representation: We return now to the setting of Exercise 4.1.8. Let H be a two-step nilpotent group, and let χ_0 be a generic character of the center Z of H. Let G be a group of automorphisms of H that fix Z. We may then construct a projective representation ω of G as follows. By the Stone-Von Neumann theorem (Exercise 4.1.8), there exists a representation (π, W) of H, unique up to isomorphism, having central character χ_0. If $g \in G$, then we obtain another such representation (π_g, W) acting on the same space by $\pi_g(h) = \pi({}^g h)$. By the uniqueness of π, these representations are isomorphic, which means that there is an intertwining map $\omega(g) : W \to W$ from (π, W) to (π_g, W) and $\omega(g)$ is unique up to constant by Schur's lemma, because π and π_g are irreducible.

Exercise 4.1.10. Prove that ω is a projective representation of G and that the defining property of ω can be written

$$\pi({}^g h) = \omega(g)\pi(h)\omega(g)^{-1}.$$

We make this projective representation explicit in a special case and relate it to the Weil representation as described in the text.

Proof. To prove projective representation, by definition, we are to prove that it is a homomorphism of G into $PGL(n, \mathbb{C})$ for some n. It is indeed to $PGL(n, \mathbb{C})$ as it is a map from G to $\mathrm{End}_H(W)$ and W as a vector space with basis. Hence we need to prove the into part. To show that it is into PGL, what we need is that it has no center, otherwise the center would be mapped to the identity and the map will not be injective. This is true because as automorphism group G that fix the center Z, the group G is isomorphic to the automorphism group G' of H/Z. Now if there exists an element f in G that is in the center, then we would have

$$f(g)f(x)f(g)^{-1} = gf(x)g^{-1}$$

and thus

$$f(x)f(g)^{-1}g = f(g)^{-1}gf(x).$$

This implies that $f(g)^{-1}g$ commutes with everything in the group, i.e., the nontrivial element $f(g)^{-1}g$ is in the center, which contradicts the fact that G' has trivial center. Hence ω is a projective representation of G. Then we are to show the identity. We have

$$\pi({}^g h)\omega(g)(w) = \pi_g(h)\omega(g)(w) = \omega(g)\pi(h)(w).$$

Thus we have

$$\pi({}^g h) = \omega(g)\pi(h)\omega(g)^{-1}$$

as wanted. □

Let F be a finite field with q elements, and assume that q is odd. We will arrive at a reinterpretation of the Weil representation. Let V be a vector space over F, and let $B : V \times V \to F$ be a nondegenerate symmetric bilinear form. Let ψ be a nontrivial additive character of F. Define a two-step nilpotent group H as follows. As a set, $H = V \oplus V \oplus F$, with multiplication

$$(v_1, v_2, x)(v_1', v_2', x') = (v_1 + v_1', v_2 + v_2', x + x' + B(v_1, v_2') - B(v_1', v_2)).$$

The center Z of H consists of the subgroup of elements of the form $(0, 0, z)$ with $z \in F$, and we define a character χ_0 of Z by

$$\chi_0(0, 0, z) = \psi(z).$$

Let A be the subgroup of elements of the form $(v_1, 0, x)$.

Exercise 4.1.11. Prove that χ_0 is generic, and that the subgroup A is polarizing.

Proof. Recall the definition of generic is that the pairing $\langle , \rangle$ is nondegenerate where the pairing is defined as

$$\langle \overline{x}, \overline{y} \rangle = \chi_0(xyx^{-1}y^{-1}).$$

Now take any elements $x = (v_1, v_2, x)$ and $y = (v_1', v_2', y)$ and their image under modulo center $\overline{x} = (v_1, v_2, 0)$ and $\overline{y} = (v_1', v_2', 0)$, so their inverse would be $(-v_1, -v_2, 0)$ and $(-v_1', -v_2', 0)$, and we check

$$\langle \overline{x}, \overline{y} \rangle = \chi_0(xyx^{-1}y^{-1}) = (v_1, v_2, x)(v_1', v_2', y)(-v_1, -v_2, -x)(-v_1', -v_2', -y)$$

$$= (v_1 + v_1', v_2 + v_2', x + y + B(v_1, v_2') - B(v_1', v_2))(-v_1, -v_2, -x)$$

$$= (0, 0, B(v_1 - v_1', v_2' - v_2))$$

by the bilinearity and nondegenerate property of B. If $\langle \overline{x}, \overline{y} \rangle = 0$, meaning that $B(v_1 - v_1', v_2' - v_2) = 0$, we have that B nondegenerate and thus $v_1 - v_1' = 0 = v_2' - v_2$. Therefore $v_1 = v_1'$ and $v_2 = v_2'$. Hence to make the form trivial, fixing one nonzero $\overline{x}$, we have only itself to output zero. That is, it is not degenerate. Thus χ_0 is generic. For A, consider any element (v_2, t, y) and we check

$$\langle (v_1, 0, x), (v_2, t, y) \rangle = \chi_0((v_1 + v_2, t, x + y + B(v_1, t) - B(0, v_2))(-v_1, 0, -x)(-v_2, -t, -y))$$

$$= (0, 0, 2B(v_1, t))$$

by the property of B. Hence this form would be zero if and only if $t = 0$ which means that this element is in A. Hence A is polarizing. $\square$

Exercise 4.1.8 shows that there exists a unique irreducible representation of H with central character χ_0 and that this representation can be obtained by extending χ_0 to A in an arbitrary way and then inducing; we induce the extension

$$(v_1, 0, x) \mapsto \psi(x)$$

and call the resulting representation ρ. Thus ρ acts on the space W_ρ of functions $\phi : H \to \mathbb{C}$ satisfying $\phi(ah) = \chi(a)\phi(h)$ for $a \in A$, and $\rho(h)\phi(h') = \phi(h'h)$. We will use a slightly different model of the representation, however.

Exercise 4.1.12. Show that if Φ is an arbitrary $\mathbb{C}-$valued function on V, there exists a unique element ϕ of W_ρ such that $\phi(0, v, 0) = \Phi(v)$. Let W be the space of all complex-valued functions on V, and let (π, W) be the representation determined by the condition that $\Phi \mapsto \phi$ is an isomorphism. Verify that

$$\pi(u, 0, 0)\Phi(v) = \psi(-2B(v, u))\Phi(v),$$

$$\pi(0, u, 0)\Phi(v) = \Phi(u + v).$$

Proof. Consider such ϕ_1 and ϕ_2 and assume that they are distinct. If we have $\phi_1(0, v, 0) = \Phi(v) = \phi_2(0, v, 0)$, then by the property of W_ρ we would have for any (a, b, c) that $\phi_1(a, b, c) = \phi_1((a, 0, x)(0, b, 0)) = \chi(a, 0, x)\phi_1(0, b, 0) = \chi(a, 0, x)\phi_2(0, b, 0) = \phi_2(a, b, c)$ and hence they define the same element. Therefore, such ϕ is unique. Therefore, any element in W can be identified with some ϕ in W_ρ. From the identity we obtained, we would have

$$\pi(u, 0, 0)\Phi(v) = \pi(u, 0, 0)\phi(0, v, 0) = \phi((0, v, 0)(u, 0, 0))$$

$$= \phi((u, v, -B(u, v))) = \phi((u, 0, -2B(u, v))(0, v, 0))$$

$$= \chi(u, 0, -2B(u, v))\Phi(v)$$

$$= \psi(-2B(v, u))\Phi(v).$$

Similarly, it is clear that

$$\pi(0, u, 0)\Phi(v) = \phi((0, v, 0)(0, u, 0)) = \phi(0, v + u, 0) = \Phi(u + v).$$

Hence we verified the two identities. $\square$

Exercise 4.1.13. Let G_1 be the group $\mathrm{SL}(2, F)$. Verify that there is an action of G_1 on H defined by

$$^g(v_1, v_2, x) = (av_1 + bv_2, cv_1 + dv_2, x),\ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Hence by Exercise 4.1.10 there is a projective representation ω_1 of $\mathrm{SL}(2, F)$, where $\omega_1 : \mathrm{SL}(2, F) \to \mathrm{GL}(W)$ is defined up to constant multiple by Eq.(1.28). Let $\chi : F^\times \to \{\pm 1\}$ be the unique quadratic character of the cyclic group $F^\times$. Verify that

$$(\omega_1 \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \Phi)(v) = \psi(xB(v, v))\Phi(v),$$

$$(\omega_1 \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \Phi)(v) = \chi(a)^{\dim V}\Phi(av),$$

and

$$\omega_1 \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \Phi = \hat{\Phi},$$

where the Fourier transform is defined by

$$\hat{\Phi}(v) = \epsilon q^{-\dim(V)/2} \sum_{u \in V} \Phi(u)\psi(2B(u, v)),$$

and ϵ is a constant to be chosen later.

Proof. To show it defines a group action, we first need to check the identity $g = I$ and then $^I(v_1, v_2, x) = (v_1, v_2, x)$. Then we check the compatibility that take g as above and $h = \begin{pmatrix} e & f \\ m & t \end{pmatrix}$ and we have $gh = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ m & t \end{pmatrix} = \begin{pmatrix} ae+bm & af+bt \\ ce+dm & cf+dt \end{pmatrix}$. Now we check

$$^{gh}(v_1, v_2, x) = ((ae + bm)v_1 + (af + bt)v_2, (ce + dm)v_1 + (cf + dt)v_2, x).$$

Then $^h(v_1, v_2, x) = (ev_1 + fv_2, mv_1 + tv_2, x)$ and then

$$^g(ev_1 + fv_2, mv_1 + tv_2, x) = ((ae + bm)v_1 + (af + bt)v_2, (ce + dm)v_1 + (cf + dt)v_2, x)$$

$$=^{gh} (v_1, v_2, x).$$

Hence the group action is checked. Now for the second part of this question, we want to prove that with the action defined in Eq(1.31) to Eq(1.33), the equation (1.28) holds for $(h,0,0)$ and $(0,h,0)$ as they are generators of H, and if so, it means that ω_1 indeed is a projective representation and since it is unique up to a constant, for instance, we have

$$\pi({}^g(h,0,0))\Phi(v) = \psi(-2B(v,h))\Phi(v),$$

and that

$$\omega_1(g)\pi((h,0,0))\omega(g)^{-1}\Phi(v) = \psi(-xB(v,v) + xB(v,v) - 2B(v,h)) = \pi({}^g(h,0,0))\Phi(v).$$

Similarly, it can be checked to hold for the other two identities. (for factors, TBD). $\square$

Exercise 4.1.14. We need to

(a). Prove that there exists a fourth root of unity ϵ_0 such that for $a \in F^\times$, the Gauss sum

$$\sum_{x\in F} \psi(ax^2) = \chi(a)\epsilon_0\sqrt{q},$$

and $\epsilon_0^2 = \chi(-1)$.

(b). Show that there exists a fourth root of unity ϵ such that

$$\sum_{v\in V} \psi(aB(v,v)) = \epsilon\chi(a)q^{\dim(V)/2}$$

and $\epsilon^2 = \chi(-1)^{\dim V}$.

Proof. We show

(a). We obtain the result from our calculation in Section 1.1, where we have that

$$G = \sum_{y\in F} \chi(y)\psi(ay)$$

which has absolute value of $q^{1/2}$. We also showed in Section 1.1 that the only nontrivial character is the quadratic character χ. Therefore we have the Gauss sum $G\overline{G} = |G|^2 = q$ and by making the change of variables $y \to -a^2y$ we would have

$$\overline{G} = \sum_{y\in F} \chi(y)\psi(-ay) = \chi(-a^2)G = \chi(-1)G$$

and thus $G^2 = \chi(-a^2)q = \chi(-1)\chi(a^2)$ and thus $G = \chi(a)\epsilon_0 q^{1/2}$.

(b). Since V is regarded as a vector space, if it is one dimensional then it is just the same as regarding V as F of V, then the summation can be regarded as $\sum_{v\in V}\psi(aB(v,v)) = \sum_{x\in F}\psi(ax^2) = \chi(a)\epsilon q^{1/2}$. Now if V is not one dimensional, that is, $V = (F)^n$, and we pick an orthogonal basis and we would have $\sum_n \psi(aB(c_1v_1 + c_2v_2 + \cdots + c_nv_n, d_1v_1 + \cdots + d_nv_n)) = \sum_{v\in F^n}\psi(aB(v,v))$ which has absolute value $q^{n/2}$ and thus with the same process we have

$$\sum_{v\in V}\psi(aB(v,v)) = \epsilon\chi(a)q^{\dim(V)/2}$$

where $\epsilon^2 = \chi(-1)^{\dim V}$.

Hence the result. □

Exercise 4.1.15. Prove that if ϵ is chosen as in Exercise 4.1.14, the projective representation ω_1 with the formulas in Exercise 4.1.13 is a true representation.

Proof. The proof is similar to the proof of Proposition 4.1.3, and we also obtain the summation $\sum_{v\in F^2}\psi(aB(v,v)) = \epsilon\chi(a)^2q^{2/2} = \epsilon q$ and the steps would follow. □

Exercise 4.1.16. As in the text, let K be a two-dimensional semisimple algebra over F, and let $V = K$ with $B(x,y) = \frac{1}{2}\mathrm{tr}(x\overline{y})$. Verify that the Weil representation as constructed in these exercises agrees with the construction in the text.

Proof. The three identities in this definition can be identified as

$$(\omega_1 n(x)\Phi)(v) = \psi(x\frac{1}{2}\mathrm{tr}(x\overline{x}))\Phi(v) = \psi(xN(x))\Phi(v)$$

which is identical with the definition of the one in the text. The second equation follows by $\dim V = 2$ and thus $\chi(a)^{\dim V} = 1$ as well as the Fourier transformation and the third equation. □

Exercise 4.1.17. Let (π, V) be an irreducible representation of $\mathrm{GL}(n,F)$, where F is a finite field. Prove that π is cuspidal if and only if π does not occur in any representation parabolically induced from a cuspidal representation of the Levi subgroup $M_\lambda(F)$ of a parabolic subgroup.

Proof. Note that being cuspidal is equivalent to saying that

$$\mathrm{Hom}_N(\pi|_N, \mathbb{C}) = 0.$$

Also Frobenius reciprocity tells us that

$$\mathrm{Hom}_N(\pi|_N, \mathbb{C}_N) \simeq \mathrm{Hom}_G(\pi, \mathbb{C}_N^G)$$

and thus being cuspidal is equivalent to having no intertwining operator between π and $\mathbb{C}_N^G$ which contains all the representations induced from the representation of N. Hence we obtained the result. □

Exercise 4.1.18 (The Bruhat decomposition for $\mathrm{GL}(n)$)**.** Now

(a). Let W be the group of permutation matrices in $\mathrm{GL}(n)$, isomorphic to the symmetric group S_n. Let B be the Borel subgroup of $\mathrm{GL}(n)$, consisting of upper triangular matrices. Show that

$$\mathrm{GL}(n) = \bigsqcup_{w \in W} BwB.$$

(b). Let $\lambda = (\lambda_1, \cdots, \lambda_h)$ and $\mu = (\mu_1, \cdots, \mu_k)$ be ordered partitions of n. Let $W_\lambda = W \cap M_\lambda \simeq W_{\lambda_1} \times \cdots \times W_{\lambda_h}$. According to (a), the canonical map $W \to \mathrm{GL}(n) \to B\backslash\mathrm{GL}(n)/B$ is a bijection. Show that the composition

$$W \to \mathrm{GL}(n) \to B\backslash\mathrm{GL}(n)/B \to P_\mu\backslash\mathrm{GL}(n)/P_\lambda$$

induces a bijection

$$W_\mu\backslash W/W_\lambda \simeq P_\mu\backslash\mathrm{GL}(n)/P_\lambda.$$

Proof. We have

(a). This can be shown by applying Gauss eliminations. Conceptually we want to have $g = b_1 w b_2$ for some $w \in W$ and b_1, b_2 upper triangular matrices. Now note that the inverse of upper triangular matrix is also upper triangular (Borel is a subgroup), hence given any $g \in \mathrm{GL}(n)$, we can always first apply row operation that is isomorphic to upper triangular matrix to it, that is, we are able to add a lower row to a higher row and scale rows. This can help us eliminate to the desired form except that it might not be upper triangular. Moreover, by applying the swapping rows, that is, the elements $w \in W$, we would finish a full Gaussian elimination, which gives us one upper triangular matrix. Hence, for any g we have $w^{-1}b_1^{-1}g = b_2$, and the Bruhat decomposition then follows as the other direction and the disjoint property is clear.

(b). By the bijection $W \simeq B\backslash \mathrm{GL}(n)/B$ we obtained, one would consider $W_\mu\backslash W/W_\lambda \simeq W_\mu\backslash(B\backslash \mathrm{GL}(n)/B)/W_\lambda$. Moreover, consider the semidirect product of B and W_μ, W_λ and hence we obtain

$$W_\mu\backslash W/W_\lambda \simeq P_\mu\backslash \mathrm{GL}(n)/P_\lambda$$

as one can compute that the result of the semidirect product gives P_λ, P_μ.

Hence the Bruhat decomposition for $\mathrm{GL}(n)$. □

Exercise 4.1.19. Let $\lambda = (\lambda_1, \cdots, \lambda_h)$ and $\mu = (\mu_1, \cdots, \mu_k)$ be ordered partitions of n. Let $\pi = \pi_1 \otimes \cdots \otimes \pi_h$ and $\theta = \theta_1 \otimes \cdots \otimes \theta_k$ be cuspidal representations of $M_\lambda(F)$ and $M_\mu(F)$, respectively, and let (Π, V_Π) and (Θ, V_Θ) be the representations of $\mathrm{GL}(n, F)$ parabolically induced from π and θ. Show that $\mathrm{Hom}_{\mathrm{GL}(n,F)}(\Pi, \Theta) = 0$ unless $k = h$, in which case the dimension of $\mathrm{Hom}_{\mathrm{GL}(n,F)}(\Pi, \Theta)$ is equal to the number of permutations σ of $\{1, 2, \cdots, h\}$ such that $\lambda_{\sigma(i)} = \mu_i$ and $\pi_{\sigma(i)} \simeq \theta_i$ as a $\mathrm{GL}(n, \mu_i)-$module for each $i = 1, \cdots, h$.

Proof. Since both Θ and Π are induced representations, from Mackey's theorem, we have that

$$\mathrm{Hom}_{\mathrm{GL}(n,F)}(\Pi, \Theta) \simeq D$$

where D is the same of functions $\triangle : \mathrm{GL}(n, F) \to \mathrm{Hom}_{\mathbb{C}}(V_\pi, V_\theta)$ that satisfy

$$\triangle(p_\mu g p_\lambda) = \theta(p_\mu) \circ \triangle(g) \circ \pi(p_\lambda)$$

for $p_\mu \in P_\mu(F), p_\lambda \in P_\lambda(F)$ as we obtained from the previous question that $\mathrm{GL}(n, F)$ has such a double coset decomposition. Suppose such a function exists and its value depends on the representative of the double coset from previous result, hence assume the function is supported on some $w \in W$. Then from the last question we know that such set has the form $P_\mu(F)wP_\lambda(F)$. Denote the image $\triangle(w)$ to be $\phi : V_\pi \to V_\theta$. We first prove a lemma that if $\mathrm{Hom}_{\mathrm{GL}(n,F)}(\Pi, \Theta)$ is not zero then

$$M_\mu(F) = wM_\lambda(F)w^{-1}.$$

Assuming this identity, we observe that the action of w and its inverse is swapping rows of the matrices. Therefore, it means that $M_\lambda(F)$ and $M_\mu(F)$ are just some "permutation away" from each other. Hence after reordering, they must be in the same form. That is, there must be some permutation σ in $\{1, \cdots, h\}$ such that $\lambda_{\sigma(i)} = \mu_i$. Note that $\sum \lambda_i = \sum \mu_i = n$, hence this condition implies that $k = h$ also.

Then first we have that V_π is an $M_{\lambda(F)}-$module. Second for V_θ, the isomorphism $m \to wmw^{-1}$ gives V_θ an $M_{\lambda(F)}-$module structure. Hence it is possible to discuss the $M_{\lambda(F)}-$module homomorphism between V_π and V_θ. Recall from for instance the result and proof of Exercise 4.1.2 that such a function $\triangle(w)$ is an intertwining operator. Hence $\phi = \triangle(w) \in \mathrm{Hom}_{M_{\lambda(F)}}(V_\pi, V_\theta)$. Now from Schur's lemma, the right hand side space can only be empty or one dimensional, which would be isomorphism up to multiplying a constant, hence if such ϕ exists(nonzero), it induces an isomorphism between π and θ and thus $\pi_{\sigma(i)} \simeq \theta_i$ as a $GL(n, \mu_i)-$module for each $i = 1, \cdots, h$.

Therefore, we are left to prove the lemma. Suppose that on the contrary we have $M_\mu(F) \neq wM_\lambda(F)w^{-1}$, then either $M_\mu(F) \nsubseteq wM_\lambda(F)w^{-1}$ or $M_\mu(F) \nsupseteq wM_\lambda(F)w^{-1}$. Let us assume, say, $M_\mu(F) \nsubseteq wM_\lambda(F)w^{-1}$, then it means that even after reordering $M_\mu(F)$ is not contained in M_λ, meaning that some square of M_μ is larger than the corresponding square of M_λ. As they are squares, there are some element t in $M_\mu(F)$ that is not in $wM_\lambda(F)w^{-1}$ that is also in the upper triangular matrix. Hence $t \in M_\mu(F) \cap wU_\lambda(F)w^{-1}$ and it means that the subgroup of all such t is the unipotent radical of $M_\mu(F) \cap wP_\lambda(F)w^{-1}$. Therefore let us consider for $v \in V_\pi$ that

$$\theta(t)\phi(v) = \theta(t)\phi(\pi(w^{-1}tw)v)$$

which holds because we extend the representation from M to P by letting U act trivially and $t = wuw^{-1}$ means that $w^{-1}tw = u \in U$. Further, as U is a subgroup, $u^{-1} = w^{-1}t^{-1}w$ is also in U and thus also act trivially. Hence with the identity that $\phi = \triangle(w)$ the above equation gives

$$\theta(t)\phi(v) = \theta(t)\phi(\pi(w^{-1}t^{-1}w)v) = \theta(t)\triangle(w)\phi(\pi(w^{-1}t^{-1}w)v)$$

where by the defining property of $\triangle$ we obtain

$$\theta(t)\phi(v) = \theta(t)\triangle(w)(\pi(w^{-1}t^{-1}w)v) = \triangle(tww^{-1}t^{-1}w)v = \phi(v).$$

Thus $\theta(t)\phi(v) = \phi(v) = v'$ implies that any linear functional Λ would give

$$\Lambda(\theta(t)v') = \Lambda(v'),$$

which is a contradiction to the fact that θ is cuspidal. Hence $\phi(v) = v'$ must be zero vector, that is, no such intertwining operator exists and hence $\mathrm{Hom}_{\mathrm{GL}(n,F)}(\Pi, \Theta) = 0$, which is a contradiction to the assumption. Therefore $M_\mu(F) \subseteq wM_\lambda(F)w^{-1}$. For similar reasons, we would have $M_\lambda(F) \subseteq w^{-1}M_\mu(F)w$, that is, $wM_\lambda(F)w^{-1} \subseteq M_\mu(F)$. Hence $M_\mu(F) = wM_\lambda(F)w^{-1}$. Therefore, we obtained the lemma. □

Exercise 4.1.20. We have

(a). Suppose that $\lambda = (\lambda_1, \cdots, \lambda_h)$ is an ordered partition of n. Let $\pi = \pi_1 \otimes \cdots \otimes \pi_h$ be a cuspidal representation of $M_\lambda(F)$. Show that the representation (Π, V_Π) is irreducible unless there exists some $i \neq j$ such that $\lambda_i = \lambda_j$ and $\pi_i \simeq \pi_j$ as $\mathrm{GL}(\lambda_i, F)-$modules.

(b). Suppose furthermore that $\mu = (\mu_1, \cdots, \mu_k)$ is another ordered partiion of n, and that $\theta = \otimes\theta_i$ is a cuspidal representation of $M_\mu(F)$. Assuming that the representations parabolically induced from π and θ is irreducible, show that they are isomorphic if and only if the μ_i and θ_i are the λ_i and π_i rearranged.

Proof. We prove as follows.

(a). From the result of Exercise 4.1.19, take $\Theta = \Pi$, we have that if there is some permutation exists, where in this case, if there is some $i \neq j$ such that $\lambda_i = \lambda_j$ and $\pi_i \simeq \pi_j$, then $\dim \mathrm{End}_G(\Pi, V_\Pi) \geq 1$ and further, $\dim \mathrm{End}_G(\Pi, V_\Pi) \geq 2$ as if such permutation exists, then together with the identity map gives the dimension higher than or equal to 2. Therefore, there exists some homomorphism between them with a nontrivial kernel which means that the representation is not irreducible. On the other hand, if no such nontrivial permutation exists, it means we have only identity permutation and hence there is only isomorphism between them, which means it is irreducible.

(b). If they are isomorphic, from previous results we know that dim of the homomorphism space is nonzero and thus there is some permutation σ (trivial or nontrivial) that has $\lambda_i = \mu_j$ where $j = \sigma(i)$. Since they are irreducicble, the dimension is one, and thus there is only one such permutation, which means that after reordering of the sequence we would have identical sequence. Hence μ_i and θ_i are just λ_i and π_i rearranged.

Hence the result. □

4.2 Smooth and Admissible Representations (10/10)

Exercise 4.2.1. Verify that the measures (Eq.2.1) are left and right Haar measures for $B(F)$.

Proof. Now for $b \in B(F)$ we have $b = \begin{pmatrix} 1 & x_{12} & \cdots & x_{1n} \\ & 1 & \cdots & x_{2n} \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix} \begin{pmatrix} y_1 & & & \\ & y_2 & & \\ & & \ddots & \\ & & & y_n \end{pmatrix}$ and we want that

$$d_R b = \prod_{1 \le i < j \le n} dx_{ij} \prod_{i=1}^{n} d^\times y_i,$$
$$d_L b = |y_1|^{1-n} |y_2|^{3-n} \cdots |y_n|^{n-1} d_R b.$$

Now let us consider $d_R b$ first, that is, the right action. It is sufficient to check it for both generators. Hence

$$\int_G bz d_R b = \int \begin{pmatrix} 1 & x_{12} & \cdots & x_{1n} \\ & 1 & \cdots & x_{2n} \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix} \begin{pmatrix} y_1 & & & \\ & y_2 & & \\ & & \ddots & \\ & & & y_n \end{pmatrix} \begin{pmatrix} z_1 & & & \\ & z_2 & & \\ & & \ddots & \\ & & & z_n \end{pmatrix} d_R b$$
$$= \int \begin{pmatrix} 1 & x_{12} & \cdots & x_{1n} \\ & 1 & \cdots & x_{2n} \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix} \begin{pmatrix} y_1 z_1 & & & \\ & y_2 z_2 & & \\ & & \ddots & \\ & & & y_n z_n \end{pmatrix} d_R b$$

from which we see that for the measure to be invariant under the diagonal part we want the measure to have factor $d^\times y_i$ over all i. Now for the unipotent part, let us consider similar

$$\int_G bt d_R b = \int \begin{pmatrix} 1 & x_{12} & \cdots & x_{1n} \\ & 1 & \cdots & x_{2n} \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & t_{12} & \cdots & t_{1n} \\ & 1 & \cdots & t_{2n} \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix} (\prod d^\times y_i) d'_R b$$

$$= \int \begin{pmatrix} 1 & t_{12}+x_{12} & \cdots & t_{1n}+x_{12}t_{2n}+...+x_{1n} \\ & 1 & \cdots & t_{2n}+...+x_{2n} \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix} (\prod d^\times y_i) d'_R b$$

from which we see that an additive Haar measure can make the contributions of t_{ab} vanish. Hence we can choose $d'_R b = \prod_{1\leq i<j\leq n} dx_{ij}$ as the product over all such entries. Therefore we have

$$d_R b = \prod_{1\leq i<j\leq n} dx_{ij} \prod_{i=1}^n d^\times y_i$$

as wanted. Now similarly let us consider the multiplication from the left, that is, we have

$$\int_G zb d_L b = \int \begin{pmatrix} z_1 & & & \\ & z_2 & & \\ & & \ddots & \\ & & & z_n \end{pmatrix} \begin{pmatrix} 1 & x_{12} & \cdots & x_{1n} \\ & 1 & \cdots & x_{2n} \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix} \begin{pmatrix} y_1 & & & \\ & y_2 & & \\ & & \ddots & \\ & & & y_n \end{pmatrix} d_L b$$

$$= \int \begin{pmatrix} z_1 & z_1x_{12} & \cdots & z_1x_{1n} \\ & z_2 & & z_2x_{2n} \\ & & \ddots & \vdots \\ & & & z_n \end{pmatrix} \begin{pmatrix} y_1 & & & \\ & y_2 & & \\ & & \ddots & \\ & & & y_n \end{pmatrix} d_L b$$

$$= \int \begin{pmatrix} 1 & z_1x_{12}/z_2 & z_1x_{13}/z_3 & \cdots & z_1x_{1n}/z_n \\ & 1 & z_2x_{23}/z_3 & \cdots & z_2x_{2n}/z_1 \\ & & \ddots & & \\ & & & & \vdots \\ & & & & 1 \end{pmatrix} \begin{pmatrix} z_1y_1 & & \cdots & \\ & z_2y_2 & \cdots & \\ & & \ddots & \\ & & & \vdots \\ & & & z_ny_n \end{pmatrix} d_L b$$

where if we take $d_R b$ from above first, we need to deal with the first component carefully. Now if we regard each coefficient of the entries we observe that the first row gives

$$\frac{z_1}{z_2}\frac{z_1}{z_3}\cdots\frac{z_1}{z_n} = \frac{z_1^{n-1}}{z_2\cdots z_n}.$$

Similarly, we have for each $j-$th row that it gives

$$\frac{z_j^{n-j}}{z_{j+1}\cdots z_n}.$$

Therefore if we multiply them together we obtain

$$z_1^{n-1}z_2^{n-3}\cdots z_{n-1}^{3-n}z_n^{1-n}.$$

Taking the inverse of this form as a part of $d_L b$ the measure is then by definition invariant under left multiplication of diagonal matrices. Then we have left the unipotent part to deal with. It is clear that the Haar measure we gave for $d_R b$ also holds for left multiplication. Hence there is nothing to add. Therefore

$$d_L b = |y_1|^{1-n} \cdots |y_n|^{n-1} d_R b$$

as wanted. □

Exercise 4.2.2. Let G be a compact totally disconnected group. Show that a subgroup of G is open if and only if it has finite index. Hence there can be at most one topology with respect to which G is compact and totally disconnected.

Proof. If a subgroup of G is open, consider the cosets G/H, that is, the sets of the form $\{gH\}$ and g acts as homeomorphism among those sets. And this is a open covering since H itself is open. Hence $\{g_i H\}$ form an open covering of G, where by the compact property of G must have finite index. Hence H has finite index. On the other hand, if H is finite index, then the disjoint union of $g_1 H, \cdots, g_r H$ covers G and hence equals G, where if H is closed then the union is closed. However G is compact meaning that such coverings also admit an open covering, which means some subgroups of $g_i H$ is open. However, no subgroup in H would be open if H itself is closed since H is union of the cosets of any of its subgroup. Hence H must be open. As open and finite index subgroup is equivalent under this condition, if a subgroup is open in one topology and closed in another topology, it can have finite and infinite index, which is a contradiction. Hence there can be at most one topology when G is compact and totally disconnected. □

Exercise 4.2.3. Prove that a compact group is unimodular.

Proof. We can consider the modular quasicharacter $\delta_G(g)$ which is from G to $\mathbb{R}$. Then we are to prove that it is constant 1. Since this character is continuous, and G is compact, the image $\delta(G)$ is also compact as a subset of $\mathbb{R}$. Now if it is not identically one, that is, for some $g \in G$ we have $\delta(g) = t > 1$, then acting g by n times gives t^n which goes to infinity if n goes to infinity. Hence due to the compactness of G it must be identically one. Therefore G is unimodular if compact. □

Exercise 4.2.4. Let F be a local field. Let $d_a g$ denote the additive Haar measure on $\mathrm{Mat}_n(F)$. Prove that the measure $dg = |\det(g)|^{-n} d_a g$ on $\mathrm{GL}(n, F)$ is both left and right invariant, and conclude that $\mathrm{GL}(n, F)$ is unimodular.

Proof. Consider the two groups t and x such that

$$\begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ t_{21} & t_{22} & \cdots & t_{2n} \\ \vdots & & & \vdots \\ t_{n1} & \cdots & & t_{nn} \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & & & \vdots \\ x_{n1} & \cdots & & x_{nn} \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^n t_{1k}x_{k1} & \cdots & \cdots & \sum_{k=1}^n t_{ik}x_{kn} \\ \vdots & \cdots & \vdots & \\ \vdots & & & \vdots \\ \sum_{k=1}^n t_{nk}x_{k1} & \cdots & & \sum_{k=1}^n t_{nk}x_{kn} \end{pmatrix},$$

that is, the induced map T from t acts as $x_{ij} \mapsto \sum_{k=1}^n t_{ik}x_{kj}$. Therefore, the Jacobian from F^{n^2} to F^{n^2} would be $\sum_{k=1}^n t_{ki}$ for each x_{ij}. Therefore, it is isomorphic to the $n^2 \times n^2$ matrix with diagonal blocks with copies of t and zeroes everywhere else. Then $|J(T)| = |\det(t)|^n$. Now if F is a measurable function on G we consider the map $H(X) = F(X)|\det(X)|^{-n}$ that by the change of variable ($\int_{T(U)} H dX = \int_U H \circ T |J_T| dX$) that

$$\int F(X)|\det(X)|^{-n} dX = \int H(T(t))|\det(t)|^n dX = \int H(T)|\det(t)|^n dt$$
$$= \int F(T)(X)|\det(X)|^{-n} dX.$$

That is, the integral is invariant under the action of t (the induced map T), hence is a left Haar measure. For the right action, the process is similar with some modification regarding the map and its Jacobian. Hence it is unimodular. $\square$

Exercise 4.2.5. Let N be the group of upper triangular unipotent matrices in $\mathrm{GL}(n, F)$, where F is a local field. Show that if

$$x = \begin{pmatrix} 1 & x_{12} & \cdots & x_{1n} \\ & 1 & & \vdots \\ & & \ddots & \\ & & & x_{n-1,n} \\ & & & 1 \end{pmatrix},$$

then (denoting by dx_{ij} additive Haar measure on F)

$$\prod_{1 \le i < j \le n} dx_{ij}$$

is the Haar measure on $N(F)$, and conclude that $N(F)$ is unimodular.

Proof. This follows directly (actually proved) from the proof of Exercise 4.2.1, the part that is not unimodular is from the diagonal elements. □

Exercise 4.2.6. Would Proposition 4.2.9 be true for an admissible representation of an arbitrary totally disconnected locally compact group?

Proof. In general it is not true. A not nice example is to take the group to be finite and consider the Weil representation in the non-split case, where the induced representation is cuspidal and irreducible with dimension some multiple of $q-1$. Taking $q>2$ we have a counterexample. □

Exercise 4.2.7. Let $V = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}$ be the direct sum of an infinite number of copies of $\mathbb{C}$. Thus an element $(a_n) \in V$ consists of a sequence of complex numbers indexed by $n \in \mathbb{Z}$ such that $a_n = 0$ for all but finitely many n. We have a representation $\rho : \mathbb{Z} \to \mathrm{End}(V)$ in which $\rho(m)$ is the shift operator that sends (a_n) to (a_{n+m}). Describe the invariant subspaces of V.

Proof. We note that V can be identified with the polynomial ring $R = \mathbb{C}[X, X^{-1}]$, whose spectrum can be seen is (0) and $(X-a)$ where $a \in \mathbb{C}^*$. This identification tells us the invariant subspaces of V can be identified with the ideals in R. Hence for example, $(X-1)$ is one such ideal and its preimage under this identification tells us the invariant subspaces can be described as the subspace such that there is two adjacent coordinates whose value is inverse to each other. □

Exercise 4.2.8. Prove that the contragredient of an irreducible admissible representation of $\mathrm{GL}(n, F)$ is irreducible.

Proof. From Theorem 4.2.2, we know that $\hat{\pi}(g) \simeq \pi_1(g) = \pi({}^T g^{-1})$ hence we just need to show that π_1 is irreducible when π is. If π is not irreducible, that is, there is some invariant subspace V under the representation π, then it is invariant under the action of $\pi(g^{-1})$ since g^{-1} is still in G. Then if a subspace is invariant under matrix M, it is also under its transpose. Hence indeed invariant property of a subspace is the preserved between π and its contragredient representation. □

Exercise 4.2.9. Prove that if (π, V) is a finite-dimensional irreducible admissible representation of $\mathrm{GL}(2, F)$, then V is one dimensional.

Proof. Consider the finite dimensional irreducible admissible representation of $\mathrm{GL}(2, F)$ namely V. Then we can pick basis for V namely $v_1, \cdots, v_n$. Now admissible representation is smooth, which by definition means that the stabilizer of each vector is open as a subgroup. Hence by its topological property, such a subgroup is a neighborhood of the identity and hence contains a compact open subgroup. Also, the intersection H of such subgroups should fix all vectors and thus have image as the identity matrix. Therefore, the intersection is in the kernel of this representation. Now since in non-archimedean topology the singleton is not open, hence each H_i being open also means that it is nontrivial. Therefore their intersection being open is also nontrivial. Hence it is an open compact subgroup that fixes everything. Since in totally disconnected locally compact topological group we have that each neighborhood has some nontrivial intersection with neighborhood basis of the identity, we can take ϵ small enough that the matrices $\begin{pmatrix} 1 & \epsilon \\ & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & \\ \epsilon & 1 \end{pmatrix}$ is also in H. Now note that no matter how small ϵ is, as long as not zero, these two generators can generate the whole $\mathrm{SL}(2, F)$. Therefore, $\mathrm{SL}(2, F)$ is in the kernel K of $\mathrm{GL}(2, F)$. This means that $\mathrm{GL}(2, F)/K \hookrightarrow \mathrm{GL}(2, F)/\mathrm{SL}(2, F) \simeq F^\times$. That is, the action of $\mathrm{GL}(2, F)$ factors through some subgroup $F^\times$ which is Abelian. Hence the irreducible representation can only be one dimensional. (This also follows from Proposition 4.2.9 that the admissible representation of $(F^\times)^k$ contains one dimensional invariant subspace and hence irreducible is one dimensional).

For the generator of $\mathrm{SL}(2, F)$ part, consider that

$$\begin{pmatrix} a & b \\ & d \end{pmatrix} = \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & a^{-1}b \\ & 1 \end{pmatrix}$$

and if $c \neq 0$ that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & c^{-1}a \\ & 1 \end{pmatrix} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \begin{pmatrix} -c & \\ & -c^{-1} \end{pmatrix} \begin{pmatrix} 1 & c^{-1}d \\ & 1 \end{pmatrix}.$$

Further, we have

$$\begin{pmatrix} y & \\ & y^{-1} \end{pmatrix} = \begin{pmatrix} 1 & \\ y^{-1} - 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ y - 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -y^{-1} \\ & 1 \end{pmatrix}$$

and $\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = \begin{pmatrix} 1 & \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ -1 & 1 \end{pmatrix}$. Moreover, for any $x \in F$ we can choose $b \in F$ such that $|bx|$ is sufficiently small that $N(bx)$ is in the kernel and

$$\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} = \begin{pmatrix} b & \\ & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & bx \\ & 1 \end{pmatrix} \begin{pmatrix} b & \\ & 1 \end{pmatrix}$$

is also in the kernel. Similarly for the lower triangular matrices. □

Exercise 4.2.10. We call a representation *indecomposable* if it does not decompose into a direct sum of irreducible representations. Before Proposition 4.2.9 we gave an example of an indecomposable irreducible admissible representation of $F^\times$ of dimension two. Show that any two-dimensional indecomposable admissible representation of $F^\times$ is obtained by tensoring this example with a quasicharacter of $F^\times$.

Proof. Since we are considering as well two dimensional indecomposable space W with V, both having basis v_1, v_2 and w_1, w_2, they are isomorphic as vector spaces. Therefore there exists some nonzero intertwining operator between them. By Schur's Lemma, as they are both irreducible, the only intertwining operator can act as complex scalars. Furthermore, as mentioned in the book that if $G = \mathrm{GL}(n, F)$ where in our case $n = 1$, then the center $Z(F)$ acts as $\omega(F^\times)$ for some quasicharacter ω. In $\mathrm{GL}(1, F) \simeq F^\times$, the center is the group itself as it is Abelian, hence $F^\times$ acts as $\omega(F^\times)$ which is the complex scalars we mentioned above. Therefore, given any required representation, we have such an intertwining operator which can be regarded as the tensor product of such indecomposable admissible representation with the quasicharacter ω of $F^\times$. □

4.3 Distributions and Sheaves (5/5)

Exercise 4.3.1. Let $\mathfrak{T}$ be the topology of the topological space X, and let $\mathfrak{T}_0 \subseteq \mathfrak{T}$ be a base for the topology. Let $\mathscr{F}$ be a sheaf on X with base $\mathfrak{T}_0$. Show that there is a sheaf $\mathscr{F}'$ on X with base $\mathfrak{T}$, together with isomorphisms $\sigma_U : \mathscr{F}(U) \to \mathscr{F}'(U)$ for $U \in \mathfrak{T}_0$ such that if $U \supseteq V$ are elements of $\mathfrak{T}_0$, we have $\rho_{U,V} \circ \sigma_U = \sigma_V \circ \rho_{U,V}$, and that $\mathscr{F}'$ is unique up to isomorphism.

Proof. Take $\mathcal{F}'$ to be that $\mathcal{F}'(U)$ is the space of sections of the etale space $\hat{\mathcal{F}}$ of $\mathcal{F}$ where the sections have input U and output elements of $\hat{\mathcal{F}}$. Then the isomorphism $\sigma_U : \mathcal{F}(U) \to \mathcal{F}'(U)$ takes the element of $\mathcal{F}(U)$ to some section of the etale space. Now let us check that

$$\rho_{U,V} \circ \sigma_U(f_U)(x) = \rho_{U,V}(s_{f_U})(x) = \rho_{U,V}\rho_{U_i,x}(f_i)$$

where s_f is some section associated with f takes U as input. Also $\sigma_V \circ \rho_{U,V}(f_U)(x) = \sigma_V(f_V)(x) = s_{f_V}(x)$ which by the gluing and local property of sheaves is equal to $\rho_{V_i,x}(f_i)$. Note that the first equation above is the same as looking at the intersection of U_i and V which can be reordered as V_i as the second equation. Hence they are identical. □

Exercise 4.3.2. In this situation, prove that if $g_i \in G_i$ satisfies $\phi_i(g_i) = 0$, then $\phi_{ij}(g_i) = 0$ for some $j > i$.

Proof. First we know that the direct limit can be realized as a quotient of $\oplus G_i$ from purely categorical results. Then the statement is clear that if the preimage is in the kernel and the map for $j > i$ would not be all zero and hence by the uniqueness of the universal property the map from G_i to G_j must be zero. □

Exercise 4.3.3. Let $\mathcal{F}$ be a sheaf on X, and let $U \subset X$ be open. Let $f \in \mathcal{F}(U)$ have the property that $\rho_{U,x}(f) = 0$ in $\mathcal{F}_x$ for all $x \in U$. Prove that $f = 0$.

Proof. For each $x \in U$ there is an open neighborhood U_x of x such that $\rho_{U,U_x}(f)$ by the previous exercise as we regard those open covers of U_i as the Abelian groups and x as the direct limit. Then by the axiom of sheaf we have that those open covers can glue such that we obtain $f = 0$. □

Exercise 4.3.4. Let X be a totally disconnected locally compact space.

(a). Let $\mathcal{F}$ be a sheaf of modules over C^∞. Show that the Abelian gourp $\mathcal{F}_c$ constructed in Proposition 4.3.9 is naturally a module over $C_c^\infty(X)$ and that it is cosmooth.

(b). Let M be a cosmooth module over $C^\infty(X)$, and let $\mathcal{M}$ be the associated sheaf of modules over C^∞, constructed in Proposition 4.3.12. Show that $\mathcal{M}_c$ is isomorphic to M.

(c). Conversely, let $\mathscr{F}$ be a sheaf of modules over C^∞ (with base $\mathfrak{T}_0$, the set of open and compact sets of X). Show that the sheaf associated with $\mathscr{F}_c$ by Proposition 4.3.9 is isomorphic to $\mathscr{F}$.

Proof. We see

(a). As construction in the proposition, we may assume $\mathscr{F}_c$ to be the union of the spaces $\mathscr{F}(U)$ with $U \in \mathfrak{T}_c$. As $\mathscr{F}(U)$ are such that modules over $C^\infty(U)$, and we can take U over all X, then $\mathscr{F}_c$ is hence a module over $C^\infty(X)$ and hence a module also over $C_c^\infty(X)$. Now for the cosmoothness, let us check the definition. For every $f \in \mathscr{F}_c$, we want to find an open compact subset U of X such that $1_U f = f$. This is clearly true as we regard them as $C_c^\infty(X)$ modules then the U can be simply the compact set that f has support on.

(b). From the previous proposition, we obtained that a cosmooth $C_c^\infty(X)-$module M' can be extended uniquely to make it a $C^\infty(X)-$module M. There is hence a bijection between M' and its corresponding $C^\infty(X)-$module M. Note that the proposition we used is regarding M' and $\mathcal{M}$ where $\mathcal{M}$ can be associated with $\mathcal{M}_c$ by Proposition 4.3.9. By those bijections we can reduce the isomorphism to the correspondence constructed in Proposition 4.3.12.

(c). TBD.

Hence the statements. □

Exercise 4.3.5. Show that

(a). Prove that if G is a compact topological group and H is an open subgroup, then H has finite index in G.

(b). Prove that if G is a locally compact topological group and H is an open subgroup, then H has nonzero volume with respect to the left Haar measure.

Proof. We have

(a). This part was proved in the previous section such that we only used the compact property in this direction.

(b). If H is open subgroup, and G is a locally compact topological group, it means that we can choose a compact cover of H, which then allows us to use the previous result as H would have finite index in this group.

Hence the result. $\square$

4.4 Whittaker Modules and the Jacquet Functor (2/2)

Exercise 4.4.1 (Waldspurger 1980, Proposition 9, p.31)**.** Let T be a maximal torus of GL(2) defined over F. (T may or may not be split - the result is valid in both cases.) Let (π, V) be an irreducible admissible representation of $\mathrm{GL}(2, F)$. Use the involution method to prove that there exists at most one linear functional $L : V \to \mathbb{C}$ such that $L(\pi(t)v) = L(v)$ for $t \in T(F)$.

Proof translated from the paper. [If $\mathcal{U}_v^\xi$ exists, we consider an isomorphism $j : V_v \to \mathcal{U}_v^\xi$, and the map $l(e) = j(e)(1)$. Reciprocally if l exists, we put $j(e)(g) = l(\rho_r(g)e)$.]

(1). Suppose $\xi \in F^{\times 2}, \xi = \eta^2$. Let us put $y_n = \begin{pmatrix} 1 & 1 \\ \eta & -\eta \end{pmatrix}$. We verify that

$$y_\eta^{-1} O_\xi y_\eta = \{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} = D.\}$$

We abandon the v indices to lighten the notations.

The map $l \mapsto l'$, with $l'(e) = l(\rho(y_\eta)e)$ provides a bijection between the considered space of functions l and the space of linear maps $l' : V \to \mathbb{C}$ as

$$l'(\rho(g)e) = l'(e) \text{ for any } e \in V, g \in D. \tag{4.9}$$

Suppose that V is the Kirillov model of ρ (relative to ψ). If $e \in V$, e is therefore a function of $F^\times$ in $\mathbb{C}$. According to [God], p.1.36, e is integrable for the Haar measure $d^\times x$ on $F^\times$. Let us set

$$l_0'(e) = \int_{F^\times} e(x) d^\times x.$$

Given the fact that ρ is trivial on the center Z, l_0' verifies the relation (4.9), and is of course not identically zero.

Let l' be another map verifying the relation (4.9). Let $e_0 = Car(\mathcal{O}^\times)$, where $\mathcal{O}$ designates the ring of integers of F, and $\mathcal{O}^\times$ its group of units. We have $e_0 \in \mathfrak{S}(F^\times) \subset V$. Let us set $c = l'(e_0)/l_0'(e_0)$, and show that $l' = cl_0'$. Let π be a uniformizer, $m \in \mathbb{Z}$ and v be a character of $\mathcal{O}^\times$. Let us set

$$e_{m,v}(x) = \begin{cases} 0 \text{ if } v(x) \neq m \\ v(\pi^{-m}x) \text{ if } v(x) = m. \end{cases}$$

Suppose $v \neq 1$, and let $y \in \mathfrak{O}^\times$ be such that $v(y) \neq 1$. So:

$$\rho \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} e_{m,v} = v(y)e_{m,v}.$$

So

$$l'(e_{m,v}) = l'(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} e_{m,v}) = v(y)l'(e_{m,v}),$$

hence $l'(e_{m,v}) = 0 = cl_0'(e_{m,v})$.

On the contrary, suppose $v = 1$. So:

$$\rho \begin{pmatrix} \pi^m & 0 \\ 0 & 1 \end{pmatrix} e_{m,v} = e_0,$$

and likewise $l'(e_{m,v}) = l'(e_0) = cl_0'(e_0) = cl_0'(e_{m,v})$.

As the functions $e_{m,v}$ generate $\mathfrak{S}(F^\times)$, we have $l' = cl_0'$ over $\mathfrak{S}(F^\times)$. According to [God], p.1-36, V is generated by $\mathfrak{S}(F^\times)$ and by a finite number of functions of the form $e_\mu = \mu.Car(\mathcal{O})$, where μ is a non-trivial quasi-character of $F^\times$. For such a function, let us choose y such that $\mu(y) \neq 1$. So:

$$\rho \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} e_\mu(x) = \mu(u)e_\mu(x) \text{ if } x \in \mathcal{O} \cap y^{-1}\mathcal{O},$$

therefore $\rho \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} e_\mu - \mu(y)e_\mu \in \mathfrak{S}(F^\times)$.

We can deduce

$$l'(\rho \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} e_\mu - \mu(y)e_\mu) = cl_0'(\rho \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} e_\mu - \mu(y)e_\mu,$$

i.e.

$$(1 - \mu(y))l'(e_\mu) = c(1 - \mu(y))l_0'(e_\mu)$$

hence $l'(e_\mu) = cl'_0(e_\mu)$, which ends the demonstration in this case.

(2). Suppose $\xi \notin F^{\times 2}$. In this case, the group $Z\backslash O_\xi$ is compact. Let V_ξ be the subspace of V forms vectors invariant by $\rho(g)$ for all $g \in O_\xi, V'_\xi$ the subspace generated by vectors of the form $\rho(g)e - e$, or $e \in V, g \in O_\xi$. This involves demonstrating that V has codimension less than or equal to 1 in V. We will prove some lemmas.

Lemma 1. The map $V_\xi \to V/V'_\xi$ is surjective.

Proof. We must show that $V = V_\xi + V'_\xi$. Let $e \in V$ and K' be a compact open subgroup of K leaving e invariant. Then $Z\backslash O_\xi K'$ is compact, and $Z\backslash ZK'$ is open, therefore $O_\xi K'/ZK'$ is finite, therefore $O_\xi/ZK' \cap O_\xi$ is finite. Let $\{g_i; i \in I\}$ be a system of representatives of this quotient. Let's set

$$e_\xi = (\sharp I)^{-1} \sum_{i \in I} \rho(g_i)e.$$

It is clear that if $g \in O_\xi$, we have $\rho(g)e_\xi = e_\xi$, therefore $e_\xi \in V_\xi$, and

$$e'_\xi = e - e_\xi = (\sharp I)^{-1} \sum_{i \in I} [\rho(g_i)e - e],$$

so $e'_\xi \in V'_\xi$.

Let H^Z be the space of locally constant functions with compact support on $Z\backslash G$. The space H^Z acts on V by

$$(f \in H^Z, e \in V), \rho(f)e = \int_{Z\backslash G} f(g)\rho(g)edg.$$

We verify that $\rho(f) \circ \rho(f') = \rho(f * f')$, or

$$f * f'(g) = \int_{Z\backslash G} f(h)f'(h^{-1}g)dh.$$

Let H^Z_ξ be the subspace of H^Z forming right and left O_ξ-invariant functions. For the convolution product, it is a subalgebra of H^Z, which annihilates V'_ξ, therefore acts on V/V'_ξ.

Lemma 2. The representation of H^Z_ξ on V/V'_ξ is irreducible.

Proof. Suppose $V/V'_\xi \neq \{0\}$. Let $e \in V$, with image $\overline{e}$, non-zero, in V/V'_ξ. We can suppose $e \in V_\xi$ (according to the Lemma 1). Let K' be a compact open subgroup of G leaving e invariant, and $g \in G$. Let

$$f = Car(Z\backslash O_\xi g K' O_\xi).$$

Then f is a compact support in $Z\backslash G$, locally constant (because K' is open), biinvariant by O_ξ. Let us calculate $\rho(f)\overline{e}$

$$\rho(f)e = \int_{Z\backslash G} f(h)\rho(h)edh.$$

But if $h = r_1 g k r_2$, with $r_1, r_2 \in O_\xi, k \in K'$, we have

$$\rho(h)e = \rho(r_1)\rho(g)e = \rho(g)e + e_h,$$

or

$$e_h = \rho(r_1)\rho(g)e - \rho(g)e \in V'_\xi.$$

We therefore obtain

$$\rho(f)\overline{e} = \int_{Z\backslash G} f(h)\overline{\rho(g)e}dh = mes(Z\backslash O_\xi g K' O_\xi)\overline{\rho(g)e}.$$

As the vectors $\rho(g)e$, for $g \in G$, generate V, the vectors $\rho(f)\overline{e}$ generate V/V'_ξ.

Lemma 3. The algebra H_ξ^Z is commutative.

Proof. Let the quadratic extension $L = F(\sqrt{\xi}), \eta = \sqrt{\xi} \in L$, and $y = \begin{pmatrix} 1 & 1 \\ \eta & -\eta \end{pmatrix} \in G_L$. The map $j : g \mapsto y_\eta^{-1} g y_\eta$ identifies

$$G_F \text{ has the set } \{\begin{pmatrix} u & v \\ \overline{v} & \overline{u} \end{pmatrix}; u, v \in L, u\overline{u} - v\overline{v} \neq 0\},$$

$$O_\xi \text{ has the set } \{\begin{pmatrix} u & 0 \\ 0 & \overline{u} \end{pmatrix}; u \in L, u\overline{u} \neq 0\},$$

where we denote $\overline{u}$ the conjugate of u. Let q be the inner automorphism associated with $\begin{pmatrix} \eta & 0 \\ 0 & \overline{\eta} \end{pmatrix}$. So:

$$q\begin{pmatrix} u & v \\ \overline{v} & \overline{u} \end{pmatrix} = \begin{pmatrix} u & -v \\ -\overline{v} & \overline{u} \end{pmatrix}.$$

Let us put $p = j^{-1} \circ q \circ j$. We say that if $g \in G_F, p(g) \in O_\xi g^{-1} O_\xi$. Indeed it is enough to see that if $u, v \in L$, we can find $a, b \in L$ such that

$$q\begin{pmatrix} u & v \\ \overline{v} & \overline{u} \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & \overline{a} \end{pmatrix}\begin{pmatrix} u & v \\ \overline{v} & \overline{u} \end{pmatrix}^{-1}\begin{pmatrix} b & 0 \\ 0 & \overline{b} \end{pmatrix}$$

which amounts to $ab\overline{u} = u, a\overline{b}v = v$. If $u \neq 0$, we set $a = u, b = \overline{u}^{-1}$. If u=0, we set $a = b = 1$.

Let us then put, for $f \in H^Z$, $\hat{f}(g) = f(g^{-1}), f_p(g) = f(p(g))$. We verify that $(f*f')_p = f_p*f'_p, (f \hat{*} f') = \hat{f}'*\hat{f}$. But for $f \in H^Z_\xi, g \in G$, let $r_1, r_2 \in O_\xi$ be such that $p(g) = r_1 g^{-1} r_2$. So:

$$f_p(g) = f(p(g)) = f(r_1 g^{-1} r_2) = f(g^{-1}) = \hat{f}(g).$$

So for $\hat{f}, f' \in H^Z_\xi$,

$$(f * f')_p = f_p * f'_p = \hat{f} * \hat{f}' = (f' \hat{*} f) = (f' * f)_p$$

so $f * f' = f' * f$.

Suppose $V/V'_\xi \neq \{0\}$, and let $f \in H^Z_\xi$ be such that $\rho(f)(V/V'_\xi) \neq \{0\}$. As $\rho(f)V$ is of finite dimension, $\rho(f)(V/V'_\xi)$ is too. According to Lemma 3, it is an invariant space under the action of H^Z_ξ, therefore equal to V/V'_ξ according to the Lemma 2. So V/V'_ξ is of finite dimension. By classical reasoning, Lemma 2 and 3 then show that V/V'_ξ has dimension 1. $\square$

Exercise 4.4.2. Prove that if F is a field and $\gamma \in \mathrm{SL}(2, F) - B(F)$, then γ and $N(F)$ generate $\mathrm{SL}(2, F)$.

Proof. For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, F) - B(F)$ we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & a/c \\ & 1 \end{pmatrix} \begin{pmatrix} & -c^{-1} \\ c & \end{pmatrix} \begin{pmatrix} 1 & d/c \\ & 1 \end{pmatrix}$$

which generates everything in $\mathrm{SL}(2, F)$ with $N(F)$ as $N(F)$ can attribute the right top corner to make it any value needed. $\square$

4.5 The Principal Series Representations (5/10)

Exercise 4.5.1. Prove that if $G = \mathrm{GL}(n, F)$, where F is a non-Archimedean local field and $\mathfrak{o}$ is its ring of integers, then the group $K = \mathrm{GL}(n, \mathfrak{o})$ is a maximal compact subgroup of G and that every compact subgroup of G is conjugate to a subgroup of K.

Proof. From Iwasawa decomposition, if we have any other compact subgroup that properly contains K, then it should have a decomposition that contains such element in $B(F)$ and the entries will not all be in $\mathfrak{o}$. This part can then be decomposed to $T(F)N(F)$. If such entries appear in T, then the group is clearly not compact. If such entries appear in N part, then since we have $\mathrm{GL}(n, \mathfrak{o})$ contained, we can choose a permutation matrix to permute it to the diagonal part. Therefore K is maximal. If we have another maximal compact subgroup, it would contain $p^{-m}\mathfrak{o}$ or not. If it does not, it is contained in K. If it does contain some $p^{-m}\mathfrak{o}$, by same decomposition, it would have the form $t = b(F)k$. However, it is contained in the group with identical first part and having K as second part and that group should be compact, which however based on our previous argument is not compact. Therefore every compact subgroup of G is conjugate to a subgroup of K. □

Exercise 4.5.2. (a). If χ_1 and χ_2 are quasicharacters of $F^\times$ such that $(\chi_1\chi_2^{-1})(y) = |y|^{-1}$, prove the irreducibility of the quotient of $B(\chi_1, \chi_2)$ by its one-dimensional invariant subspace, as asserted in Theorem 4.5.1(i).

(b). If $\chi_1\chi_2^{-1}(y) = |y|$, prove the irreducibility of invariant subspace of codimenison one in $B(\chi_1, \chi_2)$, as asserted in Theorem 4.5.1(ii).

Proof. We prove one by one.

(a). The argument is similar to the initial case. That is, assume that there is another nonzero proper invariant subspace $V''' \subseteq V/V' = V''$ and $V'''' = V''/V'''$. Now from previous argument that $J_\psi(V'')$ is not zero, hence similarly, there should be $J_\psi(V''')$ or $J_\psi(V/V''')$ being zero. Then similar argument will show that it also contains an invariant subspace. However, for one case, if it contains another one dimensional subspace, it will also be the form of $f = \chi(\det(g))$ which is zero in the quotient space. If it contains a codimension one subspace, it will contain $g = \chi^{-1}(\det(g))$ which also vanish. Hence it must be irreducible.

(b). Same arguments as above.

Hence the result. □

Exercise 4.5.3. Prove that the Jacquet module of $(\pi, V) = B(\chi_1, \chi_2)$ is always exactly two dimensional.

Proof. We consider $f \in B(\chi_1, \chi_2)$ and build a map T from $B(\chi_1, \chi_2)$ to $\delta^{1/2}\chi_1 \otimes \chi_2 =: \delta\chi$ by $T(f) = f|_B$. Then we define $V := \ker(T)$ and hence we have the short exact sequence,

$$0 \longrightarrow V \longrightarrow B(\chi_1, \chi_2) \xrightarrow{T} \delta\chi \longrightarrow 0.$$

Since the Jacquet functor is exact, we then have another exact sequence

$$0 \longrightarrow V_N \longrightarrow B(\chi_1, \chi_2)_N \longrightarrow \delta\chi_N = \delta\chi \longrightarrow 0,$$

where $B(\chi_1, \chi_2)_N$ is the Jacquet module we need. Therefore we only need to look at V_N. Those are the functions that has trivial value on B, then it is supported on $B\omega_0 N$ by the Bruhat decomposition. Now note that $V(N)$ are the vectors that vanish under the map taking f to $\int_N f(wn)dn$ and that

$$\int_N f(wnt)dn = \int_N f(wtt^{-1}nt) = \int_N \delta^{-1}(t)f(wtn)$$
$$= \delta^{-1}(t)\int_N f(wtwwn) = \delta^{-1/2}\chi_2 \otimes \chi_1(t)\int_N f(wn)$$

meaning this assignment is $\delta\chi^{-1}$ equivariant. Further, a vector f is in $V(N)$ if and only if $\int_N f(wn)dn$ is zero, with the fact that $f \in V$ viewed as functions supported on $B\omega N$. That is, as a $T-$module, the torus T acts through the map $\delta\chi^{-1}$. Therefore, the space $V/V(N) = V_N$ is one-dimensional. Therefore, as the short exact sequence gives, we have $B(\chi_1, \chi_2)_N$ is 2-dimensional.

In other words, we can illustrate in the fashion of the hints. We first want to count the linear functionals on $J(V)$. First we show that it is at most 2-dimensional, then we construct two linearly independent functionals. As the dimension of the dual space is equal to dimension of the original finite dimensional space, we conclude that $J(V)$ is two dimensional.

Note that a linear functional on $J(V)$ is exactly that satisfies $L(\pi(n)f) = L(f)$ where $n \in N(F), f \in V$. With similar proof as Theorem 4.5.2 (distribution arguments and what we did above), we can show that the space of such functionals is at most 2-dimensional. Then we need to construct two such linear functionals. Those can be

$$L_1(f) = f(1)$$

and

$$L_2(f) = \int_F [f(\begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}) - h(x)f(1)]dx$$

where

$$h(x) = \begin{cases} |x|^{-1}(\chi_1^{-1}\chi_2)(x) \text{ if } |x| > 1, \\ 0 \text{ if } |x| \leq 1 \end{cases}.$$

Also note the identity

$$f(\begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g) = |x|^{-1}(\chi_1^{-1}\chi_2)(x) f(\begin{pmatrix} 1 & \\ x^{-1} & 1 \end{pmatrix} g)$$

so

$$L_2(f) = \int_{|x|>1} |x|^{-1}\chi_1^{-1}\chi_2(x) f(\begin{pmatrix} 1 & \\ x^{-1} & 1 \end{pmatrix}) - |x|^{-1}(\chi_1^{-1}\chi_2)(x)f(1)dx + \int_{|x|\leq 1} f \cdots$$

where the second term is

$$\int_{|x|\leq 1} |x|^{-1}(\chi_1^{-1}\chi_2)(x) f(\begin{pmatrix} 1 & \\ x^{-1} & 1 \end{pmatrix})dx.$$

The second term is compactly supported. Now let us define f_1 to be some function in V, and define f_2 to be such that

$$f_2(g) = f_2(\begin{pmatrix} y_1 & z \\ & y_2 \end{pmatrix} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}) = |\frac{y_1}{y_2}|^{1/2}\chi_1(y_1)\chi_2(y_2)$$

where we require $y_1, y_2 \in F^\times, z \in F$ and $x \in \mathfrak{o}$, and if g cannot be expressed in such form (by Bruhat decomposition, it means g is not in the big cell $B\omega N$ but in B), then $f(g) = 0$. Now we see

$$L_1(f_1) = f_1(1) = 1, L_1(f_2) = f_2(1) = 0.$$

Further,

$$L_2(f_2) = \int_{|x|\leq 1} 1dx \neq 0$$

as the other part of the domain vanishes by our definition of f_2. Therefore, we constructed a linear functional L_1 and L_2 that are linearly independent. Therefore, there are at least two dimensions in $J(V)$. Hence there are exactly two. Thus $J(V)$ is 2-dimensional.

Indeed, the construction of f_1 can be that $f_1(bk) = (\delta^{1/2}\chi)(b)$ if $b \in B(F), k \in K_1(\mathfrak{a})$ and $f_1(g) = 0$ if $g \notin B(F)K_1(\mathfrak{a})$. Also $\mathfrak{a}$ is chosen so that the function is well-defined. $\square$

Exercise 4.5.4. Suppose that χ_1 and χ_2 are characters of $F^\times$ such that $\chi_1\chi_2^{-1}(y) = |y|^{-1}$, so that $B(\chi_1, \chi_2)$ is reducible. Prove that the Jacquet modules of $\pi(\chi_1, \chi_2)$ and $\sigma(\chi_1, \chi_2)$ are one dimensional and that the characters of $T(F)$ that they afford are $\delta^{1/2}\chi$ and $\delta^{1/2}\chi'$, respectively, where χ and χ' are the two diagonal characters flipped to each other. Show that the image of the intertwining integral $B(\chi_2, \chi_1) \to B(\chi_1, \chi_2)$ is the subrepresentation of $B(\chi_1, \chi_2)$ isomorphic to the one-dimensional representation $\pi(\chi_1, \chi_2)$, and that the image of the intertwining integral $B(\chi_1, \chi_2) \to B(\chi_2, \chi_1)$ is the subrepresentation of $B(\chi_2, \chi_1)$ isomorphic to $\sigma(\chi_1, \chi_2)$.

Proof. From Theorem 4.5.4, we have that the representation of $T(F)$ on the Jacquet module of principal series can only be one of the two two dimensional representations. Since $\chi_1\chi_2^{-1}(y) = |y|^{-1}$, we have that $\chi_1 \neq \chi_2$. We have the short exact sequence

$$0 \longrightarrow \pi(\chi_1, \chi_2) \longrightarrow B(\chi_1, \chi_2) \longrightarrow \sigma(\chi_1, \chi_2) \longrightarrow 0$$

and we take the Jacquet module to obtain

$$0 \longrightarrow \pi(\chi_1, \chi_2)_N \longrightarrow B(\chi_1, \chi_2)_N \longrightarrow \sigma(\chi_1, \chi_2)_N \longrightarrow 0$$

where in this case that $\chi_1 \neq \chi_2$ we have that $B_N = \{t \mapsto \begin{pmatrix} \delta^{1/2}\chi(t) & \\ & \delta^{1/2}\chi'(t) \end{pmatrix}\}$ which is clearly two dimensional and hence gives the results on the Jacquet module of π and σ.

Further, we have that $\mathrm{Hom}_T(J(V), \delta^{1/2}\chi) \simeq \mathrm{Hom}_G(V, B(\chi_1, \chi_2))$, hence take $V = B(\chi_2, \chi_1)$ we see that an intertwining integral between B_1 and B_2, it gives an isomorphism between $B(\chi_2, \chi_1)_N$ and $\delta^{1/2}\chi$. In the Jacquet module of $B(\chi_2, \chi_1)_N$, we can see that it corresponds to the σ part and hence it means that the image of $B(\chi_2, \chi_1)$ is the π part of $B(\chi_1, \chi_2)$, hence we get this relation. □

Exercise 4.5.5 (Simple Mackey Theory)**.**

Proof. TBD. □

Exercise 4.5.6. Let G be a totally disconnected locally compact group, and let H be an open subgroup. Let (π_0, V_0) be a unitarizable smooth representation of H, and let (π, V) be the representation of G obtained by compact induction. Prove that (π, V) is unitarizable.

Proof. If $\langle , \rangle$ is an $H-$invariant positive definite Hermitian inner product on V_0, then define another form by

$$\langle\langle f_1, f_2 \rangle\rangle = \int_G \langle f_1(g), f_2(g) \rangle.$$

This is first $G-$invariant, as the integral is over all $g \in G$. It is then positive definite as the integrands are positive definite. The compactness of G implies that this integral is compactly supported, hence this shows that we can unitarize π. □

Exercise 4.5.7. Let $B_1(F)$ be the group of elements of $\mathrm{GL}(2, F)$ of the form $\begin{pmatrix} a & b \\ & 1 \end{pmatrix}$. Let K be the Kirillov representation of $B_1(F)$, defined before. Show that K is isomorphic to the representation obtained from the character ψ_N of the group $N(F)$ by compact induction.

Proof. TBD. □

Exercise 4.5.8 (Transitivity of induction)**.** Let G be a totally disconnected locally compact group, and let $H_1 \subset H_2 \subset G$ be closed subgroups. Let (σ, V) be a smooth representation of H_1. We can induce σ from H_1 to G in one step, or we can induce it first from H_1 to H_2, then from H_2 to G. Show that the resulting representations are isomorphic. Prove the corresponding statement for compact induction under the hypothesis that H_2 is cocompact.

Proof. TBD. □

Exercise 4.5.9 (Twisting)**.** Let (π, V) be an admissible representation of $\mathrm{GL}(2, F)$, and let χ be a quasicharacter of $F^\times$. Define another admissible representation $\chi \otimes \pi$ acting on the same space V by

$$(\chi \otimes \pi)(g)v = \chi(\det(g))\pi(g)v, g \in \mathrm{GL}(2, F), v \in V.$$

Prove that if χ_1 and χ_2 are quasicharacters of $F^\times$, then

$$\chi \otimes B(\chi_1, \chi_2) \simeq B(\chi\chi_1, \chi\chi_2).$$

Proof. □

Exercise 4.5.10. Let f_{s_1,s_2} be a flat section of $B(\chi_1,\chi_2)$, where $\chi_i(x)=\xi_i(x)|x|^{s_i}(i=1,2)$. Show that there exist a finite number of flat sections $\widetilde{f^j_{s_2,s_1}}$ of $B(\chi_2,\chi_1)$ and meromorphic functions $\phi_j(s_1,s_2)$ that are holomorphic except at values of s_1 and s_2 such that $\chi_1=\chi_2$ such that

$$Mf_{s_1,s_2}=\sum_j \phi_j(s_1,s_2)\widetilde{f^j_{s_2,s_1}}.$$

Proof. TBD. □

4.6 Spherical Representations (0/2)

4.7 Local Functional Equations (0/3)

4.8 Supercuspidals and the Weil Representation (0/1)

4.9 The Local Langlands Correspondence