

Approximate Minimum-Weight Partial Matching under Translation

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Wolfgang Mulzer ³ Günter Rote ³ Micha Sharir ² Allen Xiao ¹

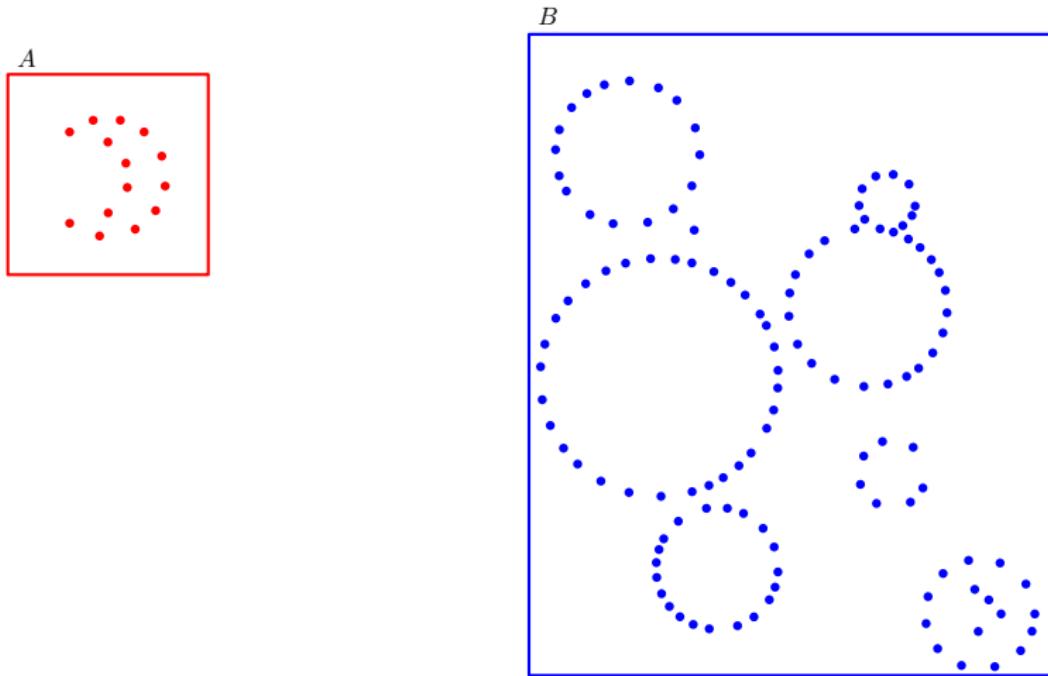
¹Duke University

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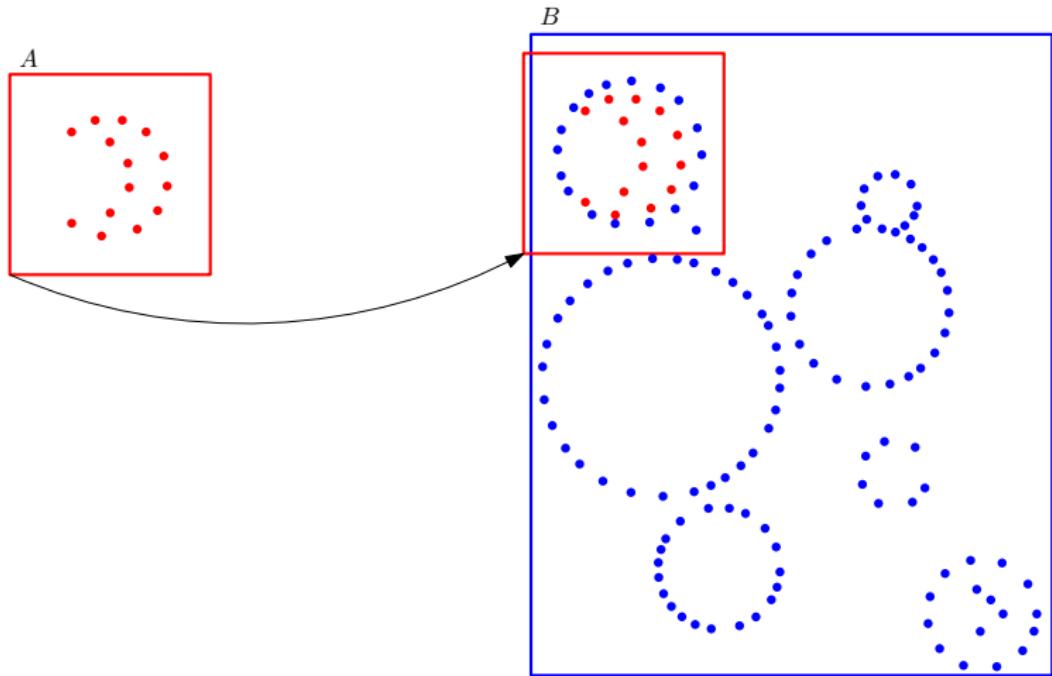
December 2018

Example

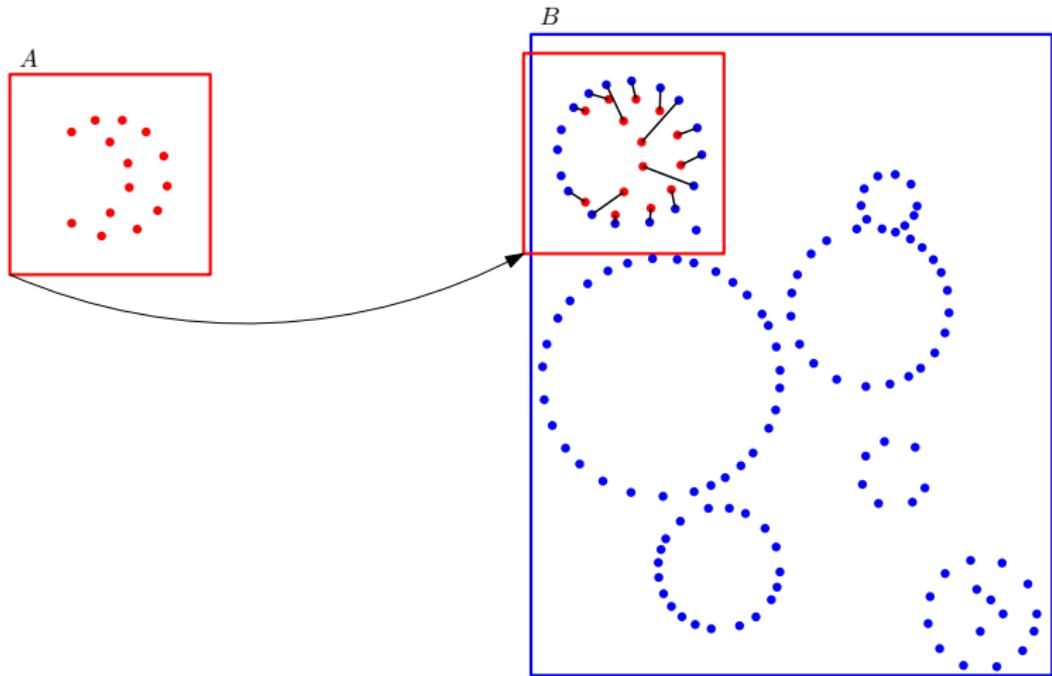


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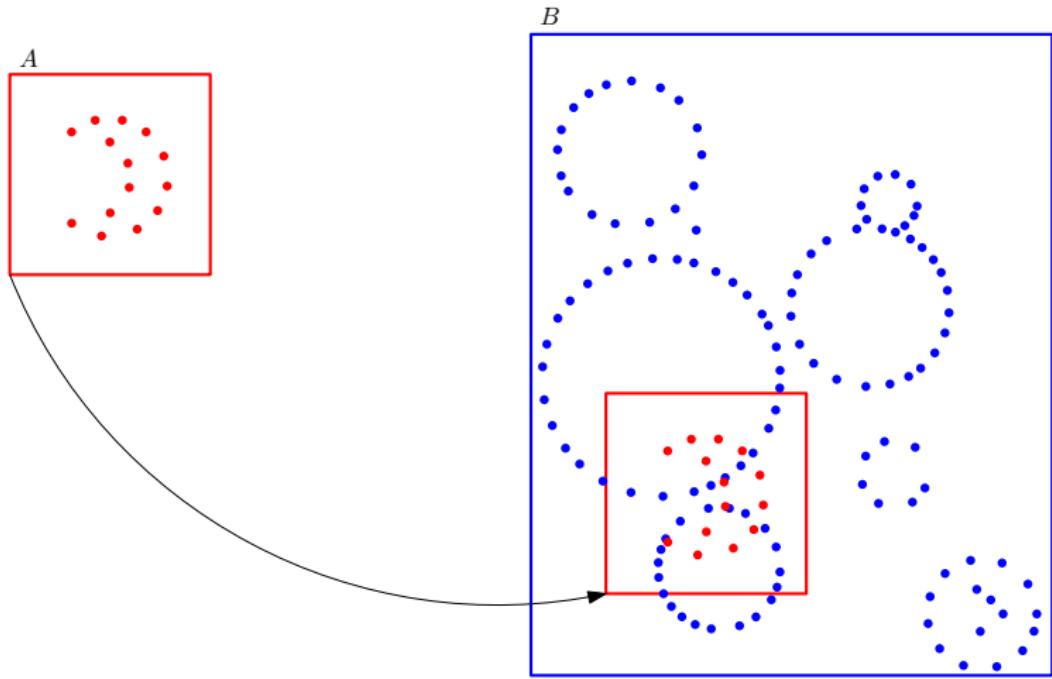
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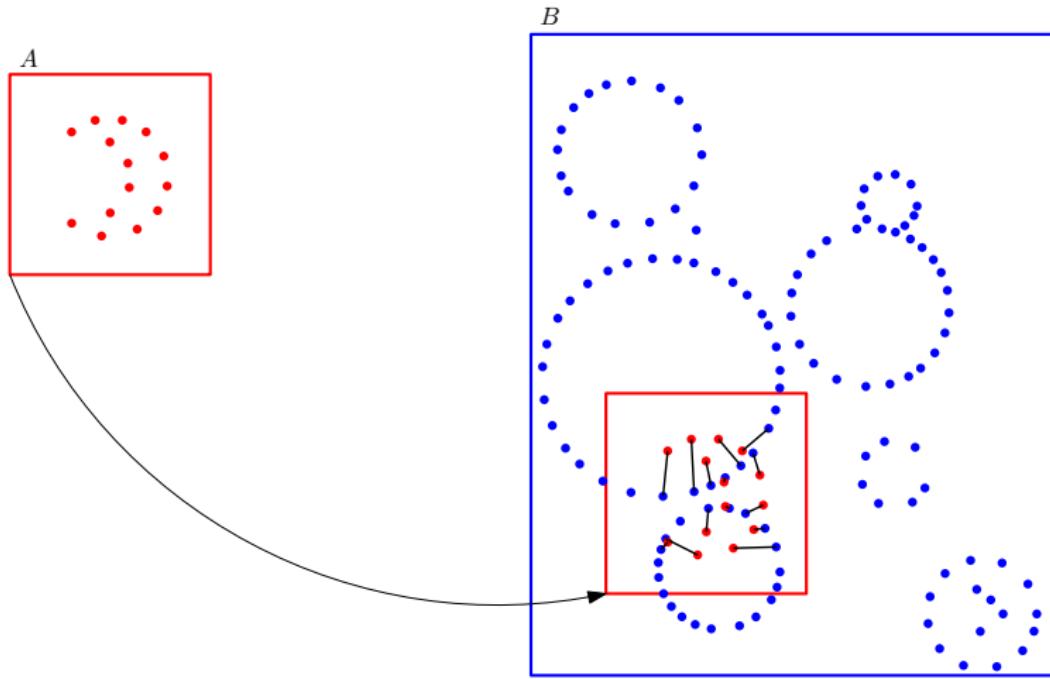


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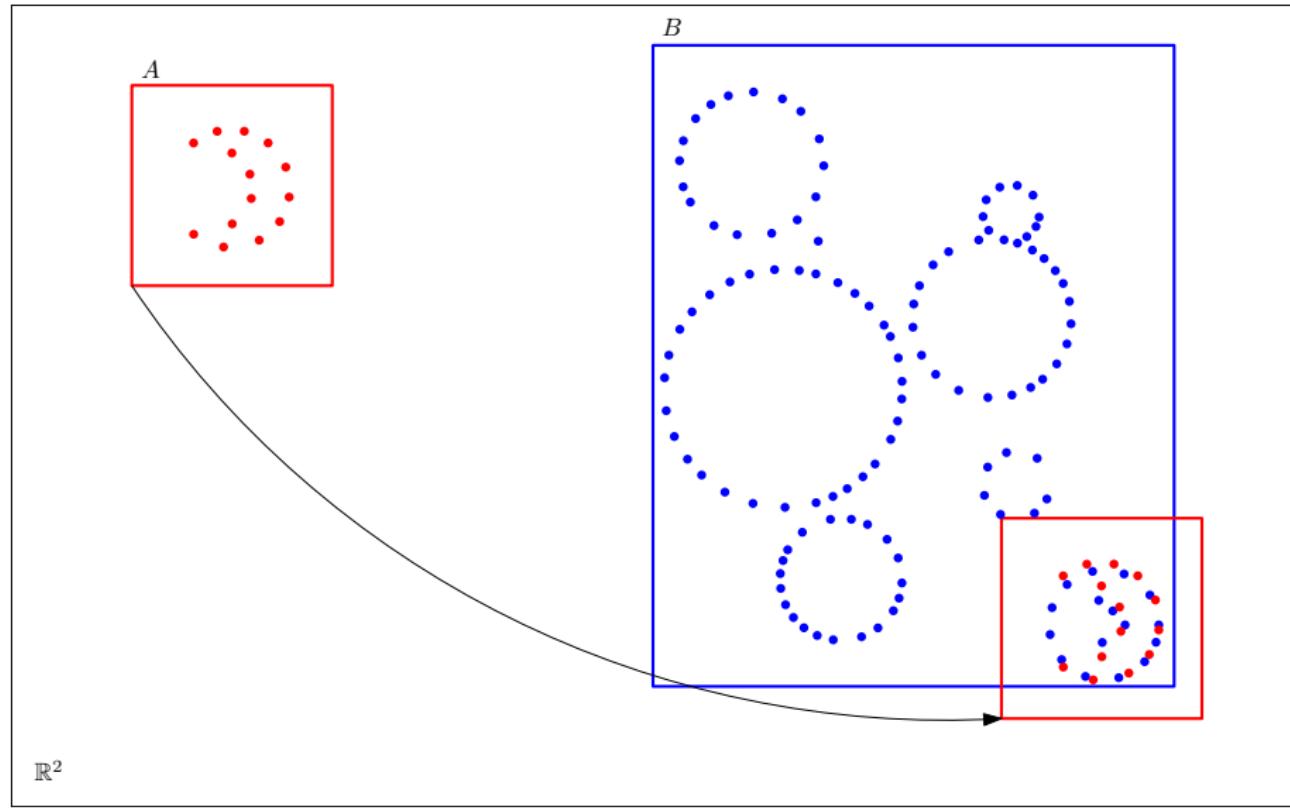
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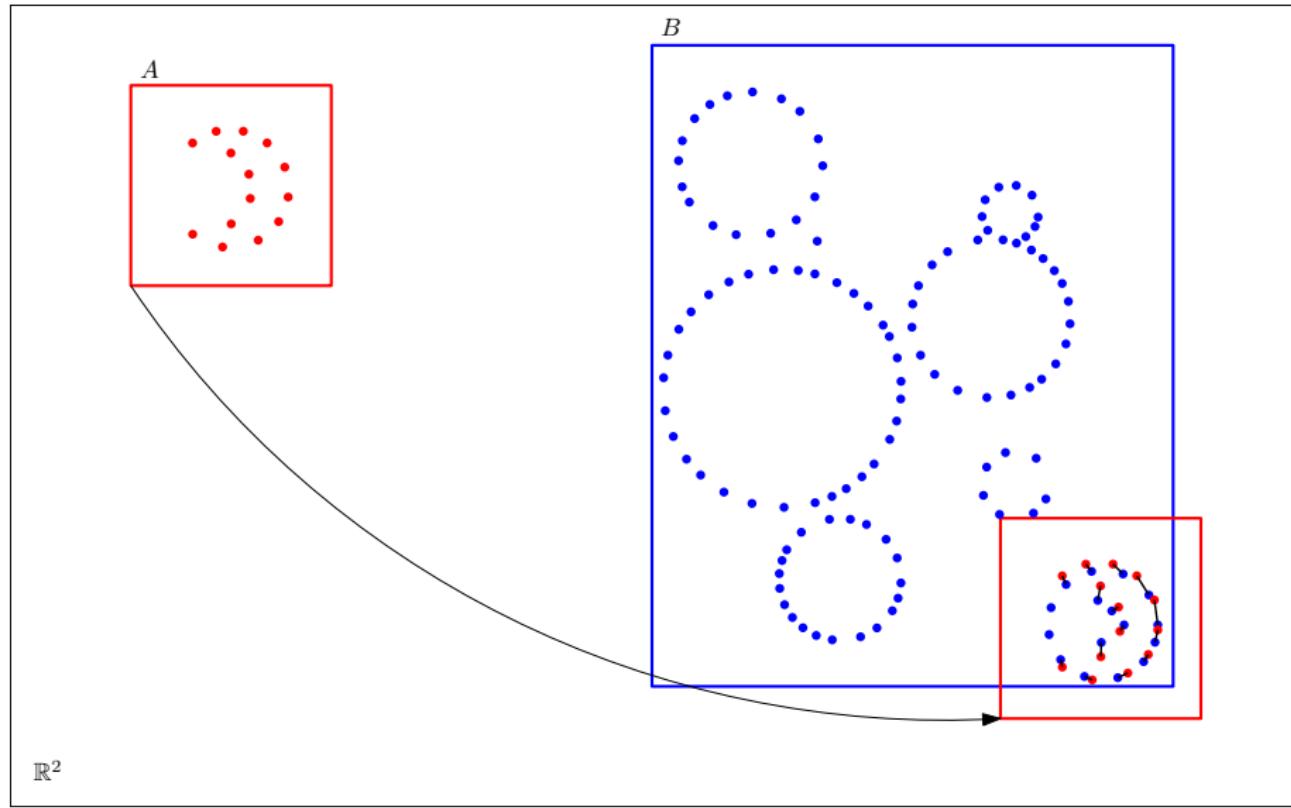


\mathbb{R}^2

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Find the minimum-cost matching over all translations

Cost function

- ▶ Given point sets A, B , with $|A| = m$ and $|B| = n$, match A into an m -subset of B after translation t .
- ▶ For matching M and translation t , define the **root-mean-square (RMS) cost** of the matching:

$$\text{cost}(M, t) := \left[\frac{1}{m} \sum_{(a,b) \in M} \|a + t - b\|^2 \right]^{1/2}$$

- ▶ Best matching cost at each translation:

$$\text{cost}^*(t) := \min_{\text{matching } M} \text{cost}(M, t)$$

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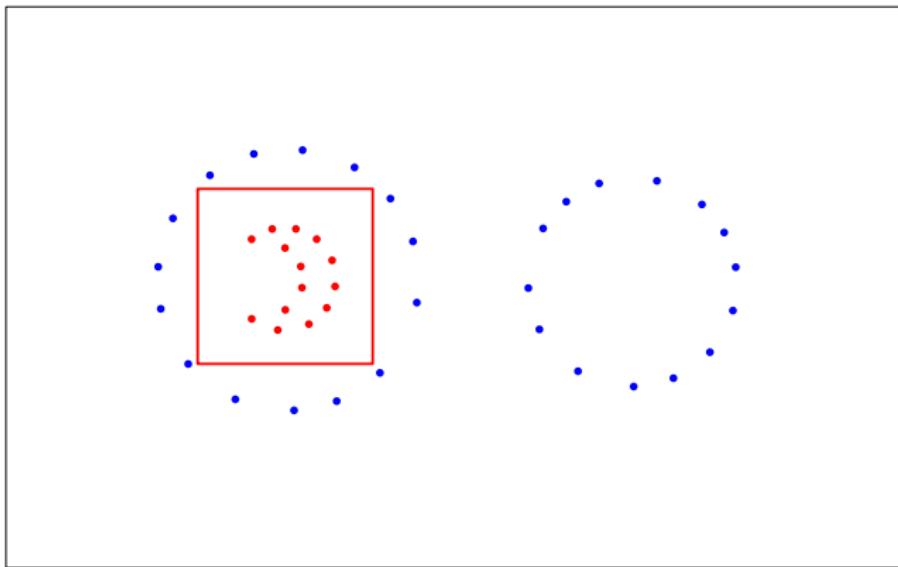
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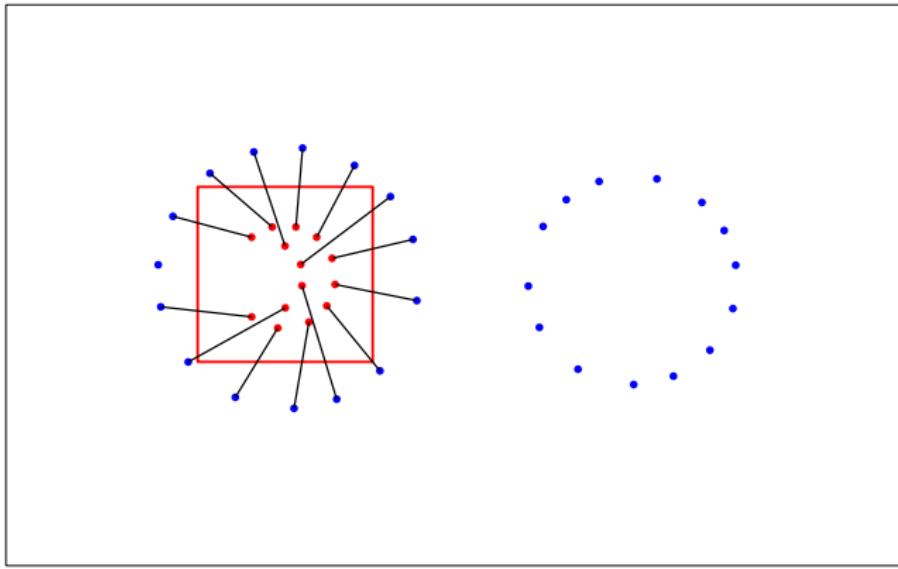
$$\text{cost}^*(t) := \min_{\text{matching } M} \text{cost}(M, t)$$



- ▶ How many distinct matchings appear in $\text{cost}^*(t)$?

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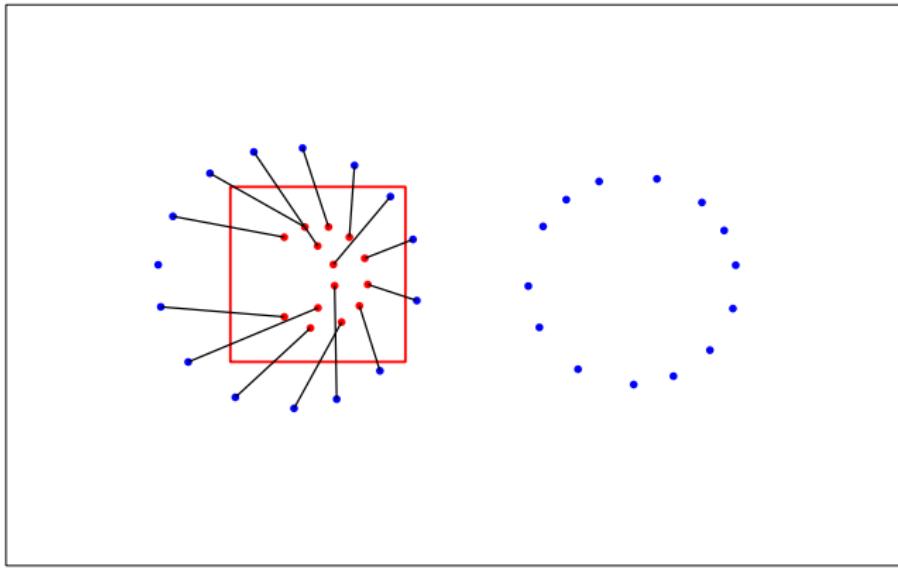
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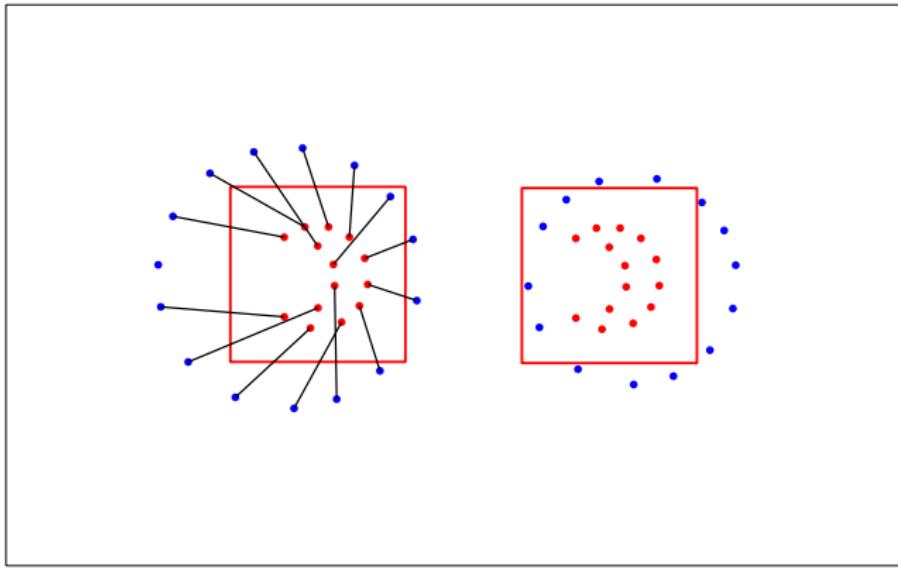
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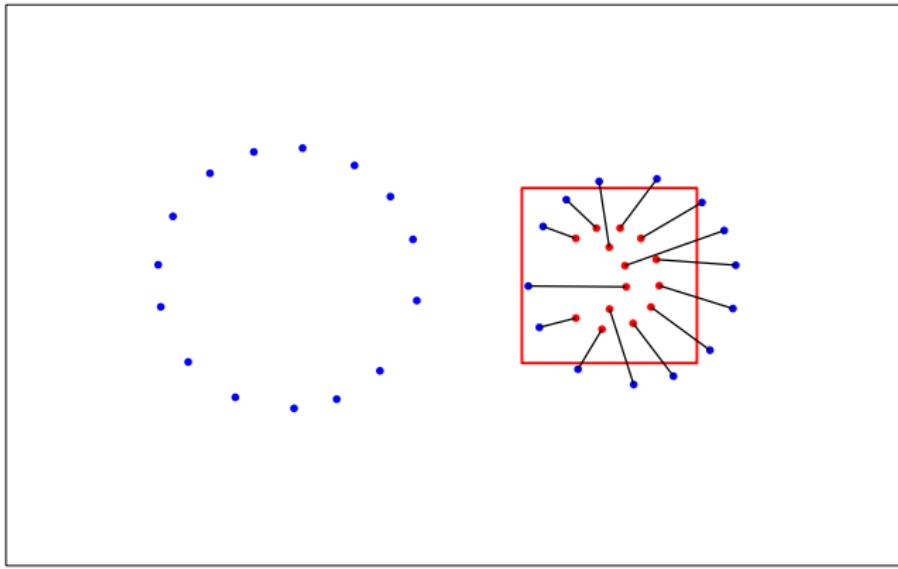
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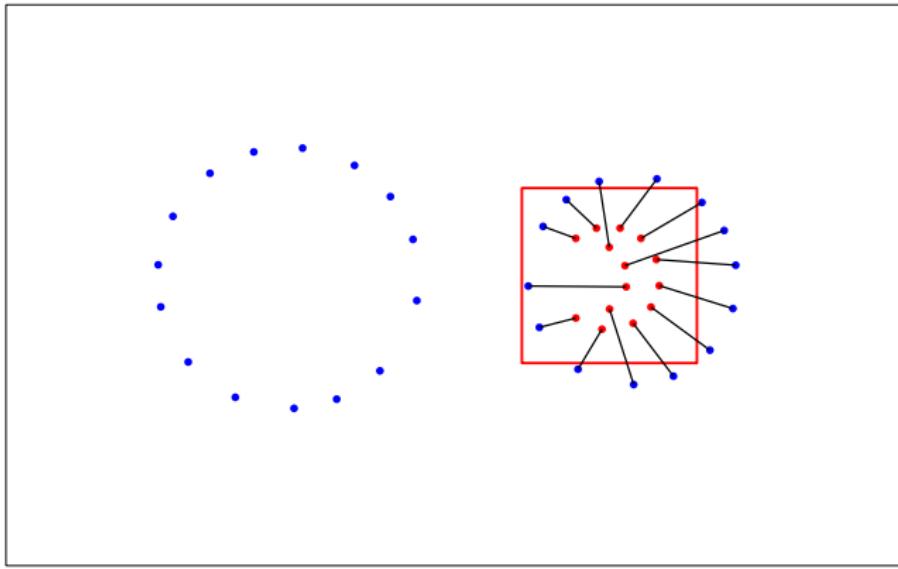
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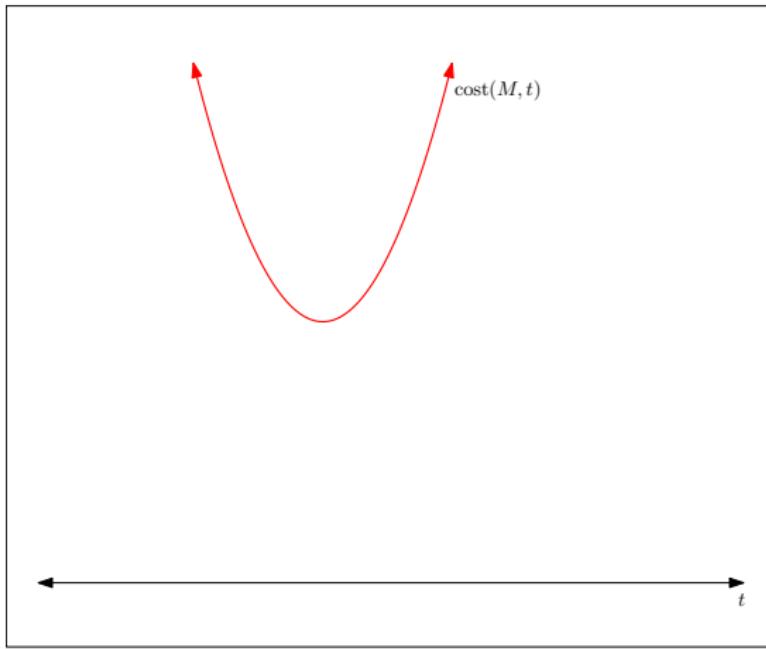
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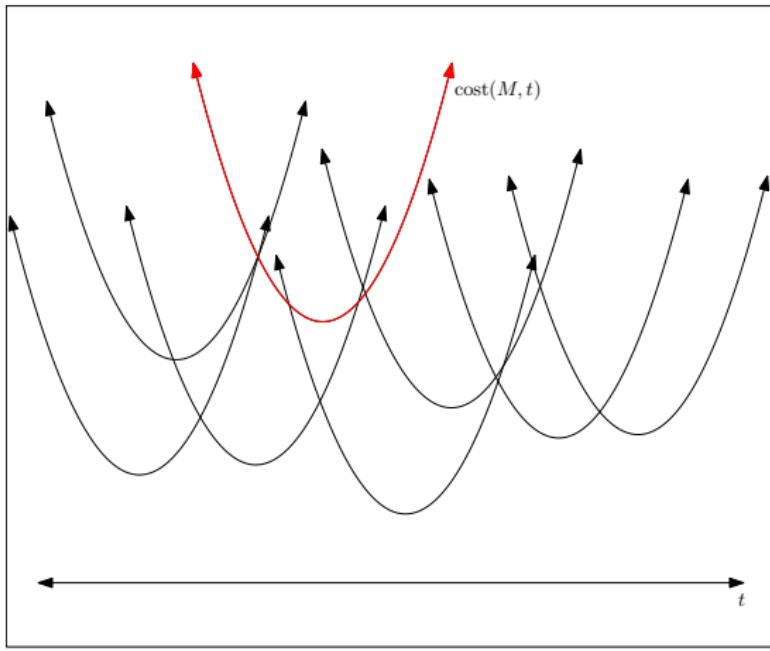
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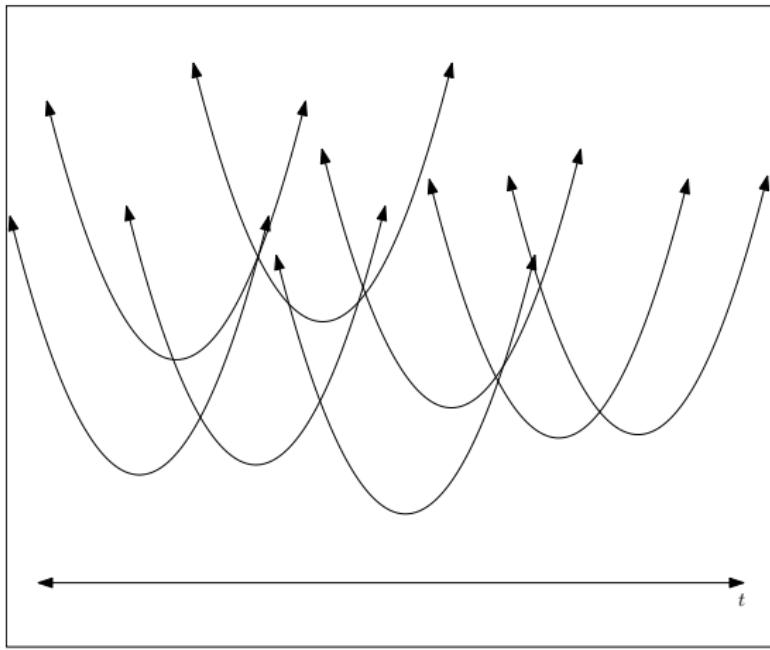
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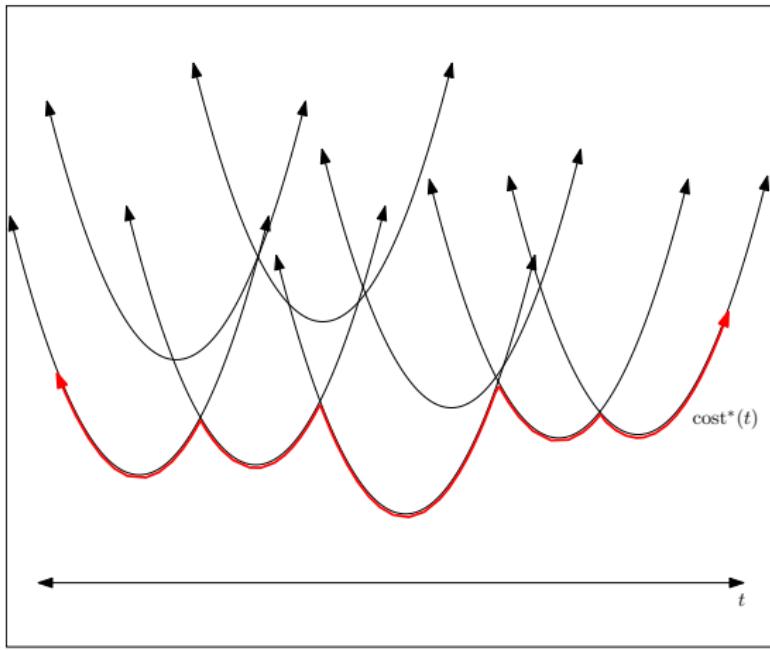
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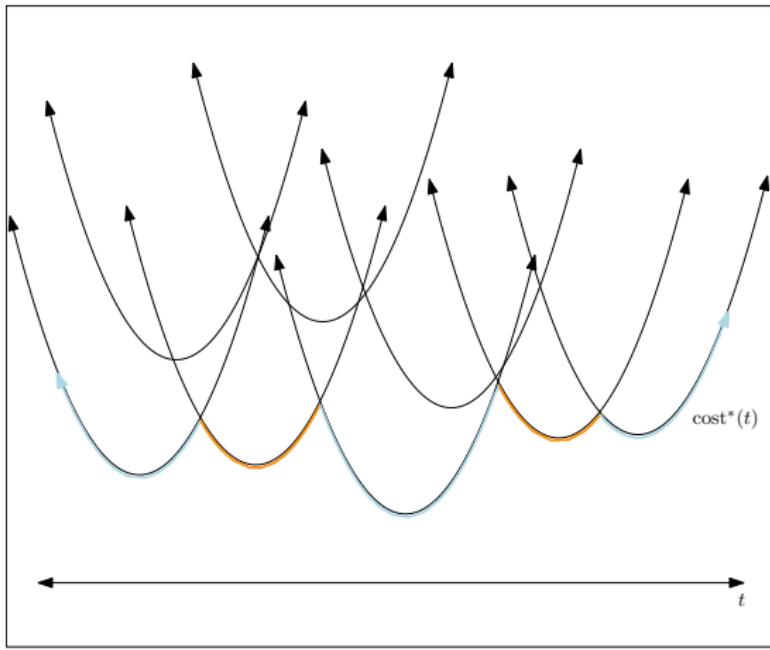
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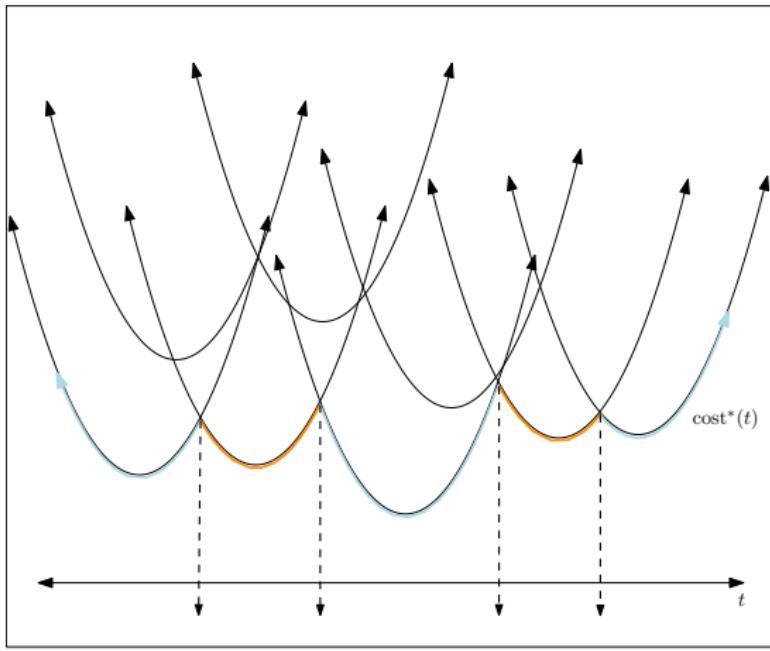
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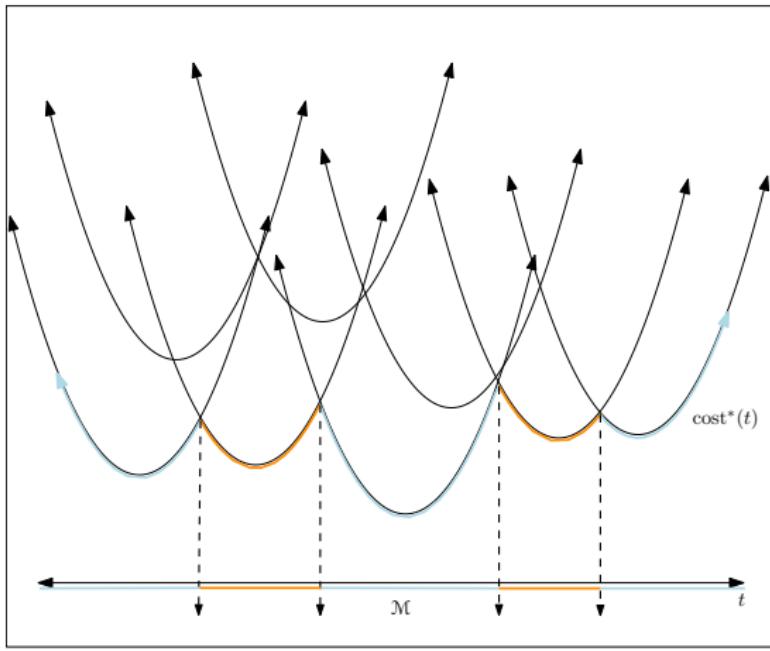
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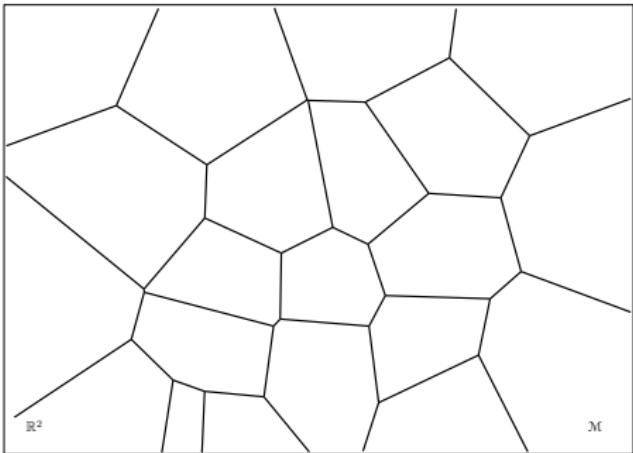
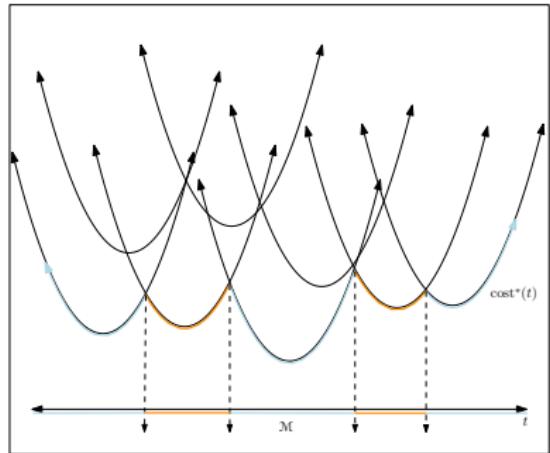


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How many distinct matchings? (2-dimensional)



- ▶ How many distinct matchings appear in $\text{cost}^*(t)$?
- ▶ What is the combinatorial complexity of \mathcal{M} ?

Questions and prior results

1. How quickly can we compute t^* , a global minimum of $\text{cost}^*(t)$?
2. What is the combinatorial complexity of \mathcal{M} ?

- ▶ [Rote 10]: in 1D, at most $m(n - m) + 1$.
- ▶ [Ben-Avraham *et al.* 14]: $O(n^2m^{3.5}(e \ln m + e)^m)$
- ▶ Open to find t^* in polytime, and whether \mathcal{M} has polynomial complexity.
- ▶ Approximation?

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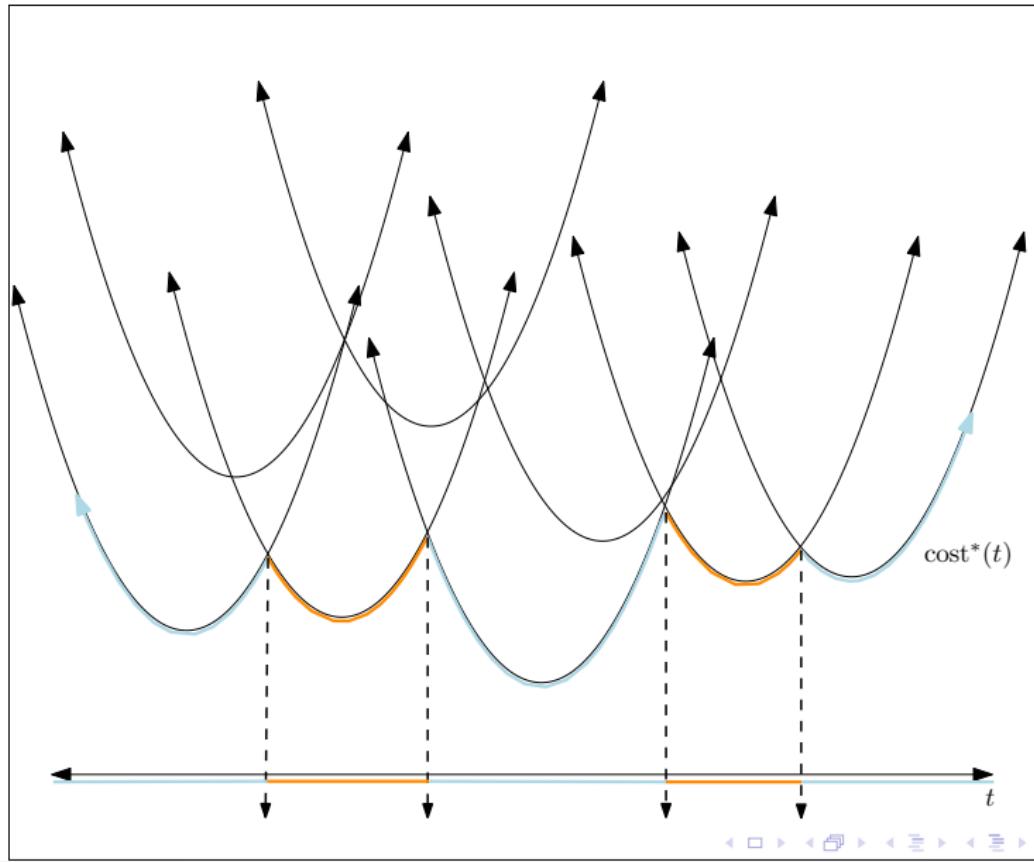
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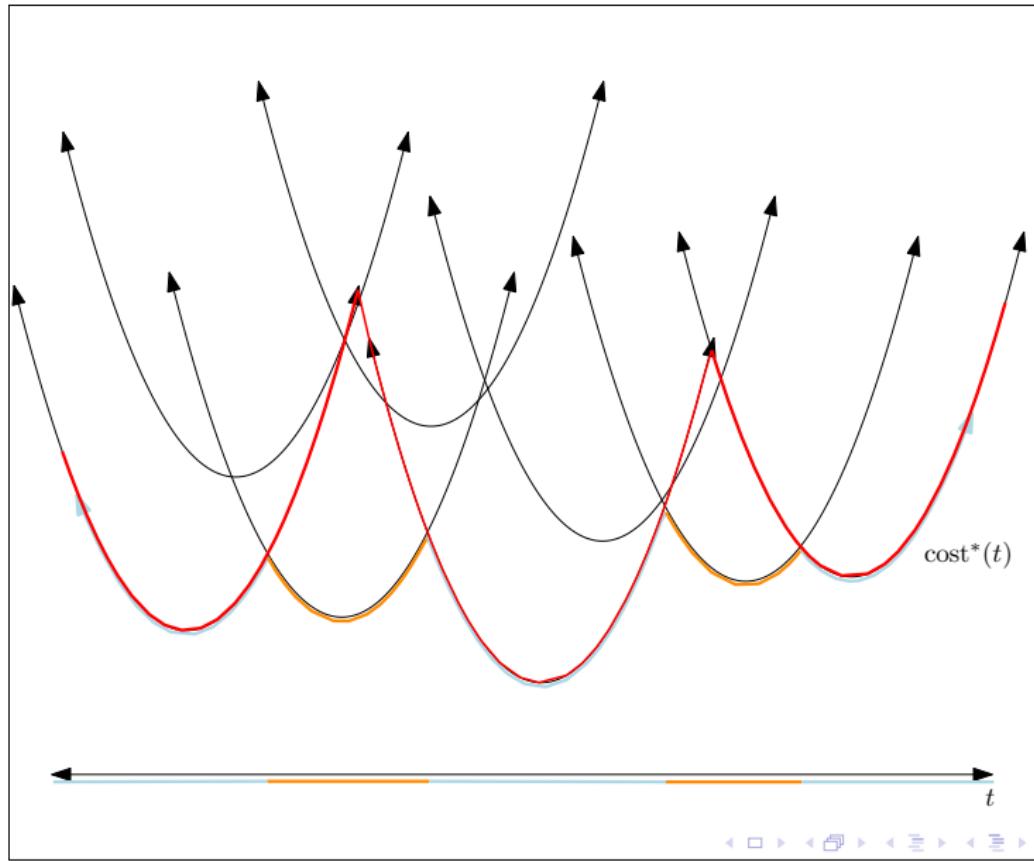
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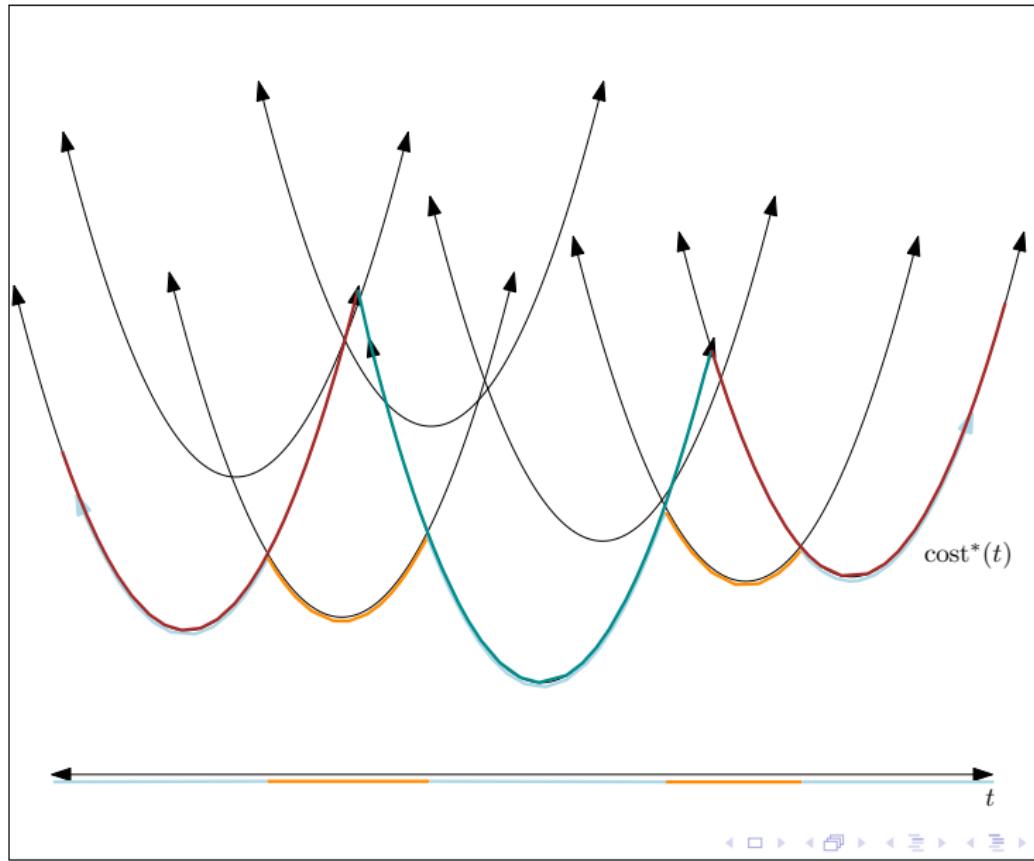
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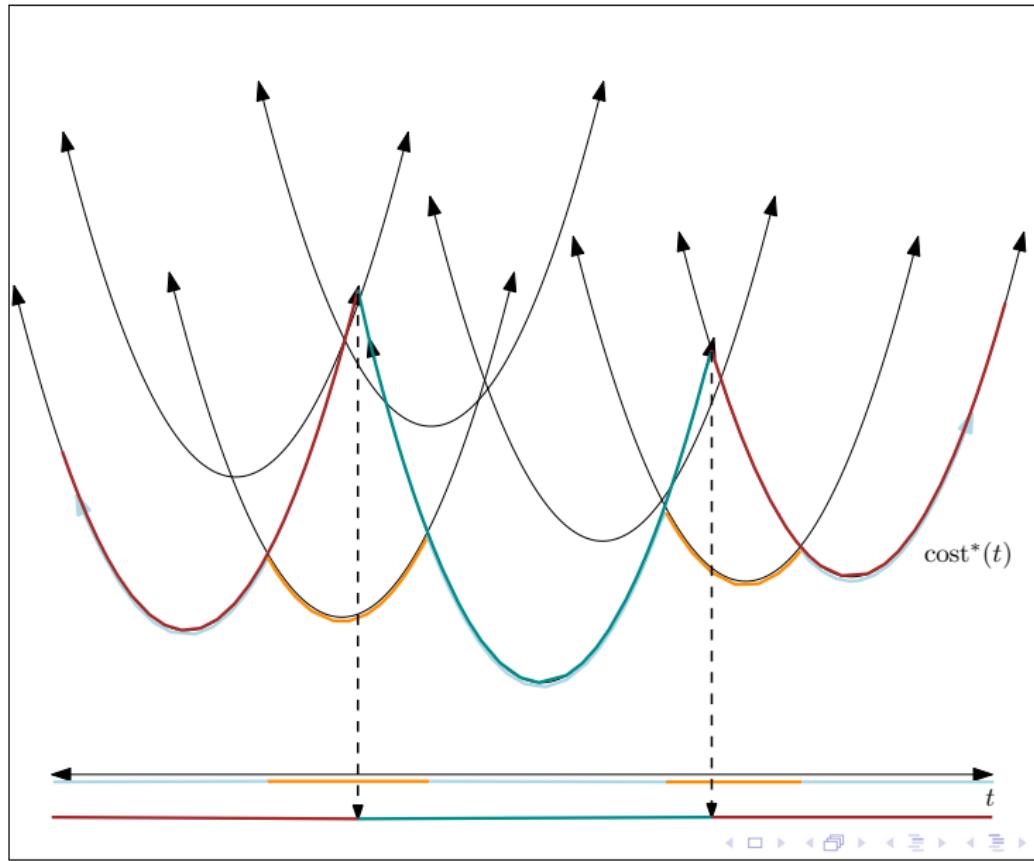
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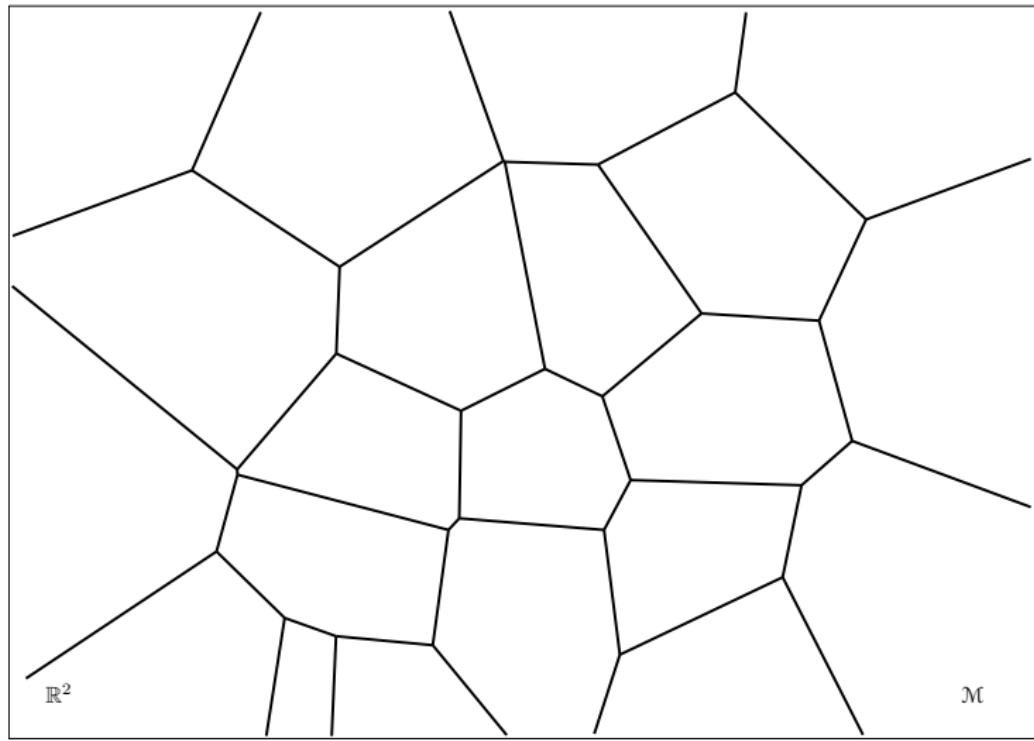
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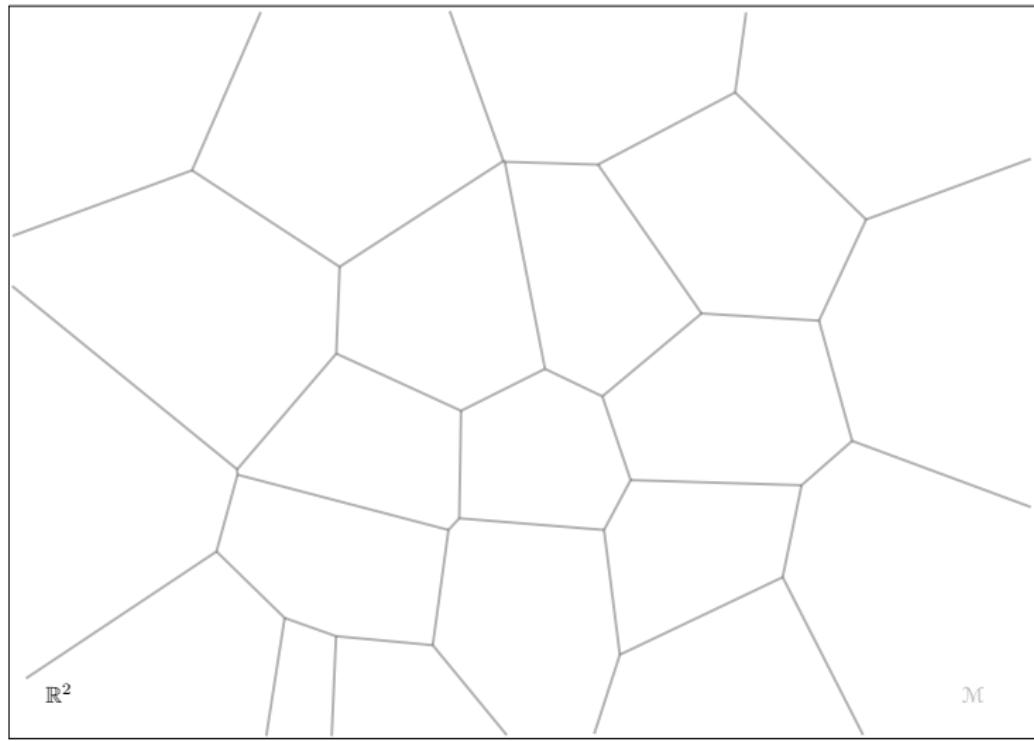
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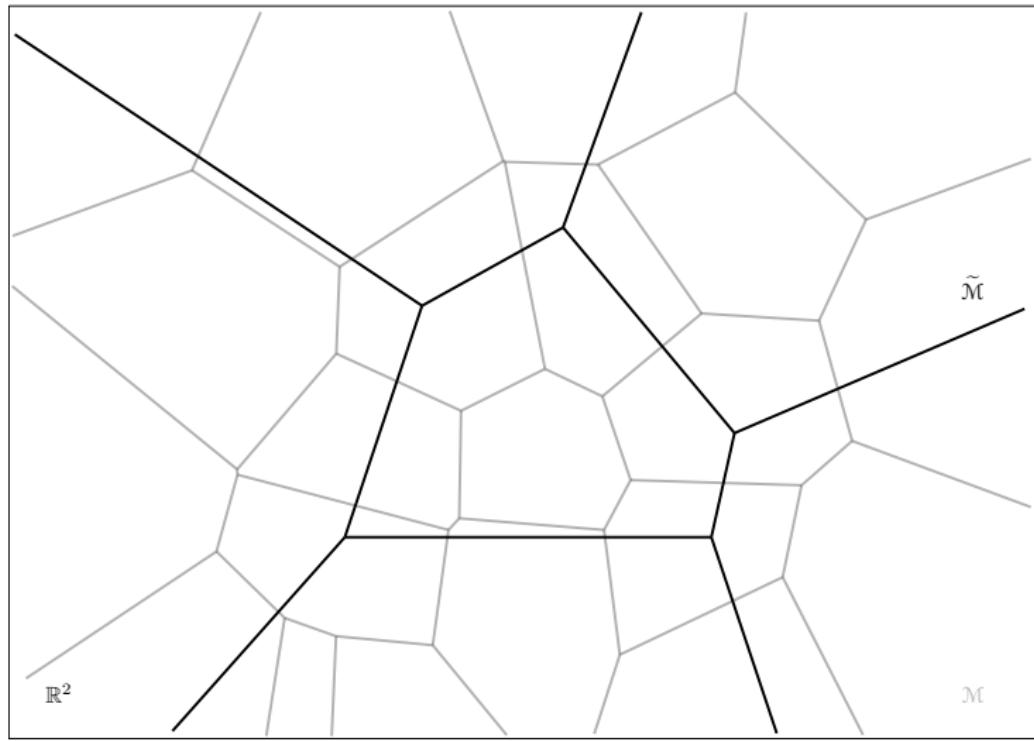
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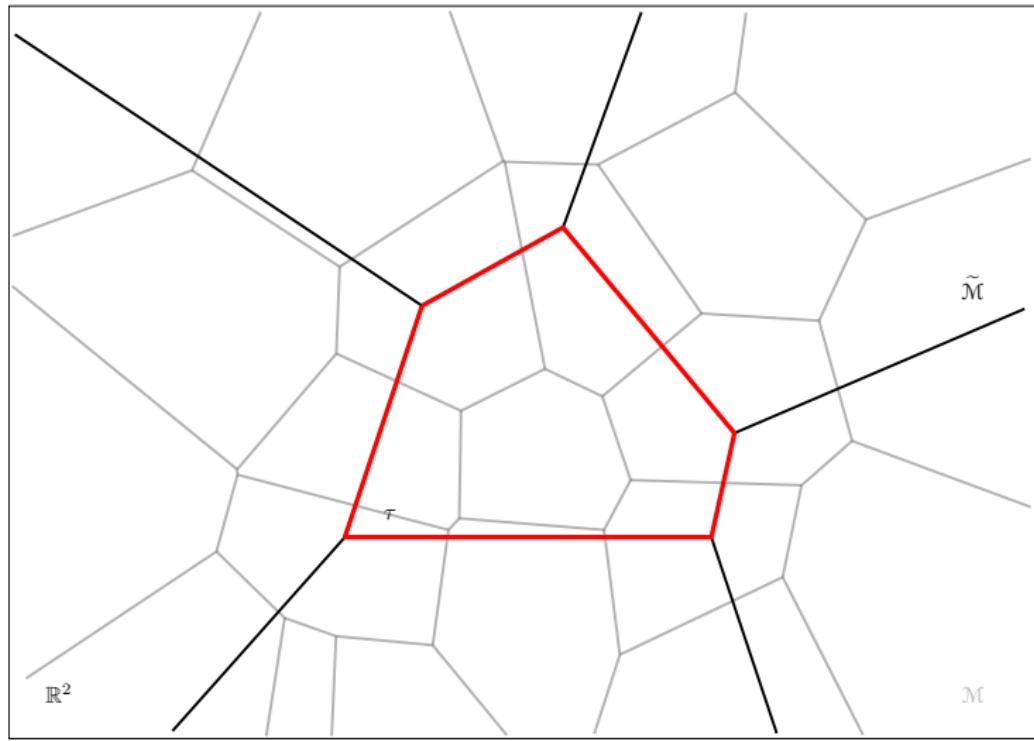
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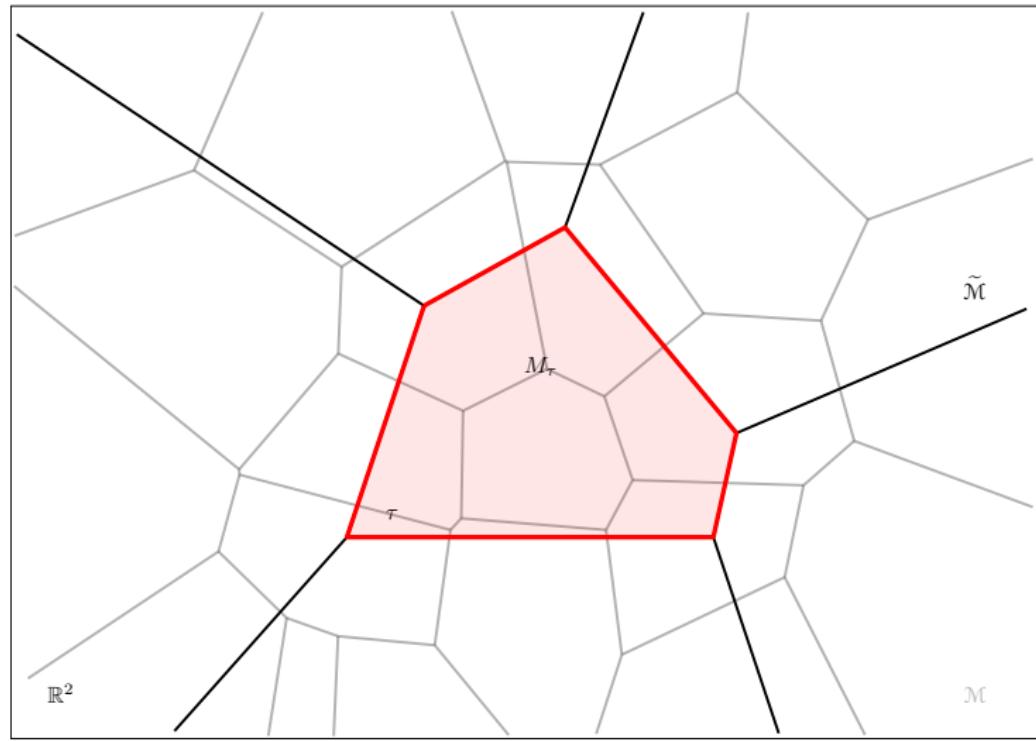
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Our results (approximation helps)

1. How quickly can we compute t^* , a global minimum of $\text{cost}^*(t)$?

Theorem

In $\text{poly}(m, n, \varepsilon^{-1})$ time, can compute a $(1 + \varepsilon)$ approximation to $\text{cost}^(t^*)$ by exploring the faces of $\tilde{\mathcal{M}}$.*

2. What is the combinatorial complexity of \mathcal{M} ?

Theorem

In $\text{poly}(m, n, \varepsilon^{-1})$ time, can construct a $(1 + \varepsilon)$ approximate diagram $\tilde{\mathcal{M}}$ of complexity $O(n\varepsilon^{-2} \log \varepsilon^{-1})$.

Overview

1. The set of point-to-point translations give a constant approximate diagram of size mn .
2. Using exponential grids, constant $\rightarrow (1 + \varepsilon)$ approximation of size $O(mn\varepsilon^{-2} \log \varepsilon^{-1})$.
3. Reduce size to $O(n\varepsilon^{-2} \log \varepsilon^{-1})$ by clustering.

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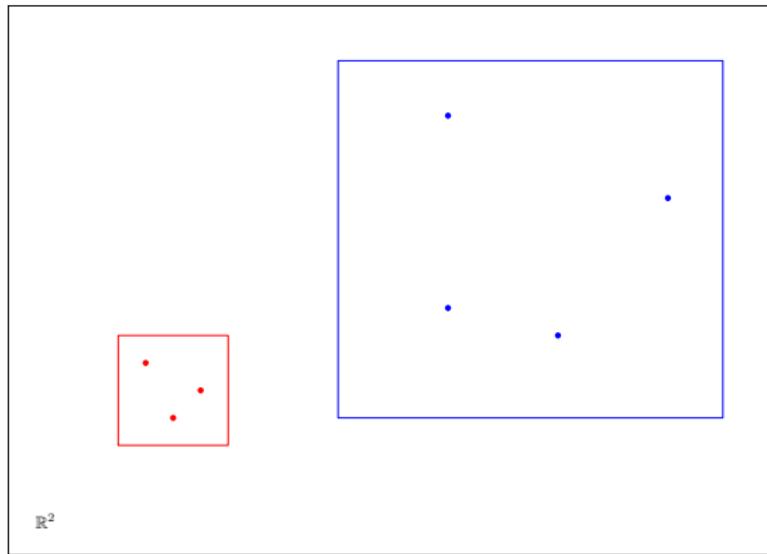
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Point-to-point translations

- ▶ Point-to-point translations [Cabello et al. 08]:

$$T := \{t_{ba} = (b - a) \mid a \in A, b \in B\}$$

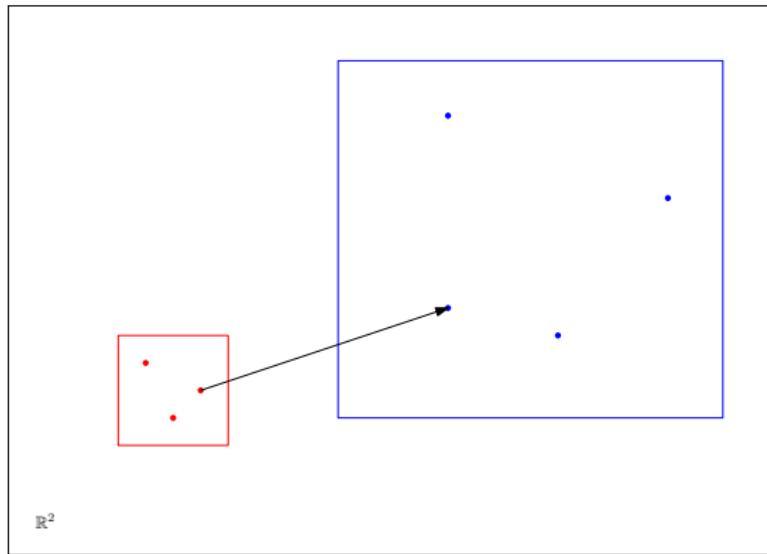


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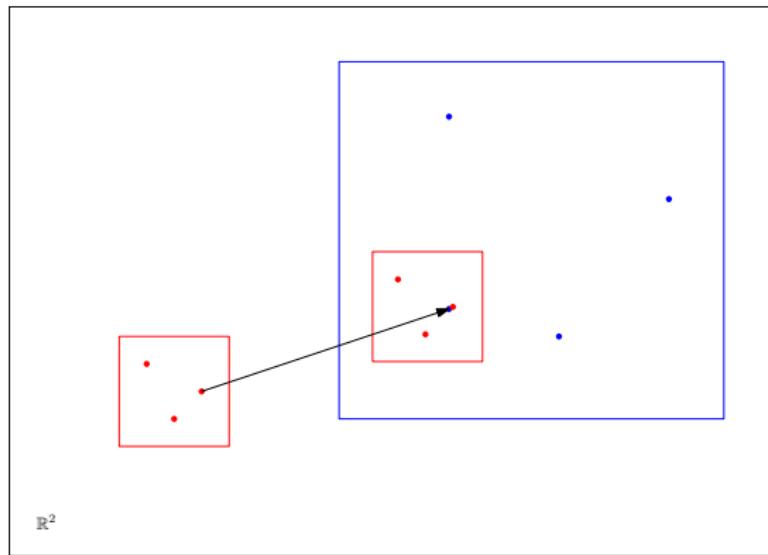


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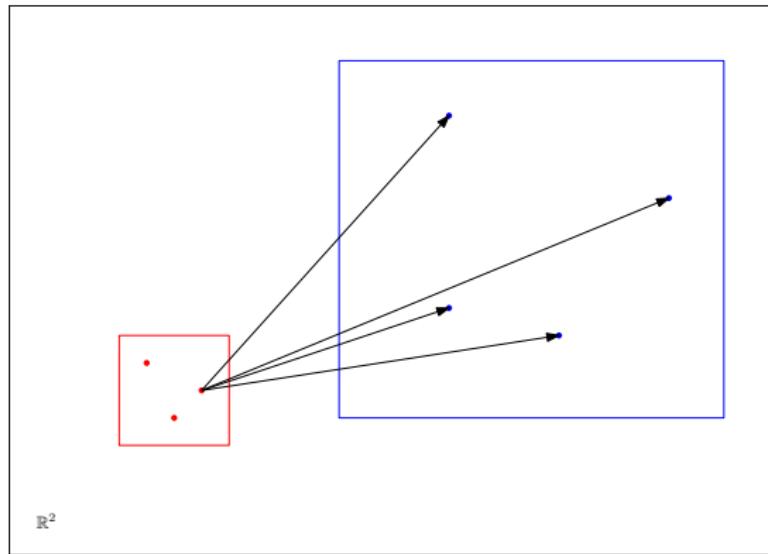


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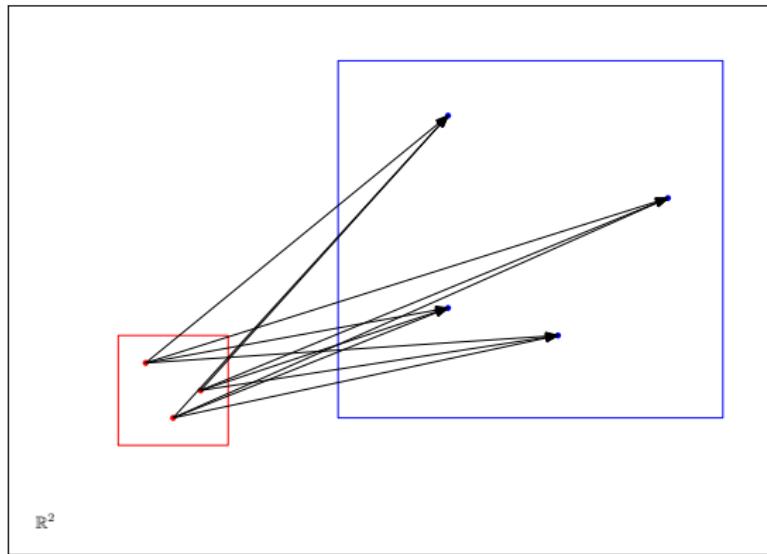


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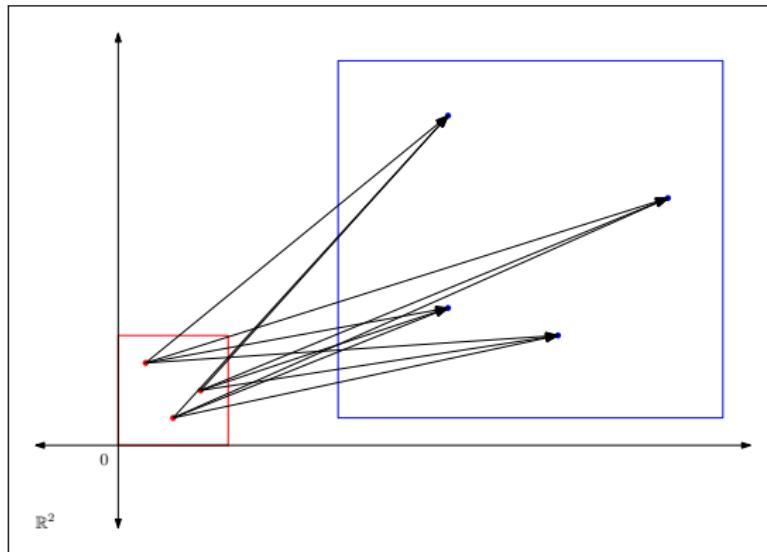


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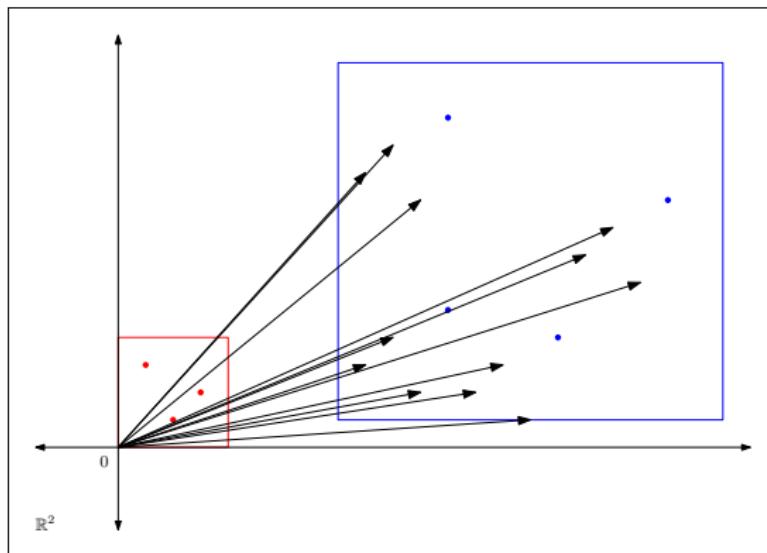


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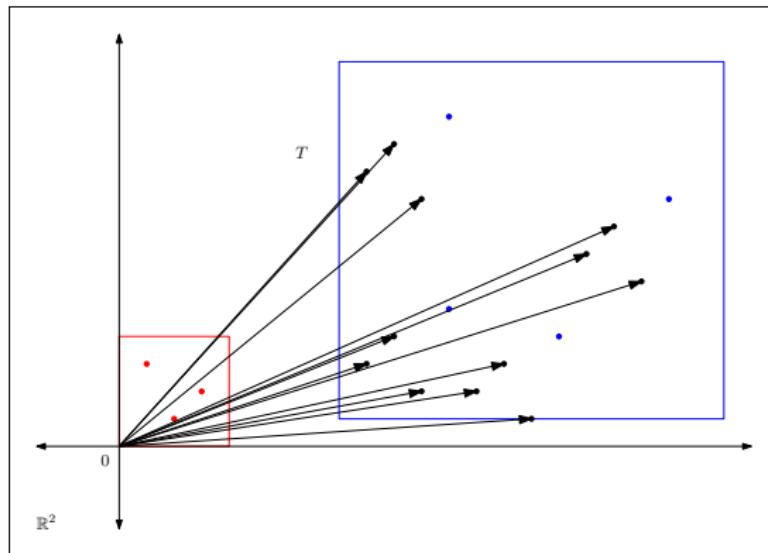


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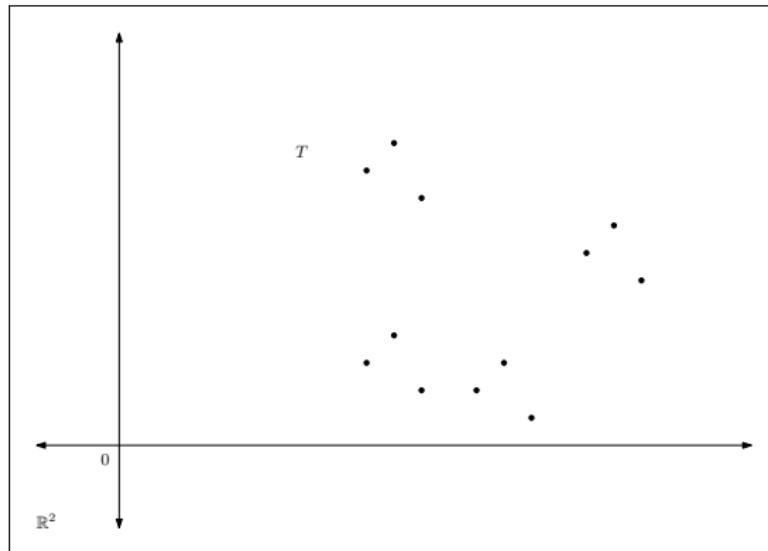


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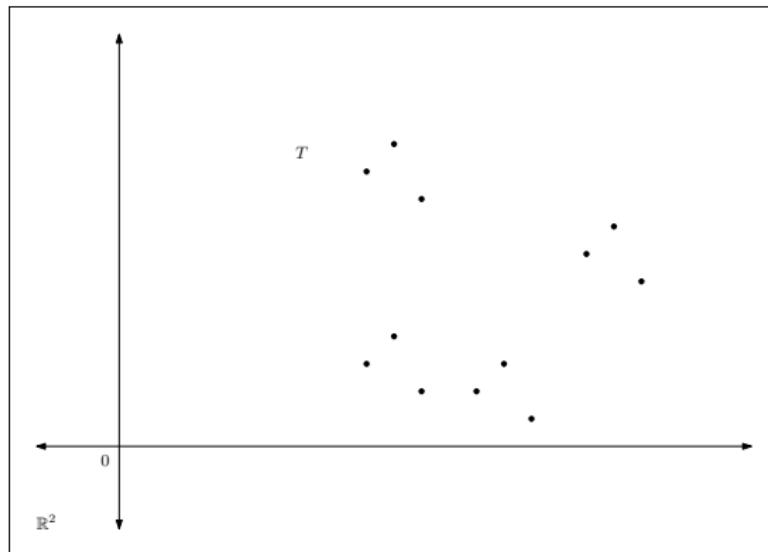


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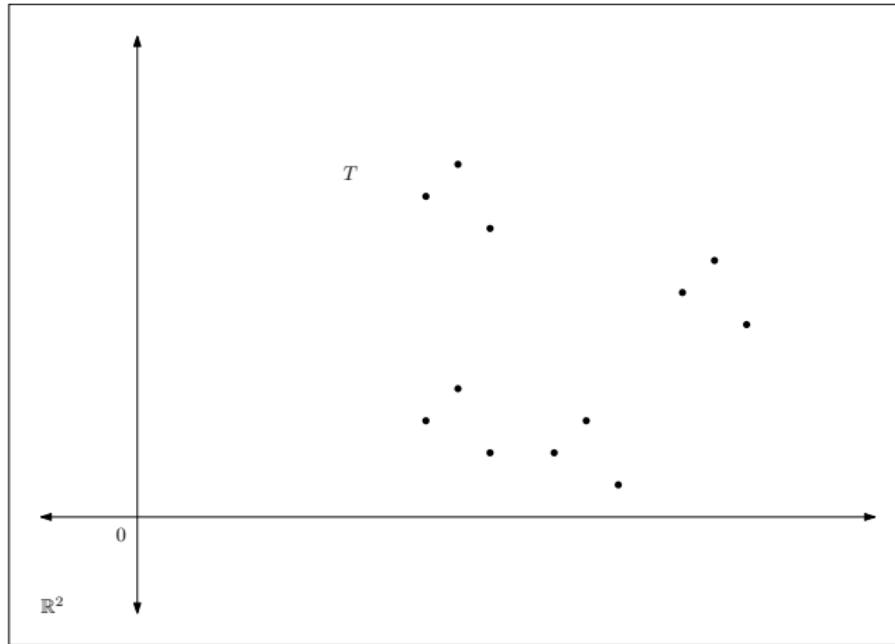
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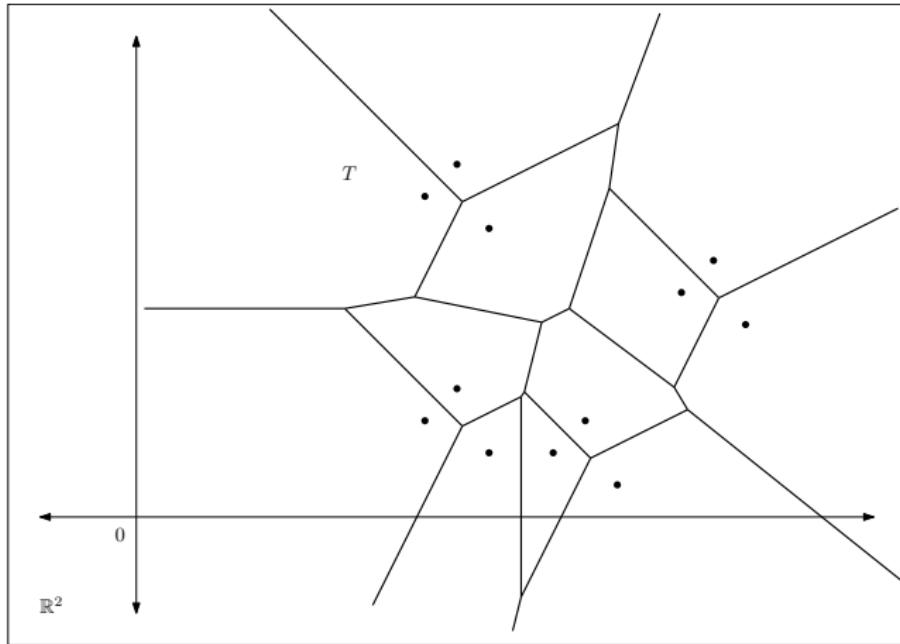
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$O(1)$ -approximate diagram from point-to-point translations



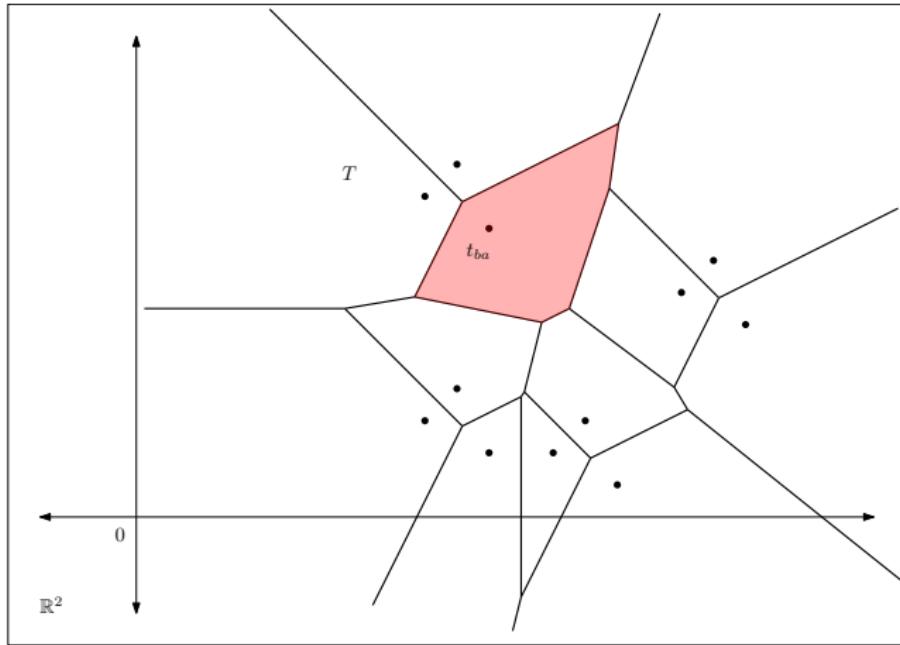
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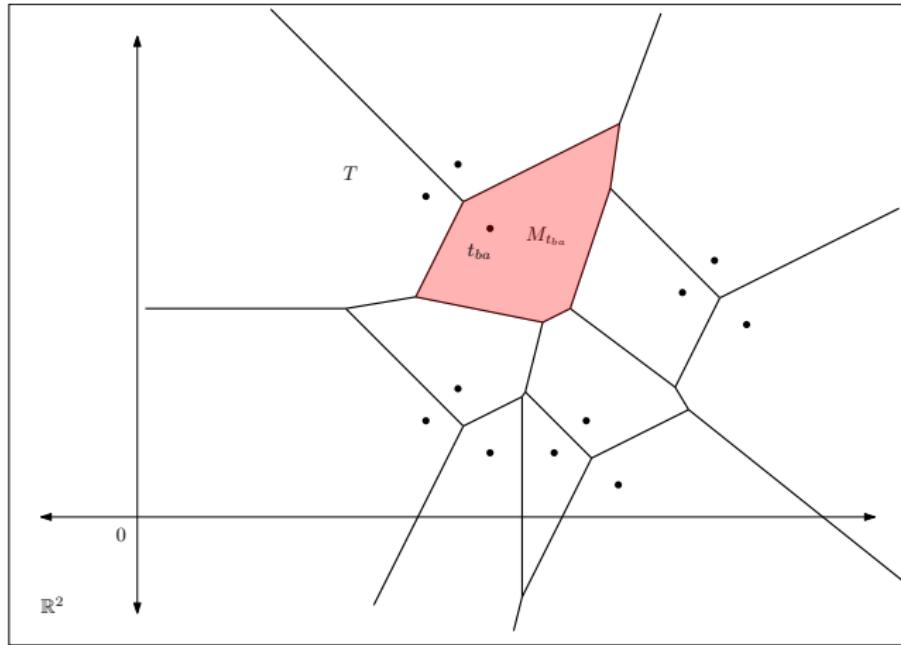
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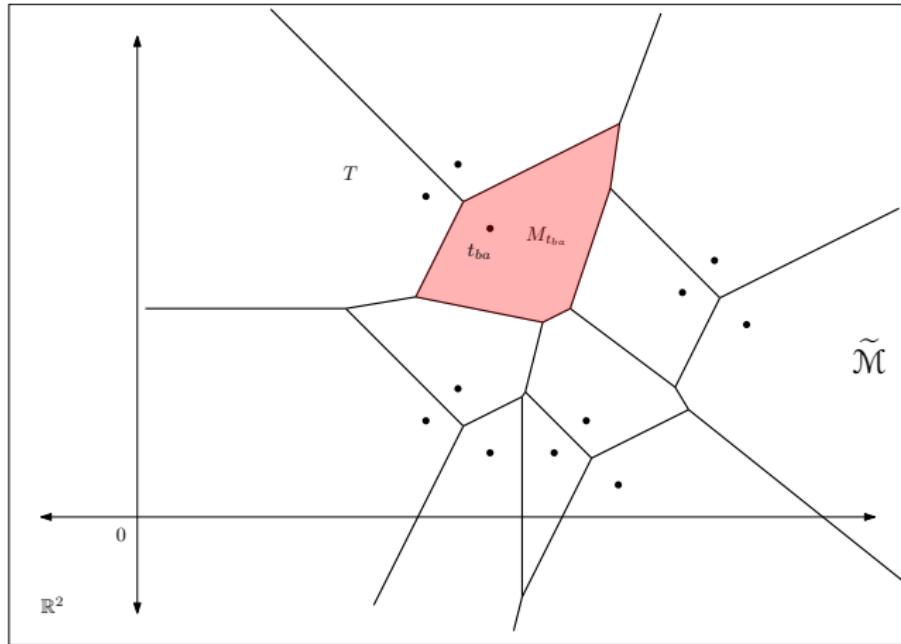
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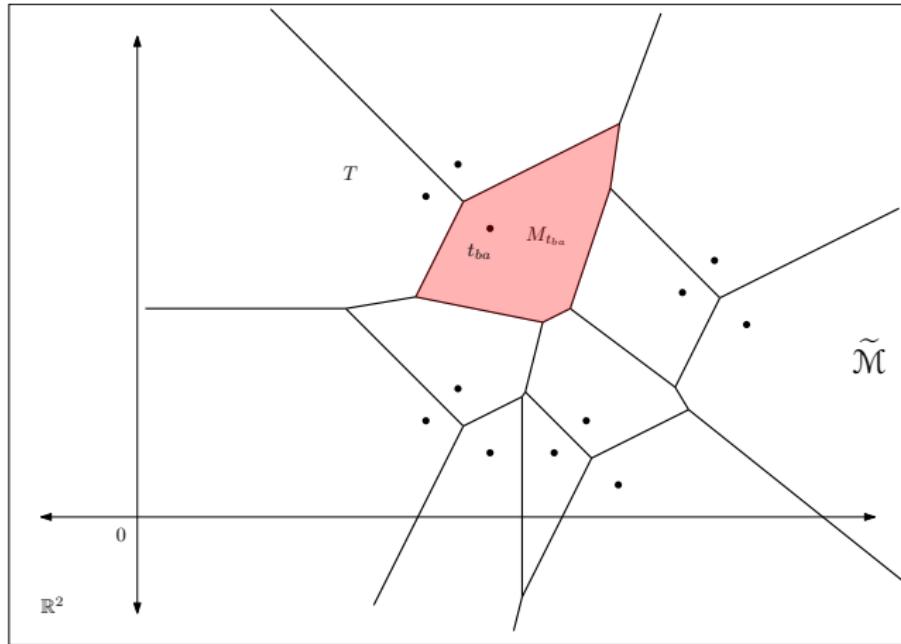
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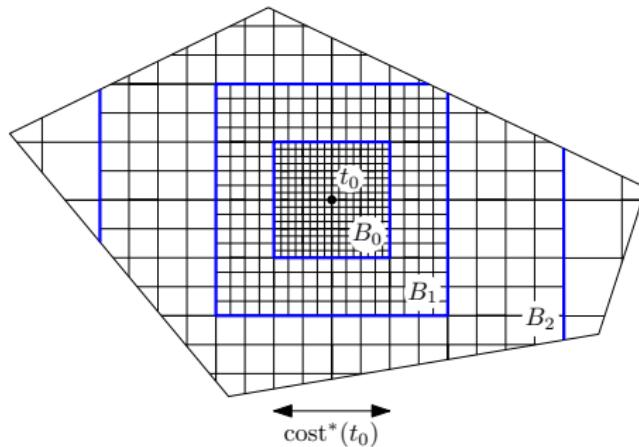
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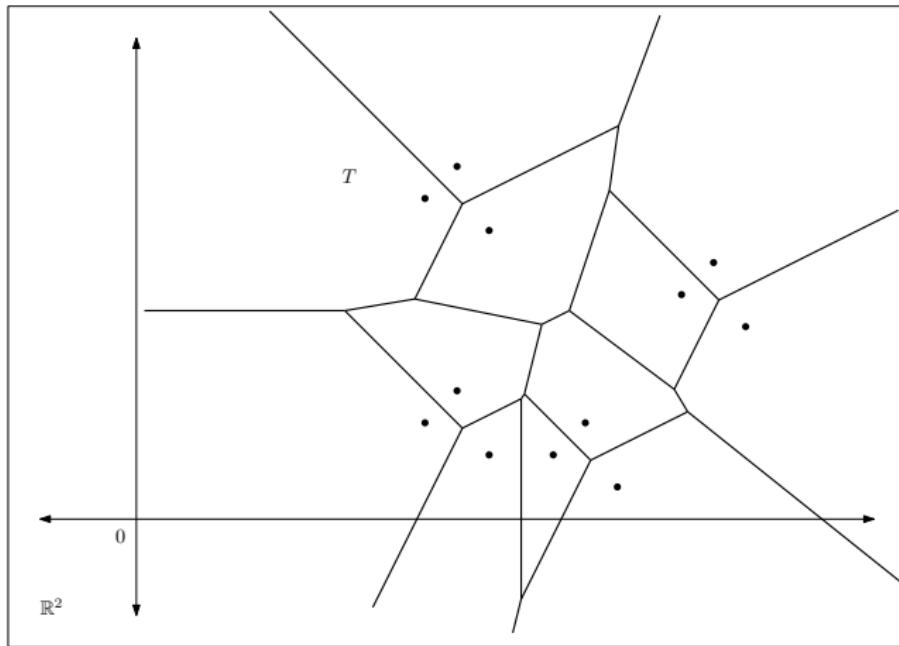
$O(1) \rightarrow (1 + \varepsilon)$ approximation



- ▶ $(1 + \varepsilon)$ approximate diagram of size
 $O(|T|\varepsilon^{-2} \log \varepsilon^{-1}) = O((mn)\varepsilon^{-2} \log \varepsilon^{-1})$

Near-linear size: clustering T

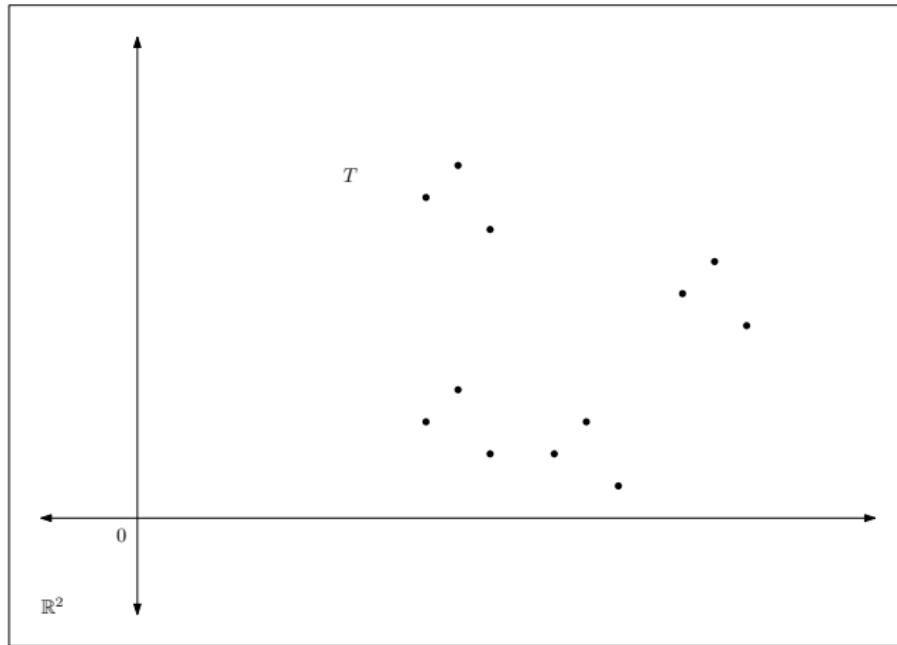
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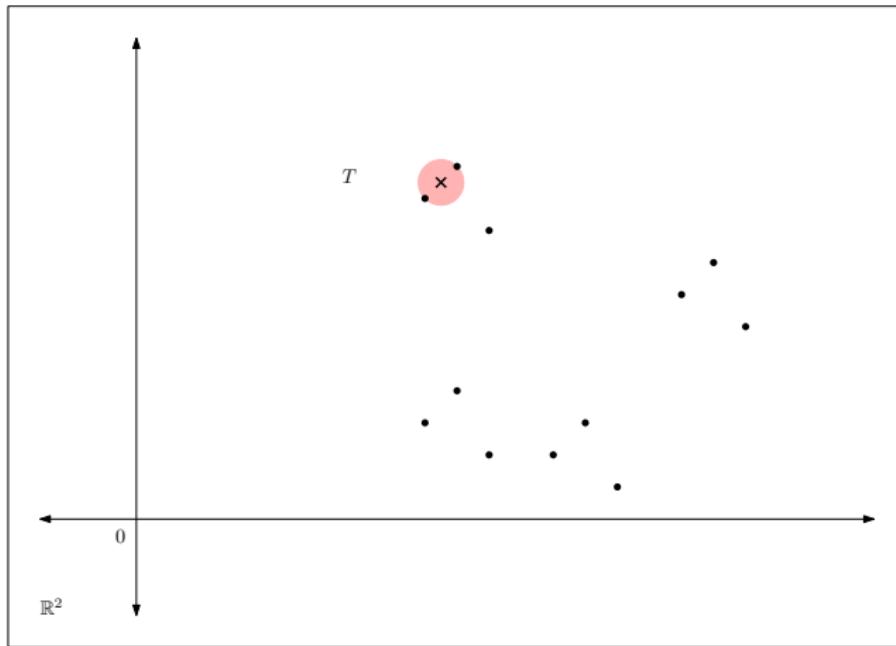
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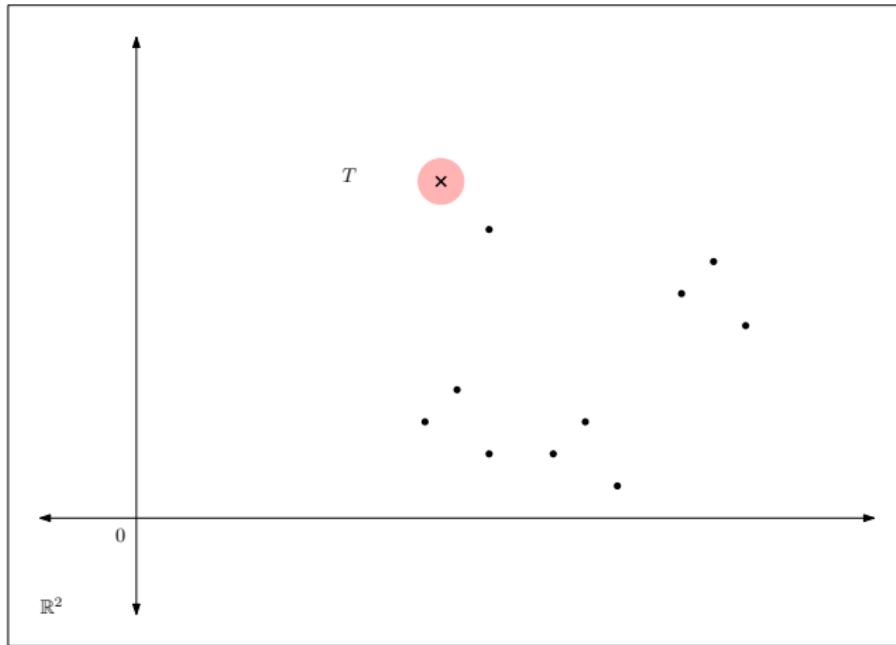
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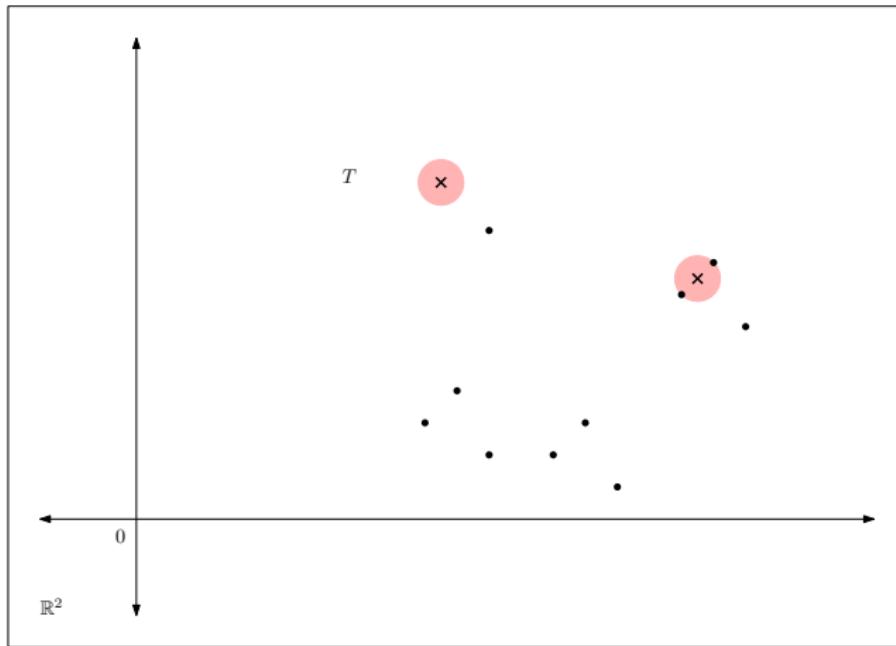
- ▶ Any $O(1)$ approx. diagram is enough for the $(1 + \varepsilon)$ diagram.
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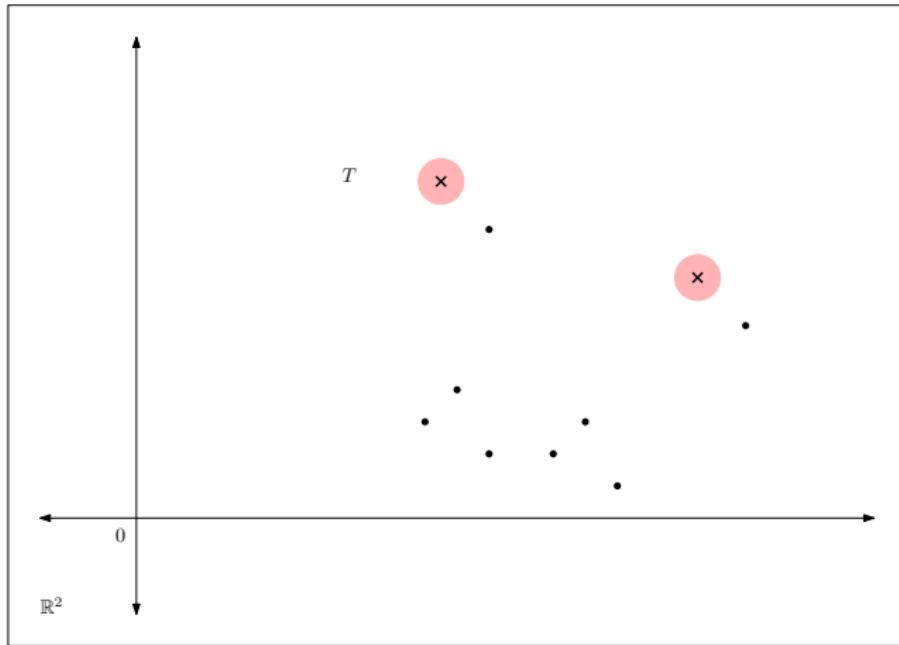
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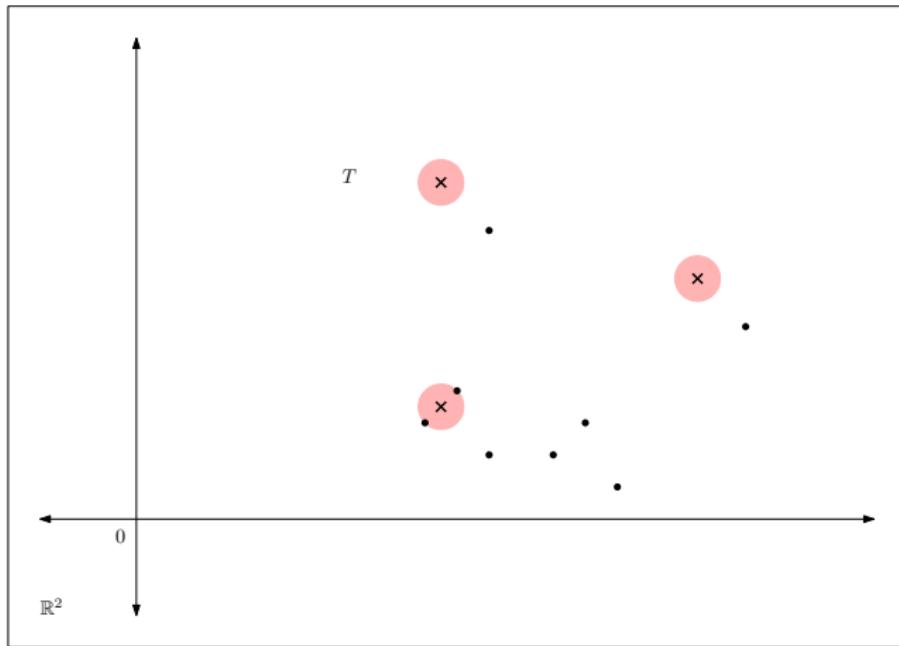
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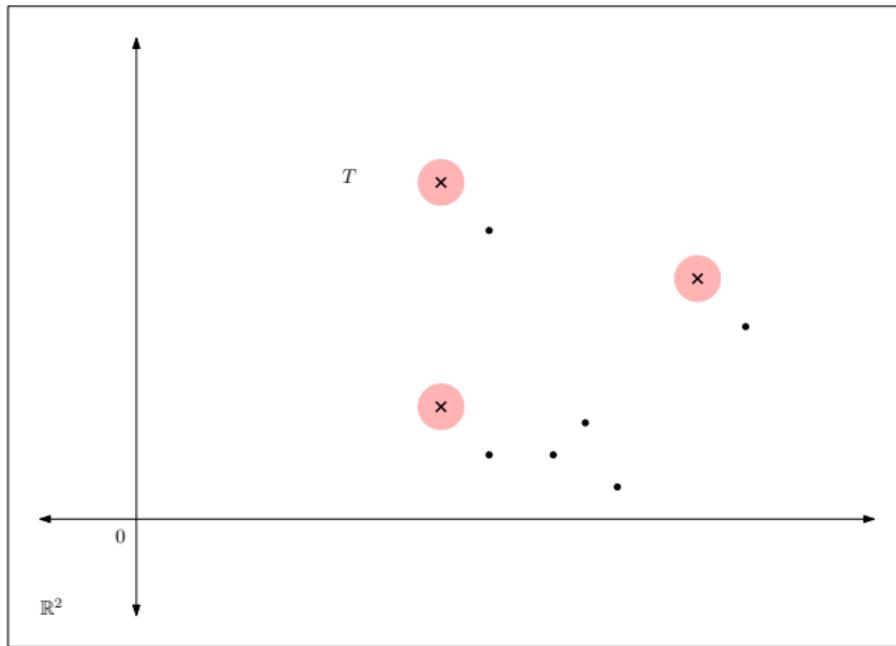
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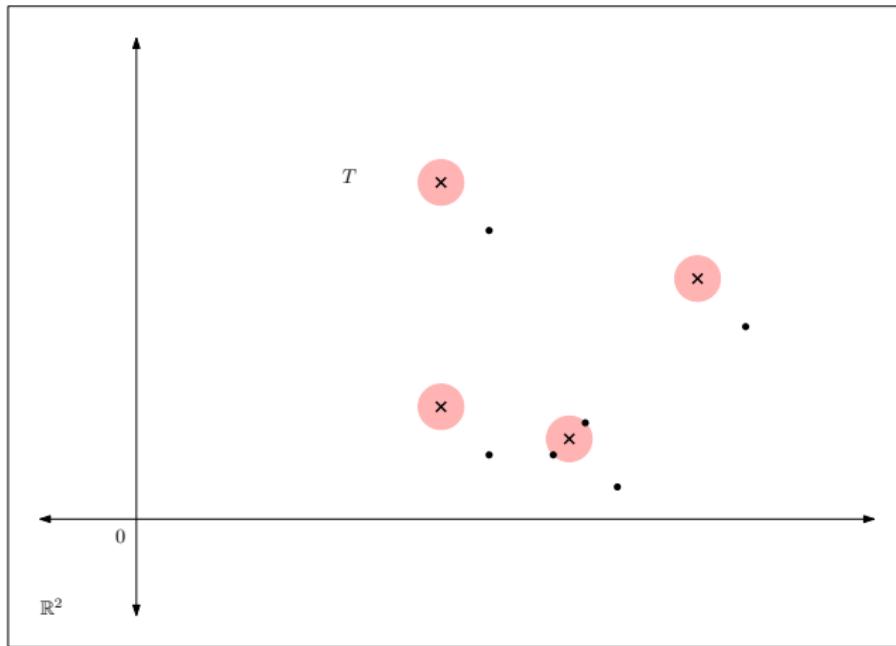
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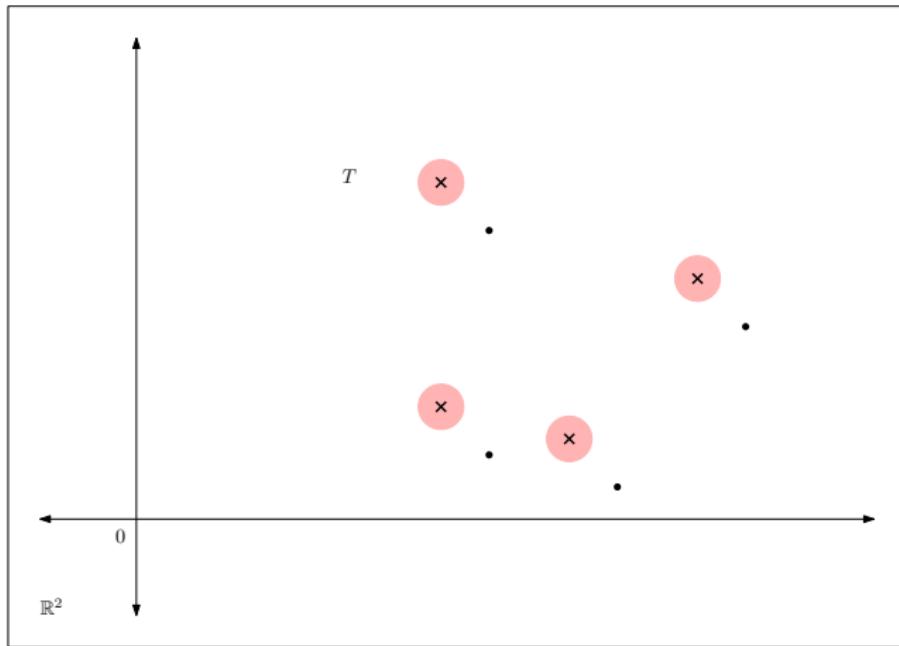
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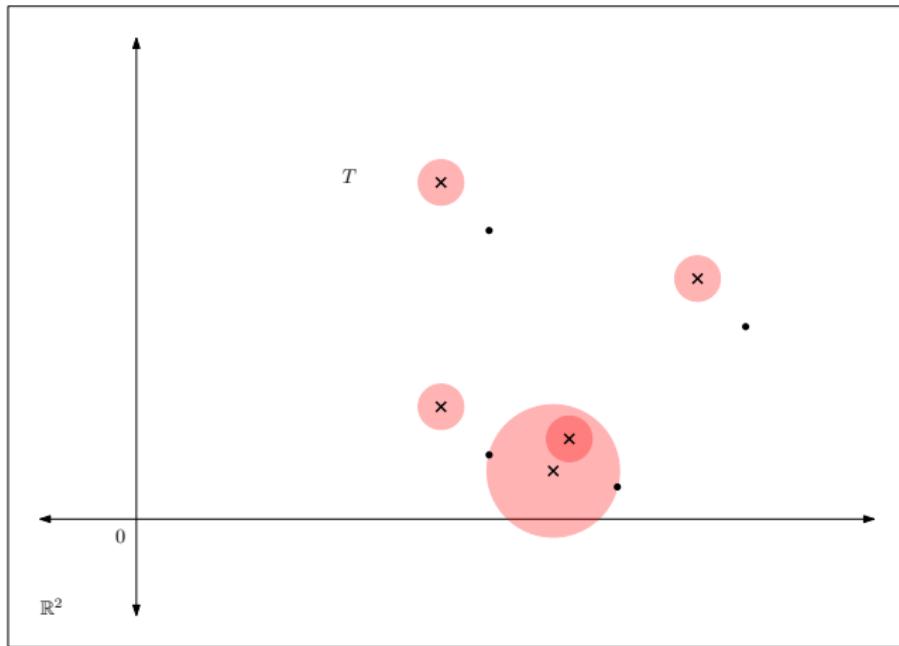
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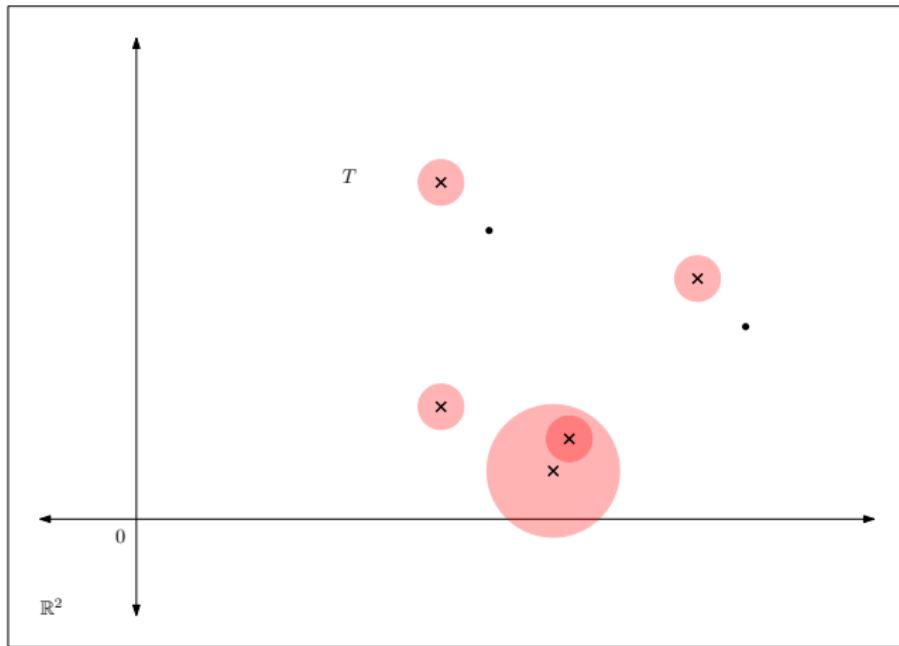
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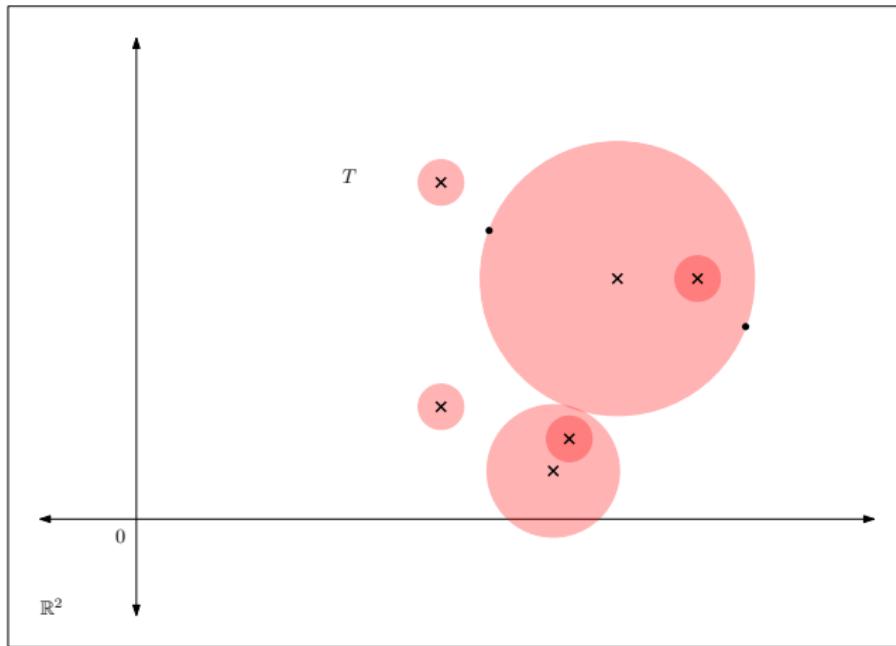
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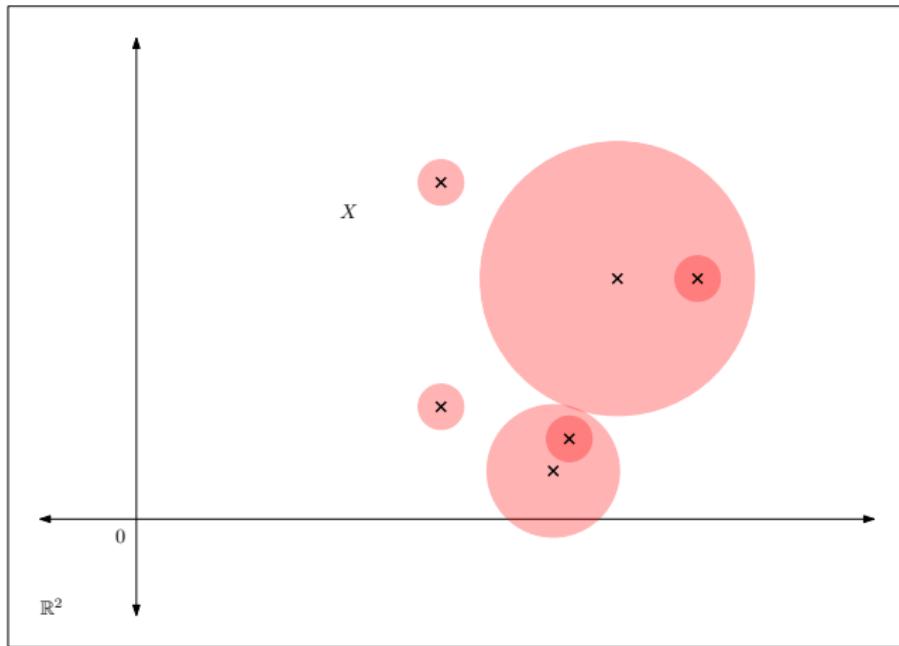
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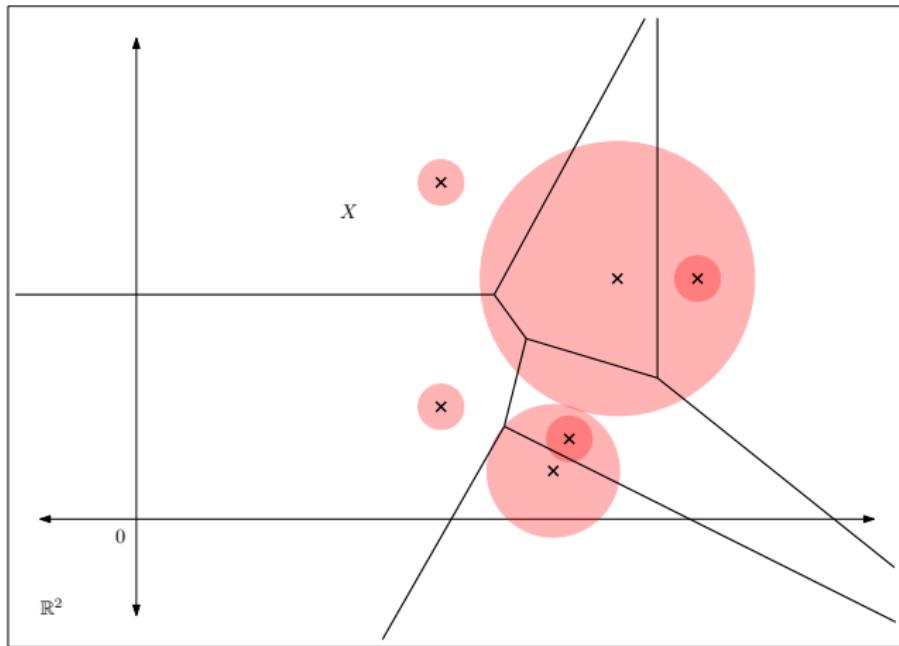
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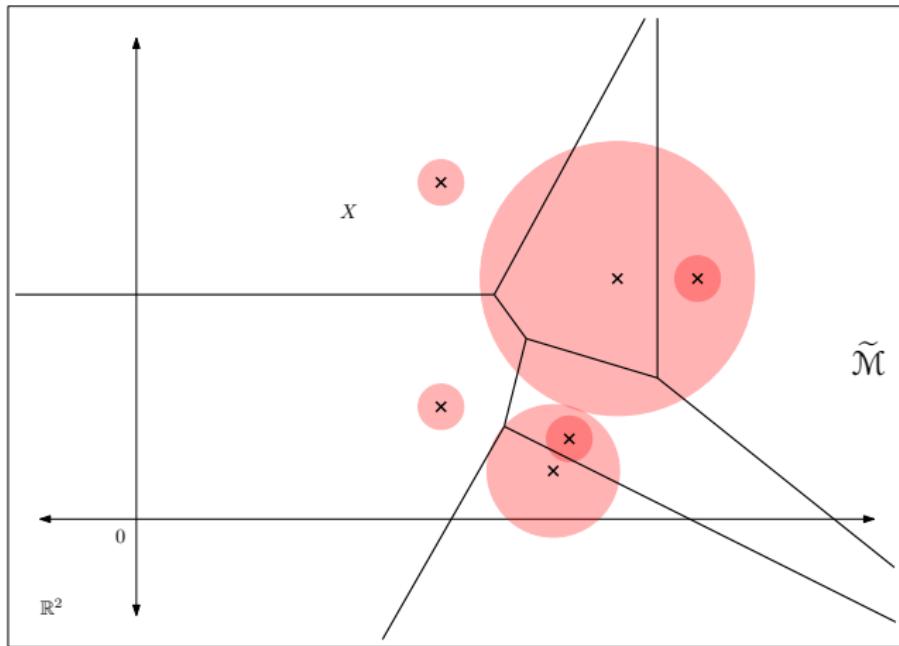
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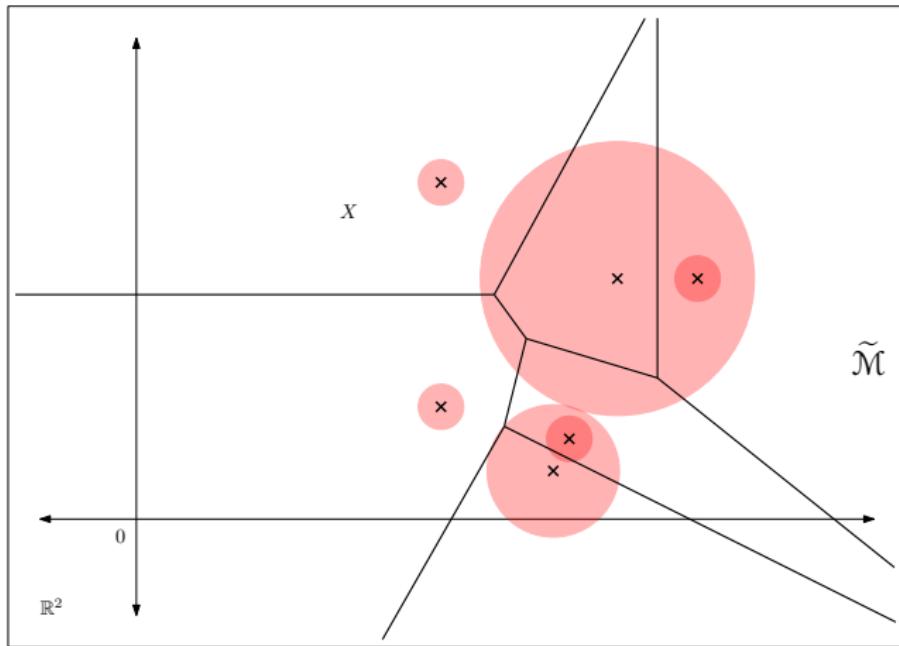
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Open questions

1. Is the complexity of \mathcal{M} polynomial?
2. Approximate diagrams for rotations? Rigid transforms?

The End

Thank you.