Approximate Minimum-Weight Partial Matching under Translation

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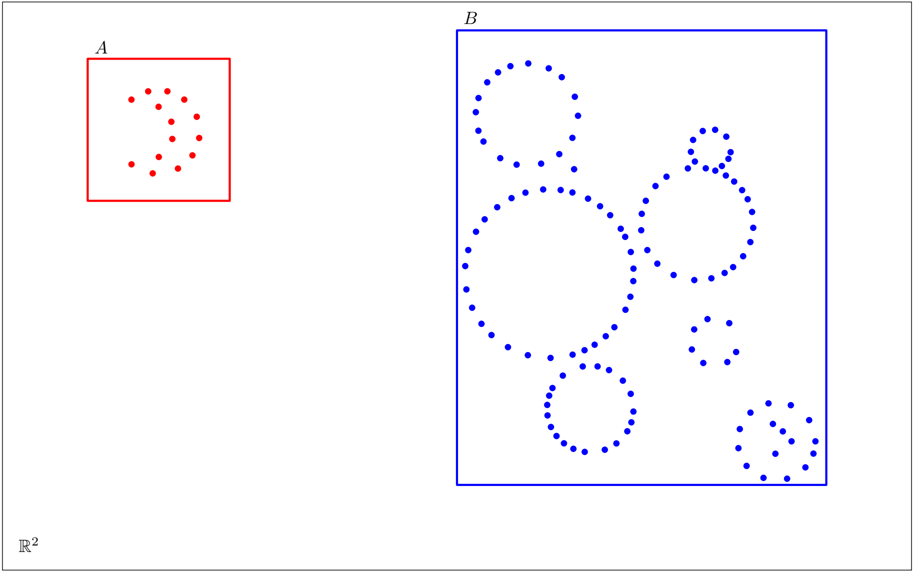
Example



- Find the minimum cost matching over all translations

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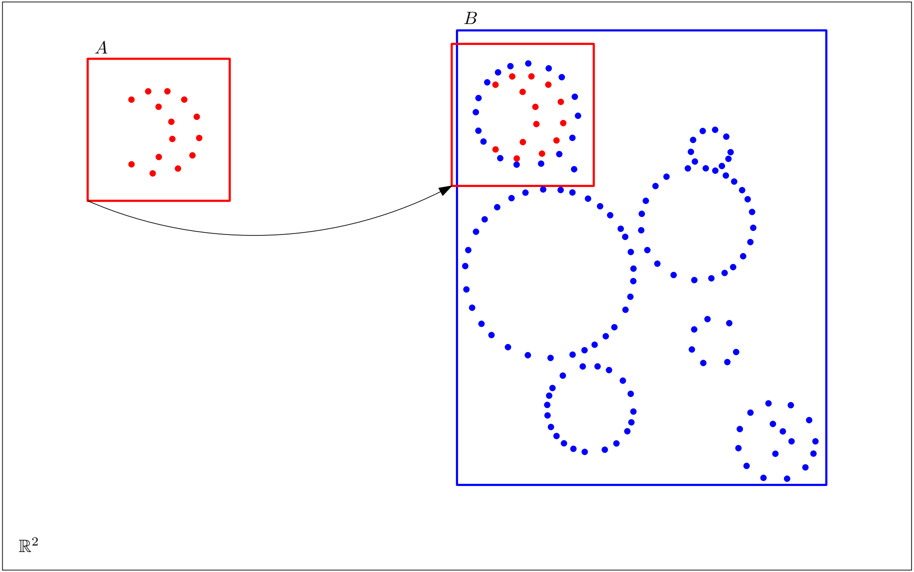

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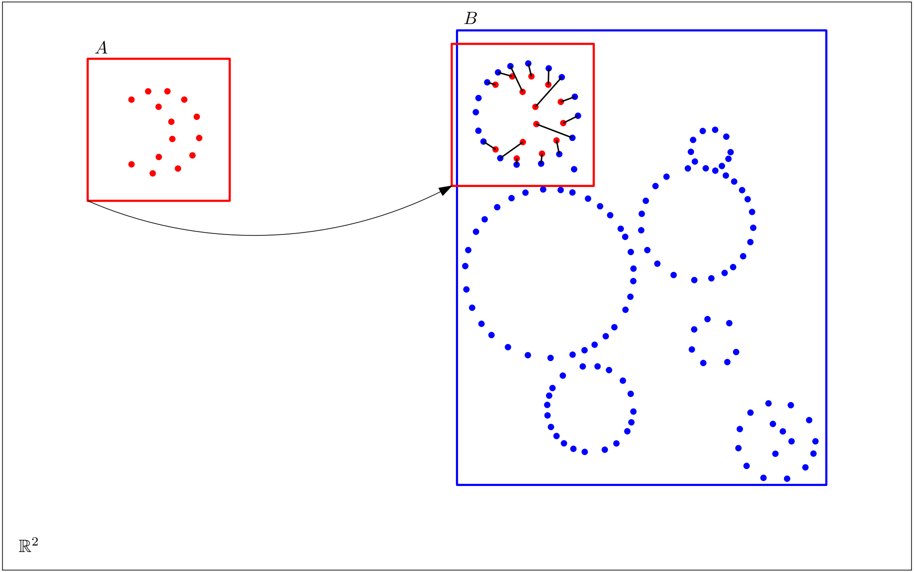

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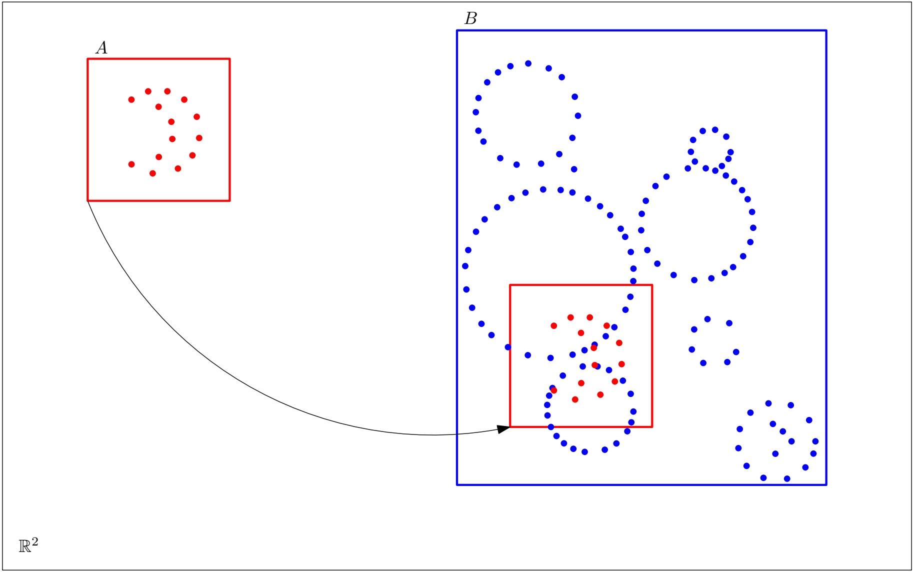

▶ Find the minimum-cost matching over all translations

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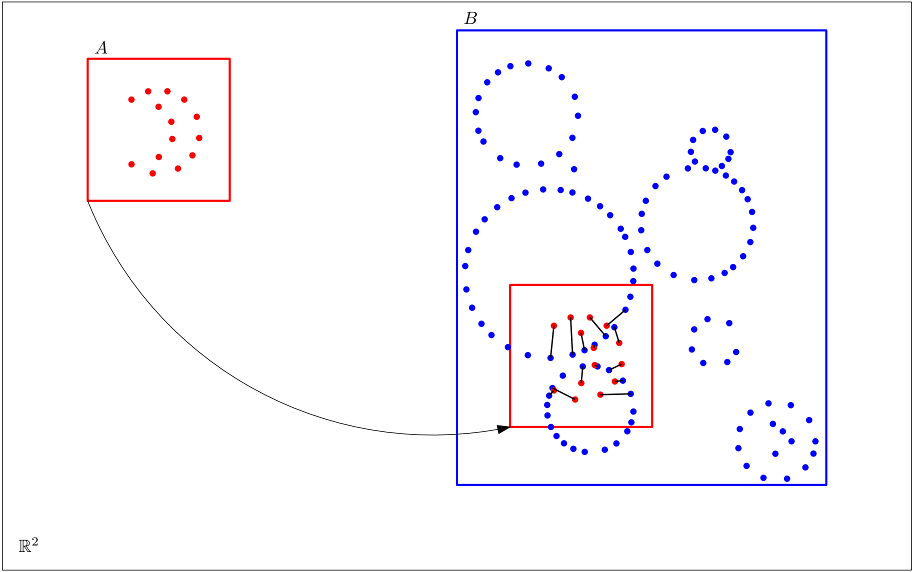

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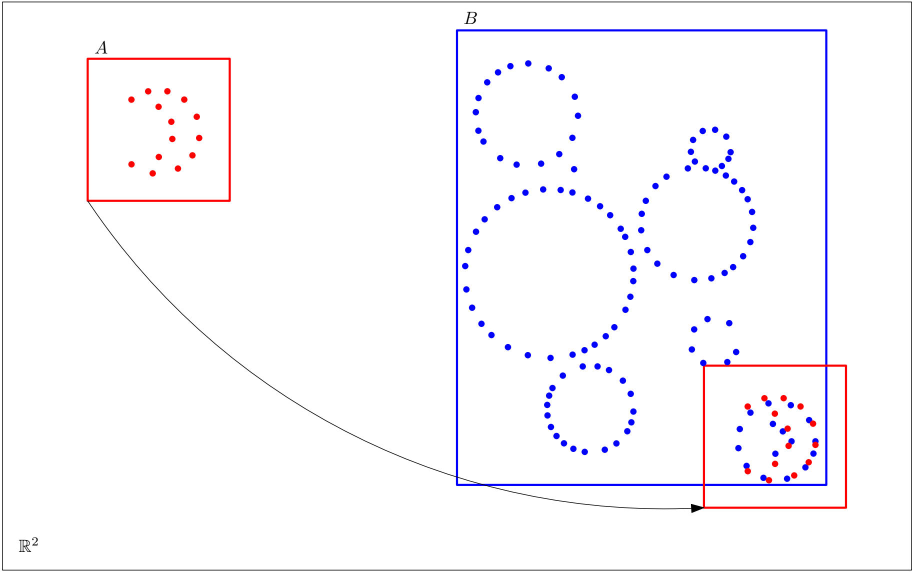

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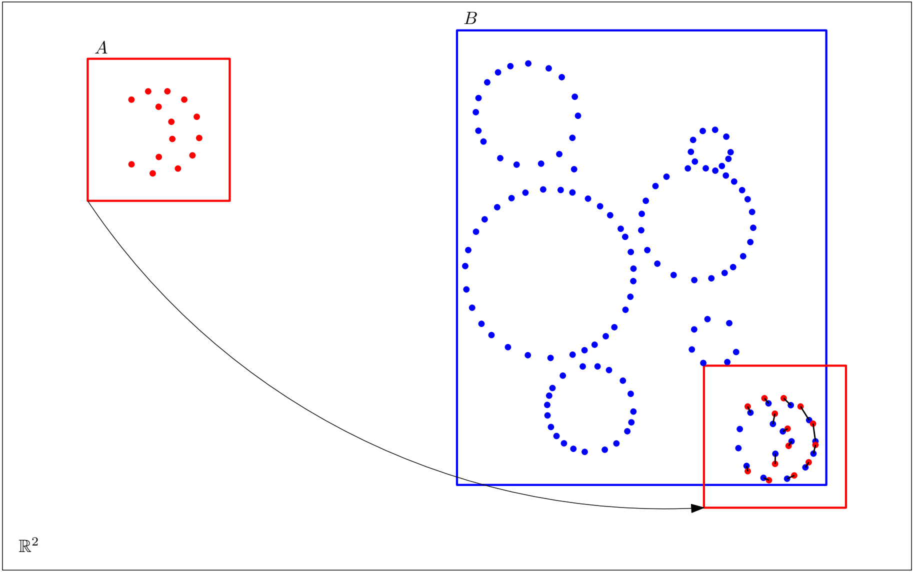

- Find the minimum cost matching over all translations.

Cost function

- Given point sets A, B, with $|A| = m$ and $|B| = n$, match A into an m-subset of B after translation t.
- For matching M and translation t, define the root-mean-sqaure (RMS) cost of the matching:

$$\operatorname{cost}(M, t) := \left[\frac{1}{m} \sum_{(a,b) \in M} \|a + t - b\|^p \right]^{1/2}$$

- Can generalize to p-th power/p-th root (versus 2), and matching of size k (versus m).
- Best matching cost at each translation:

$$\operatorname{cost}^*(t) := \min_{\text{matching } M} \operatorname{cost}(M, t)$$

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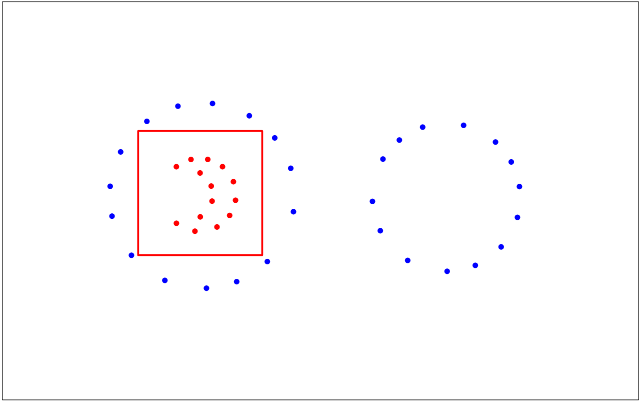

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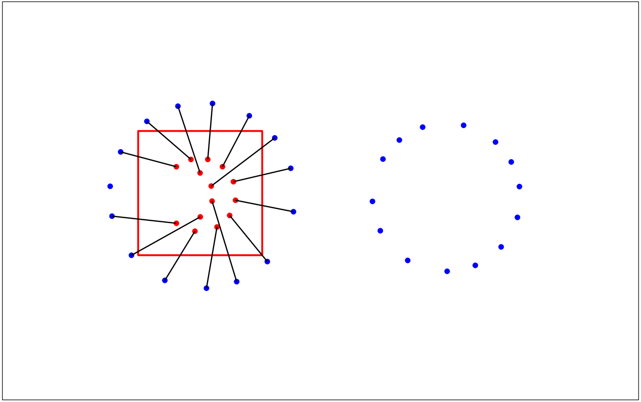

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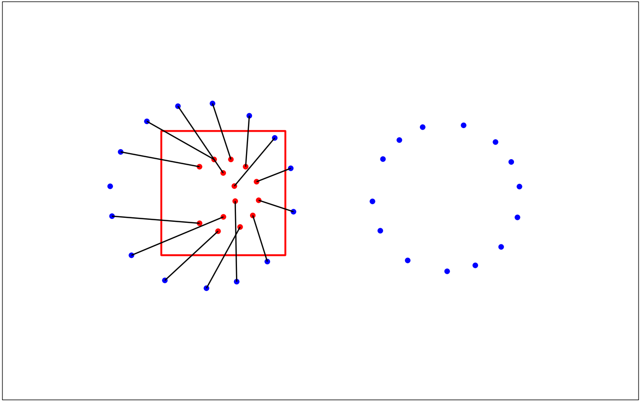

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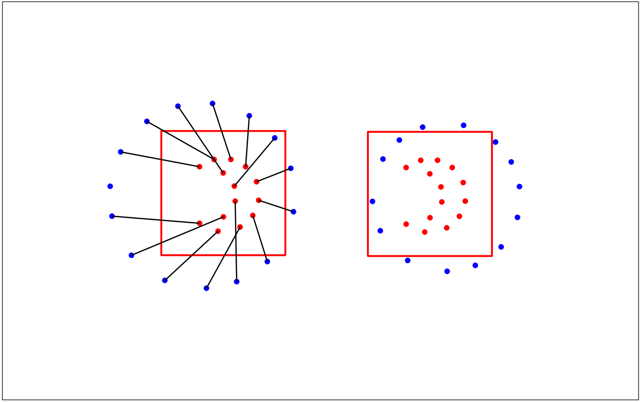

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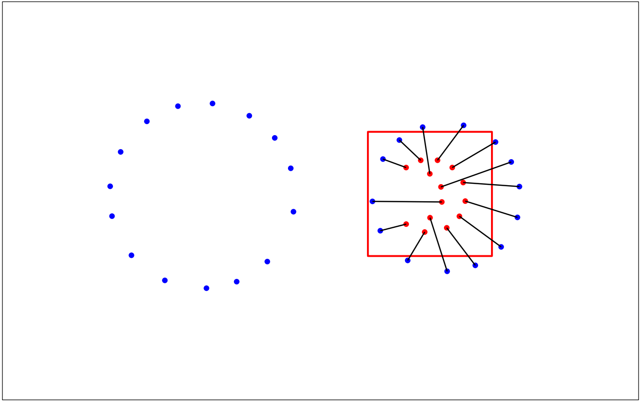

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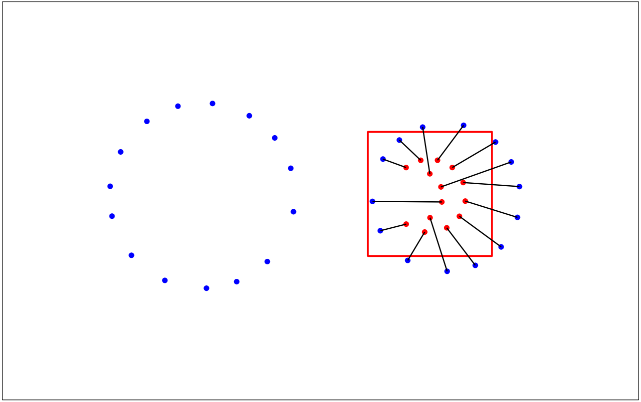

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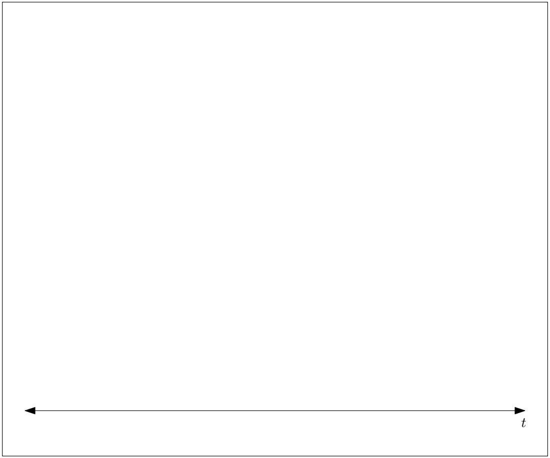

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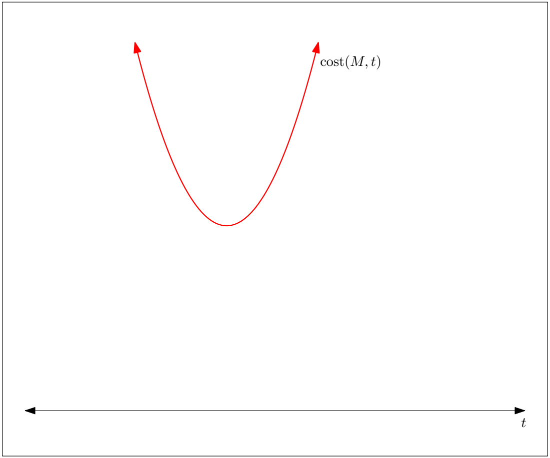

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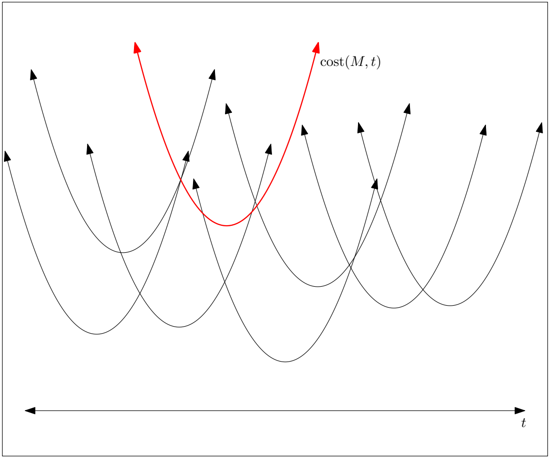

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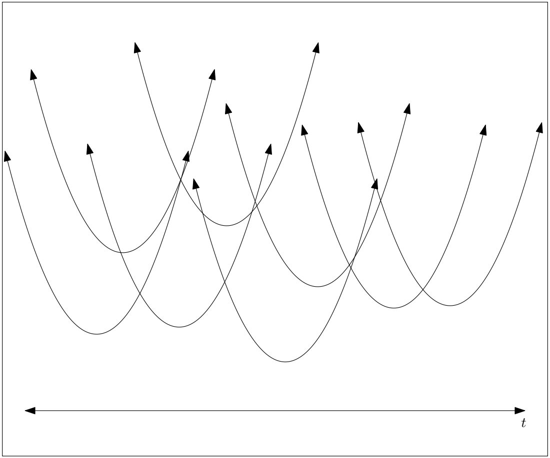

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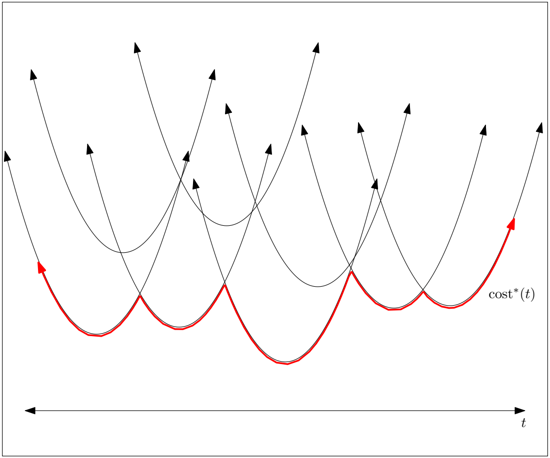

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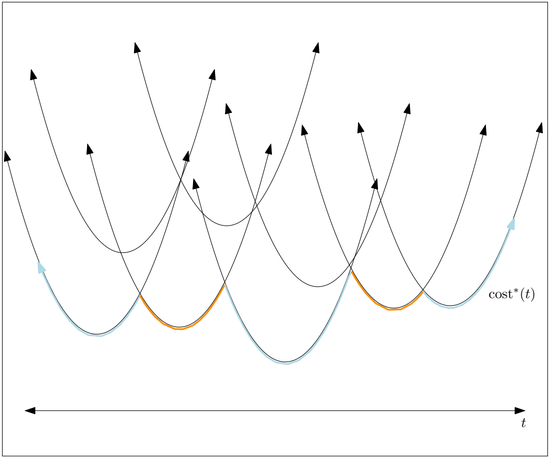

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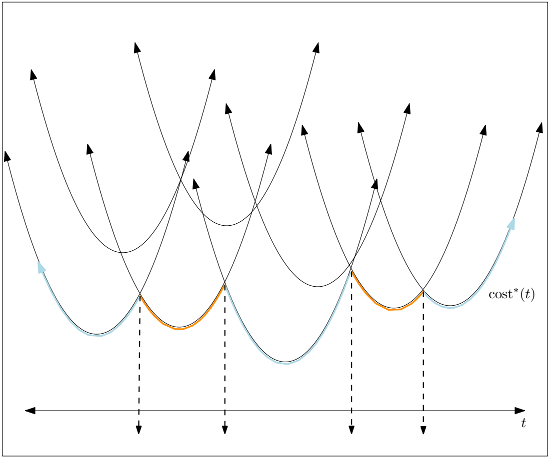

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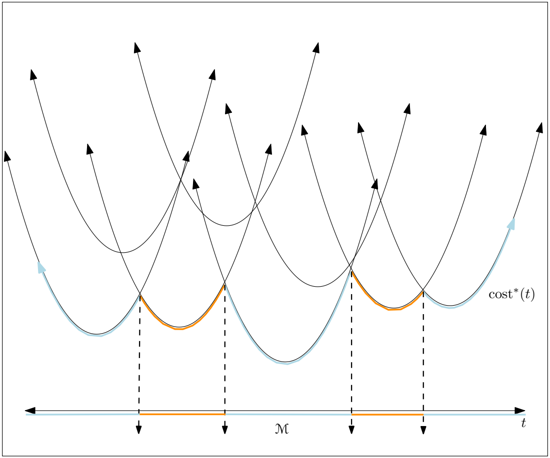

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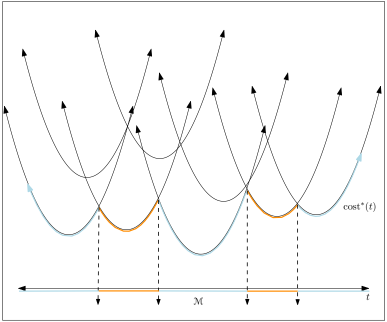

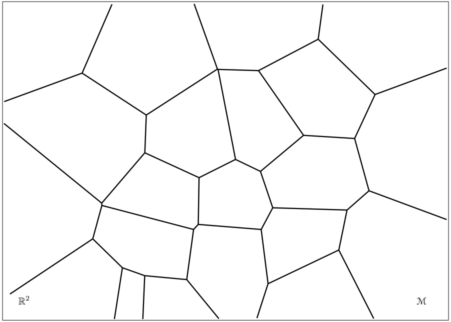

- ~~How many distinct matchings appear in $\mathrm{cost}^*(t)$?~~
- What is the combinatorial complexity of $\mathcal{M}$?

Questions and prior results

1. How quickly can we compute t^*, a global minimum of $\mathrm{cost}^*(t)$?
2. What is the combinatorial complexity of $\mathcal{M}$?
   - [Rote 10]: in 1D, at most $m(n-m)+1$.
   - [Ben-Avraham *et al.* 14]: $O(n^2 m^{3.5}(e \ln m + e)^m)$

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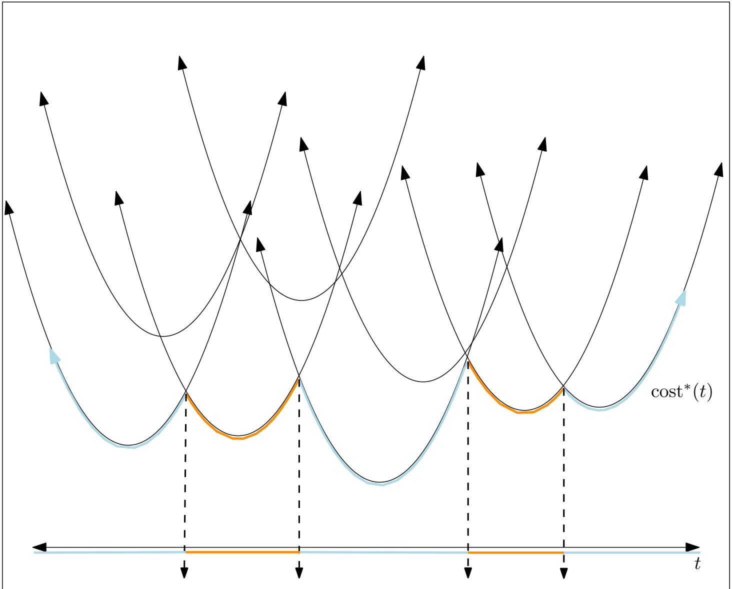

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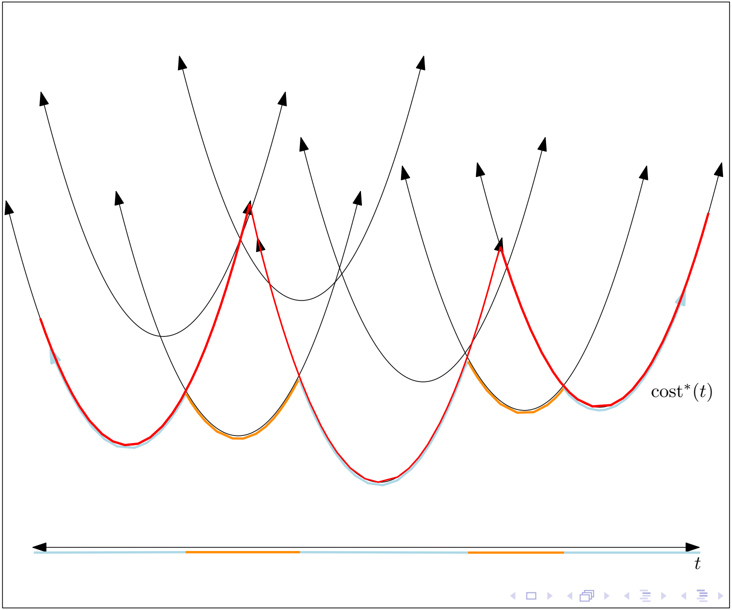

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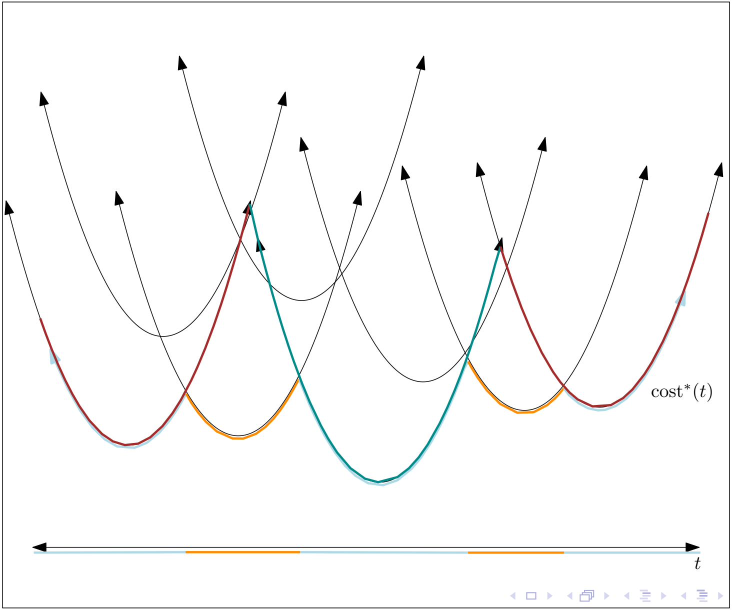

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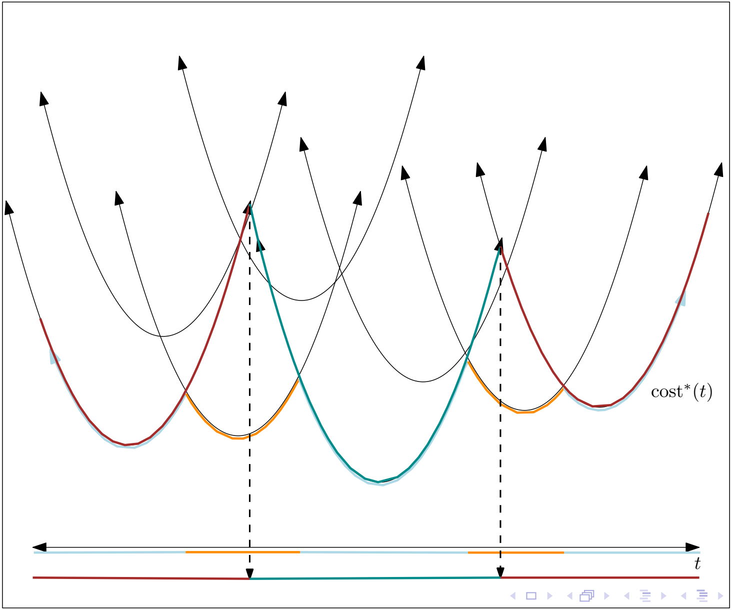

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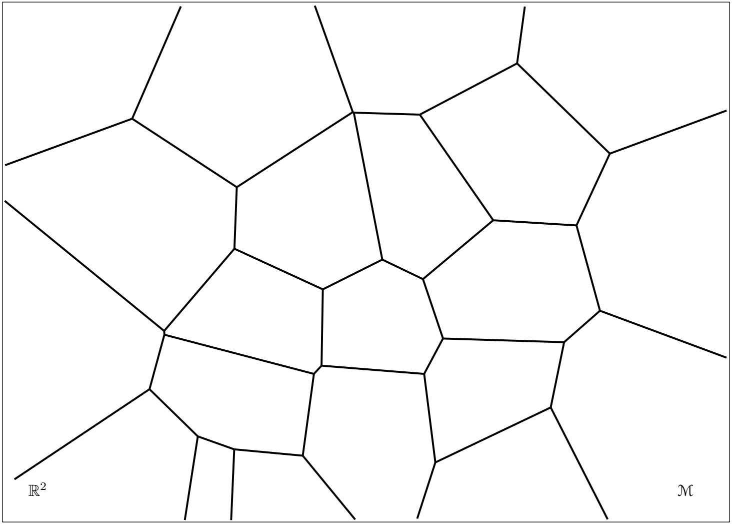

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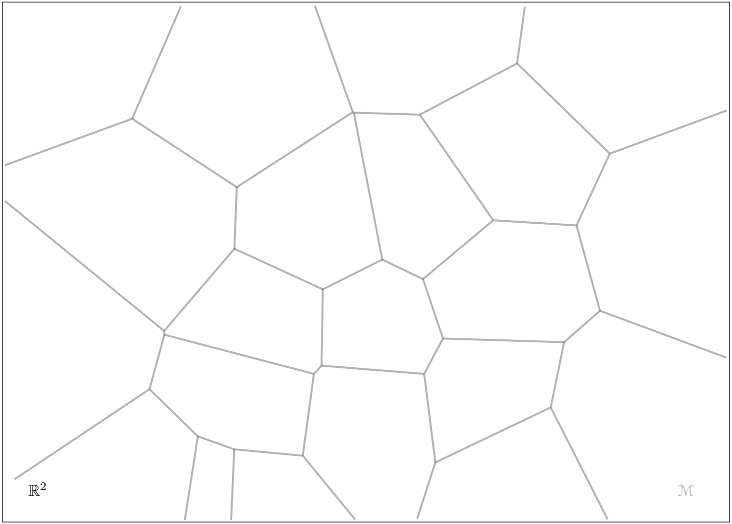

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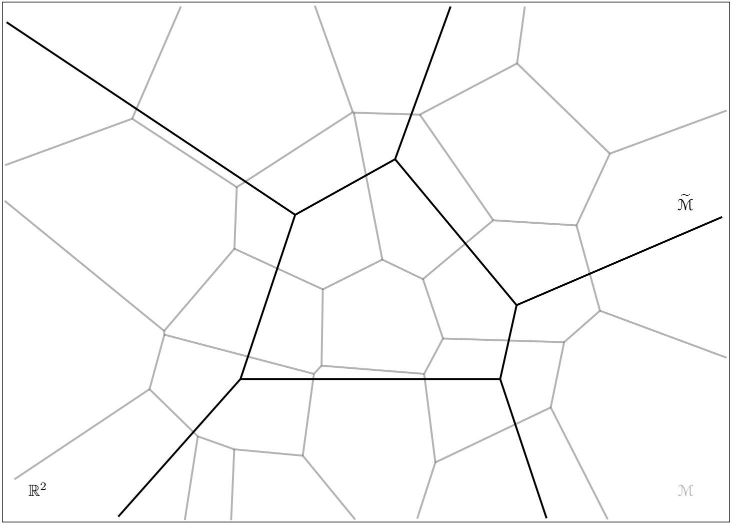

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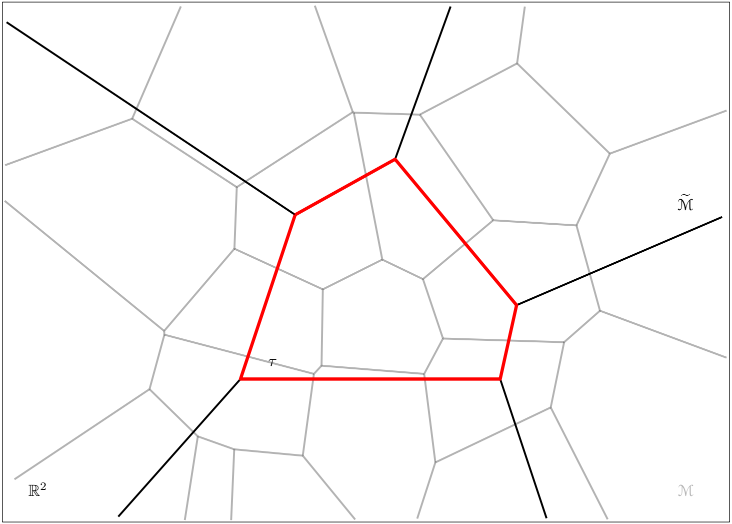

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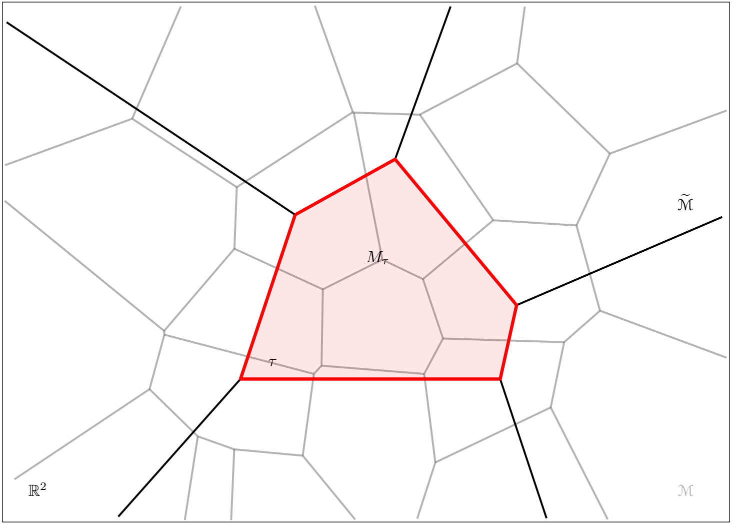

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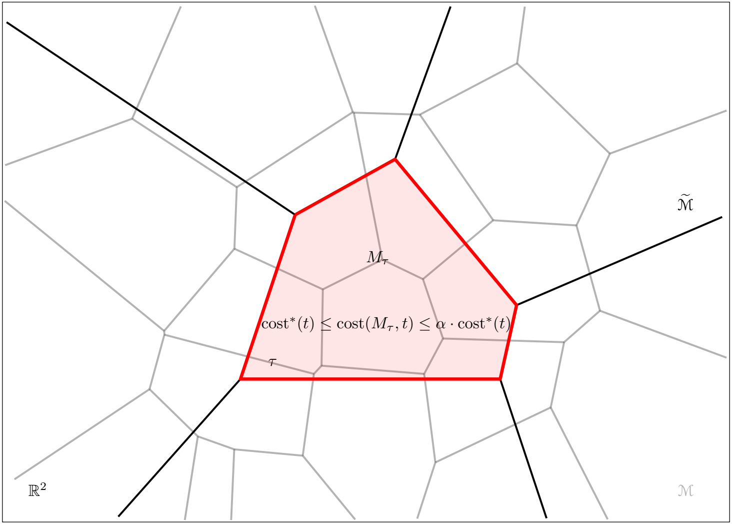

Our results (approximation helps)

1. How quickly can we compute t^*, a global minimum of $\text{cost}^*(t)$?

Theorem

In $\text{poly}(m, n, \varepsilon^{-1})$ *time, can compute a* $(1+\varepsilon)$ *approximation to* $\text{cost}^*(t^*)$ *by exploring the faces of* $\widetilde{\mathcal{M}}$.

2. What is the combinatorial complexity of $\mathcal{M}$?

Theorem

In $\text{poly}(m, n, \varepsilon^{-1})$ *time, can construct a* $(1+\varepsilon)$ *approximate diagram* $\widetilde{\mathcal{M}}$ *of complexity* $O(n\varepsilon^{-2}\log\varepsilon^{-1})$.

Overview

1. The set of point-to-point translations give a constant approximate diagram of size mn.
2. Using exponential grids, constant $\rightarrow (1+\varepsilon)$ approximation of size $O(mn\varepsilon^{-2}\log\varepsilon^{-1})$.
3. Reduce size to $O(n\varepsilon^{-2}\log\varepsilon^{-1})$ by clustering.

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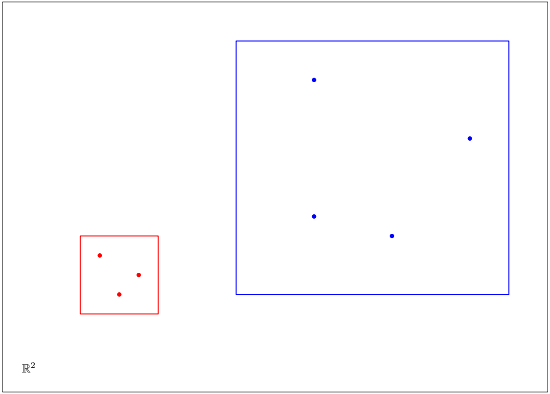
Point-to-point translations

- Point-to-point translations [Cabello *et al.* 08]:
  $T := \{t_{ba} = (b - a) \mid a \in A, b \in B\}$



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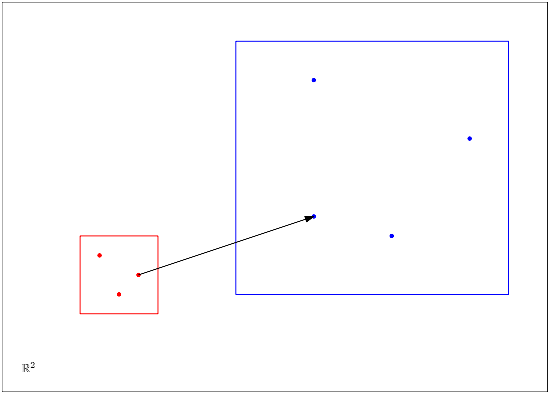

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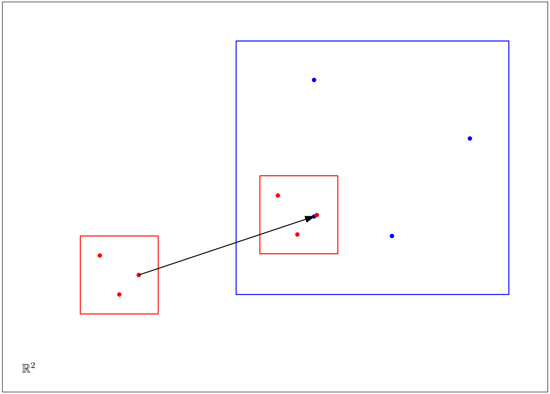

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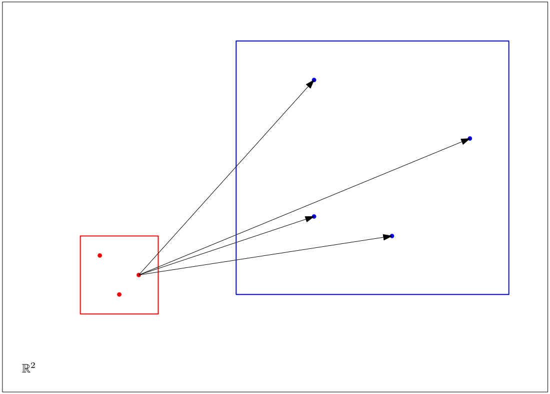

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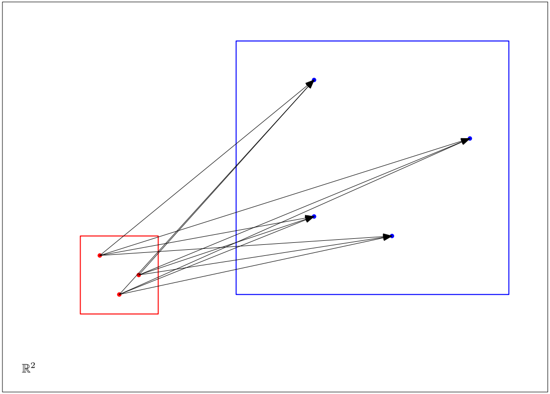

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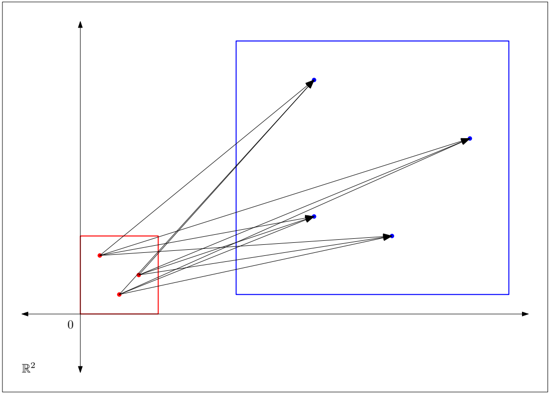

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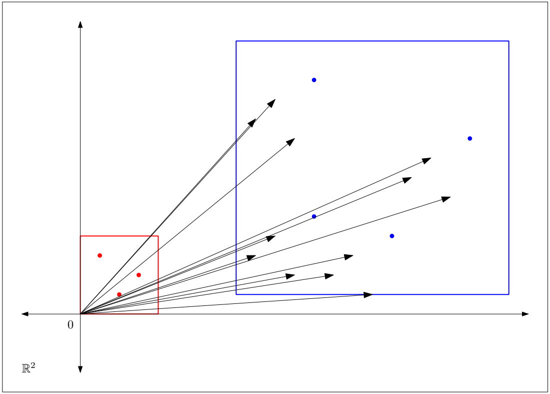

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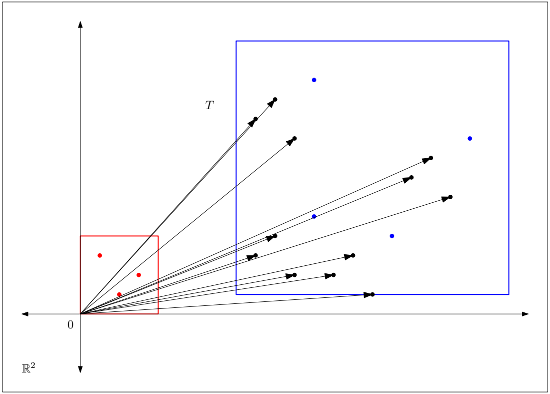

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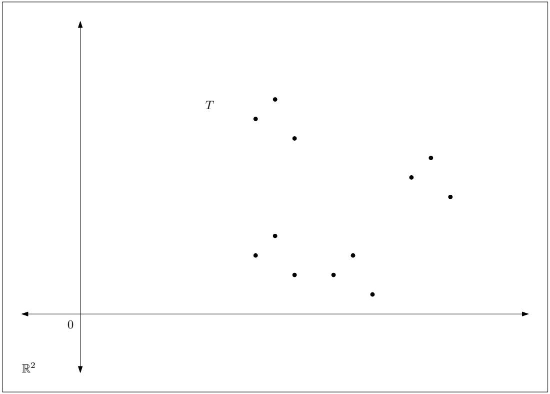

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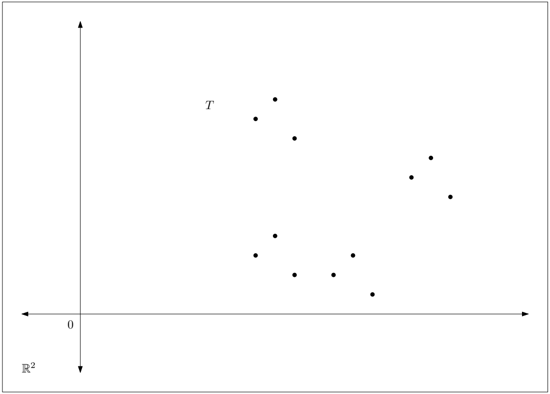

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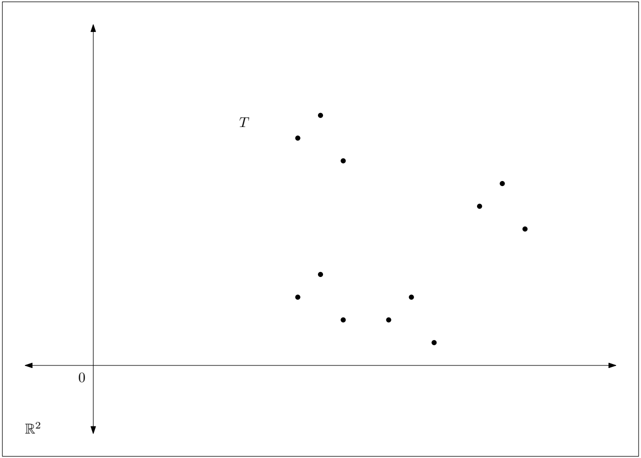
$O(1)$-approximate diagram from point-to-point translations



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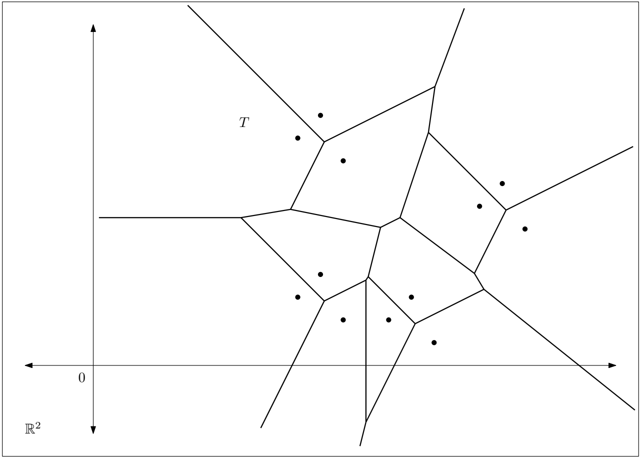

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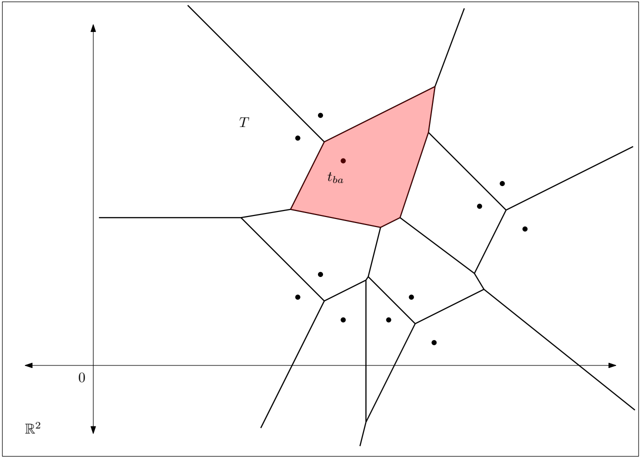

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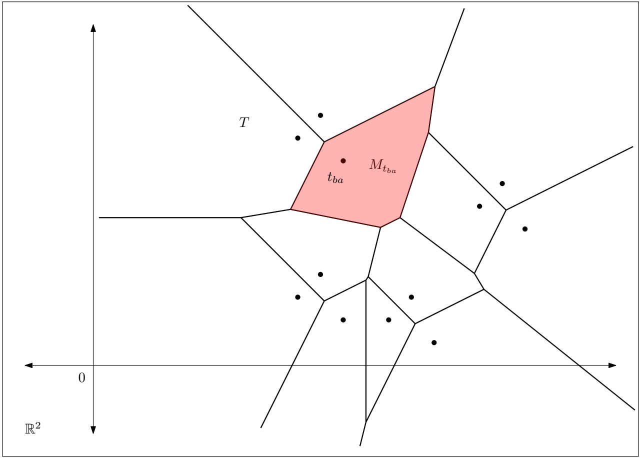

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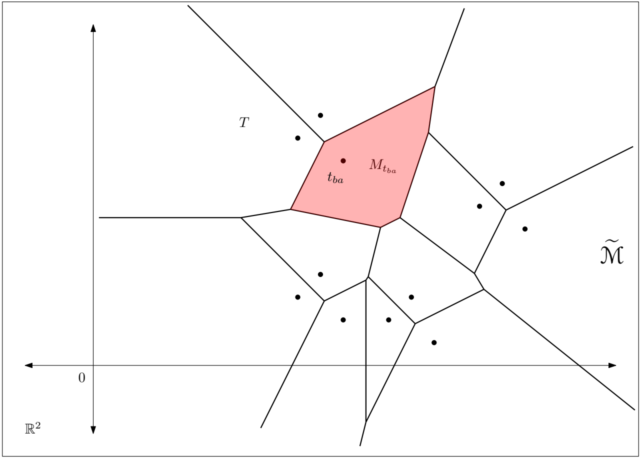

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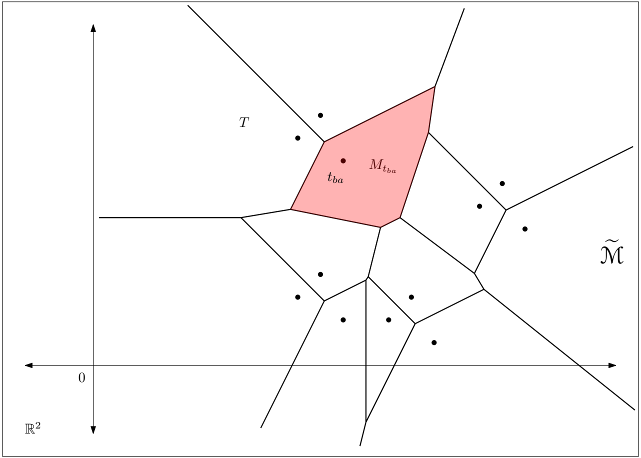

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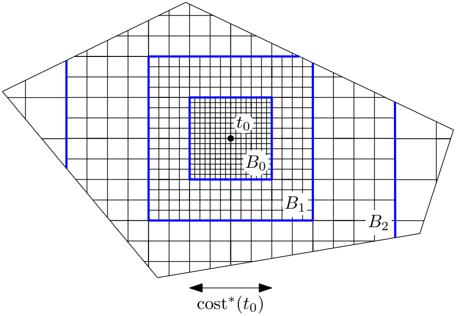
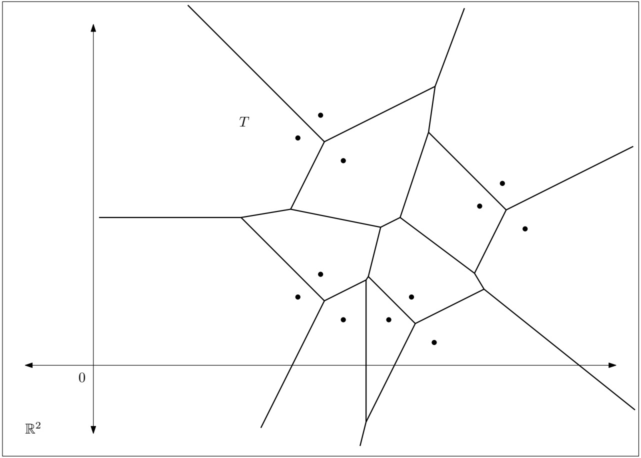
$O(1) \to (1+\varepsilon)$ approximation



- $(1+\varepsilon)$ approximate diagram of size $O(|T|\varepsilon^{-2}\log\varepsilon^{-1}) = O((mn)\varepsilon^{-2}\log\varepsilon^{-1})$

Near-linear size: clustering T

- Any $O(1)$ approx. diagram is enough for the $(1+\varepsilon)$ diagram.
- Partition T into clusters of size at least $m/2$.



- Reduces diagram size from $mn \to O(n)$.

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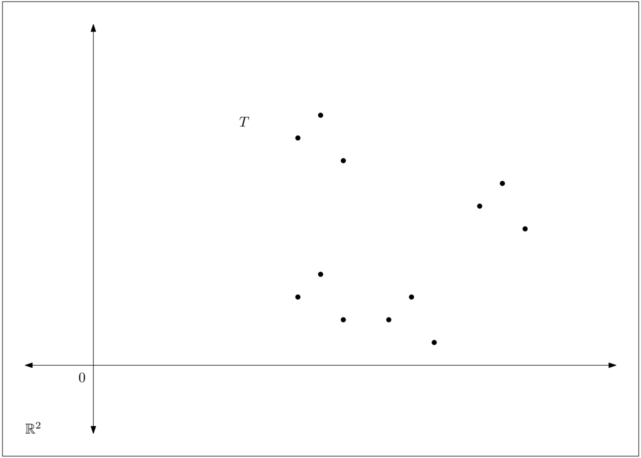

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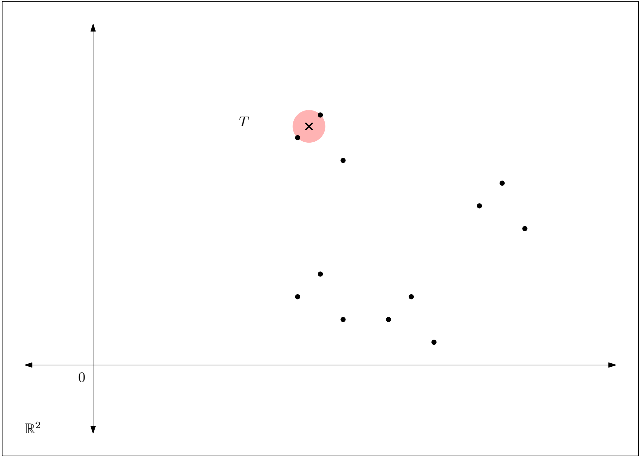

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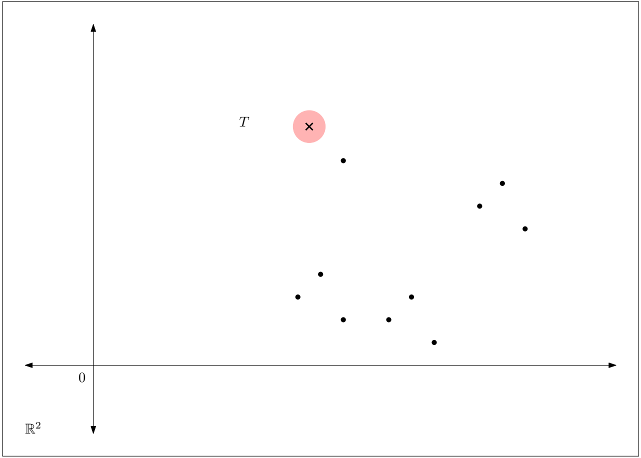

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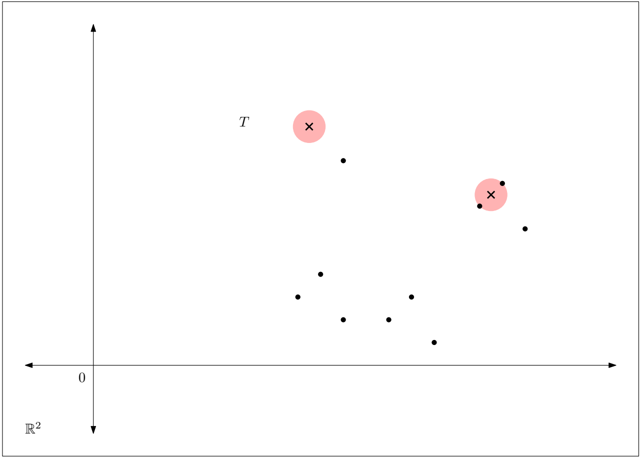

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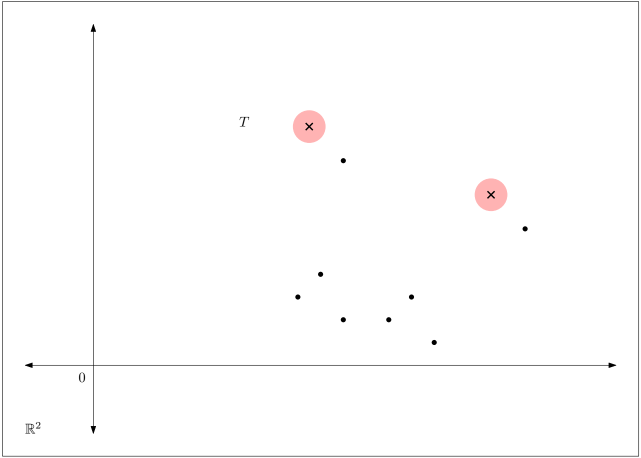

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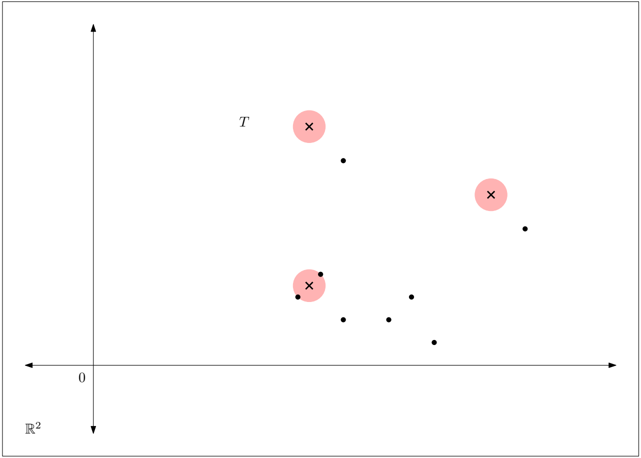

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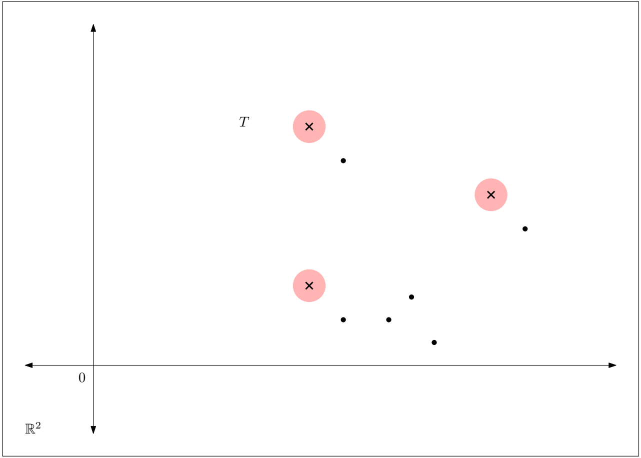

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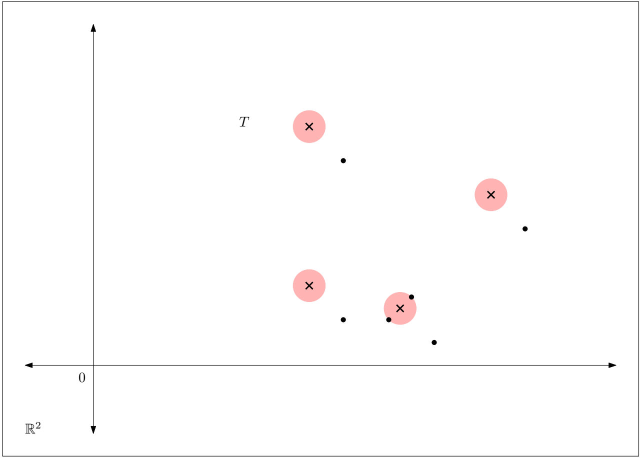

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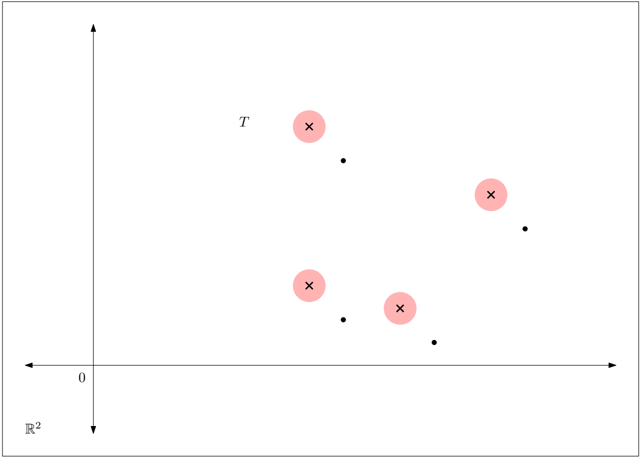

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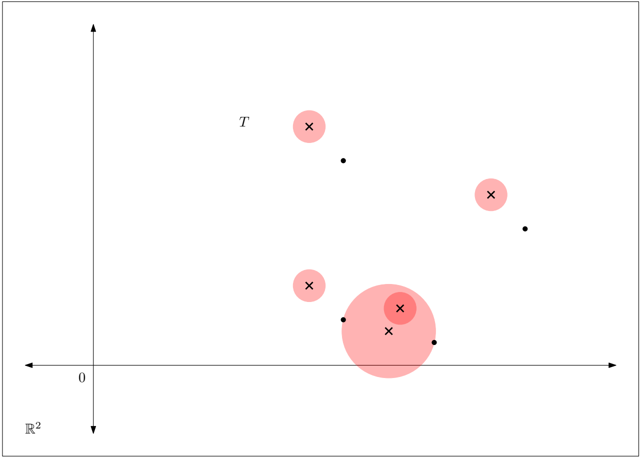

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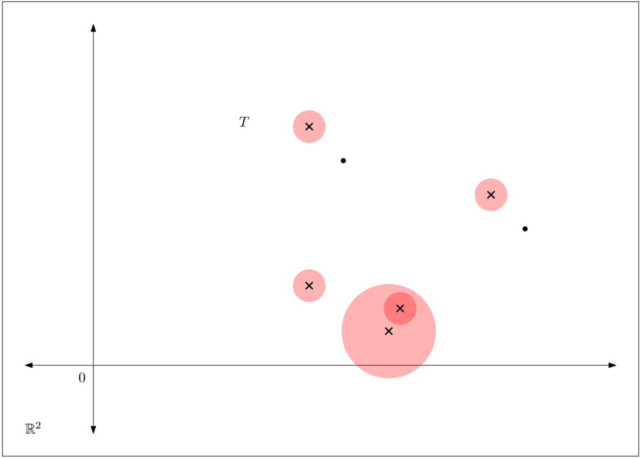

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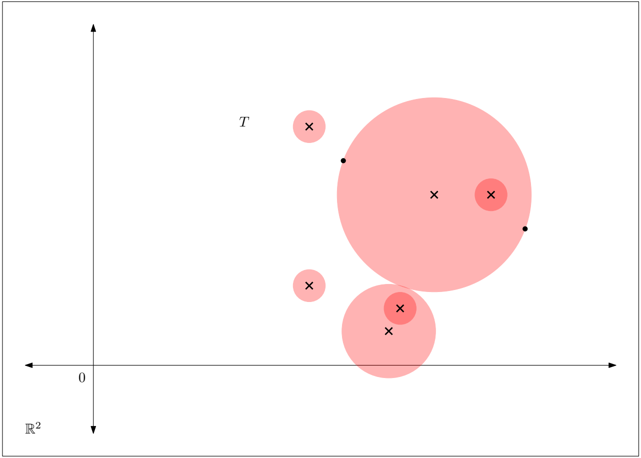

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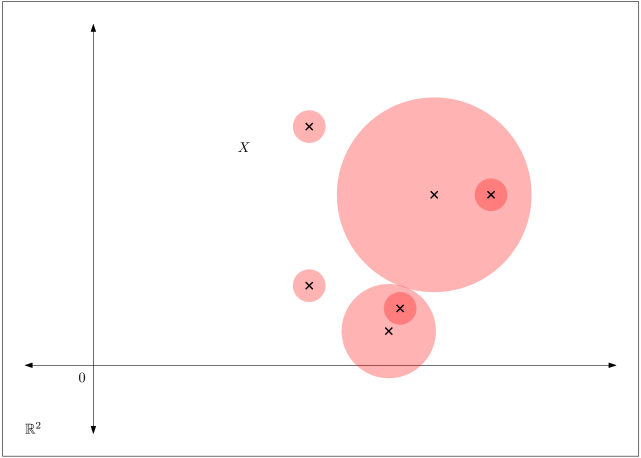

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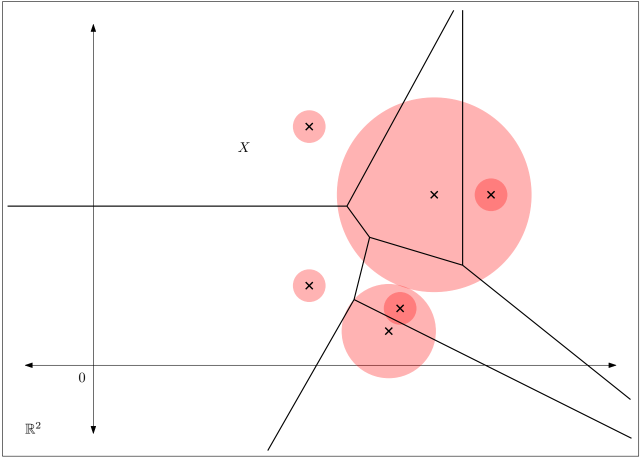

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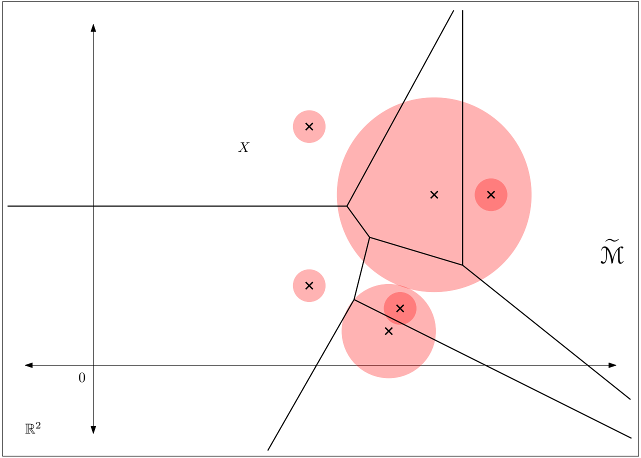

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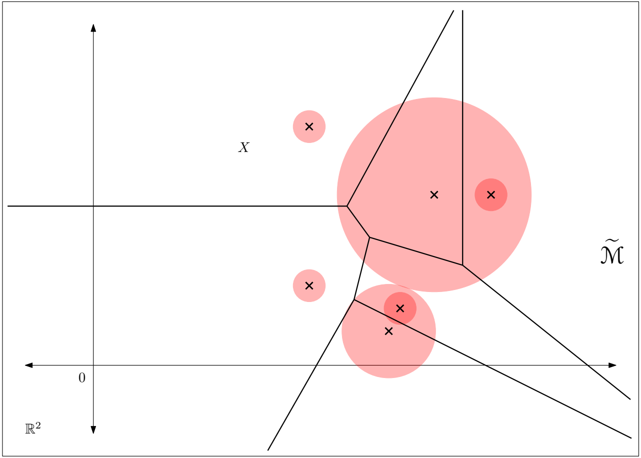

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Open questions

1. Is the complexity of $\mathcal{M}$ polynomial?
2. Approximate diagrams for rotations? Rigid transforms?

The End

Thank you.