

Geometric Partial Matchings and Unbalanced Transportation Problem

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Abstract

Let A and B be two point sets in the plane with uneven sizes r and n respectively (assuming r is at most n), and let k be a parameter. The geometric partial matching problem asks to find the minimum-cost size- k matching between A and B under powers of L_p distances. Applying combinatorial algorithms for partial matching in general graphs to our setting naïvely requires quadratic time due to existence of many edges between point sets A and B . Most previous work for geometric matching has focused on the setting when k , r , and n are all equal. The best algorithm in this setting, due to Sharathkumar and Agarwal [STOC 2012], runs in time $O(n \text{ polylog } n \cdot \text{poly } \varepsilon^{-1})$, but is limited to matching objectives that are sum-of-distances.

We present the first set of geometric algorithms which work for any powers of L_p -norm matching objective: An exact algorithm which runs in $O((n + k^2) \text{ polylog } n)$ time, and a $(1 + \varepsilon)$ -approximation which runs in $O((n + k\sqrt{k}) \text{ polylog } n \cdot \log \varepsilon^{-1})$ time. Both algorithms are based on primal-dual flow augmentation scheme; the main improvements are obtained by using dynamic data structures to achieve efficient flow augmentations. Using similar techniques, we give an exact algorithm for the planar transportation problem, which runs in $O(rn^{3/2} \text{ polylog } n)$ time. This is the first sub-quadratic time exact algorithm when $r = o(\sqrt{n})$, which improves over the state-of-art quadratic time algorithm by Agarwal *et al.* [SOCG 2016].

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1 Introduction

Given two point sets A, B in the plane, we consider the problem of finding the minimum-cost size- k matching between A and B . Formally, suppose $|A| = r$ and $|B| = n$, with $r \leq n$. Let $G(A, B)$ be the undirected complete bipartite graph between A and B , and let the cost of $(a, b) \in A \times B$ be $c(a, b) = \|a - b\|_p^q$, for some $1 \leq p < \infty$ and $q \geq 1$. Define $C := \max_{(a,b) \in A \times B} c(a, b)$. A *matching* M in $G(A, B)$ is a set of edges sharing no endpoints, and the cost of M is

$$\text{cost}(M) = \sum_{(a,b) \in A \times B} \|a - b\|_p^q.$$



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For a parameter $k \leq r$, we call the problem of finding the minimum-cost size- k matching in $G(A, B)$ the *geometric partial matching* problem. We call the respective problem in general bipartite graphs (with arbitrary edge costs) the *partial matching* problem. This has also been called the *imperfect matching* problem.

The second problem we consider is a vertex-weighted variant of bipartite matching called the *transportation problem*. Let $\lambda : A \cup B \rightarrow \mathbb{Z}$ be a *supply-demand function* with positive value on points of A , negative value on points of B , satisfying $\sum_{a \in A} \lambda(a) = -\sum_{b \in B} \lambda(b)$. Define $U := \max_{p \in A \cup B} |\lambda(p)|$. A *transportation map* is a function $\tau : A \times B \rightarrow \mathbb{R}_{\geq 0}$. A transportation map τ is *feasible* if $\sum_{b \in B} \tau(a, b) = \lambda(a)$ for all $a \in A$, and $\sum_{a \in A} \tau(a, b) = -\lambda(b)$ for all $b \in B$. In other words, the value $\tau(a, b)$ describes how much supply at a should be sent to meet demands at b , and τ is feasible if all supplies are sent and all demands are met. We define the cost of τ to be

$$\text{cost}(\tau) := \sum_{(a,b) \in A \times B} \|a - b\|_p^q \cdot \tau(a, b).$$

The *transportation problem* asks to find a feasible transportation map of minimum cost. The cost itself is called the *optimal transport cost* or the *earth mover's distance*. We focus on the setting where $r < n$, i.e. the point sets are *unbalanced* (although the total supply and demand are balanced).

1.1 Related work

Non-geometric algorithms for partial matching can be applied to $G(A, B)$, which is a graph with $r + n$ vertices and $m := rn$ edges. There is a wealth of literature on general min-cost bipartite matching between *balanced* vertex sets ($r = n$) finding a *perfect matching* ($k = r = n$). Not all algorithms for perfect matching can be easily modified to solve partial matching, with a notable exception being the celebrated primal-dual *Hungarian Algorithm* [12]. The Hungarian Algorithm can be “stopped after k iterations” to solve partial matching in $O(km + k^2 \log r) = O(knr + k^2 \log r)$ time, see e.g. [14]. Ramshaw and Tarjan [14] give a cost-scaling algorithm for partial matching which runs in time $O(m\sqrt{n} \log(nC)) = O(n^{3/2}r \log(nC))$, but only when costs are integral (for distances, they generally are not). By reduction to unit-capacity min-cost flow, Goldberg *et al.* [9] give a cost-scaling algorithm for partial matching which runs in $O(m\sqrt{k} \log(kC)) = O(nr\sqrt{k} \log(kC))$ time, again only for integral costs.

Similarly, many geometric perfect matching algorithms do not convert easily into geometric partial matching algorithms. However, the $(1 + \varepsilon)$ -approximation algorithm due to Sharathkumar and Agarwal [16] can be modified to solve partial matching by “stopping at the k -th iteration,” to solve geometric partial matching in $O(n \text{ poly}(\log n, 1/\varepsilon))$ time, when costs are the p -norm. We note that these costs are less general than the setting in this paper (“ q -th power of the p -norm”). For instance, Sharathkumar-Agarwal does not support squared Euclidean costs, i.e. *root mean squared* (RMS) matching costs.

The transportation problem can similarly be formulated as a *minimum-cost flow* problem on $G(A, B)$. Thus, the strongly polynomial uncapacitated min-cost flow algorithm by Orlin [13] solves transportation in $O((m + n \log n)n \log n) = O(n^3 \log n + n^2 \log^2 n)$ time. Lee and Sidford give a weakly polynomial algorithm which runs in $O(m\sqrt{n} \text{ polylog}(n, U)) = O(n^{3/2}r \text{ polylog}(n, U))$ time.

Another line of approximation algorithms for transportation use an entropy regularizer to pose it as a *matrix scaling problem*, following the work of Cuturi [5]. The regularized problem admits a simple iterative algorithm called the *Sinkhorn-Knopp algorithm*. Theoretical

bounds for the Sinkhorn-Knopp algorithm have emerged only recently: Sinkhorn-Knopp finds an additive ε -approximation in time $O(\frac{n^2}{\varepsilon^2} \text{polylog } n)$ due first to Altschuler *et al.* [3] and improved by Dvurechensky *et al.* [6], under the very general setting of *nonnegative cost matrices*. For costs $\|\cdot\|_2^2$ (RMS costs) Altschuler *et al.* [2] are able to modify the additive approximation to run in $O(\frac{n}{\varepsilon}(\log n/\varepsilon)^d)$ time, for points in dimension d .

In terms of geometric specialty algorithms, Agarwal *et al.* [1] give an expected $O(\log^2(1/\varepsilon))$ approximation running in $O(n^{1+\varepsilon})$ expected time, an $(1+\varepsilon)$ -approximation algorithm running in $O(n^{3/2} \text{polylog}(n, U))$ time, and an exact $O(n^2 \text{polylog } n)$ time algorithm. Whether a near-linear time algorithm exists for $(1+\varepsilon)$ -approximating the transportation problem with geometric costs is still open.

1.2 Contributions

We present two algorithms for geometric partial matching that are based on fitting nearest-neighbor (NN) and geometric closest pair (BCP) oracles into primal-dual algorithms for non-geometric bipartite matching and minimum-cost flow. This pattern is not new, see for example [1, 4, 15, 17], but allows us to push the analysis further elsewhere.

First in Section 2, we show that the Hungarian algorithm [12] combined with a BCP oracle solves geometric partial matching exactly in time $O((n+k^2) \text{polylog } n)$. Our main technique is something we call a *rewinding mechanism* to quickly initialize the (size $O(n)$) data structures in each iteration. Roughly, the initial data structures for two consecutive iterations differ by only one point, so we can generate the new initialization by “undoing” the sequence of insertions/deletions from the last iteration and performing one additional update operation.

► **Theorem 1.1.** *A minimum-cost geometric partial matching of size k can be computed between A and B in $O((n+k^2) \text{polylog } n)$ time.*

Next in Section 3, we apply a similar technique to the unit-capacity min-cost flow algorithm of Goldberg, Hed, Kaplan, and Tarjan [9]. The resulting algorithm finds a $(1+\varepsilon)$ -approximation to the optimal geometric partial matching in $O((n+k\sqrt{k}) \text{polylog } n \log(n/\varepsilon))$ time. The main issue for this algorithm is what we call *null vertices*, which do not contribute to an augmentation, but for which the algorithm may waste time examining. Instead, we run the unit-capacity min-cost flow algorithm on a *shortcut network* which has paths circumventing all null vertices. The shortcut graph itself may have $O(n^2)$ arcs, but we can query its minimum arcs efficiently using a BCP oracle without explicit construction. Once we have done this, we are able to charge the execution time of the iteration to the size of the *flow support*, the set of arcs with positive flow, which turns out to be size $O(k)$.

► **Theorem 1.2.** *A $(1+\varepsilon)$ geometric partial matching of size k can be computed between A and B in $O((n+k\sqrt{k}) \log(1/\varepsilon) \text{polylog } n)$ time.*

Our third algorithm solves the transportation problem in the unbalanced setting using the strongly polynomial uncapacitated min-cost flow algorithm by Orlin [13], adapted for geometric costs as in Agarwal *et al.* [1]. The result is an $O(r\sqrt{n}(r+\sqrt{n}) \text{polylog } n)$ time exact algorithm for unbalanced transportation. This improves over the $O(n^2 \text{polylog } n)$ time exact algorithm in Agarwal *et al.* [1] when $r = o(\sqrt{n})$. Unlike matching, the flow support may have $\omega(n)$ size even if $r = O(1)$, so it seems like we may be unable to charge to flow support as before. However, we show that most of these support arcs are degree one and partition into *stars* centered around vertices of A , and the remainder is size $O(r)$. Unfortunately,

we are not able to handle these stars as easily in our data structures, and each iteration is dominated by the time spent maintaining data structures representing stars.

► **Theorem 1.3.** *Let A and B be two point sets in the plane with $|A| = r$ and $|B| = n$ satisfying $r \leq n$, with supplies and demands given by the function $\lambda : (A \cup B) \rightarrow \mathbb{Z}$ such that $\sum_{a \in A} \lambda(a) = \sum_{b \in B} \lambda(b)$. An optimal transportation map can be computed in $O(rn^{3/2} \text{polylog } n)$ time.*

2 Minimum-Cost Partial Matchings using Hungarian Algorithm

The Hungarian algorithm [12] is a primal-dual algorithm for min-cost bipartite matching in general graphs that can be adapted to solve the partial matching problem exactly if one terminates the algorithm after k iterations (see e.g. [14]). In this section, we prove Theorem 1.1 by implementing the Hungarian algorithm in $O((n + k^2) \text{polylog } n)$ time.

2.1 Matching Terminologies

«Move some to intro» Let G be a bipartite graph between vertex sets A and B and edge set E , with costs $c(v, w)$ for each edge (v, w) in G . A *matching* $M \subseteq E$ is a set of edges where no two edges share an endpoint. A vertex v is *matched* by M if v is the endpoint of some matching edge in M ; otherwise v is *unmatched*. The *size* of a matching is the number of edges in the set, and the *cost* of a matching is the sum of costs of its edges. For a parameter k , the *minimum-cost partial matching problem (MPM)* asks to find a size- k matching of minimum cost. In the geometric partial matching setting, we have $E = A \times B$ and $c(a, b) = \|a - b\|_p^q$ for every edge (a, b) in G .

The linear program dual to the standard linear program for MPM has dual variables for each vertex, called *potentials* π . Given potentials π , we can define the *reduced cost* on the edges to be $c_\pi(v, w) := c(v, w) - \pi(v) + \pi(w)$. Potentials π are *feasible* if the reduced costs are nonnegative for all edges in G . We say that an edge (v, w) is *admissible* under potentials π if $c_\pi(v, w) = 0$.

Consider a matching M of size less than r . An *augmenting path* $\Pi = (a_1, b_1, \dots, a_\ell, b_\ell)$ is an odd-length path with unmatched endpoints $(a_1$ and $b_\ell)$ and all other points matched. The edges of Π alternate between edges outside and inside of matching M . The symmetric difference $M \oplus \Pi$ creates a new matching of size $|M| + 1$. We say that $M \oplus \Pi$ is the result of *augmenting* M by Π .

2.2 The Hungarian Algorithm

The Hungarian algorithm is initialized with $M = \emptyset$ and $\pi = 0$. Each iteration of the Hungarian algorithm augments M with an admissible augmenting path Π , discovered using a procedure called the *Hungarian search*. The algorithm terminates once M has size k ; Ramshaw and Tarjan [14] showed that M is guaranteed to be an optimal partial matching.

The Hungarian search tries grow a set of *reachable vertices* S by augmenting paths consisting of admissible edges. Initially, S is the set of unmatched vertices in A . Let the *frontier* of S be the edges in $(A \cap S) \times (B \setminus S)$. In each iteration, the Hungarian search first *relaxes* the minimum-reduced-cost edge (a, b) in the frontier, raising $\pi(a)$ by $c_\pi(a, b)$ for all $a \in S$ to make (a, b) admissible, and adding b into S . It is easy to verify that this potential change preserves feasibility. If b is already matched, then we also relax the matching edge (a', b) and add a' into S . The search finishes when b is unmatched, and an admissible augmenting path now can be recovered.

158 The remainder of this section describes an implementation of Hungarian search that runs
 159 in $O(k \text{ polylog } n)$ time after an $O(n \text{ polylog } n)$ time preprocessing (see Lemma 2.1). Our
 160 implementation of the Hungarian algorithm therefore runs in $O((n + k^2) \text{ polylog } n)$ time,
 161 which proves Theorem 1.1.

162 2.3 Fast implementation of Hungarian search

163 The most expensive step in augmentation is to find the minimum-reduced-cost frontier edge
 164 that needs to be relaxed — the search must “look at every edge”. In the geometric setting,
 165 we find the min-cost edge using a dynamic *bichromatic closest pair* (BCP) data structure, as
 166 observed in [1]. Given two point sets P and Q in the plane, the bichromatic closest pair are two
 167 points $p \in P$ and $q \in Q$ minimizing the additively weighted distance $\|p - q\| - \omega(p) + \omega(q)$ for
 168 some real-valued vertex weights ω . Thus, the minimum reduced-cost among the frontier edges
 169 is precisely the cost of the BCP of point sets $P = A \cap S$ and $Q = B \setminus S$, with $\omega(p) = \pi(p)$.

170 **«Under the assumption ... on the metric,»** The state of the art dynamic BCP data
 171 structure from Kaplan *et al.* [11] supports point insertions and deletions in $O(\text{polylog } n)$
 172 time, and answers queries in $O(\log^2 n)$ time. During each relaxation, we perform at most
 173 one query and add a vertex to S incurring one BCP insertion or deletion. Thus the running
 174 time for the search is $O(k \text{ polylog } n)$.

175 **Initial BCP sets by rewinding.** Recall that in the beginning of each iteration, S is initialized
 176 to the set of unmatched vertices in A , and therefore $Q = B \setminus S$ has size n **«what?»**. We
 177 cannot afford to take $O(n \text{ polylog } n)$ time initialize the BCP data structure at the beginning
 178 of every Hungarian search beyond the first. However, the set of unmatched A vertices has
 179 changed by exactly one vertex since the last Hungarian search — the augmentation newly
 180 matched one vertex $a^* \in A$. Thus, given the initial BCP sets P', Q' from the beginning of
 181 the last Hungarian search, we can construct P and Q for the current iteration using a single
 182 BCP deletion in $O(\text{polylog } n)$ time.

183 To acquire P' and Q' , we keep track of a list of the points added to S over the course
 184 of the Hungarian search. At the end of each Hungarian search we *rewind* the BCP data
 185 structure by tracing the list in reverse order. The number of points in the list is at most $O(k)$
 186 as it is bounded by the number of relaxations per Hungarian search. Thus, in $O(k \text{ polylog } n)$
 187 time, we can reconstruct P' and Q' for each Hungarian search beyond the first. We refer to
 188 this procedure as the *rewinding mechanism*.

189 Potential updates. «REWRITE TO BE MORE SUCCINCT.»

190 We modify a trick from Vaidya [17] for batching potential updates. Potentials have a
 191 *stored value*, i.e. the currently recorded value of $\pi(v)$, and a *true value*, which may have
 192 changed from $\pi(v)$. The resulting algorithm queries the minimum-reduced-cost under the
 193 true values of π and updates the stored value occasionally.

194 Throughout the entire Hungarian algorithm, we maintain a nonnegative scalar δ (initially
 195 set to 0) which aggregates potential changes. Vertices $a \in A$ that are added to S are inserted
 196 into BCP with weight $\omega(a) \leftarrow \pi(a) - \delta$, for whatever value δ is at the time of insertion.
 197 Similarly, vertices $b \in B$ that are added to S have $\omega(b) \leftarrow \pi(b) - \delta$ recorded ($B \cap S$ points
 198 aren't added into a BCP set). When the Hungarian search wants to raise the potentials of
 199 points in S , δ is increased by that amount instead. Thus, true value for any potential of a
 200 point in S is always $\omega(p) + \delta$. For points of $(A \cup B) \setminus S$, the true potential is equal to the
 201 stored potential. Since all the points of $A \cap S$ have weights uniformly offset from their true
 202 potentials, the minimum edge returned by the BCP does not change. **«why?»**

Once a point is removed from S (i.e. by an augmentation or the rewinding mechanism), we update its stored potential $\pi(p) \leftarrow \omega(p) + \delta$, again for the current value of δ . Most importantly, δ is not reset at the end of a Hungarian search and persists through the entire algorithm. Thus, the initial BCP sets constructed by the rewinding mechanism have true potentials accurately represented by δ and $\omega(p)$.

We update δ once per edge relaxations; thus $O(k)$ times in total per Hungarian search. There are $O(k)$ stored values updated per Hungarian search during the rewinding process. The time spent on potential updates per Hungarian search is therefore $O(k)$.

Putting everything together we obtain the following:

► **Lemma 2.1.** *Each Hungarian search can be implemented in $O(k \text{ polylog } n)$ time after a one-time $O(n \text{ polylog } n)$ preprocessing.*

3 Approximating Min-Cost Partial Matching through Cost-Scaling

The goal of section is to prove Theorem 1.2; that is, to compute a size- k geometric partial matching between two point sets A and B in the plane, with cost at most $(1 + \varepsilon)$ times the optimal matching, in time $O((n + k\sqrt{k}) \text{ polylog } n \log(1/\varepsilon))$.

«Summarize our new ideas that lead to the improvement.»

After introducing the necessary terminologies in Section 3.1, we reduce the partial matching problem to computing an approximate minimum-cost flow on a unit-capacity reduction network in Section 3.2. In Section 3.3 we outline the high-level overview of the cost-scaling algorithm. We postpone the fast implementation using dynamic data structures to Section 4.

3.1 Preliminaries on Network Flows

Due to the space restriction, we omit the definitions of standard network flow theory terminologies from the main text. For a reference see Appendix A.1, or any texts on network flows [1]. We emphasize that a directed graph $G = (V, E)$ is augmented by edge costs c and capacities u , and a supply-demand function ϕ defined on the vertices. A *network* $N = (V, \vec{E})$ turns each edge in E into a pair of *arcs* $v \rightarrow w$ and $w \rightarrow v$ in arc set \vec{E} . With the unit-capacity assumption on the network, all the pseudoflows in this section take integer values. The *support* of a pseudoflow f in N , denoted as $\text{supp}(f)$, is the set of arcs with positive flows: $\text{supp}(f) := \{v \rightarrow w \in \vec{E} \mid f(v \rightarrow w) > 0\}$. If all vertices are *balanced*, the pseudoflow is a *circulation*. The *cost* of a pseudoflow is defined to be

$$\text{cost}(f) := \sum_{v \rightarrow w \in \text{supp}(f)} c(v \rightarrow w) \cdot f(v \rightarrow w).$$

The *minimum-cost flow problem (MCF)* asks to find a circulation of minimum cost inside a given directed graph.

LP-duality and admissibility. To solve the minimum-cost flow problem, we focus on the primal-dual algorithms using linear programming. Let $G = (V, E)$ be a given directed graph with the corresponding network $N = (V, \vec{E}, c, u, \phi)$. Formally, the *potentials* $\pi(v)$ are the variables of the linear program dual to the standard linear program for the minimum-cost flow problem with variables $f(v, w)$ for each directed edge in E . Assignments to the primal variables satisfying the capacity constraints extend naturally into a pseudoflow on the network

243 N . Let $G_f = (V, \vec{E}_f)$ be the residual graph under pseudoflow f . The *reduced cost* of an arc
 244 $v \rightarrow w$ in \vec{E}_f with respect to π is defined as

$$245 \quad c_\pi(v \rightarrow w) := c(v \rightarrow w) - \pi(v) + \pi(w).$$

246 Notice that the cost function c_π is also antisymmetric.

247 The *dual feasibility constraint* says that $c_\pi(v \rightarrow w) \geq 0$ holds for every directed edge (v, w)
 248 in E ; potentials π which satisfy this constraint are said to be *feasible*. Suppose we relax the
 249 dual feasibility constraint to allow some small violation in the value of $c_\pi(v \rightarrow w)$. We say that
 250 a pair of pseudoflow f and potential π is *ε -optimal* [?, ?] if $c_\pi(v \rightarrow w) \geq -\varepsilon$ for every residual
 251 arc $v \rightarrow w$ in \vec{E}_f . Pseudoflow f is *ε -optimal* if it is ε -optimal with respect to some potentials
 252 π ; potential π is *ε -optimal* if it is ε -optimal with respect to some pseudoflow f . Given a
 253 pseudoflow f and potentials π , a residual arc $v \rightarrow w$ in \vec{E}_f is *admissible* if $c_\pi(v \rightarrow w) \leq 0$. We
 254 say that a pseudoflow g in G_f is *admissible* if all support arcs of g on G_f are admissible; in
 255 other words, $g(v \rightarrow w) > 0$ holds only on admissible arcs $v \rightarrow w$.

256 ► **Lemma 3.1.** *Let f be an ε -optimal pseudoflow in G and let f' be an admissible flow in*
 257 *G_f . Then $f + f'$ is also ε -optimal. ◀Lemma 5.3 in [10]? Also, where is this used?▶*

258 3.2 Reduction to Unit-Capacity Min-Cost Flow Problem

259 The goal of the subsection is to reduce the minimum-cost partial matching problem to
 260 the unit-capacity minimum-cost flow problem with a polynomial bound on diameter of the
 261 underlying point set. To this end we first provide an upper bound on the size of support
 262 of an integral pseudoflow on the standard reduction network between the two problems.
 263 This upper bound in turn provides an additive approximation on the cost of an ε -optimal
 264 circulation. Next we employ a technique by Sharathkumar and Agarwal [15] to transform an
 265 additive ε -approximate solution into a multiplicative $(1 + \varepsilon)$ -approximation for the geometric
 266 partial matching problem. The reduction does not work out of the box, as Sharathkumar
 267 and Agarwal were tackling a similar but different problem on geometric transportations.

268 ► **Lemma 3.2.** *Computing $(1 + \varepsilon)$ -approximate geometric partial matching can be reduced to*
 269 *the following problem in $O(n \text{ polylog } n)$ time: Given a reduction network N over a point set*
 270 *with diameter at most $K \cdot kn^3$ for some constant K , compute a $(K \cdot \varepsilon / 6k)$ -optimal circulation*
 271 *on N .*

272 **Additive approximation.** Given a bipartite graph $G = (A, B, E_0)$ for the geometric partial
 273 matching problem with cost function c , we construct the *reduction network* N_H as follows:
 274 Direct the edges in E_0 from A to B , and assign each directed edge with capacity 1. Now
 275 add a dummy vertex s with directed edges to all vertices in A , and add a dummy vertex t
 276 with directed edges from all vertices in B ; each edge added this way has cost 0 and capacity
 277 1. Denote the new graph with vertex set $V = A \cup B \cup \{s, t\}$ and edge set E as the *reduction*
 278 *graph* H . Assign vertex s with supply k and vertex t with demand k ; the rest of the vertices in
 279 H have zero supply-demand. We call the network naturally corresponds to H as the *reduction*
 280 *network*, denoted by N_H .

281 It is straightforward to show that any integer circulation f on N_H uses exactly k of the
 282 A -to- B arcs, which correspond to the edges of a size- k matching M_f . Notice that the cost of
 283 the circulation f is equal to the cost of the corresponding matching M_f .

284 First we show that the number of arcs used by any integer pseudoflow in N_H is asymp-
 285 totically bounded by the excess of the pseudoflow.

286 ► **Lemma 3.3.** *The size of $\text{supp}(f)$ is at most $3k$ for any integer circulation f in reduction*
 287 *network N_H . As a corollary, the number of residual backward arcs is at most $3k$.*

288 Using the bound on the support size, we show that an ε -optimal integral circulation gives
 289 an additive $O(k\varepsilon)$ -approximation to the MCF problem.

290 ► **Lemma 3.4.** *Let f be an ε -optimal integer circulation in N_H , and f^* be an optimal integer*
 291 *circulation for N_H . Then, $\text{cost}(f) \leq \text{cost}(f^*) + 6k\varepsilon$.*

292 **Multiplicative approximation.** Now we employ a technique from Sharathkumar and Agarwal
 293 [15] to convert the additive approximation into a multiplicative one. Here we sketch a
 294 proof to Lemma 3.2; a complete proof can be found in the Appendix. **«Is it okay if we hide**
 295 **the parametric search?»**

296 Sharathkumar and Agarwal [15, §3.5] provide a construction that partitions the input
 297 point sets A and B for the MPM problem into clusters, such that the diameter of each cluster
 298 is upper bounded by the cost of the optimal partial matching multiply by a polynomial
 299 factor. To prove Lemma 3.2, we further modify the point set by moving the clusters so
 300 that the cost of the optimal solution does not change, while the diameter of the *whole* point
 301 set is bounded. Now one can prove Lemma 3.2 by computing an $(\varepsilon \text{cost}(M^*)/6k)$ -optimal
 302 circulation f on the modified point set using additive approximation from Lemma 3.4.

303 3.3 High-Level Description of Cost-Scaling Algorithm

307 Our main algorithm for the unit-capacity minimum-cost flow problem is based on the *cost-*
 308 *scaling* technique, originally due to Goldberg and Tarjan [10]; Goldberg *et al.* [9] applied
 309 the technique on unit-capacity networks. The algorithm finds ε -optimal circulations for
 310 geometrically shrinking values of ε . Each fixed value of ε is called a *cost scale*. Once ε is
 311 sufficiently small, the ε -optimal flow is a suitable approximation according to Lemma 3.2.¹

312 The cost-scaling algorithm initializes the flow f and the potential π to be zero. Note
 313 that the zero flow is trivially a kC -optimal flow, where C is the maximum arc cost. At the
 314 beginning of each scale starting at $\varepsilon = kC$,

- 315 ■ SCALE-INIT takes the previous circulation (now 2ε -optimal) and transforms it into an
 316 ε -optimal pseudoflow with $O(k)$ excess.
- 317 ■ REFINE then reduces the excess in the newly constructed pseudoflow to zero, making it
 318 an ε -optimal circulation.

319 Thus for any $\varepsilon^* > 0$, the algorithm produces an ε^* -optimal circulation after $O(\log(kC/\varepsilon^*))$
 320 scales. Using the reduction in Lemma 3.2, we have the diameter of the point set, thus
 321 maximum cost C , bounded by $O(K \cdot kn^3)$ for some value K . By setting ε^* to be $K \cdot \varepsilon/6k$,
 322 the number of cost scales is bounded above by $O(\log(n/\varepsilon))$.

323 **Scale initialization.** Recall that H is the *reduction graph* and N_H is the *reduction network*,
 324 both constructed in Section 3.2. The vertex set of H consists of two point sets A and B , as
 325 well as two dummy vertices s and t . The directed edges in H are pointed from s to A , from
 326 A to B , and from B to t . We call those arcs in N_H whose direction is consistent with their
 327 corresponding directed edges as *forward arcs*, and those arcs that points in the opposite
 328 direction as *backward arcs*.

304 ¹ When the costs are integers, an ε -optimal circulation for a sufficiently small ε (say less than $1/n$) is itself
 305 an optimal solution [9, 10]. We present this algorithm without the integral-cost assumption because in
 306 the geometric partial matching setting (with respect to L_p norms) the costs are generally not integers.

329 **«Describe how it's different from original.»** The procedure SCALE-INIT transforms a
 330 2ε -optimal circulation from the previous cost scale into an ε -optimal flow with $O(k)$ excess,
 331 by raising the potentials π of all vertices in A by ε , those in B by 2ε , and the potential of t
 332 by 3ε . The potential of s remains unchanged. Now the reduced cost of every forward arc is
 333 dropped by ε , and thus all the forward arcs have reduced cost at least $-\varepsilon$.

334 As for backward arcs, the procedure SCALE-INIT continues by setting the flow on $v \rightarrow w$ to
 335 zero for each backward arc $w \rightarrow v$ violating the ε -optimality constraint. In other words, we
 336 set $f(v \rightarrow w) = 0$ whenever $c_\pi(w \rightarrow v) < -\varepsilon$. This ensures that all such backward arcs are no
 337 longer residual, and therefore the flow (now with excess) is ε -optimal.

338 Because the arcs are of unit-capacity in N_H , each arc desaturation creates one unit of
 339 excess. By Lemma 3.3 the number of backward arcs is at most $3k$. Thus the total amount of
 340 excess created is also $O(k)$.

341 In total, potential updates and backward arc desaturations, thus the whole procedure
 342 SCALE-INIT, take $O(n)$ time.

343 **Refinement.** The procedure REFINE is implemented using a primal-dual augmentation
 344 algorithm, which sends flows on admissible arcs to reduce the total excess, like the Hungarian
 345 algorithm. Unlike the Hungarian algorithm, it uses *blocking flows* instead of augmenting
 346 paths. We call a pseudoflow f on residual network N_g a *blocking flow* if f saturates at least
 347 one residual arc in every augmenting path in N_g . In other words, there is no admissible
 348 augmenting path in N_{f+g} from an excess vertex to a deficit vertex.

349 Each iteration of REFINE finds an admissible blocking flow that is then added to the
 350 current pseudoflow in two stages:

- 351 1. A *Hungarian search*, which increases the dual variables π of vertices that are reachable
 352 from an excess vertex by at least ε , in a Dijkstra-like manner, until there is an excess-deficit
 353 path of admissible edges.
- 354 2. A *depth-first search* through the set of admissible edges to construct an admissible
 355 blocking flow. It suffices to repeatedly extract admissible augmenting paths until no
 356 more admissible excess-deficit paths remain. By definition, the union of such paths is a
 357 blocking flow.

358 The algorithm continues until the total excess becomes zero and the ε -optimal flow is now a
 359 circulation.

360 First we analyze the number of iterations executed by REFINE. The proof follows the
 361 strategy in Goldberg *et al.* [9, Section 3.2]. Due to space constraint we omit all the proofs
 362 here; see Appendix A.3 for complete proofs. **«Explain what is new here.»**

363 ► **Lemma 3.5.** *Let f be a pseudoflow in N_H with $O(k)$ excess. The procedure REFINE runs*
 364 *for $O(\sqrt{k})$ iterations before the excess of f becomes zero.*

365 The goal of the next section is to show that after $O(n \text{ polylog } n)$ time preprocessing,
 366 each Hungarian search and depth-first search can be implemented in $O(k \text{ polylog } n)$ time.
 367 Combined with the $O(\sqrt{k})$ bound on the number of iterations we just proved, the procedure
 368 REFINE can be implemented in $O((n + k\sqrt{k}) \text{ polylog } n)$ time. Together with our analysis
 369 on scale initialization and the bound on number of cost scales, this concludes the proof to
 370 Theorem 1.2.

371 4 Fast Implementation of Refinement

372 Both Hungarian search and depth-first search are implemented in a Dijkstra-like fashion,
 373 traversing through the residual graph using admissible arcs starting from the excess vertices.

Each step of the search procedures *relaxes* a minimum-reduced-cost arc from the set of visited vertices to an unvisited vertex, until a deficit vertex is reached. At a high level, our analysis strategy is to charge the relaxation events to the support arcs of f , which has size at most $O(k)$ by Corollary A.3.

4.1 Null vertices and shortcut graph

As it turns out, there are some vertices visited by a relaxation event which we cannot charge to $\text{supp}(f)$. Unfortunately the number of such vertices can be as large as $\Omega(n)$. To overcome this issue, we replace the residual graph with an equivalent graph that excludes all the null vertices, and run the Hungarian search and depth-first search on the resulting graph instead.

Null vertices. We say a vertex v in the residual graph N_f is a *null vertex* if $\phi_f(v) = 0$ and no arcs of $\text{supp}(f)$ is incident to v . We use A_\emptyset and B_\emptyset to denote the null vertices A and B respectively. Vertices that are not null are called *normal vertices*. A *null 2-path* is a length-2 subpath in N_f from a normal vertex to another normal vertex, passing through a null vertex. As every vertex in A has in-degree 1 and every vertex in B has out-degree 1 in the residual graph, the null 2-paths must be of the form either (s, v, b) for some vertex b in $B \setminus B_\emptyset$ or (a, v, t) for some vertex a in $A \setminus A_\emptyset$. In either case, we say that the null 2-path *passes through* null vertex v . Similarly, we define the length-3 path from s to t that passes through two null vertices to be a *null 3-path*. Because reduced costs telescope for residual paths, the reduced cost of any null 2-path or null 3-path does not depend on the null vertices it passes through.

Shortcut graph. We construct the *shortcut graph* \tilde{H}_f from the reduction network H by removing all null vertices and their incident edges, followed by inserting an arc from the head of each each null path Π to its tail, with cost equals to the sum of costs on the arcs. We call this arc the *shortcut* of null path Π , denoted as $\text{short}(\Pi)$. The resulting multigraph \tilde{H}_f contains only normal vertices of H_f , and the reduced cost of any path between normal vertices are preserved. We argue now that \tilde{H}_f is fine as a surrogate for H_f . Let $\tilde{\pi}$ be an ε -optimal potential on \tilde{H}_f . Construct potentials π on H_f which extends $\tilde{\pi}$ to null vertices, by setting $\pi(a) := \tilde{\pi}(s)$ for $a \in A_\emptyset$ and $\pi(b) := \tilde{\pi}(t)$ for $b \in B_\emptyset$.

► **Lemma 4.1.** *Consider $\tilde{\pi}$ an ε -optimal potential on \tilde{H}_f and π the corresponding potential constructed on H_f . Then,*

1. *potential π is ε -optimal on H_f , and*
2. *if arc $\text{short}(\Pi)$ is admissible under $\tilde{\pi}$, then every arc in Π is admissible under π .*

«Talk about the size of shortcut graph briefly.»

4.2 Dynamic data structures for search procedures

Hungarian search. Conceptually, we are executing the Hungarian search on the shortcut graph \tilde{H}_f . We describe how we can query the minimum-reduced-cost arc leaving \tilde{S} in $O(\text{polylog } n)$ time for the shortcut graph, without constructing \tilde{H}_f explicitly. For this purpose, let \tilde{S} be a set of “reached” vertices maintained, identical to \tilde{S} except whenever a shortcut is relaxed, we add the null vertices passed by the corresponding null path to \tilde{S} in addition to its (normal) endpoints. Observe that the arcs of \tilde{H}_f leaving \tilde{S} fall into $O(1)$ categories:

- non-shortcut backward arcs (v, w) with $(w, v) \in \text{supp}(f)$;
- non-shortcut A -to- B forward arcs;

416 ■ non-shortcut forward arcs from s -to- A and from B -to- t ;
 417 ■ shortcut arcs (s, b) corresponding to null 2-paths from s to $b \in (B \setminus B_\emptyset) \setminus S$;
 418 ■ shortcut arcs (a, t) corresponding to null 2-paths from $a \in (A \setminus A_\emptyset) \cap S$ to t ; and
 419 ■ shortcut arcs (s, t) corresponding to null 3-paths.
 420 For each category of arcs we maintain a proper data structure (either heap or BCP) to
 421 answer to the min-cost arc query.

422 **Depth-first search.** Depth-first search is similar to Hungarian search in that it uses the
 423 relaxation of minimum-reduced-cost arcs/null paths, this time to identify admissible arcs/null
 424 paths in a depth-first manner. Similar to the Hungarian search, for each category of arcs
 425 in \tilde{H}_f leaving \tilde{S} , we maintain a proper data structure to answer the minimum-reduced-cost
 426 arc leaving a *fixed* vertex in \tilde{S} given by the query. Thus unlike Hungarian search which uses
 427 BCP data structures, we use dynamic nearest-neighbor data structures instead [].

428 Each of the above data structures requires $O(1)$ queries and updates per relaxation. So
 429 in collaboration each relaxation can be implemented in $O(\text{polylog } n)$ time [?, ?].

430 **Time analysis.** The complete time analysis can be found in Appendix B.2, B.3, and B.4;
 431 here we sketch the ideas. First we show (in Appendix B.2) that both Hungarian search and
 432 depth-first search performs $O(k)$ relaxations before a deficit vertex is reached, by looking at
 433 shortcut and non-shortcut relaxations separately. Both types of relaxations are eventually
 434 charged to the suppot size of f . As for the time analysis (see Appendix B.3), using the
 435 same rewinding mechanism as in Section 2.3, the running time of the Hungarian search
 436 and depth-first search, other than the potential updates, can be charged to the number of
 437 relaxations. Again using the trick by Vaidya [17] we can charge the potential updates of
 438 normal vertices to the number of relaxations in the Hungarian search. We never explicitly
 439 maintain the potentials on the null vertices; instead they are reconstructed whenever needed,
 440 either at the end of each iteration of refinement or when an augmentation sends flow through
 441 a null vertex. We show that such updates does not happen often in Appendix B.4. This
 442 completes the time analysis, which we summarize as follows:

443 ► **Lemma 4.2.** *After $O(n \text{ polylog } n)$ -time preprocessing, each Hungarian search and depth-*
 444 *first search can be implemented in $O(k \text{ polylog } n)$ time.*

445 5 Unbalanced Transportation

446 In this section, we give an exact algorithm which solves the planar transportation problem in
 447 $O(rn^{3/2} \text{ polylog } n)$ time, proving Theorem 1.3. Our strategy is to use the standard reduction
 448 to the uncapacitated min-cost flow problem, and provide a fast implementation under the
 449 geometric setting for the uncapacitated min-cost flow algorithm by Orlin [13], combined with
 450 some of the tools developed in Sections 2 and 3.

451 5.1 Uncapacitated MCF by excess scaling

452 We give an outline of the strongly polynomial algorithm for uncapacitated min-cost flow
 453 problem from Orlin [13]. Orlin's algorithm follows an *excess-scaling* paradigm originally due
 454 to Edmonds and Karp [7]. Consider the basic primal-dual framework used in the previous
 455 sections: The algorithm begins with both flow f and potentials π set to zero. Repeatedly
 456 runs a *Hungarian search* that raises potentials (while maintaining dual feasibility) to create
 457 an admissible augmenting excess-deficit path, on which we perform flow augmentations. In

terms of cost, f is maintained to be 0-optimal with respect to π and each augmentation over admissible edges preserve such property by Lemma 3.1. Thus, the final circulation must be optimal. The excess-scaling paradigm builds on top of this skeleton by specifying (i) between which excess and deficit vertices we send flows, and (ii) how much flow is sent by the augmentation.

The excess-scaling algorithm maintains a *scale parameter* Δ , initially set to U . A vertex v with $|\phi_f(v)| \geq \Delta$ is called *active*. Each augmenting path is chosen between an active excess vertex and an active deficit vertex. Once there are no more active excess or deficit vertices, Δ is halved. Each sequence of augmentations where Δ holds a constant value is called an *excess scale*. There are $O(\log U)$ excess scales before $\Delta < 1$ and, by integrality of supplies/demands, f is a circulation.

With some modifications to the excess-scaling algorithm, Orlin [13] obtains a strongly polynomial bound on the number of augmentations and excess scales. First, an *active* vertex is redefined to be one satisfying $|\phi_f(v)| \geq \alpha\Delta$, for a fixed parameter $\alpha \in (0.5, 1)$. Second, arcs with flow value at least $3n\Delta$ at the beginning of a scale are *contracted* to create a new vertex, whose supply-demand is the sum of those on the two endpoints of the contracted arc. We use $\hat{G} = (\hat{V}, \hat{E})$ to denote the resulting *contracted graph*, where each $\hat{v} \in \hat{V}$ is a contracted component of vertices from V . Intuitively, the flow is so high on contracted arcs that no set of future augmentations can remove the arc from $\text{supp}(f)$. Third, in addition to halving, Δ is aggressively lowered to $\max_{v \in V} \phi_f(v)$ if there are no active excess vertices and $f(v \rightarrow w) = 0$ holds for every arc $v \rightarrow w \in \hat{E}$. Finally, flow values are not tracked within contracted components, but once an optimal circulation is found on \hat{G} , optimal potentials π^* can be *recovered* for G by sequentially undoing the contractions. The algorithm then performs a post-processing step which finds the optimal circulation f^* on G by solving a max-flow problem on the set of admissible arcs under π^* .

► **Theorem 5.1 (Orlin [13, Theorems 2 and 3]).** *Orlin’s algorithm finds a set of optimal potentials after $O(n \log n)$ scaling phases and $O(n \log n)$ total augmentations.*

The remainder of the section focuses on showing that each augmentation can be implemented in $O(r\sqrt{n} \text{polylog } n)$ time (after preprocessing). Additionally, we show that f^* can be recovered from π^* very quickly in our setting.

Implementing contractions. **«REWRITE»** Following Agarwal *et al.* [1], our geometric data structures must deal with real points in the plane instead of the the contracted components. We will track the contracted components described in \hat{G} (e.g. with a disjoint-set data structure) and mark the arcs of $\text{supp}(f)$ that are contracted. We maintain potentials on the points A and B directly, instead of the contracted components.

When conducting the Hungarian search, we initialize S to be the set of vertices from *active excess contracted components* who (in sum) meet the imbalance criteria. **«unclear»** Upon relaxing any $v \in \hat{v}$, we immediately relax all the contracted support arcs which span \hat{v} . Since the input network is uncapacitated, each contracted component is strongly connected in the residual network by the admissible forward/backward arcs of each contracted arc. **«unparsable»** To relax arcs in \hat{E} , we relax the support arcs before attempting to relax any non-support arcs. **«mention the reason to make support acyclic»** Relaxations of support arcs can be performed without further potential changes, since they are admissible by invariant.

During the augmentations, contracted residual arcs are considered to have infinite capacity, and we do not update the value of flows on these arcs. We allow augmenting paths to begin

from any point $a \in \hat{v} \cap A$ in an active excess component \hat{v} , and end at any point $b \in \hat{w} \cap B$ in an active deficit component \hat{w} .

Recovering optimal flow. Rewinding contracted components to recover an optimal flow naïvely takes $O(rn^2)$ time. Use a strategy from Agarwal *et al.* [1], we can recover the optimal flow in time $O(n \text{ polylog } n)$. If furthermore the cost function is just the p -norm (without the q th-power), an even stronger result stands: In this case, the set of admissible arcs under an optimal potential forms a planar graph, and thus we can apply the planar maximum-flow algorithm [?, 8] which runs in $O(n \log n)$ time. For details see the appendix.

5.2 Support stars

To find an augmenting path, we again use a Hungarian search with geometric data structures to perform relaxations quickly. Our strategy is summarized as follows:

- Discard vertices which lead to dead ends in the search (not on a path to a deficit vertex).
- Cluster parts of the flow support, such that the number of support arcs outside clusters is $O(r)$. The number of relaxations we perform is proportional to the number of support arcs outside of clusters.

Querying/updating clusters degrades our amortized time per relaxation from $O(\text{polylog } n)$ to $O(\sqrt{n} \text{ polylog } n)$. Thus overall each augmentation takes $O(r\sqrt{n} \text{ polylog } n)$ time.

Support stars. The vertices of B with support degree 1 are partitioned into subsets $\Sigma_a \subset B$ by the $a \in A$ lying on the other end of their single support arc. We call Σ_a the *support star* centered at $a \in A$.

Roughly speaking, we would like to handle each support star as a single unit. When the Hungarian search reaches a or any $b \in \Sigma_a$, the entirety of Σ_a (as well as a) is also admissibly-reachable and can be included into S without further potential updates. Additionally, the only outgoing residual arcs of every $b \in \Sigma_a$ lead to a , thus the only way to leave $\Sigma_a \cup \{a\}$ is through an arc leaving a . Once a relaxation step reaches some $b \in \Sigma_a$ or a itself, we would like to quickly update the state such that the rest of $b \in \Sigma_a$ is also reached without performing relaxation steps to each individual $b \in \Sigma_a$.

5.3 Implementation details

Before describing our workaround for support stars, we analyze the number of relaxation steps for arcs outside of support stars. To this end we need to strip of some *dead* vertices—having no incident flow support edges and not an active excess or deficit vertex—that does not affect the search. We use A_ℓ and B_ℓ to denote vertices of points in A and B that are not dead. The details for handling such vertices can be found in Appendix C.3. For a proof of the following lemma, see Appendix C.4.

► **Lemma 5.2.** *Suppose we have stripped the graph of dead vertices. The number of relaxation steps in a Hungarian search outside of support stars is $O(r)$.*

Relaxations outside support stars. For relaxations that don't involve support star vertices, we can once again maintain a BCP data structure to query the minimum A_ℓ -to- B_ℓ arc. To elaborate, this is the BCP between $P = A_\ell \cap S$ and $Q = (B_\ell \setminus (\bigcup_{a \in A_\ell} \Sigma_a)) \setminus S$, weighted by potentials. Since the query is outside the support stars, there is at most one update per relaxation. Backward (support) arcs are kept admissible by the invariant, so we relax them immediately when they arrive at the frontier.

546 **Relaxing a support star.** We classify support stars into two categories: *big stars* with
 547 $|\Sigma_a| > \sqrt{n}$, and *small stars* with $|\Sigma_a| \leq \sqrt{n}$. Let $A_{big} \subseteq A$ denote the centers of big stars
 548 and $A_{small} \subseteq A$ denote the centers of small stars. We keep the following data structures
 549 to manage support stars.

- 550 ■ For each big star Σ_a , we use a data structure $\mathcal{D}_{big}(a)$ to maintain BCP between $P = A_\ell \cap S$
 551 and $Q = \Sigma_a$. We query this until $a \in S$ or any vertex of Σ_a is added to S .
- 552 ■ All small stars are added to a single BCP data structure \mathcal{D}_{small} between $P = A_\ell \cap S$ and
 553 $Q = (\bigcup_{a \in A_{small}} \Sigma_a) \setminus S$. When an $a \in A_{small}$ or any vertex of its support star is added to
 554 S , we remove the points of Σ_a from \mathcal{D}_{small} using $|\Sigma_a|$ deletion operations.

555 We will update these data structures as each support star center is added into S . If a
 556 relaxation step adds some $b \in B_\ell$ and b is in a support star Σ_a , then we immediately relax
 557 $b \rightarrow a$, as all support arcs are admissible.

558 Suppose a relaxation step adds $a \in A_\ell$ to S . We must (i) remove a from every \mathcal{D}_{big} , (ii)
 559 remove a from \mathcal{D}_{small} . If $a \in A_{big}$, we also (iii) deactivate $\mathcal{D}_{big}(a)$. If $a \in A_{small}$, we also (iv)
 560 remove the points of Σ_a from \mathcal{D}_{small} . The operations (i–iii) can be performed in $O(\text{polylog } n)$
 561 time each, but (iv) may take up to $O(\sqrt{n} \text{polylog } n)$ time. On the other hand, there are now
 562 $O(\sqrt{n})$ data structures to query during each relaxation step, which takes $O(\sqrt{n} \log^2 n)$ time
 563 in total. Together with Lemma 5.2 we bound the time for each Hungarian search.

564 ► **Lemma 5.3.** *Hungarian search takes $O(r\sqrt{n} \text{polylog } n)$ time.*

565 **Updating support stars.** As the flow support changes, the membership of support stars may
 566 shift and a big star may eventually become small (or vice versa). To efficiently support this,
 567 we introduce a soft boundary in determining whether a support star is big or small. Standard
 568 charging argument shows that the amortized update time is $O(r\sqrt{n}(r + \sqrt{n}) \text{polylog } n)$; for
 569 a complete argument see Appendix C.5.

570 Membership of support stars can only be changed by augmentations, so the number of
 571 star membership changes by a single augmenting path is bounded above by twice of its
 572 length. Thus, each membership change can be performed in $O(\text{polylog } n)$ time, and there
 573 are $O(rn \log n)$ many.

574 **Preprocessing time.** To build the very first set of data structures, we take $O(rn \text{polylog } n)$
 575 time. There are $r|\Sigma_a|$ points in each $\mathcal{D}_{big}(a)$, but the Σ_a are disjoint, so the total points
 576 to insert is $O(rn)$. \mathcal{D}_{small} also has at most $O(rn)$ points. Each BCP data structure can be
 577 constructed in $O(\text{polylog } n)$ times its size, so the total preprocessing time is $O(rn \text{polylog } n)$.

578 **Between searches.** After an augmentation, we reset the above data structures to their
 579 initial state plus the change from the augmentation using the rewinding mechanism. By
 580 reversing the sequence of insertions/deletions to each data structure over the course of
 581 the Hungarian search, we can recover the versions data structures as they were when the
 582 Hungarian search began. This takes time proportional to the time of the Hungarian search,
 583 $O(r\sqrt{n} \text{polylog } n)$ by Lemma 5.3. The most recent augmentation may have deactivated at
 584 most one active excess and at most one active deficit, which we can update in the data
 585 structures in $O(\sqrt{n} \text{polylog } n)$ time. Additionally, the augmentation may have changed the
 586 membership of some support stars, but we analyzed the time for membership changes earlier.
 587 Finally, we note that an augmenting path cannot reduce the support degree of a vertex to
 588 zero, and therefore no new dead vertices are created by augmentation.

Between excess scales. When the excess scale changes, vertices that were previously inactive may become active, and vertices that were dead may be revived (however, no active vertices deactivate, and no live vertices die as the result of Δ decreasing). If we have the data structures built on the active excesses at the end of the previous scale, then we can add in each newly active $a \in A$ and charge this insertion to the (future) augmenting path or contraction which eventually makes the vertex inactive, or absorbs it into another component. By Theorem 5.1, there are $O(n \log n)$ such newly active vertices. The time to perform data structure updates for each of them is $O(\sqrt{n} \text{polylog } n)$, so the total time spent bookkeeping newly active vertices is $O(n^{3/2} \text{polylog } n)$.

Putting it together. After $O(rn \text{polylog } n)$ preprocessing, we spend $O(r\sqrt{n} \text{polylog } n)$ time each Hungarian search by Lemma 5.3. After each augmentation, we spend the same amount of time (plus $O(\text{polylog } n)$ extra) to initialize data structures for the next Hungarian search. We spend up to $O((rn + r^2\sqrt{n}) \text{polylog } n)$ total time making big-small star switching updates. We spend $O(n^{3/2} \text{polylog } n)$ time activating and reviving vertices. Thus, the algorithm takes $O(rn(r/\sqrt{n} + \sqrt{n}) \text{polylog } n)$ time to produce optimal potentials π^* , from which we can recover f^* in $O(r\sqrt{n} \text{polylog } n)$ additional time. This completes the proof of Theorem 1.3.

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A Proofs from Section 3

A.1 Preliminaries on Network Flows

Network. Let $G = (V, E)$ be a directed graph, augmented by edge costs c and capacities u , and a supply-demand function ϕ defined on the vertices. One can turn the graph G into a *network* $N = (V, \vec{E})$: For each directed edge (v, w) in E , insert two *arcs* $v \rightarrow w$ and $w \rightarrow v$ into the arc set \vec{E} ; the *forward arc* $v \rightarrow w$ inherits the capacity and cost from the directed graph G , while the *backward arc* $w \rightarrow v$ satisfies $u(w \rightarrow v) = 0$ and $c(w \rightarrow v) = -c(v \rightarrow w)$. This we ensure that the graph (V, \vec{E}) is *symmetric* and the cost function c is *antisymmetric* on N . The positive values of $\phi(v)$ are referred to as *supply*, and the negative values of $\phi(v)$ as *demand*. We assume that all capacities are nonnegative, all supplies and demands are integers, and the sum of supplies and demands is equal to zero. A *unit-capacity* network has all its edge capacities equal to 1. In this section we assume all networks are of unit-capacity.

Pseudoflows. Given a network $N := (V, \vec{E}, c, u, \phi)$, a *pseudoflow* (or *flow* to be short) $f: \vec{E} \rightarrow \mathbb{Z}^2$ on N is an antisymmetric function on the arcs of N satisfying $f(v \rightarrow w) \leq u(v \rightarrow w)$ for every arc $v \rightarrow w$. We sometimes abuse the terminology by allowing pseudoflow to be defined on a directed graph, in which case we are actually referring to the pseudoflow on the corresponding network by extending the flow values antisymmetrically to the arcs. We say that f *saturates* an arc $v \rightarrow w$ if $f(v \rightarrow w) = u(v \rightarrow w)$; an arc $v \rightarrow w$ is *residual* if $f(v \rightarrow w) < u(v \rightarrow w)$. The *support* of f in N , denoted as $\text{supp}(f)$, is the set of arcs with positive flows:

$$\text{supp}(f) := \{v \rightarrow w \in \vec{E} \mid f(v \rightarrow w) > 0\}.$$

Given a pseudoflow f , we define the *imbalance* of a vertex (with respect to f) to be

$$\phi_f(v) := \phi(v) + \sum_{w \rightarrow v \in \vec{E}} f(w \rightarrow v) - \sum_{v \rightarrow w \in \vec{E}} f(v \rightarrow w).$$

We call positive imbalance *excess* and negative imbalance *deficit*; and vertices with positive and negative imbalance *excess vertices* and *deficit vertices*, respectively. A vertex is *balanced* if it has zero imbalance. If all vertices are balanced, the pseudoflow is a *circulation*. The *cost* of a pseudoflow is defined to be

$$\text{cost}(f) := \sum_{v \rightarrow w \in \text{supp}(f)} c(v \rightarrow w) \cdot f(v \rightarrow w).$$

The *minimum-cost flow problem (MCF)* asks to find a circulation of minimum cost inside a given directed graph.

Residual graph. Given a pseudoflow f , one can define the *residual network* as follows. Recall that the set of *residual arcs* \vec{E}_f under f are those arcs $v \rightarrow w$ satisfying $f(v \rightarrow w) < u(v \rightarrow w)$. In other words, an arc that is not saturated by f is a residual arc; similarly, given an arc $v \rightarrow w$ with positive flow value, the backward arc $w \rightarrow v$ is a residual arc.

Let $N = (V, \vec{E}, c, u, \phi)$ be a network constructed from graph G , with a pseudoflow f on N . The *residual graph* G_f of f has V as its vertex set and \vec{E}_f as its arc set. The *residual capacity*

² In general the pseudoflows are allowed to take real-values. Here under the unit-capacity assumption any optimal flows are integer-valued.

707 u_f with respect to pseudoflow f is defined to be $u_f(v \rightarrow w) := u(v \rightarrow w) - f(v \rightarrow w)$. Observe
 708 that the residual capacity is always nonnegative. We can define residual arcs differently using
 709 residual capacities:

$$710 \quad \vec{E}_f = \{v \rightarrow w \mid u_f(v \rightarrow w) > 0\}.$$

711 In other words, the set of residual arcs are precisely those arcs in the residual graph, each of
 712 which has nonzero residual capacity.

713 ► **Lemma 3.1.** *Let f be an ε -optimal pseudoflow in G and let f' be an admissible flow in*
 714 *G_f . Then $f + f'$ is also ε -optimal. ◀(Lemma 5.3 in [10]? Also, where is this used?)▶*

715 **Proof.** Augmentation by f' will not change the potentials, so any previously ε -optimal arcs
 716 remain ε -optimal. However, it may introduce new arcs $v \rightarrow w$ with $u_{f+f'}(v \rightarrow w) > 0$, that
 717 previously had $u_f(v \rightarrow w) = 0$. We will verify that these arcs satisfy the ε -optimality condition.

718 If an arc $v \rightarrow w$ is newly introduced this way, then by definition of residual capacities
 719 $f(v \rightarrow w) = u(v \rightarrow w)$. At the same time, $u_{f+f'}(v \rightarrow w) > 0$ implies that $(f + f')(v \rightarrow w) < u(v \rightarrow w)$.
 720 This means that f' augmented flow in the reverse direction of $v \rightarrow w$ ($f'(w \rightarrow v) > 0$). By
 721 assumption, the arcs of $\text{supp}(f')$ are admissible, so $w \rightarrow v$ was an admissible arc ($c_\pi(w \rightarrow v) \leq 0$).
 722 By antisymmetry of reduced costs, this implies $c_\pi(v \rightarrow w) \geq 0 \geq -\varepsilon$. Therefore, all arcs with
 723 $u_{f+f'}(v, w) > 0$ respect the ε -optimality condition, and thus $f + f'$ is ε -optimal. ◀

724 ► **Lemma 3.3.** *The size of $\text{supp}(f)$ is at most $3k$ for any integer circulation f in reduction*
 725 *network N_H . As a corollary, the number of residual backward arcs is at most $3k$.*

726 **Proof.** Because f is a circulation, $\text{supp}(f)$ can be decomposed into k paths from s to t . Each
 727 s -to- t path in N_H is of length three, so the size of $\text{supp}(f)$ is at most $3k$. As every backward
 728 arc in the residual network must be induced by positive flow in the opposite direction, the
 729 total number of residual backward arcs is at most $3k$. ◀

730 ► **Lemma 3.4.** *Let f be an ε -optimal integer circulation in N_H , and f^* be an optimal integer*
 731 *circulation for N_H . Then, $\text{cost}(f) \leq \text{cost}(f^*) + 6k\varepsilon$.*

732 **Proof.** By Lemma 3.3, the total number of backward arcs in the residual network N_f is at
 733 most $3k$. Consider the residual flow in N_f defined by the difference between f^* and f . Since
 734 both f and f^* are both circulations and N_H has unit-capacity, the flow $f - f^*$ is comprised
 735 of unit flows on a collection of edge-disjoint residual cycles $\Gamma_1, \dots, \Gamma_\ell$. Observe that each
 736 residual cycle Γ_i must have exactly half of its arcs being backward arcs, and thus we have
 737 $\sum_i |\Gamma_i| \leq 6k$.

738 Let π be some potential certifying that f is ε -optimal. Because Γ_i is a residual cycle, we
 739 have $c_\pi(\Gamma_i) = c(\Gamma_i)$ since the potential terms telescope. We then see that

$$740 \quad \text{cost}(f) - \text{cost}(f^*) = \sum_i c(\Gamma_i) = \sum_i c_\pi(\Gamma_i) \geq \sum_i (-\varepsilon) \cdot |\Gamma_i| \geq -6k\varepsilon,$$

741 where the second-to-last inequality follows from the ε -optimality of f with respect to π .
 742 Rearranging the terms we have that $\text{cost}(f) \leq \text{cost}(f^*) + 6k\varepsilon$. ◀

743 A.2 Multiplicative approximation

744 Let T be the minimum spanning tree on input graph G and order its edges by increasing
 745 length as e_1, \dots, e_{r+n-1} . Let T_ℓ denote the subgraph of T obtained by removing the heaviest
 746 ℓ edges in T . Let i be the largest index so that the optimal solution to the MPM problem has
 747 edges between components of T_i . Choose j to be the smallest index larger than i satisfying

748 $c(e_j) \geq kn \cdot c(e_i)$. For each component K of T_j , let G_K be the subgraph of G induced on
 749 vertices of K ; let $A_K := K \cap A$ and $B_K := K \cap B$, respectively. We partition A and B into
 750 the collection of sets A_K and B_K according to the components K of T_j . Since $j < i$, the
 751 optimal partial matching in G can be partitioned into edges between A_K and B_K within
 752 G_K ; no optimal matching edges lie between components.

753 ► **Lemma A.1 (Sharathkumar and Agarwal [15, §3.5]).** *Let $G = (A, B, E_0)$ be the input*
 754 *to MPM problem, and consider the partitions A_K and B_K defined as above. Let M^* be the*
 755 *optimal partial matching in G . Then,*

- 756 (i) $c(e_i) \leq \text{cost}(M^*) \leq kn \cdot c(e_i)$, and
 757 (ii) *the diameter of G_K is at most $kn^2 \cdot c(e_i)$ for every $K \in T_j$,*

758 To prove Lemma 3.2, we need to further modify the point set so that the cost of the
 759 optimal solution does not change, while the diameter of the *whole* point set is bounded. Move
 760 the points within each component in *translation* so that the minimum distances between
 761 points across components are at least $kn \cdot c(e_i)$ but at most $O(n \cdot kn^2 \cdot c(e_i))$. This will
 762 guarantee that the optimal solution still uses edges within the components by Lemma A.1.
 763 The simplest way of achieving this is by aligning the components one by one into a “straight
 764 line”, so that the distance between the two farthest components is at most $O(n)$ times the
 765 maximum diameter of the cluster.

766 Now one can prove Lemma 3.2 by computing an $(\varepsilon c(e_i)/6k)$ -optimal circulation f on the
 767 point set after translations using additive approximation from Lemma 3.4, together with the
 768 bound $c(e_i) \leq \text{cost}(M^*)$ from Lemma A.1.

769 One small problem remains: We need to show that such reduction can be performed in
 770 $O(n \text{ polylog } n)$ time. Sharathkumar and Agarwal [15] have shown that the partition of A and
 771 B into A_K s and B_K s can be computed in $O(n \text{ polylog } n)$ time, assuming that the indices i
 772 and j can be determined in such time as well. However in our application the choice of index
 773 i depends on the optimal solution of MPM problem which we do not know.

774 To solve this issue we perform a binary search on the edges e_1, \dots, e_{r+n-1} . «**Hmm, we**
 775 **have no way to check Lemma 4.5(i); but in fact a polynomial bound is good enough.**»
 776 «**UNRESOLVED ISSUE**»

777 A.3 Number of iterations during refinement.

778 To this end we need a bound on the size of the support of f right before and throughout the
 779 execution of REFINE. This bound will also be used in the analysis for the running time of
 780 REFINE.

781 ► **Lemma A.2.** *Let f be an integer pseudoflow in N_H with $O(k)$ excess. Then, the size of*
 782 *the support of f is at most $O(k)$.*

783 **Proof.** Observe that the reduction graph H is a directed acyclic graph, and thus the support
 784 of f does not contain a cycle. Now $\text{supp}(f)$ can be decomposed into a set of inclusion-maximal
 785 paths, each of which contributes a single unit of excess to the flow if the path does not
 786 terminate at t or if more than k paths terminate at t . By assumption, there are $O(k)$ units
 787 of excess to which we can associate to the paths, and at most k paths (those that terminate
 788 at t) that we cannot associate with a unit of excess. The length of any such paths is at most
 789 three by construction of the reduction graph H . Therefore we can conclude that the number
 790 of arcs in the support of f is $O(k)$. ◀

791 ► **Corollary A.3.** *The size of $\text{supp}(f)$ is at most $O(k)$ for pseudoflow f right before or during*
 792 *the execution of REFINE.*

793 ► **Lemma 3.5.** *Let f be a pseudoflow in N_H with $O(k)$ excess. The procedure REFINE runs*
 794 *for $O(\sqrt{k})$ iterations before the excess of f becomes zero.*

795 **Proof.** Let f_0 and π_0 be the flow and potential at the start of the procedure REFINE. Let f
 796 and π be the current flow and the potential. Let $d(v)$ defined to be the amount of potential
 797 increase at v , measured in units of ε ; in other words, $d(v) := (\pi(v) - \pi_0(v))/\varepsilon$.

798 Now divide the iterations executed by the procedure REFINE into two phases: The
 799 transition from the first phase to the second happens when every excess vertex v has
 800 $d(v) \geq \sqrt{k}$. At most \sqrt{k} iterations belong to the first phase as each Hungarian search
 801 increases the potential π by at least ε for each excess vertex (and thus increases $d(v)$ by at
 802 least one).

803 The number of iterations belonging to the second phase is upper bounded by the amount
 804 of total excess at the end of the first phase, because each subsequent push of a blocking
 805 flow reduces the total excess by at least one. We now show that the amount of such excess
 806 is at most $O(\sqrt{k})$. Consider the set of arcs $E^+ := \{v \rightarrow w \mid f(v \rightarrow w) < f_0(v \rightarrow w)\}$. The total
 807 amount of excess is upper bounded by the number of arcs in E^+ that crosses an arbitrarily
 808 given cut X that separates the excess vertices from the deficit vertices, when the network has
 809 unit-capacity [9, Lemma 3.6]. Consider the set of cuts $X_i := \{v \mid d(v) > i\}$ for $0 \leq i < \sqrt{k}$;
 810 every such cut separates the excess vertices from the deficit vertices at the end of first phase.
 811 Each arc in E^+ crosses at most 3 cuts of type X_i [9, Lemma 3.1]. So there is one X_i crossed
 812 by at most $3|E^+|/\sqrt{k}$ arcs in E^+ . The size of E^+ is bounded by the sum of support sizes
 813 of f and f_0 ; by Corollary A.3 the size of E^+ is $O(k)$. This implies an $O(\sqrt{k})$ bound on the
 814 total excess after the first phase, which in turn bounds the number of iterations in the second
 815 phase. ◀

816 B Proofs from Section 4

817 ► **Lemma 4.1.** *Consider $\tilde{\pi}$ an ε -optimal potential on \tilde{H}_f and π the corresponding potential*
 818 *constructed on H_f . Then,*

- 819 1. *potential π is ε -optimal on H_f , and*
- 820 2. *if arc $\text{short}(\Pi)$ is admissible under $\tilde{\pi}$, then every arc in Π is admissible under π .*

821 **Proof.** Reduced costs for any arc from a normal vertex another is unchanged under either $\tilde{\pi}$
 822 or π . Recall that a null path is comprised of one A -to- B arc, and one or two zero-cost arcs
 823 (connecting the null vertex/vertices to s and/or t). With our choice of null vertex potentials,
 824 we observe that the zero-cost arcs still have zero reduced cost. It remains to prove that an
 825 arbitrary **«residual?»** arc (a, b) **«arc or directed edge?»** satisfies the ε -optimality condition
 826 and admissibility when either a or b is a null vertex.

827 By construction of the shortcut graph, there is always a null path Π that contains (a, b) .
 828 Observe that $c_\pi(a, b) = c_\pi(\Pi)$, independent to the type of null path. Again by construction,
 829 $c_\pi(\Pi) = c_{\tilde{\pi}}(\text{short}(\Pi))$, so we have $c_\pi(a, b) = c_{\tilde{\pi}}(\text{short}(\Pi)) \geq -\varepsilon$. Additionally, if $\text{short}(\Pi)$ is
 830 admissible under $\tilde{\pi}$, then so is (a, b) under π . This proves the lemma. ◀

831 B.1 Dynamic data structures for search procedures

832 Here we formally describe in details the set of dynamic data structure we use for the
 833 Hungarian search and depth-first search procedures.

834 For Hungarian search, we maintain the following for each type of outgoing arcs of \tilde{H}_f
 835 leaving \tilde{S} :

1. Non-shortcut backward arcs (v, w) with $(w, v) \in \text{supp}(f)$. For these, we can maintain a min-heap on $\text{supp}(f)$ arcs as each v arrives in \tilde{S} .
2. Non-shortcut A -to- B forward arcs. For these, we can use a BCP data structure between $(A \setminus A_\emptyset) \cap \tilde{S}$ and $(B \setminus B_\emptyset) \setminus \tilde{S}$, weighted by potential.
3. Non-shortcut forward arcs from s -to- A and from B -to- t . For s , we can maintain a min-heap on the potentials of $B \setminus \tilde{S}$, queried while $s \in \tilde{S}$. For t , we can maintain a max-heap on the potentials of $A \cap \tilde{S}$, queried while $t \notin \tilde{S}$.
4. Shortcut arcs (s, b) corresponding to null 2-paths from s to $b \in (B \setminus B_\emptyset) \setminus S$. For these, we maintain a BCP data structure with $P = A_\emptyset$, $Q = (B \setminus B_\emptyset) \setminus S$ with weights $\omega(p) = \pi(s)$ for all $p \in P$, and $\omega(q) = \pi(q)$ for all $q \in Q$. A response (a, b) corresponds to th null 2-path (s, a, b) . This is only queried while $s \in S$.
5. Shortcut arcs (a, t) corresponding to null 2-paths from $a \in (A \setminus A_\emptyset) \cap S$ to t . For these, we maintain a BCP data structure with $P = (A \setminus A_\emptyset) \cap S$, $Q = B_\emptyset \setminus S$ with weights $\omega(p) = \pi(p)$ for all $p \in P$, and $\omega(q) = \pi(t)$ for all $q \in Q$. A response (a, b) corresponds to th null 2-path (a, b, t) . This is only queried while $t \notin S$.
6. Shortcut arcs (s, t) corresponding to null 3-paths. For these, we maintain in a BCP data structure with $P = A_\emptyset \setminus S$, $Q = B_\emptyset \setminus S$ with weights $\omega(p) = \pi(s)$ for all $p \in P$, and $\omega(q) = \pi(t)$ for all $q \in Q$. A response (a, b) corresponds to th null 3-path (s, a, b, t) . This is only queried while $s \in S$ and $t \notin S$.

For depth-first search, we maintain the following for each type of outgoing arcs of \tilde{H}_f leaving \tilde{S} :

1. Non-shortcut backward arcs (v', w') with $(w', v') \in \text{supp}(f)$. For these, we can maintain a min-heap on $(w', v') \in \text{supp}(f)$ arcs for each normal $v' \in V$.
2. Non-shortcut A -to- B forward arcs. For these, we maintain a NN data structure over $P = (B \setminus B_\emptyset) \setminus \tilde{S}$, with weights $\omega(p) = \pi(p)$ for each $p \in P$. We subtract $\pi(v')$ from the NN distance to recover the reduced cost of the arc from v' .
3. Non-shortcut forward arcs from s -to- A and from B -to- t . For s , we can maintain a min-heap on the potentials of $B \setminus \tilde{S}$, queried only if $v' = s$. For B -to- t arcs, there is only one arc to check if $v' \in B$, which we can examine manually.
4. Shortcut arcs (s, b) corresponding to null 2-paths from s to $b \in (B \setminus B_\emptyset) \setminus S$. For these, we maintain a NN data structure with $P = A_\emptyset$, $Q = (B \setminus B_\emptyset) \setminus S$ with weights $\omega(p) = \pi(s)$ for all $p \in P$, and $\omega(q) = \pi(q)$ for all $q \in Q$. A response (a, b) corresponds to th null 2-path (s, a, b) . This is only queried if $v' = s$.
5. Shortcut arcs (a, t) corresponding to null 2-paths from $a \in (A \setminus A_\emptyset) \cap S$ to t . For these, we maintain a NN data structure over $P = B_\emptyset \setminus S$ with weights $\omega(p) = \pi(t)$ for each $p \in P$. A response (v', b) corresponds to th null 2-path (v', b, t) . We subtract $\pi(v')$ from the NN distance to recover the reduced cost of the arc from v' . This is not queried if $t \in \tilde{S}$.
6. Shortcut arcs (s, t) corresponding to null 3-paths. For these, we maintain in a NN data structure with $P = A_\emptyset \setminus S$, $Q = B_\emptyset \setminus S$ with weights $\omega(p) = \pi(s)$ for all $p \in P$, and $\omega(q) = \pi(t)$ for all $q \in Q$. A response (a, b) corresponds to th null 3-path (s, a, b, t) . This is only queried while $v' = s$ and $t \notin S$.

B.2 Number of relaxations

First we bound the number of relaxations performed by both the Hungarian search and the depth-first search.

► **Lemma B.1.** *Hungarian search performs $O(k)$ relaxations before a deficit vertex is reached.*

Proof. **«TO BE REWRITTEN.»** First we prove that there are $O(k)$ non-shortcut relaxations. Each edge relaxation adds a new vertex to S , and non-shortcut relaxations only add normal vertices. The vertices of $V \setminus S$ fall into several categories: (i) s or t , (ii) vertices of A or B with 0 imbalance, and (iii) deficit vertices of A or B (S contains all excess vertices). The number of vertices in (i) and (iii) is $O(k)$, leaving us to bound the number of (ii) vertices.

An A or B vertex with 0 imbalance must have an even number of $\text{supp}(f)$ edges. There is either only one positive-capacity incoming arc (for A) or outgoing arc (for B), so this quantity is either 0 or 2. Since the vertex is normal, this must be 2. We charge 0.5 to each of the two $\text{supp}(f)$ arcs; the arcs of $\text{supp}(f)$ have no more than 1 charge each. Thus, the number of type (ii) vertex relaxations is $O(|\text{supp}(f)|)$. By Corollary A.3, $O(|\text{supp}(f)|) = O(k)$.

Next we prove that there are $O(k)$ shortcut relaxations. Recall the categories of shortcuts from the list of data structures above. We have shortcuts corresponding to (i) null 2-paths surrounding $a \in A_\emptyset$, (ii) null 2-paths surrounding $b \in B_\emptyset$, and (iii) null 3-paths, which go from s to t . There is only one relaxation of type (iii), since t can only be added to S once. The same argument holds for type (ii).

Each type (i) relaxation adds some normal $b \in B \setminus B_\emptyset$ into S . Since b is normal, it must either have deficit or an adjacent arc of $\text{supp}(f)$. We charge this relaxation to b if it is deficit, or the adjacent arc of $\text{supp}(f)$ otherwise. No vertex is charged more than once, and no $\text{supp}(f)$ edge is charged more than twice, therefore the total number of type (i) relaxations is $O(|\text{supp}(f)|)$. By Corollary A.3, $O(|\text{supp}(f)|) = O(k)$. ◀

Similarly we can prove that there are $O(k)$ relaxations during the DFS.

► **Corollary B.2.** *Depth-first search performs $O(k)$ relaxations before a deficit vertex is reached.*

B.3 Time analysis

Now we complete the time analysis by showing that each Hungarian search and depth-first search can be implemented in $O(k \text{ polylog } n)$ time after a one-time $O(n \text{ polylog } n)$ -time preprocessing.

► **Lemma B.3.** *After $O(n \text{ polylog } n)$ -time preprocessing, each Hungarian search can be implemented in $O(k \text{ polylog } n)$ time.*

Proof. Each of the constant number of data structures used by the Hungarian search can be constructed in $O(n \text{ polylog } n)$ time. For each data structure queried during a relaxation, the new vertex moved into S causes a constant number of updates, each of which can be implemented in $O(\text{polylog } n)$ time. We first prove that the number of BCP operations during the Hungarian search over is bounded by $O(k)$.

1. Let S^t denote the initial set S at the beginning of the t -th Hungarian search. Assume for now that, at the beginning of the $(t + 1)$ -th Hungarian search, we have the set S^t from the previous iteration. To construct S^{t+1} , we remove the vertices that had excess decreased to zero by the t -th blocking flow. Thus, we are able to initialize S at the cost of one BCP deletion per excess vertex, which sums to $O(k)$ over the entire course of REFINES. **«Too strong as a bound? Is it enough to look at one Hungarian search?»**
2. During each Hungarian search, a vertex entering S may incur one BCP insertion/deletion. We can charge the updates to the number of relaxations over the course of Hungarian search. The number of relaxations in a Hungarian search is $O(k)$ by Lemma B.1.

925 3. To obtain S^t , we keep track of the points added to S^t since the last Hungarian search.
 926 After the augmentation, we remove those points added to S^t . By (2) there are $O(k)$ such
 927 points to be deleted, so reconstructing S^t takes $O(k)$ BCP operations.

928 For potential updates, we use the same trick by Vaidya [17] to lazily update potentials
 929 after vertices leave S (similar to Lemma ??), but this time only for normal vertices. Normal
 930 vertices are stored in each data structure with weight $\omega(v) = \pi(v) - \delta$, and δ is increased in
 931 lieu of increasing the potential of vertices in S . When a vertex leave S (through the rewind
 932 mechanism above), we restore its potential as $\pi(v) \leftarrow \omega(v) + \delta$. With lazy updates, the
 933 number of potential updates on normal vertices is bounded by the number of relaxations in
 934 the Hungarian search, which is $O(k)$ by Lemma B.1. Note that null vertex potentials are not
 935 handled in the Hungarian search. **«then where? Lemma B.6»** ◀

936 There are no potentials to update within DFS, so the running time of DFS boils down
 937 to the time spent to querying and updating the data structures.

938 ► **Lemma B.4.** *After $O(n \text{ polylog } n)$ -time preprocessing, each depth-first search can be*
 939 *implemented in $O(k \text{ polylog } n)$ time.*

940 **Proof.** At the beginning of REFINE, we can initialize the $O(1)$ data structures used in DFS
 941 in $O(n \text{ polylog } n)$ time. We use the same rewinding mechanism as in Hungarian search
 942 (Lemma B.3) to avoid reconstructing the data structures across iterations of REFINE, so
 943 the total time spent is bounded by the $O(\text{polylog } n)$ times the number of relaxations. By
 944 Corollary B.2, the running time for depth-first search is $O(k \text{ polylog } n)$. ◀

945 B.4 Number of potential updates on null vertices

946 In our implementation of REFINE, we do not explicitly construct \tilde{H}_f ; instead we query its
 947 edges using BCP/NN oracles and min/max heaps on elements of H_f . Potentials on the null
 948 vertices are only required right before an augmentation sends a flow through a null path,
 949 making the null vertices it passes normal. We use the construction from Lemma 4.1 to obtain
 950 potential π on H_f such that the flow f is both ε -optimal and admissible with respect to π .

951 **Size of blocking flows.** Now we bound the total number arcs whose flow is updated by a
 952 blocking flow during the course of REFINE. This bounds both the time spent updating the
 953 flow on these arcs and also the time spent on null vertex potential updates (Lemma B.6).

954 ► **Lemma B.5.** *The support of each blocking flow found in REFINE is of size $O(k)$.*

955 **Proof.** Let i be fixed and consider the invocation of DFS which produces the i -th blocking
 956 flow f_i . DFS constructs f_i as a sequence of admissible excess-deficit paths, which appear as
 957 path P in Algorithm ??. Every arc in P is an arc relaxed by DFS, so N_i is bounded by the
 958 number of relaxations performed in DFS. Using Corollary B.2, we have $N_i = O(k)$. ◀

959 ► **Lemma B.6.** *The number of end-of-REFINE null vertex potential updates is $O(n)$. The*
 960 *number of augmentation-induced null vertex potential updates in each invocation of REFINE*
 961 *is $O(k \log k)$.*

962 **Proof.** The number of end-of-REFINE potential updates is $O(n)$. Each update due to flow
 963 augmentation involves a blocking flow sending positive flow through a null path, causing a
 964 potential update on the passed null vertex. We charge this potential update to the edges
 965 of that null path, which are in turn arcs with positive flow in the blocking flow. For each

blocking flow, no positive arc is charged more than twice. It follows that the number of augmentation-induced updates is at most the size of support of the blocking flow, which is $O(k)$ by Lemma B.5. According to Lemma 3.5 there are $O(\sqrt{k})$ iterations of REFINE before it terminates. Summing up we have an $O(k\sqrt{k})$ bound over the course of REFINE. ◀

Now combining Lemma B.3, Lemma B.4, and Lemma B.6 completes the proof of Theorem 1.2.

C Proofs from Section 5

C.1 Recovering the optimal flow

We use the recovery strategy from Agarwal *et al.* [1], which runs in $O(n \text{polylog } n)$ time. The main idea is that, if \mathcal{T} is an undirected *spanning tree of admissible edges* under optimal potentials π^* , then there exists an optimal flow f^* with support only on arcs corresponding to edges of \mathcal{T} . Intuitively, \mathcal{T} is a maximal set of linearly independent dual LP constraints for the optimal dual (π^*), so there exists an optimal primal solution (f^*) with support only on these arcs. To see this, we can use a perturbation argument: raising the cost of each non-tree edge by $\varepsilon > 0$ does not change $\text{cost}(\pi^*)$ or the feasibility of π^* , but does raise the cost of any circulation f using non-tree edges. Strong duality suggests that $\text{cost}(f^*) = \text{cost}(\pi^*)$ is unchanged, therefore f^* must have support only on the tree edges.

Since the arcs corresponding to edges of \mathcal{T} have no cycles, we can solve the maximum flow in linear time using the following greedy algorithm. Let $\text{par}(v)$ be the parent of vertex v in \mathcal{T} . We begin with $f^* = 0$ and process \mathcal{T} from its leaves upwards. For a supply leaf v , we satisfy its supply by choosing $f^*(v \rightarrow \text{par}(v)) \leftarrow \phi(v)$. Otherwise if leaf v is a demand vertex, we choose $f^*(\text{par}(v) \rightarrow v) \leftarrow -\phi(v)$. Once we've solved the supplies/demands for each leaf, then we can *trim* the leaves, removing them from \mathcal{T} and setting the supply/demand of each parent-of-a-leaf to its current imbalance. Then, we can recurse on this smaller tree and its new set of leaves.

► **Lemma C.1.** *Let $G(\mathcal{T})$ be the subnetwork of G corresponding to edges of the undirected spanning tree \mathcal{T} . If there exists a flow in $G(\mathcal{T})$ which satisfies every supply and demand, then the greedy algorithm finds the maximum flow in $G(\mathcal{T})$ in $O(n)$ time.*

Proof. Observe that, for any flow f in G , $\text{supp}(f)$ has no paths of length longer than one. Thus, if a flow f^* satisfying supplies/demands exists within $G(\mathcal{T})$, then each supply vertex has flow paths that terminate at its parent/children. Similarly, each demand vertex receive all its flow from its parent/children. Since there is only one option for a supply leaf (resp. demand leaf) to send its flow (resp. receive its flow), the greedy algorithm correctly identifies the values of f^* for arcs adjoining \mathcal{T} leaves. Trimming these leaves, we can apply this argument recursively for their parents. The running time of the greedy algorithm is $O(n)$, as leaves can be identified in $O(n)$ time and no vertex becomes a leaf more than once. ◀

It remains to show how we construct \mathcal{T} . We begin with a (spanning) *shortest path tree* (SPT) T in the residual network of f , under reduced costs and rooted at an arbitrary vertex r . For the SPT to span, we need the additional assumption that G is strongly connected. We can make G strongly connected by adding a 0-supply vertex s with arcs $s \rightarrow a$ for all $a \in A$ and $b \rightarrow a$ for all $b \in B$, with some high cost M . Following Orlin [13], these arcs cannot appear in an optimal flow if M is sufficiently high, and we can extend π^* to include s using $\pi^*(s) = 0$ if $M > \max_{b \in B} \pi^*(b)$. This extension to π^* preserves feasibility.

1009 The edges corresponding to arcs of T do not suffice for \mathcal{T} , since some SPT arcs may be
 1010 inadmissible. Let $d_r(v)$ be the shortest path distance of $v \in A \cup B \cup \{s\}$ from r , and consider
 1011 potentials $\pi^\# = \pi^* - d_r$.

1012 ► **Lemma C.2 Orlin [13] Lemma 3.** *Let f be a flow satisfying the optimality conditions*
 1013 *with respect to π^* . Then, (i) f satisfies the optimality conditions with respect to $\pi^\#$, and (ii)*
 1014 *all SPT arcs are admissible under $\pi^\#$.*

1015 We can use this lemma to argue that $\pi^\#$ is still optimal. Recall that f has values defined
 1016 only on the non-contracted residual arcs; we can apply the first part of Lemma C.2 on these
 1017 arcs. For arcs within contracted components, we use a different argument. Observe that
 1018 each $\hat{v} \in \hat{V}$ is spanned by a set of $\text{supp}(f)$ arcs, which are admissible by invariant. Thus, all
 1019 $v \in \hat{v}$ are equidistant from r , and they will have the same value $d_r(v)$. It follows that the
 1020 reduced costs of arcs with both endpoints in \hat{v} do not change when replacing π^* with $\pi^\#$, so
 1021 arcs contained in \hat{v} that met the optimality conditions for π^* still meet them for $\pi^\#$.

1022 From the second part of Lemma C.2, the SPT T is a spanning tree of admissible arcs
 1023 under $\pi^\#$. We set \mathcal{T} to be the set of undirected edges corresponding to T .

1024 **Computing the SPT.** We conclude by describing the procedure for building the SPT, i.e.
 1025 by running Dijkstra's algorithm in the residual network. We use a geometric implementation
 1026 that is very similar to Hungarian search. We begin with $S = \{r\}$ and $d_r(r) = 0$, where r is
 1027 our arbitrary root. For all other vertices, $d_r(v)$ is initially unknown. In each iteration, we
 1028 relax the minimum-reduced cost arc $v \rightarrow w$ in the frontier $S \times (A \cup B) \setminus S$, adding w to S , and
 1029 setting $d_r(w) = d_r(v) + c_{\pi^*}(v, w)$. Once $S = A \cup B$, the SPT T is the set of relaxed arcs.

1030 If an either direction of an arc of $\text{supp}(f)$ enters the frontier, we relax it immediately.
 1031 To detect support arcs, we build a list for each $v \in A \cup B$ of the support arcs which use
 1032 v as an endpoint, and once $v \in S$ we check its list. There are $O(n)$ support arcs in total
 1033 (by acyclicity of $E(\text{supp}(f))$), so the total time spent searching these lists is $O(n)$. Such
 1034 relaxations are correct for the shortest path tree, since the support edges are admissible and
 1035 reduced costs are nonnegative.

1036 Other edges appearing in the frontier can be split into three categories:

- 1037 1. Forward A -to- B arcs. We query these using a BCP with $P = A \cap S$ and $Q = B \setminus S$.
- 1038 2. B -to- s arcs. These will never have flow support. We can query the minimum with a
 1039 max-heap on potentials of $B \cap S$. We query these while $s \in S$.
- 1040 3. s -to- A arcs. These will also never have flow support. We can query the minimum with a
 1041 min-heap on potentials of $A \setminus S$. We query these while $s \in S$.

1042 We perform $O(n)$ relaxations and takes $O(\text{polylog } n)$ time per relaxation, for non-support
 1043 relaxations. An additional $O(n)$ time is spent relaxing support edges. The total running
 1044 time of Dijkstra's algorithm is $O(n \text{ polylog } n)$. Combining with Lemma C.1, we obtain the
 1045 following.

1046 ► **Lemma C.3.** *Given optimal potentials π^* and an optimal contracted flow f , the optimal*
 1047 *flow f^* can be computed in $O(n \text{ polylog } n)$ time.*

1048 C.2 Recovering the optimal flow for sum-of-distances.

1049 When the matching objective uses the just the p -norms (that is, when $q = 1$), we can prove
 1050 that the subgraph formed by admissible arcs is in fact *planar*. Planarity gives us two things
 1051 towards a simple f^* recovery: there are only a linear number of admissible arcs, and the
 1052 max-flow on them can be solved in near-linear time with planar graph max-flow algorithms.

Up until now, we have not placed restrictions on coincidence between A and B , but for the next proof it is useful to do so. We can assume that all points within $A \cup B$ are distinct, otherwise we can replace all points coincident at $x \in \mathbb{R}^2$ with a single point whose supply/demand is $\sum_{v \in A \cup B: v=x} \lambda(v)$. This is roughly equivalent to transporting as much as we can between coincident supply and demand, and is optimal by triangle inequality.

Without loss of generality, assume π^* is nonnegative (raising π^* uniformly on all points does not change the objective or feasibility). Recall that π^* is feasibility if for all $a \in A$ and $b \in B$

$$c_{\pi^*}(a, b) = \|a - b\|_p - \pi^*(a) + \pi^*(b) \geq 0.$$

An arc $a \rightarrow b$ is admissible when

$$c_{\pi^*}(a, b) = \|a - b\|_p - \pi^*(a) + \pi^*(b) = 0.$$

We note that these definitions have a nice visual: Place disks D_q of radius $\pi(q)$ at each $q \in A \cup B$. Feasibility states that for all $a \in A$ and $b \in B$, D_a cannot contain D_b with a gap between their boundaries. The arc $a \rightarrow b$ is admissible when D_a contains D_b and their boundaries are tangent.

► **Lemma C.4.** *Let π^* be a set of optimal potentials for the point sets A and B , under costs $c(a, b) = \|a - b\|_p$. Then, the set of admissible arcs under π^* form a planar graph.*

Proof. We assume the points of $A \cup B$ are in general position (e.g. by symbolic perturbation) such that no three points are collinear. Let $a_1 \rightarrow b_1$ and $a_2 \rightarrow b_2$ be any pair of admissible arcs under π^* . We will isolate them from the rest of the points, considering π^* restricted to the four points $\{a_1, a_2, b_1, b_2\}$. Clearly, this does not change whether the two arcs cross. Observe that we can raise $\pi^*(a_2)$ and $\pi^*(b_2)$ uniformly, until $c_{\pi^*}(a_2, b_1) = 0$, without breaking feasibility or changing admissibility of $a_1 \rightarrow b_1$ and $a_2 \rightarrow b_2$. Henceforth, we assume that we have modified π^* in this way to make $a_2 \rightarrow b_1$ admissible. Given positions of a_1, a_2 , and b_1 , we now try to place b_2 such that $a_1 \rightarrow b_1$ and $a_2 \rightarrow b_2$ cross. Specifically, b_2 must be placed within a region \mathcal{F} that lies between the rays $\overrightarrow{a_2 a_1}$ and $\overrightarrow{a_2 b_1}$, and within the halfplane bounded by $\overleftrightarrow{a_1 b_1}$ that does not contain a_2 .

Let $g_a(q) := \|a - q\| - \pi^*(a)$ for $a \in A$ and $q \in \mathbb{R}^2$. Let the *bisector* between a_1 and a_2 be $\beta := \{q \in \mathbb{R}^2 \mid g_{a_1}(q) = g_{a_2}(q)\}$. β is a curve subdividing the plane into two open faces, one where g_{a_1} is minimized and the other where g_{a_2} is. From these definitions, admissibility of $a_1 \rightarrow b_1$ and $a_2 \rightarrow b_1$ imply that b_1 is a point of the bisector.

We show that \mathcal{F} lies entirely on the g_{a_1} side of the bisector. First, we prove that the closed segment $\overline{a_1 b_1}$ lies entirely on the g_{a_1} side, except b_1 which lies on β . Any $q \in \overline{a_1 b_1}$ can be written parametrically as $q(t) = (1 - t)b_1 + ta_1$ for $t \in [0, 1]$. Consider the single-variable functions $g_{a_1}(q(t))$ and $g_{a_2}(q(t))$.

$$\begin{aligned} g_{a_1}(q(t)) &= (1 - t)\|a_1 - b_1\| - \pi(a_1) \\ g_{a_2}(q(t)) &= \|(a_2 - b_1) - t(a_1 - b_1)\| - \pi(a_2) \end{aligned}$$

At $t = 0$, these expressions are equal. Observe that the derivative with respect to t of $g_{a_1}(q(t))$ is less than $g_{a_2}(q(t))$. Indeed, the value of $\frac{d}{dt}\|(a_2 - b_1) - t(a_1 - b_1)\|$ is at least $-\|a_1 - b_1\| = \frac{d}{dt}g_{a_1}(q(t))$, which is realized if and only if $\frac{(a_2 - b_1)}{\|a_2 - b_1\|} = \frac{(a_1 - b_1)}{\|a_1 - b_1\|}$. This corresponds to $\overrightarrow{a_2 b_1}$ and $\overrightarrow{a_1 b_1}$ being parallel, but this is disallowed since a_1, a_2, b_1 are in general position. Thus, $g_{a_1}(q(t)) \leq g_{a_2}(q(t))$ with equality only at b_1 .

Now, we parameterize each point of \mathcal{F} in terms of points on $\overline{a_1 b_1}$. Every $q \in \mathcal{F}$ can be written as $q(t') = q' + t'(q' - a_2)$ for some $q' \in \overline{a_1 b_1}$ and $t \geq 0$, i.e. $q' = \overline{a_1 b_1} \cap \overrightarrow{a_2 q}$.

We call q' the *projection* of q onto $\overline{a_1 b_1}$. We can write g_{a_1} and g_{a_2} in terms of t' and observe that $\frac{d}{dt'} g_{a_1}(q(t')) \leq \frac{d}{dt'} g_{a_2}(q(t'))$, as the derivative of $g_a(q(t'))$ is maximized if $(q(t') - a)$ is parallel to $(q(t') - a_2)$ and lower otherwise. Notably, $q(t')$ with projection b_1 have $\frac{d}{dt'} g_{a_1}(q(t')) < \frac{d}{dt'} g_{a_2}(q(t'))$, since a_1, a_2, b_1 are in general position. Any $q(t')$ with a different projection do not have strict inequality, but the projection itself has $g_{a_1}(q') < g_{a_2}(q')$ for $q' \neq b_1$ since it lies on $\overline{a_1 b_1}$. Therefore, for all $q \in \mathcal{F} \setminus \{b_1\}$, $g_{a_1}(q') < g_{a_2}(q')$, and \mathcal{F} lies on the g_{a_1} side of the bisector except for b_1 which lies on β . We can eliminate b_1 as a candidate position for b_2 , since points of B cannot coincide.

Observe that $g_{a_1}(b) < g_{a_2}(b)$ for $b \in B$ implies that $c_\pi(a_1, b) < c_\pi(a_2, b)$, and $c_\pi(a_1, b) = c_\pi(a_2, b)$ if and only if b lies on β . This holds for all $b \in \mathcal{F}$ including our prospective b_2 , but then $c_\pi(a_1, b_2) < c_\pi(a_2, b_2) = 0$ since $a_2 \rightarrow b_2$ is admissible. This violates feasibility of $a_1 \rightarrow b_2$, so there is no feasible placement of b_2 which also crosses $a_1 \rightarrow b_1$ with $a_2 \rightarrow b_2$. ◀

We can construct the entire set of admissible arcs by repeatedly querying the minimum-reduced-cost outgoing arc for each $a \in A$ until the result is not admissible. By Lemma C.4 the resulting arc set forms a planar graph, so by Euler's formula the number of arcs to query is $O(n)$. We can then find the maximum flow in time $O(n \log n)$ time, using for example the planar maximum-flow algorithm by Erickson [8]. ◀◀cite others like Klein▶▶

► **Lemma C.5.** *If the transportation objective is sum-of-costs, then given the optimal potentials π^* , we can compute an optimal flow f^* in $O(n \text{ polylog } n)$ time.*

C.3 Dead vertices

Let the *support degree* of a vertex be its degree in the graph induced by the underlying edges of $\text{supp}(f)$. We call a vertex $b \in B$ *dead* if b has support degree 0 and is not an active excess or deficit vertex; call it *living* otherwise. Dead vertices are essentially equivalent to the *null vertices* of Section 3. However, since the reduction in this section does not use a super-source/super-sink, we can simply remove these from consideration during a Hungarian search — they will not terminate the search, and have no outgoing residual arcs. Like the null vertices, we ignore dead vertex potentials and infer feasible potentials when they become live again. We use A_ℓ and B_ℓ to denote the living vertices of points in A and B , respectively. Note that being dead/alive is a notion strictly defined only for vertices, and not for contracted components.

We say a dead vertex is *revived* when it stops meeting either condition of the definition. Dead vertices are only revived after Δ decreases (at the start of a subsequent excess scale) as no augmenting path will cross a dead vertex and they cannot meet the criteria for contractions. When a dead vertex is revived, we must add it back into each of our data structures and give it a feasible potential. For revived $b \in B$, a feasible choice of potential is $\pi(b) \leftarrow \max_{a \in A} (\pi(a) - c(a, b))$ which we can query by maintaining a weighted nearest neighbor data structure on the points of A . The total number of revivals is bounded above by the number of augmentations: since the final flow is a circulation on \hat{G} and a newly revived vertex v has no incident arcs in $\text{supp}(f)$ and cannot be contracted, there is at least one subsequent augmentation which uses v as its beginning or end. Thus, the total number of revivals is $O(n \log n)$.

C.4 Number of relaxations

By prioritizing the relaxation of support arcs, we also have the following lemma.

1139 ► **Lemma C.6 (Agarwal et al. [1]).** *If arcs of $\text{supp}(f)$ are relaxed first as they arrive on*
 1140 *the frontier, then $E(\text{supp}(f))$ is acyclic.*

1141 **Proof.** Let f_i be the pseudoflow after the i -th augmentation, and let T_i be the forest of
 1142 relaxed arcs generated by the Hungarian search for the i -th augmentation. Namely, the i -th
 1143 augmenting path is an excess-deficit path in T_i , and all arcs of T_i are admissible by the time
 1144 the augmentation is performed. Let $E(T_i)$ be the undirected edges corresponding to arcs of
 1145 T_i . Notice that, $E(\text{supp}(f_{i+1})) \subseteq E(\text{supp}(f_i)) \cup E(T_i)$. We prove that $E(\text{supp}(f_i)) \cup E(T_i)$
 1146 is acyclic by induction on i ; as $E(\text{supp}(f_{i+1}))$ is a subset of these edges, it must also be
 1147 acyclic. At the beginning with $f_0 = 0$, $E(\text{supp}(f_0))$ is vacuously acyclic.

1148 Let $E(\text{supp}(f_i))$ be acyclic by induction hypothesis. Since T_i is a forest (thus, acyclic),
 1149 any hypothetical cycle Γ that forms in $E(\text{supp}(f_i)) \cup E(T_i)$ must contain edges from
 1150 both $E(\text{supp}(f_i))$ and $E(T_i)$. To give a visual analogy, we will color $e \in \Gamma$ *purple* if
 1151 $e \in E(\text{supp}(f_i)) \cap E(T_i)$, *red* if $e \in E(\text{supp}(f_i))$ but $e \notin E(T_i)$, and *blue* if $e \in E(T_i)$ but
 1152 $e \notin E(\text{supp}(f_i))$. Then, Γ is neither entirely red nor entirely blue. We say that red and purple
 1153 edges are *red-tinted*, and similarly blue and purple edges are *blue-tinted*. Roughly speaking,
 1154 our implementation of the Hungarian search prioritizes relaxing red-tinted admissible arcs
 1155 over pure blue arcs.

1156 We can sort the blue-tinted edges of Γ by the order they were relaxed into S during the
 1157 Hungarian search forming T_i . Let $(v, w) \in \Gamma$ be the last pure blue edge relaxed, of all the
 1158 blue-tinted edges in Γ — after (v, w) is relaxed, the remaining unrelaxed, blue-tinted edges
 1159 of Γ are purple.

1160 Let us pause the Hungarian search the moment before (v, w) is relaxed. At this point,
 1161 $v \in S$ and $w \notin S$, and the Hungarian search must have finished relaxing all frontier support
 1162 arcs. By our choice of (v, w) , $\Gamma \setminus (v, w)$ is a path of relaxed blue edges and red-tinted edges
 1163 which connect v and w . Walking around $\Gamma \setminus (v, w)$ from v to w , we see that every vertex of
 1164 the cycle must be in S already: $v \in S$, relaxed blue edges have both endpoints in S , and any
 1165 unrelaxed red-tinted edge must have both endpoints in S , since the Hungarian search would
 1166 have prioritized relaxing the red-tinted edges to grow S before relaxing (v, w) (a blue edge).
 1167 It follows that $w \in S$ already, a contradiction.

1168 No such cycle Γ can exist, thus $E(\text{supp}(f_i)) \cup E(T_i)$ is acyclic and $E(\text{supp}(f_{i+1})) \subseteq$
 1169 $E(\text{supp}(f_i)) \cup E(T_i)$ is acyclic. By induction, $E(\text{supp}(f_i))$ is acyclic for all i . ◀

1170 Let $E(\Sigma_a)$ *«only used once»* be the underlying edges of the support star centered at a
 1171 and $F := E(\text{supp}(f)) \setminus \bigcup_{a \in A} E(\Sigma_a)$. Using Lemma C.6, we can show that the number of
 1172 support arcs outside support stars ($|F|$) is small.

1173 ► **Lemma C.7.** $|B_\ell \setminus \bigcup_{a \in A} \Sigma_a| \leq r$.

1174 **Proof.** F is constructed from $E(\text{supp}(f))$ by eliminating edges in support stars, therefore all
 1175 edges in F must adjoin vertices in B of support degree at least 2. By Lemma C.6, $E(\text{supp}(f))$
 1176 is acyclic and therefore forms a spanning forest over $A \cup B_\ell$, so F is also a bipartite forest.
 1177 All leaves of F are therefore vertices of A .

1178 Pick an arbitrary root for each connected component of F to establish parent-child
 1179 relationships for each edge. As no vertex in B is a leaf, each vertex in B has at least one
 1180 child. Charge each vertex in B to one of its children in F , which must belong to A . Each
 1181 vertex in A is charged at most once. Thus, the number of B_ℓ vertices outside of support
 1182 stars is no more than r . ◀

1183 ► **Lemma 5.2.** *Suppose we have stripped the graph of dead vertices. The number of relaxation*
 1184 *steps in a Hungarian search outside of support stars is $O(r)$.*

1185 **Proof.** If there are no dead vertices, then each non-support star relaxation step adds either
 1186 (i) an active deficit vertex, (ii) a non-deficit vertex $a \in A_\ell$, or (iii) a non-deficit vertex $b \in B_\ell$
 1187 of support degree at least 2. There is a single relaxation of type (i), as it terminates the
 1188 search. The number of vertices of type (ii) is r , and the number of vertices of type (iii) is at
 1189 most r by Lemma C.7. The lemma follows. ◀

1190 ▶ **Lemma 5.3.** *Hungarian search takes $O(r\sqrt{n} \text{polylog } n)$ time.*

1191 **Proof.** The number of relaxation steps outside of support stars is $O(r)$ by Lemma 5.2. The
 1192 time per relaxation outside of support stars is $O(\sqrt{n} \text{polylog } n)$. The time spent processing
 1193 relaxations within a support star is $O(\sqrt{n} \text{polylog } n)$, and at most r are relaxed during the
 1194 search. The total time is therefore $O(r\sqrt{n} \text{polylog } n)$. ◀

1195 C.5 Updating support stars

1196 Initially, we label stars big or small according to the \sqrt{n} threshold. A star that is currently
 1197 big is turned into a small star once $|\Sigma_a| \leq \sqrt{n}/2$. A star that is currently small is turned into
 1198 a big star once $|\Sigma_a| \geq 2\sqrt{n}$. This way, the time spent rebuilding/updating the respective
 1199 data structures can be amortized to the insertions/deletions that preceded the switch, plus
 1200 some $O(r)$ extra work if the the update is small-to-big.

1201 A star Σ_a that is switching from big-to-small has size $|\Sigma_a| \leq \sqrt{n}/2$. When switching, we
 1202 delete $\mathcal{D}_{\text{big}}(a)$ and insert Σ_a into $\mathcal{D}_{\text{small}}$. Thus, the time spent for big-to-small update is
 1203 $O(\sqrt{n} \text{polylog } n)$, and there were at least $\sqrt{n}/2$ points removed from Σ_a since it was last big.

1204 **«A MESS; NEED TO BE REWRITTEN»** A star Σ_a that is switching from small-
 1205 to-big has size $|\Sigma_a| = \sqrt{n} + x \geq 2\sqrt{n}$, for some integer $x \geq \sqrt{n}$. Rearranging, we have
 1206 $|\Sigma_a| \leq 2x$. When switching, we delete all $|\Sigma_a|$ points from $\mathcal{D}_{\text{small}}$ and construct a new $\mathcal{D}_{\text{big}}(a)$.
 1207 Constructing $\mathcal{D}_{\text{big}}(a)$ requires inserting $O(r)$ points of A (into P) and the $|\Sigma_a|$ points of the
 1208 star (into Q). Thus, the time spent for a small-to-big update is $O((r + x) \text{polylog } n)$, and
 1209 there were at least $x \geq \sqrt{n}$ points added to Σ_a since it was last small.