Efficient Algorithms for Geometric Partial Matching

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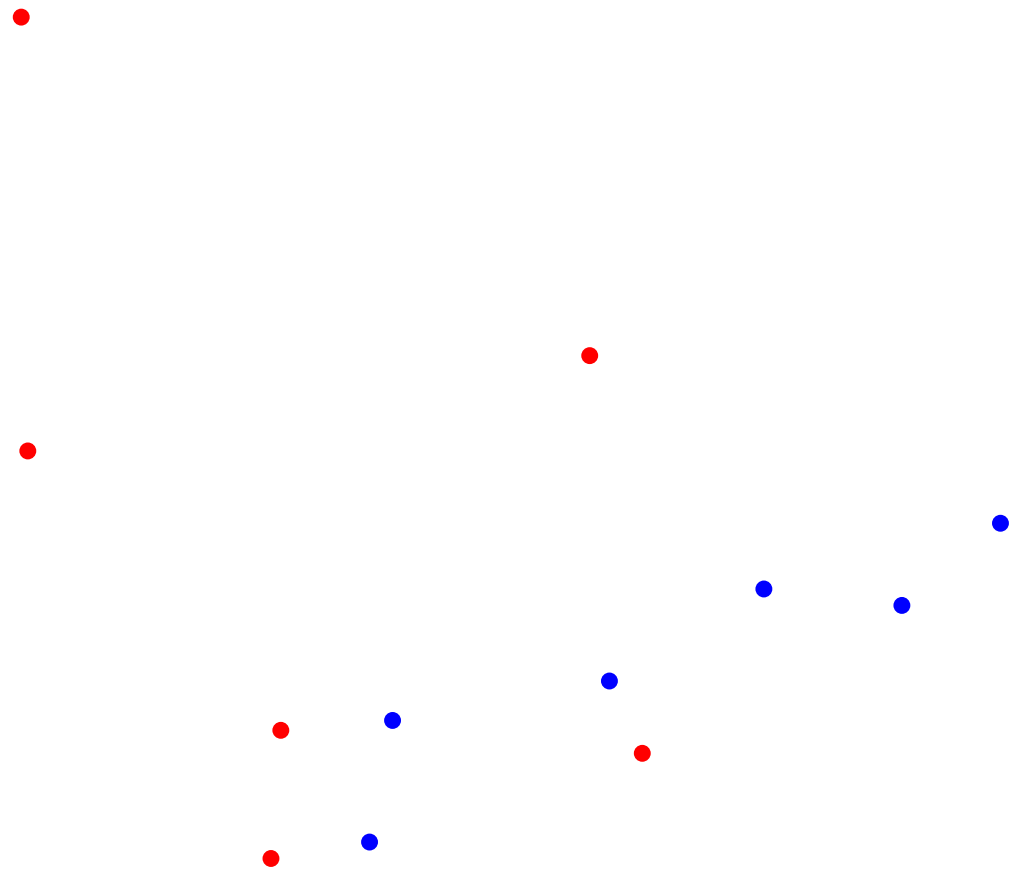
Geometric (bipartite) matching

A, B



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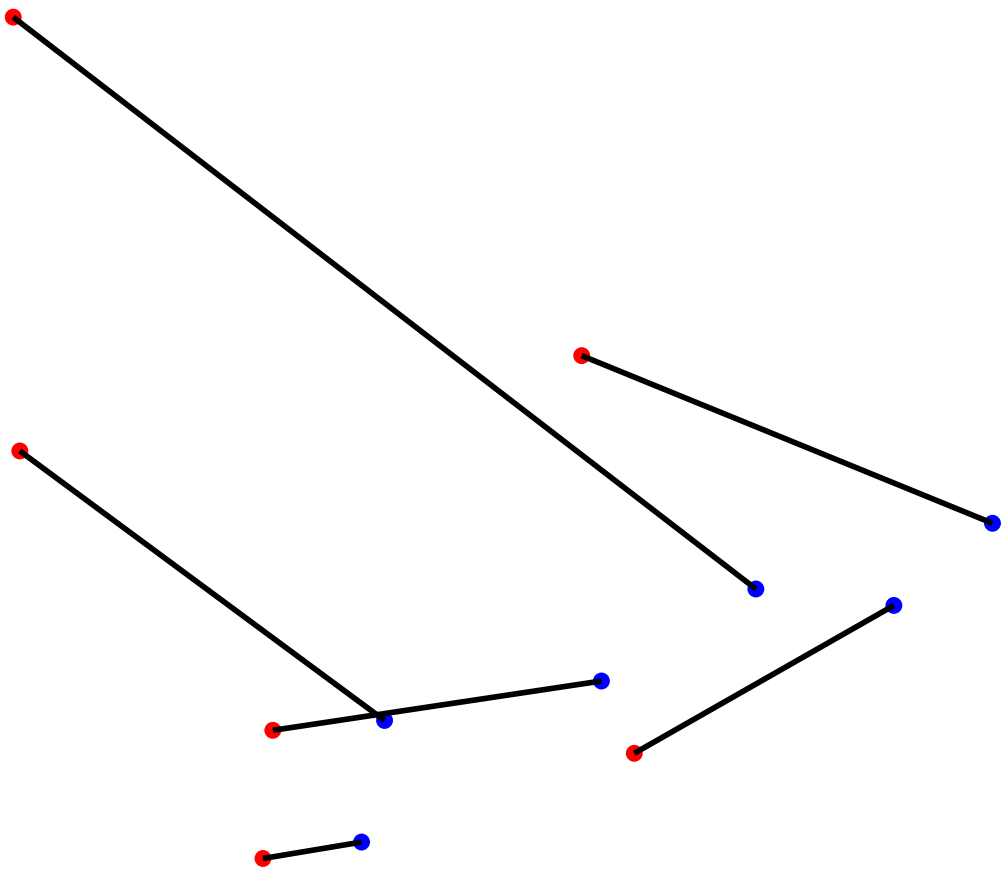

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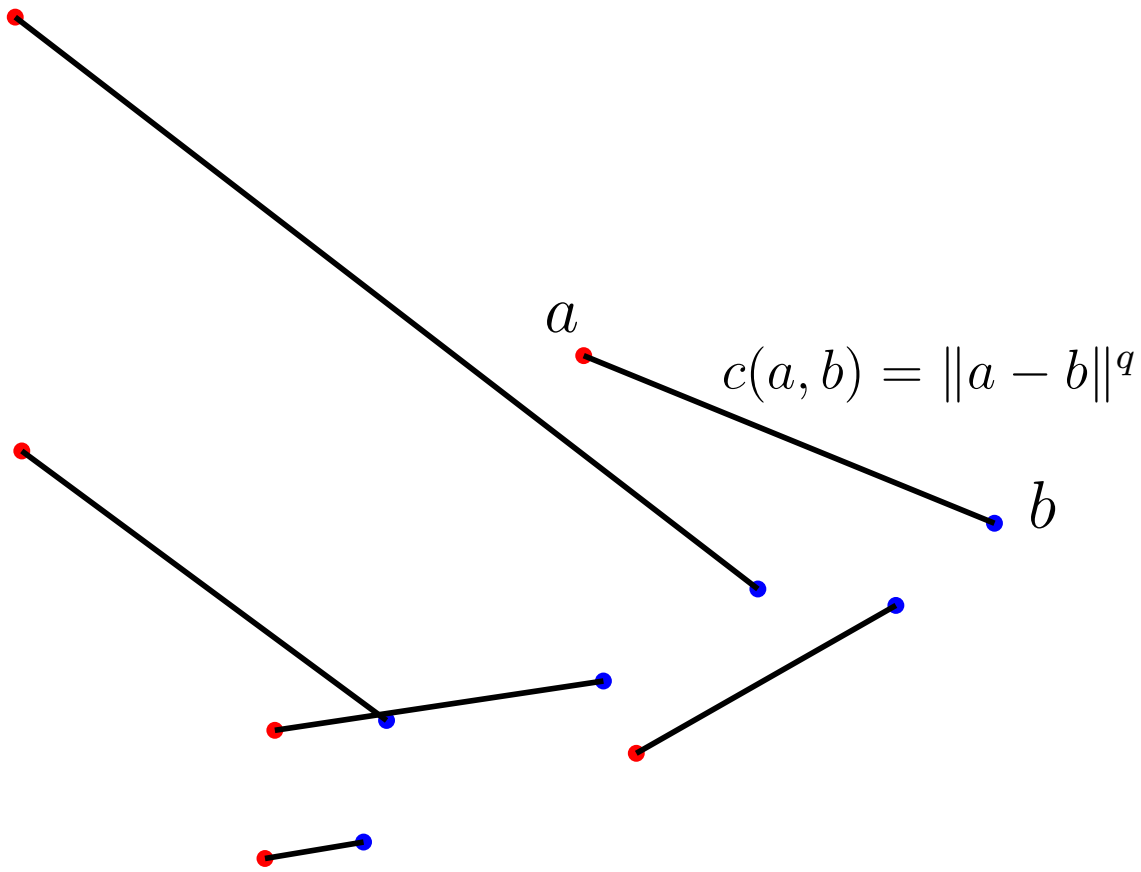

Geometric (bipartite) partial matching

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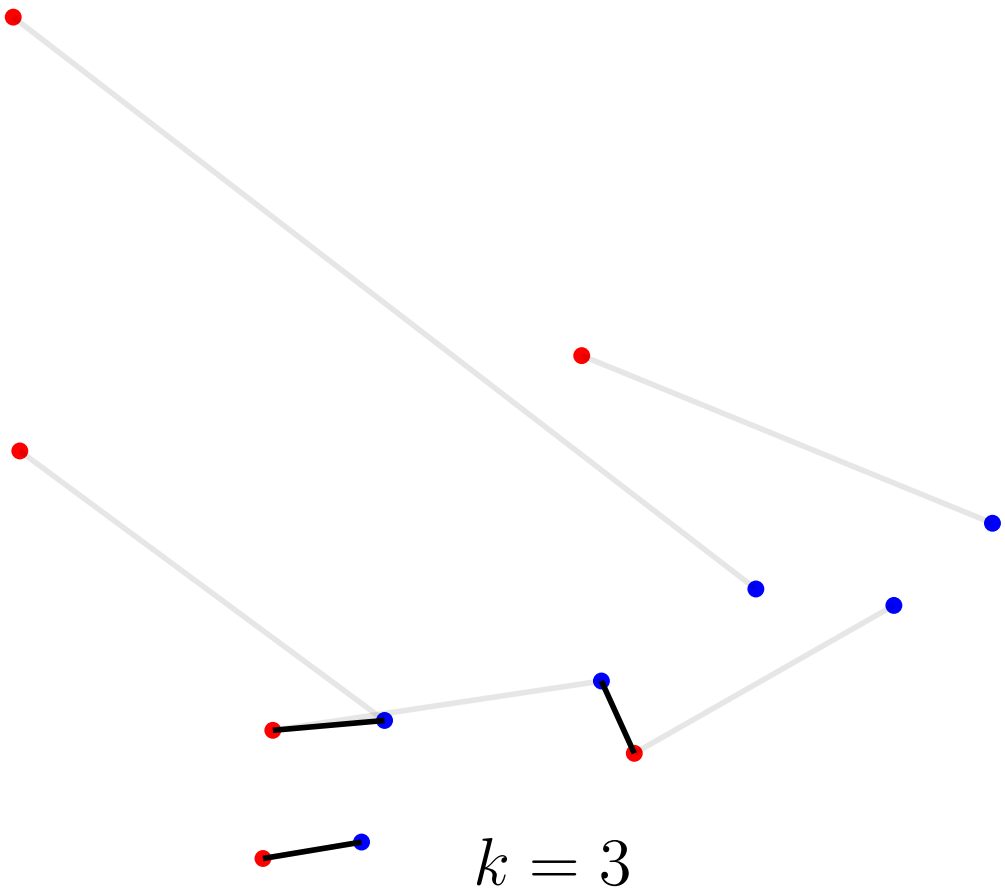
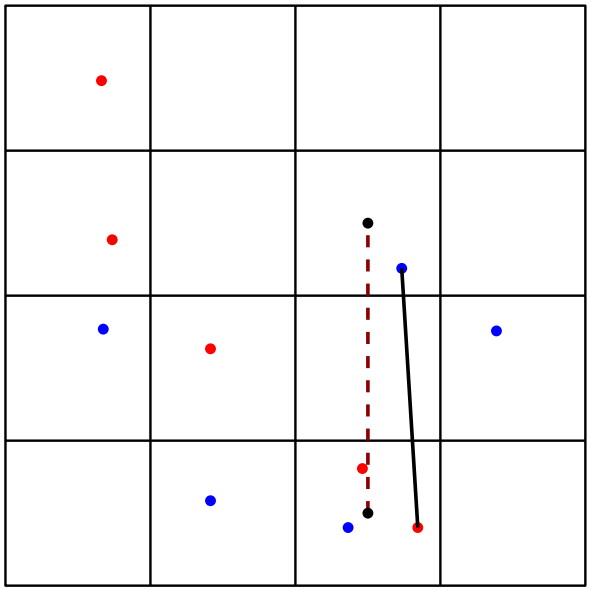


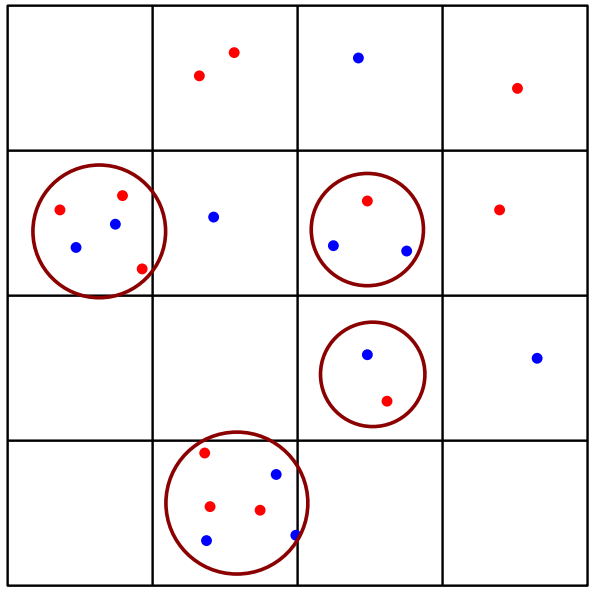
$k = 3$

Issues for geometric perfect matching algorithms

$$|A| = r, \quad |B| = n, \quad A, B \subset \mathbb{R}^2, \quad q \geq 1$$

$$\min_{k\text{-matching } M} \sum_{(a,b)\in M} \|a - b\|^q$$





For $q = 1$, [Sharathkumar-Agarwal]: $(1+\varepsilon)$-approx. in $O(n \operatorname{poly}(1/\varepsilon, \log n))$ time.

Partial matching algorithms (non-geometric)

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- Hungarian algorithm [Kuhn]: $O(krn + k^2 \log r)$
- Cost-scaling [Ramshaw-Tarjan, Goldberg *et al.*]: $O(rn\sqrt{k}\log(kC))$ (integer costs), or $O(rn\sqrt{k})$ time per cost scale.

Using dynamic bichromatic closest pair (BCP) data structures:

- Hungarian algorithm: $O(kn\,\mathrm{polylog}(n))$
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Our results

Previous:

- Hungarian algorithm w/ BCP: $O(kn \operatorname{polylog}(n))$
- Cost-scaling w/ BCP: $O(n\sqrt{k} \operatorname{polylog}(n))$ time per cost scale.

New:

1. Hungarian algorithm: $O((n + k^2) \operatorname{polylog}(n))$ time, exact.
2. Cost-scaling: $O((n + k\sqrt{k}) \operatorname{polylog}(n))$ time per cost scale, $O(\log(n^q/\varepsilon))$ scales, $(1 + \varepsilon)$-approx.
3. Geometric Transportation: $O(\min(n^2, nr^{3/2}) \operatorname{polylog}(n))$ time, exact.

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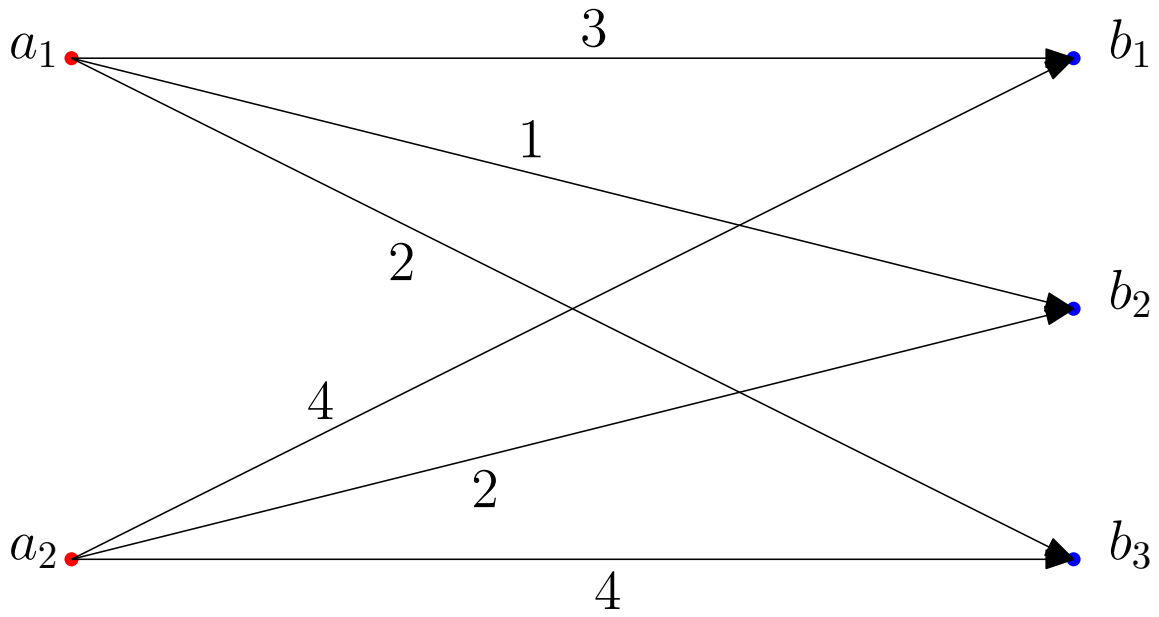

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1. Hungarian search: Dijkstra-like search changing π until $\exists$ an admissible augmenting path.
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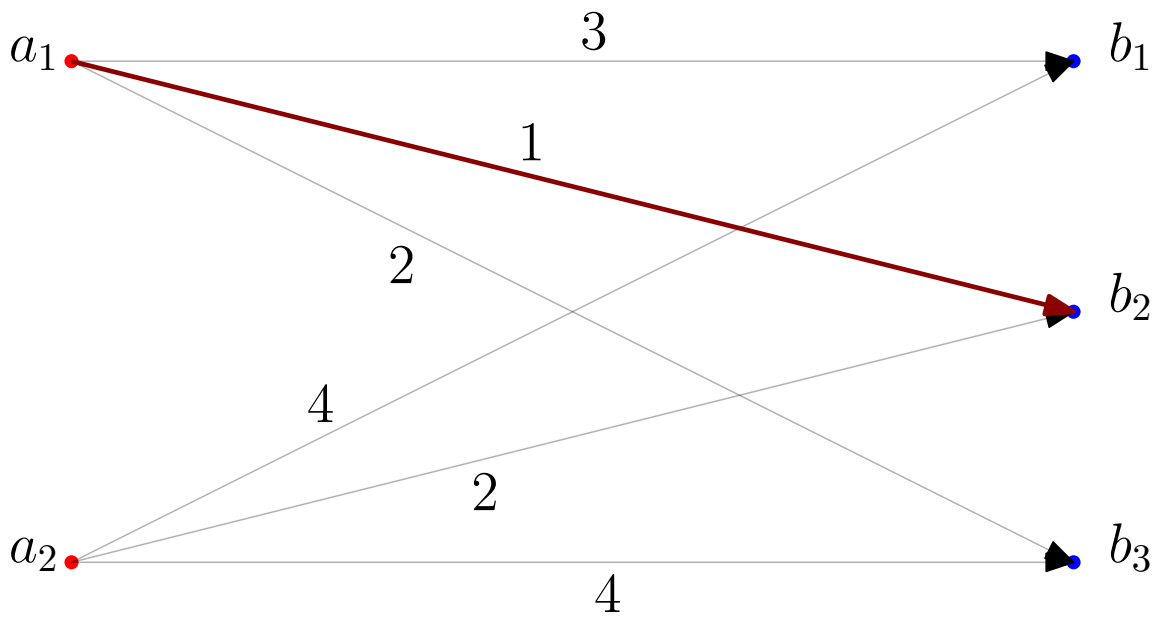

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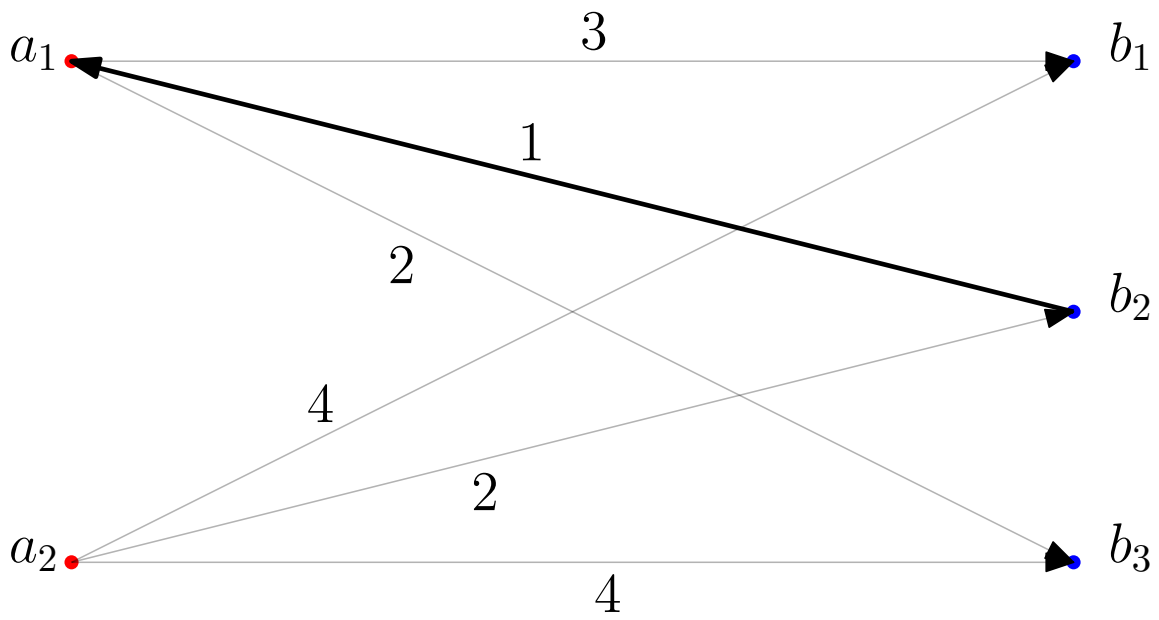

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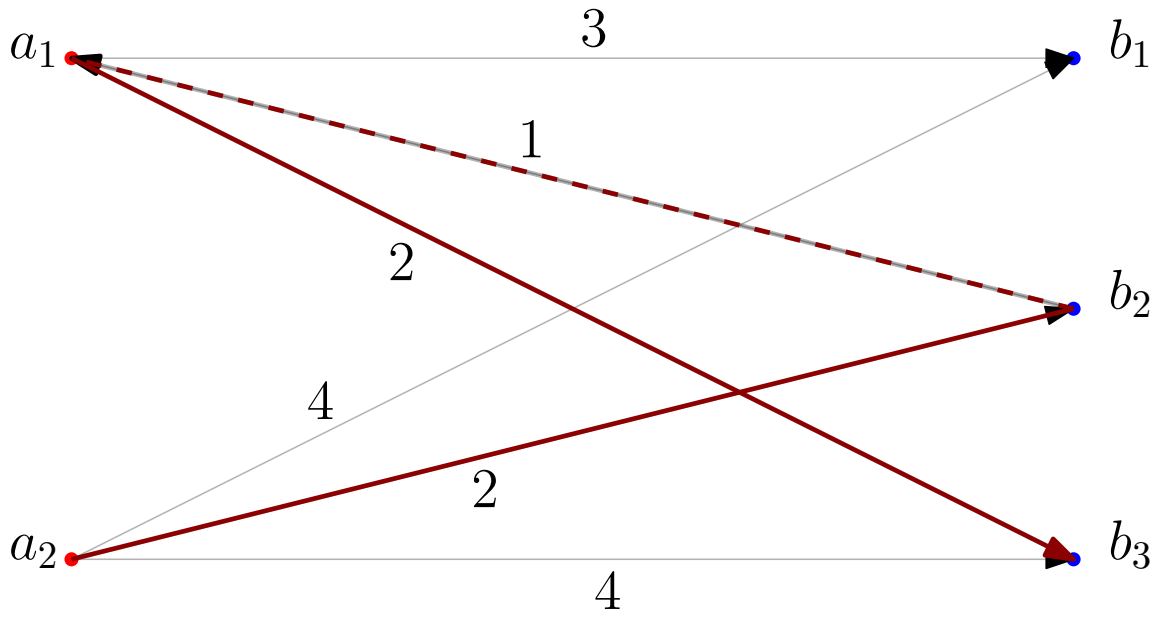

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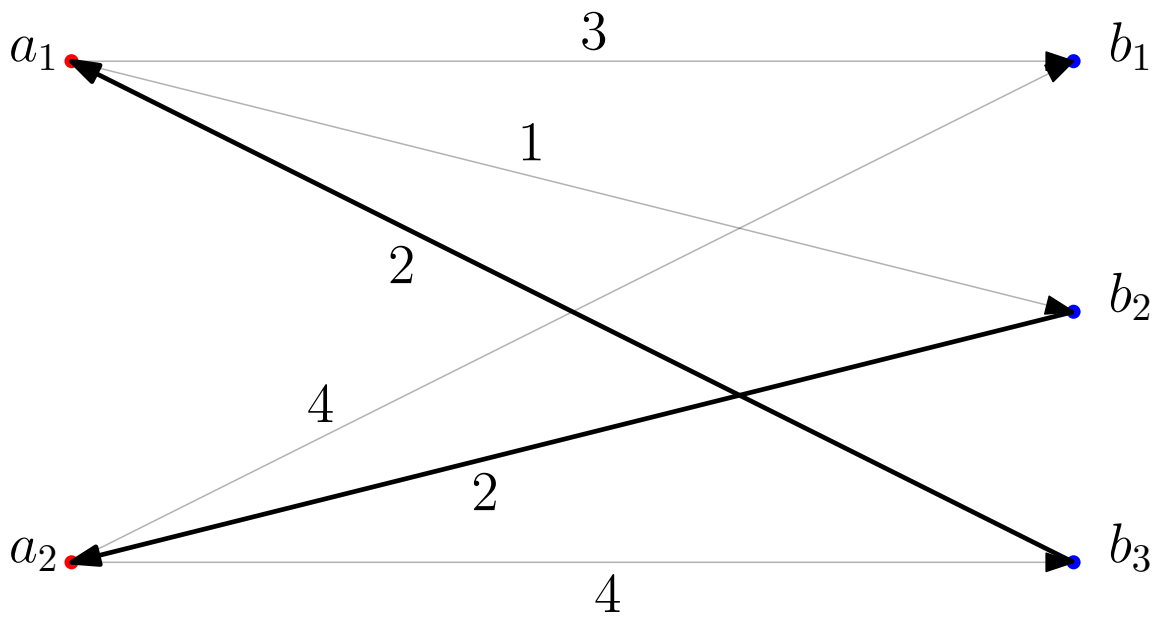

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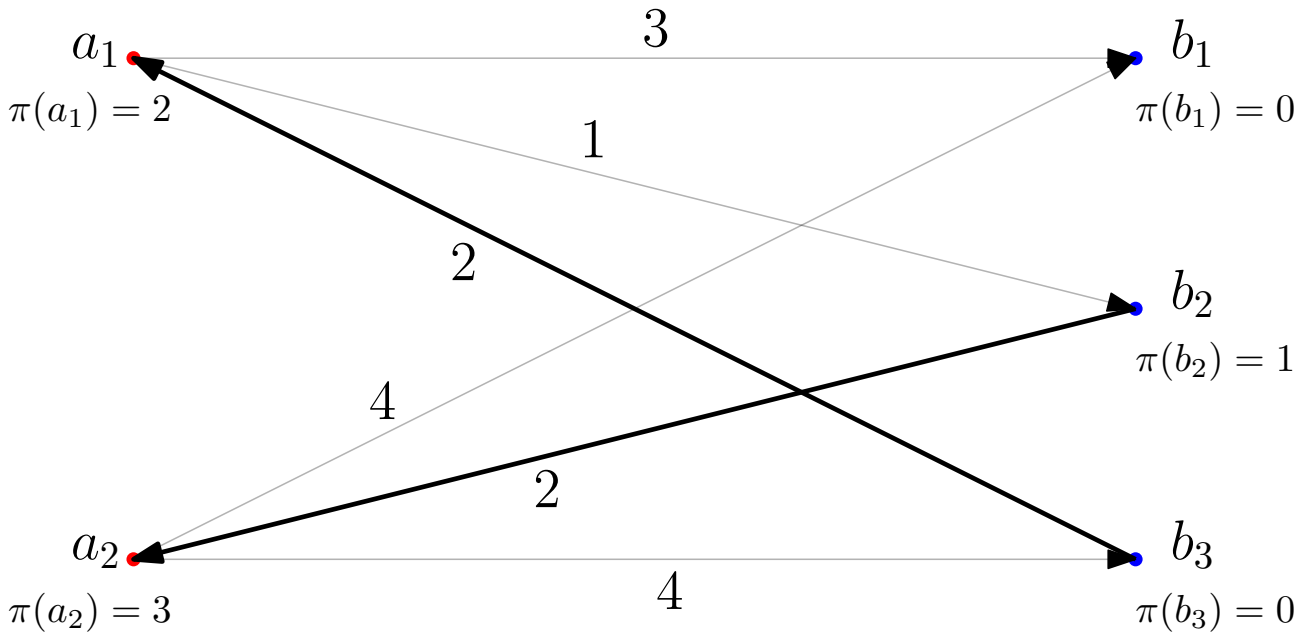

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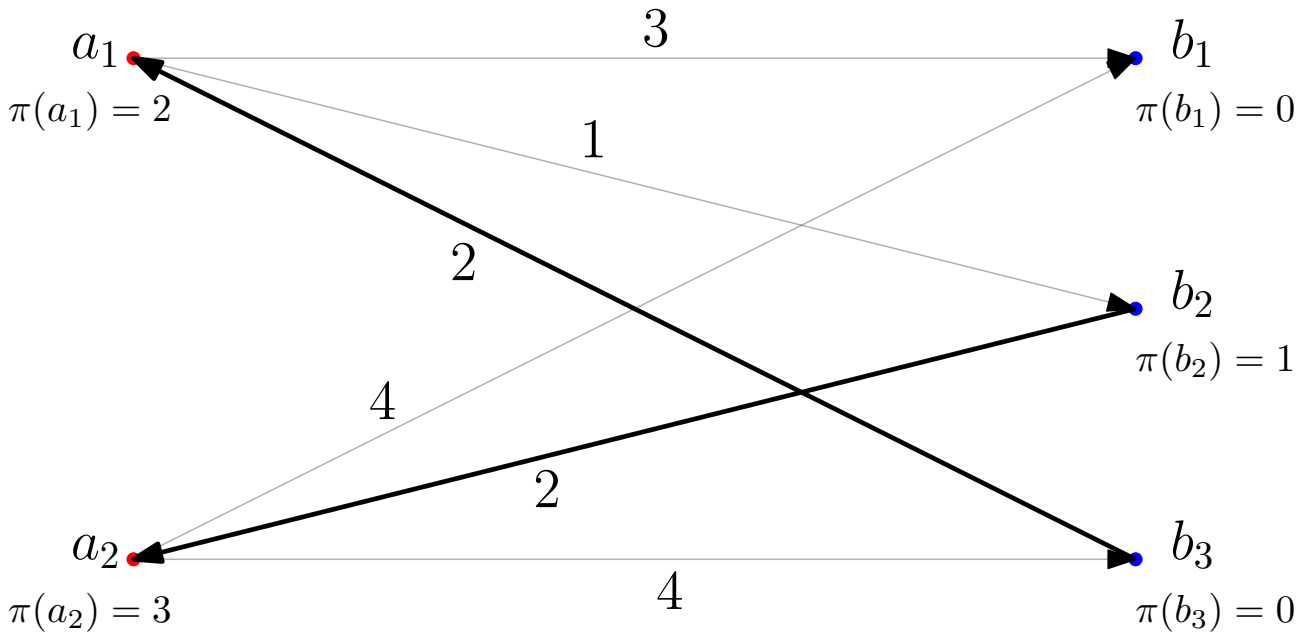

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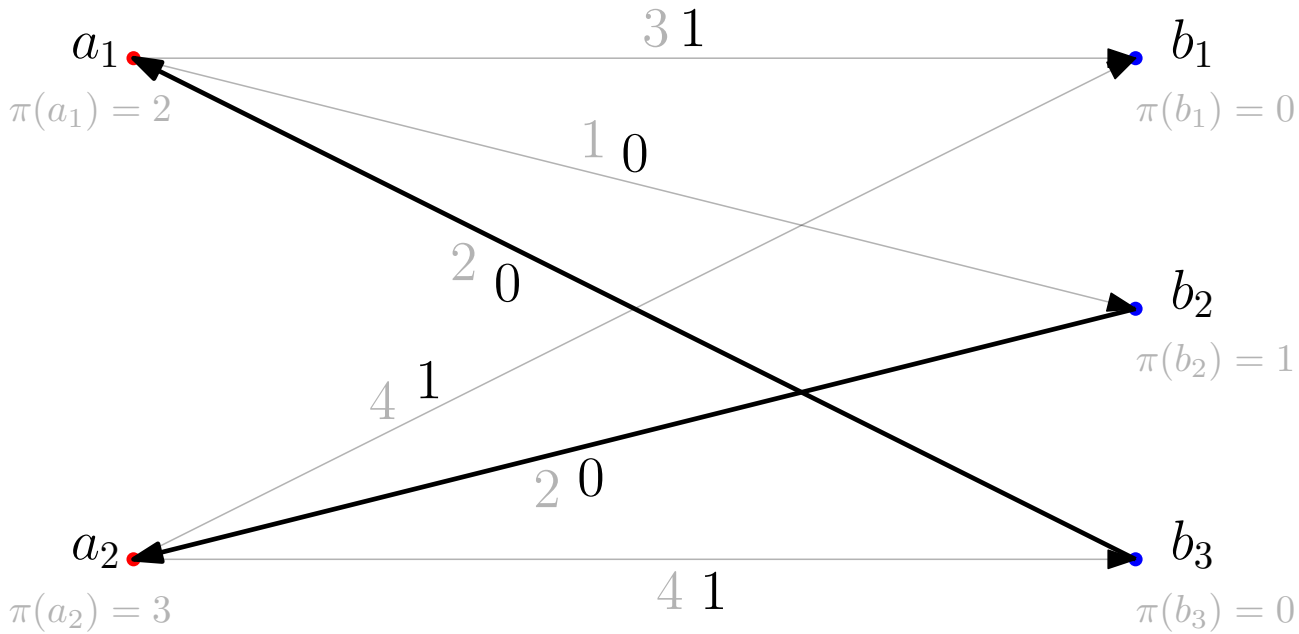

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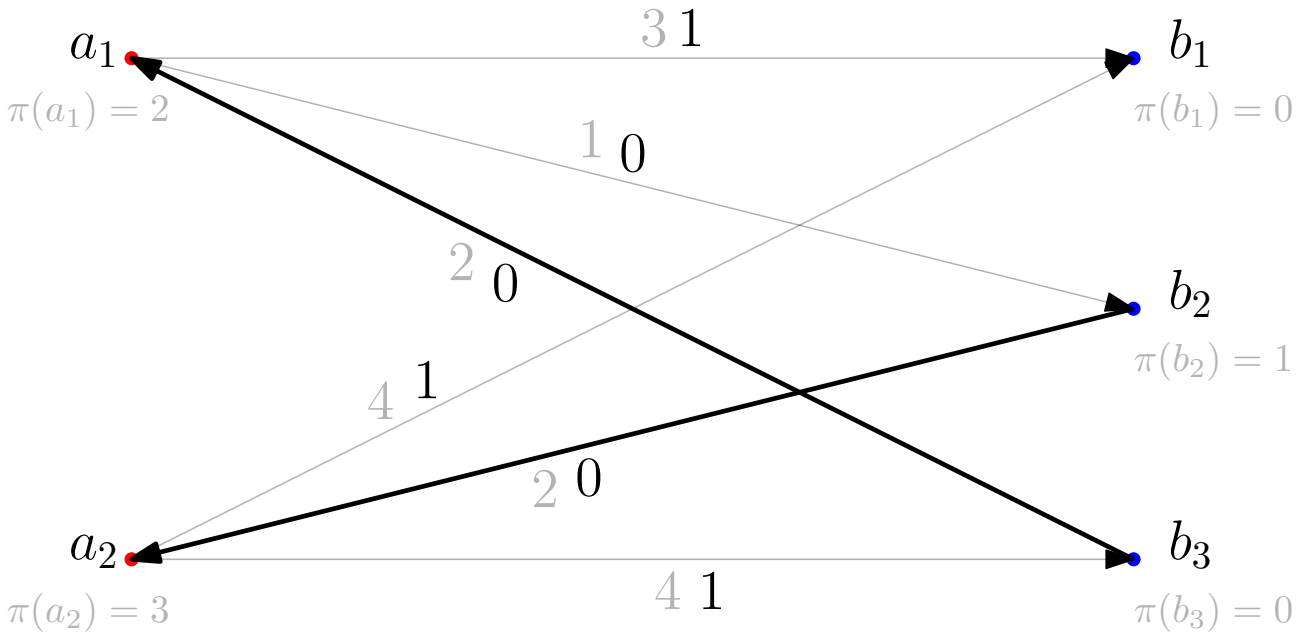

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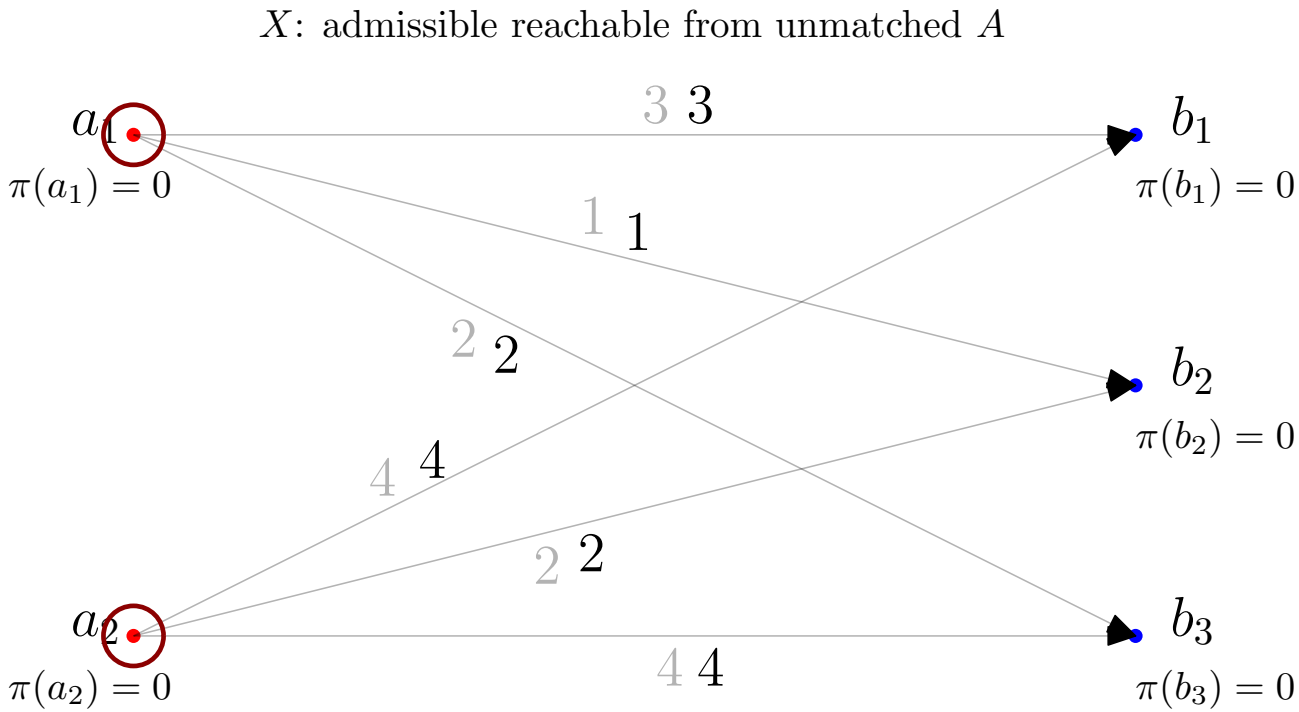

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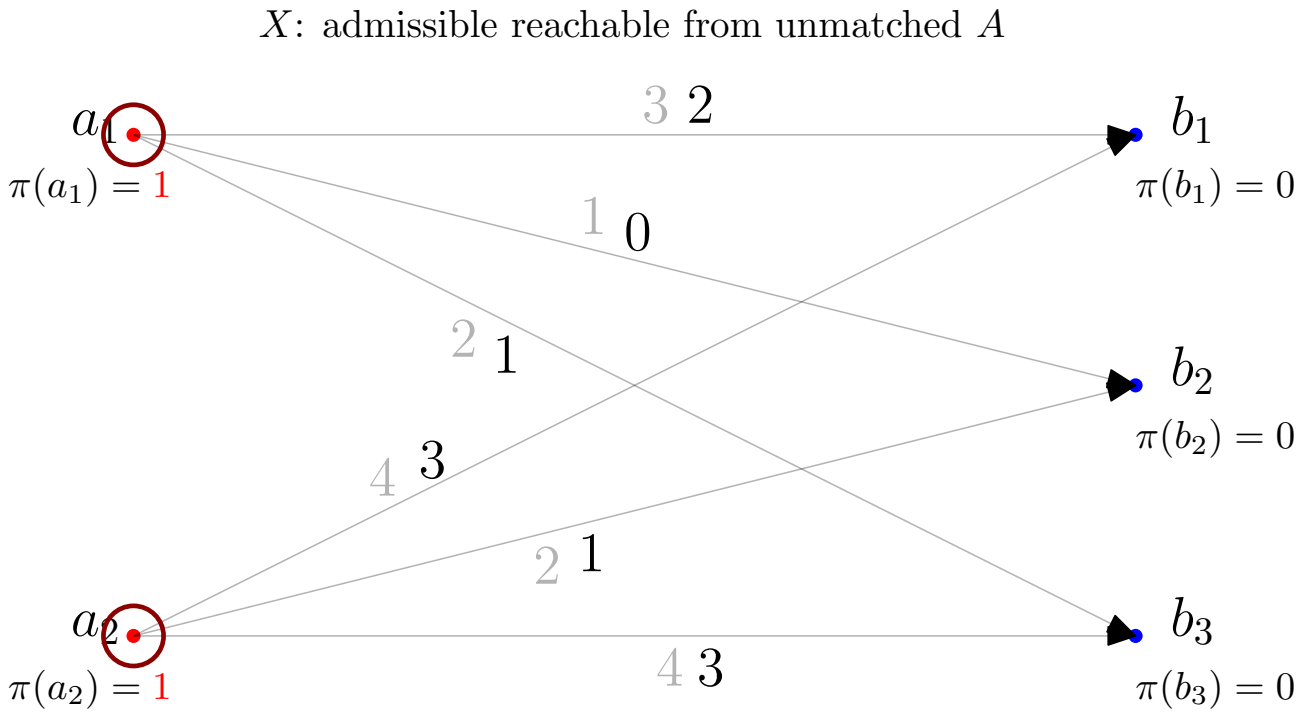

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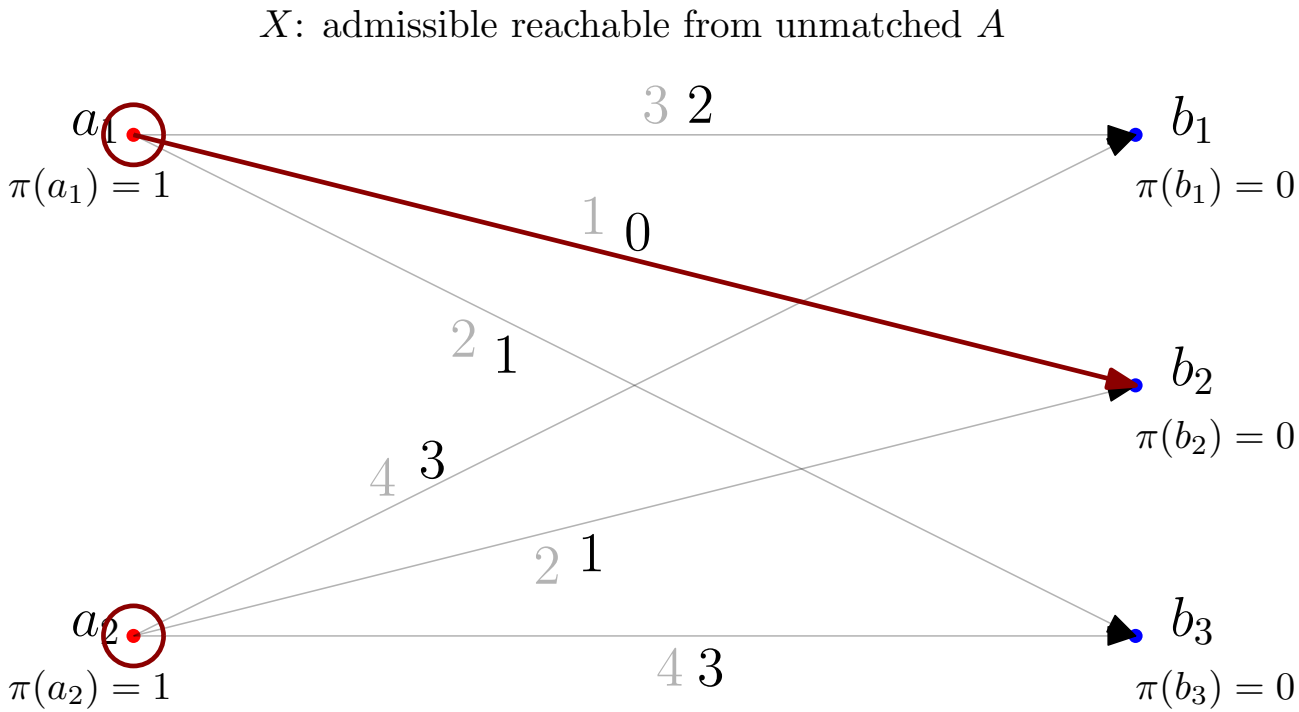

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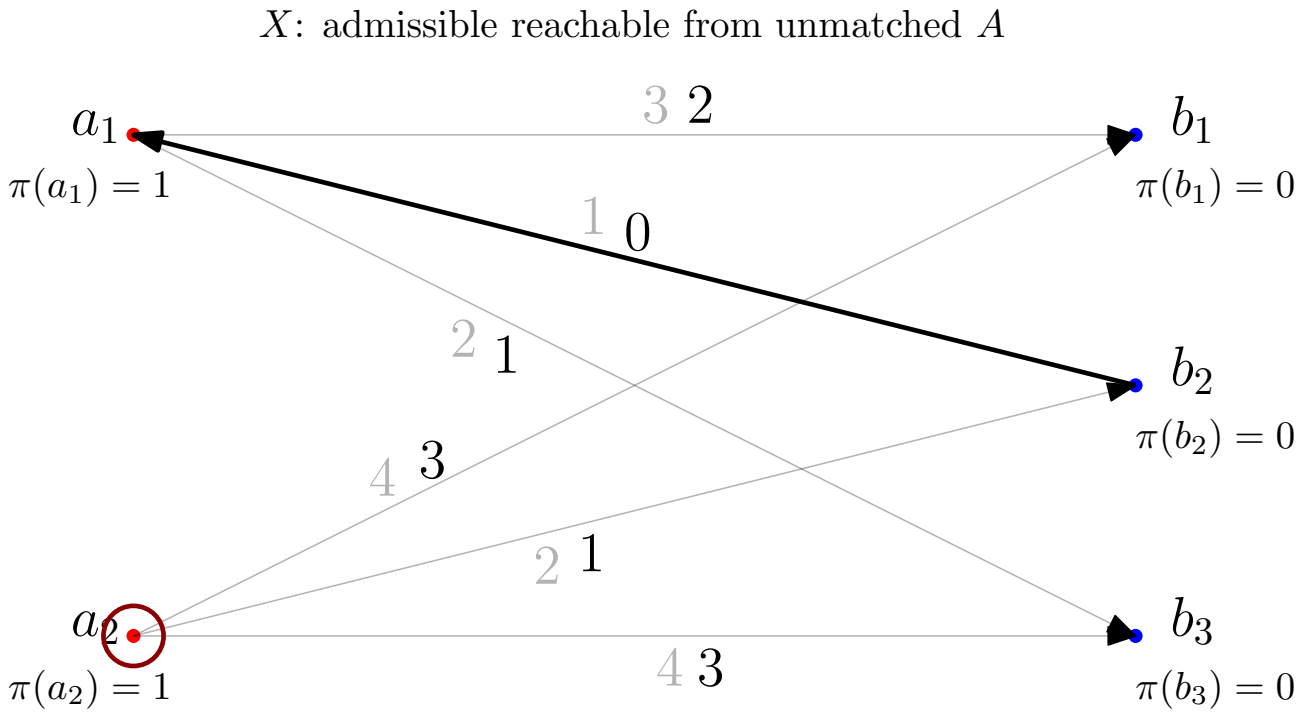

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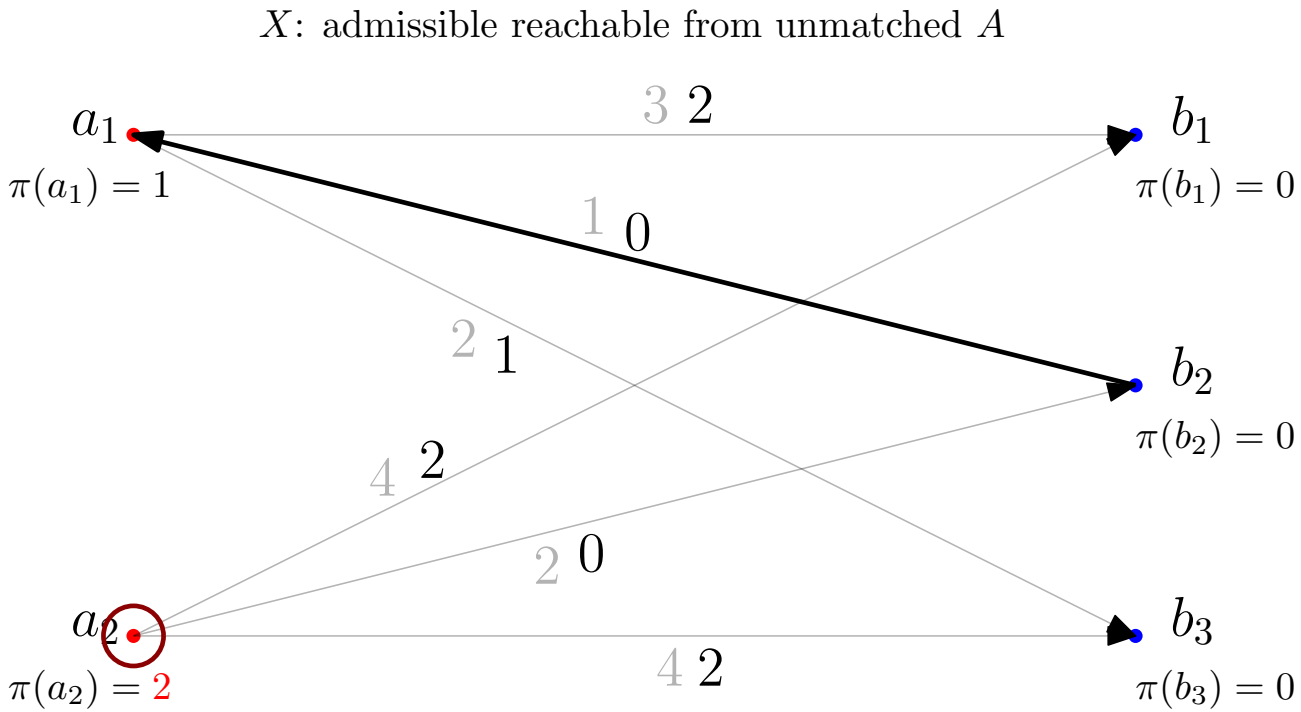

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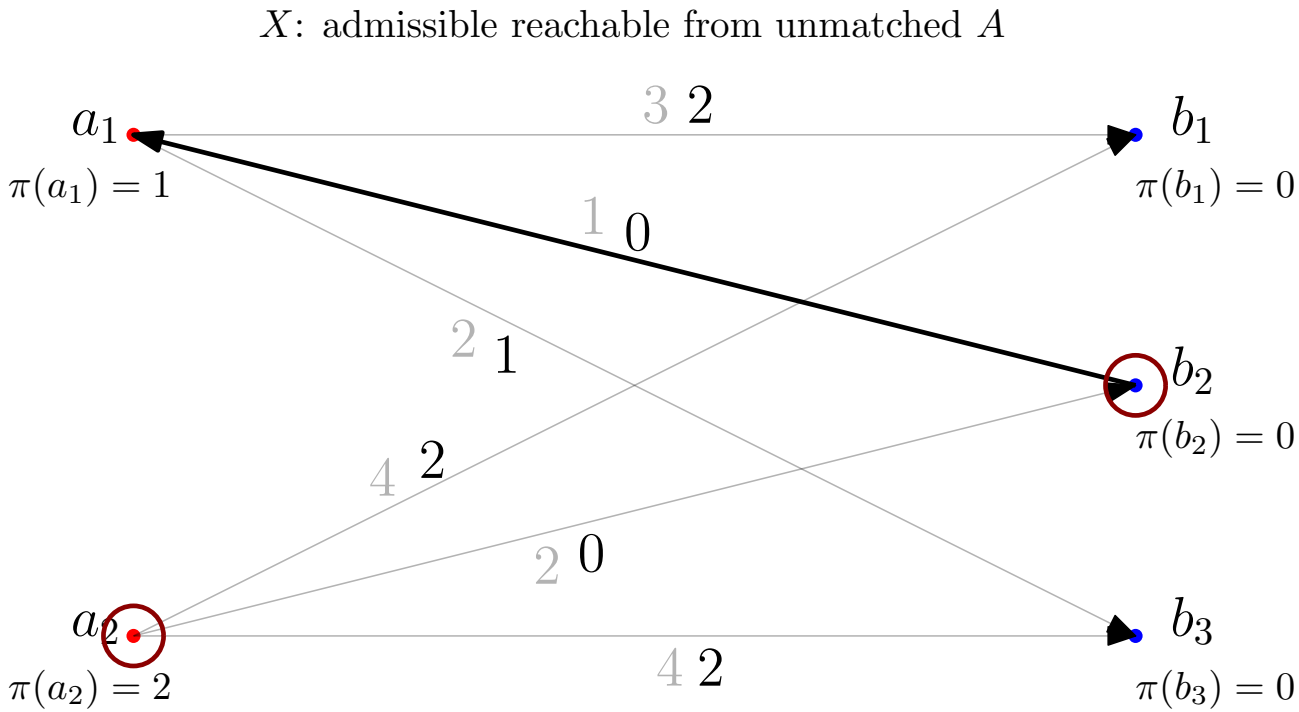

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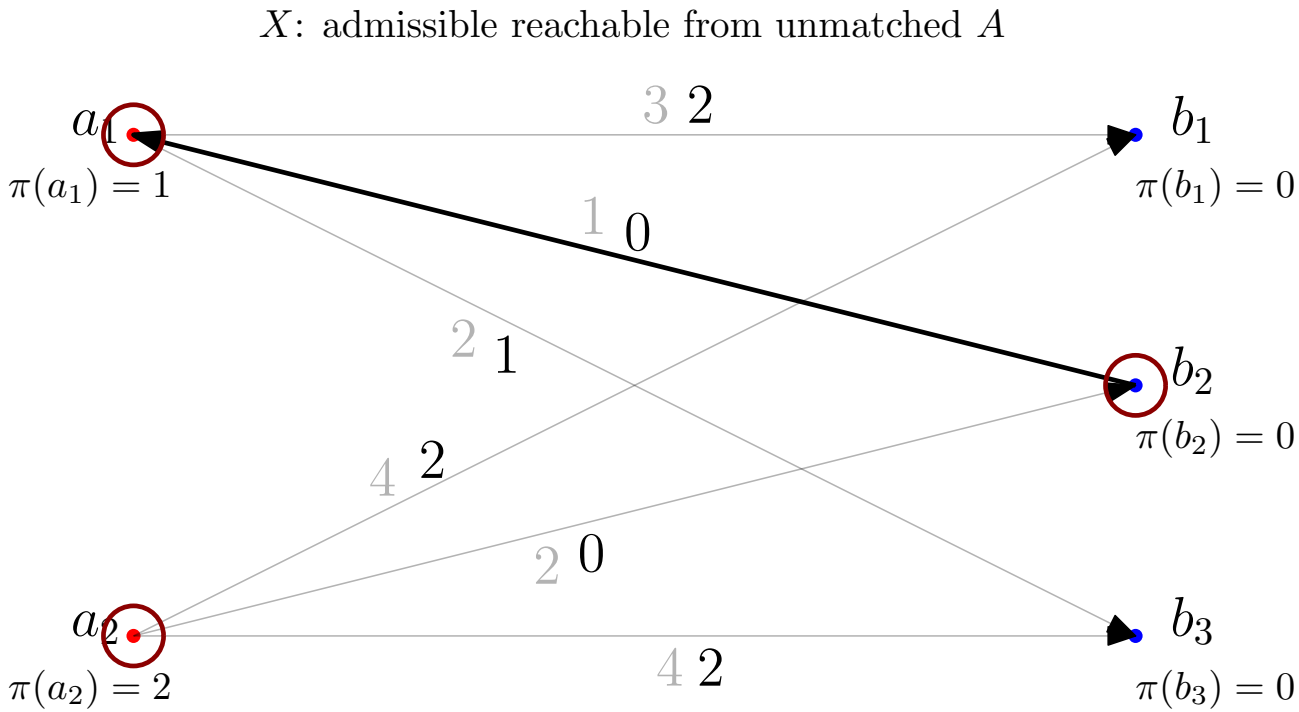

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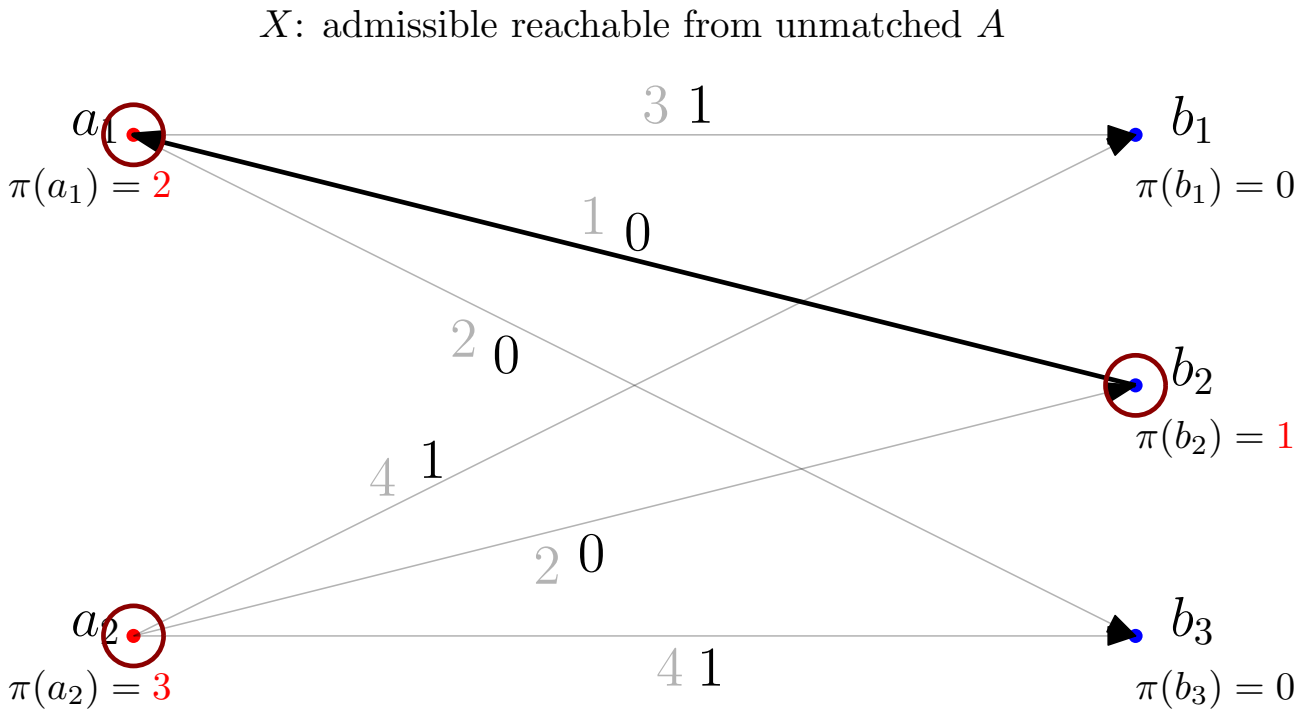

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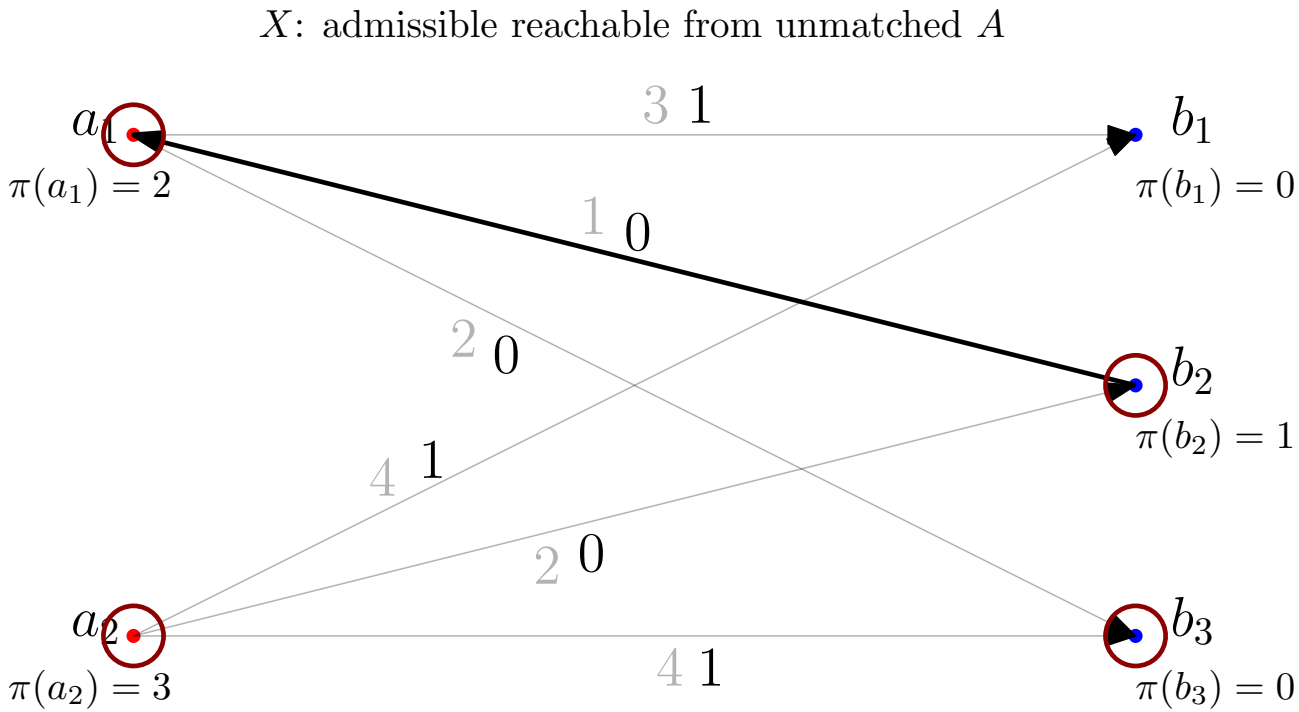

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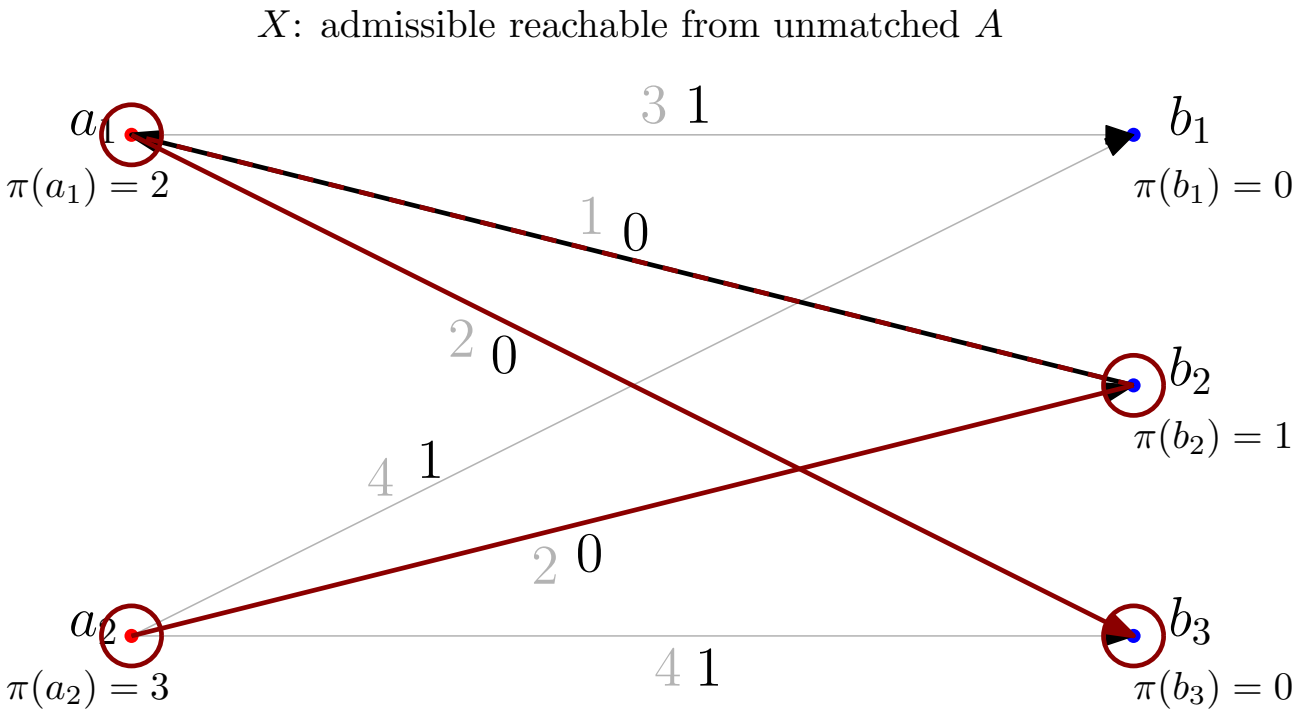

Hungarian search, with BCP

- "What is the minimum reduced cost residual arc leaving X?"
- For $B \to A$ residual arcs, min-heap over matching edges (at most k).
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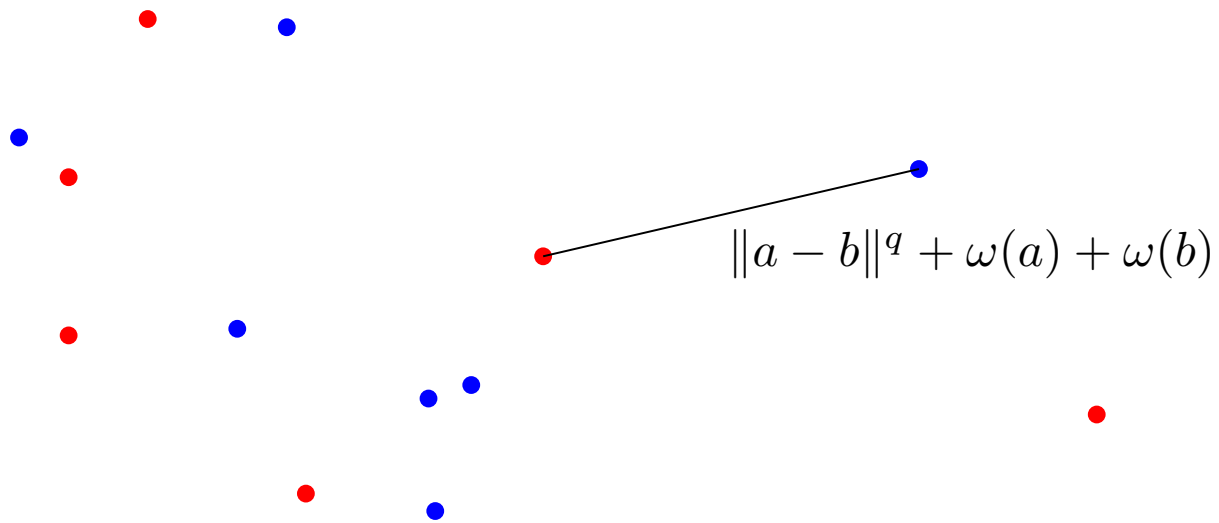

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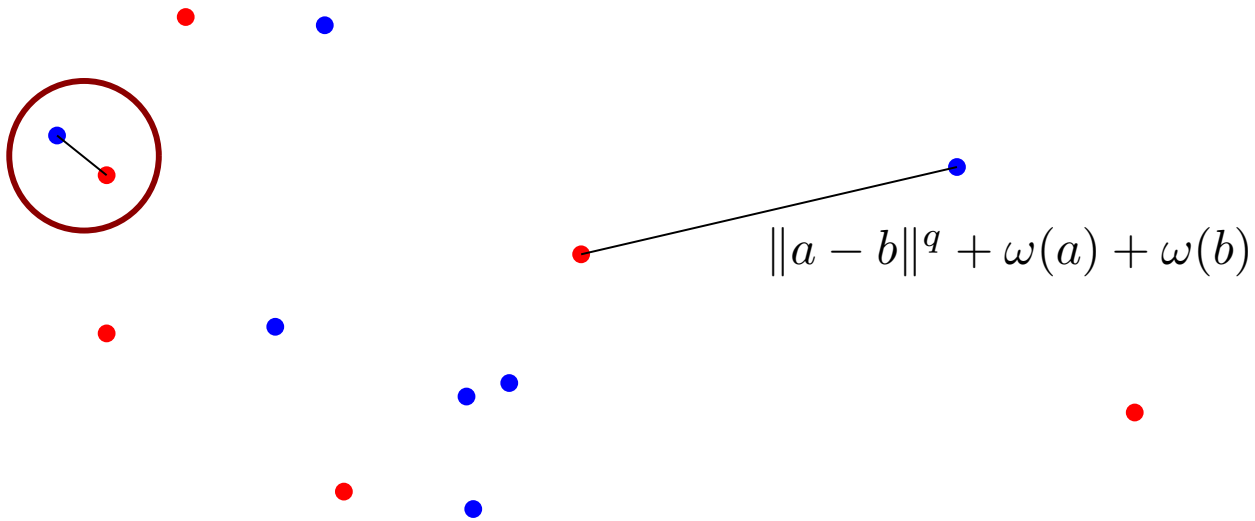

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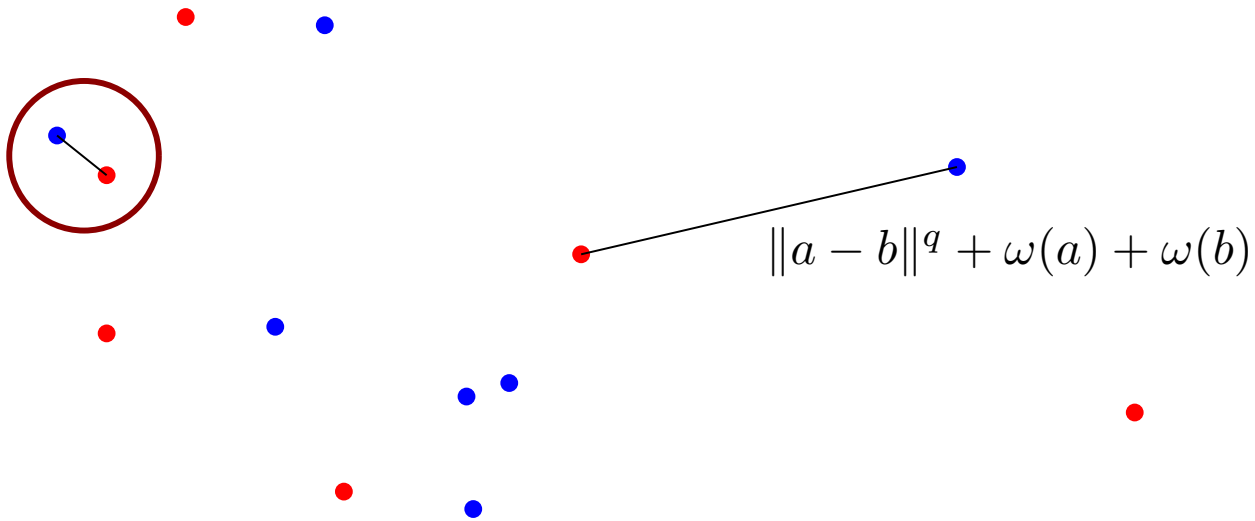

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Problem: BCP initialization

- k augmentations
- Goal: $O(k \operatorname{polylog} n)$ time per augmentation
- $O(k)$ relaxations per Hungarian search (at most k matching edges)
- But, the initial BCP data structure of each Hungarian search has size $O(n)$.
- $\ldots O(kn \operatorname{polylog} n)$ time so far.

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Rewinding

- Initial X is the set of unmatched A vertices.
- Initial X loses exactly one vertex after an augmentation.

1. Record the BCP operations we did during the Hungarian search, then rewind them to recover the initial BCP state.
2. Delete one more point from the BCP to get the next initial state.

- $O(k \operatorname{polylog} n)$ time to rewind.
- Build the whole BCP once ($O(n \operatorname{polylog} n)$), reuse every augmentation.
- Persistence?
  Unfortunately, the BCP insert/delete bounds are amortized.

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  Unfortunately, the BCP insert/delete bounds are amortized.

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- Initial X is the set of unmatched A vertices.
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- During the Hungarian search, how do we have time to raise π for every $v \in X$?
- [Vaidya 89]: batch changes into a single variable δ.
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- When a point enters X we save it with $\gamma(v) \leftarrow \pi(v) - \delta$.
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- Summary: Build the BCP once ($O(n \operatorname{polylog} n)$). Each Hungarian search and augmentation takes $O(k \operatorname{polylog} n)$ time.
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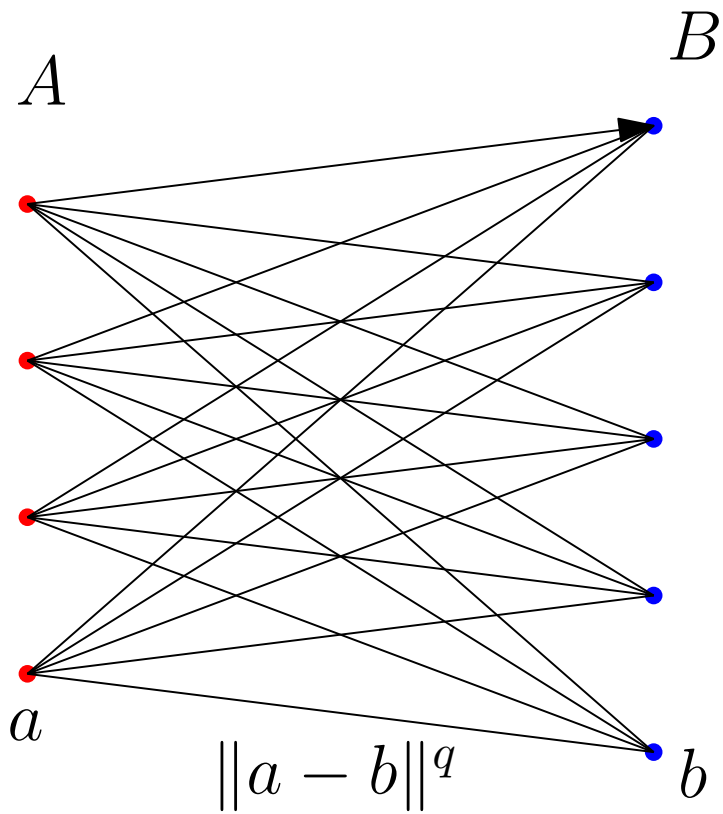
Our results

New:

1. Hungarian algorithm: $O((n + k^2)\,\mathrm{polylog}(n))$ time, exact.
2. Cost-scaling: $O((n + k\sqrt{k})\,\mathrm{polylog}(n))$ time per cost scale, $O(\log(n^q/\varepsilon))$ scales, $(1+\varepsilon)$-approx.
3. Geometric Transportation: $O(\min(n^2, nr^{3/2})\,\mathrm{polylog}(n))$ time, exact.

Geometric partial matching as unit-capacity min-cost flow



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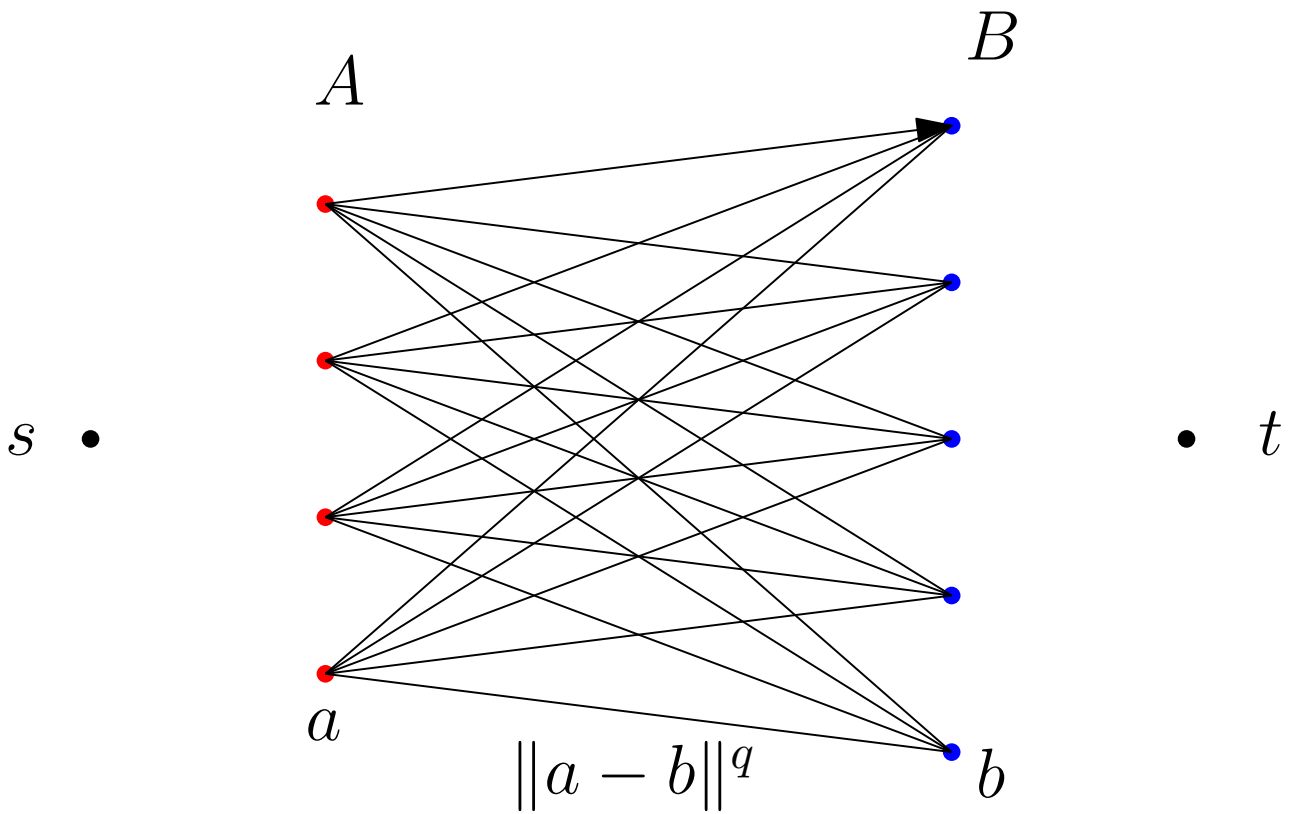

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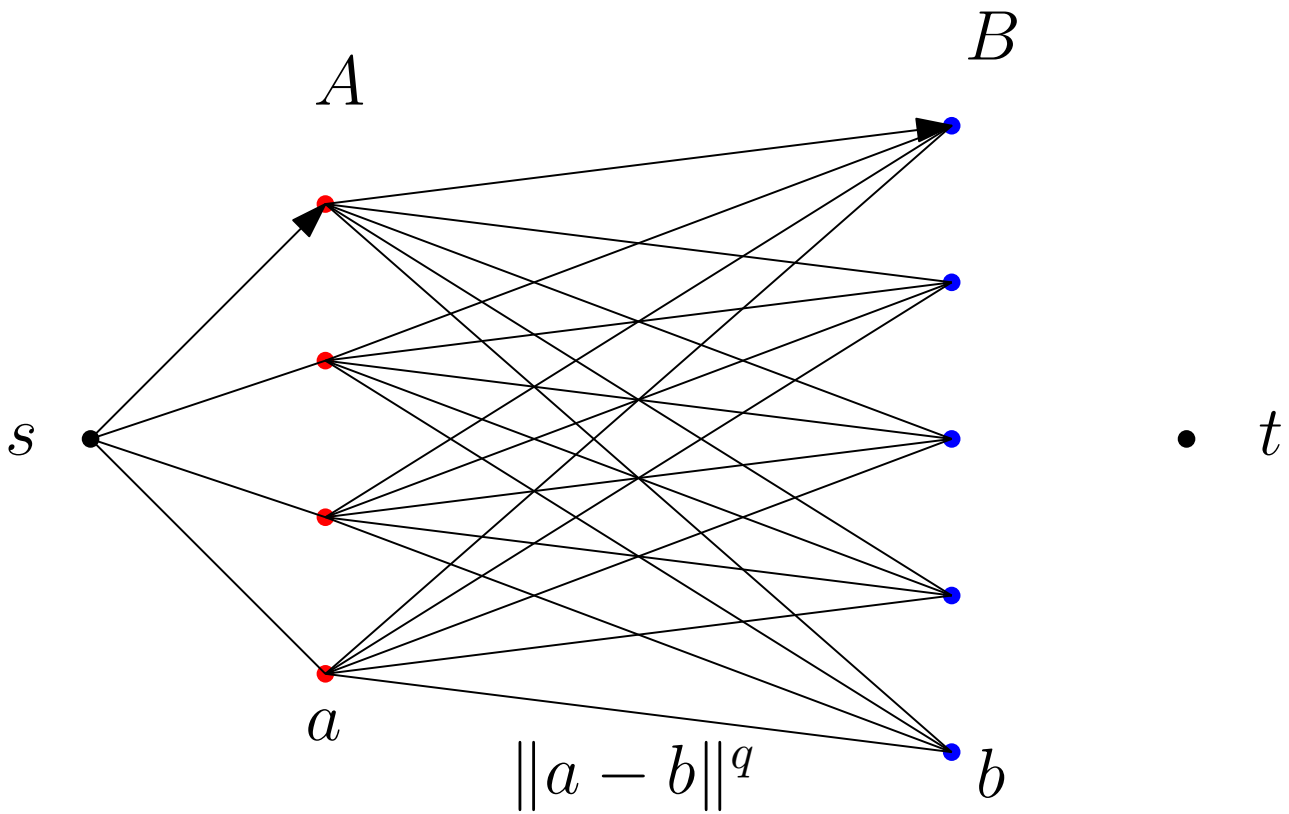

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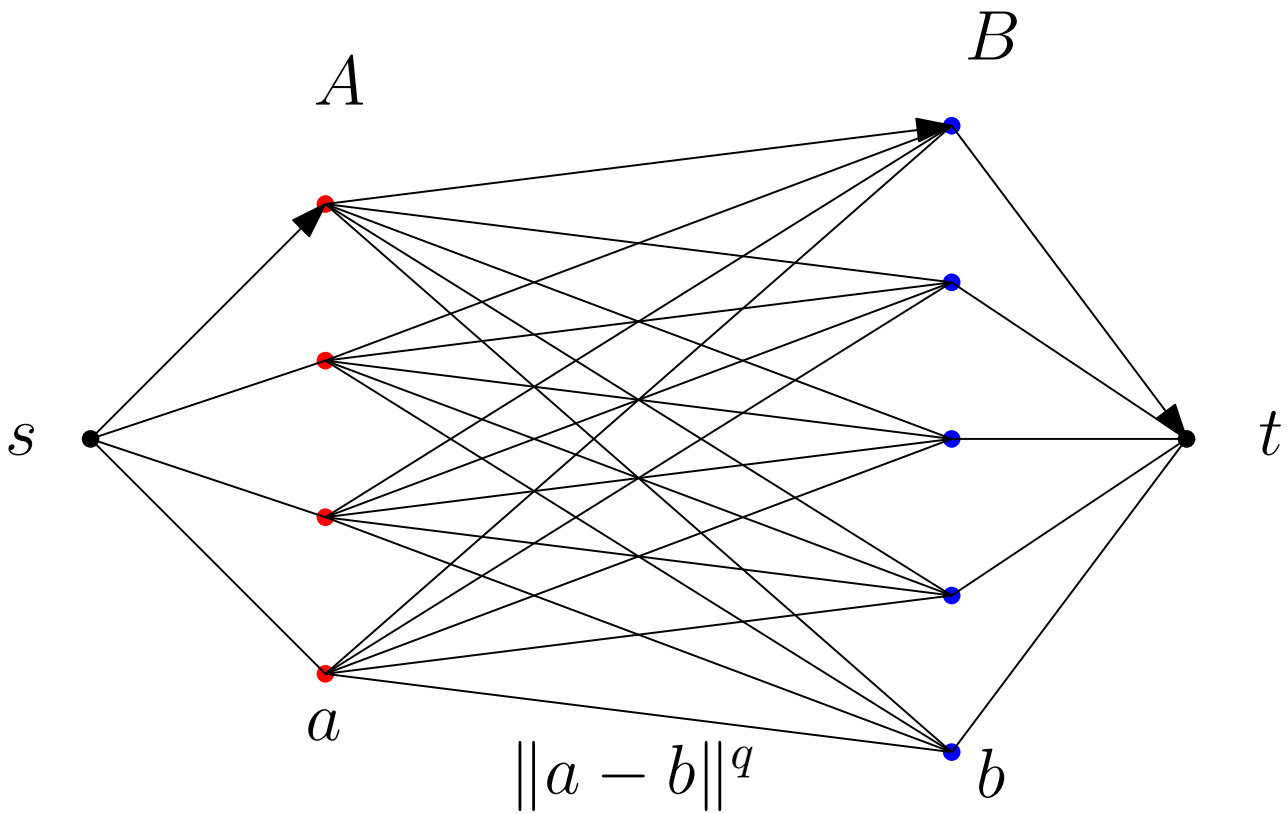

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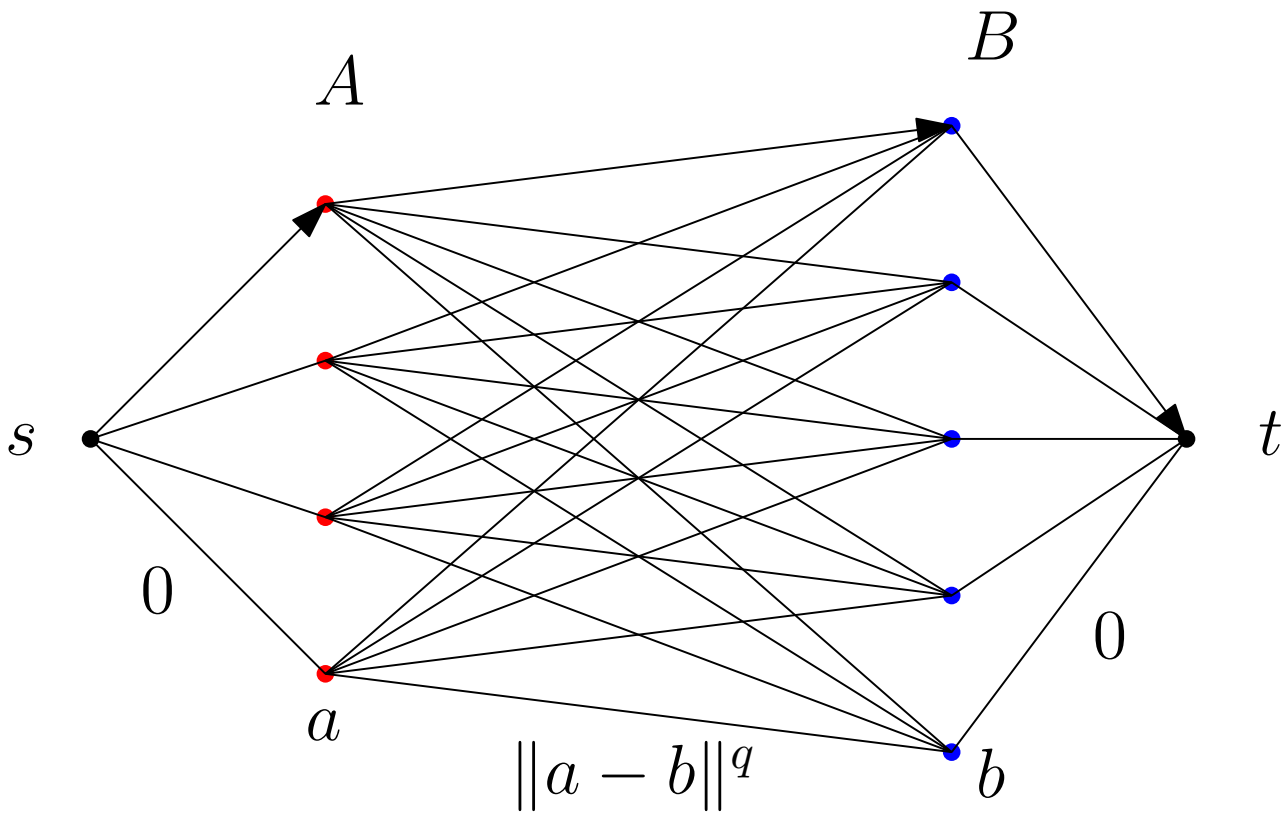

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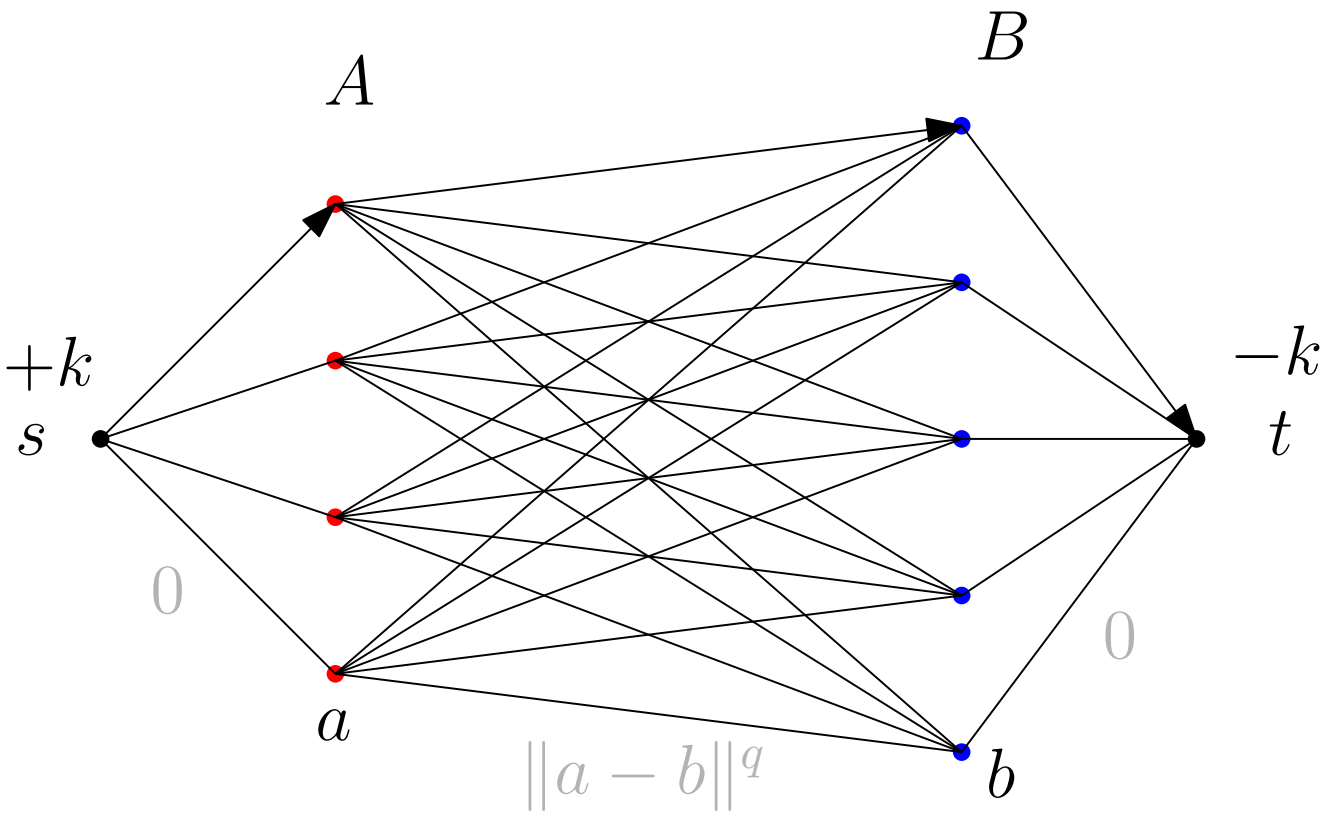

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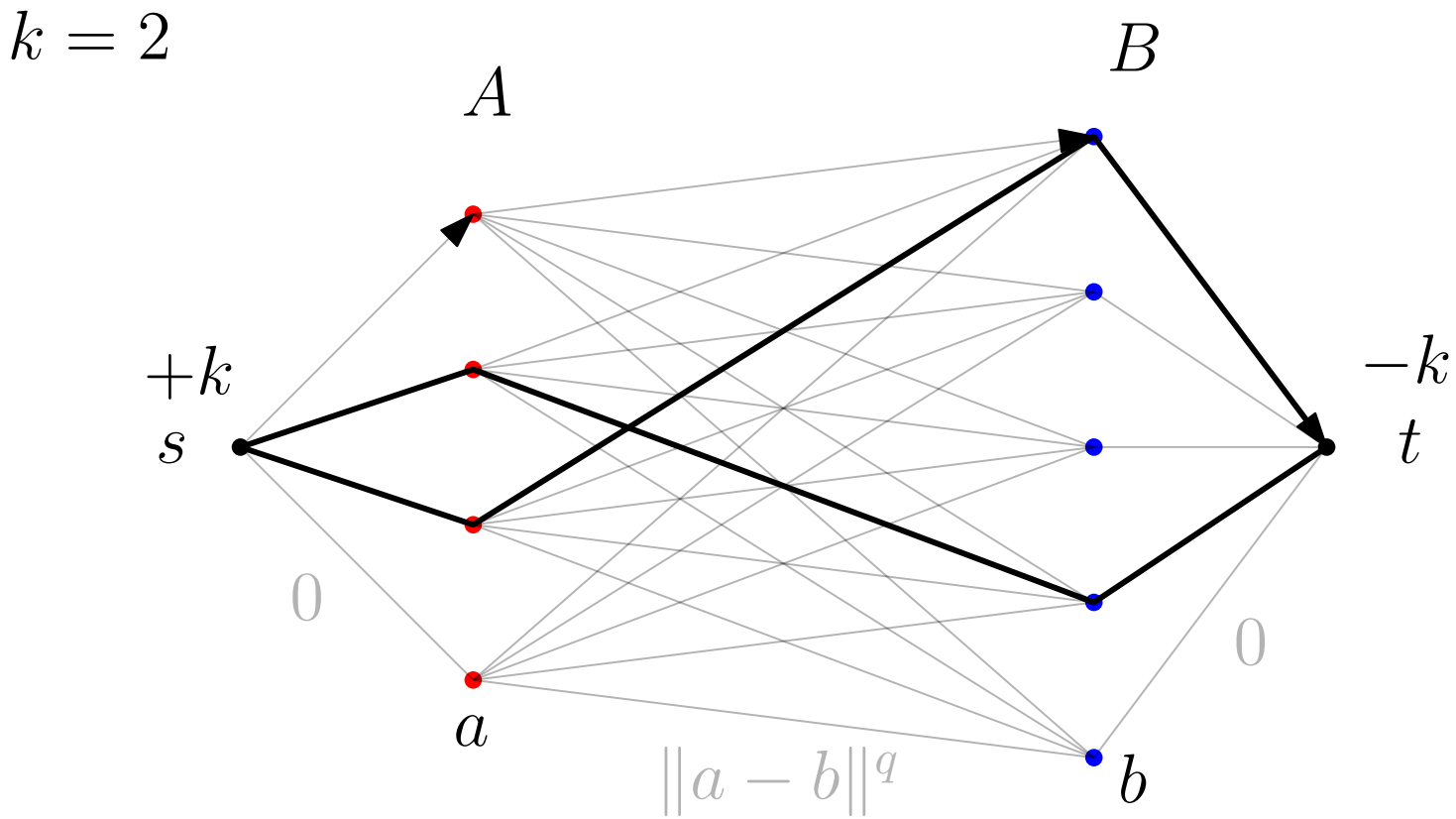

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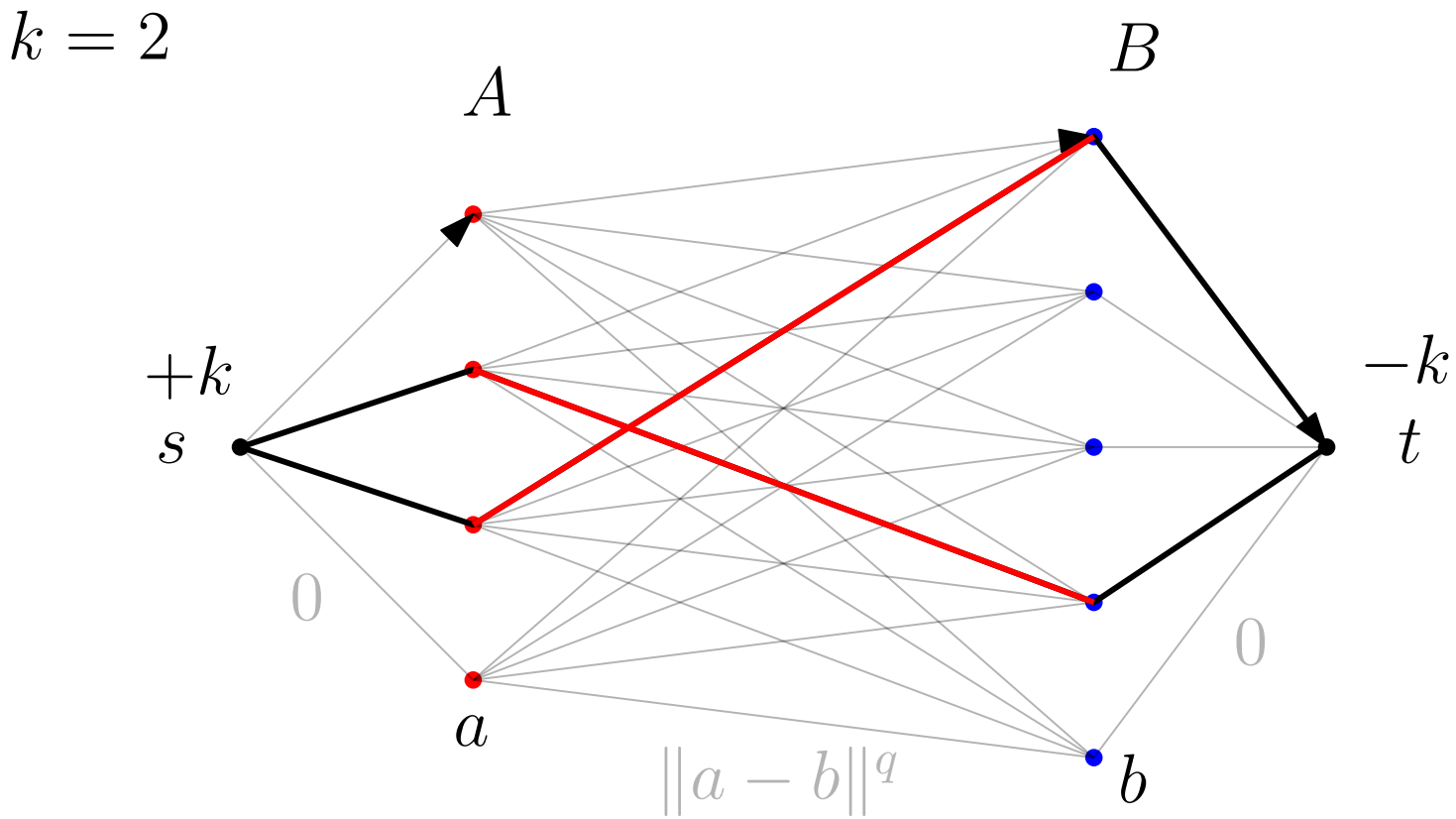

Partial matching by cost-scaling

- Reduced cost: $c_\pi(v, w) = c(v, w) - \pi(v) + \pi(w)$
- Keep π feasible — $c_\pi(v, w) \geq 0$ for all residual arcs (v, w).
- θ-optimal: $c_\pi(v, w) \geq -\theta$ for all residual arcs (v, w).
- Admissible: $c_\pi(v, w) \leq 0$ for residual arc (v, w).

Lemma

A θ-optimal circulation for the partial matching network has cost at most $c(f^*) + 6k\theta$.

- Find θ-optimal circulations for shrinking values of scale θ.

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Given a 2θ-optimal circulation from the previous scale,

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Can reduce $(1+\varepsilon)$-approx. geometric partial matching to executing $O(\log(n^q/\varepsilon))$ scales of the cost-scaling algorithm.

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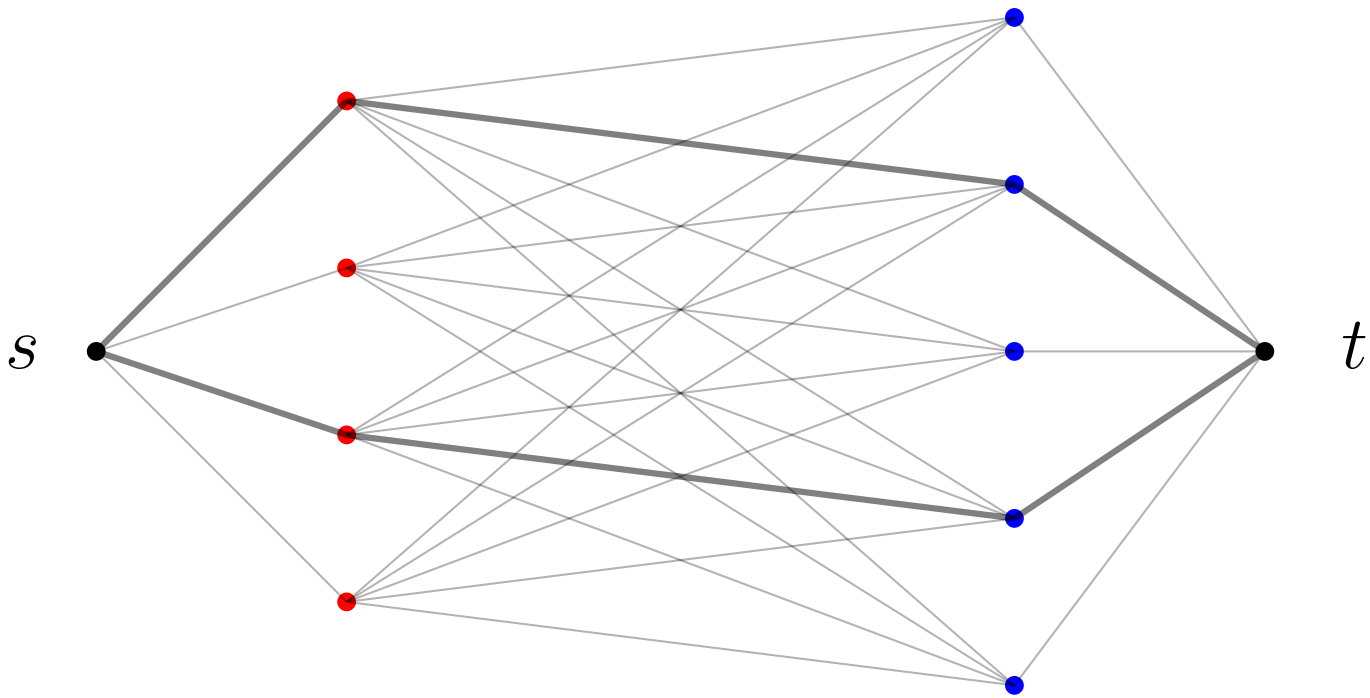

- Input circulation: $c_\pi(v, w) \geq -2\theta$ for all residual arcs.
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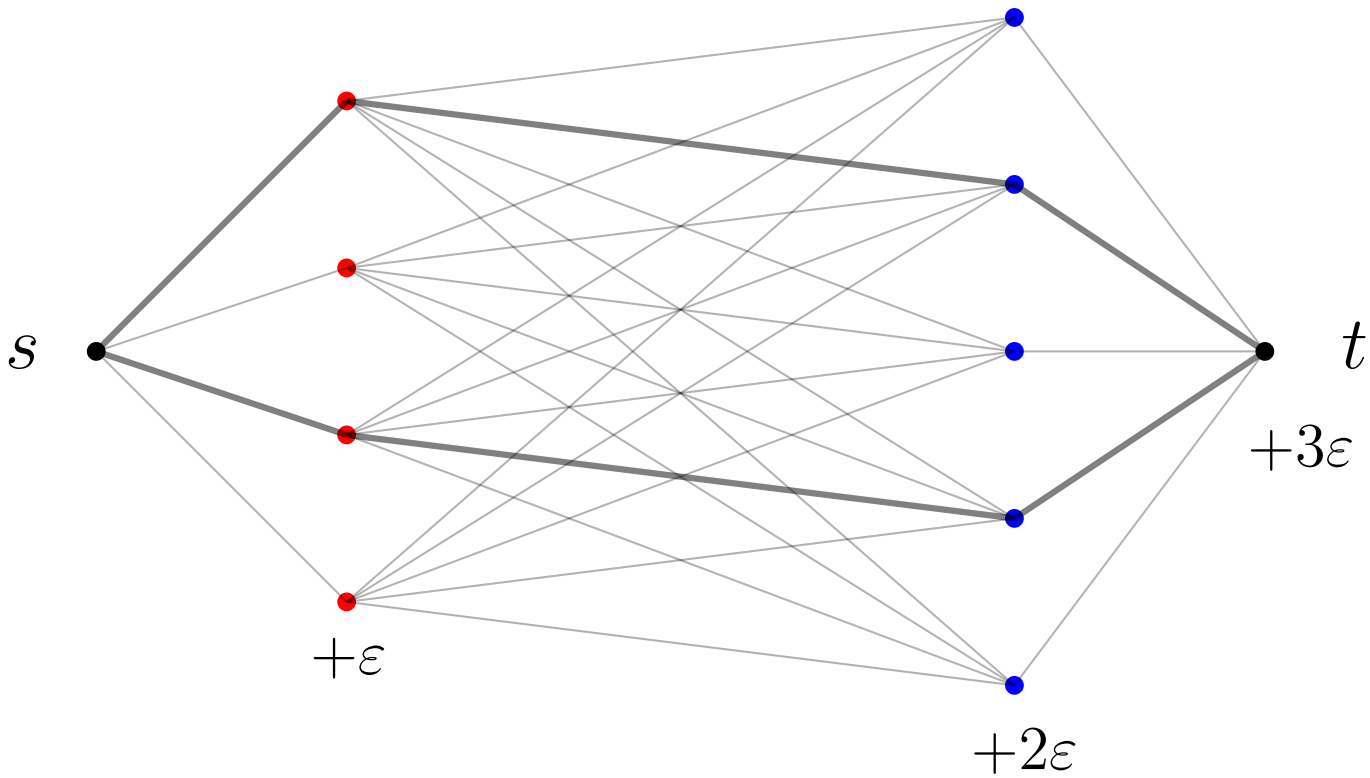

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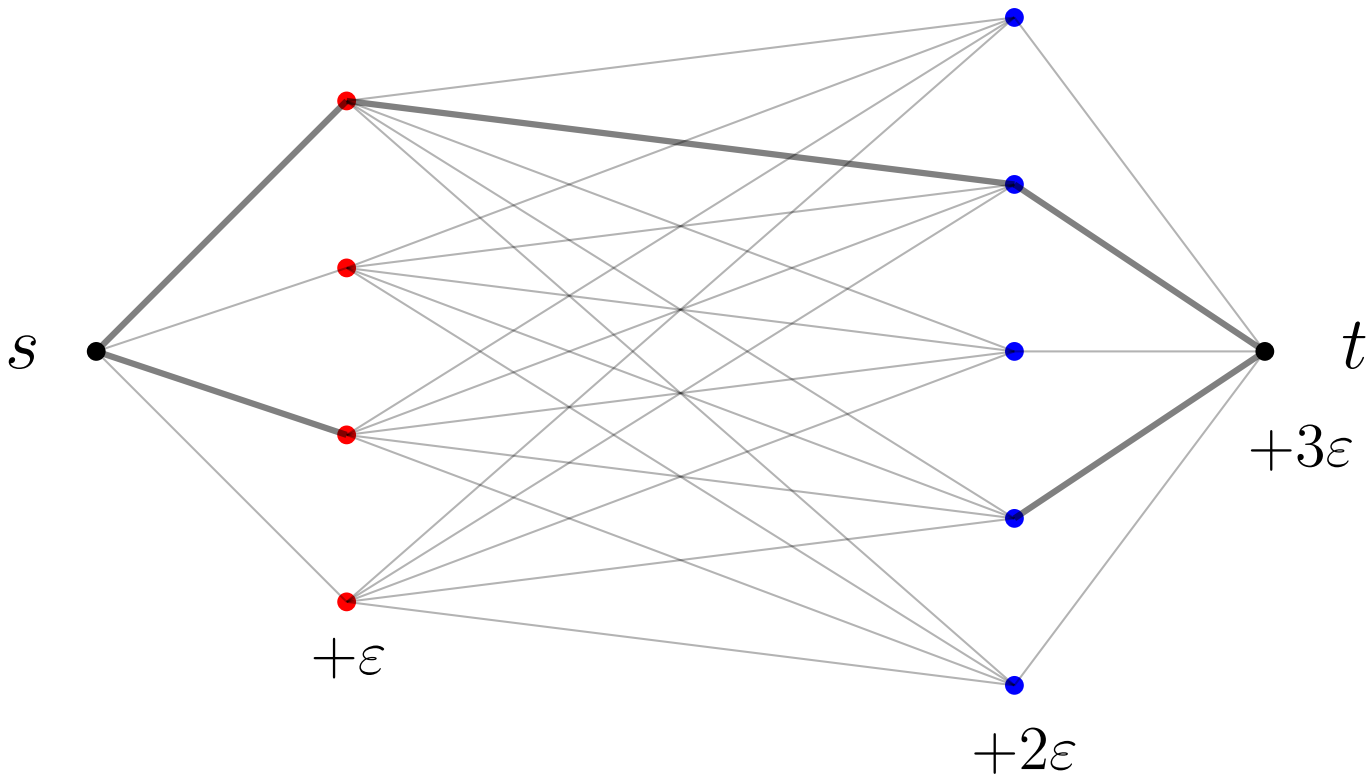

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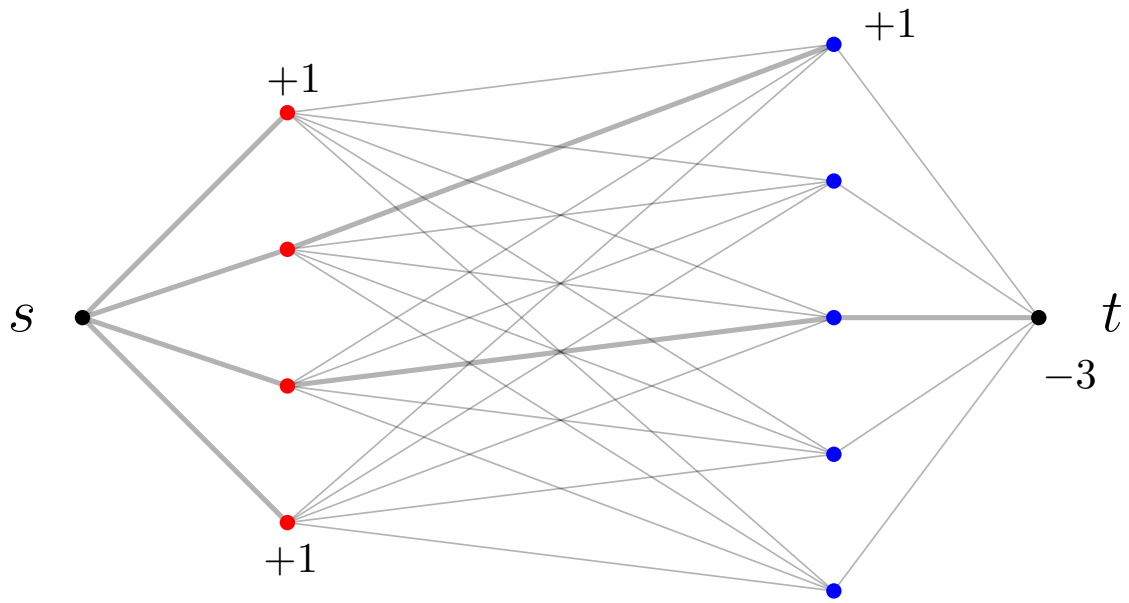
Refinement by blocking flows

1. Hungarian search: Dijkstra-like search changing π until $\exists$ an admissible augmenting path.

   Dijkstra-like search changing π in units of θ until $\exists$ an admissible augmenting path.
2. Augment by the admissible augmenting path you found.

   Augment by an admissible blocking flow g.



- Find by DFS through the admissible residual arcs.

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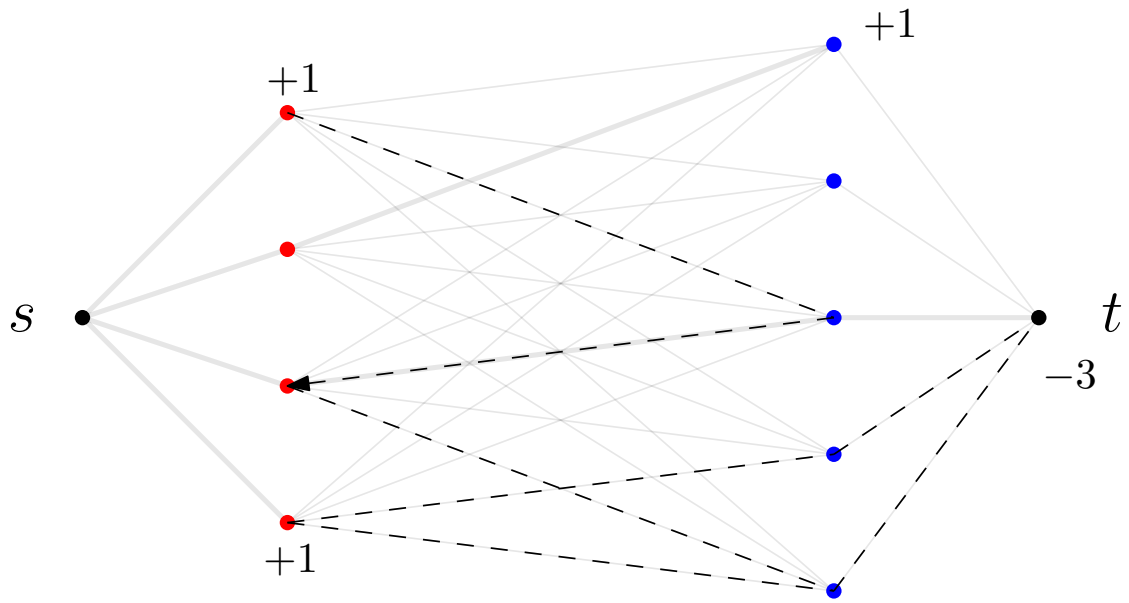

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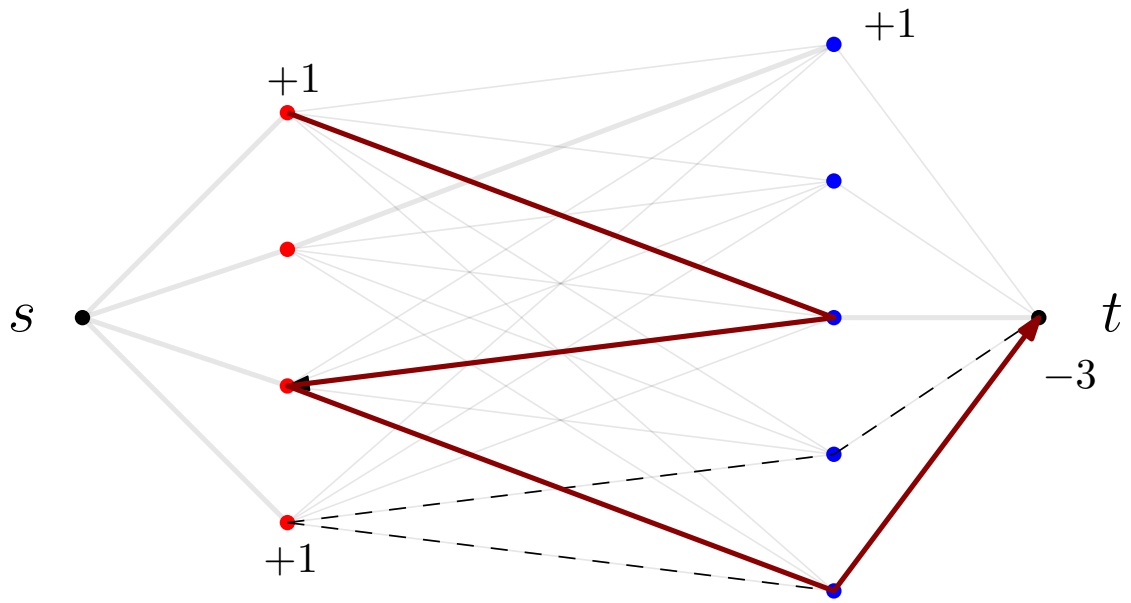

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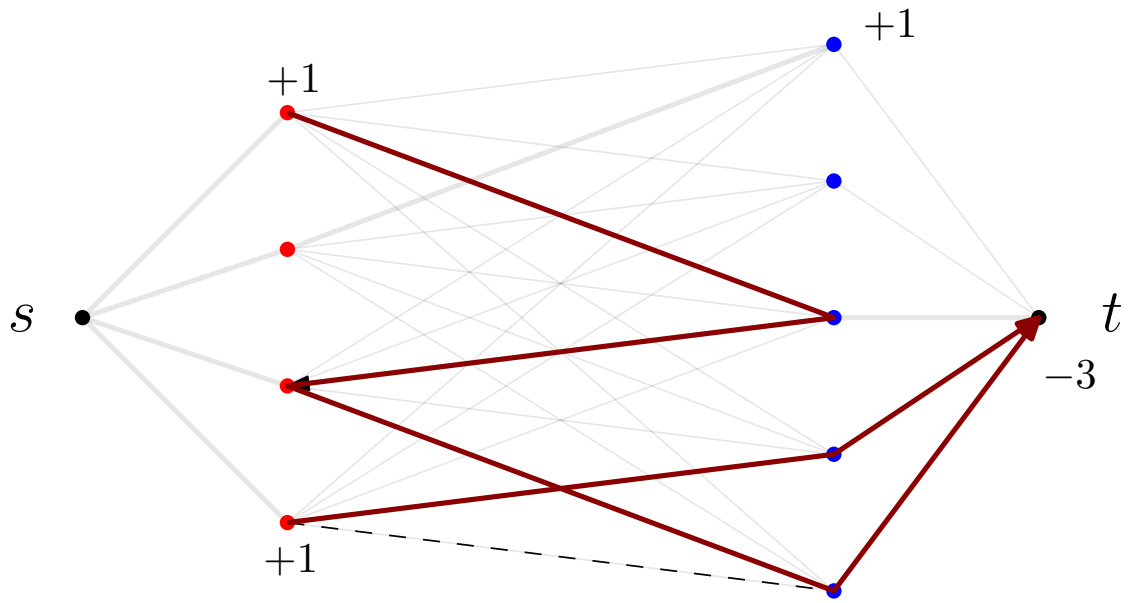

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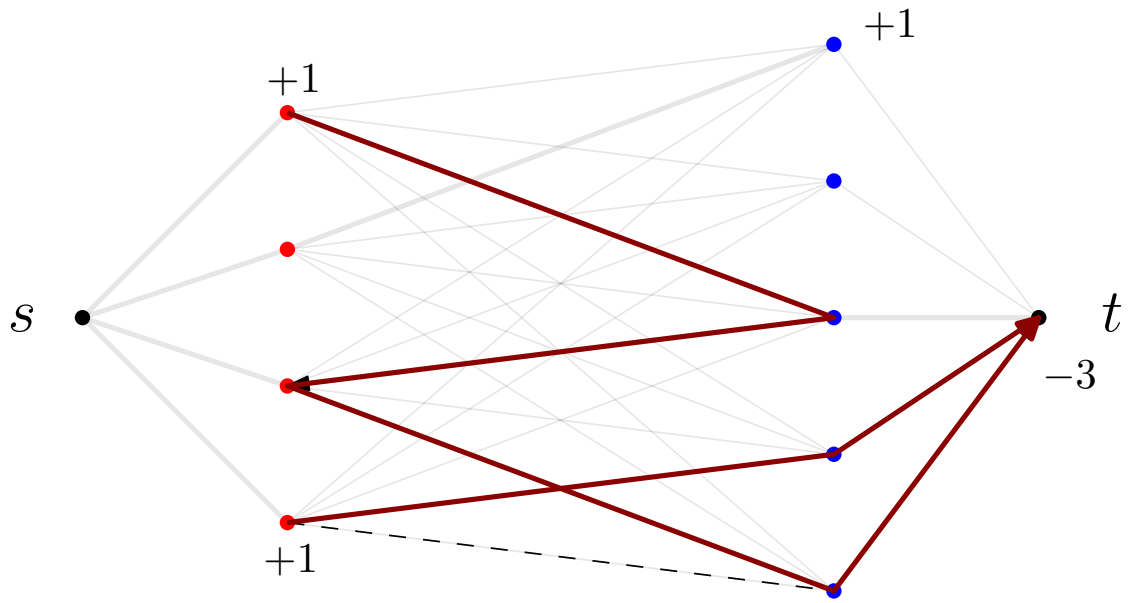

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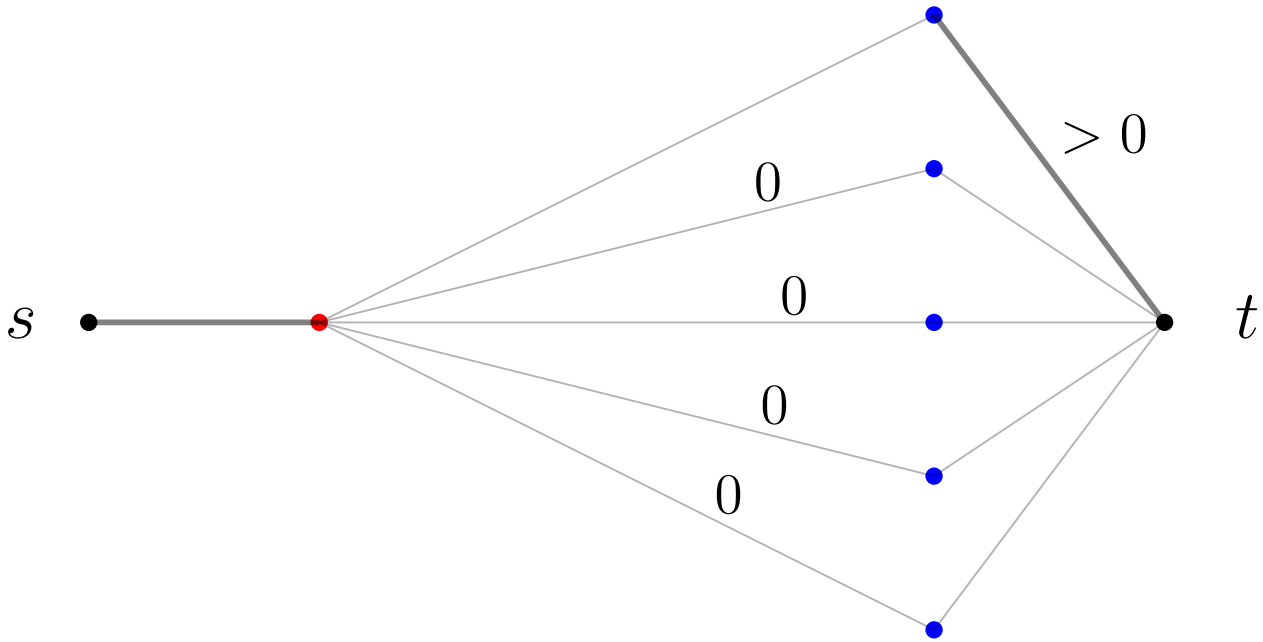
Lemma

After $O(\sqrt{k})$ blocking flows, f is a circulation.

Hungarian search dead ends

- There can be $\Omega(n)$ relaxations even if k is very small.



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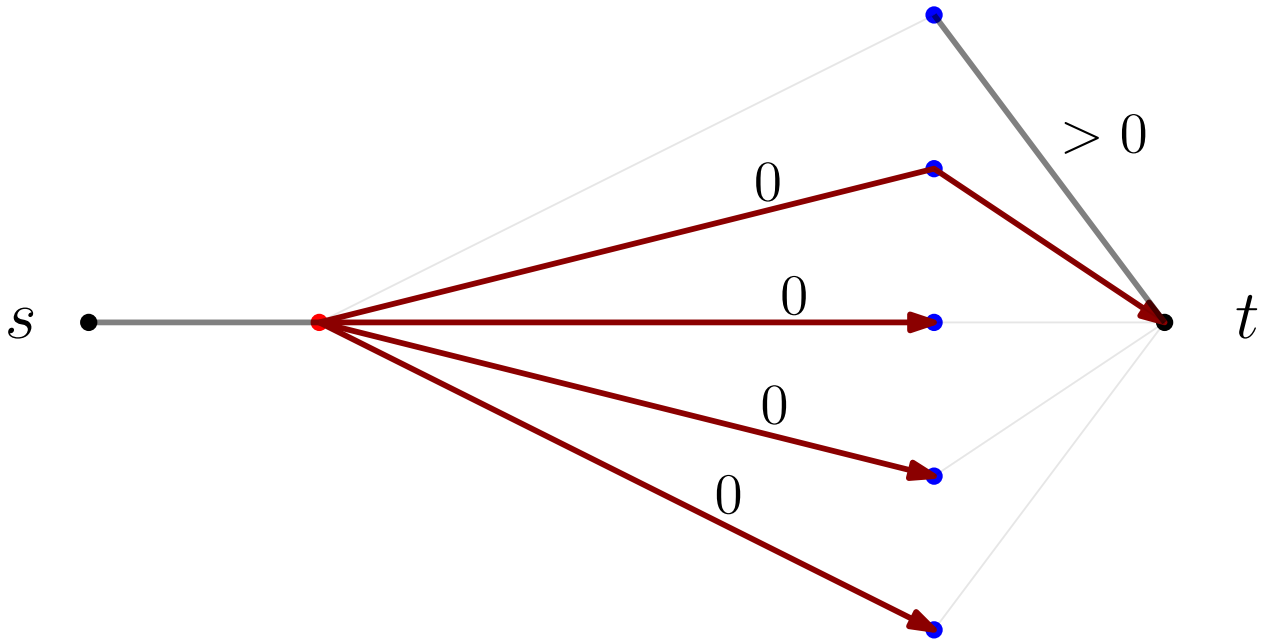

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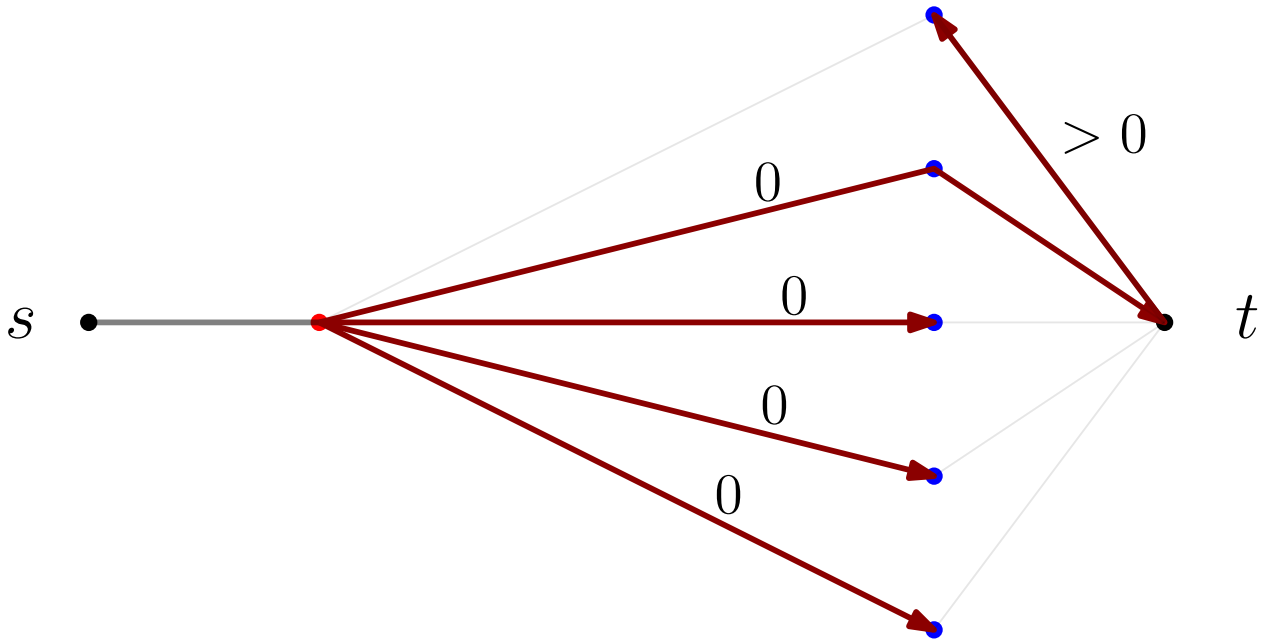

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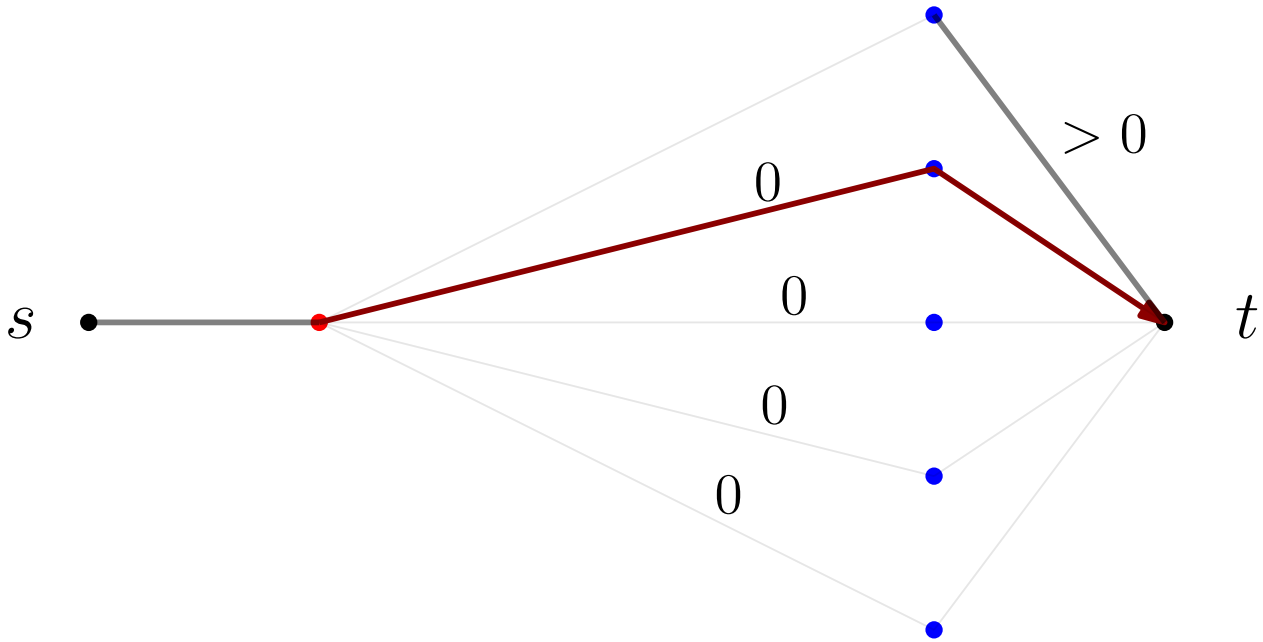
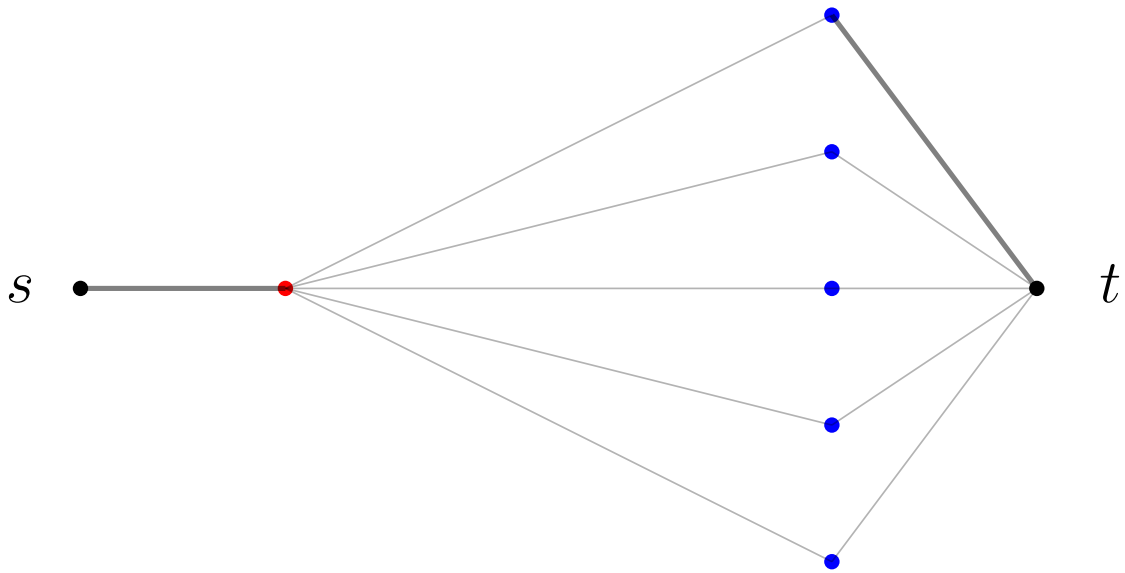


Dead or alive

- Alive nodes: nonzero excess/deficit, or adjoining flow support arcs.
- Dead nodes: ones which aren't alive.



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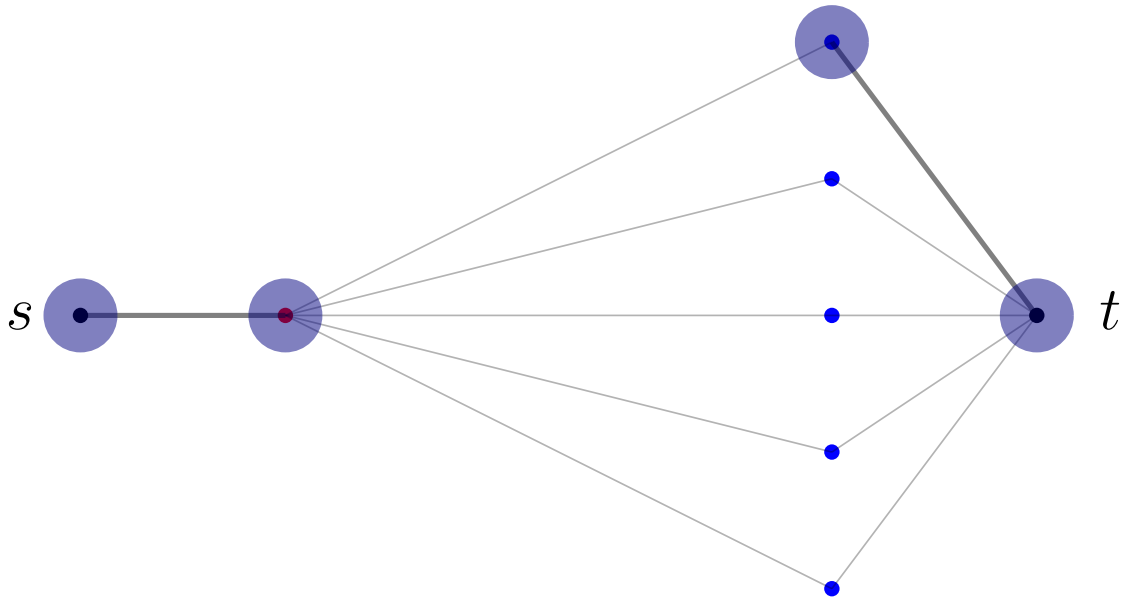

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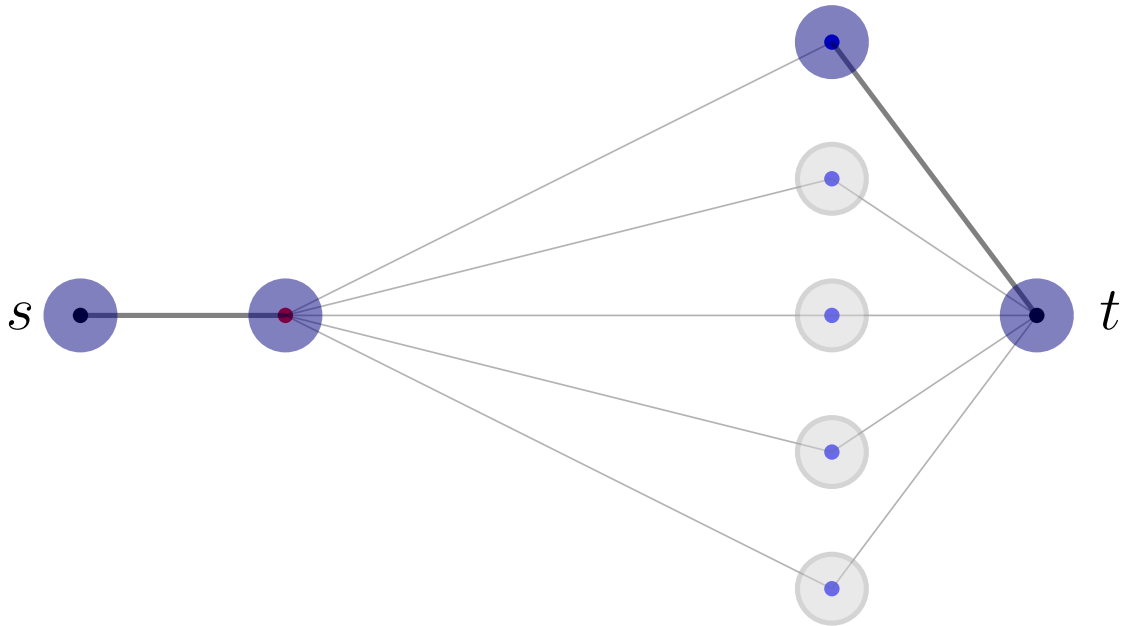

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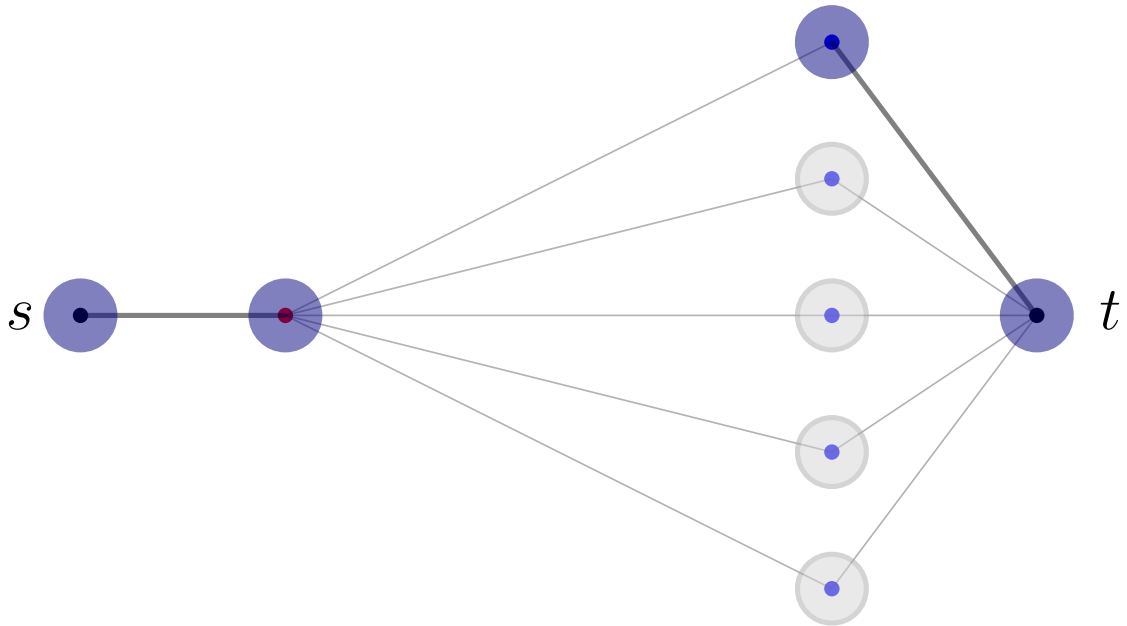

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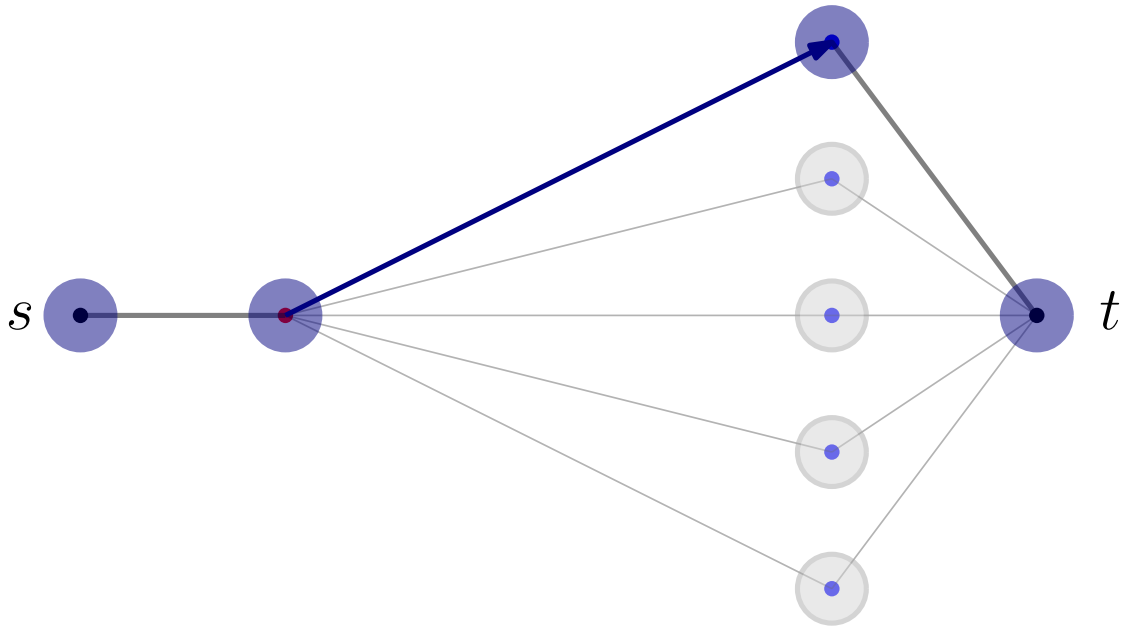

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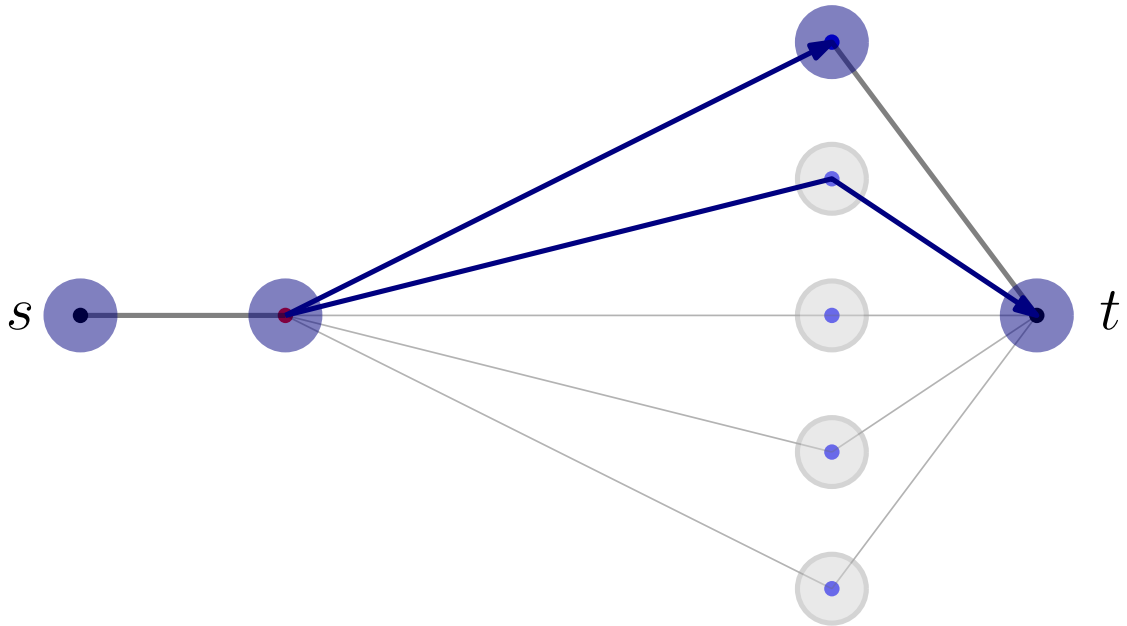

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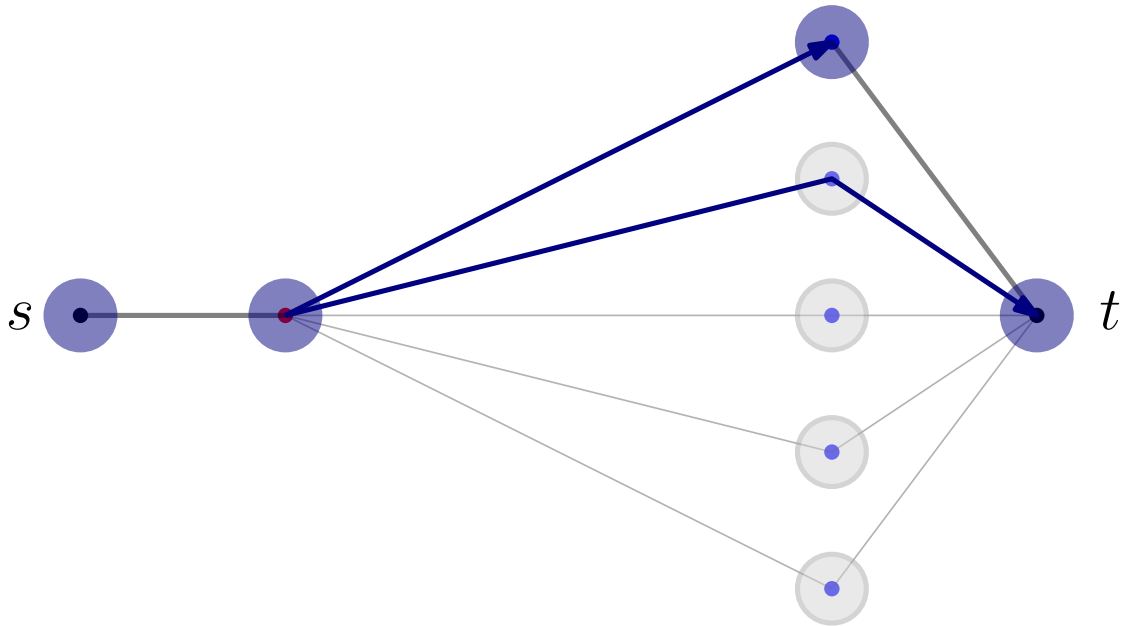

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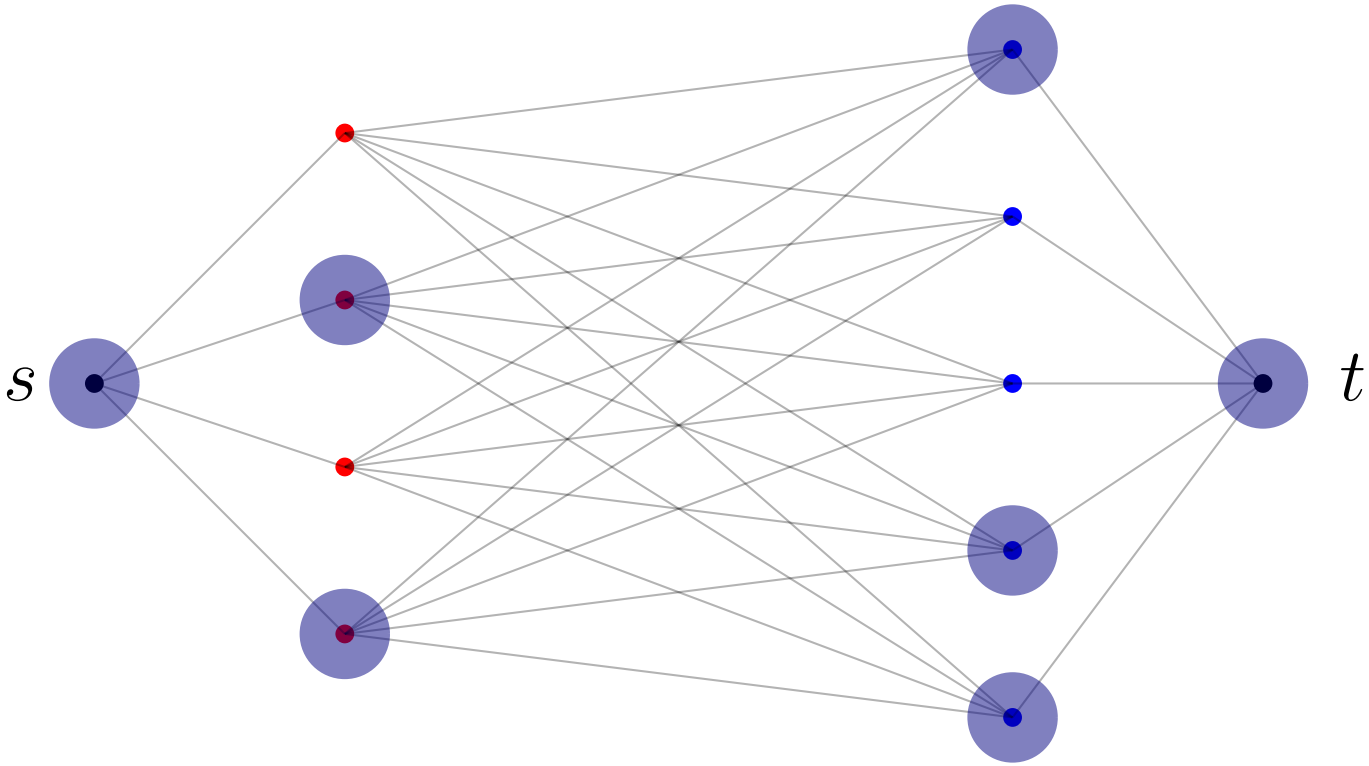
Hungarian search over alive paths

- Alive paths have length 1, 2, or 3.



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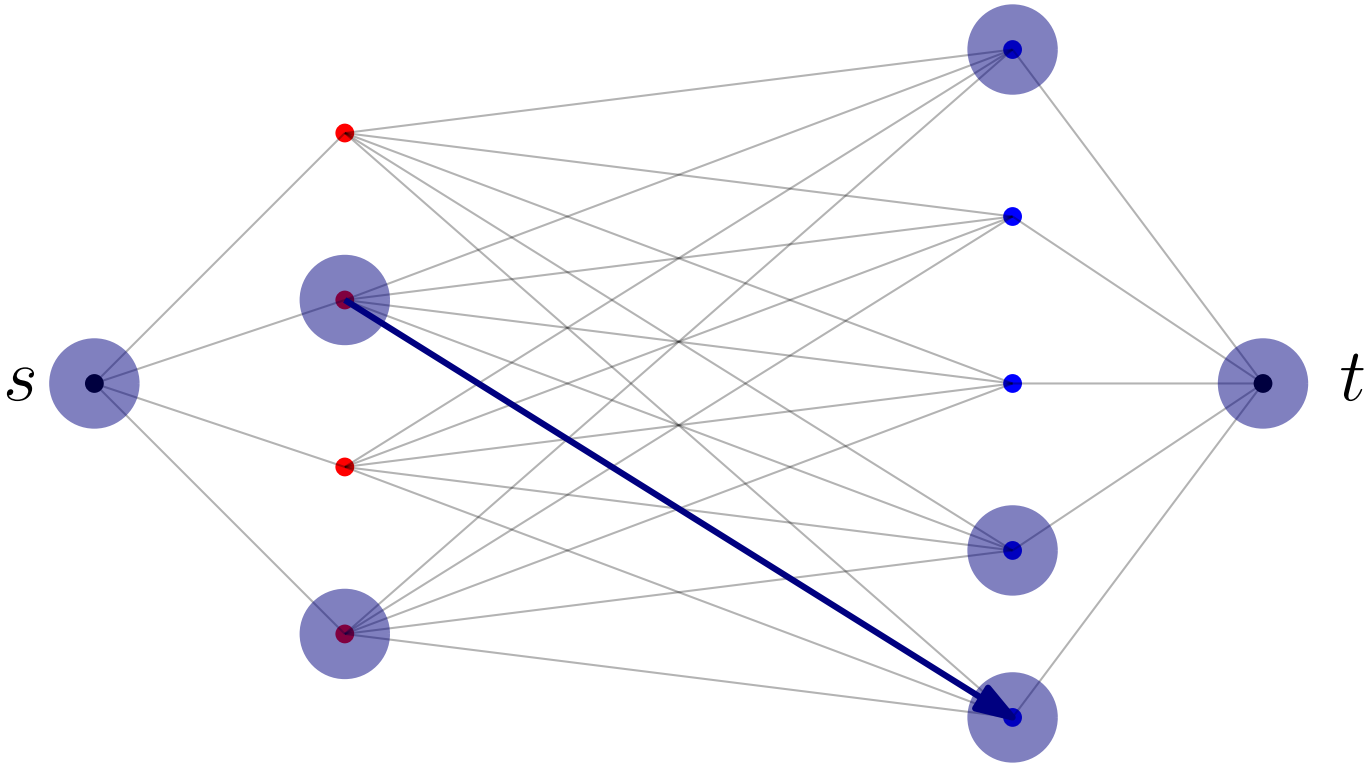

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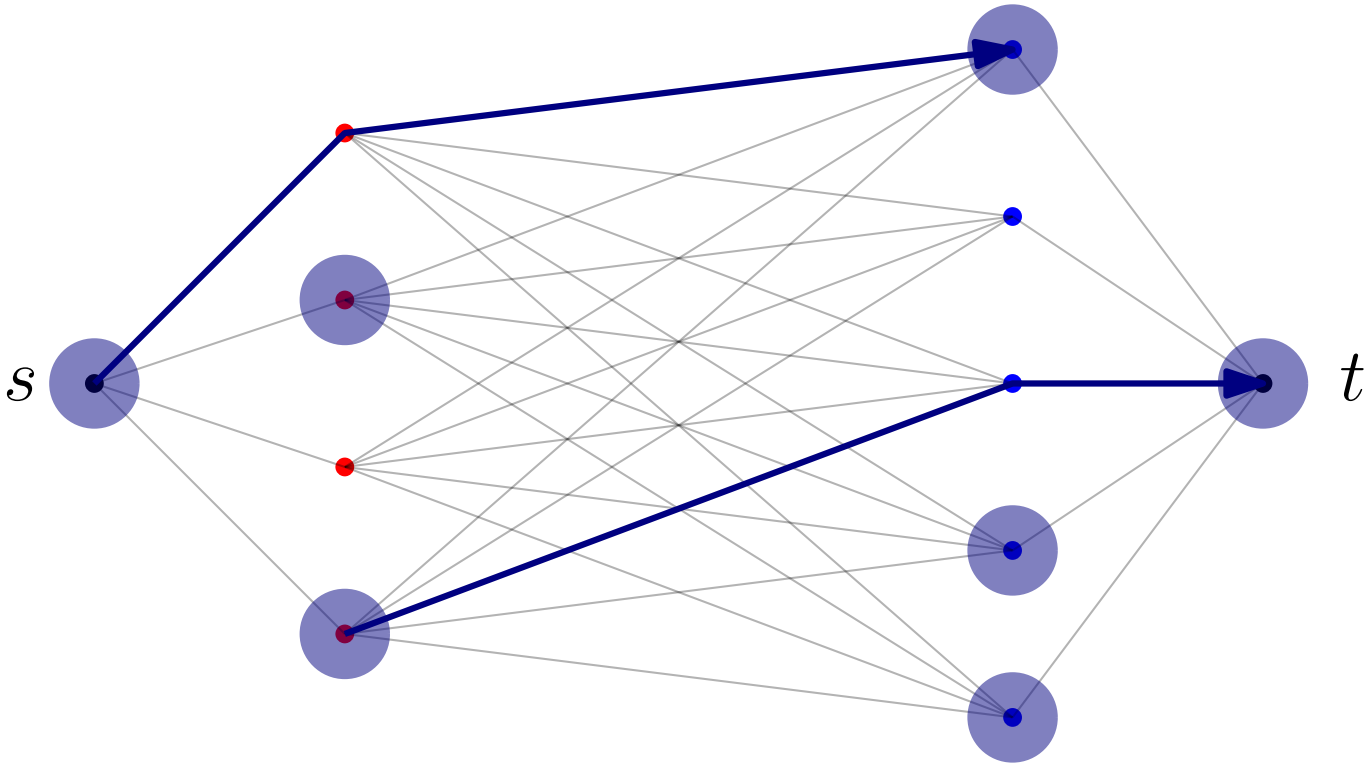

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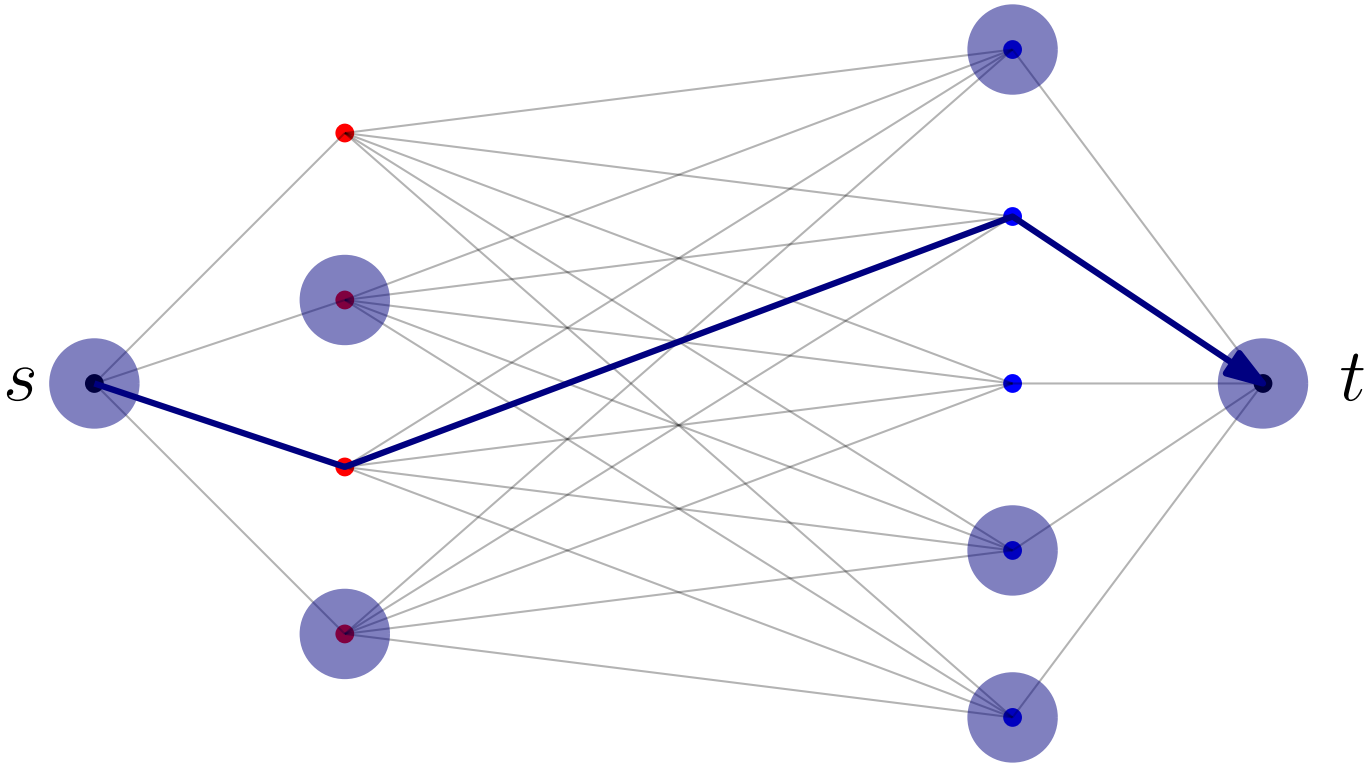

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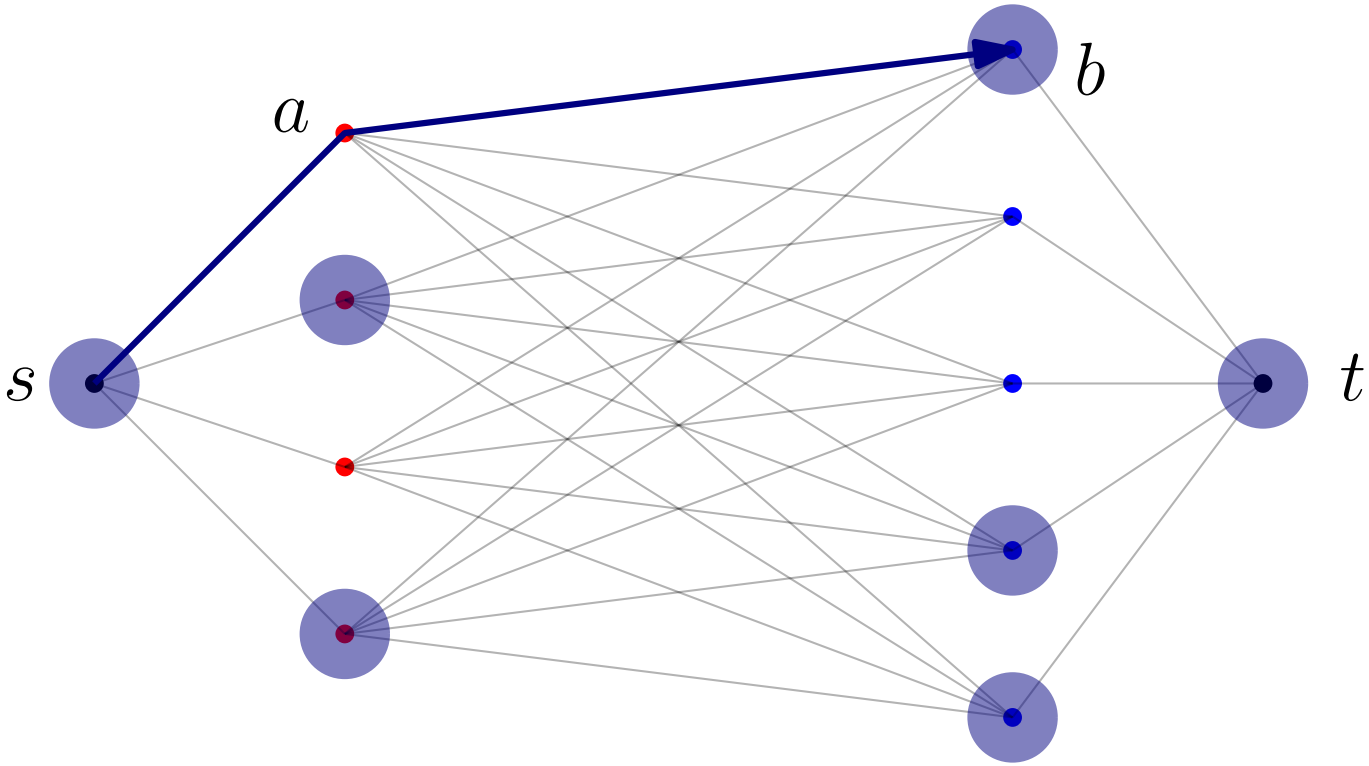

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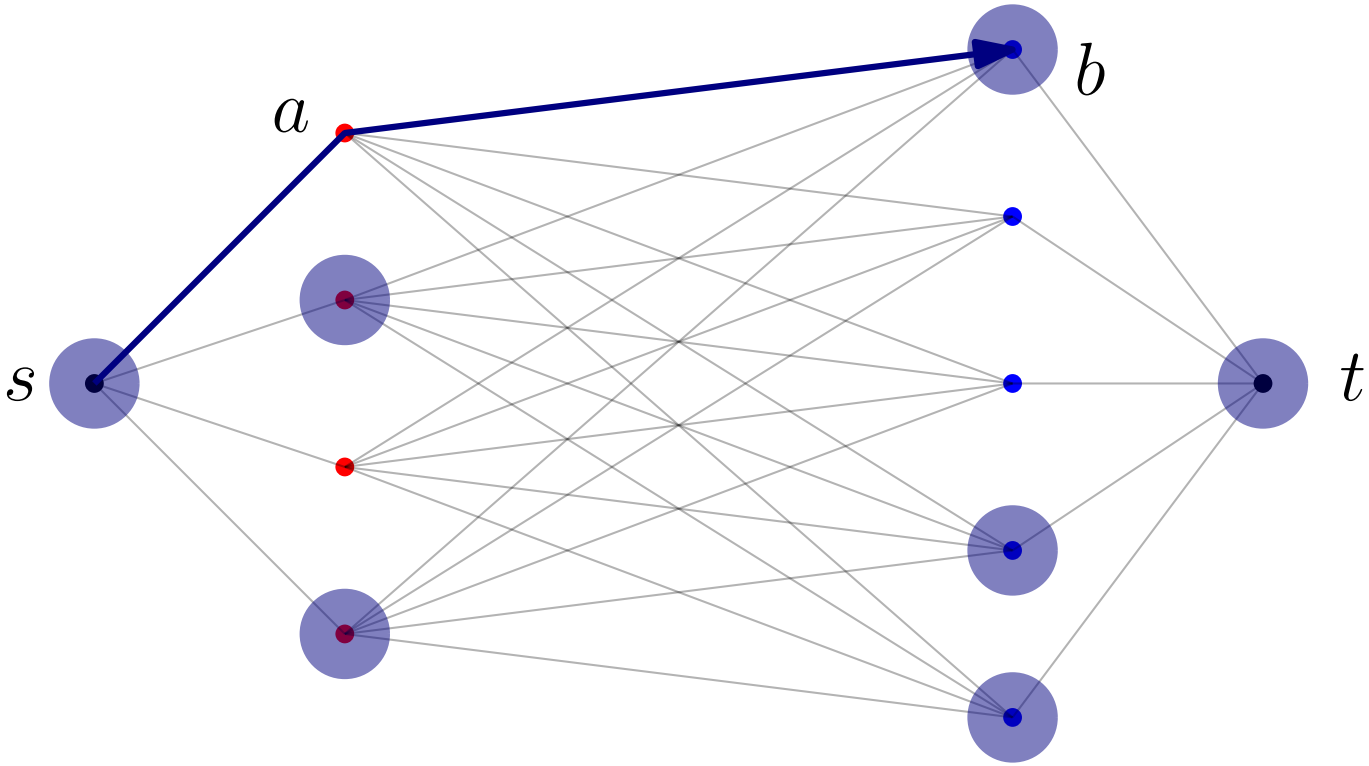

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How many relaxations now?

- $O(k)$ alive nodes.
- Every alive path relaxation adds another alive node to X.
- Thus, $O(k)$ relaxations.

Dead node potentials

- Dead nodes may become alive again after an augmentation.
- Also, at the end of a scale, we need potentials for all nodes.
- What's the right value for a dead node's potential?

$$\pi(v) = \begin{cases} \pi(s) & v \in A \\ \pi(t) & v \in B \end{cases}$$

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Cost-scaling time summary

- $O(\log(n^q/\varepsilon))$ scales.
- $O(\sqrt{k})$ iterations of refinement.
- Each blocking flow has size $O(k)$.

- Combine alive paths with previous ideas: rewinding and batched potential updates.
- $O(k \operatorname{polylog} n)$ time for each Hungarian search and DFS (finding blocking flow).
- $O(n \operatorname{polylog} n)$ to build the initial BCP data structures per scale.

- $O((n + k\sqrt{k}) \operatorname{polylog} n)$ time per scale, $O(\log(n^q/\varepsilon))$ scales, $(1+\varepsilon)$-approx.

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The End

Thank you.