A Comprehensive Study of Options Pricing by Monte Carlo Methods

Project Group No. 11

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December 11, 2022

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Outline/Contents/Overview

(1) Introduction

- Monte Carlo simulation is a method of analysis that includes intentionally reproducing a random process (typically with a computer), executing it several times, and immediately seeing the outcomes.
- The Black-Scholes formula is exact when the underlying follows a lognormal distribution. However, that is not the case in real life, and hence we shall use numerical methods instead.
- Different methods that have been developed for option pricing, MC method is one of the best of them.
- Since most of the quantitative analysis in finance or risk management involves computing quantities that are indeed expected values, MC simulation is widely used in the financial industry.

(2) Learning Outcomes

- In this slide, we will define some common Options such as European options, American options and Asian options, and also discuss about how to valuate them using Monte Carlo method, also using Black-Scholes model. Moreover, we will calculate Greeks(Delta,Theta) for European Options. .
- We used LaTeX for writing and '*Python*' programming to illustrate the figures.

(3) Preliminaries

Definition (Derivatives)

A derivative can be defined as a financial contract whose value is based on, or "derived" from, a traditional security (such as a stock or bond), an asset (such as a commodity), or a market index.

Types of Derivatives

- Forward Contracts
- Future Contracts
- Options

(3) Preliminaries Contd.

Options

There are two kinds of options:

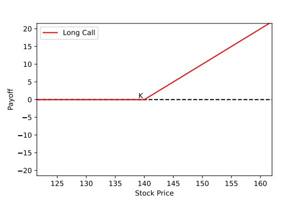
- **Call Option**
  A call option authorizes the holder the right to purchase the underlying asset by a specific date (**T**) at a specific price (**K**).
- **Put Option**
  A put option authorizes the holder the right to sell the underlying asset by a certain date(**T**) for a certain price(**K**).

(3) Preliminaries Contd.

Call Option Payoff

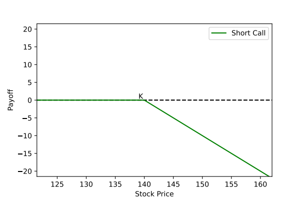
- **Long Call**

$$C(S,\quad T) = \max(S-K,0) \\ = \begin{cases} 0, & \text{if} \quad S_T < K \\ S_T - K, & \text{if} \quad S_T \geq K \end{cases}$$



- **Short Call**

$$C(S,\quad T) = -\max(S-K,0) \\ = \begin{cases} K - S_T, & \text{if} \quad S_T \geq K \\ 0, & \text{if} \quad S_T < K \end{cases}$$

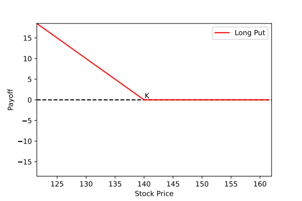


(3) Preliminaries Contd.

Put Option Payoff

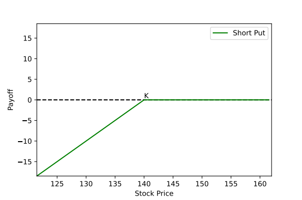
- **Long Put**

$$P(S,\quad T) = \max(K - S, 0) = \begin{cases} K - S_T, & if \quad S_T \leq K \\ 0, & if \quad S_T > K \end{cases}$$



- **Short Put**

$$P(S,\quad T) = -\max(K - S, 0) = \begin{cases} S_T - K, & if \quad S_T \leq K \\ 0, & if \quad S_T > K \end{cases}$$



(3) Preliminaries Contd.

European Options Pricing with the Black-Scholes Model(BSM)

- **BSM model PDE**

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0$$

- **Call Option Pricing Formula**

$$C = S_0 N(d_1) - Ke^{-rT} N(d_2)$$

- **Put Option Pricing Formula**

$$P = Ke^{-r(T-t)} N(-d_2) - S_0 N(-d_1)$$

(3) Preliminaries Contd.

European Options Pricing with the Black-Scholes Model(BSM)

where,

$$
\begin{aligned}
d_1 &= \frac{\ln\left(S_0/K\right) + \left(r + \sigma^2/2\right) T}{\sigma\sqrt{T}} \\
d_2 &= \frac{\ln\left(S_0/K\right) + \left(r - \sigma^2/2\right) T}{\sigma\sqrt{T}} \\
&= d_1 - \sigma\sqrt{T}
\end{aligned}
$$

$$
\text{and} \quad N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2} dy
$$

(3) Preliminaries Contd.

European Options Pricing with the Black-Scholes Model(BSM)

To visualize valuation, we illustrate the following Figure with value $T = 10$, $K = 50$, $r = 0.1$, $\sigma = 0.4$, $t = 8.5, 9, 9.5, 9.9, 10$ using above formulas:

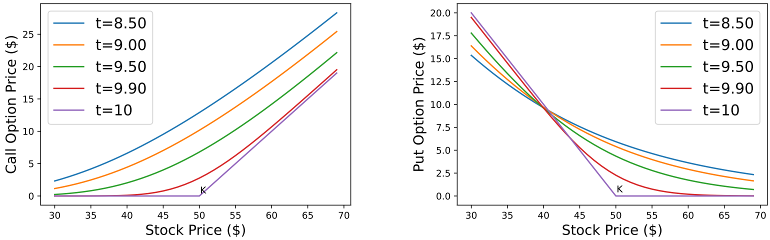


Figure: Call & Put options valuation.

(3) Preliminaries Contd.

American Options

An American call option gives the holder the right to buy an underlying asset for a strike price of K at any time prior to the specified expiry date T. The holder can exercise American options at any time until they expire. The option payoff can be written as follows:

- **Long Call**

$$C_A(S, T) = \max_{0<t\leq T}(S_T - T, 0)$$

- **Long Put**

$$P_A(S, T) = \max_{0<t\leq T}(K - S_T, 0)$$

(3) Preliminaries Contd.

Asian Options

- Originally employed in 1987 by the Tokyo office of Banker's Trust to price average options on crude oil contracts
- The underlying variable is the average price over a given time period
- Less expensive than their European counterparts due to their reduced volatility
- They are typically traded on low-volume currencies and commodities
- Arithmetic average Asian call option price:

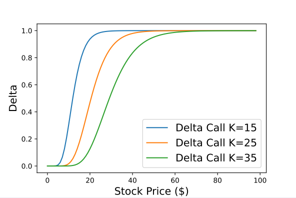
$$\Phi(S) = \left(\frac{1}{T} \int_0^T S(t)dt - K \right)^+$$

(3) Preliminaries Contd.

Delta: European Options

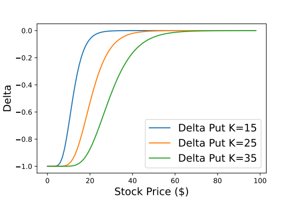
- **Delta: Call Option**

$$\Delta = \frac{\partial C}{\partial S} = N(d_1)$$



- **Delta: Put Option**

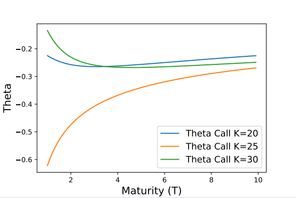
$$\Delta = \frac{\partial P}{\partial S} = N(d_1) - 1$$



(3) Preliminaries Contd.

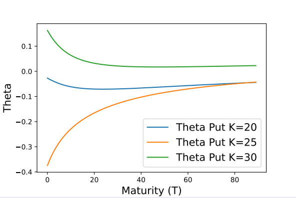
Theta: European Options

- **Theta: Call Option**



$$\Theta = \frac{\partial C}{\partial t}$$
$$= -\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT} N(d_2)$$

- **Theta: Put Option**



$$\Theta = \frac{\partial C}{\partial t}$$
$$= -\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} + rKe^{-rT} N(-d_2)$$

(3) Preliminaries Contd.

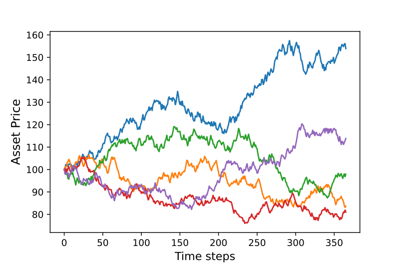
Geometric Brownian motion (GBM)

A stochastic process $S(t)$ is said to follow a Geometric Brownian motion (GBM) if

$$dS = \mu S dt + \sigma S dW.$$

where W is a Wiener process, and μ and σ are constants. We have an exact formula for the stock price at time t is given by:

$$S(t) = S_0 e^{(\mu - \frac{\sigma^2}{2})t} + \sigma W(t)$$



(4) Our Model(s), Results and Discussion

Monte Carlo Method

It provides an approximate solution to a mathematical problems by performing statistical sampling experiments on computer simulated data using different stochastic models. Simulation steps are:

1. In a risk-neutral world, simulate one stock price path.
2. Determine the value of the stock option.
3. Steps 1 and 2 should be repeated several times to obtain a large number of sample payoffs.
4. Repeat steps 1 and 2 several times to obtain a large number of sample payoffs.
5. Calculate the average payoff.
6. To assess the option's value, discount the mean payment at the risk-free rate.

(4) Our Model(s), Results and Discussion

Pricing European Options

The solution of the Geometric Brownian motion path can be written as:

$$S_t = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma X \sqrt{t}} \qquad \text{where } X \sim N(0,1)$$

For the European option, and the corresponding Monte Carlo procedure is as follows:

1. Set sum $= 0$
2. Generate value for stock price S_t using Equation **??**
3. Calculate sum $=$ sum $+ \max(0, S_t - K)$ for all S_t
4. Monte Carlo option price $= \frac{\text{sum}}{n} * e^{-rT}$ where n is the total simulation number.

Pricing European Options

We make a table for European Call option pricing by Monte Carlo method(with 100000 simulations):

Data Set:

$\{S_0 = 100, K = 90,\ r = 0.05,\ \sigma = 0.2,\ T = 2, 4, 6, 8, 10\}$

K	T	BSM Call Price	MC Call Price	Abs. Differ.
90	2	22.033	21.985	0.048
	4	30.664	30.783	0.119
	6	37.826	37.689	0.136
	8	44.030	44.147	0.117
	10	49.504	49.443	0.061

Table: Comparison of BSM values and Monte Carlo estimated values against Maturity(T)

Pricing European Options

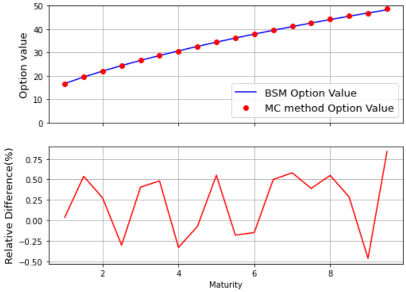


Figure: Comparison of BSM values and Monte Carlo estimated values against Maturity(T)

◇Every valuation difference is less than 1%, including both positive and negative deviations.

Pricing European Options

A similar analysis can be present for different *strike price* instead of different maturity (with 100000 simulations):

Data Set: $\{S_0 = 100,\ T = 6,\ r = 0.05,\ \sigma = 0.2\}$

T	K	BSM Call Price	MC Call Price	Abs. Difference
2	60	45.823	45.631	0.192
	70	37.163	37.205	0.042
	80	29.120	29.243	0.123
	90	22.033	22.011	0.022
	100	16.127	16.127	0.000
	110	11.455	11.398	0.057
	120	7.928	7.904	0.024
	130	5.367	5.377	0.010

Table: Comparison of BSM values and Monte Carlo estimated values.

Pricing European Options

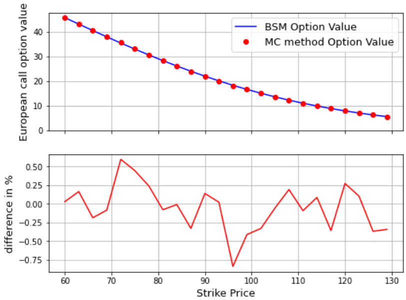


Figure: Comparison of BSM values and Monte Carlo estimated values against Strike Price(K)

◇Likewise, all valuation discrepancies, positive and negative, are less than 1%.

Pricing European Options

A similar analysis can be present for different *strike price* instead of different maturity (with 100000 simulations):
Data Set: $\{S_0 = 30,\ T = 1,\ r = 0.01,\ \sigma = 0.3\}$

T	K	BSM	MC	Abs. Diff
1	40	0.98	0.78	0.2
	45	0.48	0.36	0.12
	50	0.23	0.26	0.03
	55	0.11	0.16	0.05
	60	0.05	0.04	0.01

Table: BSM vs MC for European Call

Pricing European Options using Antithetic

Data Set: $\{S_0 = 30,\ T = 1,\ r = 0.01,\ \sigma = 0.3\}$

T	K	BSM	Antithetic	Abs. Diff
1	40	0.98	0.9	0.08
	45	0.48	0.41	0.07
	50	0.23	0.25	0.02
	55	0.11	0.11	0.0
	60	0.05	0.03	0.02

Table: BSM vs Antithetic

Pricing European Options using Antithetic

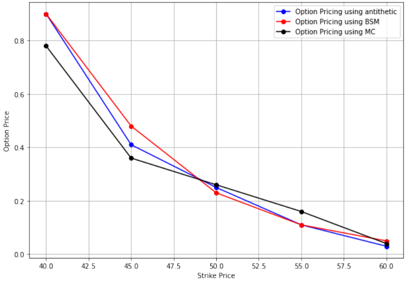


Figure: Option Pricing Using Different Method

◇We can see that the MC and Antithetic is showing good results. In fact, Antithetic is giving better approximation.

(4) Our Model(s), Results and Discussion

Pricing American Options

We make a table for American Put option values calculated using LSM method(with 50 exercisable times & 10000 simulations)
Data Set: $\{S_0 = 100,\ r = 0.05,\ \sigma = 0.2,\ T = 1\}$

T	K	European Put	American Put	Early Exer. Premium
1	85	1.2958	1.2997	0.0039
	90	2.3921	2.4819	0.0897
	95	3.7161	4.0270	0.3109
	100	5.5877	5.9574	0.3697
	105	7.9323	8.7286	0.7964
	110	10.4625	11.9199	1.4575
	115	13.7574	15.6874	1.9300

Pricing American Options

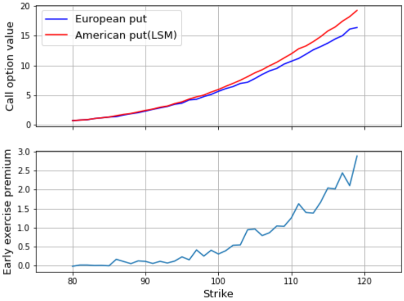


Figure: Comparison of European and American(LSM) Monte Carlo estimator values against Strike Price(K)

◇American put option price is generally higher than the price of the corresponding European put option.

Pricing Asian Options

We use BSM & MC method to apply Arithmetic Average method for Asian call option and construct table with m sample points:

Data Set: $\{S_0 = 100;\ r = 0.05;\ \sigma = 0.2;\ T = 1\}$

K	m	Euro.(BSM)	Asian(BSM)	Asian(MC)	Abs.Differ.
80	10	24.5888	19.9857	22.5423	2.5566
	20		21.1000	22.5706	1.4706
	40		21.6572	22.5403	0.8831
	60		21.8430	22.5377	0.6947
	80		21.9359	22.5498	0.6139
	100		21.9916	22.5476	0.5560
	130		22.0430	22.5444	0.5014
	150		22.0659	22.5426	0.4767
	200		22.1031	22.5467	0.4436

Table: Arithmetic average Asian Call option prices using BSM and MC

Pricing Asian Options

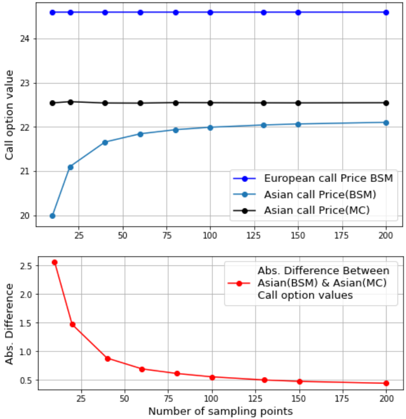


Figure: Arithmetic average Asian option prices using BSM and MC

Delta

For a European call option, Delta is given by:

$$\Delta = \frac{\partial C}{\partial S} \approx \frac{C(t, S+\delta) - C(t,S)}{\delta}$$

Data Set: $\{K = 25;\ r = 0.01;\ \sigma = 0.1;\ T = 10;\ t = 0;\ \delta > 0\}$

K	T	S	Delta(BSM)	Delta(MC)	Abs. Diff.
25	10	22	0.5279	0.5072	0.0207
		23	0.5834	0.5506	0.0328
		24	0.6350	0.6196	0.0154
		25	0.6824	0.6457	0.0367
		26	0.7252	0.6993	0.0259
		27	0.7635	0.7662	0.0027
		28	0.7975	0.7736	0.0239

Delta

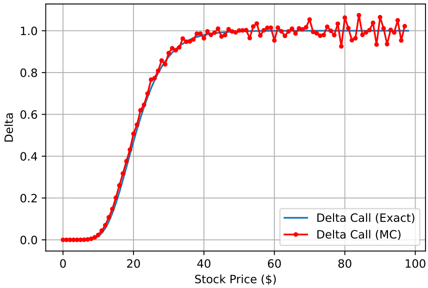


Figure: Graphical representation of Delta using BSM and MC method

◇The Monte Carlo method gives good approximated values for Delta which is very similar to BSM model values.

Theta

For a European call option, theta is :

$$\Theta = \frac{\partial C}{\partial t} \approx \frac{C(t+\delta, S) - C(t, S)}{\delta}$$

Data Set: $\{S = 10;\ r = 0.01;\ \sigma = 0.1;\ K = 8;\ t = 0;\ \delta > 0\}$

K	S	T	Theta(BSM)	Theta(MC)	Abs. Diff.
8	10	4	-0.1049	0.0042	0.1091
		5	-0.1026	0.0015	0.1041
		6	-0.1000	0.0005	0.1005
		7	-0.0973	0.0002	0.0974
		8	-0.0946	0.0001	0.0947
		9	-0.0921	0.0000	0.0922
		10	-0.0898	0.0000	0.0898

Theta

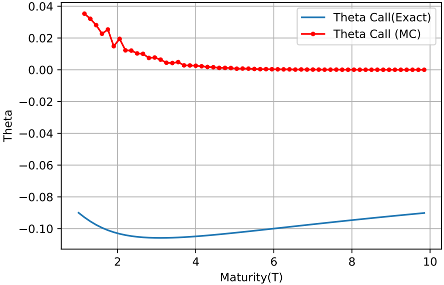


Figure: Graphical representation of Theta using BSM and MC method

◇Trends of both resulted data is nearly same. Absolute error with compared to BSM model is also very low.

(5) Conclusion

- Throughout this project we learned various basic topics and elements relative to 'Option Pricing' & 'Monte Carlo Method'.
- We got some ideas about European options, American options, Asian options, Delta and Theta.
- We have used '**Python**' code to calculate & plot the required payoff and profit diagrams, which helped us enough to enrich our '*Python*' programming skills.
- We used LaTeX which is a renowned most scientific typesetting.
- Although our project gives a basic idea of 'Pricing Options using MC method', it could be an ideal guideline for further research.

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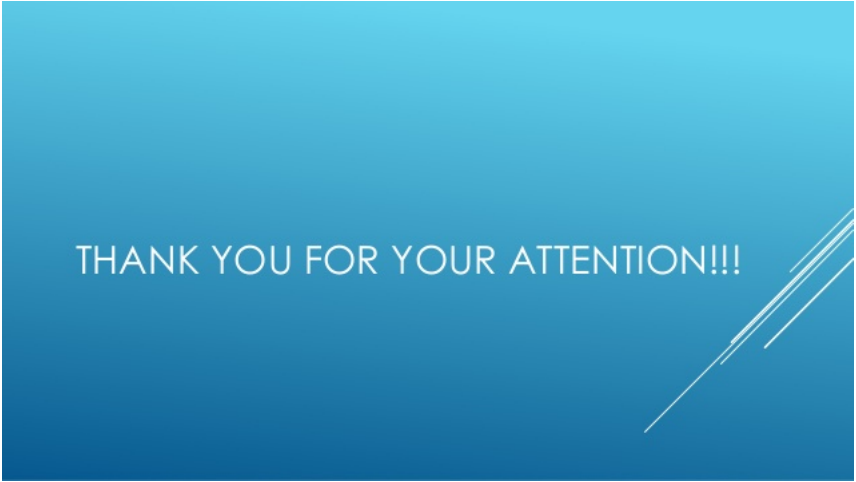
THANK YOU FOR YOUR ATTENTION!!!