FIRST ORDER LANGUAGE

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Introduction

The main issue in first order logic is how to work with quantifiers. This statement is of course almost superfluous, since, for example, first order logic is obtained from sentential logic by adding quantifiers to the sentential connectives. In turn, the universal instantiation rule of inference is, arguably, the most substantive property of universal quantifiers. This rule, stated informally, is that from the universal statement: All objects have the property P. , we can infer, for any particular object, say a, that : a has the property P. While, perhaps no one would question, that the rule, as stated above, is quite reasonable, or, one might say, even correct, it is nevertheless only an informal way of expressing one of the fundamental steps of logical reasoning.

A direct reformulation of such a rule for the setting of first order languages is as follows. Let φ be a wff of a first order language, and τ be a term. Then the rule would be, from $\forall x\varphi$, we can infer φ^x_t , where φ^x_t is the result of substituting t for x in φ, or using standard notation,

$$\frac{\forall x\varphi}{\varphi^x_\tau} \tag{0.1}$$

This however cannot be a correct formulation, since every inference rule must be valid, in order for the soundness theorem to hold. And, we can see by an example, that the above formulation is not valid. Namely, let φ be $\exists y(y \neq x)$. Then the wff $\forall x\varphi$ i.e. $\forall x\exists y(y \neq x)$ is satisfied in all structures that have at least two elements. However, φ^x_y is $\exists y(y \neq y)$, which is false in every structure. Thus, for the φ above, we cannot infer φ^x_y from $\forall x\varphi$, hence the rule (1.1) is not correct in general, as an inference rule for first order logic. One might comment that, in a sense, the above difficulty will not occur reasoning carried out in English. Namely, the statement $\forall x\varphi$, translated into English, simply says that : For every object, there exists another object, different from the first object. But, from this statement, we can clearly infer (at least at our level of informality) that: Given any particular object, there exists another object, different from this given particular object.

The preceding example suggests, that the difficulty that we identified, has something to do with the use of variables, which one does not do in very informal English. However, the use of variables in first order languages is a part of the essence of first order language, which allows one to, we might say mathematicize logic, and, there is really no alternative viable approach to first order logic. Thus we have no choice but to grapple with this issue, and attempt to formulate the rule (1.1) correctly. We will do this in this paper, essentially following the book [E], but adding more details. For example, some such details are left as exercises in [E]. The key notion is the notion of substitutability. Namely, the rule (1.1) is correct, i.e. valid, under the assumption that t is substitutable for x in φ. That the rule (1.1), with this additional assumption is, in fact, the right formulation of universal instantiation, is born out by the fact that the completeness theorem holds with this formulation of universal instantiation. In other words, since the completeness theorem holds under this formulation of universal instantiation, it means that this formulation is not too restrictive.

Section 1

The notion of substitutability, and an auxiliary notion of free variables, are defined on pages 112, 113 in [E]. We shall modify this approach somewhat, which allows to introduce the notion of which occurrences of variable x in α are free, and which are bound rather than whether x occurs free in α at all. Furthermore, we will only discuss the substitutability of one variable for another, rather than the substitutability of arbitrary terms. This is sufficient to illustrate the essential issues.

We begin with an analysis of wffs based on the Unique Readability Theorem (URT) which is as follows.

The Unique Readability Theorem.

Let α be a wff of a first order language. Then exactly one of the following cases, 0 - 7 happens:

0. α is atomic ;
1. There exists a unique wff β such that α is $(\neg\beta)$;

2. There exist unique wffs β and γ such that α is $(\beta \wedge \gamma)$;
3. There exist unique wffs β and γ such that α is $(\beta \vee \gamma)$;
4. There exist unique wffs β and γ such that α is $(\beta \to \gamma)$;
5. There exist unique wffs β and γ such that α is $(\beta \leftrightarrow \gamma)$;
6. There exist unique wffs β and a unique positive integer i such that α is $\forall v_i \beta$;
7. There exist unique wffs β and a unique positive integer i such that α is $\exists v_i \beta$;

Let α be a wff of the FOL under consideration. We will define the tree of α, which we will denote by $T(\alpha)$. Essentially, this will be the construction tree of α. Informally speaking, $T(\alpha)$ will include objects, which are related to the wffs from which α is built by the formula building operations. To distinguish these objects from from the actual wffs from which α is build we will put a bar above them.

The Case 0, When α is atomic, is simple. In this case, we will define $\overline{\alpha}_1 = \alpha$, and set $T(\alpha) = \{\{\overline{\alpha}_1\}\} = \{\{\alpha\}\}$ (Comment : The double bracketing is in order for uniform treatment of all cases.)

If α is not atomic, the first step of building $T(\alpha)$ involves applying the URT to identify the final formula building operation of α, and include appropriate objects in $T(\alpha)$ accordingly. We state how we do this by considering the cases (1)-(7) of URT.

Case 1: There exists a unique wff β such that α is $(\neg\beta)$. Suppose furthermore that β is the expression $\beta = s_1 s_2 .... s_n$. We define $\overline{\alpha}_1 = t_3 t_4 .... t_{n+2}$ where $t_3 = s_1, t_4 = s_2, ...t_{n+2} = s_n$. i.e. $\overline{\alpha}_1$ is a finite sequence numbered not by 1, 2,..., n but by integers i, such that 3 $\leq$ i $\leq$ n+2. Hence $\overline{\alpha}_1$ shows the position of β within α, not just β itself in isolation. We now put, into $T(\alpha)$, The pair $\{\neg, \overline{\alpha}_1\}$. We note that $\overline{\alpha}_1$ allows us to reconstruct β itself - simply by changing the indexing from 3, 4, ..., n+2 back to 1, 2,..., n. Thus α itself is the finite sequence $t_1 t_2 .... t_n + 3$ where $t_1 =$ (, $t_2 = \neg$, t_3 through $t_n + 2$

Case 2: There exist unique β, γ such that α is $(\beta \wedge \gamma)$. Suppose further that β is the expression $\gamma_1 \gamma_2 .... \gamma_m$ and γ is the expression $s_1 s_2 .... s_n$. We define

$\overline{\alpha}_1 = t_2 t_3 .... t_{m+1}$, where $t_2 = r_1, t_3 = r_2, ...t_{m+1} = r_m$ and

$\overline{\alpha}_2 = t_{m+3} t_{m+4} .... t_{m+2+n}$ where $t_{m+3} = s_1, t_{m+4} = s_2, ...t_{m+2+n} = s_n$

We not put into $T(\alpha)$, the three-element set $\{\wedge, \overline{\alpha}_1, \overline{\alpha}_2\}$ Thus $\overline{\alpha}_1$, $\overline{\alpha}_2$ are just reindexed β , γ respectively, the new indexes being the subscripts of the symbols of β, γ within α (rather than the original subscripts that range from 1 to m, or 1 to n, respectively). Thus, similarly as in Case 1, α its the finite sequence $t_1 t_2 .... t_{m+n+3}$, where $t_1 = (, t_{m+2} = \wedge, t_{m+n+2} =)$ and the remains t_i are as defined above.

Case 3: There exist unique β, γ such that α is $(\beta \vee \gamma)$. Suppose further that β is the expression $\gamma_1 \gamma_2 .... \gamma_m$ and γ is the expression $s_1 s_2 .... s_n$. We define

$\overline{\alpha}_1 = t_2 t_3 .... t_{m+1}$, where $t_2 = r_1, t_3 = r_2, ... t_{m+1} = r_m$ and

$\overline{\alpha}_2 = t_{m+3} t_{m+4} .... t_{m+2+n}$ where $t_{m+3} = s_1, t_{m+4} = s_2, ... t_{m+2+n} = s_n$

We not put into $T(\alpha)$, the three-element set $\{\vee, \overline{\alpha}_1, \overline{\alpha}_2\}$. We note $\overline{\alpha}_1$ allows us to reconstruct β. $\overline{\alpha}_2$ allows us to reconstruct γ, simply by changing the indexing to start with 1.

Case 4: There exist unique β, γ such that α is $(\beta \rightarrow \gamma)$. Suppose further that β is the expression $\gamma_1 \gamma_2 .... \gamma_m$ and γ is the expression $s_1 s_2 .... s_n$. We define

$\overline{\alpha}_1 = t_2 t_3 .... t_{m+1}$, where $t_2 = r_1, t_3 = r_2, ... t_{m+1} = r_m$ and

$\overline{\alpha}_2 = t_{m+3} t_{m+4} .... t_{m+2+n}$ where $t_{m+3} = s_1, t_{m+4} = s_2, ... t_{m+2+n} = s_n$

We not put into $T(\alpha)$, the three-element set $\{\rightarrow, \overline{\alpha}_1, \overline{\alpha}_2\}$. We note $\overline{\alpha}_1$ allows us to reconstruct β. $\overline{\alpha}_2$ allows us to reconstruct γ, simply by changing the indexing to start with 1.

Case 5: There exist unique β, γ such that α is $(\beta \leftrightarrow \gamma)$. Suppose further that β is the expression $\gamma_1 \gamma_2 .... \gamma_m$ and γ is the expression $s_1 s_2 .... s_n$. We define

$\overline{\alpha}_1 = t_2 t_3 .... t_{m+1}$, where $t_2 = r_1, t_3 = r_2, ... t_{m+1} = r_m$ and

$\overline{\alpha}_2 = t_{m+3} t_{m+4} .... t_{m+2+n}$ where $t_{m+3} = s_1, t_{m+4} = s_2, ... t_{m+2+n} = s_n$

We not put into $T(\alpha)$, the three-element set $\{\leftrightarrow, \overline{\alpha}_1, \overline{\alpha}_2\}$. We note $\overline{\alpha}_1$ allows us to reconstruct β. $\overline{\alpha}_2$ allows us to reconstruct γ, simply by changing the indexing to start with 1.

Case 6: There exist unique wff β and a unique positive integer i α is $\forall v_i \beta$. Suppose further that β is the expression $\beta = s_1 s_2 ..... s_n$, we define

$\overline{\alpha}_1 = t_3 t_4 .... t_{n+2}$, where $t_3 = s_1, t_4 = s_2, ... t_{n+2} = s_n$

i.e. $\overline{\alpha}_1$ is a finite sequence numbered not by 1, 2, ... n but by integers i such that $3 \leq i \leq n+2$. Hence $\overline{\alpha}_1$ shows the position of β within α, not just β itself in isolation. We not put into $T(\alpha)$, the pair $\{\forall v_i \overline{\alpha}_1\}$.

<u>Case 7</u>: There exist unique wff β and a unique positive integer i α is $\exists v_i \beta$. Suppose further that β is the expression $\beta = s_1 s_2 .....s_n$, we define

$\overline{\alpha}_1 = t_3 t_4 ....t_{n+2}$, where $t_3 = s_1, t_4 = s_2, ...t_{n+2} = s_n$

i.e. $\overline{\alpha}_1$ is a finite sequence numbered not by 1, 2, ... n but by integers i such that $3 \leq i \leq n+2$. Hence $\overline{\alpha}_1$ shows the position of β within α, not just β itself in isolation. We not put into $T(\alpha)$, the pair $\{\exists v_i \overline{\alpha}_1\}$.

We expand the tree $T(\alpha)$ inductively. That is, every element of $T(\alpha)$ is a set of one of the following kinds : 0. $\{\{\overline{\alpha}\}\}$, where $\overline{\alpha}$ is an atomic wff

1. $\{\neg, \overline{\alpha}_{a1}\}$, where a is a finite sequence of 1's and 2's

2. - 5. $\{\vee, \overline{\alpha}_{a1}, \overline{\alpha}_{a2}\}$, $\{\wedge, \overline{\alpha}_{a1}, \overline{\alpha}_{a2}\}$, $\{\rightarrow, \overline{\alpha}_{a1}, \overline{\alpha}_{a2}\}$, $\{\leftrightarrow, \overline{\alpha}_{a1}, \overline{\alpha}_{a2}\}$, where again a is a finite sequence of 1's and 2's ;

6...7. $\{\forall v_i, \overline{\alpha}_{a1}\}$, $\{v_i, \overline{\alpha}_{a1}\}$, is a finite sequence of 1's and 2's, i is a positive integer.

The Case 0. can only occur when α itself is atomic, in which case $\overline{\alpha} = \alpha$.

It follows inductively that the subexpression of α of the form $\overline{\alpha}_a$, in cases 1. - 7., have the property that, if reindexed by integers begining with 1, we obtain a wff. If this wff is not atomic, it falls under one of the cases 1. - 7., and we apply the procedure described in cases 1.- 7. for α itself, adding additional sets of the type in cases 1. - 7. to $T(\alpha)$.

By a symbol of our FOL, we mean the symbols listed at the bottom of p.69, and top of p.70, but we also add $\wedge, \vee, \leftrightarrow, \exists$. If α is any wff, then α is a finite sequence $t_1 t_2 ....t_n$ of symbols, and for any symbol s of our FOL, and any j such that $1 \leq j \leq n$, we say that t_j is an occurrence of s in α if $t_j = s$

Consider the case when t_j is an occurrence of a variable v_i, i.e. $t_j = v_i$. We then say that this occurrence of v_i in α is a bound occurrence of v_i if either $t_{j-1} = \forall$ or $t_{j-1} = \exists$, or the following, more involved, situation occurs : (Note that, the only other way that v_i can occur in α, is as being a part of an atomic wff.)

There exists a finite sequence a of 1s and 2s such that some set belonging to $T(\alpha)$ has the form $\{\exists v_i, \overline{\alpha}_a\}$ or $\{\forall v_i, \overline{\alpha}_a\}$ and j is one of the indices of the subexpression $\overline{\alpha}_a$. I.e., α itself being $t_1t_2....t_n, \overline{\alpha}_a$ is $t_lt_{l+1}....t_m$ and we must have $l \leq j \leq m$. An occurrence t_j, of v_i, in α is said to be free if it is not bound.

We define $\alpha(v_i/v_j)$ to be the wff obtained from α by replacing all free occurrences of v_i in α by v_j. (We omit the proof that $\alpha(v_i/v_j)$ is indeed a wff.)

We say that substituting v_j for v_i in α is legitimate if all occurrences of v_j in $\alpha(v_i/v_j)$ that result from the substitution, are free in $\alpha(v_i/v_j)$.

We can now state a rigorous version of Universal Instantiation : Namely, if substituting v_j for v_i in α is legitimate, then inferring $\alpha(v_i/v_j)$ from $\forall v_i\alpha$ is a valid inference. I.e. for all structures $\mathcal{A}$, and all assignments s, if $\underset{\mathcal{A}}{\models} \forall\alpha(v_i/v_j)[s]$, then $\underset{\mathcal{A}}{\models} \alpha(v_i/v_j)[s]$

<u>The Main Lemma</u> if v_j is substitutable for v_i in α, and $\underset{\mathcal{A}}{\models} \alpha[s(v_i \leftarrow s(v_j))]$, then $\underset{\mathcal{A}}{\models} \alpha(v_i/v_j)[s]$

Comment : We will write $s(v_i \leftarrow s(v_j)) = s^{'}$. Proof : We use the Induction Principle. Thus we first consider the case when α is atomic. Let's being with the case when α is Exy.

Case 1. Neither x nor y is v_i. Then $\alpha(v_i/v_j)$ is again Exy. Thus the assignments s and $s^{'} = s(v_i \leftarrow s(v_j))$ agree on x and y, and therefore $< s^{'}(x), s^{'}(y) >\in E^{\mathcal{A}}$ if and only if $< s(x), s(y) >\in E^{\mathcal{A}}$. Thus $\underset{\mathcal{A}}{\models} Exy[s^{'}]$ if and only if $\underset{\mathcal{A}}{\models} Exy[s]$. But $\alpha(v_i/v_j)$ is Exy. Thus $\underset{\mathcal{A}}{\models} \alpha[s^{'}]$ if and only if $\underset{\mathcal{A}}{\models} \alpha(v_i/v_j)[s]$.

Case 2. Either x or y, or both are v_i. Let's first consider the case when x is v_i, but y is not v_i, i.e, α is Ev_iy, with y being v_k for some $k \neq i$. Then $\alpha(v_i/v_j)$ is Ev_jy.

Set $s^{'} = s(v_i \leftarrow s(v_j))$. Thus $s^{'}(v_i) = s(v_j)$ and $s^{'}(y) = s(y)$. $\underset{\mathcal{A}}{\models} \alpha[s^{'}]$ then means $\underset{\mathcal{A}}{\models} Ev_iy[s^{'}]$ which means $< s^{'}(v_i), s^{'}(y) >\in E^{\mathcal{A}}$, i.e. $< s(v_j), s(y) >\in E^{\mathcal{A}}$, which in turn means $\underset{\mathcal{A}}{\models} Ev_jy[s]$, i.e. $\underset{\mathcal{A}}{\models} \alpha(v_i/v_j)[s]$, as desired.

We will now consider the quantifier cases of the Induction Principle. First let α be $\forall x\beta$, and we assume that the statement of the lemma hold for β.

Case 1. x is v_k for some k such that $k \neq i$ and $k \neq j$. Thus $\forall x\beta$ is $\forall v_k\beta$ and $(\forall v_k\beta)(v_i/v_j)$ is $\forall v_k\beta(v_i/v_j)$.

$\underset{\mathcal{A}}{\models} v_k\beta[s^{'}]$ means : for all d in $|\mathcal{A}|$, $\underset{\mathcal{A}}{\models} \beta[s^{'}(v_k \leftarrow d)]$. But $\underset{\mathcal{A}}{\models} \beta[s^{'}(v_k \leftarrow d)]$ means $\underset{\mathcal{A}}{\models} \beta[s(v_i \leftarrow s(v_j))(v_k \leftarrow d)]$, i.e. $\underset{\mathcal{A}}{\models} \beta[s(v_k \leftarrow d)(v_i \leftarrow s(v_j))]$ which yields (by the assumption of the Induction Principle)

$\models_{\mathcal{A}} \beta(v_i/v_j)[s(v_k \leftarrow d)]$.

This in turn holds for every d in $|\mathcal{A}|$. Thus from $\models_{\mathcal{A}} \forall v_k \beta[s']$ we obtain:

For all d in $|\mathcal{A}|$, $\models_{\mathcal{A}} \beta(v_i/v_j)[s(v_k \leftarrow d)]$, i.e. $\models_{\mathcal{A}} \forall v_k \beta(v_i/v_j)[s]$, which is the desired conclusion.

References

[1] Herbert B. Enderton. A mathematical Introduction to Logic, 2nd edition, Harcourt/Academic Press.