Axiom Education

Name:	
Student ID:	

Circular Functions [9.0]

Year 10A **Sample Worksheet**

Time: 3 hours

Mark
/90

- This Question and Answer Book consists of **15 questions**, with a total of **90 marks** available.

Approved materials

- Pens, pencils, highlighters, erasers, sharpeners, and rulers
- Protractors, set squares and aids for curve sketching

Materials supplied

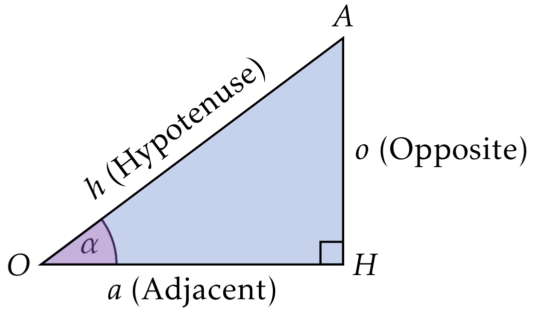
- Question and Answer Book of 29 pages.
- Formula Sheet
- Working space is provided throughout the book

Instructions

- If additional space is needed to complete your answer, space is available at the end of this question book. Clearly write the number of the question that you are answering in this space.
- If you have any questions at any time during the examination, you must raise your hand and wait for a supervisor. Do **not** attempt to communicate, by any means, with any other student before or during the examination.
- Mobile phones, unauthorised electronic devices (including calculators or software), notes of any kind, blank sheets of paper, and/or correction fluid/tape are **not** permitted.
- Do not begin until instructed by your supervisor.

Definitions & the Unit Circle

Consider a **right-angled triangle** $\triangle OAH$ with sides o, a, and Hypotenuse h, where the angle $\angle OHA$ is $90°$, and the angle $\angle AOH$ is denoted as α:



We use the ratios between the three sides of the triangle to define three functions of α:

Definition 1.1 Trigonometric Functions

- The **sine** of the angle α is $\sin(\alpha) = \frac{o}{h}$,
- the **cosine** of the angle α is $\cos(\alpha) = \frac{a}{h}$, and
- the **tangent** of the angle α is $\tan(\alpha) = \frac{o}{a}$, which in turn is equal to $\frac{\sin(\alpha)}{\cos(\alpha)}$.

π

By definition, we also know that the hypotenuse is the longest side of a triangle. This means that the ranges of both $\sin(\alpha)$ and $\cos(\alpha)$ are both $[0, 1]$! Can you work out why?

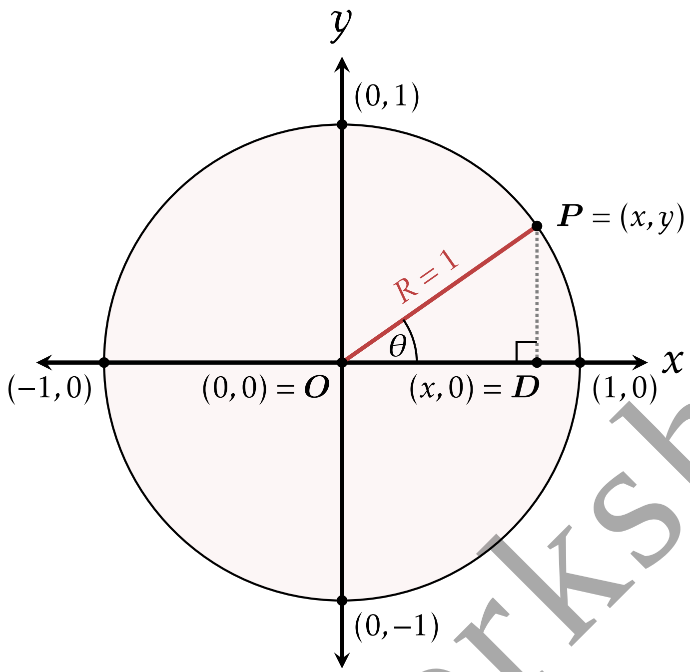
Note 1.1 A Useful Acronym

Remembering the definitions can be hard, try remembering the following phrase instead!

SOH CAH TOA

!

You may be wondering "okay, but that's a triangle, aren't we meant to be studying circular functions?" Well, turns out these same functions are widely applicable in all of geometry, and to see this you have to look no further than a **unit circle**!



A unit circle is simply a circle that has a radius of 1 and is centered at the origin. We use it as a reference point because it is the "simplest" circle we can analyse.

Challenge 1.1 Trigonometric Functions on a Circle

The triangle $\triangle OPD$ is a right triangle. Therefore, try finding $\sin\theta$ and $\cos\theta$ given that $\theta = \angle POD$!

?

This idea is so useful that we can use what you found as our new definitions for sin and cos!

Definition 1.2 Circular Functions

Given a point $\boldsymbol{P} = (x, y)$ on a unit circle, let θ be the angle formed between $\boldsymbol{P}$, the origin, and the positive side of the x-axis. Then

- The **sine** of the angle θ is $\sin(\theta) = y$, and
- the **cosine** of the angle θ is $\cos(\theta) = x$.

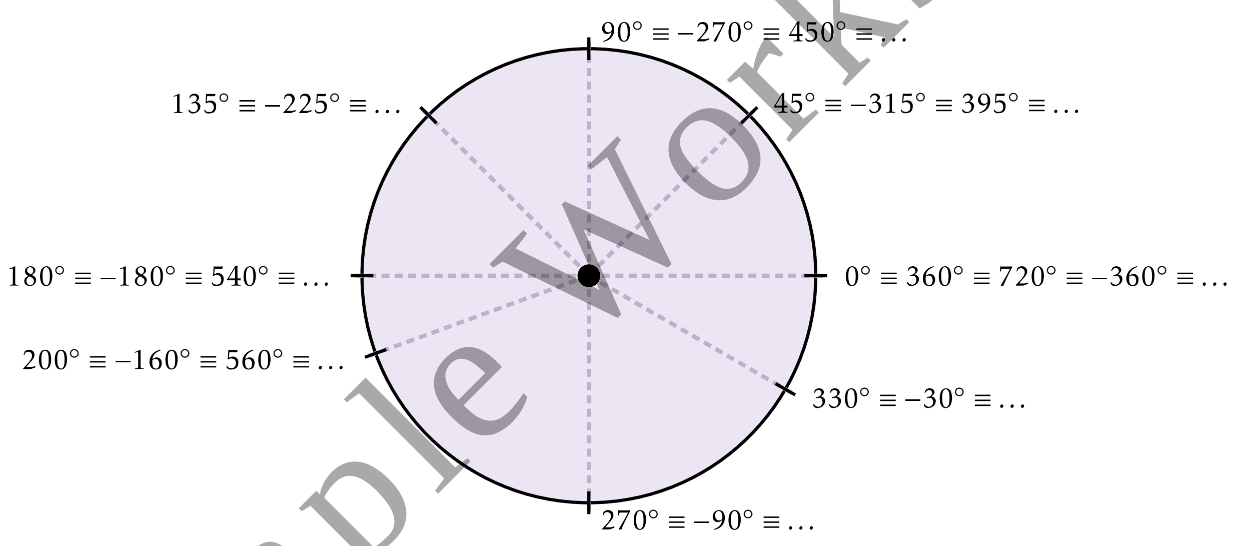
π

Note 1.2 Another Trick for Another Definition

If you like Pokémon, try remembering the word **shiny**; this means that sine is y!

!

This has the added benefit of **extending** our functions, meaning that we've expanded their domains and ranges! Consider the diagram below.



As you can see, any real value of degrees is actually equivalent to some value in the interval $[0°, 360°]$, meaning that we have *expanded* our domain from $[0°, 90°]$ to $\mathbb{R}$. Given that the x and y values can also be negative, our ranges have also *expanded* from $[0, 1]$ to $[-1, 1]$. This **equivalency** is also what we mean by the **period** of our functions.

Definition 1.3 Period

The period of a function is how often it repeats itself. What do you think the periods of sin and cos are?

π

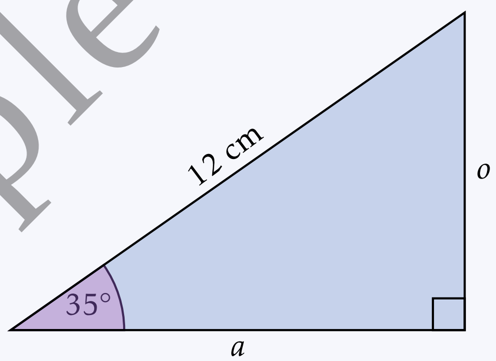
Finally, here's a table of exact values for the main three circular functions. Don't worry too much about the tan column as we'll get back to that later!

$\theta[^\circ]$	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$
0	**0**	**1**	**0**
30	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90	1	0	undefined
180	0	−1	0
270	−1	0	undefined
360	0	1	0

Example 1.1 Right-Angled Triangle

A right-angled triangle has an acute angle 35° with a hypotenuse 12 cm. Find the length of the sides adjacent and opposite to the 35° angle.

Solution:
Drawing a diagram of this triangle with opposite side o and adjacent side a



We get $\sin(35^\circ) = \frac{o}{12} \implies o = 12\sin(35^\circ) \approx 6.88$ cm
and
$\cos(35^\circ) = \frac{a}{12} \implies a = 12\cos(35^\circ) \approx 9.83$ cm

Question 1.1 (4 marks)

Consider a triangle with a hypotenuse 3 cm and acute angle $60°$.

a) Find the length of the side opposite to the angle. **2 marks**

b) Find the length of the side adjacent to the angle. **2 marks**

Question 1.2 (4 marks)

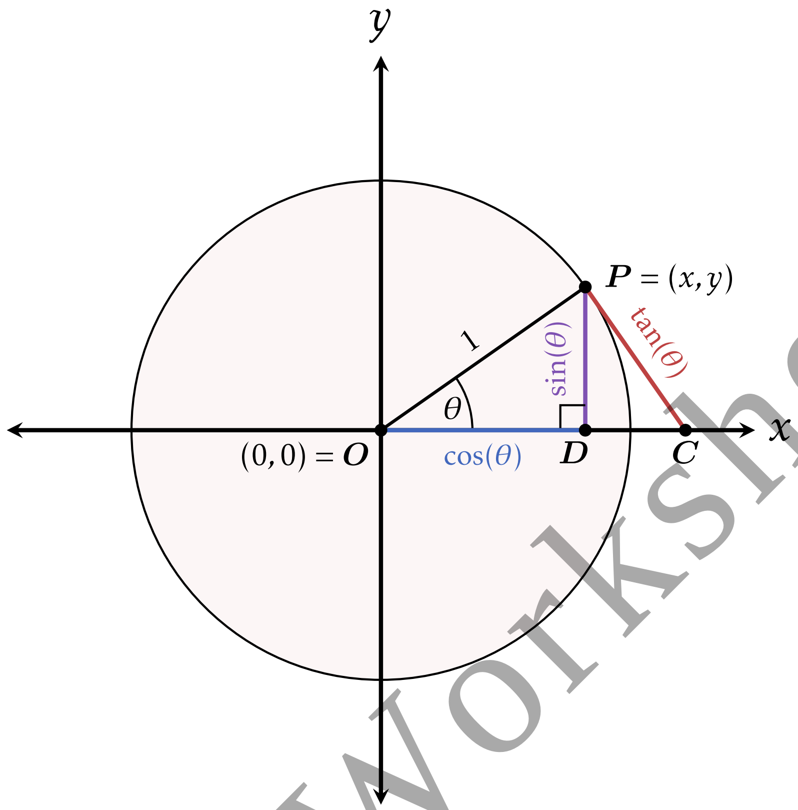
Consider the point on a unit circle $P(m, n)$, where $m = n$. Let $\theta \in [0°, 360°]$ be the angle formed between P, the origin, and the x-axis.

a) Find the possible values of m and n. **2 marks**

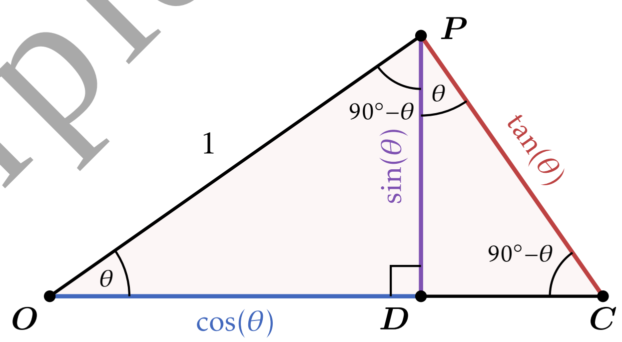
b) State the corresponding values of θ for the values of m and n found in **part a**. **2 marks**

The Tangent Function

Where does the tangent function get its name from? It is the length of the tangent line to the unit circle!



Here, we are defining $\tan(\theta)$ to be length of the red line segment, and it's an interesting endeavour to see if we can relate this new length back to $\sin\theta$ and $\cos\theta$! Let's zoom into the triangle we have created and analyse it a bit further.



Using the simple fact that the interior angles in a triangle sum up to 180°, we discover that the outer triangle $\triangle OPC$ and the inner triangle $\triangle OPD$ are actually similar triangles. This means that they share the same ratios for the side lengths! Hence

- For the **inner triangle** $\triangle OPD$, the ratio is $\dfrac{\text{opposite}}{\text{adjacent}} = \dfrac{\sin(\theta)}{\cos(\theta)}$
- For the **outer triangle** $\triangle OPC$, the ratio is $\dfrac{\text{opposite}}{\text{adjacent}} = \dfrac{\tan(\theta)}{1}$

Definition 2.1 Algebraic Definition of Tangent Function

Since these two triangles are similar, the ratios must equal each other, and hence we have the following relationship, which was our algebraic definition from before!

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

As you can see, both the geometric definition and the algebraic definition end up being equivalent.

Note 2.1 Length and Gradient

This leads to the interesting observation that the length of PC is equal to the gradient of OP!

One final thing to note about $\tan(\theta)$ is that it is intrinsically linked to the gradient of a line, because $\frac{\sin(\theta)}{\cos(\theta)}$ is the same as $\frac{\text{rise}}{\text{run}}$. This makes it a very useful tool to relate gradients to angles!

Challenge 2.1 Relating Gradient and Angle

Consider the line given by the graph of $y = mx + c$. If θ represents the angle the line makes with the positive side of the x-axis, find an expression for $\tan(\theta)$.

Example 2.1 Angle Between Two Lines

Consider the two lines L_1 and L_2 with equations given by $y = \sqrt{3}x + 2$ and $\sqrt{3}y = x + \sqrt{3}$ respectively. Find the acute angle between L_1 and L_2.

Solution:

Let θ be the acute angle between L_1 and L_2. Let α be the angle between L_1 and the positive direction of the x-axis. Let β be the angle between L_2 and the positive direction of the x-axis.

Then $\theta = \alpha - \beta$.

The gradient of L_1 is $m_1 = \sqrt{3}$ and the gradient of L_2 is $m_2 = \dfrac{1}{\sqrt{3}}$.

Therefore $\tan(\alpha) = \sqrt{3}$ and $\tan(\beta) = \dfrac{1}{\sqrt{3}}$.

Using our exact values, we get $\alpha = 60°$ and $\beta = 30°$.

Therefore $\theta = 60° - 30° = 30°$.