

$$1) A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad B = \begin{bmatrix} 4 \\ 5 \\ b \end{bmatrix} \quad C = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$$

$$1) 2a - b = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \\ b \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 6 \end{bmatrix}$$

$$2) \hat{a} = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \frac{a}{|a|} \rightarrow \sqrt{1+4+9} = \sqrt{14}$$

$$3) ||a|| = \sqrt{14} \quad \text{angle of } A \text{ rel. to pos. } x \text{ axis is } \arccos\left(\frac{1}{\sqrt{14}}\right)$$

$$4) \cos \alpha = \frac{1}{|a|}, \quad \cos \beta = \frac{2}{|a|}, \quad \cos \gamma = \frac{3}{|a|}$$

$$= \frac{1}{\sqrt{14}}, \quad \frac{2}{\sqrt{14}}, \quad \frac{3}{\sqrt{14}}$$

$$5) \cos \theta = \frac{\vec{u} \cdot \vec{v}}{||\vec{u}|| ||\vec{v}||} = \frac{4 + 10 + 18}{\sqrt{14} \cdot \sqrt{77}} = \frac{32}{\sqrt{1078}} = \cos \theta$$

$\vec{u} = a, \vec{v} = b$   
 $||\vec{u}|| = \sqrt{14}, ||\vec{v}|| = \sqrt{77}$   
 $\theta = \cos^{-1}\left(\frac{32}{\sqrt{1078}}\right)$

$$6) a \cdot b = 32$$

$$b \cdot a = a \cdot b = 32$$

$$7) a \cdot b = |a| |b| \cos \theta = \sqrt{14} \sqrt{77} \cdot \frac{32}{\sqrt{1078}} = 32$$

$$8) \text{ scalar projection of } b \text{ onto } \hat{a} \quad \hat{a} = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \end{bmatrix}$$

$$= \frac{\vec{b} \cdot \hat{a}}{||\hat{a}||} = \frac{\vec{b} \cdot \hat{a}}{1} = \frac{32}{\sqrt{14}}$$

$$\sqrt{\left(\frac{1}{\sqrt{14}}\right)^2 + \left(\frac{2}{\sqrt{14}}\right)^2 + \left(\frac{3}{\sqrt{14}}\right)^2} = \sqrt{\frac{1}{14} + \frac{4}{14} + \frac{9}{14}} = 1$$

$$4 \cdot \frac{1}{\sqrt{14}} + 5 \cdot \frac{2}{\sqrt{14}} + 6 \cdot \frac{3}{\sqrt{14}} = \frac{4}{\sqrt{14}} + \frac{10}{\sqrt{14}} + \frac{18}{\sqrt{14}} = \frac{32}{\sqrt{14}} = 1$$

⑨ a vector perpendicular to a

$$[1 \ 4 \ 2] \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 0 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \neq$$

\* if two vectors are perpendicular then dot products are zero

(10)  $a \times b =$

$$a = \begin{matrix} 1 \\ 1 \\ 3 \end{matrix} \quad b = \begin{matrix} 4 \\ 5 \\ 6 \end{matrix}$$

$$(2 \cdot 6 - 3 \cdot 5)\hat{i} + -(6 - 3 \cdot 4)\hat{j} + (5(1) - 8)$$

$$-3\hat{i} + 6\hat{j} - 3\hat{k} \quad \begin{pmatrix} -3 \\ 6 \\ -3 \end{pmatrix}$$

$$b \times a = \begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 5 & 6 \\ 1 & 1 & 3 \end{matrix}$$

$$(5 \cdot 3) - (6 \cdot 2) - ((4 \cdot 3) - (6 \cdot 1)) + (4(2) - 5(1))$$

$$(15 - 12) - (6) + (3)$$

$$3 - 6 + 3 = \begin{pmatrix} 3 \\ -6 \\ 3 \end{pmatrix}$$

(11) \* vector perpendicular to two vectors is calculated by the determinant (i.e. cross product)

$$\begin{pmatrix} -3 \\ 6 \\ -3 \end{pmatrix} \text{ is perpendicular}$$

(12) a and b are linearly dependent  $a + 3 = b$   
 a and c are linearly independent  
 b and c 1 1 1 1 1

(13)  $a^T b$

$$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = 32$$

$$a^T b = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} (4 \ 5 \ 6) = \begin{pmatrix} 4 & 5 & 6 \\ 8 & 10 & 12 \\ 12 & 15 & 18 \end{pmatrix} \neq$$

$3 \times 1 \quad 1 \times 3 = 3 \times 3$



$$(5) \quad A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 5 & -1 \end{pmatrix}_{3 \times 3} \quad B = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{pmatrix}_{3 \times 3} \quad C = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -1 & 1 & 3 \end{pmatrix} \quad d = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

(1)  $2A - B$

$$\begin{pmatrix} 2 & 4 & 6 \\ 8 & -4 & 6 \\ 0 & 10 & -2 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 5 \\ 6 & -5 & 10 \\ -3 & 12 & -3 \end{pmatrix} \neq$$

(2)  $AB$ :  $\begin{pmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 14 & -2 & -4 \\ 9 & 0 & 15 \\ 7 & 7 & -21 \end{pmatrix}$

$BA$ :  $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{pmatrix} = \begin{pmatrix} 9 & 3 & 8 \\ 6 & -18 & 13 \\ -5 & 15 & 2 \end{pmatrix}$

(3)  $(AB)^T$  and  $B^T A^T$ .  $B^T = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & -2 \\ 1 & -4 & 1 \end{pmatrix}$   $A^T = \begin{pmatrix} 1 & 4 & 0 \\ 2 & -2 & 5 \\ 3 & 3 & -1 \end{pmatrix}$

$L_7 = \begin{pmatrix} 14 & 9 & 7 \\ -2 & 0 & 7 \\ -4 & 15 & -21 \end{pmatrix} \rightarrow B^T A^T = \begin{pmatrix} 14 & 9 & 7 \\ -2 & 0 & 7 \\ -4 & 15 & -21 \end{pmatrix} \neq$

(4)  $|A|$  and  $|C|$

$|A| = 55$ ,  $A$ -co we get  $|C| = 0$

(5) matrix  $B$

(6)  $A^{-1} = \frac{1}{\det(A)} \cdot \text{Adj}(A)$

$\frac{1}{55} \begin{pmatrix} -13 & -17 & 12 \\ -7 & -1 & 9 \\ 20 & -5 & -10 \end{pmatrix}$

cofactors

1:  $A_{11} = \begin{vmatrix} -2 & 3 \\ 5 & -1 \end{vmatrix} = -13$

2:  $A_{12} = \begin{vmatrix} 4 & 3 \\ 0 & -1 \end{vmatrix} = -7$

$\det(A) = 1 \cdot \det \begin{pmatrix} -2 & 3 \\ 5 & -1 \end{pmatrix} - 2 \cdot \det \begin{pmatrix} 4 & 3 \\ 0 & -1 \end{pmatrix} + 3 \cdot \det \begin{pmatrix} 4 & -2 \\ 0 & 5 \end{pmatrix}$

$(2-15) - 2(-4-0) + 3(20-0)$   
 $-13 + 8 + 60 = 55 \neq$

3:  $A_{13} = \begin{vmatrix} 4 & -2 \\ 0 & 5 \end{vmatrix} = 20 \neq$

4:  $A_{21} = \begin{vmatrix} 2 & 3 \\ 5 & -1 \end{vmatrix} = -17$



$$-2 = A_{21}: \begin{vmatrix} 1 & 3 \\ 0 & -1 \end{vmatrix} = -1 - 0 = -1$$

$$3 = A_{23}: \begin{vmatrix} 1 & 2 \\ 0 & 5 \end{vmatrix} = 5$$

$$0 = A_{31}: \begin{vmatrix} 2 & 3 \\ -2 & 3 \end{vmatrix} = 12$$

$$5 = A_{32}: \begin{vmatrix} 1 & 4 \\ 0 & 3 \end{vmatrix} = 3$$

$$-9 = A_{33}: \begin{vmatrix} 1 & 2 \\ 4 & -2 \end{vmatrix} = -10$$

Cofactor matrix

$$= \begin{bmatrix} -13 & -7 & 20 \\ -17 & -1 & 5 \\ 0 & 3 & -10 \end{bmatrix}$$

$$A^T = \text{adj}(A)$$

$$= \begin{bmatrix} -13 & -17 & 0 \\ -7 & -1 & 3 \\ 20 & 5 & -10 \end{bmatrix}$$

(7)

$B^{-1}$ : for an inverse matrix  
if it's orthogonal then its inverse is the  
transpose and then you normalize to  
yield an orthonormal vector.

$$\begin{bmatrix} 1/6 & 2/21 & 3/14 \\ 2/6 & 1/21 & -2/14 \\ 1/6 & -1/21 & 1/14 \end{bmatrix}$$

(7)

$$C^{-1} = \frac{1}{|C|} \cdot \text{adj}(C)$$

$|C| = 0$  (from previous problem) so

$C^{-1}$  does not exist

(8)

Ad

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 3 \\ 0 & 5 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 14 \\ 9 \\ 7 \end{bmatrix}$$

(9)

Proj of  $\vec{a}$  onto  $\vec{b} = \frac{\vec{b} \cdot \vec{a}}{|\vec{b}|}$

rows A  
 $A_1 \cdot d = 14$

$$A_1 = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \quad d = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

$$|d| = \sqrt{14}$$

row 1  
scalar proj

$$\frac{14}{\sqrt{14}}$$

$$A_2 = \begin{bmatrix} 4 & -2 & 3 \end{bmatrix}$$

$$A_2 \cdot d = 9$$

~~$$\begin{bmatrix} 9 & 12 & 21 \\ 12 & 16 & 24 \\ 21 & 24 & 33 \end{bmatrix}$$~~

$$\left( \frac{9}{\sqrt{14}} \right) \#$$

row = 2  
Scalar  
proj.

$$A_3 = \begin{bmatrix} 0 & 5 & -1 \end{bmatrix}$$

$$A_3 \cdot d = 7$$

~~$$\begin{bmatrix} 7 & 14 & -7 \\ 14 & 28 & -14 \\ -7 & -14 & 7 \end{bmatrix}$$~~

~~$$\begin{bmatrix} 7 & 14 & -7 \\ 14 & 28 & -14 \\ 21 & 42 & -21 \end{bmatrix}$$~~

~~$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ 3 & 6 & -3 \end{bmatrix}$$~~

row = 3  
Scalar  
proj.

$$\left( \frac{7}{\sqrt{14}} \right) \#$$

~~(10B)~~ or (10B) vector proj.

$$\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2} \vec{b}$$

proj of  $\vec{a}$  onto  $\vec{b}$ .

$$\vec{d} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

row 1 =  $[1 \ 2 \ 3]$

Proj of  
row 1 onto  $\vec{d}$ :  $\frac{14}{14} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_\#$

row 2 =  $[4 \ -2 \ 3]$

proj of  
row 2 onto  $\vec{d}$ :  $4(1) - (4) + 9 = 9 \Rightarrow \frac{9}{14} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_\#$

row 3 =  $[0 \ 5 \ -1]$

proj of  
row 3 onto  $\vec{d}$ :  $0 + 10 - 3 = 7 \Rightarrow \frac{7}{14} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_\#$



$$\textcircled{11} \quad \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} \cdot 1 + \begin{bmatrix} 2 \\ -2 \\ 5 \end{bmatrix} \cdot 2 + \begin{bmatrix} 3 \\ 3 \\ -1 \end{bmatrix} \cdot 3 = \begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix} + \begin{bmatrix} 4 \\ -4 \\ 10 \end{bmatrix} + \begin{bmatrix} 9 \\ 9 \\ -3 \end{bmatrix} = \begin{bmatrix} 14 \\ 9 \\ 7 \end{bmatrix}$$

$$\textcircled{12} \quad Bx = d \quad x = B^{-1}d$$

$$\frac{1}{6} + \frac{2}{4} + \frac{9}{14} = 1 \quad \frac{1}{6} + \frac{4}{4} + \frac{5}{14} = 1$$

$$= \begin{bmatrix} 1/6 & 2/4 & 3/14 \\ 2/6 & 1/2 & -2/14 \\ 1/6 & -4/21 & 1/14 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \neq$$

$$\frac{2}{6} + \frac{2}{4} - \frac{6}{14} = 0$$

$$1/6 \neq \frac{8}{21} + \frac{3}{14} = 0$$

$$\textcircled{13} \quad x = C^{-1}d$$

$\rightarrow$  there's no sol'n for  $x$  since  $C$  can't be inverted. The determinant of  $C$  is 0 and it's a singular matrix.

$$\textcircled{C} \quad D = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \quad E = \begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix} \quad F = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

1. eigenvalues & corresponding eigenvectors of  $D$ .

$$\det \begin{pmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{pmatrix} = (1-\lambda)(2-\lambda) - 6 = 0 \quad \rightarrow \lambda^2 - 3\lambda - 4 = 0$$

$$2 - 3\lambda + \lambda^2 - 6 = 0 \quad \rightarrow (\lambda - 4)(\lambda + 1) = 0$$

$$\lambda = 4, \lambda = -1$$

$$\lambda = 4 \quad \begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{rref} \left( \begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix} \middle| \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right)$$

$$= \text{rref} \left( \begin{pmatrix} 1 & -2/3 \\ 0 & 0 \end{pmatrix} \middle| \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) \quad \rightarrow \begin{cases} x_1 - \frac{2x_2}{3} = 0 \\ x_1 = \frac{2x_2}{3} \end{cases}$$

Let  $x_1 = 2$   
 $x_2 = 3$

$Q_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$\lambda = -1$

$A - \lambda I = \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}$

singular matrix

$e_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$x_1 + x_2 = 0$

$x_1 - \frac{2x_2}{3} = 0 \quad x_1 = \frac{2x_2}{3}$

$x_1 = \frac{2}{3} x_2$

$3x_1 = 2x_2$

Let

$3(2) = 2(3)$

$x_1 = 2, x_2 = 3$

$\begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$x_1 + x_2 = 0$

Let  $x_1 = 1$

$x_2 = -1$

(2)  $e_1 \cdot e_2 = 2(1) + 3(-1) = -1$

(3)  $E = \begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix}$

$\det \begin{pmatrix} 2-\lambda & -2 \\ -2 & 5-\lambda \end{pmatrix} = 0$

$(2-\lambda)(5-\lambda) - 4 = 0$

$10 - 7\lambda + \lambda^2 - 4 = 0$

$\lambda^2 - 7\lambda + 6 \rightarrow (\lambda - 6)(\lambda - 1) = 0$

$\lambda = 6$

$\lambda = 1$

$\begin{pmatrix} 2-6 & -2 \\ -2 & 5-6 \end{pmatrix} = \begin{pmatrix} -4 & -2 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\text{rref} \begin{pmatrix} -4 & -2 & 0 \\ -2 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$x_1 = 0$   
 $x_2 = 0$

$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

The dot product MUST be 0, since one of the eigen-vectors is indeed orthogonal to the other.



(4) the dot product of two vectors  $\Rightarrow$  it means the vectors are orthogonal.

Therefore the eigenvectors of  $E$  are orthogonal and  $E$  is a ~~Hermitian~~ symmetric matrix

(5)  $Fx = 0$   $x = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \neq$   $x_1 = -2x_2$   $0, 0$

(6)  $Fx = 0$  two nontrivial sol'n's

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \quad \text{ref} \left( \begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 4 & 0 \end{array} \right) = \begin{array}{cc|c} x_1 & x_2 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}$$

$$4x_1 - 4 = -2(2)$$

1 sol'n =  $\begin{pmatrix} -4 \\ 2 \end{pmatrix} \neq$  2nd sol'n =  $\begin{pmatrix} -2 \\ 1 \end{pmatrix} \neq$

(7) Since row of  $D$  is linearly independent then  $x$  must  $\neq 0$ ,  $\Rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix} \neq$

(D)

(1)  $f(x) = x^2 + 3$

$f'(x) = 2x$   $f''(x) = 2$

$f(x) = x^2 + 3$

$g(x) = x^2 \Rightarrow f(g(x)) = x^4 + 3$

(2)  $\frac{\partial a}{\partial x} = 2x$   $\frac{\partial e}{\partial y} = 2y$

(3)  $\nabla q(x, y) = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$

(4)  $\frac{\partial f(g(x))}{\partial x}$  w/ chain rule:  $\frac{\partial f(g(x))}{\partial g(x)} \cdot \frac{\partial g(x)}{\partial x} =$

W/o chain rule:

$\frac{\partial}{\partial x} (x^4 + 3) = 4x^3 \neq$

$2x^2 \cdot 2x = 4x^3 \neq$