CS 577

Homework 0

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CS 577 HW 0. Anas Puthawala.

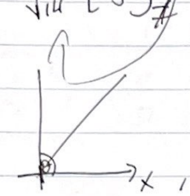
A) $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $B = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ $C = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$

1) $2a - b$

$= \begin{matrix} 2 \\ 4 \\ 6 \end{matrix} - \begin{matrix} 4 \\ 5 \\ 6 \end{matrix} = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}$ #

2) $\hat{a} = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ # $\quad \frac{a}{|a|} \rightarrow \sqrt{1+4+9} = \sqrt{14}$

3) $||a||,$
$= \sqrt{14}$ #



angle of A rel. to pos. X axis is $\arccos\left(\frac{1}{\sqrt{14}}\right)$

4) $\cos\alpha = \frac{\alpha}{|a|}$, $\cos\beta = \frac{\beta}{|a|}$, $\cos\gamma = \frac{\gamma}{|a|}$ $\quad \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$

$= \frac{1}{\sqrt{14}}$, $\frac{2}{\sqrt{14}}$, $\frac{3}{\sqrt{14}}$ #

5) $\cos\theta = \frac{\overset{a}{\vec{u}} \cdot \overset{b}{\vec{v}}}{||\vec{u}|| \cdot ||\vec{v}||} = 4 + 10 + 18 = 32 = \frac{32}{\sqrt{14} \cdot \sqrt{77}} = \frac{32}{\sqrt{1078}} = \cos\theta$

$\# u = a$
$v = b$

$||\vec{u}|| \hookrightarrow = \sqrt{14}$, $||\vec{v}|| \rightarrow \sqrt{77}$

$\theta = \cos^{-1}\left(\frac{32}{\sqrt{1078}}\right)$ #

6) $a \cdot b = 32$
$b \cdot a = a \cdot b = 32$ #

7) $a \cdot b = |A||B| \cos\theta = (\sqrt{14})(\sqrt{77}) \cdot \frac{32}{\sqrt{1078}} = 32$

8) scalar projection of b onto $\hat{a}$ $\quad \hat{a} = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ # $= \begin{bmatrix} \frac{1}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \end{bmatrix}$

$= \frac{\vec{b} \cdot \vec{\hat{a}}}{|\hat{a}|} = \frac{\vec{b} \cdot \hat{a}}{1} = \frac{32}{\sqrt{14}}$ #

$\sqrt{\left(\frac{1}{\sqrt{14}}\right)^2 + \left(\frac{2}{\sqrt{14}}\right)^2 + \left(\frac{3}{\sqrt{14}}\right)^2} = \sqrt{\frac{1}{14} + \frac{4}{14} + \frac{9}{14}} = 1$

$4 \cdot \frac{1}{\sqrt{14}} + 5 \cdot \frac{2}{\sqrt{14}} + 6 \cdot \frac{3}{\sqrt{14}} = \frac{4}{\sqrt{14}} + \frac{10}{\sqrt{14}} + \frac{18}{\sqrt{14}} = \frac{32}{\sqrt{14}}$

(9) a vector perpendicular to a

$$[x \quad y \quad z]\begin{bmatrix}1\\2\\3\end{bmatrix} = 0 = \begin{bmatrix}1\\1\\-1\end{bmatrix} \#$$

* if two vectors are perpendicular their dot products are zero ✓

$(1)(1) + (1)(2) - 3 = 0$ ✓

(10) $a \times b =$

$$\begin{array}{cccc} & \hat{i} & \hat{j} & \hat{k} \\ a = & 1 & 2 & 3 \\ b = & 4 & 5 & 6 \end{array}$$

$(2 \cdot 6 - 3 \cdot 5)\hat{i} + -(6 - 3(4))\hat{j} + (5(1) - 8)$

$-3\hat{i} + 6\hat{j} - 3\hat{k} \qquad \begin{pmatrix}-3\\6\\-3\end{pmatrix}$

$$\begin{array}{cccc} & \hat{i} & \hat{j} & \hat{k} \\ b \times a = & 4 & 5 & 6 \\ & 1 & 2 & 3 \end{array}$$

$(5 \cdot 3) - (6(2)) - ((4 \cdot 3) - (6)) + (4(2) - 5(1))$

$(15 - 12) - (6) + (3)$

$3 \quad -6 \quad 3 = \begin{pmatrix}3\\-6\\3\end{pmatrix}$

(11) * vector perpendicular to two vectors is calculated by the determinant (i.e. cross product)

$\begin{pmatrix}-3\\6\\-3\end{pmatrix}$ is perpendicular

(12) a and b are linearly dependent $a + \cancel{2} = b$

a and c are linearly independent

b and c 〃 〃 〃

(13) $a^T b \quad (1 \quad 2 \quad 3)\begin{bmatrix}4\\5\\6\end{bmatrix} = \underline{32}$

$\quad 1\times3 \qquad 3\times1$

$$ab^T = \begin{bmatrix}4\\8\\3\end{bmatrix}(4\ 5\ 6) = \begin{pmatrix}4 & 5 & 6\\8 & 10 & 12\\12 & 15 & 18\end{pmatrix} \#$$

$3\times1 \quad 1\times3 = 3\times3$

(B) $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{pmatrix}$ 3×3 $B = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{pmatrix}$ 3×3 $C = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -1 & 1 & 3 \end{pmatrix}$ $d = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

(1) $2A - B$

$$\begin{pmatrix} 2 & 4 & 6 \\ 8 & -4 & 6 \\ 0 & 10 & -2 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 5 \\ 6 & -5 & 10 \\ -3 & 12 & -3 \end{pmatrix} \#$$

(2) AB:
$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 14 & -2 & -4 \\ 9 & 0 & 15 \\ 7 & 7 & -21 \end{pmatrix}$$

BA:
$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{pmatrix} = \begin{pmatrix} 9 & 3 & 8 \\ 6 & -18 & 13 \\ -5 & 15 & 2 \end{pmatrix}$$

(3) $(AB)^T$ and $B^T A^T$. $B^T = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & -2 \\ 1 & -4 & 1 \end{pmatrix}$ $A^T = \begin{pmatrix} 1 & 4 & 0 \\ 2 & -2 & 5 \\ 3 & 3 & -1 \end{pmatrix}$

$$\hookrightarrow = \begin{pmatrix} 14 & 9 & 7 \\ -2 & 0 & 7 \\ -4 & 15 & -21 \end{pmatrix}$$

$$B^T A^T = \begin{pmatrix} 14 & 9 & 7 \\ -2 & 0 & 7 \\ -4 & 15 & -21 \end{pmatrix} \#$$

(4) $|A|$ and $|C|$

$|A| = 55$, A - Co we get $|C| = 0$

(5) matrix B

(6) $A^{-1} = \frac{1}{\det(A)} \cdot \operatorname{Adj}(A)$

$$\frac{1}{55} \begin{pmatrix} -13 & -17 & 12 \\ -7 & -1 & 9 \\ 20 & -5 & -10 \end{pmatrix}$$

$\det(A) = 1 \cdot \det\begin{pmatrix} -2 & 3 \\ 5 & -1 \end{pmatrix} - 2 \cdot \det\begin{pmatrix} 4 & 3 \\ 0 & -1 \end{pmatrix} + 3 \cdot \det\begin{pmatrix} 4 & -2 \\ 0 & 5 \end{pmatrix}$

$1(2-15) - 2(-4-0) + 3(20-0)$

$-13 + 8 + 23 = 55 \#$

cofactors

$1: A_{11} = \begin{vmatrix} -2 & 3 \\ 5 & -1 \end{vmatrix} = -13$

$2: A_{12} = \begin{vmatrix} 4 & 3 \\ 0 & -1 \end{vmatrix} = -7$

$3: A_{13} = \begin{vmatrix} 4 & -2 \\ 0 & 5 \end{vmatrix} = 20 \#$

$4: A_{21} = \begin{vmatrix} 2 & 3 \\ 5 & -1 \end{vmatrix} = -17$

$-2 = A_{22}: \begin{vmatrix} 1 & 3 \\ 0 & -1 \end{vmatrix} -1 - \cancel{3} = -1$ $\quad 3 = A_{23}: \begin{vmatrix} 1 & 2 \\ 0 & 5 \end{vmatrix} 5$

$0 = A_{31}: \begin{vmatrix} 2 & 3 \\ -2 & 3 \end{vmatrix} = 12$ $\quad 5 = A_{32}: \begin{vmatrix} 1 & 3 \\ 4 & 3 \end{vmatrix} = -9$ $\quad -1 = A_{33}: \begin{vmatrix} 1 & 2 \\ 4 & -2 \end{vmatrix} = -10$

cofactor matrix

$$= \begin{bmatrix} -13 & -7 & 20 \\ -17 & -1 & 5 \\ 12 & -1 & -10 \end{bmatrix} = \hat{A} \quad \hat{A}^T = adj(A) = \boxed{\begin{pmatrix} -13 & -17 & 12 \\ -7 & -1 & 9 \\ 20 & -5 & -10 \end{pmatrix}} \#$$

(6) B^{-1}: for an inverse matrix
if it's orthogonal then it's inverse is the transpose and then you normalize to yield an orthonormal vector.

$$\begin{bmatrix} 1/6 & 2/21 & 3/14 \\ 2/6 & 1/21 & -2/14 \\ 1/6 & -1/21 & 1/14 \end{bmatrix} \#$$

(7) $C^{-1} = \frac{1}{|C|} \cdot adj(C)$

$\to |C| = 0$ (from previous problem) so

$\boxed{C^{-1} \text{ does not exist}}$

(8) Ad

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 14 \\ 9 \\ 7 \end{bmatrix}$$

3×3   3×1

(9) Proj of $\vec{a}$ onto $\vec{b} = \frac{\vec{b} \cdot \vec{a}}{|\vec{b}|}$ rows A $A_1 = [1 \; 2 \; 3]$ $d = [1 \; 2 \; 3]$

$|d| = \sqrt{14}$

$A_1 \cdot d = 14$

$\frac{14}{\sqrt{14}}$ #

row 1 scalar proj

$A_2 = [4 \ -2 \ 3]$

$A_2 \cdot d = 9$

$\frac{9}{\sqrt{14}}$ # row 2 scalar proj.

$A_3 = [0 \quad 5 \quad -1]$

$A_3 \cdot d = 7$

row 3 scalar proj $= \frac{7}{\sqrt{14}}$ #

~~10B~~ 5n 10B vector proj. $\frac{a \cdot b}{|b|^2}\vec{b}$ proj of $\vec{a}$ onto $\vec{b}$.

$d = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

row 1 = [1 2 3]

Proj of row 1 onto D: $\frac{14}{14}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ #

row 2 = [4 -2 3]

proj of row 2 onto D: $4(1) - (4) + 9 = \frac{9}{14}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ #

row 3 = [0 5 -1]

proj of row 3 onto D: $0 + 10 - 3 = \frac{7}{14}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ #

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$$\begin{bmatrix}1\\4\\0\end{bmatrix}\cdot 1 + \begin{bmatrix}2\\-2\\5\end{bmatrix}\cdot 2 + \begin{bmatrix}3\\3\\-1\end{bmatrix}\cdot 3 = \begin{bmatrix}1\\4\\0\end{bmatrix} + \begin{bmatrix}4\\-4\\10\end{bmatrix} + \begin{bmatrix}9\\9\\-3\end{bmatrix} = \begin{bmatrix}14\\9\\7\end{bmatrix}$$

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$Bx = d \qquad x = B^{-1} d$

$$= \begin{bmatrix}1/6 & 2/21 & 3/14\\ 2/6 & 1/21 & -2/14\\ 1/6 & -4/21 & 1/14\end{bmatrix} \cdot \begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix} \#$$

x.

$\frac{1}{6} + 2(2/21) + 9/14$

$\frac{1}{6} + \frac{4}{21} + \frac{9}{14} = 1$

$\frac{2}{6} + \frac{2}{21} - \frac{6}{14} = 0$

$1/6 - \frac{8}{21} + \frac{3}{14} = 0$

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$x = C^{-1} d$

→ there's no sol'n for x since C can't be inverted. The determinant of C is 0 and it's a singular matrix

C

$D = \begin{pmatrix}1 & 2\\3 & 2\end{pmatrix} \quad E = \begin{pmatrix}2 & -2\\-2 & 5\end{pmatrix} \quad F = \begin{pmatrix}1 & 2\\2 & 4\end{pmatrix}$

1. eigenvalues & corresponding eigenvectors of D.

$\det\begin{pmatrix}1-\lambda & 2\\3 & 2-\lambda\end{pmatrix} = (1-\lambda)(2-\lambda) - 6 = 0$

$2 - 3\lambda + \lambda^2 - 6 = 0$

$\lambda^2 - 3\lambda - 4 = 0$

$(\lambda - 4)(\lambda + 1) = 0$

$\lambda = 4,\ \lambda = -1$

$\underline{\lambda = 4}$

$\begin{pmatrix}-3 & 2\\3 & -2\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$ rref$\left(\begin{array}{cc|c}-3 & 2 & 0\\3 & -2 & 0\end{array}\right)$

$\Rightarrow \left(\begin{array}{cc|c}1 & -2/3 & 0\\0 & 0 & 0\end{array}\right)$

$1x_1 - \frac{2x_2}{3} = 0$

$x_1 = \frac{2x_2}{3}$

Let $x_1 = \cancel{4}\ 2$

$x_2 = \cancel{3/4}\ 3$

$$\boxed{e_1 = \begin{bmatrix} \cancel{4} \\ \cancel{3/4} \end{bmatrix} \quad \begin{pmatrix} 2 \\ 3 \end{pmatrix}}$$ #

$x_1 + x_2 = 0$

$x_1 - \frac{2x_2}{3} = 0 \quad x_1 = \frac{2x_2}{3}$

$x_1 = \cancel{3} \quad x_2 =$

$3x_1 = 2x_2$

$3(2) = 2(3)$

$x_1 = 2,\ x_2 = 3$

$\lambda = -1$

$A - \lambda I = \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}$ singular Matrix

$$\begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$1 \quad 1 \quad 0$

$0 \quad 0 \quad 0$

$x_1 + x_2 = 0$

let $x_1 = 1$

$x_2 = -1$ #

$$\boxed{e_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

(2) $e_1 \cdot e_2 = 2(1) + 3(-1) = -1$

(3) $E = \begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix}$

$$\det\begin{pmatrix} 2-\lambda & -2 \\ -2 & 5-\lambda \end{pmatrix} = 0$$

$(2-\lambda)(5-\lambda) - 4 = 0$

$10 - 7\lambda + \lambda^2 - 4 = 0$

$\lambda^2 - 7\lambda + 6 \longrightarrow (\lambda - 6)(\lambda - 1) = 0$

$\lambda = 6,\ \lambda = 1$

$\lambda = 6$

$$\begin{pmatrix} 2-6 & -2 \\ -2 & 5-6 \end{pmatrix} = \begin{pmatrix} -4 & -2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\text{rref}\left(\begin{array}{cc|c} -4 & -2 & 0 \\ -2 & 1 & 0 \end{array}\right) = \begin{array}{l} x_1 = 0 \\ x_2 = 0 \end{array}$

$\neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

The dot product MUST be 0, since one of the $\boxed{e_1 \cdot e_2 \text{ for } \vec{E}}$ $\overset{?}{=}$ 0 eigen-vectors is indeed $\vec{0}$.

(4) the dot product of two vectors $= 0$ it means
when the vectors are orthogonal.

Therefore the eigenvectors of E are orthogonal
and E is a ~~Hermitian matrix~~ symmetric matrix

(5) $Fx = 0 \quad x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ #

$x_1 = -2x_2$ $\boxed{0, 0}$

(6) $Fx = 0$ two nontrivial sol'ns

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \qquad \text{rref}\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 4 & 0 \end{array}\right) = \begin{array}{ccc} x_1 & x_2 & \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}$$

$x_1 = -2x_2$

$-4 = -2(2)$

1 sol'n $= \begin{pmatrix} -4 \\ 2 \end{pmatrix}$ # 2nd sol'n $= \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ #

(7) Since row of D is linearly independent
then x must $= 0$. $\Rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ #

D

(1) $f(x) = x^2 + 3$

$f'(x) = 2x \qquad f''(x) = 2$

$f(x) = x^2 + 3$

$g(x) = x^2 \Rightarrow f(g(x)) = x^4 + 3$

(2) $\dfrac{\partial q}{\partial x} = 2x \qquad \dfrac{\partial q}{\partial y} = 2y$

(3) $\nabla q(x, y) = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$

(4) $\dfrac{\partial}{\partial x} f(g(x))$ w/ chain rule: $\dfrac{\partial}{\partial g(x)} f(g(x)) \cdot \dfrac{\partial (g(x))}{\partial x} =$

$2x^2 \cdot 2x = 4x^3$ #

W/o chain rule:

$\dfrac{\partial}{\partial x}(x^4 + 3) = 4x^3$ #