MATH 410, FALL 2020: HOMEWORK 7

Due: <u>November 5 before class (1:30 pm US eastern time)</u>

Read Section 3.3 in Stein-Shakarchi and Chapter 10 in the textbook. You can use any results we have covered so far from the book or from class for the following problems, but please state which results you are using when writing your solutions.

Exercise 1. Let $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ be the extended complex plane. A function is called **linear fractional**, if it is of the form $f(z) = \dfrac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$.

(1) Show that a linear fractional function $f(z)$ is a bijection $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$ by finding the inverse function. Is the inverse function also linear fractional?

(2) Show that any bijective meromorphic function on $\overline{\mathbb{C}}$ must be linear fractional. (Hint: How many zeros and poles does this function have? What happens if their orders are not equal to 1?)

Exercise 2. Determine the singularities and associated residues of

$$(1)\ \frac{1}{z^5 - z^2} \qquad (2)\ \sec(z) \qquad (3)\ z^5 e^{\left(z^{-3}\right)}$$

Exercise 3. Write $z = re^{i\theta}$ for $z \neq 0$, where $r = |z| > 0$ and $\theta = \arg(z)$.

(1) Recall $\theta = \arctan\left(\frac{y}{x}\right) + k\pi$ for some $k \in \mathbb{Z}$. Set $\mathrm{d}\theta = P(x, y)\mathrm{d}x + Q(x, y)\mathrm{d}y$. Find the formulas of $P(x, y)$ and $Q(x, y)$.

(2) Set $\mathrm{d}z = R(r, \theta)\mathrm{d}r + S(r, \theta)\mathrm{d}\theta$. Find the formulas of $R(r, \theta)$ and $S(r, \theta)$.

(3) Show that $\frac{\mathrm{d}z}{z} = i\mathrm{d}\theta + \mathrm{d}f(r)$ for some differentiable function $f(r)$. Express f as a function of x and y.

(4) From this, conclude that for any closed curve γ not containing $(0, 0) \in \mathbb{R}^2$,

$$\oint_\gamma \frac{\mathrm{d}z}{z} = i \cdot \oint_\gamma \mathrm{d}\theta.$$

Exercise 4. Let C_r be the circle $z(t) = re^{it}, 0 \leq t \leq 2\pi$. Use Cauchy's Residue Theorem to evaluate the integrals:

$$(1)\ \oint_{C_5} \frac{\mathrm{d}z}{z^2 - 2z - 8} \qquad (2)\ \oint_{C_3} \frac{\mathrm{d}z}{z^2 + 2z - 8} \qquad (3)\ \oint_{C_3} \frac{e^{\pi z}}{z^2 + 4}\mathrm{d}z$$

Exercise 5. Let C be the unit circle $z(t) = e^{it}, 0 \leq t \leq 2\pi$.

(1) Use Cauchy's Residue Theorem to show that

$$\oint_C \frac{e^z}{z}\mathrm{d}z = 2\pi i.$$

(2) Evaluate the integral in (1) directly and use (1) to show that:

$$\int_0^{2\pi} e^{\cos t}\cos(\sin t)\mathrm{d}t = 2\pi, \quad \int_0^{2\pi} e^{\cos t}\sin(\sin t)\mathrm{d}t = 0.$$

$\boxed{\textbf{Exercise 6.}}$ Recall $e \approx 2.718$. Use Rouché's theorem to find the number of zeros of

(1) $f_1(z) = 32e^z - z^2$ in $|z| \leq 2$.
(2) $f_2(z) = e^z - 3z$ in $|z| \leq 1$.
(3) $f_3(z) = z^4 + 5z - 1$ in $1 \leq |z| \leq 2$.