Weak Law of Large Numbers

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Overview: Apply Chebyshev's inequality to the average $\bar{X}_n$ where we use $E[\bar{X}_n] = \mu$ and $Var(\bar{X}_n) = \sigma^2/n$ and where where $\epsilon > 0$.

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$$\begin{aligned} P\left(|\bar{X}_n - \mu| > \epsilon\right) &= P\left(|\bar{X}_n - E[\bar{X}_n]| > \epsilon\right) \\ &\leq \frac{1}{\epsilon^2} Var(\bar{X}_n) \\ &\leq \frac{\sigma^2}{n\epsilon^2} \end{aligned}$$

Definition: Weak Law of Large Numbers

Weak Law of Large Numbers - if $\hat{X}_n$ is the average of n independent random variables with expectation μ and variance σ^2, then for any $\epsilon > 0$:

$$\lim_{n \to \infty} P\left(|\bar{X}_n - \mu| > \epsilon\right) = 0$$

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Measuring Things

- Assume we conduct a random experiment n times. Let S_n be the number of times event A occurs.
- We intuitively assume

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- We can't "know" $P[A] = p$ precisely because we can only know what we have seen. How can we estimate p?

- Assuming each observation is independent, then S_n has a Binomial distribution with $E[S_n] = np$ and $Var[S_n] = np(1-p)$

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- Weak Law of Large Numbers - we can increase the precision of measurements by measuring more:

$$P[|\frac{S_n}{n} - p| \geq \delta] \leq \frac{p(1-p)}{n\delta^2}$$

Weak Law of Large Numbers - II

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Thus..

$$E[S_n/n] = 1/nE[S_n] = p$$

$$Var[S_n/n] = 1/n^2 Var[S_n] = \frac{p(1 - p)}{n}$$

Weak Law of Large Numbers - III

- Chebyshev:

$$P[|X - E[X]| \geq t] \leq \frac{\sigma^2}{t^2}$$

- Compute bound for S_n/n:

$$P[|\frac{S_n}{n} - p| \geq \delta] \leq \frac{p(1-p)}{n\delta^2}$$

- We can make the R.H.S. arbitrarily small by increasing n (measuring more)
- Thus, we can estimate $P[A]$ from S_n/n.

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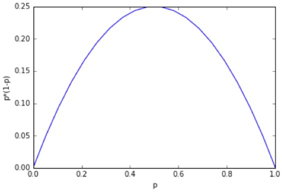

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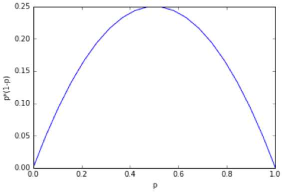

- We can be certain that no matter what the value of p,

$$P[|\frac{S_n}{n} - p| \geq \delta] \leq \frac{1}{4n\delta^2}$$

Weak Law of Large Numbers - How large should n be?

- We often want to say "I know that 95% of the time, my estimate will be within δ of the mean".
- Or, "I am willing to have my estimate be more than δ from the mean no more 5% of the time"

$$P[|\frac{S_n}{n} - p| \geq \delta] \leq \epsilon$$

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Or, since I may not know what p is (since I'm estimating it),

$$n \geq \frac{1}{4\epsilon\delta^2}$$

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- In other words, my estimate will be no more than 0.1 off 95% of the time.

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- In other words, my estimate will be no more than 0.1 off 95% of the time.
- Since I don't know p then $n \geq 1/(4\epsilon\delta^2)$ says I need

$$n \geq 1/(4 * 0.05 * 0.01) = 500$$

measurements

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- If we're willing to make more assumptions about our measurements, we can make even fewer measurements as shown by the Central Limit Theorem.