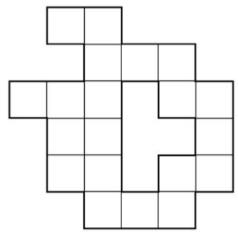
PROBLEM 1

In this problem, a tile is a 1x1 square and an n-tiling is a group of n tiles placed so that every tile is contiguous along at least one side to another tile. (The tiles cannot be offset. That is, they must be placed into a grid like Scrabble tiles.) The *edge length* of a tiling is the total length of edges that are not touching other tiles. For instance, in the following 20-tiling, the edge length is 36.



Prove that for all n-tilings where $n \geq 1$, the edge length is even.

__Given__: A tile is a 1x1 square and an n-tiling is a group of n tiles placed contiguously along at least one side to another tile.

__Base Case__: First for the base case, let's consider P(1).
If $n = 1$, then edge lengths = 4, an even number and thus true.

__Induction Hypothesis__: For all n-tilings, where $n \geq 1$, each has a $2k$ even edge length.

__Induction Step__: Assume that the claim holds for all n square tiles.

Now let's take any arbitrary $n + 1$ square tiling.

Adding one square tile that intersects the rest of the figure at one contiguous exterior edges decreases the total edge length by 1 and increases the total edge length by 3. Thus, -1 + 3 = 2.
Since 2 is a multiple of 2, the edge length is even.

Moreover, now let's consider two intersections.
The exterior edge length decreases by 2 and increases by 2.
Thus, the total edge length = -2 + 2 = 0.
Since 0 is a multiple of 2, the edge length is even.

Furthermore, now let's consider three intersections.
The exterior edge length decreases by 3 and increases by 1.
Thus, the total edge length = -3 + 1 = -2.
Since -2 is a multiple of 2, the edge length is even.

Finally, now let's consider four intersections.
The exterior edge length decreases by 4 and increases by 0.
Thus, the total edge length = -4 + 0 = -4.
Since -4 is a multiple of 2, the edge length is even.

__Conclusion__: In conclusion, by the induction hypothesis we just proved that for all n-tilings, where $n \geq 1$, each has a $2k$ even edge length.

1 Problem 1 **34 / 34**

✓ **- 0 pts** Correct

- 2 pts Error in base case

- 4 pts Incorrect or missing base case

- 2 pts Error in inductive hypothesis—assumes the claim is true for all/every n >= 1

- 4 pts Incorrect or missing inductive hypothesis

- 2 pts Missing/incorrect case where (n+1) tile is touching on 1 side

- 2 pts Missing/incorrect case where (n+1) tile is touching on 2 sides

- 2 pts Missing/incorrect case where (n+1) tile is touching on 3 sides

- 2 pts Missing/incorrect case where (n+1) tile is touching on 4 sides

- 1 pts Error in one of the cases

- 3 pts Error across all of the cases

- 1 pts Does not use the inductive hypothesis in the inductive step

- 1 pts Confuses n with the edge length of an n-tiling

- 1 pts Error in arithmetic, propositional logic, or variables

- 20 pts Proof lacks any structure, but there is a base case and a single case of the inductive step that are correctly proved, so partial credit is warranted

- 3 pts Proof requires further explanation or more clarity

PROBLEM 2

Prove using strong induction that for any positive integer n and any $x \in \mathbb{R}$ where $\neq 0$ that if $x + 1/x$ is an integer, then $x^n + 1/x^n$ is also an integer.

Given: Any positive integer n and any $x \in$ where $\neq 0$ that if $x + 1/x$ is an integer then $x^n + 1/x^n$ is also an integer.

Base Case: First for the base case let's consider P(1) .
For all n = 1, if $1/x$ is an integer, then $x^n + 1/x^n$ is true.

Induction Hypothesis: For all positive integers n and any $x \in \mathbb{R}$ where $\neq 0$
if $x + 1/x$ is an integer, then $x^n + 1/x^n$ is also an integer.
Assume that for all integers less than or equal to n, the statement holds.
We want to show that $x^{n+1} + 1/x^{n+1} \in \mathbb{Z}$ or is an integer.

Strong Inductive Step: By the inductive assumption if $(x + 1/x)$ is an integer, then $(x^n + 1/x^n)$ is an integer.

Furthermore, the product of two integers is an integer:
$(x + 1/x)\ (x^n + 1/x^n) = x^{n+1} + 1/x^{n-1} + x^{n-1} + 1/x^{n+1}$

$(x + 1/x)\ (x^n + 1/x^n)$ - x^{n-1} - $1/x^{n-1} = x^{n+1} + 1/x^{n+1}$

Conclusion: In conclusion, multiplying together two integers (the $n = 1$ case and the n case) and subtracting an integer (the n - 1 case), results in an integer.
More formally, by the strong induction hypothesis we have proved that for any positive integer n and any $x \in$ where $\neq 0$ that
if $x + 1/x$ is an integer, then $x^n + 1/x^n$ is also an integer.
More specifically, $x^{n+1} + 1/x^{n+1} \in \mathbb{Z}$ or is an integer.

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- **- 0 pts** Correct
- **- 11 pts** Major error(s) in the base cases or both base cases are missing
- ✓ **- 5 pts Minor error(s) in the base cases**
- **- 5 pts** Major error(s) in inductive hypothesis
- **- 2 pts** Minor error(s) in inductive hypothesis
- **- 5 pts** Major error(s) in inductive step
- **- 2 pts** Minor error(s)/unclear explanations in inductive step
- **- 11 pts** Major error(s) in conclusion
- **- 5 pts** Minor error(s) in conclusion
- **- 20 pts** Proof claims to use induction but doesn't use the formula for induction: intro, base case(s), inductive hypothesis, inductive step, conclusion. There are good ideas here so partial credit
- **- 1 pts** Error in defining a predicate for the claim
- **- 33 pts** Solution contains some arithmetic/algebraic/logical steps but do not form a coherent proof or explanation
- **- 3 pts** Incomplete conclusion

Missing second base case

PROBLEM 3

Prove that for all nonnegative integers n. You may use induction or well-ordering (or do it both ways for practice!)

Given:

$$\sum_{i=0}^{n} i^3 = \left(\sum_{i=0}^{n} i\right)^2$$

Hint: the following identity may be useful

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$$

Base Case: First for the base case let's consider the base case $n = 0$,

$$\sum_{i=0}^{n} i^3 = \left(\frac{0(0+1)}{2}\right) = 0$$

The base case holds true for $n = 0$.

Induction Hypothesis: We will prove by induction that for all non-negative integers n

$$\sum_{i=0}^{n+1} i^3 = \left(\sum_{i=0}^{n+1} i\right)^2$$

or

$$\sum_{i=0}^{n+1} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Induction Step: Assume that the claim holds true for any n, in other words

$$\sum_{i=0}^{n+1} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Consider $\sum_{i=0}^{n+1} i^3 = \sum_{i=0}^{n} i^3 + (n+1)^3$

$$\begin{aligned}
&= \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 \\
&= \left(\frac{n^2(n+1)^2}{4}\right) + (n^3 + 3n^2 + 3n + 1) \\
&= \left(\frac{n^2(n^2 + 2n + 1) + 4(n^3 + 3n^2 + 3n + 1)}{4}\right) \\
&= \left(\frac{n^4 + 2n^3 + n^2 + 4n^3 + 12n^2 + 12n + 4)}{4}\right) \\
&= \left(\frac{n^4 + 6n^3 + 13n^2 + 12n + 4)}{4}\right) \\
&= \left(\frac{(n^2 + 2n + 1)(n^2 + 4n + 4)}{4}\right) \\
&= \left(\frac{(n+1)^2(n+2)^2}{4}\right) \\
&= \left(\frac{(n+1)(n+2)}{2}\right)^2 \\
&= \left(\frac{n(n+1)}{2}\right)^2
\end{aligned}$$

Conclusion: In conclusion, we proved by induction that for all nonnegative integers n

$$\sum_{i=0}^{n+1} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

or

$$\sum_{i=0}^{n+1} i^3 = \left(\sum_{i=0}^{n+1} i\right)^2$$

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✓ **- 0 pts** **Correct intro**

✓ **- 0 pts** **Correct base case**

✓ **- 0 pts** **Correct proof that claim holds for base case**

✓ **- 0 pts** **Correct Inductive hypothesis**

- 0 pts Correct inductive step

- 0 pts Correct proof of inductive step

- 0 pts Correct conclusion

✓ **- 4 pts** **No introduction**

- 2 pts Introduction fails to state that the proof will be by induction

- 6 pts Missing base case.

- 4 pts Incorrect proof of base case.

- 2 pts Missed base case (the base case should be 0, since the claim is that P(n) holds for all nonnegative integers)

- 4 pts Incorrect inductive hypothesis. Example: assume P(n) for *all* nonnegative integers.

- 2 pts Never explicitly defines the inductive hypothesis, but uses it correctly

- 2 pts Never explicitly states the inductive step, but proof shows that the correct choice was made about what needed to be shown.

- 2 pts Confused definition of inductive hypothesis or inductive step. The hypothesis is that the claim is true for n, i.e. P(n). The inductive step is the proof that P(n) -> P(n+1)

✓ **- 3 pts** **Correct elements misplaced in proof.**

- 2 pts Minor error in proof of inductive step. Example: proof needs more explanations or has some problems with clarity. It may not be clear what is assumed and why.

✓ **- 4 pts** **Error in proof of inductive step. Example: P(n) is a predicate, but proof uses it as a number**

- 4 pts Error in inductive step. P(k+1) is the proposition that the sum of cubes equals the square of the triangular numbers. P(k+1) is not equal to the sum of squares of the triangular numbers.

- 6 pts Major error in proof of inductive step. Algebra mistakes or false claims, but proof still gets to the correct final answer when the mistakes cancel out, i.e. student knows where they are trying to go, but doesn't know how to get there.

- 8 pts Completely incorrect proof of inductive step. Examples: Circular logic. Assumes claim is true in proof of claim. Proves different claim.

- 5 pts No conclusion

What you have labeled as the inductive hypothesis should be the introduction. What you label as the inductive step is actually the inductive hypothesis. You never explicitly state what you need to prove in the inductive step.

1 Everything was right until here, but then this is not equivalent to the prior step.

2 This is not correct. You want (n+1)(n+2)/2.

3 Once again, this is not a correct statement of the claim you need to prove.