Midterm 1 Math 370 F 2020 (Take-Home)

Instructions: You may consult any written source in notes, books, or on the internet, but all work must be your own. Create a pdf of your solutions and upload it to canvas by midnight on Monday, Oct. 12.

1. (15 points) Recall that S_n denotes the symmetric group on n letters. Let $V = \{e, (12)(34), (13)(24), (14)(23)\}$ be the subgroup of S_4 consisting of the identity and all permutations whose cycle type is 2^2 (i.e., two cycles of size 2).

a) Show that V is a normal subgroup of S_4.

b) Show that for any $\beta \in S_4$, there is exactly one element of the right coset $V\beta$ which fixes 4.

c) Prove that S_4/V is isomorphic to S_3.

2. (15 points)

a) Let a, b be two elements of a group G. Prove that ab and ba are conjugate in G.

b) Let p, q be two permutations in S_n. Prove that pq and qp have the same cycle type.

c) Prove that the map $A \to (A^T)^{-1}$ (the inverse of the transpose map) is an automorphism of $\mathrm{GL}_n(\mathbb{R})$.

3. (10 points) Does the symmetric group S_7 contain an element of order 5? of order 10? of order 15? What is the largest possible order of an element of S_7?

4. (10 points) Prove that S_n is generated by the cycles $(123 \cdots n)$ and (12).

5. (20 points) a) Show the dihedral group D_4 of order 8 is isomorphic to the subgroup of S_8 generated by the permutations (24) and (1234).

b) By a *lattice* we mean a collection of subsets partially ordered by inclusion. Determine the lattice of subgroups of D_4, indicating the orders of each subgroup and which subgroups are normal subgroups of other subgroups. Use the representation of D_4 (and its subgroups) as subgroups of S_4 from part a above.

c) Describe the quotient group of D_4 by each proper normal subgroup of D_4.

d) Determine all subgroups of D_4 fixed by every automorphism of D_4.