MATH 410, FALL 2020: HOMEWORK 4

Due: October 1 before class (1:30 pm US eastern time)

Read Chapters 4 and 5. You can use any results we have covered so far from the book or from class for the following problems, but please state which results you are using when writing your solutions.

Exercise 1. Calculate $\int_C |z|^2 \mathrm{d}z$ where

(1) C denotes the path that goes vertically from 0 to i then horizontally from i to $1+i$.
(2) C denotes the path that goes horizontally from 0 to 1 then vertically from 1 to $1+i$.

Is it possible to find an anti-derivative for $f(z) = |z|^2$ and why?

Exercise 2. Evaluate $\int_C (z^2 + i)\mathrm{d}z$, where C is a parabola segment:

$$z(t) = t^2 - it, \quad t \in [-1, 1]$$

(1) by finding the anti-derivative of the integrand.
(2) by integrating along the straight line between the end points of C and then using the Closed Curve Theorem.

Exercise 3. (1) Let C be a closed curve and f be an entire function. Write $f(z) = u(x, y)+iv(x, y)$ where u and v are real valued functions. Set $\mathrm{d}z = \mathrm{d}x+i\mathrm{d}y$. Rewrite the line integral $\int_C f \mathrm{d}z$ as:

$$\int_C f\mathrm{d}z = \int_C (P\mathrm{d}x + Q\mathrm{d}y) + i \cdot \int_C (R\mathrm{d}x + S\mathrm{d}y),$$

where P, Q, R, S are real valued functions. Find their formulas in terms of u and v.

(2) Use Green's theorem and the Cauchy-Riemann equations for u and v to show that $\int_C f(z)\mathrm{d}z = 0$

Exercise 4. Find the power series expansions of the functions at given points:

(1) $f(z) = e^z$ and $z = a$ for any complex number a.
(2) $f(z) = \sin z$ at $z = i$.

Exercise 5. (Sometimes one can use Cauchy's Integral formula even in the case when f is not analytic.)

Let $f(z) = |z+2|^2$. Let C be a circle of radius 2 centered at $z = 0$, with counterclockwise orientation.

(1) Show that f is not analytic on any domain that contains C.

(2) Find a function g that is analytic on some domain that contains C and such that $f(z) = g(z)$ at all points on the circle C. It follows that $\int_C f \mathrm{d}z = \int_C g \mathrm{d}z$. (Hint: $z\bar{z} = |z|^2 = 4$ on C.)

(3) Use Cauchy's Integral formula to show that

$$\int_C |z+2|^2 \mathrm{d}z = \int_C g(z) \mathrm{d}z = 16\pi i$$

$\boxed{\textbf{Exercise 6.}}$ (1) Use mathematical induction and the Fundamental Theorem of Algebra[1] to show that any polynomial $P(z)$ with complex coefficients of degree n has n complex roots, counted in multiplicity.

(2) $P(z)$ is called a real polynomial if all of its coefficients are real. Show that a real polynomial factorizes as a product of linear and quadratic real polynomials. (Hint: check that when $P(z)$ is real, a complex number w is a root of P iff its conjugate $\bar{w}$ is. Then check that $(z-w)(z-\bar{w})$ is a real quadratic polynomial.)

(3) Show that a real polynomial of odd degree must have a real root.

[1] Theorem 5.12 in the textbook