LGIC 010/PHIL 005

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Lecture 17

Binary Relations: Some Properties

Definition

L^A is *reflexive* if and only if

$$A \models (\forall x)Lxx.$$

Definition

L^A is *irreflexive* if and only if

$$A \models (\forall x)\neg Lxx.$$

Definition

L^A is *symmetric* if and only if

$$A \models (\forall x)(\forall y)(Lxy \supset Lyx).$$

Binary Relations: Some Properties

Definition

L^A is *asymmetric* if and only if

$$A \models (\forall x)(\forall y)(Lxy \supset \neg Lyx).$$

Definition

L^A is *transitive* if and only if

$$A \models (\forall x)(\forall y)(\forall z)(Lxy \supset (Lyz \supset Lxz)).$$

Definition

A is a *simple graph* if and only if L^A is irreflexive and symmetric.

Binary Relations: Further Properties

Definition

L^A is *connected* if and only if

$$A \models (\forall x)(\forall y)(x \neq y \supset (Lxy \vee Lyx)).$$

Definition

L^A is a *tournament* if and only if L^A is asymmetric and L^A is connected.

Definition

L^A is a *strict linear order* if and only if L^A is transitive and L^A is a tournament.

Counting Tournaments and Linear Orders

- Let T be a schema such that $A \models T$ if and only if T is a tournament.
- Observe that $|\mathbf{mod}(T, n)| = 2^{\binom{n}{2}}$.
- Let S be a schema such that $A \models S$ if and only if A is a strict linear order.
- Observe that $|\mathbf{mod}(S, n)| = n!$.

Counting Isomorphism Types

- Let R be a schema. We define $\mathbf{iso}(R, n) = \{\mathbb{I}(A) \mid A \models R \text{ and } U^A = [n]\}$.
- Let's compute $|\mathbf{iso}(T, 3)|$ and $|\mathbf{iso}(T, 4)|$.

Functional Relations

Definitions

- Let F be an $n+1$-adic predicate.
- F^A is total if and only if $A \models (\forall x_1)\dots(\forall x_n)(\exists y)Fx_1\dots x_n y$.
- F^A is single-valued if and only if $A \models (\forall x_1)\dots(\forall x_n)(\forall y)(\forall z)((Fx_1\dots x_n y \wedge Fx_1\dots x_n z) \supset y = z)$.
- F^A is a an n-ary functional relation if and only if F^A is total and single-valued.
- Observe that F^A is an n-ary functional if and only if F^A is the graph of an n-ary total function mapping U^A to U^A.
- In order to perspicuously express properties of functional relations, we introduce a new syntactic category of symbols into our logical language, namely, function symbols, $f, g, h, \dots$ with specified number of argument places.

Functional Relations and Functions

Example: Associativity

- Let F be a ternary predicate and suppose we wish to formulate a schema S such that $A \models S$ if and only if F^A is the graph of a binary function f^A and f^A is associative.
- In order to do so we can conjoin the schema that expresses that F^A is a binary functional relation with the following schema.

$$(\forall x)(\forall y)(\forall z)(\forall u)(\forall v)(\forall w)((Fxyz \land Fzuv \land Fyuw) \supset Fxwv)$$

- By use of the function symbol f, we may express this condition by the schema: $f(f(x, y), z) = f(x, f(y, z))$.
- By use of another function symbol such as $\circ$, with infix notation, we may express associativity yet more perspicuously by the schema: $(x \circ y) \circ z = x \circ (y \circ z)$.

Structures for Functional Vocabularies

Example: Groups

- Let A be a structure interpreting a binary function symbol $\circ$, a unary function symbol $^{-1}$ and a 0-ary function symbol (also known as a constant symbol) e.
- A is a *group* if and only if A satisfies the conjunction of the universal closures of the following schemata.
- $(x \circ y) \circ z = x \circ (y \circ z)$
- $x \circ e = x$
- $x \circ x^{-1} = e$
- Recall that $\mathbb{S}_n$, the collection of permutations of $[n]$, equipped with the operations of function composition and function inverse is a group.

Counting: An Example

- Let S be the schema $f(f(x)) = x$.
- Let's compute $\mathbf{mod}(S, 5)$ and $\mathbf{iso}(S, 5)$.

Example: Peano-structures

- Let A be a structure interpreting a unary function symbol s and a constant symbol 0. A is a *Peano-structure* if and only if A satisfies the conjunction of the following schemata.
- $(\forall x)s(x) \neq 0$
- $(\forall x)(\forall y)(s(x) = s(y) \supset x = y$
- $(\forall x)(x \neq 0 \supset (\exists y)s(y) = x)$
- What do Peano-structures look like?