Chebyshev Inequalities

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Chebyshev's Inequailty

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Markov Inequality is $P[X \geq t] \leq \frac{E[X]}{t}$.

Now, substitute $X = (Y - E[Y])^2$ and $w^2 = t$ into Markov Inequality.

$$P[(Y - E[Y])^2 \geq w^2] \quad \leq \quad \frac{E[(Y - E[Y])^2}{w^2}$$

Chebyshev's Inequailty - Absolute Value

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Chebyshev's Inequailty - Absolute Value

$$
\begin{aligned}
P[(Y - E[Y])^2 \geq w^2] &\leq \frac{E[(Y - E[Y])^2}{w^2} \\
&\leq \frac{Var[Y]}{w^2}
\end{aligned}
$$

And $E[(Y - E[Y])^2]$ is $Var[Y]$

Chebyshev's Inequailty - Absolute Value

$$
\begin{aligned}
P[(Y - E[Y])^2 \geq w^2] &\leq \frac{E[(Y - E[Y])^2}{w^2} \\
&\leq \frac{Var[Y]}{w^2} \\
P[(Y - E[Y])^2 \geq w^2] &= P[|(Y - E[Y])| \geq w]
\end{aligned}
$$

And $E[(Y - E[Y])^2]$ is $Var[Y]$

Rearrange terms noting that $(Y - E[Y])^2 \geq w^2$ whenever $|Y - E[Y]| \geq w$.

Chebyshev's Inequailty - Rearranged

$$P[|Y - E[Y]| \geq w] \quad \leq \quad \frac{Var[Y]}{w^2}$$

Chebyshev’s Inequailty - Rearranged

$$
\begin{aligned}
P[|Y - E[Y]| \geq w] &\leq \frac{Var[Y]}{w^2} \\
P[|Y - E[Y]| \geq w' \sqrt{Var[Y]}] &\leq \frac{Var[Y]}{(w' \sqrt{Var[Y]})^2}
\end{aligned}
$$

Substitue $w = w' \sqrt{Var[Y]}$

Chebyshev's Inequailty - Rearranged

$$\begin{aligned} P[|Y - E[Y]| \geq w] &\leq \frac{Var[Y]}{w^2} \\ P[|Y - E[Y]| \geq w' \sqrt{Var[Y]}] &\leq \frac{Var[Y]}{(w' \sqrt{Var[Y]})^2} \\ P[|Y - E[Y]| \geq w' \sqrt{Var[Y]}] &\leq \frac{1}{w'^2} \end{aligned}$$

Cancel $Var[Y]$ on r.h.s.
The probability of drawing a sample greater than 2 standard deviations from the mean is less than 25%.

Markov and Chebyshev's Inequailty - Example

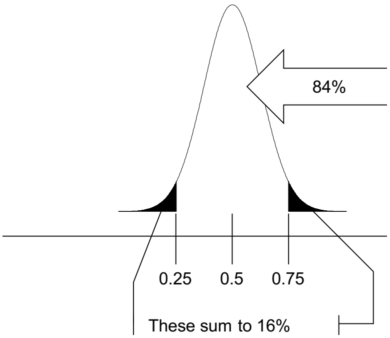
- Suppose an interactive computer is proposed for which it is estimated that the mean response time $E[T] = 0.5$ seconds.
- By Markov's inequality, the probability of the response time be more than 2 seconds would be $P[T > 2] \leq \frac{0.5}{2}$ or 25%. This is very conservative.

Markov and Chebyshev's Inequailty - Example

- Suppose an interactive computer is proposed for which it is estimated that the mean response time $E[T] = 0.5$ seconds.
- By Markov's inequality, the probability of the response time be more than 2 seconds would be $P[T > 2] \leq \frac{0.5}{2}$ or 25%. This is very conservative.
- If the estimated standard deviation is 0.1 seconds, then Chebyshev's inequality tells us $P[(T \leq 0.25) \cup (T \geq 0.75)] = P[|T - 0.5| \geq 0.25]$ which is $\leq \frac{0.1^2}{0.25^2}$ or ≤ 016.
- So, there's an 84% probabilty of the response time being between 0.25 and 0.75 seconds.

Markov and Chebyshev's Inequailty - Example

- We'll see relations like $P[|T - 0.5| \geq 0.25]$ often. We expanded this to $P[(T \leq 0.25) \cup (T \geq 0.75)]$ above.
- We're excluding the symmetric lower and upper portions of the event space.



Chebyshev's Inequailty - One Sided

- It's also possible to derive <u>one sided</u> inequalities...
- $P[X \leq t] \leq \frac{\sigma^2}{\sigma^2+(t-E[X])^2}$ if $t < E[X]$
- $P[X > t] \leq \frac{\sigma^2}{\sigma^2+(t-E[X])^2}$ if $t \geq E[X]$
- but we won't go through derivation.