PRINCETON LECTURES IN ANALYSIS

II

COMPLEX ANALYSIS

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and

$$\begin{aligned}|\cosh \pi z| &= \left|\frac{e^{\pi z} + e^{-\pi z}}{2}\right| \\ &\geq \frac{1}{2}\left|\, |e^{\pi z}| - |e^{-\pi z}|\, \right| \\ &\geq \frac{1}{2}(e^{\pi R} - e^{-\pi R}) \\ &\to \infty \qquad \text{as } R \to \infty,\end{aligned}$$

which shows that the integral over the vertical segment on the right goes to 0 as $R \to \infty$. A similar argument shows that the integral of f over the vertical segment on the left also goes to 0 as $R \to \infty$. Finally, we see that if I denotes the integral we wish to calculate, then the integral of f over the top side of the rectangle (with the orientation from right to left) is simply $-e^{4\pi\xi}I$ where we have used the fact that $\cosh \pi\zeta$ is periodic with period $2i$. In the limit as R tends to infinity, the residue formula gives

$$\begin{aligned}I - e^{4\pi\xi}I &= 2\pi i\left(\frac{e^{\pi\xi}}{\pi i} - \frac{e^{3\pi\xi}}{\pi i}\right) \\ &= -2e^{2\pi\xi}(e^{\pi\xi} - e^{-\pi\xi}),\end{aligned}$$

and since $1 - e^{4\pi\xi} = -e^{2\pi\xi}(e^{2\pi\xi} - e^{-2\pi\xi})$, we find that

$$I = 2\frac{e^{\pi\xi} - e^{-\pi\xi}}{e^{2\pi\xi} - e^{-2\pi\xi}} = 2\frac{e^{\pi\xi} - e^{-\pi\xi}}{(e^{\pi\xi} - e^{-\pi\xi})(e^{\pi\xi} + e^{-\pi\xi})} = \frac{2}{e^{\pi\xi} + e^{-\pi\xi}} = \frac{1}{\cosh \pi\xi}$$

as claimed.

A similar argument actually establishes the following formula:

$$\int_{-\infty}^{\infty} e^{-2\pi i x\xi}\frac{\sin \pi a}{\cosh \pi x + \cos \pi a}\,dx = \frac{2\sinh 2\pi a\xi}{\sinh 2\pi\xi}$$

whenever $0 < a < 1$, and where $\sinh z = (e^z - e^{-z})/2$. We have proved above the particular case $a = 1/2$. This identity can be used to determine an explicit formula for the Poisson kernel for the strip (see Problem 3 in Chapter 5 of Book I), or to prove the sum of two squares theorem, as we shall see in Chapter 10.

3 Singularities and meromorphic functions

Returning to Section 1, we see that we have described the analytical character of a function near a pole. We now turn our attention to the other types of isolated singularities.

Let f be a function holomorphic in an open set Ω except possibly at one point z_0 in Ω. If we can define f at z_0 in such a way that f becomes holomorphic in all of Ω, we say that z_0 is a **removable** singularity for f.

Theorem 3.1 (Riemann's theorem on removable singularities) *Suppose that f is holomorphic in an open set Ω except possibly at a point z_0 in Ω. If f is bounded on $\Omega - \{z_0\}$, then z_0 is a removable singularity.*

Proof. Since the problem is local we may consider a small disc D centered at z_0 and whose closure is contained in Ω. Let C denote the boundary circle of that disc with the usual positive orientation. We shall prove that if $z \in D$ and $z \neq z_0$, then under the assumptions of the theorem we have

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta. \tag{4}$$

Since an application of Theorem 5.4 in the previous chapter proves that the right-hand side of equation (4) defines a holomorphic function on *all* of D that agrees with $f(z)$ when $z \neq z_0$, this give us the desired extension.

To prove formula (4) we fix $z \in D$ with $z \neq z_0$ and use the familiar toy contour illustrated in Figure 4.

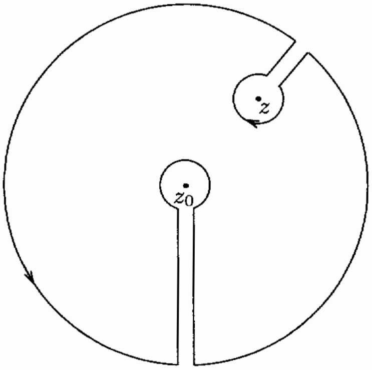

Figure 4. The multiple keyhole contour in the proof of Riemann's theorem

The multiple keyhole avoids the two points z and z_0. Letting the sides of the corridors get closer to each other, and finally overlap, in the limit we get a cancellation:

$$\int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta + \int_{\gamma_\epsilon} \frac{f(\zeta)}{\zeta - z}\, d\zeta + \int_{\gamma'_\epsilon} \frac{f(\zeta)}{\zeta - z}\, d\zeta = 0,$$

where γ_ϵ and γ'_ϵ are small circles of radius ϵ with negative orientation and centered at z and z_0 respectively. Copying the argument used in the proof of the Cauchy integral formula in Section 4 of Chapter 2, we find that

$$\int_{\gamma_\epsilon} \frac{f(\zeta)}{\zeta - z}\, d\zeta = -2\pi i f(z).$$

For the second integral, we use the assumption that f is bounded and that since ϵ is small, ζ stays away from z, and therefore

$$\left| \int_{\gamma'_\epsilon} \frac{f(\zeta)}{\zeta - z}\, d\zeta \right| \leq C\epsilon.$$

Letting ϵ tend to 0 proves our contention and concludes the proof of the extension formula (4).

Surprisingly, we may deduce from Riemann's theorem a characterization of poles in terms of the behavior of the function in a neighborhood of a singularity.

Corollary 3.2 *Suppose that f has an isolated singularity at the point z_0. Then z_0 is a pole of f if and only if $|f(z)| \to \infty$ as $z \to z_0$.*

Proof. If z_0 is a pole, then we know that $1/f$ has a zero at z_0, and therefore $|f(z)| \to \infty$ as $z \to z_0$. Conversely, suppose that this condition holds. Then $1/f$ is bounded near z_0, and in fact $1/|f(z)| \to 0$ as $z \to z_0$. Therefore, $1/f$ has a removable singularity at z_0 and must vanish there. This proves the converse, namely that z_0 is a pole.

Isolated singularities belong to one of three categories:

- Removable singularities (f bounded near z_0)
- Pole singularities ($|f(z)| \to \infty$ as $z \to z_0$)
- Essential singularities.

By default, any singularity that is not removable or a pole is defined to be an **essential singularity**. For example, the function $e^{1/z}$ discussed at the very beginning of Section 1 has an essential singularity at

0. We already observed the wild behavior of this function near the origin. Contrary to the controlled behavior of a holomorphic function near a removable singularity or a pole, it is typical for a holomorphic function to behave erratically near an essential singularity. The next theorem clarifies this.

Theorem 3.3 (Casorati-Weierstrass) *Suppose f is holomorphic in the punctured disc $D_r(z_0) - \{z_0\}$ and has an essential singularity at z_0. Then, the image of $D_r(z_0) - \{z_0\}$ under f is dense in the complex plane.*

Proof. We argue by contradiction. Assume that the range of f is not dense, so that there exists $w \in \mathbb{C}$ and $\delta > 0$ such that

$$|f(z) - w| > \delta \quad \text{for all } z \in D_r(z_0) - \{z_0\}.$$

We may therefore define a new function on $D_r(z_0) - \{z_0\}$ by

$$g(z) = \frac{1}{f(z) - w},$$

which is holomorphic on the punctured disc and bounded by $1/\delta$. Hence g has a removable singularity at z_0 by Theorem 3.1. If $g(z_0) \neq 0$, then $f(z) - w$ is holomorphic at z_0, which contradicts the assumption that z_0 is an essential singularity. In the case that $g(z_0) = 0$, then $f(z) - w$ has a pole at z_0 also contradicting the nature of the singularity at z_0. The proof is complete.

In fact, Picard proved a much stronger result. He showed that under the hypothesis of the above theorem, the function f takes on every complex value infinitely many times with at most one exception. Although we shall not give a proof of this remarkable result, a simpler version of it will follow from our study of entire functions in a later chapter. See Exercise 11 in Chapter 5.

We now turn to functions with only isolated singularities that are poles. A function f on an open set Ω is **meromorphic** if there exists a sequence of points $\{z_0, z_1, z_2, \ldots\}$ that has no limit points in Ω, and such that

(i) the function f is holomorphic in $\Omega - \{z_0, z_1, z_2, \ldots\}$, and

(ii) f has poles at the points $\{z_0, z_1, z_2, \ldots\}$.

It is also useful to discuss functions that are meromorphic in the extended complex plane. If a function is holomorphic for all large values of

z, we can describe its behavior at infinity using the tripartite distinction we have used to classify singularities at finite values of z. Thus, if f is holomorphic for all large values of z, we consider $F(z) = f(1/z)$, which is now holomorphic in a deleted neighborhood of the origin. We say that f has a **pole at infinity** if F has a pole at the origin. Similarly, we can speak of f having an **essential singularity at infinity**, or a **removable singularity** (hence holomorphic) **at infinity** in terms of the corresponding behavior of F at 0. A meromorphic function in the complex plane that is either holomorphic at infinity or has a pole at infinity is said to be **meromorphic in the extended complex plane**.

At this stage we return to the principle mentioned at the beginning of the chapter. Here we can see it in its simplest form.

Theorem 3.4 *The meromorphic functions in the extended complex plane are the rational functions.*

Proof. Suppose that f is meromorphic in the extended plane. Then $f(1/z)$ has either a pole or a removable singularity at 0, and in either case it must be holomorphic in a deleted neighborhood of the origin, so that the function f can have only finitely many poles in the plane, say at $z_1, \ldots, z_n$. The idea is to subtract from f its principal parts at all its poles including the one at infinity. Near each pole $z_k \in \mathbb{C}$ we can write

$$f(z) = f_k(z) + g_k(z)\,,$$

where $f_k(z)$ is the principal part of f at z_k and g_k is holomorphic in a (full) neighborhood of z_k. In particular, f_k is a polynomial in $1/(z - z_k)$. Similarly, we can write

$$f(1/z) = \tilde{f}_\infty(z) + \tilde{g}_\infty(z)\,,$$

where $\tilde{g}_\infty$ is holomorphic in a neighborhood of the origin and $\tilde{f}_\infty$ is the principal part of $f(1/z)$ at 0, that is, a polynomial in $1/z$. Finally, let $f_\infty(z) = \tilde{f}_\infty(1/z)$.

We contend that the function $H = f - f_\infty - \sum_{k=1}^{n} f_k$ is entire and bounded. Indeed, near the pole z_k we subtracted the principal part of f so that the function H has a removable singularity there. Also, $H(1/z)$ is bounded for z near 0 since we subtracted the principal part of the pole at ∞. This proves our contention, and by Liouville's theorem we conclude that H is constant. From the definition of H, we find that f is a rational function, as was to be shown.

Note that as a consequence, a rational function is determined up to a multiplicative constant by prescribing the locations and multiplicities of its zeros and poles.

The Riemann sphere

The extended complex plane, which consists of $\mathbb{C}$ and the point at infinity, has a convenient geometric interpretation, which we briefly discuss here.

Consider the Euclidean space $\mathbb{R}^3$ with coordinates (X, Y, Z) where the XY-plane is identified with $\mathbb{C}$. We denote by $\mathbb{S}$ the sphere centered at $(0, 0, 1/2)$ and of radius $1/2$; this sphere is of unit diameter and lies on top of the origin of the complex plane as pictured in Figure 5. Also, we let $\mathcal{N} = (0, 0, 1)$ denote the north pole of the sphere.

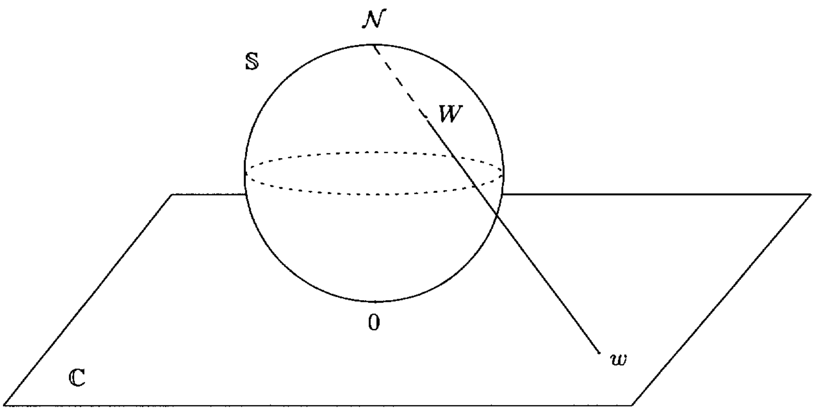

Figure 5. The Riemann sphere $\mathbb{S}$ and stereographic projection

Given any point $W = (X, Y, Z)$ on $\mathbb{S}$ different from the north pole, the line joining $\mathcal{N}$ and W intersects the XY-plane in a single point which we denote by $w = x + iy$; w is called the **stereographic projection** of W (see Figure 5). Conversely, given any point w in $\mathbb{C}$, the line joining $\mathcal{N}$ and $w = (x, y, 0)$ intersects the sphere at $\mathcal{N}$ and another point, which we call W. This geometric construction gives a bijective correspondence between points on the punctured sphere $\mathbb{S} - \{\mathcal{N}\}$ and the complex plane; it is described analytically by the formulas

$$x = \frac{X}{1-Z} \quad \text{and} \quad y = \frac{Y}{1-Z},$$

giving w in terms of W, and

$$X = \frac{x}{x^2+y^2+1}, \quad Y = \frac{y}{x^2+y^2+1}, \quad \text{and} \quad Z = \frac{x^2+y^2}{x^2+y^2+1}$$

giving W in terms of w. Intuitively, we have wrapped the complex plane onto the punctured sphere $\mathbb{S} - \{\mathcal{N}\}$.

As the point w goes to infinity in $\mathbb{C}$ (in the sense that $|w| \to \infty$) the corresponding point W on $\mathbb{S}$ comes arbitrarily close to $\mathcal{N}$. This simple observation makes $\mathcal{N}$ a natural candidate for the so-called "point at infinity." Identifying infinity with the point $\mathcal{N}$ on $\mathbb{S}$, we see that the extended complex plane can be visualized as the full two-dimensional sphere $\mathbb{S}$; this is the **Riemann sphere**. Since this construction takes the unbounded set $\mathbb{C}$ into the compact set $\mathbb{S}$ by adding one point, the Riemann sphere is sometimes called the **one-point compactification** of $\mathbb{C}$.

An important consequence of this interpretation is the following: although the point at infinity required special attention when considered separately from $\mathbb{C}$, it now finds itself on equal footing with all other points on $\mathbb{S}$. In particular, a meromorphic function on the extended complex plane can be thought of as a map from $\mathbb{S}$ to itself, where the image of a pole is now a tractable point on $\mathbb{S}$, namely $\mathcal{N}$. For these reasons (and others) the Riemann sphere provides good geometrical insight into the structure of $\mathbb{C}$ as well as the theory of meromorphic functions.

4 The argument principle and applications

We anticipate our discussion of the logarithm (in Section 6) with a few comments. In general, the function $\log f(z)$ is "multiple-valued" because it cannot be defined unambiguously on the set where $f(z) \neq 0$. However it is to be defined, it must equal $\log |f(z)| + i \arg f(z)$, where $\log |f(z)|$ is the usual real-variable logarithm of the positive quantity $|f(z)|$ (and hence is defined unambiguously), while $\arg f(z)$ is some determination of the argument (up to an additive integral multiple of 2π). Note that in any case, the derivative of $\log f(z)$ is $f'(z)/f(z)$ which is single-valued, and the integral

$$\int_\gamma \frac{f'(z)}{f(z)}\, dz$$

can be interpreted as the change in the argument of f as z traverses the curve γ. Moreover, assuming the curve is closed, this change of argument is determined entirely by the zeros and poles of f inside γ. We now formulate this fact as a precise theorem.

We begin with the observation that while the additivity formula

$$\log(f_1 f_2) = \log f_1 + \log f_2$$

fails in general (as we shall see below), the additivity can be restored to the corresponding derivatives. This is confirmed by the following