Probability

Tuesday, September 03, 2019 8:24 AM

$8^{\underline{40}} - 9^{\underline{55}}$
$10^{\underline{10}} - 11^{\underline{25}}$ } 15 min break

Quiz is on September 24, $10^{\underline{10}}$

one page of notes for every test, keep the old ones & add a page every time. Double sided.

Midterm is on October 1 (the entire class time, $8^{\underline{40}} - 11^{\underline{25}}$ am)

Final is on October 17 (the entire class time, $8^{\underline{40}} - 11^{\underline{25}}$ am).

My office hours will be on Tuesdays $11^{\underline{30}} - 1^{\underline{30}}$, SSW 1017 & online

TA office hours will be announcement soon.

One of the main objectives of a statistician is to draw conclusions (or inference) about a population by conducting an experiment.

examples 1) Toss 2 coins, and record the outcome, that is we note the upfacing sides

H = heads, T = tails

$\{HT, HH, TH, TT\}$

2) Observe an outcome of a 7 horse race, label horses 1, 2, ..., 7

(3 1 7 2 5 6 4), ...

3) Recording the lifetime of a transistor in hours, so

22.3, 40.7, ...

any positive real number

$\mathbb{R}_+$ = positive real half line.

Def An experiment is any action or process that generates observations.

Def The set of all possible outcomes of an experiment is called the sample space, S.

examples 1) 2 coin example

$S = \{(HH)(HT)(TH)(TT)\}$

4 possible outcomes.

2) 7 horse race

S = { any permutation of 7 labels }

7! possible outcomes in this S.

3) Transistor lifetime

$S = \mathbb{R}_+ = [0, \infty)$

# of outcomes can not be enumerated

Assume for now that S, the sample space, is known in advance.

Def An event: E, F, G is any collection of outcomes in the sample space, any subset of the sample space.

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examples 1) 2 coins

E = there is at least one tail up among 2 coins.

$E = \{ (HT) (TH) (TT) \}$

2) 7 horse race

E = horse #3 comes in 1st

$E = \{ (3 \underbrace{\ldots\ldots}_{\text{any permutation of the other 6 labels}}) \}$

$6!$ outcomes in E.

3) Transistor lifetime

E = "transistor survives at least 5 hours"

$E = [5, \infty)$. //

Quick Review of Basic Set Theory.

1) $E \cup F = \{ x \in S : x \in E$ or $x \in F$ or both $\}$

"union" = union of events E and F.

2) $E \cap F = \{ x \in S : x \in E$ and $x \in F \}$

"intersection" = intersection of events E and F

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examples 1) 2 coins

E = first coin comes up heads,

F = first coin comes up tails.

$E = \{(HH)(HT)\}$, $F = \{(TH)(TT)\}$

$E \cup F = S$

$E \cap F = \emptyset \overset{\text{def}}{\equiv}$ "empty set" =

= set that does not contain any elements

$\emptyset \subset F$ - subset of any other set.

"

"belongs to" or "subset of"

left hand side

$\in$ ← used when LHS is one outcome

$\subset$ ← used when LHS is a set

For any event $F \subset S$. } always.

$E \cap F \subset E \cup F$

Def If E and F are events such that $E \cap F = \emptyset$ then E and F are mutually exclusive or disjoint.

Let $F_1, F_2, \ldots, F_n, \ldots$ be a sequence of events

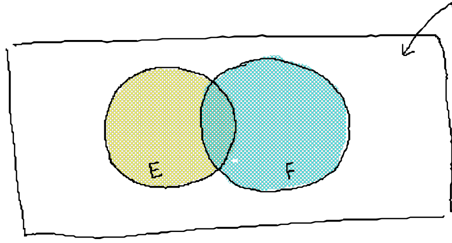
Union : $\bigcup_{n=1}^{\infty} F_n = \{ x \in S : x \in F_n$ for at least one $n \in \mathbb{N} \}$

natural numbers

Intersection : $\bigcap_{n=1}^{\infty} F_n = \{ x \in S : x \in F_n$ for all $n \in \mathbb{N} \}$

Venn Diagram

points inside represent S



= E

= F

$E \cup F$ = anything colored

$E \cup F$ = anything colored

$E \cap F$ =

Several Laws:

1) Commutative Law: $E \cup F = F \cup E$

$E \cap F = F \cap E$

2) Associative Law: $(E \cup F) \cup G =$

$= E \cup (F \cup G)$

$(E \cap F) \cap G =$

$= E \cap (F \cap G)$

3) Distributive Law:

① $(E \cup F) \cap G = (E \cap G) \cup (F \cap G)$

② $(E \cap F) \cup G = (E \cup G) \cap (F \cup G)$

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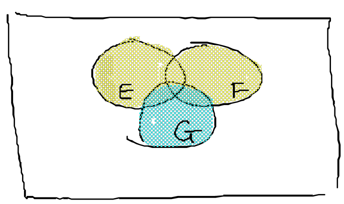
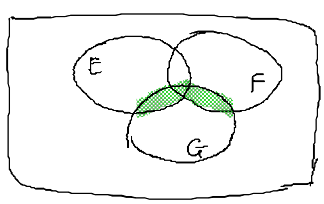
①

LHS:



$= (E \cup F) \cap G$

RHS:



$= (E \cap G) \cup (F \cap G)$

$F \cup F = F \cap F = F$
$\emptyset \subset F \subset S$

"any" $F \equiv$

$\equiv \forall F$

De Morgen's Laws

"does not belong to"

Def $E^c = \{ x \in S : x \notin E \}$ is

the compliment of event E.

example $S^c = \emptyset$, $\emptyset^c = S$

$$\left.\begin{array}{l} 1) \quad \left(\bigcup_{n=1}^{\infty} E_n \right)^c = \bigcap_{n=1}^{\infty} E_n^c \\ 2) \quad \left(\bigcap_{n=1}^{\infty} E_n \right)^c = \bigcup_{n=1}^{\infty} E_n^c \end{array} \right\} \text{De Morgen's Laws}$$

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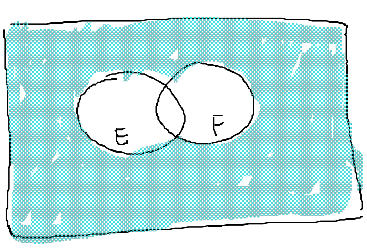
example 1) $(E \cup F)^c$

$E_1 = E$, $E_2 = F$, $E_3 = E_4 = \ldots = \emptyset = \ldots$

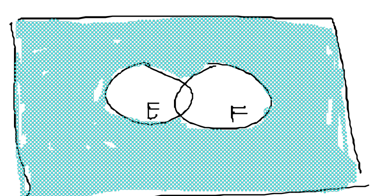
$$(E \cup F)^c = \left(\bigcup_{n=1}^{\infty} E_n \right)^c = \bigcap_{n=1}^{\infty} E_n^c =$$

$$= E^c \cap F^c \cap S \cap S \cap \ldots$$

$$= E^c \cap F^c \quad //$$



$= (E \cup F)^c = E^c \cap F^c$



Rigorous proof that $(E \cup F)^c = E^c \cap F^c$

Rigorous proof that $(E \cup F)^c = E^c \cap F^c$

1) $(E \cup F)^c \subset E^c \cap F^c$ and

2) $E^c \cap F^c \subset (E \cup F)^c$

Proof 1) Suppose $x \in (E \cup F)^c \Rightarrow$

$\Rightarrow x \notin E \cup F \Rightarrow x \notin E$ and $x \notin F$

$\Rightarrow x \in E^c$ and $x \in F^c \Rightarrow$

$\Rightarrow x \in E^c \cap F^c$.

2) Suppose $x \in E^c \cap F^c \Rightarrow$

$\Rightarrow x \in E^c$ and $x \in F^c \Rightarrow$

$\Rightarrow x \notin E$ and $x \notin F \Rightarrow$

$\Rightarrow x \notin E \cup F \Rightarrow x \in (E \cup F)^c$. //

Axioms of Probability

Def $P(\cdot)$ = probability function

is a function defined on the subsets of the sample space S such that

axiom 1 : $0 \le P(E) \le 1$ for $\forall E \subset S$

axiom 2 : $P(S) = 1$.

axiom 3 : For any sequence $E_1, E_2, \ldots E_n, \ldots$

of events that mutually exclusive

$E_i \cap E_j = \phi$ for any $i \neq j$

$$P\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} P(E_n).$$

So, we say that $P(E)$ is the probability of event E.

Some simple implications.

(properties of probability functions)

1) $\phi \cap S = \phi \Rightarrow \phi$ and S mutually exclusive

Moreover, $\phi \cup S = S$

$$1 \overset{\text{axiom 2}}{=} P(S) = P(\phi \cup S) \overset{\text{axiom 3}}{=}$$

convince yourself that axiom 3 works for 2 events

$$= P(\phi) + P(S) = P(\phi) + 1$$

$$\Rightarrow P(\phi) = 0$$

2) $E \cup E^c = S$, $E \cap E^c = \phi$

$$1 = P(S) = P(E \cup E^c) = P(E) + P(E^c)$$

$$\Rightarrow P(E^c) = 1 - P(E).$$

3) Suppose $E \subset F$.

$$F = F \cap S = F \cap (E \cup E^c) =$$

$$= (F \cap E) \cup (F \cap E^c)$$

$$= E \cup (F \cap E^c)$$

disjoint

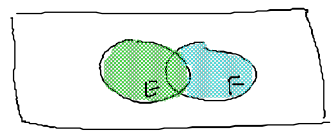
$$P(F) = P(E \cup (F \cap E^c)) =$$

$$= P(E) + \underbrace{P(F \cap E^c)}_{\geq 0}$$

$\Rightarrow$ If $E \subset F$ then $P(E) \leq P(F)$

4) $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

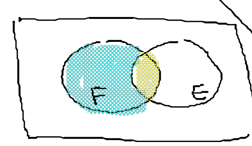
$$E \cup F = E \cup (F \cap E^c)$$



$$P(E \cup F) = P(E \cup (F \cap E^c)) =$$

$$= P(E) + P(F \cap E^c)$$

$$F = (F \cap E) \cup (F \cap E^c)$$



$$P(F) = P(F \cap E) + P(F \cap E^c)$$

$$\Rightarrow P(F \cap E^c) = P(F) - P(F \cap E)$$

$$P(E \cup F) = P(E) + P(F) - P(F \cap E) \quad //$$

<u>examples</u> 1) 2 coin example

$$S = \{(HH)(HT)(TH)(TT)\}$$

$$E_1 = HH, \quad E_2 = HT, \quad E_3 = TH, \quad E_4 = TT$$

$$P(S) = 1$$

$$S = E_1 \cup E_2 \cup E_3 \cup E_4$$

$$\Rightarrow \quad 1 = P(S) = P(E_1 \cup E_2 \cup E_3 \cup E_4)$$

$$= P(E_1) + P(E_2) + P(E_3) + P(E_4)$$

If coins are fair then we can assume that $P(E_1) = P(E_2) = P(E_3) = P(E_4) = \frac{1}{4}$

E = 1st coin comes up heads $= \{(HH)(HT)\}$

F = 2nd coin comes up heads $= \{(HH)(TH)\}$

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

$$P(E) = P(E_1 \cup E_2) = P(E_1) + P(E_2) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$P(F) = P(E_1 \cup E_3) = P(E_1) + P(E_3) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$P(E \cap F) = P(E_1) = \frac{1}{4}$$

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

$$= \frac{1}{2} + \frac{1}{2} - \frac{1}{4} = 3/4$$

$$E \cup F = \{(HH)(HT)(TH)\} = E_1 \cup E_2 \cup E_3$$

$$\Rightarrow \quad P(E \cup F) = P(E_1) + P(E_2) + P(E_3) = 3/4 \quad //$$

$$\Rightarrow \quad P(E \cup F) = P(E_1) + P(E_2) + P(E_3) = 3/4 \quad //$$

<u>example</u> Probability that a randomly selected person subscribes to any of 2 newspapers (or both) is .8. The chance of subscribing to 1^{st} paper is $1/2$, the chance of subscribing to 2^{nd} is .6.

What is the chance that a randomly selected person subcribes to both papers?

A_1 = event that a randomly selected person subcribes to the 1^{st} newspaper.

A_2 = event that a randomly selected person subscribes to the 2^{nd} newspaper.

$P(A_1) = \frac{1}{2}$, $P(A_2) = .6$

$P(A_1 \cup A_2) = .8$

$$P(A_1 \cap A_2) = P(A_1) + P(A_2) - P(A_1 \cup A_2)$$

$$= .5 + .6 - .8 = .3 \quad //$$

Sample Spaces with Equally Likely Outcomes

$S = \{1, 2, \ldots, N\}$ (finitely many outcomes)

Often natural to assume that all outcomes are equally likely

$$E_1 = \{1\},\quad E_2 = \{2\}, \ldots,\quad E_N = \{N\}$$

$$S = \bigcup_{n=1}^{N} E_n$$

$$1 = P(S) = P\left(\bigcup_{n=1}^{N} E_n\right) = \sum_{n=1}^{N} P(E_n)$$

If $P(E_1) = P(E_2) = \ldots = P(E_N) = \frac{1}{N}$

$$P(E) = P\left(\bigcup_{\substack{n:\ \text{outcomes} \\ \text{in } E}} E_n\right) = \sum_{\substack{n:\ \text{outcomes} \\ \text{in } E}} P(E_n)$$

$$= \frac{\#\ \text{outcomes in } E}{N} = \frac{\#\ \text{outcomes in } E}{\#\ \text{outcomes in } S} \quad //$$