PROBLEM 1

Prove that there exist two distinct natural numbers m and n with m, $n \leq 2049$ such that 19^m - 19^n is divisible by 2019.
The truth of this claim does not depend on the exact values 19, 2019, and 2049. So, your solution should give an argument that is easy to adapt to other values for these constants rather than, say, using a computer program to find specific values for m and n.

Claim: There exist two distinct natural numbers m and n with m, $n \leq 2049$ such that 19^m - 19^n is divisible by 2019.

(Pigeonhole Principle)

Proof: There are 2019 different values that $19^m - 19^n$ mod 2019 could have.
Therefore, some $19^m - 19^n$ mod 2019 $= 19^{m'} - 19^{n'}$ mod 2019.

$$19^m - 19^n - 19^{m'} - 19^{n'} \equiv 0$$

There are 2049 choices for m and 2019 possible values of 19^m,
so there must be some m, $n \leq 2049$ such that
$19^m \equiv 19^n$ mod 2019 → $19^m - 19^n \equiv 0$ mod 2019.
Thus, $19^m - 19^n$ is divisible by 2019.

1 Problem 1 **16.5 / 22**

- **- 0 pts** Correct
- **- 5.5 pts** Incorrectly identified pigeons
- ✓ **- 5.5 pts Incorrectly identified pigeonholes**
- **- 22 pts** No answer
- **- 11 pts** Incorrect final reasoning
- **- 2.75 pts** Identified two sets correctly but the remainders are the pigeonholes and the possible values of 19^i are the pigeons
- **- 3 pts** Incorrectly treated the statement that had to be proven as an assumption
- **- 5 pts** Didn't quite follow through on the pigeonhole argument
- **- 5 pts** The gist of the proof is correct, but you included some false statements
- **- 19 pts** Some reasoning, but no answer

PROBLEM 2

Determine whether the following sets are finite, countably infinite, or uncountable. Justify your answers.
(A) The set of all total functions from domain $\{0, 1\}$ to co-domain $\{0, 1\}$
(B) The set of all total functions from domain $\mathbb{N}$ to co-domain $\{0, 1\}$
(C) The set of all total functions from domain $\{0, 1\}$ to co-domain $\mathbb{N}$

A. The set of all total functions from domain $\{0, 1\}$ to co-domain $\{0, 1\}$ are finite because it contains a limited number of elements and thus it is possible to list every element.

B. The set of all total functions from domain $\mathbb{N}$ to co-domain $\{0, 1\}$ are uncountable because it is not possible to list all $\mathbb{N}$ numbers.

C. The set of all total functions from domain $\{0, 1\}$ to co-domain $\mathbb{N}$ are countably infinite because there is a bijection between all points.

2 Problem 2 **18 / 27**

- **- 0 pts** Correct
- **- 4.5 pts** A incorrect (answer should be "finite")
- **- 4.5 pts** A explanation missing / unclear
- **- 2 pts** A) partially correct explanation
- **- 4.5 pts** B incorrect (answer should be "uncountable")
- ✓ **- 4.5 pts B explanation missing / unclear**
- **- 2 pts** B) partially correct explanation
- **- 4.5 pts** C incorrect (answer should be "countable")
- ✓ **- 4.5 pts C explanation missing / unclear**
- **- 2 pts** C) partially correct explanation
- **- 2 pts** C) Restating the in class problem solution is not sufficient

PROBLEM 3

Prove by contradiction that $\sqrt{3} + \sqrt{2}$ is irrational. ***Hint:*** **Consider ($\sqrt{3} + \sqrt{2}$) ($\sqrt{3}$ - $\sqrt{2}$)**

Claim: $\sqrt{3} + \sqrt{2}$ is irrational

Proof by Contradiction:

Suppose $\sqrt{3} + \sqrt{2}$ is rational.
Since the product of any rational number multiplied by any irrational number is an irrational number, then
($\sqrt{3} + \sqrt{2}$) ($\sqrt{3}$ - $\sqrt{2}$) = irrational number.
However, ($\sqrt{3} + \sqrt{2}$) ($\sqrt{3}$ - $\sqrt{2}$) $= 3 - 2 = 1$,
which is a rational number. This is a contradiction.
Therefore, $\sqrt{3} + \sqrt{2}$ is irrational.

3 Problem 3 **4 / 22**

- **- 0 pts** Correct
- **- 1 pts** Minor typo in proof
- **- 1 pts** Minor order of operations error
- **- 2 pts** Leaps in logic while showing a contradiction
- **- 2 pts** Incorrect justification of a statement by closure of the integers or rational numbers
- **- 2 pts** Minor incorrect usage of the initial assumption for proof by contradiction
- **- 3 pts** Did not specify that $$p$$ and $$q$$ (or other integer numerators and denominators) are integers
- **- 4 pts** Did not explain why $$\sqrt{3} - \sqrt{2}$$ is rational (because it is equal to the reciprocal of the fractional representation of a rational number)
- **- 4 pts** Did not explain why the sum of two rational numbers is rational (by closure of the integers under addition and multiplication)
- **- 3 pts** Minor assumptions that require more explanation
- **- 7 pts** Major assumptions that require more explanation or are incorrect
- **- 7 pts** Incorrect/missing assumption for purposes of contradiction
- **- 4 pts** Arithmetic/algebraic errors
- **- 2 pts** Clearly state what is being assumed for the sake of contradiction.
- ✓ **- 18 pts Incorrect proof**
- **- 7 pts** need a lemma to prove sqrt(6) is irrational
- **- 4 pts** need a lemma to show that rationals are closed under multiplication

PROBLEM 4

Recall that we saw that that the set $\mathbb{R}_{(0,1)} = \{r \in \mathbb{R}: < 0 < r < 1\}$ is uncountably infinite using Cantor's diagonalization method.

(A) The Schroder-Bernstein Theorem states that for sets S and T, if there exist injective functions $f: S \rightarrow T$ and $g: T \rightarrow S$, then S and T have the same cardinality. Using the Schroder-Bernstein Theorem show that the cardinality of the set of real numbers in the closed interval [0, 1] is the same as the cardinality of the set of all real numbers in the open interval (0, 1). To receive full credit on this problem you must formally define two total injective functions $f : \mathbb{R}_{(0,1)} \rightarrow \mathbb{R}_{[0,1]}$ and $g : \mathbb{R}_{[0,1]} \rightarrow \mathbb{R}_{(0,1)}$
(B) Using what we have proved about intervals of the real number line, prove that there are at least a countably infinite number of uncountably infinite sets.

A) In general, to show that two sets have the same cardinality, show that they are $1-1$ functions, which means there is a bijection between the two sets.

More specifically, we need to show that there exists a bijection between the two sets, [0, 1] and (0, 1).

Suppose, $|\mathbb{N}| = |\mathbb{Z}|$
then
$f: \mathbb{R}_{(0,1)} \rightarrow \mathbb{R}_{[0,1]}$ is $\mathbb{N} \rightarrow \mathbb{Z}$ where $f(x) = x$
$g: \mathbb{R}_{[0,1]} \rightarrow \mathbb{R}_{(0,1)}$ is $\mathbb{Z} \rightarrow \mathbb{N}$ where $g(x) = 2x$ if $x \geq 0$
$2|x| + 1$ if $x < 0$

Thus, f and g are both 1-1 functions and $|\mathbb{N}| = |\mathbb{Z}|$.

B) A set is countably infinite if it's elements can be 1-1 with the set of natural numbers, $\mathbb{N}$. A set is uncountably infinite if it's elements cannot be 1-1 with the set of integers, $\mathbb{Z}$.

Using these definitions, now suppose there exists 2 sets of numbers such that:

<u>Set 1</u>		<u>Set 2</u>
1	→	(0,1)
2	→	(1,2)
3	→	(2,3)

... and so on and so forth.

There exists a bijection from natural numbers, $\mathbb{N}$, in set 1 into the set of sets, Set 2. This means there are at least a countably infinite number (Set 2) of uncountably infinite sets (Set 1).

4.1 A **5 / 20**

- 0 pts Correct

- 5 pts Failure to prove/show that one or more of the injective functions is injective -- that is, never maps two elements of the domain to the same element of the co-domain.

- 20 pts No solution given

- 2 pts Insufficient explanation/explication of the method chosen. Why are the functions defined this way and how do they work? How can we be sure they are injective? Total?

- 3 pts Minor error in definition of one of the injective functions. Idea was correct but minor error in implementation.

- 10 pts Missing/incorrect injective function from closed interval to open interval.

- 0.5 pts Not typeset in LaTeX

✓ **- 15 pts Major error in interpretation of problem.**

- 10 pts Missing/incorrect function from open to closed interval.

- 5 pts Errors and lack of clarity in explanation.

The domain and co-domain in this problem are intervals on the real line, so not the natural numbers or the integers. The functions you define seem to be defined with two domains, which doesn't make sense and seem to be referring to inputs that are naturals or integers. Your piecewise function for the first interval would work to map the reals from (0,1) -> [0,1] because it is the identity function (if we ignore the issues w/ the domain). But the second one doesn't make sense. First of all, there are no negative inputs in the interval [0,1], so only the first case gets used. Second, it will map 0 to 0, but 0 is not in the co-domain. And it will map all elements of the domain larger than 1/2 to elements not in the co-domain. A better strategy would have been to scale the domain into a smaller interval of the co-domain. Something like g(x) = x/2 + 1/4.

PROBLEM 4

Recall that we saw that that the set $\mathbb{R}(0,1) = \{r \in \mathbb{R}: < 0 < r < 1\}$ is uncountably infinite using Cantor's diagonalization method.

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(B) Using what we have proved about intervals of the real number line, prove that there are at least a countably infinite number of uncountably infinite sets.

A) In general, to show that two sets have the same cardinality, show that they are $1-1$ functions, which means there is a bijection between the two sets.

More specifically, we need to show that there exists a bijection between the two sets, [0, 1] and (0, 1).

Suppose, $|\mathbb{N}| = |\mathbb{Z}|$
then

f : $\mathbb{R}(0,1) \rightarrow \mathbb{R}[0,1]$ is $\mathbb{N} \rightarrow \mathbb{Z}$ where $f(x) = x$

g : $\mathbb{R}[0,1] \rightarrow \mathbb{R}(0,1)$ is $\mathbb{Z} \rightarrow \mathbb{N}$ where $g(x) = \begin{cases} 2x & \text{if } x \geq 0 \\ 2|x| + 1 & \text{if } x < 0 \end{cases}$

Thus, f and g are both 1-1 functions and $|\mathbb{N}| = |\mathbb{Z}|$.

B) A set is countably infinite if it's elements can be 1-1 with the set of natural numbers, $\mathbb{N}$. A set is uncountably infinite if it's elements cannot be 1-1 with the set of integers, $\mathbb{Z}$.

Using these definitions, now suppose there exists 2 sets of numbers such that:

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... and so on and so forth.

There exists a bijection from natural numbers, $\mathbb{N}$, in set 1 into the set of sets, Set 2. This means there are at least a countably infinite number (Set 2) of uncountably infinite sets (Set 1).

4.2 B **9 / 9**

✓ **- 0 pts** **Correct**

- 9 pts No solution given. (send me a regrade request if I missed it b/c of page marking issues or some other reason)