# PSet 8

***Focus: bilinear maps and tensor products.***
***Deadline: April 14, 11:00pm ET.*** *The problems can be submitted orally to an instructor or in written form to Canvas. Problems submitted on time cost 2 points; after the deadline 1 point; and there is no partial credit.*

**Notation.** Let $M$ and $N$ be to $R$-modules. Denote by $G(M \times N)$ the free module whose basis is labeled by elements of $M \times N$:

$$G(M \times N) \stackrel{\text{def}}{=} \bigoplus_{(m,n) \in M \times N} Re_{(m,n)}.$$

Denote by $Rel(M \times N)$ the submodule of $G(M \times N)$ spanned by the following elements, for each $m, m' \in M$, $n, n' \in N$, $r \in R$:

- $e_{(m+m',n)} - e_{(m,n)} - e_{(m',n)}$;
- $e_{(m,n+n')} - e_{(m,n)} - e_{(m,n')}$;
- $e_{(rm,n)} - re_{(m,n)}$;
- $e_{(m,rn)} - re_{(m,n)}$;

**Construction.** The *tensor product* $M \otimes_R N$ (or just $M \otimes N$ when the base ring is understood) can be defined by generators and relations:

$$M \otimes_R N = G(M \times N)/Rel(M \times N).$$

**Theorem.** The above construction of tensor product satisfies the following universal property:

$$\mathrm{Hom}_R\left(M \otimes_R N, T\right) \cong \mathrm{Bil}_R\left(M, N\,;\,T\right).$$

**Definition.** Elements of the module $M \otimes_R N$ are called *tensors*. Tensors of the form $m \otimes n$ are called *elementary*, or *decomposable*.

**Properties of tensor products.** The key to staying sane when working with tensor products for the first time is to compute them not from the definition, but from elementary properties. Here is the list of most useful of these. Below, we fix a ring $R$ and a triple of $R$-modules $L$, $M$, $N$, and we write $\otimes$ meaning $\otimes_R$.

- $M \otimes N \cong N \otimes M$;
- $(L \otimes M) \otimes N \cong L \otimes (M \otimes N)$;
- $(L \oplus M) \otimes N \cong (L \otimes N) \oplus (M \otimes N)$;
- if $I$ is an ideal of $R$, then $M \otimes (R/I) \cong M/IM$;
- in particular, $R \otimes M \cong M$;
- if $x_1, \ldots, x_m$ generate $M$ and $y_1, \ldots, y_n$ generate $N$, then the set $\{x_i \otimes y_j \mid 1 \le i \le m, 1 \le j \le n\}$ generates $M \otimes N$.

1. *(Block multiplication of matrices.)* Let $R$ be a ring.

(a) Take $R$-modules $M_1, M_2$ and $N_1, N_2$. Explain how you can write homomorphisms

$$\varphi \in \mathrm{Hom}_R(M_1 \oplus M_2, N_1 \oplus N_2)$$

as $2 \times 2$ matrices with entries in $\mathrm{Hom}_R(M_i, N_j)$.

(b) Generalize the previous part to any number of $M_i$'s and $N_j$'s.

(c) Explain how multiplication on the right corresponds to column operations using block multiplication:

$$\mathrm{Hom}_R(R^n, R^m) \times \mathrm{Hom}_R(R^n, R^n) \cong \mathrm{Hom}_R(R, R^m)^{\oplus n} \times \mathrm{Hom}_R(R^n, R^n),$$

where you take $M_1 = \cdots = M_n = R$ and only one factor $N_1 = R^m$.

(d) In a similar fashion, explain how multiplication on the left corresponds to row operations.

2. *(Examples of bilinear maps.)* Prove that the following maps are bilinear:

(a) external product $\mathbb{R}^3 \times \mathbb{R}^3 \xrightarrow{\times} \mathbb{R}^3$;

(b) $\varphi \circ B$ is $R$-bilinear, where we fix a quadruple of $R$-modules $M$, $N$, $P$, $Q$, and $B : M \times N \to P$ is $R$-bilinear and $\varphi : P \to Q$ is $R$-linear;

(c) $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ that sends $(v, w) \mapsto v^T M w$, where $M$ is an $n \times n$-matrix.

3. *(Dual modules.)* Let $M$ be an $R$-module. Recall that we can define its *dual* $M^*$ as the module of homomorphisms $\mathrm{Hom}_R(M, R)$. If we repeat this, we get the *double dual* $M^{**} = \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R)$.

(a) Define a homomorphism $\mathrm{ev} : M \to M^{**}$ that sends an element $m \in M$ to the evaluation homomorphism $\mathrm{ev}_m : M^* \to R$, so that if $\varphi \in M^*$, then $\mathrm{ev}_m(\varphi) = \varphi(m)$. Prove that ev is a homomorphism of $R$-modules.

(b) If $\psi : M \to P$ is any homomorphism to a torsion-free module, prove that any torsion element $m \in \mathrm{Tors}\, M$ lies in the kernel of $\psi$.

(c) Prove that $M^*$ is torsion-free, as long as the base ring $R$ is an integral domain.

(d) Using previous parts, conclude that over an integral domain $R$, if $m \in M$ is a torsion element, then $\mathrm{ev}_m$ is the zero homomorphism: $\mathrm{ev}_m = 0 : M^* \to R$. (So $\mathrm{Tors}\, M \subset \mathrm{Ker}\, \mathrm{ev}$.)

(e) If $R$ is a field and $M$ is a finite-dimensional vector space $R$-module, prove that $\mathrm{ev} : M \to M^{**}$ is an isomorphism. *(Hint: you can prove that $M^{**}$ is a vector space of the same dimension and that* ev *is injective.)*

4. *(Examples of tensor products.)* Calculate the following tensor products (or prove given isomorphisms) over a ring $R$. In this problem, when we write $\otimes$, we mean $\otimes_R$. *Remember that you have a nice list of properties of tensor products on the front page!*

(a) $R = \mathbb{Z}$, $\mathbb{Z}/(2) \otimes \mathbb{Z}/(3)$;

(b) $R = \mathbb{Z}$, $\mathbb{Z}/(2) \otimes \mathbb{Z}/(4)$;

(c) $R = \mathbb{Z}$, $\mathbb{Z}/(n) \otimes \mathbb{Q} \cong 0$;

(d) $R$ ring, $M$ is an $R$-module, $R^n \otimes M \cong M^n$;

(e) $R$ ring, then the tensor product of two finite free modules is finite free and its rank is the product of ranks: $R^m \otimes R^n \cong R^{mn}$;

(f) $R = \mathbb{C}[x, y]$, $I = (x, y)$ is an ideal in $R$, $M \cong \mathbb{C}$ is the $R$-module where multiplication by $x$ or $y$ is zero, that is $M \cong R/I$. What is $I \otimes M$? What is its dimension as a $\mathbb{C}$-vector space?