LGIC 010/PHIL 005

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Lecture 14

PQT

Satisfaction

- Thus, we may introduce without further ado the notion of a schema S of polyadic quantification being true in (or satisfied by) a structure A that assigns extensions to all the polyadic predicate letters appearing in S (written $A \models S$).
- If S contains free variables, we must of course supplement the structure S with assignments of elements from U^A to those free variables.
- For example, we write

$$A \models S[(x|a)(y|b)]$$

  for "the structure A satisfies the schema S with respect to the assignments of a to x and b to y."
- This notation is used with the understanding that no variables other than x and y occur free in S and that $a, b \in U^A$.

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Satisfiability, Validity, Implication, and Equivalence

- With this definition of satisfaction, our definitions of satisfiability, validity, implication, and equivalence for closed monadic quantificational schemata generalize immediately to PQT.
  - A polyadic schema S is *valid* if and only if for every structure A, $A \models S$.
  - A polyadic schema S is *satisiable* if and only if for some structure A, $A \models S$.
  - A polyadic schema S *implies* a polyadic schema T if and only if for every structure A, if $A \models S$, then $A \models T$.
  - polyadic schemata S and T are equivalent if and only if S implies T, and T implies S.
- Thus, in order to show that a schema S fails to imply a schema T, it suffices to exhibit a *counterexample* to the implication, that is, a structure A such that $A \models S$ and $A \not\models T$.

Examples of Counterexamples

- Recall the four schemata we discussed in our last meeting.
  - $S_1 : \quad (\forall x)(\exists y)(Lxy)$
  - $S_2 : \quad (\exists y)(\forall x)(Lxy)$
  - $S_3 : \quad (\forall y)(\exists x)(Lxy)$
  - $S_4 : \quad (\exists x)(\forall y)(Lxy)$
- We proceed to illustrate this technique by showing that among the four schemata $S_1, \ldots, S_4$ discussed above, if $i \neq j$, then S_i does not imply S_j except in case $i = 2$ and $j = 1$, or $i = 3$ and $j = 4$.
- We begin by specifying three structures A, B, C which act as a counterexamples to various of these implications.
- First we let $U^A = U^B = U^C = \{1, 2\}$.
- We then specify the extension of L in each structure as follows.
  - $L^A = \{\langle 1, 1\rangle, \langle 2, 2\rangle\}$.
  - $L^B = \{\langle 2, 2\rangle, \langle 1, 2\rangle\}$.
  - $L^C = \{\langle 2, 2\rangle, \langle 2, 1\rangle\}$.

Examples of Counterexamples

- Note that $A \models S_1$ and $A \models S_3$, while $A \not\models S_2$ and $A \not\models S_4$, from which it follows, by definition, that S_1 does not imply S_2, nor does S_3 imply S_4.
- Moreover $B \models S_2$, but $B \not\models S_3$, and $C \models S_4$, but $C \not\models S_1$; thus S_2 does not imply S_3, and S_4 does not imply S_1.
- Failure of the remaining implications now follows.
- For example, S_1 does not imply S_4.
- To see this, suppose, *ad reductio*, that S_1 implies S_4.
- Then since S_2 implies S_1 and S_4 implies S_3, it follows, by the transitivity of implication, that S_2 implies S_3.
- But we have already seen that S_2 does not imply S_3 (B was the counterexample), a contradiction.

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Summary of Implications

- We summarize the results of this discussion in the following matrix $\langle a_{ij} \mid 1 \leq i, j \leq 4 \rangle$, where $a_{ij} = 1$ if and only if the schema in the i-th row implies the schema in the j-th column.

S_i implies S_j	S_1	S_2	S_3	S_4
S_1	1	0	0	0
S_2	1	1	0	0
S_3	0	0	1	0
S_4	0	0	1	1

Quantificational Ambiguity

- Consider the statement,

  "everybody loves a lover."

- This statement is ambiguous, as can be seen by considering two schematizations, each of which corresponds to a natural reading of the statement.

Quantificational Ambiguity

- First, schematize "x is a lover" as $(\exists y)Lxy$.
- Now, consider the following two schemata.

$$(\forall z)(\exists x)((\exists y)Lxy \wedge Lzx)$$

$$(\forall x)((\exists y)Lxy \supset (\forall z)Lzx)$$

- The first schema corresponds to the reading "everybody loves someone who is a lover," while the second corresponds to the reading "if someone is a lover, then everybody loves her."

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Quantificational Ambiguity

- Observe that a structure A satisfies the second schema if and only if either L^A is empty or $L^A = U^A \times U^A$, the cartesian product of the universe of A with itself.
- On the other hand, if a structure B satisfies the first schema, then L^B is non-empty; moreover, if B consists of a pair of requiting lovers at least one of whom is not a narcissist, B satisfies the first, but not the second, schema.
- Thus, neither disambiguation of the original sentence implies the other.

Binary Relations: Some Properties

Definition

L^A is *reflexive* if and only if

$$A \models (\forall x)Lxx.$$

Definition

L^A is *irreflexive* if and only if

$$A \models (\forall x)\neg Lxx.$$

Definition

L^A is *symmetric* if and only if

$$A \models (\forall x)(\forall y)(Lxy \supset Lyx).$$

Binary Relations: Some Properties

Definition

L^A is *asymmetric* if and only if

$$A \models (\forall x)(\forall y)(Lxy \supset \neg Lyx).$$

Definition

L^A is *transitive* if and only if

$$A \models (\forall x)(\forall y)(\forall z)(Lxy \supset (Lyz \supset Lxz)).$$

Definition

A is a *simple graph* if and only if L^A is irreflexive and symmetric.

Identity and Numerical Quantification

- The identity relation "$=$" has a uniform interpretation over all structures A namely $=^A$ is equal to $\{\langle a, a\rangle \mid a \in U^A\}$.
- For each integer $k \geq 1$, the quantifiers "there are at least k x's such that $S(x)$", "there are at most k x's such that $S(x)$", and "there are exactly k x's such that $S(x)$" as follows.

$$
\begin{array}{ll}
(\exists^{k\leq} x)S(x): & (\exists x_1)\ldots(\exists x_k)(\bigwedge_{1\leq i<j\leq k} x_i \neq x_j \wedge \bigwedge_{1\leq i\leq k} S(x_i)) \\
(\exists^{\leq k} x)S(x): & \neg(\exists^{k+1\leq} x)S(x) \\
(\exists^{=k} x)S(x): & (\exists^{\leq k} x)S(x) \wedge (\exists^{k\leq} x)S(x)
\end{array}
$$

- Let S be a schema with a single free variable x. $S[A]$ is the set of members of U^A that satisfy $S(x)$ in A, that is,

$$[A] = \{a \in U^A \mid A \models S[x|a]\}.$$

- Observe that $A \models (\exists^{k\leq} x)S(x)$ if and only if $k \leq |S[A]|$, and similarly for the other two numerical quantifiers.

k-regular Simple Graphs

Definition

A simple graph A is *k-regular* if and only if

$$A \models (\forall y)(\exists^{=k} x))Lyx.$$

Examples

- A 1-regular simple graph consists of a "perfect matching" of the nodes - each node has exactly one neighbor.
- It follows at once that if a finite simple graph A is 1-regular, then $|U^A|$ is even.
- A finite 2-regular simple graph consists of a disjoint union of simple cycles.