MATH 410, FALL 2020: HOMEWORK 9

Due: <u>November 24 before class (1:30 pm US eastern time)</u>

Read Sections 7.1, 13.1, and 13.2 in the textbook. You can use any results we have covered so far from the book or from lectures for the following problems, but please state which results you are using when writing your solutions. All numbered exercises are from the the textbook.

Exercise 1. Find the images of

(1) the rectangle $[-1,1] \times [0,\pi]$ under the map $f(z) = e^z$.
(2) the circle $|z| = R$ under $f(z) = \frac{1}{z-R}$.
(3) the half disk $\{z \in \mathbb{C} \mid |z| < 1, \mathrm{Im}\ z > 0\}$ under $f(z) = -\frac{1}{2}(z + \frac{1}{z})$. (Hint: Set $f(z) = w$ and solve for z. For which values of w does this equation have a solution in the half disk?)

Exercise 2. Exercises 13.4, 13.5, 13.6.

Exercise 3. Let $\mathrm{GL}_2(\mathbb{C})$ be the group of invertible 2×2 complex matrices. Show that the map $\Phi : \mathrm{GL}_2(\mathbb{C}) \to \mathrm{Aut}\left(\overline{\mathbb{C}}\right), \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \frac{az+b}{cz+d}$ is a group homomorphism, i.e.

(1) $\Phi(AB) = \Phi(A) \circ \Phi(B)$.
(2) $\Phi(A^{-1}) = (\Phi(A))^{-1}$.

Is Φ surjective? What is $\ker \Phi := \Phi^{-1}(\mathrm{id})$? You may use the result from Ex 1 in HW 7.

Exercise 4. Find all conformal automorphisms of the first octant

$$\mathcal{O} := \left\{ z \in \mathbb{C} \backslash \{0\} \,\middle|\, 0 < \arg z < \frac{\pi}{4} \right\}.$$

Exercise 5. Let D be the unit disk $\{z \in \mathbb{C} \mid |z| < 1\}$.

(1) Show that if a conformal mapping $f : D \to D$ has two distinct fixed points, then it must be the identity mapping $f(z) = z$.
(2) Is it possible that a conformal mapping $f : D \to D$ has no fixed point?

Exercise 6. Use the Schwarz's lemma to show that if $f : D(0; R) \to \mathbb{C}$ is holomorphic with $|f(z)| \le M$ for some $M > 0$, then

$$\left| \frac{f(z) - f(0)}{M^2 - \overline{f(0)} f(z)} \right| \le \frac{|z|}{MR}.$$

Find all possible forms of $f(z)$ when the equality holds.