PROBLEM 1

Define a relation $\lhd$ on $\mathbb{Z} \times \mathbb{Z}$ by $(a, b) \lhd (c, d)$ if and only if either $a < c$ or else $a = c$ and $b \leq d$.

(A) Prove that $\lhd$ is transitive.
(B) Prove that $\lhd$ is antisymmetric.

Lexicographic Order

A. <u>Transitive</u>

If a Δ b and b Δ c, then a Δ c

Assume: For all such that 3 arbitrary points a, b, and c:

	(a1, b1) Δ (a2, b2)	Either a1 < a2 or a1 = a2 and b1 <= b2
	(a2, b2) Δ (a3, b3)	Either a2 < a3 or a2 = a3 and b2 <= b3

We are trying to prove that either a1 < a3 or a1 = a3 and b1 <= b3.

4 Cases:

- Case 1: If a1 < a2 and a2 < a3, then a1 Δ a3, which means (a1, b1) Δ (a3, b3)
- Case 2: If a1 <a2 and a2 =a2 and b2 <= b3, then a1 < a3, which means (a1, b1) Δ (a3, b3)
- Case 3: If a1 = a2 and b1 <= b2, and a2 <=a3, then a1 <a3, which means (a1, b1) Δ (a3, b3)
- Case 4: If a1 = a2 and b1 <= b2 and a2 = a3 and b2 <= b3, then a1 = a3 and b1 <= b3, which means (a1, b1) Δ (a3, b3).

In conclusion, for all (a1, b1), (a2, b2), and (a3, b3) such that (a1, b1) Δ (a2, b2) and (a2, b2) Δ (a3, b3), all cases lead to the same conclusion (a1, b1) Δ (a3, b3). Therefore, Δ is transitive.

B. <u>Antisymmetric</u>

If (a1, b1) Δ (a2, b2), then (a2, b2) not Δ (a1, b1)
Assume either a1 < a2 or a1 = a2 and b1<= b2
We want to prove that not(a2 < a1 or a2 = a1 and b2 <= b1)

In other words, we want to prove not(a2 < a1) and not(a2 = a1 and b2 <= b1), which is equivalent to a2 >= a1 and (not(a2 = a1) or not (b2<=b1)), which is equivalent to a2 >= a1 and (a2 not = a1 or b2 > b1

By assumption, we know that in either case a2 >= a1
In the first case, a2 not = 1.
In the second case, b1 < b2.
Therefore, (a2, b2) not Δ (a1, b1)
As long as (a1, b1) is not equal to (a2, b2), we have that (a2, b2) not Δ (a1, b1)
Thus, Δ is antisymmetric.

PROBLEM 2

Bulgarian solitaire is a game played by one player. The game starts with 6 coins distributed in 1-6 piles. Then the player repeats the following step:

- Remove one coin from each existing pile and form a new pile.

The order of the piles doesn't matter, so the state can be described as a sequence of positive integers in non-increasing order adding up to 6. For example, the first two moves when a player begins with two piles of 3 coins are $(3,3) \to (2,2,2)$ and $(2,2,2) \to (3,1,1,1)$. On the next move, the last three piles disappear, creating piles of 4 and 2 coins.

(A) Trace the sequence of moves starting from two initial piles of 3 until it repeats.
(B) Draw as a directed graph the complete state space with six coins and initial piles of various sizes.
(C) Show that if the stacks are of heights $n, n-1, \dots 1$ for any n, the next configuration is the same.

A. (3,3) → (2, 2, 2) → (3, 1, 1, 1) → (4, 2) → (3, 2, 1) → (3, 2, 1)
B. (1, 1, 1, 1, 1, 1) → (6) → (5, 1) → (4, 2) → (3, 2, 1)
   (2, 1, 1, 1, 1) → (5, 1) → (4, 2)
   (2, 2, 1, 1) → (4, 1, 1) → (3, 3)
C. (n, n-1, …, 2, 1)
   We have n piles, so we pick up one coin from each pile.
   We have n coins, which we form into a pile of n coins.
   For each i between 1 and n, the pile with I coins becomes one with i – 1 coins
   For each i between 1 and n – 1 inclusive, we know that i + ith pile becomes that pile after that coin is removed.
   The nth pile comes from the n coins we picked up.
   There are no other piles because we formed a pile when we picked up all the coins and put them together.
   We deleted a pile when we took 1 coin from the pile with 1 coin.

PROBLEM 3

A robot named Wall-E wanders around a two-dimensional grid. He starts out at $(0,0)$ and is allowed to take four different types of steps:

1. $(+2, -1)$
2. $(-1, +2)$
3. $(+1, +1)$
4. $(-3, +0)$

Thus, for example, Wall-E might walk as follows. The types of his steps are listed above the arrows:

$$(0,0) \xrightarrow{1} (2,-1) \xrightarrow{3} (3,0) \xrightarrow{2} (2,2) \xrightarrow{4} (-1,2)$$