LGIC 010/PHIL 005

Scott Weinstein

University of Pennsylvania

weinstein@cis.upenn.edu

Lecture 10

Types

Definitions

- For concreteness, we focus on pure schemata involving only the monadic predicate letters "F" and "G." The generalization to pure schemata involving additional monadic predicate letters is straightforward.
- We call each of the following four one variable open schemata *types*.

$$T_1 : (Fx \wedge Gx) \quad T_2 : (Fx \wedge \neg Gx) \quad T_3 : (\neg Fx \wedge Gx) \quad T_4 : (\neg Fx \wedge \neg Gx)$$

- We say that a structure A *realizes* a given type T if and only if $A \models (\exists x)T$.

Types

A Very Important Example

- The following structure realizes all four of the types listed above.
- $A : U^A = \{1, 2, 3, 4\}, F^A = \{1, 3\}, G^A = \{1, 2\}$
- Moreover, the 14 proper substructures of A realize exactly the fourteen proper nonempty subsets of the types listed above.

Types and Monadic Similarity

Theorem

If A and B realize the same types, then they are monadically similar.

Proof

- If A and B realize the same types, then there is a single structure C which is a surjective homomorphic image of both A and B.
- Therefore, by our earlier result, A is monadically similar to C and B is monadically similar to C.
- It follows at once that A is monadically similar to B. $\square$

Corollary (The Small Model Theorem)

There is a collection $\mathcal{W}$ of $2^{(2^n)} - 1$ structures, each of size $\leq 2^n$, such that for any structure A interpreting the monadic predicate letters "F_1," ... "F_n," there is a structure $B \in \mathcal{W}$ such that $A \approx_M B$.

Fundamental Properties of Monadic Quantification Theory

Corollary (Small Model Property)

There is a collection $\mathcal{W}$ *of* $2^{(2^n)} - 1$ *structures each of size* $\leq 2^n$ *such that for any pure monadic schema* S *involving only the predicate letters "*F_1*,"... "*F_n*," if* S *is satisfiable, then there is a structure* $A \in \mathcal{W}$ *such that* $A \models S$*.*

Corollary (Decidability of Satisfiability)

There is a mechanical decision procedure to determine whether a pure monadic schema is satisfiable.

Computing $|\text{mod}(S, n)|$

Definitions

- If X is a finite set, we write $|X|$ for the number of members of X.
- In the context of today's class, all structures interpret the three monadic predicate letters F, G, and H and all schemata are built from these predicate letters.
- We write $\mathcal{M}_n$ for the set of structures A such that $U^A = \{1, \ldots, n\}$.
- If S is a schema, we write $\text{mod}(S, n)$ for the set of structures $A \in \mathcal{M}_n$ such that $A \models S$.

Computing $|\text{mod}(S, n)|$

An Example

- Compute $|\text{mod}((\forall x)(Fx \supset (Gx \supset Hx)), 10)|$.
- Recall that $(\forall x)(Fx \supset (Gx \supset Hx))$ is equivalent to $(\forall x)(Fx \vee Gx \vee Hx)$.
- Observe that $A \models (\forall x)(Fx \vee Gx \vee Hx)$ if and only if A omits the type $\neg Fx \wedge \neg Gx \wedge \neg Hx$.
- It follows that $|\text{mod}((\forall x)(Fx \supset (Gx \supset Hx)), 10)| = 7^{10}$.

Computing prob(S, n)

Definition

- For S a schema,
$$\text{prob}(S, n) = \frac{|\text{mod}(S, n)|}{|\mathcal{M}_n|}.$$
- prob(S, n) is the probability with respect the the "uniform distribution" that S is satisfied by a structure in $\mathcal{M}_n$.
- We can think of this distribution in the following way. To determine a structure $A \in \mathcal{M}_n$, each $1 \leq i \leq n$ flips three fair coins, the F-coin, the G-coin, and the the H-coin, to decide whether or not it belongs to the extension of F, G, and H, respectively.
- Thus, a structure $A \in \mathcal{M}_n$ is determined by $3 \cdot n$ flips of a fair coin. Hence, the probability of the "event" $\{A\}$, with respect to this distribution, is 2^{-3n}.

Computing prob(S, n)

An Example

- Compute $\text{prob}((\forall x)(Fx \supset (Gx \supset Hx)), 10)$.
- Note the $|\mathcal{M}_n| = 2^n \cdot 2^n \cdot 2^n = 2^{3n}$, corresponding to three independent choices of extension (a subset of $\{1, \ldots, n\}$) to the predicate letters F, G, and H.
- It now follows from our earlier computation of $|\text{mod}((\forall x)(Fx \supset (Gx \supset Hx)), 10)|$ that

$$\text{prob}((\forall x)(Fx \supset (Gx \supset Hx)), 10) = \frac{7^{10}}{2^{30}} = \left(\frac{7}{8}\right)^{10}.$$

- Alternatively, if we think of the structure A as determined by coin flips, we see that for each $1 \leq i \leq 10$, the probability that i realizes a type other than $\neg Fx \wedge \neg Gx \wedge \neg Hx$ is $\frac{7}{8}$. Thus, the probability that A omits this type is $\left(\frac{7}{8}\right)^{10}$.

Complete Schemata

Definitions

- A pure monadic schema S is *complete* if and only if S is satisfiable, and for all structures A and B,

$$\text{if } A \models S \text{ and } B \models S, \text{ then } A \approx_M B.$$

- A list of pure monadic schemata is *succinct* if and only if no two schemata on the list are equivalent.

Complete Schemata

Observations

- A pure monadic schema S is complete if and only if S is satisfiable, and for all pure monadic schemata T, either S implies T or S implies the negation of T.
- It is a corollary to the Small Model Property that the length of a longest succinct list of complete pure monadic schemata (recall our earlier **context declaration**) is $2^8 - 1$ (=255).
- Every pure monadic schema is equivalent to a disjunction of a subset of the complete schemata that form such a list. This is analogous to the Disjunctive Normal Form Theorem for truth functional logic.
- Thus, the maximal length of a succinct list of pure monadic schemata is 2^{255}.

Expressive Completeness?

- In the context of monadic quantification theory, we may think of $\text{mod}(S, n)$ as the proposition expressed by S with respect to structures with universe $\{1, \ldots, n\}$.
- By way of inquiring into the expressive completeness of monadic quantification theory, we might ask whether for every n, and every $\mathcal{P} \subseteq \mathcal{M}_n$, there is a schema S such that $\text{mod}(S, n) = \mathcal{P}$.
- It follows directly from the Homomorphism Theorem that for every $n > 1$ this is not the case, that is, for some $\mathcal{P} \subseteq \mathcal{M}_n$, and for every schema S, $\text{mod}(S, n) \neq \mathcal{P}$.
- Further light is cast by the consideration that $|\mathcal{M}_3| = 8^3 (= 512)$, and thus 2^{512} distinct "propositions" over $\mathcal{M}_3$ only 2^{255} of which are expressed by monadic schemata.

Expressive Completeness?

- The most obvious deficiency in the expressive power of monadic quantification theory, is its inability to count, that is, although it can express the quantifier "there is at least one x such that . . . ," it cannot express the quantifier "there are at least two x such that . . . ."
- We will remedy this deficiency when we pass to a logic that includes the identity relation.
- But for now, let us consider further what constitutes a legitimate "proposition" in the context of monadic quantification theory.

Isomorphism

Definitions

- A function $h : U \mapsto V$ is an *injection* if and only if for all $a, b \in U$, if $a \neq b$, then $h(a) \neq h(b)$.
- A function $h : U \mapsto V$ is a *bijection* if and only if it is both an injection and a surjection.
- Let A and B be structures. A function $h : U^A \mapsto U^B$ is an *isomorphism* of A onto B if and only if h is a bijection, and a homomorphism of A onto B.
- Let A and B be structures. A is *isomorphic to* B ($A \cong B$) if and only if there is an isomorphism of A onto B.

Isomorphism Invariance Principle

- If S is a schema of ANY LOGIC, and $A \cong B$, then $A \models S$ if and only if $B \models S$.
- Thus, we should only use the term “proposition” for collections of structures that are isomorphism invariant.

Comparing $\approx_M$ and $\cong$

Definitions

- For A a structure

$$\mathbb{E}(A, n) = \{B \in \mathcal{M}_n \mid B \approx_M A\},$$

and

$$\mathbb{I}(A, n) = \{B \in \mathcal{M}_n \mid B \cong A\}.$$

- Let T be a type and A be a structure.

$$T[A] = \{c \in U^A \mid A \models T[c]\}.$$

- Observe that $A \cong B$ if and only if for every type T

$$|T[A]| = |T[B]|.$$

Comparing $\approx_M$ and $\cong$

Preparing to Compute $|\mathbb{E}(A, n)|$ and $|\mathbb{I}(A, n)|$

In our current context there are eight types.

- $T_1 : \ (Fx \wedge Gx \wedge Hx)$
- $T_2 : \ (Fx \wedge \neg Gx \wedge Hx)$
- $T_3 : \ (\neg Fx \wedge Gx \wedge Hx)$
- $T_4 : \ (\neg Fx \wedge \neg Gx \wedge Hx)$
- $T_5 : \ (Fx \wedge Gx \wedge \neg Hx)$
- $T_6 : \ (Fx \wedge \neg Gx \wedge \neg Hx)$
- $T_7 : \ (\neg Fx \wedge Gx \wedge \neg Hx)$
- $T_8 : \ (\neg Fx \wedge \neg Gx \wedge \neg Hx)$

Comparing $\approx_M$ and $\cong$

Example: Computing $|\mathbb{E}(A, 6)|$

.Let A be the following structure

- $A : U^A = \{1, 2, 3, 4, 5, 6\}$
- $F^A = \{1, 3\}$
- $G^A = \{1, 2\}$
- $H^A = \{1, 2, 4, 5, 6\}$

Note that a structure B is monadically similar to A if and only B realizes the same types as realized by A. In this case B must realize T_1, T_3, T_4, and T_6, and omit the remaining four types.
On the basis of this observation, we will show that

$$|\mathbb{E}(A, 6)| = 4^6 - \binom{4}{1}3^6 + \binom{4}{2}2^6 - \binom{4}{3}1^6.$$

Comparing $\approx_M$ and $\cong$

Example: Computing $|\mathbb{I}(A, 6)|$

.Let A be the structure defined on the preceding slide. Recall that a structure B is isomorphic to A if and only $|T_i[B]| = |T_i[A]|$ for all $1 \leq i \leq 8$, that is, $|T_1[B]| = 1, |T_3[B]| = 1, |T_4[B]| = 3, |T_6[B]| = 1$, and $|T_2[B]| = |T_5[B]| = |T_7[B]| = |T_8[B]| = 0$.
On the basis of this observation, we will show that

$$|\mathbb{I}(A, 6)| = \frac{6!}{1! \cdot 1! \cdot 3! \cdot 1!}.$$