LGIC 010/PHIL 005

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Lecture 16

Binary Relations: Some Properties

Definition

L^A is *reflexive* if and only if

$$A \models (\forall x)Lxx.$$

Definition

L^A is *irreflexive* if and only if

$$A \models (\forall x)\neg Lxx.$$

Definition

L^A is *symmetric* if and only if

$$A \models (\forall x)(\forall y)(Lxy \supset Lyx).$$

Binary Relations: Some Properties

Definition

L^A is *asymmetric* if and only if

$$A \models (\forall x)(\forall y)(Lxy \supset \neg Lyx).$$

Definition

L^A is *transitive* if and only if

$$A \models (\forall x)(\forall y)(\forall z)(Lxy \supset (Lyz \supset Lxz)).$$

Definition

A is a *simple graph* if and only if L^A is irreflexive and symmetric.

k-regular Simple Graphs

Definition

A simple graph A is *k-regular* if and only if

$$A \models (\forall y)(\exists^{=k} x))Lyx.$$

Examples

- A 1-regular simple graph consists of a "perfect matching" of the nodes - each node has exactly one neighbor.
- It follows at once that if a finite simple graph A is 1-regular, then $|U^A|$ is even.
- A finite 2-regular simple graph consists of a disjoint union of simple cycles.

Counting Simple Graphs

Examples

- Let S be the schema

$$(\forall x)\neg Lxx \wedge (\forall x)(\forall y)(Lxy \supset Lyx).$$

- $A \models S$ if and only if A is a simple graph.

$$|\mathrm{mod}(S, n)| = 2^{\binom{n}{2}}.$$

- Let T be the schema obtained by conjoining S with the schema

$$(\forall y)(\exists^{=k} x))Lyx.$$

- $A \models T$ if and only if A is a 1-regular simple graph.

$$|\mathrm{mod}(T, 2n)| = \frac{2n!}{2^n \cdot n!}.$$

Counting Simple Graphs

Isomorphisms and Automorpisms

- We present two methods to compute $|\text{mod}(T, 2n)|$
- The first will deploy isomorphisms and automorphisms.
- Let A and B be directed graphs with edge relation L. A function h is an isomorphism of A onto B if and only if
  - h is a bijection of U^A onto U^B, and
  - for all $c, d \in U^A$, $\langle c, d \rangle \in L^A$ if and only if $\langle h(c), h(d) \rangle \in L^B$.
- $A \cong B$ (A is isomorphic to B) if and only if there is an isomorphism of A onto B. $\mathbb{I}(A) = \{B \mid U^B = U^A \text{ and } A \cong B\}$.
- A function h is an automorphism of A if and only if h is an isomorphism of A on A.
- $\mathbb{A}(A)$ is the set of automorphisms of A.
- Let A be a structure with $U^A = [n]$. Then $|\mathbb{I}(A)| \cdot |\mathbb{A}(A)| = n!$.

Preliminaries to an Exemplary Application of the Orbit-Stabilizer Theorem

- Let A be a directed graphs with edge relation L and with $U^A = [n]$, and let $h : [n] \mapsto [n]$. The structure $h[A]$, the isomorphic copy of A via h, is defined as follows.
  - $U^{h[A]} = [n]$, and
  - $L^{h[A]} = \{\langle h(c), h(d)\rangle \mid \langle c, d\rangle \in L^A\}$.
- Let $\mathbb{S}_n = \{h \mid h \text{ is a permutation of } [n]\}$. S_n is called the symmetric group on $[n]$. It is a group, in the sense of algebra, with the operations of function composition and inverse of a function.
- Note that $\mathbb{I}(A) = \{h[A] \mid h \in \mathbb{S}_n\}$, if $U^A = [n]$.

Interlude: Counting Simple Chains of Length 3

Preliminaries to an Exemplary Application of the Orbit-Stabilizer Theorem

- Let S be a schema that is true in a structure A if and only if A is a simple graph with exactly two nodes of degree one and one node of degree two.
- We compute $\mathrm{mod}(S, 3)$ via application of
- The Orbit-Stabilizer Theorem: If A is a structure with $U^A = [n]$, then $|\mathbb{I}(A)| \cdot |\mathbb{A}(A)| = |\mathbb{S}_n|$.

Interlude: Counting Simple Chains of Length 3

An Exemplary Application of the Orbit-Stabilizer Theorem

- To better understand this technique, we analyze the action of $\mathbb{S}_3$ on the simple chain A of length 3 with midpoint 2.
- To this end, we list the members of $\mathbb{S}_3$.

$\mathbb{S}_3$	1	2	3
h_1	1	2	3
h_2	2	3	1
h_3	3	1	2
h_4	1	3	2
h_5	3	2	1
h_6	2	1	3

Counting Simple Graphs

Returning to 1-regular graphs

- All 1-regular graphs of the same size are isomorphic.
- It follows that if $A \in \text{mod}(T, 2n)$, then $|\text{mod}(T, 2n)| = |\mathbb{I}(A)|$.
- Hence, $|\text{mod}(T, 2n)| = 2n!/|\mathbb{A}(A)|$.
- We will show that for $A \in \text{mod}(T, 2n)$, $|\mathbb{A}(A)| = 2^n \cdot n!$, thereby concluding the computation.