Book: https://nms.kcl.ac.uk/osvaldo.simeone/bert.pdf

Chapter: 2.1 (Optimization over a Convex Set)

Problem 1

2.1.3 (Kepler's Planimetric Problem)

Among all rectangles contained in a given circle show that the one that has maximal area is a square.

Problem 2

2.1.6

Consider the problem

$$\begin{aligned} &\text{maximize} \quad x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \\ &\text{subject to} \quad \sum_{i=1}^{n} x_i = 1, \qquad x_i \geq 0, \quad i = 1, \ldots, n, \end{aligned}$$

where a_i are given positive scalars. Find a global maximum and show that it is unique.

Problem 3

2.1.7

Let Q be a positive definite symmetric matrix. State and prove a generalization of the projection theorem that involves the cost function $(z-x)'Q(z-x)$ in place of $\|z-x\|^2$. *Hint*: Use a transformation of variables.

Problem 4

2.1.12 (Projection on a Simplex)

(a) Develop an algorithm to find the projection of a vector z on a simplex. This algorithm should be almost as simple as a closed form solution.

(b) Modify the algorithm of part (a) so that it finds the minimum of the cost function

$$f(x) = \sum_{i=1}^{n} \left(\alpha_i x_i + \tfrac{1}{2} \beta_i x_i^2 \right)$$

over a simplex, where α_i and β_i are given scalars with $\beta_i > 0$. Consider also the case where β_i may be zero for some indices i.

Problem 5

2.1.17 (Fractional Programming)

Consider the problem

$$\begin{aligned} &\text{minimize} \quad \frac{f(x)}{g(x)} \\ &\text{subject to} \quad x \in X, \end{aligned}$$

where $f : \Re^n \mapsto \Re$ and $g : \Re^n \mapsto \Re$ are given functions, and X is a given subset such that $g(x) > 0$ for all $x \in X$. For $\lambda \in \Re$, define

$$Q(\lambda) = \min_{x \in X} \{ f(x) - \lambda g(x) \},$$

and suppose that a scalar λ^* and a vector $x^* \in X$ satisfy $Q(\lambda^*) = 0$ and

$$x^* = \arg \min_{x \in X} \{ f(x) - \lambda^* g(x) \}.$$

Show that x^* is an optimal solution of the original problem. Use this observation to suggest a solution method that does not require dealing with fractions of functions.