PSet 7

Focus: generators and relations; structure of modules over PIDs.
Deadline: March 26, 11:00pm ET. *The problems can be submitted orally to an instructor or in written form to Canvas. Problems submitted on time cost 2 points; after the deadline 1 point; and there is no partial credit.*

Definition. An element x of an R-module M is *torsion* if there is a nonzero element $r \in R$ such that $rx = 0$ in M. If there are nonzero torsion elements in M, we say that M *contains torsion*; otherwise we call M *torsion-free.*

Warning. In general, the subset of torsion elements does not form a submodule. However, we will see later that over integral domains, the set of torsion elements in a module forms a submodule.

Definition. Let A be any set. The *free module on A* is the direct sum of free rank one modules indexed by A:

$$R^A = \bigoplus_{a \in A} R.$$

Denote the standard basis vectors in R^A by e_a, then any tuple in R^A can be conveniently written as a finite linear combination of e_a's. For simplicity, you can only work with finite A, but this works for infinite sets, too.

Definition. A generating set $A \subset M$ of an R-module M defines a surjection $\pi : R^A \to M$. We call R^A a *module of generators* of M.

Definition. The kernel of the surjection π from the previous paragraph is another R-module, so $\operatorname{Ker} \pi$ has a set of generators $B \subset \operatorname{Ker} \pi$, and we get a surjection $R^B \to \operatorname{Ker} \pi$. We call R^B the *module of relations* of M.

Notation. Given a subset $A \subset M$ of an R-module M, we denote the submodule generated by A by any of the following: RA, $\langle A \rangle$, $\operatorname{Span} A$.

Definition. An R-module M is called *cyclic* if it is generated by one element.

1. Let R be a ring, and let K, L, M, N be R-modules. Prove:
   (a) $\operatorname{Hom}_R(R, M) \cong M$ as R-modules;
   (b) $\operatorname{Hom}_R(K \oplus L, M) \cong \operatorname{Hom}_R(K, M) \oplus \operatorname{Hom}_R(L, M)$ as R-modules;
   (c) $\operatorname{Hom}_R(L, M \oplus N) \cong \operatorname{Hom}_R(L, M) \oplus \operatorname{Hom}_R(L, N)$ as R-modules.
2. In this problem, all objects in sight will be $\mathbb{Z}$-modules. In particular, we view $\mathbb{S}^1$ as an abelian group with respect to complex multiplication:

$$\mathbb{S}^1 \overset{\text{def}}{=} \{z \in \mathbb{C} \mid |z| = 1\} = \left\{e^{2\pi i t} \mid t \in \mathbb{R}\right\} \subset \mathbb{C}.$$

   (a) Prove that $\mathbb{R}/\mathbb{Z} \cong \mathbb{S}^1$.
   (b) Construct an injection $\mathbb{Z}/n\mathbb{Z} \to \mathbb{S}^1$. Denote its image by μ_n (it is isomorphic to $\mathbb{Z}/n\mathbb{Z}$) — this is the subgroup of nth roots of unity.
   (c) Construct an injection $\mathbb{Q}/\mathbb{Z} \to \mathbb{S}^1$. Denote its image by μ_∞ (it is isomorphic to $\mathbb{Q}/\mathbb{Z}$) — this is the subgroup of all roots of unity.

(d) Show that $\mu_\infty = \bigcup_{n>0} \mu_n$, where the union is taken inside $\mathbb{S}^1$.

(e) Describe all torsion elements in $\mathbb{R}/\mathbb{Z}$.

3. *(Generators and relations.)* Let R be a ring, and let M be an R-modules. Pick a set of generators A and relations B of M, then we have surjections $\pi : R^A \to M$ and $\rho : R^B \to \operatorname{Ker}\pi$. The homomorphism ρ naturally defines a homomorphism $R^B \to R^A$. Prove that $M \cong \operatorname{Coker}(R^B \to R^A)$, and thus any module can be constructed by generators and relations. Feel free to assume that all modules in sight are finitely generated and A, B are finite sets.

4. *(Presentation of a module via generators and relations.)* Write the following R-module M as a cokernel of a map of two free modules. Write explicitly the matrix that defines the homomorphism.

   (a) $R = \mathbb{C}[t^2, t^3]$, $M = \mathbb{C}[t]$;

   (b) $R = \mathbb{C}[x, y]$, $M = (x, y) \subset R$;

   (c) $R = \mathbb{Z}$, $M = \mathbb{Z}^3/\langle v, w\rangle$, $v = (2, 0, 5)^T$, $w = (3, 10, 0)^T$. (Here T denotes the transpose so that the vectors are written vertically.)

5. *(Torsion-free modules over integral domains.)*

   (a) Let $R = \mathbb{C}[t^2, t^3]$ and consider an R-module $M = \mathbb{C}[t]$. Prove that it is torsion-free.

   (b) Choose a maximal linear independent set $Y \subset M$.

   (c) Find an element $a \in R$ such that $aM \subset RY$.

   (d) With R and M as above, find an embedding of M in a finite free R-module.

6. *(Structure of finitely generated modules over PIDs.)* In view of problem 2, we can think of matrices as presenting modules via generators and relations, by taking the cokernel of the homomorphism of free modules defined by the matrix. Diagonalize matrix C below (by applying invertible row and column operations), and find a direct sum of cyclic R-modules isomorphic to the module presented by C.

   (a) *(Artin, problem 14.4.8.)* $R = \mathbb{Z}[i]$, $C = \begin{bmatrix} 3 & 2+i \\ 2-i & 9 \end{bmatrix}$;

   (b) *(Artin, problem 14.7.1.)* $R = \mathbb{Z}$, $C = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$;

   (c) $R = \mathbb{Z}$ and C is the matrix from 4(c);

   (d) $R = \mathbb{C}[[t]]$, $C = \begin{bmatrix} 3t^2 \cdot (1-t)^{-1} & t \\ 2it^3 + 2it^5 & t^6 \end{bmatrix}$. Review which elements of the ring of formal power series are units, and denote bulky unit factors by letters to simplify calculations, e.g. you can write $2it^3 + 2it^5 = u_1 \cdot t^3$, where $u_1 = 2i(1+t^2)$ is a unit in $\mathbb{C}[[t]]$. You don't have to calculate these units in the end, because unit factors do not affect the result of taking quotient modules.