NAME: ....

MATH E-156 MATHEMATICAL STATISTICS

ASSIGNMENT 3

1. A biased coin with probability of head $P(H) = p$ is tossed seven times.
   (a) Find the probability of getting the sequence $HHHHTTT$, i.e. in this particular order.

   (b) How many ways four heads and three tails can be obtained (i.e. please count the number of ways 4 H's and 3 T's can occur)?

   (c) Find the probability of getting exactly four heads regardless of the order.

2. Consider a discrete random variable X with the following probability mass function (pmf):

k	1	2	3	4
$P(X = k)$	.1	.2	.3	.4

   Let now $Y = X^2$.

   (a) Find the distribution of Y. Please make sure to specify what values Y takes.

   (b) Using the distribution you obtained in (a), find $E[Y]$.

   (c) Compute $\sum_{k=1}^{4} k^2 P(X = k)$. Compare it with $E[Y]$. Discuss.

3. Which is more likely: 9 heads in 10 tosses of a fair coin or 18 heads in 20 tosses? Please justify.

4. Fix some $\lambda > 0$. Let X be a continuous random variable with the following probability density function (pdf):

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$$

Find $E[X]$. (**Hint**: Integrate by parts.)

5. Let $X \sim \text{Unif}[0,1]$. Find

(a) $E[X]$

(b) $E[X^2]$

(c) $E[\sqrt{X}]$

6. If $f(x)$ and $g(x)$ are two probability density functions, show that for any $\alpha \in [0,1]$,

$$\alpha f(x) + (1-\alpha)g(x)$$

is a probability density function.

Hint:
A function $\psi(x)$ is a density of some continuous random variable if and only if

- $\psi(x) \geq 0$ for all $x \in \mathbb{R}$ and
- $\int_{-\infty}^{+\infty} \psi(x)dx = 1$

7. (a) Let X be a continuous random variable with a probability density function (pdf) that is symmetric about some point, μ.
Provided $E[X]$ exists, show that $E[X] = \mu$.

(b) Fix some $\mu \in \mathbb{R}$ and $\sigma \in (0, +\infty)$. Let $X \sim N(\mu, \sigma^2)$, that is, it has the following probability density function (pdf):

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad \text{for all } x \in \mathbb{R}.$$

Find $E[X]$.

8. The gamma function, $\Gamma(x)$, is defined as

$$\Gamma(x) = \int_0^\infty u^{x-1} e^{-u} du \quad \text{for } x > 0.$$

(a) Calculate $\Gamma(1)$.

(b) Calculate $\Gamma(2)$.

(c) Show that $\Gamma(x+1) = x\Gamma(x)$ for any $x > 0$.