Midterm 2 Math 370 F 2020 (Take-Home)

Instructions: You may consult any written source in notes, books, or on the internet, but all work must be your own. Create a pdf of your solutions and upload it to canvas by midnight on Wednesday, December 2.

1. (15 points) Show that $\mathbb{Z}[\sqrt{15}]$ is not a Unique Factorization Domain (UFD). *Hint:* One way is to mimic the method used at the end of the Week 10 Lecture Notes on canvas where we showed $\mathbb{Z}[\sqrt{10}]$ is not a UFD.

2. (5 points). Prove that if p is a prime number of the form $4n+3$, then there is no x such that $x^2 \equiv -1 \mod p$. (This is exercise 4 on page 152 of the book).

3. (10 points) If R is a commutative ring, let

$$N = \{x \in R : x^n = 0 \text{ for some } n \in \mathbb{N}\}.$$

Prove

a) N is an ideal of R

b) In the ring $\overline{R} = R/N$, if $(\overline{x})^m = 0$ for some m, then $\overline{x} = 0$.

4. (10 points)

a) If G is a group of odd order, prove that for any non-identity element $x \in G$, that x and x^{-1} are not conjugate in G.

b) Determine all finite groups which have exactly two conjugacy classes. (Hint: use the class equation for a group).

5. (10 points). The Fibonacci numbers are a sequence of numbers defined as follows: $F_0 = 0$, $F_1 = 1$, and for $N \geq 2$,

$$F_N = F_{N-1} + F_{N-2}.$$

Experiment with Euclid's algorithm as applied to consecutive Fibonacci numbers F_N and F_{N+1} to answer the following. What is $\gcd(F_{N+1}, F_N)$? Prove your answer.

6. (20 points). The quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ where $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, and $ki = -ik = j$.

a) Recall that by a *lattice* we mean a collection of subsets partially ordered by inclusion. Draw the lattice of subgroups of Q_8 and determine all subgroups fixed by every automorphism of Q_8.

b) Denote the group of inner automorphisms of Q_8 by $\mathrm{Inn}(Q_8)$. Prove that

$$\mathrm{Inn}(Q_8 \equiv C_2 \times C_2.$$

c) Denote the group of automorphisms of Q_8 by $\mathrm{Aut}(Q_8)$. Prove that

$$\mathrm{Aut}(Q_8 \equiv S_4.$$