MATH 410, FALL 2020: HOMEWORK 6

Due: October 29 before class (1:30 pm US eastern time)

Read Section 8.2 and Chapter 9 in the textbook. You can use any results we have covered so far from the book or from class for the following problems, but please state which results you are using when writing your solutions.

Exercise 1. Let $f(z) = \sin(z)$.

(1) Show that $f(z) : \mathbb{C} \to \mathbb{C}$ is surjective.
(2) Show that $f(z_1) = f(z_2)$ iff $z_1 - z_2 = 2k\pi$ or $z_1 + z_2 = (2k+1)\pi$ for some $k \in \mathbb{Z}$. (Hint: Use the sum-to-product formula. Why does it hold for complex numbers?)
(3) Define an analytic branch $g(z)$ for arcsin such that $g(x) = \arcsin(x)$ for $x \in (-1, 1)$. Determine a maximal domain for $g(z)$. (Hint: Start by defining $g'(z)$ first.)
(4) Let $h(z) = f\left(\frac{1}{z}\right)$. Show that $z = 0$ is an essential singularity of $h(z)$.
(5) Verify Picard's Great Theorem for $h(z)$ at the essential singularity $z = 0$, i.e. check that for any $w \in \mathbb{C}$, $r > 0$, $h^{-1}(w) \cap D^*(0; r)$ is an infinite set.

Exercise 2. Find the poles of the following functions and determine their orders:

$$(1)\ \frac{1}{z^4+16} \qquad (2)\ \frac{1}{z^2+4z+4} \qquad (3)\ \frac{1}{z^2-z+1}$$

Exercise 3. Classify the singularity at $z = 0$ for each of the following functions:

$$(1)\ \frac{e^z-1}{z} \qquad (2)\ \cot(z) \qquad (3)\ \tan\left(\frac{1}{z}\right)$$

Exercise 4. Find the Laurent expansions for the following functions at given points:

$$(1)\ \frac{1}{z^2+1} \text{ at } z = i. \qquad (2)\ (1+2z)\cos\left(\frac{1}{z}\right) \text{ at } z = 0.$$

Exercise 5. Let $f(z) = \dfrac{1}{(z-2)^2}$. Find the Laurent series for f valid on the following annuli:

(1) $\{z \in \mathbb{C} \mid 0 < |z-2| < \infty\}$.
(2) $\{z \in \mathbb{C} \mid 0 < |z| < 2\}$.
(3) $\{z \in \mathbb{C} \mid 2 < |z| < \infty\}$.