MATH 410, FALL 2020: HOMEWORK 5

Due: October 22 before class (1:30 pm US eastern time)

Read the supplementary materials (§3.5 in Stein-Shakarchi), and §6.3 and Chapter 8 in the textbook. You can use any results we have covered so far from the book or from class for the following problems, but please state which results you are using when writing your solutions.

Exercise 1. Let $f(z) = \sinh(z)$. Find and classify all critical points of $|f(z)|^2$. Check they satisfy:

- No local maximum.
- Local minimum only when $f(z) = 0$.
- Other critical points are all saddle points.

Exercise 2. Sketch the following topological spaces and determine whether they are simply-connected or not. (No proofs needed.)

(1) $\{z \in \mathbb{C} \mid |\text{Re } z| \geq 1\}$.
(2) $\{z \in \mathbb{C} \mid 2 \leq |z| \leq 4\} - \{z \in \mathbb{C} \mid 2 \leq |z| \leq 4, \text{Re } z < 0, |\text{Im } z| < 1\}$.
(3) $\{z \in \mathbb{C} \mid |z| \leq 3, |z-1| \geq 1\}$.
(4) (Optional) $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$.

Exercise 3. Show that homotopy $\gamma_1(t) \sim \gamma_2(t)$ between paths with the same end points is an equivalence relation via the following:

- (Reflexive) For any path $\gamma(t)$, find a homotopy $H(s,t) : \gamma(t) \sim \gamma(t)$.
- (Symmetric) Suppose $H(s,t) : \gamma_1(t) \sim \gamma_2(t)$ is a homotopy, find a homotopy $H'(s,t) : \gamma_2(t) \sim \gamma_1(t)$.
- (Transitive) Suppose $H'(s,t) : \gamma_1(t) \sim \gamma_2(t), H''(s,t) : \gamma_2(t) \sim \gamma_3(t)$ are two homotopies, find a homotopy $H(s,t) : \gamma_1(t) \sim \gamma_3(t)$.

Exercise 4. Let Log be an analytic branch of complex logarithm. Show that for $z_1, z_2 \in \mathbb{C}\backslash\{0\}$,

$$\text{Log}(z_1 z_2) = \text{Log}(z_1) + \text{Log}(z_2) + 2n\pi i,$$

for some integer n. For the principal branch Log, give an example of a pair of complex numbers z_1, z_2 such that $\text{Log}(z_1 z_2) \neq \text{Log}(z_1) + \text{Log}(z_2)$.

Exercise 5. Define a function f analytic in the plane minus the non-positive real axis and such that $f(x) = x^x$ on the positive axis. Find $f(i)$ and $f(-i)$.