LGIC 010/PHIL 005

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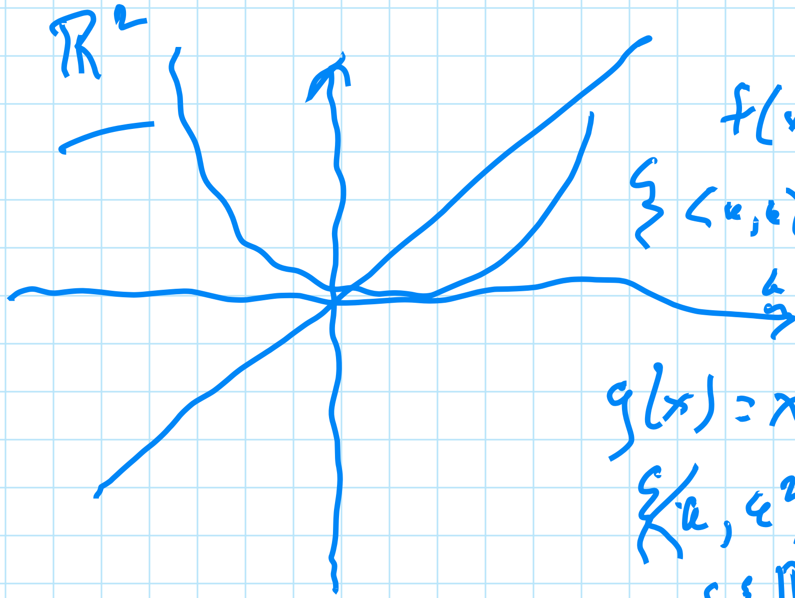
Lecture 18

Functional Relations

Definitions

- Let F be an $n+1$-adic predicate.
- F^A is total if and only if $A \models (\forall x_1)\dots(\forall x_n)(\exists y)Fx_1\dots x_n y$.
- F^A is single-valued if and only if $A \models (\forall x_1)\dots(\forall x_n)(\forall y)(\forall z)((Fx_1\dots x_n y \land Fx_1\dots x_n z) \supset y = z)$.
- F^A is a an n-ary functional relation if and only if F^A is total and single-valued.
- Observe that F^A is an n-ary functional if and only if F^A is the graph of an n-ary total function mapping U^A to U^A.
- In order to perspicuously express properties of functional relations, we introduce a new syntactic category of symbols into our logical language, namely, function symbols, $f, g, h, \dots$ with specified number of argument places.



$f(x) = x$

$\{\langle a, a\rangle \mid a \in \mathbb{R}\}$

$g(x) = x^2$

$\{\langle a, a^2\rangle \mid a \in \mathbb{R}\}$

F^A ① ② is a functional relation

F^A is the graph of a <u>binary</u> function g^A

$g^A(b, c) = d$ iff

$\langle b, c, d\rangle \in F^A$

F^A ① ②

$g^A(b) = c$ iff

$\langle b, c\rangle \in F^A$.

Functional Relations and Functions

Example: Associativity

- Let F be a ternary predicate and suppose we wish to formulate a schema S such that $A \models S$ if and only if F^A is the graph of a binary function f^A and f^A is associative.
- In order to do so we can conjoin the schema that expresses that F^A is a binary functional relation with the following schema.

$$(\forall x)(\forall y)(\forall z)(\forall u)(\forall v)(\forall w)((Fxyz \wedge Fzuv \wedge Fyuw) \supset Fxwv)$$

- By use of the function symbol f, we may express this condition by the schema: $f(f(x,y),z) = f(x,f(y,z))$.
- By use of another function symbol such as $\circ$, with infix notation, we may express associativity yet more perspicuously by the schema: $(x \circ y) \circ z = x \circ (y \circ z)$. ← associativity .

Structures for Functional Vocabularies

Example: Groups

- Let A be a structure interpreting a binary function symbol $\circ$, a unary function symbol $^{-1}$ and a 0-ary function symbol (also known as a constant symbol) e.
- A is a *group* if and only if A satisfies the conjunction of the universal closures of the following schemata.
- $(x \circ y) \circ z = x \circ (y \circ z)$ associative
- $x \circ e = x$ e is a right identity
- $x \circ x^{-1} = e$ is a right inverse
- Recall that $\mathbb{S}_n$, the collection of permutations of $[n]$, equipped with the operations of function composition and function inverse is a group. e = identity perm.

$\mathcal{A} : U^{\mathcal{A}} \neq \emptyset$

$o^{\mathcal{A}}$: binary functional relation

$a \, o^{\mathcal{A}} \, b = c \;, H$

$\langle \bar{a}, b, c \rangle \in o^{\mathcal{A}}$

$-1 \, A$

$e^{\mathcal{A}} \in U^{\mathcal{A}}$

Counting: An Example

- Let S be the schema $(\forall x)(f(f(x)) = x)$.
- Let's compute **mod**$(S, 5)$ and **iso**$(S, 5)$.

$$(\forall x)(f(f(x)) = x).$$

$$f(g(x)) = f \circ g(x)$$

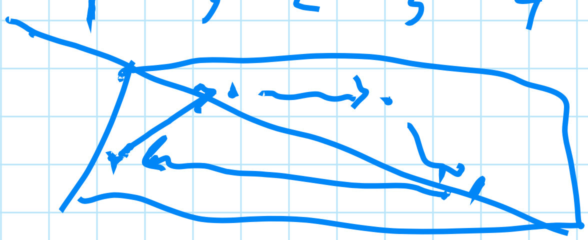
$$(\forall x)\big((f \circ f)(x)\big) = x$$

$$f \circ f = id \qquad id(x) = x$$

$$f = f^{-1}$$

f is a bijection

	1	2	3	4	5
f_1	1	2	3	4	5
f_2	1	2	3	5	4
f_3	1	3	2	5	4



$$\binom{5}{2} \cdot \binom{3}{2}$$

$$iso(5, 5) = 3$$

$$\mathbb{I}(f_1) = 1 \qquad \mathbb{I}(f_3) = \frac{\binom{5}{1} \cdot \binom{4}{2}}{2}$$

$$\mathbb{I}(f_2) = \binom{5}{3}$$

$$(\forall x)\left(f(f(f(x))) = x\right).$$

$f \circ f(x) \neq x$ unless $f = id$.

Problem 1 on PS 4.

2021 and all previous
vos years.

$$\binom{5}{2} = \frac{5 \cdot 4}{2} \quad \frac{3 \cdot \cancel{2}}{\cancel{2}}$$

$$\frac{30}{2}$$

$$\binom{5}{1} \qquad 15$$

Any problem worth doing
will have many possible
routes to its solution!

Example: Peano-structures

- Let A be a structure interpreting a unary function symbol s and a constant symbol 0. A is a *Peano-structure* if and only if A satisfies the conjunction of the following schemata.
- $(\forall x)s(x) \neq 0$
- $(\forall x)(\forall y)(s(x) = s(y) \supset x = y$
- $(\forall x)(x \neq 0 \supset (\exists y)s(y) = x)$
- What do Peano-structures look like?

A: 0^A, $s^A(0^A)$, $s^A(s^A(0))$

$S^n(0)$ $\qquad \bar{n} = \underbrace{S(S\ldots(0)))}_{n\text{-time}}$

$0^A, 1^A, 2^A, \ldots$

$0^A \xrightarrow{S^A} 1^A$ $\quad \ldots$

What else?

$$\boxed{\neg\neg p = p}$$

	0	1
p	0	1
$\neg p$	1	0
$\bot$	0	0
$\top$	1	1

$f : \mathbb{R} \to \mathbb{R}$

$f^{-1} = f$.

$$\binom{5}{1} \cdot \frac{\binom{4}{2}}{2}$$