Harvard University
Computer Science 20

Problem Set 7

Due Monday, March 29, 2021 at 11:59pm.

PROBLEM 1

Define a relation $\lhd$ on $\mathbb{Z} \times \mathbb{Z}$ by $(a, b) \lhd (c, d)$ if and only if either $a < c$ or else $a = c$ and $b \leq d$.

(A) Prove that $\lhd$ is transitive.
(B) Prove that $\lhd$ is antisymmetric.

PROBLEM 2

Bulgarian solitaire is a game played by one player. The game starts with 6 coins distributed in 1-6 piles. Then the player repeats the following step:

- Remove one coin from each existing pile and form a new pile.

The order of the piles doesn't matter, so the state can be described as a sequence of positive integers in non-increasing order adding up to 6. For example, the first two moves when a player begins with two piles of 3 coins are $(3, 3) \to (2, 2, 2)$ and $(2, 2, 2) \to (3, 1, 1, 1)$. On the next move, the last three piles disappear, creating piles of 4 and 2 coins.

(A) Trace the sequence of moves starting from two initial piles of 3 until it repeats.
(B) Draw as a directed graph the complete state space with six coins and initial piles of various sizes.
(C) Show that if the stacks are of heights $n, n-1, \dots 1$ for any n, the next configuration is the same.

PROBLEM 3

A robot named Wall-E wanders around a two-dimensional grid. He starts out at $(0, 0)$ and is allowed to take four different types of steps:

1. $(+2, -1)$
2. $(-1, +2)$
3. $(+1, +1)$
4. $(-3, +0)$

Thus, for example, Wall-E might walk as follows. The types of his steps are listed above the arrows:

$$(0, 0) \xrightarrow{1} (2, -1) \xrightarrow{3} (3, 0) \xrightarrow{2} (2, 2) \xrightarrow{4} (-1, 2)$$

Wall-E's true love, the fashionable and high-powered robot, Eve, awaits at $(0, 2)$

(A) Let Wall-E's movements be modeled by a state machine $M = \{\Sigma, S, \delta, s_0, F\}$. Σ is the set of four actions defined above. Recall that $\delta : (s_1, \sigma) \to s_2$, where $s_1, s_2 \in S$ and $\sigma \in \Sigma$. And since we'll define success as Wall-E getting to Eve, we'll say that $F = \{(0, 2)\}$. Provide definitions for the state space S, and the transition relation δ, and s_0. (For δ, you may have multiple cases.)

(B) Sadly, you can see that Wall-E will never be able to reach Eve. But Wall-E doesn't believe it. What preserved invariant could you use to prove to Wall-E that he can never reach Eve at $(0, 2)$? *Hint:* The value $x - y$ is not invariant, but how does it change?

(C) Prove to Wall-E that he cannot reach Eve using your preserved invariant and Floyd's invariant principle.

Problem set by **FILL IN YOUR NAME HERE**

Collaboration Statement: **FILL IN YOUR COLLABORATION STATEMENT HERE (See the syllabus for information)**