LGIC 010/PHIL 005

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Lecture 15

Binary Relations: Some Properties

Definition

L^A is *reflexive* if and only if

$$A \models (\forall x)Lxx.$$

Definition

L^A is *irreflexive* if and only if

$$A \models (\forall x)\neg Lxx.$$

Definition

L^A is *symmetric* if and only if

$$A \models (\forall x)(\forall y)(Lxy \supset Lyx).$$

Binary Relations: Some Properties

Definition

L^A is *asymmetric* if and only if

$$A \models (\forall x)(\forall y)(Lxy \supset \neg Lyx).$$

Definition

L^A is *transitive* if and only if

$$A \models (\forall x)(\forall y)(\forall z)(Lxy \supset (Lyz \supset Lxz)).$$

Definition

A is a *simple graph* if and only if L^A is irreflexive and symmetric.

k-regular Simple Graphs

Definition

A simple graph A is *k-regular* if and only if

$$A \models (\forall y)(\exists^{=k} x))Lyx.$$

Examples

- A 1-regular simple graph consists of a "perfect matching" of the nodes - each node has exactly one neighbor.
- It follows at once that if a finite simple graph A is 1-regular, then $|U^A|$ is even.
- A finite 2-regular simple graph consists of a disjoint union of simple cycles.

Counting Simple Graphs

Examples

- Let S be the schema

$$(\forall x)\neg Lxx \wedge (\forall x)(\forall y)(Lxy \supset Lyx).$$

- $A \models S$ if and only if A is a simple graph.

$$|\mathrm{mod}(S, n)| = 2^{\binom{n}{2}}.$$

- Let T be the schema obtained by conjoining S with the schema

$$(\forall y)(\exists^{=k} x))Lyx.$$

- $A \models T$ if and only if A is a 1-regular simple graph.

$$|\mathrm{mod}(T, 2n)| = \frac{2n!}{2^n \cdot n!}.$$

Counting Simple Graphs

Isomorphisms and Automorpisms

- We present two methods to compute $|\mathrm{mod}(T, 2n)|$
- The first will deploy isomorphisms and automorphisms.
- Let A and B be directed graphs with edge relation L. A function h is an isomorphism of A onto B if and only if
  - h is a bijection of U^A onto U^B, and
  - for all $c, d \in U^A$, $\langle c, d\rangle \in L^A$ if and only if $\langle h(c), h(d)\rangle \in L^B$.
- $A \cong B$ (A is isomorphic to B) if and only if there is an isomorphism of A onto B. $\mathbb{I}(A) = \{B \mid U^B = U^A \text{ and } A \cong B\}$.
- A function h is an automorphism of A if and only if h is an isomorphism of A on A.
- $\mathbb{A}(A)$ is the set of automorphisms of A.
- Let A be a structure with $U^A = [n]$. Then $|\mathbb{I}(A)| \cdot |\mathbb{A}(A)| = n!$.

Counting Simple Graphs

Returning to 1-regular graphs

- All 1-regular graphs of the same size are isomorphic.
- It follows that if $A \in \mathrm{mod}(T, 2n)$, then $|\mathrm{mod}(T, 2n)| = |\mathbb{I}(A)|$.
- Hence, $|\mathrm{mod}(T, 2n)| = 2n!/|\mathbb{A}(A)|$.
- We will show that for $A \in \mathrm{mod}(T, 2n)$, $|\mathbb{A}(A)| = 2^n \cdot n!$, thereby concluding the computation.