Homework 3

Assigned: February 23; Due: March 2

(1) Consider the following game: you throw a pair of fair dice. You get paid \$4 if both dice turn up six, \$2 if exactly one six turns up, but have to pay \$1 if no six turns up. What are the expected value and the standard deviation of your winnings when playing the game?

(2) You have six identical pieces of rope. The top ends of the ropes are randomly paired up and the pairs are tied together. The same procedure is done with the bottom ends of the ropes. What is the probability that, as a result of this process, the six pieces of rope will be connected in a single closed loop of rope?

(3) Each box of cereal contains a coupon. If there are n kinds of coupons, how many boxes of cereal have to be bought on average to obtain at least one coupon of each kind?

(4) Eight boys and seven girls went to the movies and sat in the same row of 15 seats. Assuming that all the 15! permutations of their seating arrangements are equally likely, compute the expected number of pairs of neighbors of different genders.

(5) An ant starts a walk from a cube vertex, it walks on the edges and at every vertex it chooses to walk along one of the available edges (including the edge it came from) with an equal probability. How many edges will the ant walk on average to come back to the starting vertex?

(6) (i) Two standard normal variables Z_1 and Z_2 have covariance 0.3. Find the mean and variance of the random variable $X = 3Z_1 - Z_2$.

(ii) Two standard normal variables Z_1 and Z_2 are independent. Find the probability density function of the *normal* random variable $X = 3Z_1 - Z_2$.

(7) Calculate $\mathbb{E}\left[N(X)\right]$, where $N(\cdot)$ is the cdf of the standard normal distribution, and X is a standard normal random variable.

(8) What is the maximal possible variance of a random variable that takes values in the set $[-1, 1]$?

(9) Let X and Y be jointly normal random variables with $X, Y \sim N(0, 1)$ and correlation $\frac{1}{2}$. Evaluate $\mathbb{E}\left[e^X \,|\, Y = 1\right]$.

(10) A stock is worth 100 today. Assume zero interest rates. The strategist for this sector calls you and tells you that the stock can be worth 90 or 110 tomorrow, with probabilities 0.6 and 0.4 respectively. You need to price an ATM call for a client. How do you proceed?

(11) Team A and team B play in a playoff series of 7 games; whoever wins 4 games first wins the series. You want to bet 100 that your team wins the series, in which case you receive 200, or 0 if they lose. However the broker only allows bets on individual games. You can bet X on any individual game that day before it occurs to receive $2X$ if it wins and 0 if it

loses. How do you achieve the desired pay-out? In particular, what do you bet on the first match?

(12) Denote by r the risk–free interest rate, assumed to be constant. The value of a European asset–or–nothing put with strike K and maturity T on a lognormally distributed underlying asset with volatility σ and paying dividends continuously at rate q is

$$P_{AoN} \;=\; Se^{-qT}N(-d_1),$$

where

$$d_1 \;=\; \frac{\ln\left(\frac{S}{K}\right) \;+\; \left(r-q+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}.$$

Find the limit of its value as volatility goes to 0 and to infinity. Finally, find the Delta of the asset–or–nothing put.

(13) In the Black–Scholes framework, what are the values of the following plain vanilla European options as time to maturity goes to infinity:

(i) at–the–money call?
(ii) at–the–money put?
(iii) ten percent in–the–money call?
(iii) ten percent in–the–money put?
(iii) ten percent out–of–the–money call?
(iii) ten percent out–of–the–money put?

(14) Let $V(S) = P_{BS}(S) - \max(K-S,0)$ be the premium of the value of a European put option on a non–dividend–paying asset over its intrinsic value $\max(K-S,0)$, where $P_{BS}(S)$ is the Black–Scholes value of the plain vanilla European put option with strike K and spot price S.

(i) Show that the maximum value of $V(S)$ is obtained at the money, i.e., for $S = K$.

(ii) What is the asymptotic behavior of $V(S)$ as $S \to 0$?

(iii) Plot $V(S)$ as a function of S.

(15) In the Black–Scholes framework, compute

$$\frac{\partial C}{\partial K} \quad \text{and} \quad \frac{\partial^2 C}{\partial K^2}.$$

Then, use the Put–Call parity to compute $\frac{\partial P}{\partial K}$ and $\frac{\partial^2 P}{\partial K^2}$.