# PHIL 005 – Problem Set 3

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## Problem 1

Consider the schema $S_1$ given by

$$(\forall x)\neg Lxx \wedge (\forall x)(\forall y)(Lxy \supset Lyx) \wedge (\forall x)(\forall y)(Lxy \supset (Fx \oplus Fy))$$

**a)** *Task:* Specify a structure $A_1$ which is a member of $\mathbf{mod}(S_1, 5)$.

*Solution:* Note that this schema expresses a simple graph such that for any two connected nodes, one of them is colored uniquely, i.e., this schema expresses a simple graph with no odd-length cycles. Thus, a possible structure which is a member of $\mathbf{mod}(S_1, 5)$ could be one where there are no edges, and all of the nodes are colored by the predicate $F$:

$$\begin{aligned} U^{A_1} &: [5] \\ L^{A_1} &: \phi \quad \text{(Empty Set)} \\ F^{A_1} &: [5] \end{aligned}$$

**b)** *Task:* What is the value of $|\mathbf{mod}(S_1, 5)|$?

*Solution:* Now, we want to count all of the ways of connecting 5 nodes such that there are no odd-length cycles. It is a well-known result of graph theory that a graph has no odd-length cycles if and only if the graph is bipartite. So, we effectively want to count the number of possible bipartitions of a graph with 5 nodes. Let's do so by considering the coloring of $F$ casewise:

1. *All Five Nodes Colored*:
   In this case, the only possible graph can be the one specified by the structure above; namely, no edges. Thus, there is 1 such graph.

2. *Four Nodes Colored, One Node Uncolored*:
   Firstly, there are $\binom{5}{4}$ ways of choosing which four nodes are colored. Then, each of those four nodes could either form an edge with the uncolored node or note, giving us $2^4$ possible choices. All in all, we have $\binom{5}{4} \cdot 2^4$ possible graphs in this case.

3. *Three Nodes Colored, Two Nodes Uncolored*:
   Similarly, there are $\binom{5}{3}$ ways of choosing which three nodes are colored. Then, each of those three nodes could either connect or not connect with the first uncolored node, and either connect or not connect with the second uncolored node, giving us $2^2$ choices for each of the three colored nodes, i.e., $(2^2)^3$ possible connections. Thus, we have $\binom{5}{3} \cdot (2^2)^3$ possible graphs in this case.

4. *Two Nodes Colored, Three Nodes Uncolored*:
   Just as before, there are $\binom{5}{2}$ ways of choosing which two nodes are colored. Then, each of those two nodes could either connect or not connect with the first uncolored node, either connect or not connect with the second uncolored node, and either connect or not connect with the third uncolored node, giving us $2^3$ choices for each of the two colored nodes, i.e., $(2^3)^2$ possible connections. Thus, we have $\binom{5}{2} \cdot (2^3)^2$ possible graphs in this case.

5. *One Node Colored, Four Nodes Uncolored*:
   Just as before, there are $\binom{5}{1}$ ways of choosing which node is colored. Then, this node could either connect or not connect with the first uncolored node, either connect or not connect with the second uncolored node, either connect or not connect with the third uncolored node, and either connect or not connect with the fourth uncolored node, giving us $2^4$ choices for this one colored node, i.e., $2^4$ possible connections. Thus, we have $\binom{5}{1} \cdot 2^4$ possible graphs in this case.

6. *All Five Nodes Uncolored*:
   Using the same reasoning as in the first case, there is 1 such graph.

So, all in all, we have found that

$$|\mathbf{mod}(S_1, 5)| = 1 + \binom{5}{4}2^4 + \binom{5}{3}2^6 + \binom{5}{2}2^6 + \binom{5}{1}2^4 + 1 = 1442 \checkmark$$

**c)** *Task:* Let $T_1$ be the conjugation of $S_1$ and the schema

$$(\forall x)(\exists y)(\forall z)(Lxz \equiv z = y).$$

What is the value of $|\mathbf{mod}(T_1, 5)|$?

*Solution:* The additional constraint this imposes on the work done above is that every vertex must have degree 1. However, since there are only 5 nodes in the graph, there must be some vertex with no edge connecting to it, i.e., some $x$ does not have a $y$ which satisfies the additional constraint described above due to there being an odd number of nodes in this graph. Thus, $|\mathbf{mod}(T_1, 5)| = 0 \checkmark$