MATH 410, FALL 2020: HOMEWORK 8

Due: November 17 before class (1:30 pm US eastern time)

Read Sections 10.2, 11.1, and 11.2 in the textbook. You can use any results we have covered so far from the book or from lectures for the following problems, but please state which results you are using when writing your solutions.

All numbered exercises are from the the textbook.

Exercise 1. Exercise 10.13. (Hint for Part (a): Rouché's theorem does not apply directly to $(1-z)P(z)$ on $|z|=1$, since the function has a zero $z=1$ on $|z|=1$. Instead, apply the theorem to $(1-z)P(z)$ on $|z|=r$ for any $r>1$. The unit disk in the problem means the closed unit disk $|z|\leq 1$.)

Exercise 2. Exercise 11.4.

Exercise 3. Exercise 11.6.

Exercise 4. Let $0<a<b$. Evaluate the integral using the Residue Theorem.

$$\int_{-\infty}^{\infty} \frac{x \sin x}{(x^2+a^2)(x^2+b^2)} \mathrm{d}x$$

Exercise 5. Suppose $a>0$ is a positive real number. Compute the following sums using the Residue Theorem:

$$\sum_{n=1}^{\infty} \frac{1}{n^2+a^2} \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+a^2}$$

Exercise 6. Suppose $f(z)$ is analytic at $w \in \overline{\mathbb{C}}$ or w is a pole. Recall in Lecture 17, when $w \neq \infty$, we define $\mathrm{ord}_w(f)$ to be the integer k such that $f(z) = (z-w)^k g(z)$, where $g(z)$ is analytic at w and $g(w) \neq 0$. For $w=\infty$, we define $\mathrm{ord}_\infty(f) := \mathrm{ord}_0(f(\frac{1}{z}))$.

(1) Check that $\mathrm{ord}_w(f) = \mathrm{Res}\left(\frac{f'}{f}; w\right)$ for any $w \in \overline{\mathbb{C}}$ as above.[1]

(2) Suppose $f(z)$ is a meromorphic function on $\overline{\mathbb{C}}$. Use (1) to conclude that:

$$\sum_{z \in \overline{\mathbb{C}}} \mathrm{ord}_z(f) = 0.$$

(3) Prove the identity (2) directly using the fact $f(z) = \frac{P(z)}{Q(z)}$ for some polynomials $P(z)$ and $Q(z)$ with no common zeros and the Fundamental Theorem of Algebra.

[1] At the beginning of Lecture 18, we defined $\mathrm{Res}(f;\infty) := \mathrm{Res}(-\frac{1}{z^2} f(\frac{1}{z}); 0)$.