# Probability

Thursday, September 05, 2019 8:19 AM

## Sample Spaces with Equally Likely Outcomes.

$S = \{1, 2, \dots, N\}$ $\quad S = \bigcup_{i=1}^{N} \{i\}$

$P(\{1\}) = P(\{2\}) = \dots = P(\{N\})$

$$P(S) = \sum_{i=1}^{N} P(\{i\}) = 1$$

$\Rightarrow \quad P(\{i\}) = \frac{1}{N}, \quad \forall\, i \in \{1, \dots, N\}.$

$$P(E) = \sum_{i \in E} P(\{i\}) = \frac{\#\text{ of outcomes in } E}{\#\text{ of outcomes in } S}$$

<u>example</u> 2 fair dice are rolled

sum of upturned faces.

$(i, j)\ , \quad 1 \le i \le 6\ , \quad 1 \le j \le 6.$

36 outcomes in $S$.

Equally likely outcomes.

$E$ = sum is equal to 7.

$= \{i + j = 7\}$

$\{1,6\}, \{2,5\}, \{3,4\}, \{4,3\}, \{5,2\}, \{6,1\}$

$$P(i+j=7) = \frac{6}{36} = \frac{1}{6}. \quad //$$

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## Counting Methods

### Multiplication Rule

$k$ experimental stages s.t.

- $1^{st}$ stage results in $n_1$ possible outcomes
- $2^{nd}$ stage results in $n_2$ possible outcomes
  for each outcome of stage 1

$\vdots$

- $k^{th}$ stage resuts in $n_k$ possible outcomes
  for each outcomes of the first $(k-1)$ stages

Then there are

$n_1 \times n_2 \times \dots \times n_k$ possible outcomes

of this $k$-stage experiment.

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examples 1) 2 fair dice

1 stage = roll $1^{st}$ die $\Rightarrow n_1 = 6$
2 stage = roll $2^{nd}$ die $\Rightarrow n_2 = 6$

$\Rightarrow$ # outcomes $= n_1 \times n_2 = 36.$

$k$ dice ? $6^k$

2) a) How many different license plates with 7 places are possible if the first 3 places are letters and the last 4 places are digits?

$\underbrace{\underline{Y}\ \underline{C}\ \underline{Z}}_{\text{letters}}\ \underbrace{\underline{5}\ \underline{0}\ \underline{9}\ \underline{\ \ }}_{\text{digits}}$ $k = 7$ stages

$n_1 = 26, \quad n_2 = 26, \quad n_3 = 26$

$n_4 = 10, \quad n_5 = n_6 = n_7 = 10$

(allow reuse of letters & digits)

$n_1 \times n_2 \times n_3 \times n_4 \times n_5 \times n_6 \times n_7 = 26^3 \times 10^4$

is the total # of <u>such</u> license plates

b) Still 7-place license plates

still 3 letters followed by 4 digits.

But now, repetitions are not allowed.

$n_1 = 26, \ n_2 = 25, \ n_3 = 24$

$\_\_ \ \_\_ \ \_\_ \ \_\_ \ \_\_ \ \_\_ \ \_\_$

$n_4 = 10, \ n_5 = 9, \ n_6 = 8,$
$n_7 = 7$

\# such license plates

$26 \times 25 \times 24 \times 10 \times 9 \times 8 \times 7$ //

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## Permutations

\# of ways in which we can arrange $N$ numbers (distinct items) in order?

$\{1, 2, \ldots, N\}$

$$\underset{1}{\underbrace{\overset{n_1 =}{N}}} \quad \underset{2}{\underbrace{\overset{n_2 =}{N-1}}} \quad \underset{3}{\underbrace{\overset{n_3 =}{N-2}}} \quad \cdots\cdots \quad \underset{N}{\underbrace{\overset{n_N}{1}}}$$

$n_1 \times n_2 \times \ldots \times n_N = N(N-1)(N-2)\ldots 1 \equiv N!$
(N factorial)

= # of ways to arrange $N$ distinct items in order.

examples

10 runners, 4 are women, 6 are men

a) # of rankings w/o regard to gender

a) # of rankings w/o regard to gender

$10!$

b) # of rankings w/ regard to gender

$4! \times 6!$

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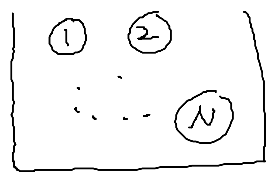
Arrange $k < N$ distinct object out of $N$




balls in an urn

Selecting $k$ out of $N$.

1 Without replacement, put them in order

# possible outcomes?

$\underset{n_1 = N}{\underbrace{\quad}_{①}}$ $\underset{n_2 = N-1}{\underbrace{\quad}_{②}}$ $\cdots\cdots$ $\underset{n_k = N-k+1}{\underbrace{\quad}_{ⓚ}}$

$$N \times (N-1) \times \ldots \times (N-k+1) = \frac{N!}{(N-k)!} \equiv$$

$$\equiv P_{N,k} \text{ permutations}$$

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example Club of 25 members

Need to choose the president (P)

& the vice president (VP)

$N = 25$, $k = 2$

$$\underset{25}{\underline{\text{P}}} \times \underset{24}{\underline{\text{VP}}} = P_{25,2} = \frac{25!}{23!}$$

$\underset{\{7\}}{\underline{\text{P}}}$ $\underset{\{5\}}{\underline{\text{VP}}}$ not the same selection as

$\underset{\{5\}}{\underline{\text{P}}}$ $\underset{\{7\}}{\underline{\text{VP}}}$ $\Rightarrow$ order matters!

Now, imagine we are choosing 2 VPs

$\underline{\text{VP}}$ $\underline{\text{VP}}$ $\{5, 7\}$ $\{7, 5\}$

$\Rightarrow$ order no longer matters!

$$\frac{25!}{23!\,2!} = \frac{24 \times 25}{2} \text{ — \# of outcomes w/o order!}$$

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(2) Choose $k$ out of $N$ distinct objects

w/o replacement, order doesn't matter.

$$\frac{N!}{(N-k)!\,k!} \equiv \binom{N}{k} \equiv C_N^k$$

( "N choose k" )

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example: Group of 5 men & 7 women is selecting a comittee of 2 women & 3 men.

How many such commitees can be formed?

$$\binom{7}{2} \times \binom{5}{3} \quad //$$

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Choosing k out of N distinct objects

Scenario 1: Choose k out N with replacement & order matters

$$\underset{N}{\underline{①}} \times \underset{N}{\underline{②}} \times \cdots \times \underset{N}{\underline{ⓚ}} = N^k =$$

= # possible arrangements.

Scenario 2: Choose k out of N without replacement & order matters

$$\frac{N!}{(N-k)!} = P_{N,k}$$

Scenario 3 Choose k out of N

$(N-k)!$

Scenario 3 Choose $k$ out of $N$ without replacement & order doesn't matter

$$\frac{P_{N,k}}{k!} = \frac{N!}{(N-k)!\,k!} = \binom{N}{k} = C_N^k$$

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Scenario 4 Choose $k$ out of $N$ distinct objects with replacement and order doesn't matter.

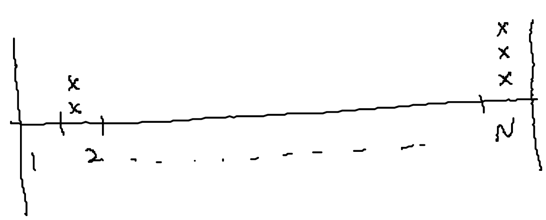
example $k=3$, $N=6$

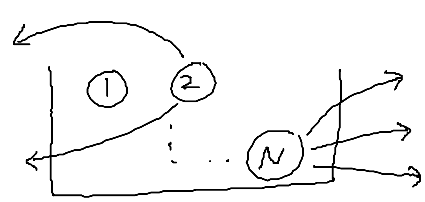
$(4\ 5\ 6)$ — $6 = 3!$ orderings

$$\left\{\begin{array}{l}(4\ 5\ 5)\\(5\ 4\ 5)\\(5\ 5\ 4)\end{array}\right. \quad \text{— 3 orderings}$$

$(5\ 5\ 5)$ — 1 order

Can not simply divide by $k!$ as before

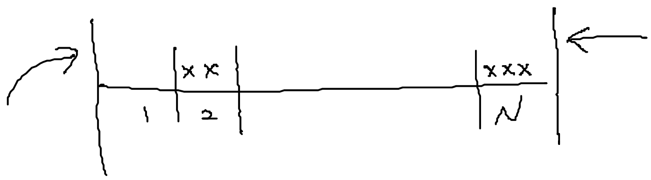


Arrange $k$ "x"'s into $N$ bins =

= counting selections of $k$ w/replacement

& no order.




Counting # of ways to select spots for "x"'s.

$k$ "x"'s, $N-1$ "movable" bin separators

$$\binom{N-1+k}{k} = \text{\# of ways to put "x"'s in places.}$$

# of places (top) — # of "x" (bottom)

$$\binom{N-1+k}{k} = \binom{N-1+k}{N-1} \quad //$$

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Summary

# of ways to choose $k$ distinct objects out of $N$

| | order matters | order doesn't matter |
|---|---|---|
| with replacement | $N^k$ | $\binom{k+N-1}{k} = \binom{k+N-1}{N-1}$ |

| | | |
|---|---|---|
| replacement | $N$ | $\binom{\quad}{k} = \binom{\quad}{N-1}$ |
| without replacement | $P_{N,k} = \frac{N!}{(N-k)!}$ | $C_N^k = \binom{N}{k} = \frac{N!}{(N-k)!k!}$ |

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<u>examples</u> ( equally likely outcomes)

1) A bowl contains 6 black balls & 5 white balls.

Choose 3 without replacement

What is the probability that 2 are white & 1 is black?

$$P(E) = \frac{\#\text{ in } E}{\#\text{ in } S} =$$

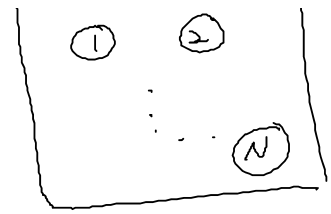
$$= \frac{\binom{5}{2} \times \binom{6}{1}}{\binom{11}{3}}.$$

2) $k$ balls are chosen out $N$ without replacement, 1 out of $N$ is "special"

... are black)

special

(1 is red, rest are black)




What's the chance that the special ball is among the selected $k$?

$\#$ of outcomes in $S = \binom{N}{k}$

$\#$ of outcomes in $E = \binom{N-1}{k-1}$

$$P(E) = \frac{\binom{N-1}{k-1}}{\binom{N}{k}} = \frac{\dfrac{(N-1)!}{\cancel{(N-k)!}\,(k-1)!}}{\dfrac{N!}{\cancel{(N-k)!}\,k!}}$$

$$= \frac{k}{N}.$$

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<u>Another</u> <u>solution</u> <u>to</u> <u>the</u> <u>same</u> <u>problem</u>

(using indicator events)

$A_i$ = $i^{th}$ draw results in the special ball, $1 \le i \le k$

$A_i \cap A_j = \emptyset \qquad i \neq j$

mutually exclusive events

$$E = \bigcup_{i=1}^{k} A_i$$

$$P(E) = P\left(\bigcup_{i=1}^{k} A_i\right) = \sum_{i=1}^{k} P(A_i) =$$

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$$P(A_1) = \frac{1}{N}, \quad P(A_i) = \frac{1}{N}$$

by symmetry

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$$= \frac{k}{N}$$

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example: A poker hand consists of 5 cards, assume playing with a standard 52 card deck.

$52 = 13 \times 4$

13 values: 2, 3, ..., 10, J, Q, K, A

4 suits: ♣ clubs, ♦ diamonds, ♥ hearts, ♠ spades

1. What's the chance of getting

a straight?
i.e. values line up in order as long as suits are not all the same (b/c it is a straight flush then)

(3, 4, 5, 6, 7) (2, 3, 4, 5, 6)

also allowed (A 2 3 4 5) (10 J Q K A)

(K A 2 3 4) is not a straight

Dealing 5 cards out of a well shuffled deck.

$$\# \text{ in } S = \binom{52}{5}$$

A 2 3 4 5, 2 3 4 5 6, 3 4 5 6 7, ...., 10 J Q K A

10 choices for values.

$$(4^5 - 4)$$

$$P(E) = \frac{10 \times (4^5 - 4)}{\binom{52}{5}} \quad //$$

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(2) What is the probability of getting

a full house?

3 + 2 combination $\begin{cases} QQ \quad KKK \\ KK \quad QQQ \end{cases}$ e.g.

$$P(E) = \frac{13 \times 12 \times \binom{4}{2} \times \binom{4}{3}}{\binom{52}{5}}$$

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example Bridge, 52 card deck

4 players, 13 cards per player.

S: total # of possible outcomes?

$$\binom{52}{13} \times \binom{39}{13} \times \binom{26}{13} \times \binom{13}{13} =$$

$$= \frac{52!}{\cancel{39!}\,13!} \times \frac{\cancel{39!}}{\cancel{26!}\,13!} \times \frac{\cancel{26!}}{13!\,13!} = \frac{52!}{(13!)^4}$$

Note: (show at your own time)

More generally, there are

$(n_1 + n_2 + \ldots + n_r)$ distinct objects without replacement

without order within groups into r groups

of sizes $n_1, n_2, \ldots, n_r$ then there are

$$\binom{n_1 + n_2 + \ldots + n_r}{n_1, n_2, \ldots, n_r} = \frac{(n_1 + \ldots + n_2)!}{n_1!\, n_2! \ldots\ldots n_r!}$$

The chance of one player getting all spades?

$$P(E) = \frac{4 \times \binom{39}{13,\,13,\,13}}{\binom{52}{13,\,13,\,13,\,13}}$$