MATH 410, FALL 2020: HOMEWORK 2

Due: September 17 before class (1:30 pm US eastern time)

Read Chapter 2. You can use any results we have covered so far from the book or from class for the following problems, but please state which results you are using when writing your solutions.

Exercise 1. Check that the Cauchy-Riemann equations hold for the following pairs of functions $u(x,y)$, $v(x,y)$. Then find a complex differentiable function $f(z) = u(x,y) + v(x,y)i$ in each case.

(1) $u(x,y) = x^3 - 3xy^2, v(x,y) = 3x^2y - y^3$.

(2) $u(x,y) = \frac{x^2-y^2}{(x^2+y^2)^2}, v(x,y) = \frac{-2xy}{(x^2+y^2)^2}$, where $(x,y) \neq (0,0)$.

Exercise 2. Let $f(x,y)$ be a complex differentiable function in a region D. Show that if $f(x,y) \in \mathbb{R} \subseteq \mathbb{C}$ for all $x+iy \in D$, then $f(x,y)$ must be a constant function.

Exercise 3. Suppose $f(z) = u(x,y) + iv(x,y)$ is a complex differentiable function. Suppose we know $u(x,y) = x^4 + y^4 - 6x^2y^2$. Use the Cauchy-Riemann equations to find all possible forms of $v(x,y)$.

Exercise 4. Use definition to compute derivatives of the following functions:

(1) $f(z) = z^2 + 2z$

(2) $f(z) = \frac{1}{z^2} \quad (z \neq 0)$.

Exercise 5. Find the radius of convergence for the following series.

(1) $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{2n\pi}{m}\right) \cdot z^n$, where $m > 1$ is a fixed integer.

(2) $\displaystyle\sum_{n=1}^{\infty} e^{-n^2+n} \cdot z^n$

(3) $\displaystyle\sum_{n=1}^{\infty} 3^n \cdot z^{n^2}$

Exercise 6. It is known that for any $\varepsilon > 0$, there are positive integers n, m, such that $|n - m\pi| < \varepsilon$. (This fact is highly non-trivial!)

(1) (Optional) Show that $\limsup_{n\to\infty} |\cos n| = 1$.

(2) Assuming (1), show that $\limsup_{n\to\infty} |\cos n|^{\frac{1}{n}} = 1$.

(3) Find the radius of convergence of $\sum_{n=1}^{\infty} \cos(n) \cdot z^n$.

Exercise 7. Show that the following identities hold for $|z| < 1$:

(1) $\sum_{n=1}^{\infty} z^{2n} = \frac{z^2}{1 - z^2}$

(2) $\sum_{n=1}^{\infty} n z^{n-1} = \frac{1}{(1 - z)^2}$

Exercise 8. Find the *domain* of convergence of the following series:

(1) $\sum_{n=1}^{\infty} \frac{(z-2)^n}{n}$

(2) $\sum_{n=1}^{\infty} \sqrt{n} \cdot (3z + 1)^n$