(1) Compute $\left(2^{\cos(x)}\right)'$ and $\left(x^{x^2}\right)'$.

(2) Let $f(x,y) = 4x^3y^2 + x(x^2+2y^2)^2 + e^{2x-3y}$. Compute

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}, \quad \int f(x,y)dx, \quad \text{and} \quad \int f(x,y)dy.$$

(3) Compute

$$\lim_{x\to\infty} 3x - \sqrt{9x^2 - 2x - 3}.$$

(4) Compute

$$\lim_{x\to 0} \frac{N(x) - \frac{1}{2} - \frac{x}{\sqrt{2\pi}}}{x^3},$$

where

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}}\, dy$$

is the cumulative density function of the standard normal variable.

(5) The value of a bond with cash flows c_i, $i = 1 : n$, at time t_i, $i = 1 : n$, and yield y (discretely compounded m times a year) is

$$B(y) = \sum_{i=1}^{n} c_i \left(1 + \frac{y}{m}\right)^{-mt_i}.$$

The modified duration and Macaulay duration of the bond are, respectively:

$$D_{mod} = -\frac{1}{B}\frac{\partial B}{\partial y};$$

$$D_{Mac} = \frac{\sum_{i=1}^{n} t_i c_i \left(1 + \frac{y}{m}\right)^{-mt_i}}{\sum_{i=1}^{n} c_i \left(1 + \frac{y}{m}\right)^{-mt_i}}.$$

Show that

$$D_{mod} = \frac{D_{Mac}}{1 + \frac{y}{m}}.$$

(6) Let $f(x_1, x_2, x_3) = 2x_1^2 + 3x_2^2 - 6x_1x_3 + 4x_2x_3$. and let $a = (1, -1, 2)$.

(i) Compute $Df(a)$, the gradient of the function $f(x)$ at the point a;

(ii) Compute $D^2f(a)$, the Hessian of the function $f(x)$ at the point a;

(iii) The quadratic Taylor expansion of $f(x)$ around the point a can be written in terms of $Df(a)$ and $D^2f(a)$ as follows:

$$f(x) \approx f(a) + Df(a)\,(x-a) + \frac{1}{2}\,(x-a)^t\, D^2f(a)\,(x-a).$$

For the point $x_0 = (1.1, -1.2, 2.1)$, compute

$$f(x_0) \quad \text{and} \quad f(a) + Df(a)\,(x_0 - a) + \frac{1}{2}\,(x_0-a)^t\, D^2f(a)\,(x_0-a).$$