Section 3B. Eigenvalues and Eigenvectors

Statistics for Data Science

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Vector Algebra

- We denote by $\mathbf{x} \in \mathbb{R}^n$ a vector with n real entries (we reserve bold letters for vectors)
  - By convention, an n-dimensional vector $\mathbf{x}$ is a matrix with n rows and 1 column
  - The i-th element of a vector $\mathbf{x}$ is denoted by x_i
  - The *inner product* of two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ is a scalar value defined as

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^\mathsf{T} \mathbf{y} = \sum_{k=1}^{n} x_k y_k = \mathbf{y}^\mathsf{T} \mathbf{x} = \langle \mathbf{y}, \mathbf{x} \rangle \in \mathbb{R}$$

  - The *outer product* of two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ is a matrix defined as $\mathbf{x}\mathbf{y}^\mathsf{T} \in \mathbb{R}^{n \times n}$; hence, $[\mathbf{x}\mathbf{y}^\mathsf{T}]_{ij} = x_i y_j$.
  - Given a matrix $A \in \mathbb{R}^{m \times n}$ and a vector $\mathbf{x} \in \mathbb{R}^n$, their product $\mathbf{y} = A\mathbf{x} \in \mathbb{R}^m$ is a vector defined entry-wise as

$$y_i = \sum_{k=1}^{n} a_{ik} x_k$$

Eigenvalues and Eigenvectors

Given a $n \times n$ matrix M, we say that the vector $\mathbf{v} \in \mathbb{C}^n$ is an *eigenvector* of M if there exists a number $\lambda \in \mathbb{C}$, called the *eigenvalue* of $\mathbf{v}$, such that the following equation is satisfied:

$$M\mathbf{v} = \lambda \mathbf{v} \text{ (Eigen-equation)}$$

It turns out that for a symmetric matrix $M = M^\intercal$, there are n possible choices for eigenvalues and eigenvectors, i.e., we have n equations of the form

$$M\mathbf{v}_i = \lambda_i \mathbf{v}_i \text{ for } i = 1, \dots, n$$

where $\lambda_i \in \mathbb{R}$, $\mathbf{v}_i \in \mathbb{R}^n$, and $\mathbf{v}_i^\intercal \mathbf{v}_j = 0$ for $i \neq j$. In other words, the eigenvalues and eigenvectors are real and the eigenvectors are perpendicular to each other