Harvard University
Computer Science 20

Problem Set 1

PROBLEM 1

Prove by truth table the first of the two distributive laws:

$$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$$

PROBLEM 2

A half dozen different operators may appear in propositional formulas, but just $\land$, $\lor$, and $\neg$ are enough to express every proposition. That is because each of the operators is equivalent to a simple formula using only these three operators. For example, $A \rightarrow B$ is equivalent to $\neg A \lor B$. So all occurrences of $\rightarrow$ in a formula can be replaced using just $\neg$ and $\lor$.

(A) Write a proposition using only $\land$, $\lor$, $\neg$ that is equivalent to $A \oplus B$. Prove your answer.
(B) Write a proposition using only $\land$, $\lor$, $\neg$ that is equivalent to $A \iff B$. Prove your answer.
(C) Prove that we don't even need $\land$, that is, write a proposition using **only** $\lor$ and $\neg$ that is equivalent to $A \land B$. Prove your answer.
(D) Prove that we can get by with the single operator NAND, written $\uparrow$, where $A \uparrow B$ is equivalent by definition to $\neg(A \land B)$. To do so, write propositions using **only** $\uparrow$ that are equivalent to 1) $\neg A$ and 2) $A \lor B$. Prove your answer. Because NAND is sufficient and easy to build in a digital circuit, in practice it is often actually the case that NAND is the only operator.

PROBLEM 3

For each of the following propositions:

(A) $\forall x \exists y. 2x - y = 0$
(B) $\forall x \exists y. x - 2y = 0$
(C) $\forall x. x < 10 \rightarrow (\forall y. y < x \rightarrow y < 9)$
(D) $\forall x \exists y. [y > x \land \exists z. y + z = 100]$

determine which propositions are valid when the variables range over:

1. the nonnegative integers (i.e., the natural numbers, $\mathbb{N}$)
2. the integers ($\mathbb{Z}$)
3. the real numbers ($\mathbb{R}$)

If the proposition is true in a domain, make an argument for its validity. If it is false, provide a counterexample.