NAME: ....
MATH E-156 MATHEMATICAL STATISTICS
ASSIGNMENT 9

1. Assume that X, Y, and W are independent normal random variables with means 3, 4, and 5, respectively. Suppose also that each of these random variables has standard deviation 7. Use X, Y, and W to construct
   (a) a chi-squared random variable with 3 degrees of freedom, χ^2_3

   (b) a t random variable with 2 degrees of freedom, t_2

2. Let $U \sim \chi^2_n$. Find $E[U]$.

3. Assume that $X_1, X_2, \ldots, X_n \stackrel{\text{iid}}{\sim} \text{Normal}(\mu, \sigma^2)$. Show that the sample variance, $S^2 \doteq \frac{1}{n-1} \sum_{k=1}^{n} \left(X_k - \bar{X}\right)^2$, is an unbiased estimator of σ^2, that is, $E[S^2] = \sigma^2$.

4. Let $X \sim \text{Normal}(\mu, \sigma^2)$. Find the moment generating function (mgf) of X.

5. Use moment generating functions to show that

   (a) $X \sim \text{Normal}(\mu, \sigma^2) \quad \Longrightarrow \quad aX + b \sim \text{Normal}(a\mu + b, a^2\sigma^2)$

   (b) $X_i \overset{\text{iid}}{\sim} \text{Normal}(\mu_i, \sigma_i^2)$, $i = 1, 2 \quad \Longrightarrow \quad X_1 + X_2 \sim \text{Normal}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

6. Let N, μ, σ^2 be a population size, population mean, and population variance, respectively. Assuming simple random sampling (SRS), let n, $\bar{X}$, S^2 be a sample size, sample mean, and sample variance, respectively. Assuming $n < N$, identify whether the given quantity is random

   (a) N

   (b) n

   (c) μ

   (d) σ

   (e) $\bar{X}$

   (f) S^2

   (g) $1.96\frac{\sigma}{\sqrt{n}}$

   (h) $\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}$

   (i) $\frac{\bar{X}-\mu}{S/\sqrt{n}}$

7. Consider a population consisting of $N = 3$ values: $-2, 0, 2$. A simple random sample (SRS) of size $n = 2$ is drawn from the population.
   (a) List all possible values $\bar{X}$ can take and specify corresponding probabilities.

   (b) Using the distribution of $\bar{X}$ you obtained in (a), compute $E[\bar{X}]$ and $\mathrm{Var}(\bar{X})$.

   (c) Calculate the population mean μ and population variance σ^2. Compare μ and σ^2/n with the results you obtained in (b). Discuss.

8. We know that in the case of simple random sampling (SRS) from a population of size N, $\mathrm{Var}(X_i) = \sigma^2$ for all i and $\mathrm{Cov}(X_i, X_j) = -\frac{\sigma^2}{N-1}$ for $i \neq j$, where σ^2 denotes the population variance. Use these results to prove that

$$\mathrm{Var}(\bar{X}) = \frac{\sigma^2}{n}\left(1 - \frac{n-1}{N-1}\right).$$