BONUS PSet $3\frac{1}{2}$

Focus: the language of categories, first examples of universal properties.
No deadline. *Every problem part on this PSet gives 1 BONUS point.*

Disclaimer. Category theory is commonly called "abstract nonsense". In one of the textbooks, "Abstract and Concrete Categories. The Joy of Cats", the epigraph to the introductory chapter is this poem:
There's a tiresome young man in Bay Shore.
When his fiancée cried, 'I adore
The beautiful sea',
He replied, 'I agree,
It's pretty, but what is it for?'
Morris Bishop

Motivation. There are however many reasons why it is widely used in mathematics, among those: economical thinking (most general patterns of mathematics are encoded in the language of categories), insight into similar constructions and universality (e.g. products of sets, groups, rings, direct sum of modules are all examples of categorical product), convenient symbolism (diagrams), duality (every concept in category theory has its "dual"). A more inspiring epigraph to the study of category theory thus would be a quote by Henri Poincaré:
Mathematics is the art of giving the same name to different things.

Definition. A *category* $\mathcal{C}$ consists of the following data:

- a class $\mathrm{Ob}\,\mathcal{C}$ whose elements are called *objects* of the category $\mathcal{C}$;
- for every pair $x, y \in \mathcal{C}$, a set $\mathrm{Hom}_{\mathcal{C}}(x, y)$ of *morphisms*; elements $f \in \mathrm{Hom}_{\mathcal{C}}(x, y)$ are graphically depicted as $f : x \to y$ or $x \xrightarrow{f} y$;
- for every object $x \in \mathcal{C}$, the *identity morphism* denoted by id_x or $1_x \in \mathrm{Hom}_{\mathcal{C}}(x, x)$;
- for every triple x, y, z of objects, a map of sets called *composition law*:

$$\mathrm{Hom}_{\mathcal{C}}(y, z) \times \mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{C}}(x, z),$$

  and denoted by $(f, g) \to f \circ g$ or $(f, g) \to fg$.

This is subject to the following conditions:

(i) Composition is associative: for morphisms $x \xrightarrow{f} y$, $y \xrightarrow{g} z$, $z \xrightarrow{h} w$, we have: $(h \circ g) \circ f = h \circ (g \circ f)$;

(ii) Identity morphisms acts as identities with respect to composition: for any $x \xrightarrow{f} y$ and $y \xrightarrow{g} z$, we should have: $1_y \circ f = f$ and $g \circ 1_y = g$.

Notation. The set of morphisms $\mathrm{Hom}_{\mathcal{C}}(x, y)$ is also denoted by $\mathcal{C}(x, y)$; people do it especially when the names of the categories become too long.

Definition. A morphism $f \in \mathcal{C}(x, y)$ is an *isomorphism* if there exists another morphism $g \in \mathcal{C}(y, x)$ such that $fg = 1_y$ and $gf = 1_x$.

1. Check that the following define categories by writing explicitly composition law and identity morphisms, and verifying axioms (i) and (ii):

(a) objects are sets and morphisms are set maps, this category is denoted by $\mathcal{S}et$ and is called the category of sets;

(b) objects are groups and morphisms are group homomorphisms, this category is denoted by $\mathcal{G}p$ and is called the category of groups;

(c) objects are commutative associative unital rings and morphisms are ring homomorphisms that preserve multiplicative identities of rings, this category can be denoted by $\mathcal{C}omm$, $\mathcal{C}omm\mathcal{R}ing$, $\mathcal{CR}ing$;

(d) objects are R-modules (over some ring R, e.g. $R \in \mathcal{CR}ing$) and morphisms are homomorphisms of R-modules; this category is denoted by R-$\mathcal{M}od$.

2. Let $\mathcal{C}$ be a category and $e : y \to y$ is a morphism that satisfies $ef = f$ and $ge = g$ for any $x \xrightarrow{f} y$ and $y \xrightarrow{g} z$. Prove that then $e = 1_y$, i.e. that the identity morphism is unique for a fixed object y.

3. Let $\mathcal{C}$ be a category. Define the *opposite category* $\mathcal{C}^\circ$ as follows: objects are the same as in $\mathcal{C}$, that is

$$\operatorname{Ob}\mathcal{C}^\circ = \operatorname{Ob}\mathcal{C},$$

and morphisms "go in the opposite direction", i.e.

$$\mathcal{C}^\circ(x, y) = \mathcal{C}(y, x).$$

Prove that $\mathcal{C}^\circ$ is a category.

4. *(Universal properties: initial and final objects.)* An object $x \in \operatorname{Ob}\mathcal{C}$ is called *initial* if for any object $t \in \operatorname{Ob} C$ there exists a unique morphism $x \to t$ that starts in x. Dually, an object $z \in \operatorname{Ob}\mathcal{C}$ is called *final* if for any object $t \in \operatorname{Ob}\mathcal{C}$ there is a unique morphism $t \to z$ to z.

(a) If $x \in \mathcal{C}$ is an initial object, then $x \in \mathcal{C}^\circ$ is a final object, and vice versa.

(b) If x and x' in $\mathcal{C}$ are initial objects, then they are isomorphic. Using this and part (a), conclude that two final objects are also isomorphic.

(c) Prove that the empty set $\varnothing$ is the initial object in $\mathcal{S}et$ and a singleton $* \stackrel{\text{def}}{=} \{*\}$ is the final object in $\mathcal{S}et$.

(d) Prove that the trivial group (i.e. consisting of one element) is both initial and final in $\mathcal{G}p$.

(e) Find initial and final objects in $\mathcal{CR}ing$.

(f) Find initial and final objects in R-$\mathcal{M}od$, for a ring R.

5. *(Universal properties: product.)* Let $\mathcal{C}$ be a category and $x, y \in \mathcal{C}$ are two objects. The *product* of x and y, if it exists, is such an object $x \times y$ with "projection morphisms" $p_1 : x \times y \to x$ and $p_2 : x \times y \to y$ that for any test object $t \in \operatorname{Ob}\mathcal{C}$ and any pair of morphisms $f_1 : t \to x$ and $f_2 : t \to y$, there exists a unique morphism $\varphi : t \to x \times y$ that factors f_1 and f_2, i.e. $f_1 = p_1\varphi$ and $f_2 = p_2\varphi$. It can be conveniently expressed as a *diagram*:

$$\begin{tikzcd}
 & t \arrow[ddl, "f_1"'] \arrow[d, dashed, "\exists!\varphi"] \arrow[ddr, "f_2"] & \\
 & x \times y \arrow[dl, "p_1"] \arrow[dr, "p_2"'] & \\
x & & y
\end{tikzcd}$$

(a) Prove that if product P of x and y exists, then it is unique up to an isomorphism. I.e., if there is another product P' satisfying the same universal property, then $P \cong P'$.

(b) Prove that direct product is the product in $\mathcal{S}et$, $\mathcal{G}p$,

(c) and in $\mathcal{CR}ing$;

(d) prove that direct sum is the product in R-$\mathcal{M}od$.

(e) Define *coproduct* in $\mathcal{C}$ as the product in $\mathcal{C}^\circ$. Formulate the universal property for the coproduct $x \sqcup y$ of x and y in terms of morphisms in the category $\mathcal{C}$, and draw the corresponding diagram.

(f) Explain that the universal property of the product can be stated as follows: for any test object $t \in \mathcal{C}$, we have a natural bijection of sets $\mathcal{C}(t, x \times y) \cong \mathcal{C}(t, x) \times \mathcal{C}(t, y)$. In a similar fashion, restate the universal property of the coproduct: $\mathcal{C}(x \sqcup y, t) \cong ???$.

(g) Prove that disjoint union is the coproduct in $\mathcal{S}et$.

(h) What is the coproduct in $\mathcal{G}p$? Describe the coproduct of $\mathbb{Z}$ and $\mathbb{Z}$ in $\mathcal{G}p$.

(i) Prove that direct sum is the coproduct in R-$\mathcal{M}od$.