LGIC 010/PHIL 005

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Lecture 11

Types

Definitions

- For concreteness, we focus on pure schemata involving only the monadic predicate letters "F" and "G." The generalization to pure schemata involving additional monadic predicate letters is straightforward.
- We call each of the following four one variable open schemata *types*.

$$T_1 : (Fx \wedge Gx) \quad T_2 : (Fx \wedge \neg Gx) \quad T_3 : (\neg Fx \wedge Gx) \quad T_4 : (\neg Fx \wedge \neg Gx)$$

- We say that a structure A *realizes* a given type T if and only if $A \models (\exists x)T$.

A Very Important Example

- The following structure realizes all four of the types listed above.
- $A : U^A = \{1, 2, 3, 4\}, F^A = \{1, 3\}, G^A = \{1, 2\}$
- Moreover, the 14 proper substructures of A realize exactly the fourteen proper nonempty subsets of the types listed above.

Types and Monadic Similarity

Theorem

If A and B realize the same types, then they are monadically similar.

Proof

- If A and B realize the same types, then there is a single structure C which is a surjective homomorphic image of both A and B.
- Therefore, by our earlier result, A is monadically similar to C and B is monadically similar to C.
- It follows at once that A is monadically similar to B. $\square$

Corollary (The Small Model Theorem)

There is a collection $\mathcal{W}$ of $2^{(2^n)} - 1$ structures, each of size $\leq 2^n$, such that for any structure A interpreting the monadic predicate letters "F_1," ... "F_n," there is a structure $B \in \mathcal{W}$ such that $A \approx_M B$.

Fundamental Properties of Monadic Quantification Theory

Corollary (Small Model Property)

There is a collection $\mathcal{W}$ of $2^{(2^n)} - 1$ structures each of size $\leq 2^n$ such that for any pure monadic schema S involving only the predicate letters "F_1," ... "F_n," if S is satisfiable, then there is a structure $A \in \mathcal{W}$ such that $A \models S$.

Corollary (Decidability of Satisfiability)

There is a mechanical decision procedure to determine whether a pure monadic schema is satisfiable.

Computing $|\text{mod}(S, n)|$

Definitions

- If X is a finite set, we write $|X|$ for the number of members of X.
- In the context of today's class, all structures interpret the three monadic predicate letters F, G, and H and all schemata are built from these predicate letters.
- We write $\mathcal{M}_n$ for the set of structures A such that $U^A = \{1, \ldots, n\}$.
- If S is a schema, we write $\text{mod}(S, n)$ for the set of structures $A \in \mathcal{M}_n$ such that $A \models S$.

An Example

- Compute $|\text{mod}((\forall x)(Fx \supset (Gx \supset Hx)), 10)|$.
- Recall that $(\forall x)(Fx \supset (Gx \supset Hx))$ is equivalent to $(\forall x)(\neg Fx \vee \neg Gx \vee Hx)$.
- Observe that $A \models (\forall x)(\neg Fx \vee \neg Gx \vee Hx)$ if and only if A omits the type $Fx \wedge Gx \wedge \neg Hx$.
- It follows that $|\text{mod}((\forall x)(Fx \supset (Gx \supset Hx)), 10)| = 7^{10}$.

Computing prob(S, n)

Definition

- For S a schema,
$$\text{prob}(S, n) = \frac{|\text{mod}(S, n)|}{|\mathcal{M}_n|}.$$
- prob(S, n) is the probability with respect the the "uniform distribution" that S is satisfied by a structure in $\mathcal{M}_n$.
- We can think of this distribution in the following way. To determine a structure $A \in \mathcal{M}_n$, each $1 \leq i \leq n$ flips three fair coins, the F-coin, the G-coin, and the the H-coin, to decide whether or not it belongs to the extension of F, G, and H, respectively.
- Thus, a structure $A \in \mathcal{M}_n$ is determined by $3 \cdot n$ flips of a fair coin. Hence, the probability of the "event" $\{A\}$, with respect to this distribution, is 2^{-3n}.

Computing prob(S, n)

An Example

- Compute $\text{prob}((\forall x)(Fx \supset (Gx \supset Hx)), 10)$.
- Note the $|\mathcal{M}_n| = 2^n \cdot 2^n \cdot 2^n = 2^{3n}$, corresponding to three independent choices of extension (a subset of $\{1, \ldots, n\}$) to the predicate letters F, G, and H.
- It now follows from our earlier computation of $|\text{mod}((\forall x)(Fx \supset (Gx \supset Hx)), 10)|$ that

$$\text{prob}((\forall x)(Fx \supset (Gx \supset Hx)), 10) = \frac{7^{10}}{2^{30}} = \left(\frac{7}{8}\right)^{10}.$$

- Alternatively, if we think of the structure A as determined by coin flips, we see that for each $1 \leq i \leq 10$, the probability that i realizes a type other than $Fx \wedge Gx \wedge \neg Hx$ is $\frac{7}{8}$. Thus, the probability that A omits this type is $\left(\frac{7}{8}\right)^{10}$.

Complete Schemata

Definitions

- A pure monadic schema S is *complete* if and only if S is satisfiable, and for all structures A and B,

$$\text{if } A \models S \text{ and } B \models S, \text{ then } A \approx_M B.$$

- A list of pure monadic schemata is *succinct* if and only if no two schemata on the list are equivalent.

Complete Schemata

Observations

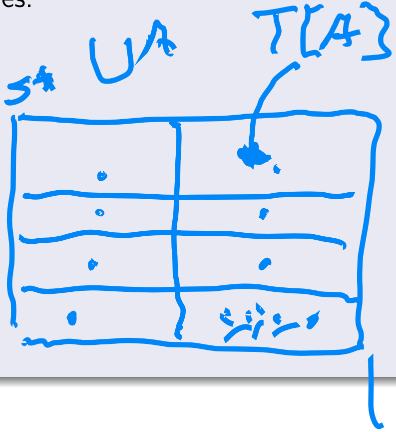
- A pure monadic schema S is complete if and only if S is satisfiable, and for all pure monadic schemata T, either S implies T or S implies the negation of T.
- It is a corollary to the Small Model Property that the length of a longest succinct list of complete pure monadic schemata (recall our earlier **context declaration**) is $2^8 - 1$ (=255).
- Every pure monadic schema is equivalent to a disjunction of a subset of the complete schemata that form such a list. This is analogous to the Disjunctive Normal Form Theorem for truth functional logic.
- Thus, the maximal length of a succinct list of pure monadic schemata is 2^{255}.

Complete Schemata

Further Observations: The Types

In our current context there are eight types.

- T_1 : $(Fx \wedge Gx \wedge Hx)$
- T_2 : $(Fx \wedge \neg Gx \wedge Hx)$
- T_3 : $(\neg Fx \wedge Gx \wedge Hx)$
- T_4 : $(\neg Fx \wedge \neg Gx \wedge Hx)$
- T_5 : $(Fx \wedge Gx \wedge \neg Hx)$
- T_6 : $(Fx \wedge \neg Gx \wedge \neg Hx)$
- T_7 : $(\neg Fx \wedge Gx \wedge \neg Hx)$
- T_8 : $(\neg Fx \wedge \neg Gx \wedge \neg Hx)$



Complete Schemata

Further Observations: Constructing Complete Sentences

- Let $A_1, \ldots A_{255}$ be a complete list of small models.
- For each of the structures A_i we can construct a complete sentence S_i of the form

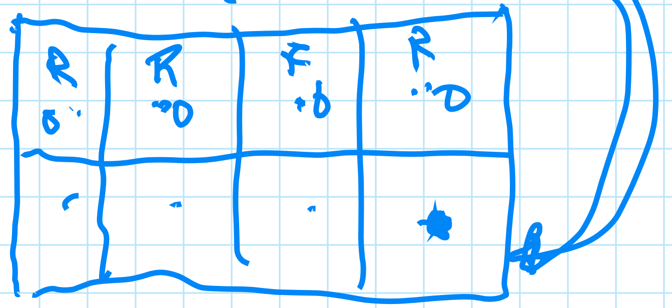
$$\bigwedge_{1 \leq k \leq 8} (\neg)(\exists x) T_k$$

such that $A_i \models S_i$ and $A_j \not\models S_j$ for $j \neq i$. S_l describes exactly which of the types are realized, and which omitted, by A_i.
- Note that for a pure monadic schema S, $A_i \models S$ if and only if S_i implies S.
- Thus, if S is the schema, $(\forall x)(Fx \supset (Gx \supset Hx))$, then 127 of the complete schemata S_i imply S and 128 imply its negation.

$A \not\models (\forall x)(Fx \supset (Gx \supset Hx))$

A omits ~~…~~ $Fx \wedge Gx \wedge \neg Hx$

realize Z_e



Expressive Completeness?

- In the context of monadic quantification theory, we may think of mod(S, n) as the proposition expressed by S with respect to structures with universe $\{1, \ldots, n\}$.
- By way of inquiring into the expressive completeness of monadic quantification theory, we might ask whether for every n, and every $\mathcal{P} \subseteq \mathcal{M}_n$, there is a schema S such that mod$(S, n) = \mathcal{P}$.
- It follows directly from the Homomorphism Theorem that for every $n > 1$ this is not the case, that is, for some $\mathcal{P} \subseteq \mathcal{M}_n$, and for every schema S, mod$(S, n) \neq \mathcal{P}$.
- Further light is cast by the consideration that $|\mathcal{M}_3| \overset{?}{=} 8^3 (= 512)$, and thus $\boxed{2^{512}}$ distinct "propositions" over $\mathcal{M}_3$ only 2^{255} of which are expressed by monadic schemata.

F, G, H

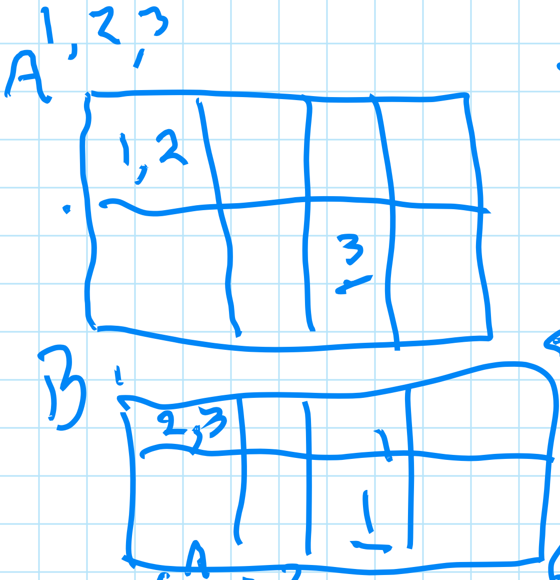
Expressive Completeness?

$T(x)$ $Fx \wedge Gx \wedge Hx$

- The most obvious deficiency in the expressive power of monadic quantification theory, is its inability to count, that is, although it can express the quantifier "there is at least one x such that . . . ," it cannot express the quantifier "there are at least two x such that . . . ."
- We will remedy this deficiency when we pass to a logic that includes the identity relation.
- But for now, let us consider further what constitutes a legitimate "proposition" in the context of monadic quantification theory.

$(\exists x)(\exists y)(Tx \wedge Ty)$

$(\exists x)(\exists y)(x \neq y \wedge Tx \wedge Ty)$



Logic Preserves Proximity

$B \cong A$

$c^{A} = 3$

$\mathrm{Dim}^{A}(c) = 3$

$c^{B} = 2$

Isomorphism

Definitions

- A function $h : U \mapsto V$ is an *injection* if and only if for all $a, b \in U$, if $a \neq b$, then $h(a) \neq h(b)$.
- A function $h : U \mapsto V$ is a *bijection* if and only if it is both an injection and a surjection.
- Let A and B be structures. A function $h : U^A \mapsto U^B$ is an *isomorphism* of A onto B if and only if h is a bijection, and a homomorphism of A onto B.
- Let A and B be structures. A is *isomorphic to* B ($A \cong B$) if and only if there is an isomorphism of A onto B.

Isomorphism Invariance Principle

- If S is a schema of ANY LOGIC, and $A \cong B$, then $A \models S$ if and only if $B \models S$.
- Thus, we should only use the term "proposition" for collections of structures that are isomorphism invariant.

$f(c^A) = c^B$ Ext $\to$ D U^B

v^A $\{1, \ldots, 5\}$, John, Gerri, Joe, Jill

Comparing $\approx_M$ and $\cong$

Definitions

- For A a structure

$$\mathbb{E}(A, n) = \{B \in \mathcal{M}_n \mid B \approx_M A\},$$

and

$$\mathbb{I}(A, n) = \{B \in \mathcal{M}_n \mid B \cong A\}.$$

- Let T be a type and A be a structure.

$$T[A] = \{c \in U^A \mid A \models T[c]\}.$$

- Observe that $A \cong B$ if and only if for every type T

$$|T[A]| = |T[B]|.$$

Comparing $\approx_M$ and $\cong$

Preparing to Compute $|\mathbb{E}(A, n)|$ and $|\mathbb{I}(A, n)|$

In our current context there are eight types.

- T_1 : $(Fx \wedge Gx \wedge Hx)$
- T_2 : $(Fx \wedge \neg Gx \wedge Hx)$
- T_3 : $(\neg Fx \wedge Gx \wedge Hx)$
- T_4 : $(\neg Fx \wedge \neg Gx \wedge Hx)$
- T_5 : $(Fx \wedge Gx \wedge \neg Hx)$
- T_6 : $(Fx \wedge \neg Gx \wedge \neg Hx)$
- T_7 : $(\neg Fx \wedge Gx \wedge \neg Hx)$
- T_8 : $(\neg Fx \wedge \neg Gx \wedge \neg Hx)$

Example: Computing $|\mathbb{E}(A, 6)|$

Let A be the following structure

- $A : U^A = \{1, 2, 3, 4, 5, 6\}$
- $F^A = \{1, 3\}$
- $G^A = \{1, 2\}$
- $H^A = \{1, 2, 4, 5, 6\}$

Note that a structure B is monadically similar to A if and only B realizes the same types as realized by A. In this case B must realize T_1, T_3, T_4, and T_6, and omit the remaining four types.

On the basis of this observation, we will show that

$$|\mathbb{E}(A, 6)| = 4^6 - \binom{4}{1}3^6 + \binom{4}{2}2^6 - \binom{4}{3}1^6.$$

Comparing $\approx_M$ and $\cong$

Example: Computing $|\mathbb{E}(A, 6)|$

- Thus, our problem reduces to counting the number of ways of sorting the six members of U^A into the four types T_1, T_3, T_4, and T_6, in such a way that each of the types is realized. Equivalently, counting the number of surjections from U^A, a set of size 6, onto the foregoing set of four types.
- Let X be the set of all such functions, and let X_i be the set of these functions that omit assigning an element to the type T_i.
- Let Y be the set of surjections. Then

$$Y = X \setminus (X_1 \cup X_3 \cup X_4 \cup X_6)$$

- We apply the Principle of Inclusion-Exclusion to compute $|Y|$.

Example: Computing $|\mathbb{E}(A, 6)|$

Since,

$$Y = X \setminus (X_1 \cup X_3 \cup X_4 \cup X_6),$$

and for each of the six pairs $i < j$, $X_i \cap X_j$ is nonempty, it follows that

$$|Y| \geq |X| - (|X_1| + |X_3| + |X_4| + |X_6|) = 4^6 - 4 \cdot 3^6.$$

Continuing in like fashion, we see that,

$$|Y| \leq 4^6 - \binom{4}{1} \cdot 3^6 + \binom{4}{2} \cdot 2^6,$$

restoring the contribution of the pairwise intersections of the distinct X_i's.

Example: Computing $|\mathbb{E}(A, 6)|$

Finally,

$$|Y| = 4^6 - \binom{4}{1} \cdot 3^6 + \binom{4}{2} \cdot 2^6 - \binom{4}{1} \cdot 1^6,$$

by removing the contribution of the intersection of distinct triples of the X_i's.

Computing the Number of Surjections

We may generalize the preceding argument to show that the number of surjections from a set of size n onto a set of size k equals

$$\sum_{i=0}^{k} -1^i \binom{k}{i} (k-i)^n.$$

Our previous reasoning concerning the contribution of each member of X to the sum is generalized by the following identity,

$$\sum_{j=0}^{m} -1^j \binom{m}{j} = 0,$$

for $m > 0$.

Comparing $\approx_M$ and $\cong$

Computing the Number of Surjections

The last identity is a corollary to the Binomial Theorem

$$(1+x)^m = \sum_{j=0}^{m} \binom{m}{j} x^j,$$

by setting $x = -1$.
The Binomial Theorem itself may be proven by application of the following identity

$$\binom{m}{j} = \binom{m-1}{j} + \binom{m}{j-1},$$

which can itself be established by a combinatorial argument.

Example: Computing $|\mathbb{I}(A, 6)|$

.Let A be the structure defined on the preceding slide. Recall that a structure B is isomorphic to A if and only $|T_i[B]| = |T_i[A]|$ for all $1 \leq i \leq 8$, that is, $|T_1[B]| = 1, |T_3[B]| = 1, |T_4[B]| = 3, |T_6[B]| = 1$, and $|T_2[B]| = |T_5[B]| = |T_7[B]| = |T_8[B]| = 0$.
On the basis of this observation, we will show that

$$|\mathbb{I}(A, 6)| = \frac{6!}{1! \cdot 1! \cdot 3! \cdot 1!}.$$

Example: Computing $|\mathbb{I}(A, 6)|$

- .For each of the 6! permutations p of U^A define the image $p[A]$ of A as follows.
  - $U^{p[A]} = U^A = \{1, 2, 3, 4, 5, 6\}$
  - $F^{p[A]} = \{p(1), p(3)\}$
  - $G^{p[A]} = \{p(1), p(2)\}$
  - $H^{p[A]} = \{p(1), p(2), p(4), p(5), p(6)\}$
- Note that p is an isomorphism from A onto $p[A]$.
- Observe that for each permutation r, $p[A] = r[A]$ if and only if $r(1) = p(1), r(2) = p(2)$ and $r(3) = p(3)$.
- It follows that for every permutation p of U^A there are 3! permutations r such that $p[A] = r[A]$.
- It now follows that

$$|\mathbb{I}(A, 6)| = \frac{6!}{1! \cdot 1! \cdot 3! \cdot 1!}.$$