Harvard University
Computer Science 20

Problem Set 1

2nd Distributive Law

$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

PROBLEM 1

Prove by truth table the first of the two distributive laws:

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

p	q	r	p	$\vee$	$(q$	$\wedge$	$r)$	$(p$	$\vee$	$q)$	$\wedge$	$(p$	$\vee$	$r)$
T	T	T	T	T	T	T	T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	F	T	T	T	T	T	T	F
T	F	T	T	T	F	F	T	T	T	F	T	T	T	T
T	F	F	T	T	F	F	F	T	T	F	T	T	T	F
F	T	T	F	T	T	T	T	F	T	T	T	F	T	T
F	T	F	F	F	T	F	F	F	T	T	F	F	F	F
F	F	T	F	F	F	F	T	F	F	F	F	F	T	T
F	F	F	F	F	F	F	F	F	F	F	F	F	F	F

The truth tables are equivalent if the following 2 rows match; - they do match.

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- ✓ - **0 pts** Correct
- - **3 pts** Incorrect/blank row for TTT
- - **3 pts** Incorrect/blank row for TTF
- - **3 pts** Incorrect/blank row for TFT
- - **3 pts** Incorrect/blank row for TFF
- - **3 pts** Incorrect/blank row for FTT
- - **3 pts** Incorrect/blank row for FTF
- - **3 pts** Incorrect/blank row for FFT
- - **3 pts** Incorrect/blank row for FFF
- - **1 pts** Wrong intermediate column
- - **1 pts** Wrong intermediate column for FTF
- - **1 pts** T = 1 and F = 0 in your chart
- - **5 pts** Missing explanation

PROBLEM 2

A half dozen different operators may appear in propositional formulas, but just $\wedge$, $\vee$, and $\neg$ are enough to express every proposition. That is because each of the operators is equivalent to a simple formula using only these three operators. For example, $A \to B$ is equivalent to $\neg A \vee B$. So all occurrences of $\to$ in a formula can be replaced using just $\neg$ and $\vee$.

(A) Write a proposition using only $\wedge$, $\vee$, $\neg$ that is equivalent to $A \oplus B$. Prove your answer.
(B) Write a proposition using only $\wedge$, $\vee$, $\neg$ that is equivalent to $A \iff B$. Prove your answer.
(C) Prove that we don't even need $\wedge$, that is, write a proposition using **only** $\vee$ and $\neg$ that is equivalent to $A \wedge B$. Prove your answer.
(D) Prove that we can get by with the single operator NAND, written $\uparrow$, where $A \uparrow B$ is equivalent by definition to $\neg(A \wedge B)$. To do so, write propositions using **only** $\uparrow$ that are equivalent to 1) $\neg A$ and 2) $A \vee B$. Prove your answer. Because NAND is sufficient and easy to build in a digital circuit, in practice it is often actually the case that NAND is the only operator.

(A)

A	B	$A \oplus B$	$(A \vee B) \wedge \neg(A \wedge B)$
T	T	F	T F F T
T	F	T	T T T F
F	T	T	T T T F
F	F	F	F F F F

(B)

$A \iff B$	$(\neg A \vee B) \wedge (\neg B \vee A)$
T	F T T T F T T
F	F F F F T T T
F	T T T F F F F
T	T T F T T T F

(C)

A	B	$A \wedge B$	$\neg(\neg A \vee \neg B)$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

$\neg(A \vee B) \equiv \neg A \wedge \neg B$

$\neg(\neg A \vee \neg B) = \neg\neg A \wedge \neg\neg B$

$\equiv$

$A \wedge B$

(D) ① $A \uparrow B \equiv \neg(A \wedge B) \equiv \neg A \vee \neg B$

$A \uparrow A \equiv \neg(A \wedge A) \equiv \neg A$

②

$\neg A \uparrow \neg B \equiv \neg(\neg A \wedge \neg B)$

$=$

$\neg\neg A \vee \neg\neg B \equiv A \vee B$

$\equiv$

$(A \uparrow A) \uparrow (B \uparrow B)$

2.1 A **6 / 8**

- **- 0 pts** Correct
- **- 2 pts** missing parentheses
- ✓ **- 2 pts** **truth table is unclear**

PROBLEM 2

A half dozen different operators may appear in propositional formulas, but just $\wedge$, $\vee$, and $\neg$ are enough to express every proposition. That is because each of the operators is equivalent to a simple formula using only these three operators. For example, $A \rightarrow B$ is equivalent to $\neg A \vee B$. So all occurrences of $\rightarrow$ in a formula can be replaced using just $\neg$ and $\vee$.

(A) Write a proposition using only $\wedge$, $\vee$, $\neg$ that is equivalent to $A \oplus B$. Prove your answer.
(B) Write a proposition using only $\wedge$, $\vee$, $\neg$ that is equivalent to $A \iff B$. Prove your answer.
(C) Prove that we don't even need $\wedge$, that is, write a proposition using **only** $\vee$ and $\neg$ that is equivalent to $A \wedge B$. Prove your answer.
(D) Prove that we can get by with the single operator NAND, written $\uparrow$, where $A \uparrow B$ is equivalent by definition to $\neg(A \wedge B)$. To do so, write propositions using **only** $\uparrow$ that are equivalent to 1) $\neg A$ and 2) $A \vee B$. Prove your answer. Because NAND is sufficient and easy to build in a digital circuit, in practice it is often actually the case that NAND is the only operator.

(A)

A	B	$A \oplus B$	$(A \vee B) \wedge \neg (A \wedge B)$
T	T	F	T F F T
T	F	T	T T T F
F	T	T	T T T F
F	F	F	F F F F

(B)

$A \iff B$	$(\neg A \vee B) \wedge (\neg B \vee A)$
T	F T T T F T T
F	F F F F T T T
F	T T T F F F F
T	T T F T T T F

(C)

A	B	$A \wedge B$	$\neg(\neg A \vee \neg B)$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

$\neg(A \vee B) \equiv \neg A \wedge \neg B$

$\neg(\neg A \vee \neg B) = \neg\neg A \wedge \neg\neg B$

$\equiv$

$A \wedge B$

(D) ① $A \uparrow B \equiv \neg(A \wedge B) \equiv \neg A \vee \neg B$

$A \uparrow A \equiv \neg(A \wedge A) \equiv \neg A$

②

$\neg A \uparrow \neg B \equiv \neg(\neg A \wedge \neg B)$

$=$

$\neg\neg A \vee \neg\neg B \equiv A \vee B$

$\equiv$

$(A \uparrow A) \uparrow (B \uparrow B)$

2.2 B **4 / 8**

- **- 0 pts** Correct
- **- 4 pts** Missing explanation. May be correct, but not a proof.
- **- 4 pts** Proposition is not correct, or is not limited to the operators as instructed.
- ✓ **- 2 pts** **Insufficient proof -- a proof should guide a reader to your conclusion like an essay.**
- ✓ **- 2 pts** **Truth tables are inaccurate, or not totally clear**
- **- 1 pts** Inaccurate / inconsistent symbols, but logic is clear

PROBLEM 2

A half dozen different operators may appear in propositional formulas, but just $\land$, $\lor$, and $\neg$ are enough to express every proposition. That is because each of the operators is equivalent to a simple formula using only these three operators. For example, $A \to B$ is equivalent to $\neg A \lor B$. So all occurrences of $\to$ in a formula can be replaced using just $\neg$ and $\lor$.

(A) Write a proposition using only $\land$, $\lor$, $\neg$ that is equivalent to $A \oplus B$. Prove your answer.
(B) Write a proposition using only $\land$, $\lor$, $\neg$ that is equivalent to $A \iff B$. Prove your answer.
(C) Prove that we don't even need $\land$, that is, write a proposition using **only** $\lor$ and $\neg$ that is equivalent to $A \land B$. Prove your answer.
(D) Prove that we can get by with the single operator NAND, written $\uparrow$, where $A \uparrow B$ is equivalent by definition to $\neg(A \land B)$. To do so, write propositions using **only** $\uparrow$ that are equivalent to 1) $\neg A$ and 2) $A \lor B$. Prove your answer. Because NAND is sufficient and easy to build in a digital circuit, in practice it is often actually the case that NAND is the only operator.

(A)

A	B	$A \oplus B$	$(A \lor B) \land \neg(A \land B)$
T	T	F	T F F T
T	F	T	T T T F
F	T	T	T T T F
F	F	F	F F F F

(B)

$A \iff B$	$(\neg A \lor B) \land (\neg B \lor A)$
T	F T T T F T T
F	F F F F T T T
F	T T T F F F F
T	T T F T T T F

(C)

A	B	$A \land B$	$\neg(\neg A \lor \neg B)$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

$\neg(A \lor B) \equiv \neg A \land \neg B$

$\neg(\neg A \lor \neg B) = \neg\neg A \land \neg\neg B$

$\equiv$

$A \land B$

(D) ① $A \uparrow B \equiv \neg(A \land B) \equiv \neg A \lor \neg B$

$A \uparrow A \equiv \neg(A \land A) \equiv \neg A$

②

$\neg A \uparrow \neg B \equiv \neg(\neg A \land \neg B)$

$=$

$\neg\neg A \lor \neg\neg B \equiv A \lor B$

$\equiv$

$(A \uparrow A) \uparrow (B \uparrow B)$

2.3 C **6 / 8**

- **- 0 pts** Correct
- ✓ **- 2 pts** **Insufficient proof. You should bring your reader along like an essay.**
- **- 4 pts** Incorrect proposition or not restricted to the symbols as instructed.
- **- 4 pts** Unclear, incomplete, or missing proof
- **- 1 pts** Unclear explanation, though the proper pieces are all there

PROBLEM 2

A half dozen different operators may appear in propositional formulas, but just $\wedge$, $\vee$, and $\neg$ are enough to express every proposition. That is because each of the operators is equivalent to a simple formula using only these three operators. For example, $A \to B$ is equivalent to $\neg A \vee B$. So all occurrences of $\to$ in a formula can be replaced using just $\neg$ and $\vee$.

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(B) Write a proposition using only $\wedge$, $\vee$, $\neg$ that is equivalent to $A \iff B$. Prove your answer.
(C) Prove that we don't even need $\wedge$, that is, write a proposition using **only** $\vee$ and $\neg$ that is equivalent to $A \wedge B$. Prove your answer.
(D) Prove that we can get by with the single operator NAND, written $\uparrow$, where $A \uparrow B$ is equivalent by definition to $\neg(A \wedge B)$. To do so, write propositions using **only** $\uparrow$ that are equivalent to 1) $\neg A$ and 2) $A \vee B$. Prove your answer. Because NAND is sufficient and easy to build in a digital circuit, in practice it is often actually the case that NAND is the only operator.

(A)

A	B	$A \oplus B$	$(A \vee B) \wedge \neg(A \wedge B)$
T	T	F	T F F T
T	F	T	T T T F
F	T	T	T T T F
F	F	F	F F F F

(B)

$A \iff B$	$(\neg A \vee B) \wedge (\neg B \vee A)$
T	F T T T F T T
F	F F F F T T T
F	T T T F F F F
T	T T F T T T T F

(C)

A	B	$A \wedge B$	$\neg(\neg A \vee \neg B)$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

$\neg(A \vee B) \equiv \neg A \wedge \neg B$

$\neg(\neg A \vee \neg B) = \neg\neg A \wedge \neg\neg B$

$\equiv$

$A \wedge B$

(D) ① $A \uparrow B \equiv \neg(A \wedge B) \equiv \neg A \vee \neg B$

$A \uparrow A \equiv \neg(A \wedge A) \equiv \neg A$

②

$\neg A \uparrow \neg B \equiv \neg(\neg A \wedge \neg B)$

$=$

$\neg\neg A \vee \neg\neg B \equiv A \vee B$

$\equiv$

$(A \uparrow A) \uparrow (B \uparrow B)$

②

2.4 D **16 / 16**

✓ **- 0 pts** Correct

- 1 pts Part 1) Correct explanation but never stated (A NAND A) only (A NAND B)

- 6 pts Part 1) Incorrect or missing proposition, but some explanation made

- 4 pts Part 1) Correct proposition, but no explanation

- 1 pts Part 1) Slightly wrong explanation

- 8 pts Part 1) Incorrect or missing proposition

- 1 pts Part 1) Technically correct, but it can be simplified

- 8 pts Part 2) Incorrect or missing proposition

- 6 pts Part 2) Incorrect or missing proposition, but some explanation made

- 1 pts Part 2) Technically correct, but it can be simplified

- 4 pts Part 2) Correct proposition, but no explanation

- 2 pts Part 2) Correct proposition, but incomplete explanation

- 2 pts Part 2) Missing parens

PROBLEM 3

For each of the following propositions:

(A) $\forall x \exists y . 2x - y = 0$
(B) $\forall x \exists y . x - 2y = 0$
(C) $\forall x . x < 10 \rightarrow (\forall y . y < x \rightarrow y < 9)$
(D) $\forall x \exists y . [y > x \wedge \exists z . y + z = 100]$

determine which propositions are valid when the variables range over:

1. the nonnegative integers (i.e., the natural numbers, $\mathbb{N}$)
2. the integers ($\mathbb{Z}$)
3. the real numbers ($\mathbb{R}$)

If the proposition is true in a domain, make an argument for its validity. If it is false, provide a counterexample.

(A) 1. True If $x=3$, then pick $=6$	2. True Same as Part-I.	3. True. Same as Part-I.
(B) 1. False Because if $x=1$, then there is no y such that $1-2y=0$.	2. False <- Same as Part-I.	3. True. $x=+2y$ $\frac{x}{2}=y$
(C) 1. True For any number.	2. True Same as Part-I	3. False Pick $x=9.9$ $y=9.8$
(D) 1. False $x=100$	2. True Pick any number.	3. True. Same as Part-II

3.1 A 3 / 9

- ✓ **- 0 pts** Correct
- **- 3 pts** Incorrect for nonnegative integers. Let y = 2x and the claim is always true.
- **- 3 pts** Incorrect for integers. Let y = 2x and the claim is always true.
- **- 3 pts** Incorrect for real numbers. Let y = 2x and the claim is always true.
- **- 1 pts** One domain is not addressed in explanation
- **- 2 pts** Explanation is incorrect/insufficient for one domain
- **- 4 pts** Explanation is incorrect/insufficient for two domains
- ✓ **- 6 pts Explanation is incorrect/insufficient for three domains**

1 Giving an example is not sufficient to argue that the proposition is valid

PROBLEM 3

For each of the following propositions:

(A) $\forall x \exists y. 2x - y = 0$
(B) $\forall x \exists y. x - 2y = 0$
(C) $\forall x. x < 10 \rightarrow (\forall y. y < x \rightarrow y < 9)$
(D) $\forall x \exists y. [y > x \wedge \exists z. y + z = 100]$

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If the proposition is true in a domain, make an argument for its validity. If it is false, provide a counterexample.

(A) 1. True If $x=3$, then pick $=6$	2. True Same as Part-I.	3. True. Same as Part-I.
(B) 1. False Because if $x=1$, then there is no y such that $1-2y=\emptyset$.	2. False <- Same as Part-I.	3. True. $x=+2y$ $\frac{x}{2}=y$
(C) 1. True For any number.	2. True Same as Part-I	3. False Pick $x=9.9$ $y=9.8$
(D) 1. False $x=100$	2. True Pick any number.	3. True. Same as Part-II

3.2 B **9 / 9**

✓ **- 0 pts** Correct

- 3 pts Incorrect for nonnegative integers. Consider an odd number x. y = x/2 is not an integer.

- 3 pts Incorrect for integers. Consider an odd number x. y = x/2 is not an integer.

- 3 pts Incorrect for real numbers

- 1 pts One domain is not addressed in explanation

- 2 pts Explanation is incorrect/insufficient for one domain

- 4 pts Explanation is incorrect/insufficient for two domains

- 6 pts Explanation is incorrect/insufficient for three domains

PROBLEM 3

For each of the following propositions:

(A) $\forall x \exists y. 2x - y = 0$
(B) $\forall x \exists y. x - 2y = 0$
(C) $\forall x. x < 10 \rightarrow (\forall y. y < x \rightarrow y < 9)$
(D) $\forall x \exists y. [y > x \wedge \exists z. y + z = 100]$

determine which propositions are valid when the variables range over:

1. the nonnegative integers (i.e., the natural numbers, $\mathbb{N}$)
2. the integers ($\mathbb{Z}$)
3. the real numbers ($\mathbb{R}$)

If the proposition is true in a domain, make an argument for its validity. If it is false, provide a counterexample.

	1.	2.	3.
(A)	True If $x=3$, then pick $=6$	True Same as Part-I.	True. Same as Part-I.
(B)	False Because if $x=1$, then there is no y such that $1-2y=0$.	False <- Same as Part-I.	True. $x=+2y$ $\frac{x}{2}=y$
(C)	True For any number.	True Same as Part-I	False Pick $x=9.9$ $y=9.8$
(D)	False $x=100$	True Pick any number.	True. Same as Part-II

3.3 C 5 / 9

- **- 0 pts** Correct
- **- 3 pts** Incorrect/missing for nonnegative integers
- **- 3 pts** Incorrect/missing for integers
- **- 3 pts** Incorrect/missing for real numbers. Consider x = 9.9, y = 9.5. y > 9, so the proposition is false.
- **- 2 pts** Explanation is incorrect/insufficient for one domain
- ✓ **- 4 pts Explanation is incorrect/insufficient for two domains**
- **- 1 pts** Reasoning about implication not included or reasoning about cases of antedecent being true or false.
- **- 1 pts** Reasoning is partially incorrect/unclear
- **- 2 pts** missing/incorrect counter example

PROBLEM 3

For each of the following propositions:

(A) $\forall x \exists y. 2x - y = 0$
(B) $\forall x \exists y. x - 2y = 0$
(C) $\forall x. x < 10 \rightarrow (\forall y. y < x \rightarrow y < 9)$
(D) $\forall x \exists y. [y > x \land \exists z. y + z = 100]$

determine which propositions are valid when the variables range over:

1. the nonnegative integers (i.e., the natural numbers, $\mathbb{N}$)
2. the integers ($\mathbb{Z}$)
3. the real numbers ($\mathbb{R}$)

If the proposition is true in a domain, make an argument for its validity. If it is false, provide a counterexample.

(A) 1. True If $x=3$, then pick $= 6$	2 True Same as Part-I.	3. True. Same as Part-I.
(B) 1. False Because if $x=1$, then there is no y such that $1-2y=\emptyset$.	2. False <- Same as Part-I.	3. True. $x = +2y$ $\frac{x}{2} = y$
(C) 1. True For any number.	2. True Same as Part-I	3. False Pick $x = 9.9$ $y = 9.8$
(D) 1. False $x = 100$	2. True Pick any number.	3. True. Same as Part-II

3.4 D **5 / 9**

- **- 0 pts** Correct
- **- 3 pts** 1) The claim is actually false for the Natural numbers. Consider x = 101. Then y>101 and z = 100 - y, but z cannot be negative, so the claim does not hold
- **- 3 pts** The claim is actually true for the integers. Let y = x+1. Then z = 100 - y.
- **- 3 pts** The claim is actually true for the reals. Let y = x+1. Then z = 100 - y.
- **- 6 pts** Lacks justification for all three cases
- ✓ **- 4 pts lacks justification for 2 cases**
- **- 2 pts** \sqrt(-1) is not a real number. The proposition is actually true for the reals
- **- 3 pts** (A). The claim is actually false.
- **- 2 pts** Correct reasoning for the integers and the reals. However, when you want to show that a universal (for all) claim is true, it is not sufficient to provide an example. You need to make an argument for why the claim is true for every x.
- **- 2 pts** The claim is invalid for natural numbers but 2 is not greater than 2 so y is not greater than x.