example (Monte Hall Game)

You are on a game show facing 3 doors. Behind one of the doors is a car, behind 2 other doors are goats.

You choose door #1.

Before opening the door, the host opens door #2, showing a goat.

Host offers you to choose again — stay with #1 or switch to #3.

Should you switch?

Let C_1, C_2, C_3 = events that the car is behind door #1, 2 or 3

E = host opens door #2 and a goat is shown.

$$P(C_3 \mid E) =$$

$$= \frac{P(E \mid C_3)\,\overbrace{P(C_3)}}{\sum_{i=1}^{3} P(E \mid C_i)\,P(C_i)}$$

$$= \frac{1 \times 1/3}{\frac{1}{2} \times \frac{1}{3} + 0 \times \frac{1}{3} + 1 \times \frac{1}{3}}$$

$$= 2/3$$

$\Rightarrow$ switch! //

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google more advanced versions of Monte Hall, when host is more intelligent.

A quick note on conditional probabilities.

Let F s.t. $P(F) > 0$. Then

axiom 1) $0 \le P(E \mid F) \le 1$;

axiom 2) $P(S \mid F) = 1$;

axiom 3) If E_i, $i = 1, 2, \ldots$, are mutually exclusive then

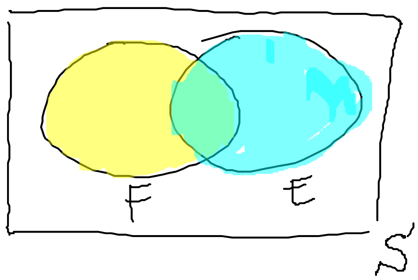
∞ ∞

mutually exclusive then

$$P\left(\bigcup_{i=1}^{\infty} E_i \mid F\right) = \sum_{i=1}^{\infty} P(E_i \mid F)$$

Conditional Probabilities "act" similarly to unconditional probabilities.

$$S \longrightarrow \tilde{S} = F$$



Now, all probabilities are relative frequencies of E within F

Random Variables & Their Distributions.

example Toss 2 fair dice

May be interested in their sum rather then individual values on each die.

Def Real-valued functions defined on the sample space are random variables, are random variables,

$$X : S \to R$$

X ← random variable.

example 1) $S = \{1, 2, 3\}$

* X is the identity fn,

$X(i) = i \quad , \quad i = 1, 2, 3$

$\Rightarrow$ random variable

* $X(2) = X(3) = 5$

$X(1) = 1$

$\Rightarrow$ random variable

2) 2 dice example

$$X(\{i, j\}) = i + j$$

$1 \le i \le 6, \quad 1 \le j \le 6$

Note : r.v. are functions

For any $w \in S$ there exists only one $x \in R$ s.t. $X(w) = x$

Convention : We will use capitol letters for random variables, and script letters for specific values.

<u>example</u> 1) Suppose we toss 3 fair coins

$$S = \{HHH, HHT, \ldots\ldots \}$$

$$2^3 = 8 \text{ outcomes}$$

Let $Y : S \to \mathbb{R}$

$= \#$ of heads in the outcome

Values are $0, 1, 2, 3$

$$P(Y=0) = \frac{1}{8} = P(Y=3)$$

$$P(Y=1) = \frac{3}{8} = P(Y=2)$$

$$\sum_{i=0}^{3} P(Y=i) = 1$$

a way to check your computations!

$HHH \to 3$

$\left.\begin{matrix} HHT \\ HTH \\ THH \end{matrix}\right\} \to 2$

$\left.\begin{matrix} HTT \\ THT \\ TTH \end{matrix}\right\} \to 1$

$TTT \to 0$

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2) Independent tosses of a coin until a "head" occurs.

$$P(H) = p \in (0,1)$$

$X = \#$ of trials until first "heads"

Values of X? 1, 2, 3, ..., i.e. $\mathbb{N}$

$\underbrace{H\ldots\ldots}_{\downarrow\; 1}$, $\underbrace{TH\ldots\ldots}_{\downarrow\; 2}$, ...

$\underbrace{\underbrace{TTT\ldots T}_{k-1} H \ldots}_{\downarrow\; k}$, ...

$$P(X=1) = p$$

$$P(X=2) = (1-p) \times p$$

$$P(X=3) = (1-p)^2 \times p$$

$$\vdots$$

$$P(X=k) = (1-p)^{k-1} \times p$$

$$\vdots$$

$$\sum_{k=1}^{\infty} P(X=k) = \sum_{k=1}^{\infty} (1-p)^{k-1} \cdot p$$

$$= p \cdot \frac{1}{1-(1-p)} = 1 \quad //$$

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3 Urn contains 11 balls:

3 white, 3 red, 5 black

A player wins \$ 1 each time

a ball selected is white & looses \$1 if a red ball is selected. Suppose 3 ball are chosen w/o replacement.

Let X = winnings of the player

Possible values of X? $0, \pm 1, \pm 2, \pm 3$

$$P(X=0) = P(\text{all black}) + P(R, W, B)$$

$$= \frac{\binom{5}{3}}{\binom{11}{3}} + \frac{\binom{3}{1}\binom{3}{1}\binom{5}{1}}{\binom{11}{3}}$$

$$= \frac{55}{165}$$

$$P(X=1) = P(2B, 1W) + P(2W, 1R)$$

$$= \frac{\binom{5}{2}\binom{3}{1}}{\binom{11}{3}} + \frac{\binom{3}{2}\binom{3}{1}}{\binom{11}{3}}$$

$$= \frac{39}{165} = P(X=-1)$$

$$P(X=2) = P(X=-2) =$$

$$= P(2W, 1B) =$$

$$= \frac{\binom{3}{2}\binom{5}{1}}{\binom{11}{3}} = 15/165$$

$$P(X=3) = P(X=-3) =$$

$$= P(3W) = \frac{1}{\binom{11}{3}} \quad \frac{1}{165}$$

$$\sum_{x=-3}^{3} P(X=x) =$$

$$= \frac{55 + (39+15+1)\times 2}{165} = 1 \quad //$$

Discrete Random Variables

Def A random variable that can take on at most countably many values is called discrete r.v. ← random variable ;

we say that this r.v. has discrete distribution.

(i.e. list of all possible values & their probabilities)

For discrete r.v. X we define a probability mass function (pmf)

$$p(x) = P(X = x)$$

$$p : \mathbb{R} \to [0, 1]$$

Clearly, 1) $0 \le p(x) \le 1$;

2) $\sum_{x : \text{all possible values}} p(x) = 1$;

3) If x is not a possible value of X

$\Rightarrow p(x) = 0$;

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examples 1) Uniform distribution on $\{1, 2, \ldots, N\}$

All outcomes (or values $1, 2, \ldots, N$) are equally likely.

X is a r.v. that takes values $1, 2, \ldots N$ for any $1 \le x \le N$ (x is integer)

$$P(X = x) = \frac{1}{N}.$$

Shorthand, $X \sim \text{Unif}\{N\}$

"distributed as"

"parameter" (# of values here)

(# of values here)

example $X \sim Unif\{5\}$

$$P(X \geq 3) = 3/5$$

2) Binomial Distribution

Shorthand $X \sim Bin(N, p)$ parameters

X = number of successes in N trials where the following holds:

1) Fixed (known in advance) # of trials, N
2) Trials are independent;
3) Each trial results in "success" with probability p.

examples a) A fair coin is tossed 10 times.

X = # of heads among 10 tosses.

$$X \sim Bin(10, \tfrac{1}{2})$$

b) # of phone calls received in one hour by a phone operator? No

c) Household with 5 adults

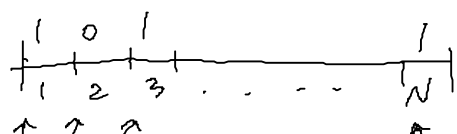
X = # among them that will vote for a democrat in the next election.

Likely not a binomial since independence may fail.

X = # of successes among N independent trials, P("success") is p.

Values of X? $\{0, 1, \dots, N\} \ni x$

pmf $\quad p(x) = P(X = x) = \binom{N}{x} p^x (1-p)^{N-x}$



$p(1-p)p \cdots\cdots p \;=\; p^x (1-p)^{N-x}$

$\binom{N}{x}$ such outcomes $\Rightarrow$ add.

$$\sum_{x=0}^{N} P(X = x) = \sum_{x=0}^{N} \binom{N}{x} p^x (1-p)^{N-x}$$

$$\boxed{\begin{array}{c}\text{Binomial formula} \\ (a+b)^N = \sum_{k=0}^{N} \binom{N}{k} a^k b^{N-k}\end{array}}$$

$$= \left(p + (1-p)\right)^N = 1$$

More on important distributions later

We first study common features useful for describing distributions

<u>Def</u> The cumulative distribution function (cdf) of a random variable X is

$$F : \mathbb{R} \to [0,1] \text{ such that}$$

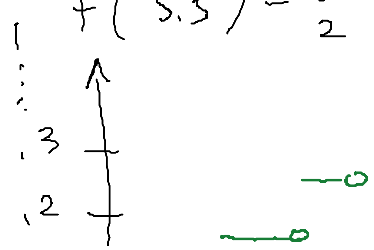
$$F(x) = P(X \le x)$$

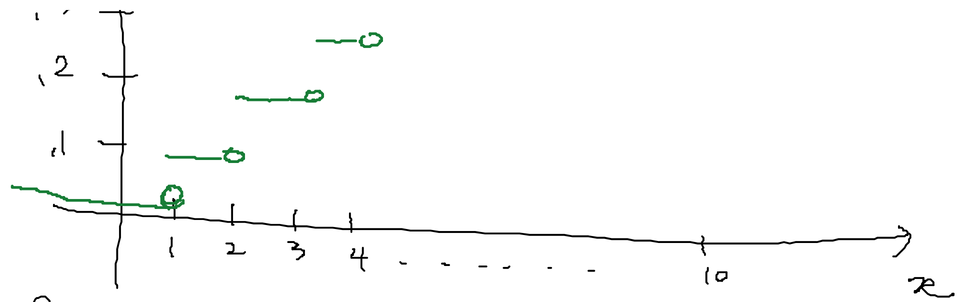
<u>example</u> Consider $X \sim \text{Unif}\{10\}$

$$F(x) = P(X \le x) = \sum_{y \le x} p(y)$$

$$F(-1) = 0 \; ; \quad F(2) = \frac{1}{5} = p(1) + p(2)$$

$$F(5.5) = \frac{1}{2} \; , \quad F(10) = 1$$





1) Cdfs completely describe distribution (unique description);

2) $F(x)$, cdf, are nondecreasing

Let $x \geq y$ $\quad \{X \leq x\} \supset \{X \leq y\}$

$$F(x) = P(X \leq x) \geq P(X \leq y) = F(y)$$

3) $F(x)$ is a right continuous fn

$\lim_{y \downarrow x} F(y) = F(x)$ and

$$\lim_{x \to -\infty} F(x) = 0 ;$$

$$\lim_{x \to +\infty} F(x) = 1 .$$

<u>Note</u>: pmfs are defined for discrete dist'ns only while cdfs are defined for all real-valued r.v.'s.

Continuous Distributions

Generally, random variables that can values in an entire interval, bounded or not, have continuous distributions; and r.v.'s are called continuous or not, have continuous distributions; and r.v.'s are called continuous r.v.'s

Def X has a continuous dist'n

if there exists a non-negative function $f: \mathbb{R} \to [0, \infty)$ s.t. for any $A \subset \mathbb{R}$ (measurable)

$$P(X \in A) = \int_A f(x)\,dx$$

example $A = [a, b)$

$$P(X \in A) = P(a \le X < b)$$
$$= \int_{[a,b)} f(x)\,dx =$$
$$= \int_a^b f(x)\,dx$$

Such function f is called a probability density function (pdf) density

1) $f(x) \geq 0$, all $x \in \mathbb{R}$

2) $\int f(x)\,dx = \int^{\infty} f(x)\,dx = 1$

2) $\int_{\mathbb{R}} f(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx = 1$

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example: Uniform distribution on an interval $[a, b]$

$X \sim \mathrm{Unif}[a, b]$

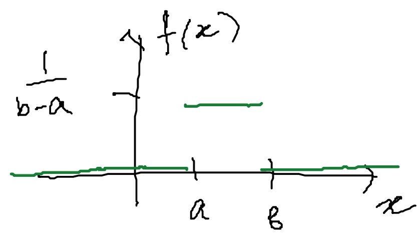
pdf, density $f(x) = \begin{cases} \text{const}, & a \le x \le b \\ 0, & \text{o.w.} \end{cases}$

$1 = \int_{-\infty}^{\infty} f(x)\,dx = \int_a^b \text{const}\,dx =$

$= \text{const}\,(b-a) \Rightarrow \text{const} = \frac{1}{b-a}$

$\Rightarrow f(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b \\ 0, & \text{o.w.} \end{cases}$

"otherwise"



A

area $= \int_A f(x)\,dx$

If $a = 0$, $b = 1$ (standard uniform)

$P(X \in (\frac{1}{2}, \frac{3}{4})) = \int_{1/2}^{3/4} 1\,dx = \frac{1}{4}$

$$P\left(X \in \left(\tfrac{1}{2}, \tfrac{3}{4}\right)\right) = \int_{\frac{1}{2}} 1\, dx = \frac{1}{4}$$

$$A = (.2, 1.3]$$

$$P(X \in A) = P(.2 < X \le 1.3)$$

$$= \int_{.2}^{1.3} f(x)\,dx = \int_{.2}^{1} 1\,dx + \int_{1}^{1.3} 0\,dx$$

$$= .8$$

For any $x \in \mathbb{R}$, if X is a cont. r.v. then

$$P(X = \{x\}) = \int_{x}^{x} f(u)\,du = 0, \quad \forall x \in \mathbb{R}$$

$$X \sim \text{Unif } [a, b]$$

$$F(x) = \begin{cases} 0, & x < a \\ \dfrac{x-a}{b-a}, & a \le x \le b \\ 1, & x > b \end{cases}$$

$a \le x \le b$

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(u)\,du$$

$$= \int_a^x \frac{1}{b-a}\,du = \frac{x-a}{b-a}$$