Harvard University
Computer Science 20

Problem Set 8

Due Monday, April 5, 2021 at 11:59pm.

PROBLEM 1

The following is a map of South America.

(A) Draw a graph that represents the adjacency of countries. (There should be a vertex for each country and an edge should connect two countries if and only if the two countries are adjacent.)
(B) Find a four coloring of the graph.
(C) Find a three coloring, or explain why no three coloring can exist.
(D) Suppose you are tasked with building a freight rail system for the continent (excluding the Falkland Islands). Your engineers tell you that the default cost to build between any two adjacent countries is 1. However, the cost of a rail from Brazil to any of its neighbors *except Peru* is 5 because of the distance. Because of the Andes mountains, the costs of several other legs are also high. The cost of a rail between Chile and Argentina is 10. The cost of any rail to/from Ecuador (*except to Peru*) is 6 and the cost of any rail to/from Peru is 8. Augment your graph with the costs of each leg.

You decide to use Prim's Algorithm to find a lowest cost spanning tree of your graph that will determine which legs of rail to build. Prim's Algorithm works as follows:

1. Start with a tree T that contains an arbitrary vertex from G.
2. Grow the tree one edge at a time by adding a minimum weight edge and its endpoint vertices to T. Select the edge by selecting among the edges that have exactly one endpoint in the tree already.

To receive full credit for this part, provide an ordered list of the vertices and edges you add at each step of the algorithm. Providing a graphical representation of the finished spanning tree may be helpful to your grader but is not necessary.
(E) Prove that Prim's algorithm (applied to any connected graph, not just this specific one) will always produce a spanning tree (not necessarily the minimum one). Hint: This is actually quite easy if you work from the definition of a spanning tree.
(F) Consider the invariant: The tree T is a subtree of a minimum weight spanning tree of G. Prove that Prim's algorithm is correct by proving that this invariant is true at the beginning of the algorithm and is preserved at each step of the algorithm. Hint: Consider the relationship between the minimum weight spanning tree of which T is a subtree and the edge that Prim's algorithm adds at a given step. Could the minimum weight spanning tree have a better edge to add than the one Prim's selects?

Solution.

Solution.

PROBLEM 2

Let $m, n \in \mathbb{Z}$ where $m \neq 0$ and $n \neq 0$. Define a set of integers L as follows:

- **Base cases:** $m, n \in L$
- **Constructor cases:** If $j, k \in L$, then
  1. $-j \in L$

2. $j + k \in L$

Prove by structural induction that every common divisor of m and n also divides every member of L

Solution.

PROBLEM 3

Bonus Problem (Extra Credit). This problem counts for up to 16 points of extra credit but cannot increase your problem set score above 100. The town of Leafton has a private country club, all of whose members are among the town's highest earners. This association among people with a shared characteristic is an example of homophily.

(A) How might the homogeneity of the club's class composition have come about as a result of choice homophily? How might it have come about as a result of induced homophily?

(B) The Leafton town council has become concerned about the social divisions in town that the country club embodies. They are considering requiring the club (and any similar club) to offer free memberships to lower income residents of the community. Does the likelihood that this proposal would be effective at reducing the social divide along class lines in town depend on whether the class homogeneity of the club is a result of induced or choice homophily? Why or why not?

Solution.

Solution.

Problem set by **FILL IN YOUR NAME HERE**

Collaboration Statement: **FILL IN YOUR COLLABORATION STATEMENT HERE (See the syllabus for information)**