Advanced Calculus, Homework 1

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Problem 1. Let $\{A_\alpha\}_{\alpha\in I}$ be an indexed collection of subsets of a set X. Prove:
a. $X \setminus (\bigcup_{\alpha\in I} A_\alpha) = \bigcap_{\alpha\in I}(X \setminus A_\alpha)$
b. $X \setminus (\bigcap_{\alpha\in I} A_\alpha) = \bigcup_{\alpha\in I}(X \setminus A_\alpha)$

(a.) We prove containment in both directions. For $X \setminus (\bigcup_{\alpha\in I} A_\alpha) \subseteq \bigcap_{\alpha\in I}(X \setminus A_\alpha)$ start by letting $x \in X \setminus (\bigcup_{\alpha\in I} A_\alpha)$. Then certainly $x \in X$ and $x \notin \bigcup_{\alpha\in I} A_\alpha$. This second fact entails that $x \notin A_\alpha$ for every $\alpha \in I$. Hence $x \in X \setminus A_\alpha$ for every $\alpha \in I$. Then $x \in \bigcap_{\alpha\in I}(X \setminus A_\alpha)$.

We can reproduce this in the other direction. Let $x \in \bigcap_{\alpha\in I}(X \setminus A_\alpha)$. Then we have $x \in X \setminus A_\alpha$ for every $\alpha \in I$. Therefore for each α we have $x \in X$ and $x \notin A_\alpha$. In particular we now know $x \in X$. Also because $x \notin A_\alpha$ for each α then $x \in \bigcap_{\alpha\in I} A_\alpha$. Now we can infer $x \in X \setminus (\bigcap_{\alpha\in I} A_\alpha)$.

(b.) The proof is nearly identical to part (a.). If $x \in X \setminus (\bigcap_{\alpha\in I} A_\alpha)$ then $x \in X$ and $x \notin \bigcap_{\alpha\in I} A_\alpha$. So $x \notin A'_\alpha$ for at least one $\alpha' \in I$. Then $x \in X \setminus A'_\alpha$ for that α'. Then $x \in \bigcup_{\alpha\in I}(X \setminus A_\alpha)$.

Going in the other direction, $x \in \bigcup_{\alpha\in I}(X \setminus A_\alpha)$. So there is at least one $\alpha \in I$ such that $x \in X \setminus A_\alpha$. Thus $x \in X$ and for some $\alpha' \in I$ we have $x \notin A_{\alpha'}$. Then $x \notin \bigcap_{\alpha\in I} A_\alpha$ and we have arrived at $x \in X \setminus (\bigcap_{\alpha\in I} A_\alpha)$.

Problem 2. Compute:

(a.) $\bigcap_{n=1}^{\infty}[n, \infty)$
(b.) $\bigcup_{n=1}^{\infty}[0, 2-1/n]$
(c.) $\limsup_{n\to\infty}(-1+(-1)^n/n, 1+(-1)^n/n)$
(d.) $\liminf_{n\to\infty}(-1+(-1)^n/n, 1+(-1)^n/n)$

(a.) $\emptyset$

For any $x \in \mathbb{R}$ there is some $n' \in \mathbb{N}$ such that $x < n'$ by the Archimedian property. So $x \notin [n', \infty)$ and therefore $x \notin \bigcap_{n=1}^{\infty}[n, \infty)$.

(b.) $[0, 2)$

Certainly for all $x < 0$ then $x \notin [0, 2-1/n]$ for any $n \in \mathbb{N}$ and just as clearly this is true for $x \geq 2$.

For every $0 \leq x < 2$ we have that $x < 2 - 1/n$ if and only if $1/n < 2 - x$ so that by the Archimedian property there must be some n' satisfying this. For such a choice we have $x \in [0, 2-1/n']$ and therefore $x \in \bigcup_{n=1}^{\infty}[0, 2-1/n]$.

(c.) $[-1, 1]$

First let's see what these intervals are. If we call

$$I_k = \left(-1 + \frac{(-1)^k}{k}, \quad 1 + \frac{(-1)^k}{k}\right)$$

then

$$\begin{aligned} I_1 &= (-2, 0) \\ I_2 &= \left(-\frac{1}{2}, \frac{3}{2}\right) \\ I_3 &= \left(-\frac{4}{3}, \frac{2}{3}\right) \\ I_4 &= \left(-\frac{3}{4}, \frac{5}{4}\right) \\ I_5 &= \left(-\frac{6}{5}, \frac{4}{5}\right) \end{aligned}$$

We first want to union these starting at an arbitrary index. For any $x \in \mathbb{R}$ such that $x < -1$ or $x > 1$ eventually x will lie outside of every interval if the starting index n is chosen large enough. On the other hand every $-1 \leq x \leq 1$ is eventually in some interval. Thus x is in the union no matter where n starts, and then x is in the intersection over all of these.

(d.) $(-1, 1)$

We use the same intervals as before except this time start with intersections and then take unions over them. I claim that for each $x \in \mathbb{R}$ where $x \leq -1$ there is some interval which excludes x and therefore $x \notin I_k$ for some sufficiently large k. Similarly if $x \geq 1$. Hence all such values will not be in the intersection, regardless of the starting index n. Since they will not be in any such intersection thereafter, x will not be in the union over them.

On the other hand, if $-1 < x < 1$ then for some sufficiently large n we will have $x \in I_n$. And then for all $k \geq n$ we will similarly have $x \in I_k$ so that $x \in \limsup_{n \to \infty} I_k$.

Problem 3. Rudin page 21 problem 1. (A rational plus irrational, and a rational times irrational, is always irrational.

Let $r \in \mathbb{Q}$ and $r \neq 0$. Also let $x \in \mathbb{R} \setminus \mathbb{Q}$. We first prove that $r + x \notin \mathbb{Q}$ by contradiction.

Suppose $r + x \in \mathbb{Q}$ so that there are $p, q \in \mathbb{Z}$ with $q \neq 0$ and $r + x = p/q$. Since also $r \in \mathbb{Q}$ we know there are $a, b \in \mathbb{Z}$ with $b \neq 0$ such that $r = a/b$. Then

$$x = \frac{p}{q} - \frac{a}{b} = \frac{pb - qa}{qb}$$

Now $pb - qa \in \mathbb{Z}$ and $qb \in \mathbb{Z}$. Since $q \neq 0 \neq b$ then $qb \neq 0$ and hence $x \in \mathbb{Q}$ contrary to our assumption that $x \notin \mathbb{Q}$. $\lightning$

Now to show that $rx \notin \mathbb{Q}$ we again assume $rx \in \mathbb{Q}$ for contradiction. Then $rx = p/q$ with the obvious constraints on p and q. We again use $r = a/b$. Then

$$x = \frac{pb}{qa}$$

Now $q \neq 0 \neq a$ so $qa \neq 0$ and hence $x \in \mathbb{Q}$ contradicting $x \notin \mathbb{Q}$. $\lightning$

Problem 4. Rudin page 22, problem 2. (Prove there is no rational whose square is 12.)

Suppose for contradiction that there are $p, q \in \mathbb{Z}$ with $q \neq 0$ and both numbers p and q are coprime. And assume $(p/q)^2 = 12$ so that

$$p^2 = 12q^2$$

hence p^2 has a factor of 3 since $12q^2$ does. But then p has a factor of 3 and therefore p^2 has two factors of 3. Say $p^2 = 9k$ so that

$$9k = 12q^2$$

and hence

$$3k = 4q^2$$

This then entails that q^2 has a factor of 3, since 4 does not. But then q has a factor of 3. We then get the contradiction that $(p, q) = 1$ and $(p, q) \geq 3$. $\lightning$

Problem 5. Suppose $f : X \to Y$ and $B \subseteq Y$. Prove that $f\Big(f^{-1}(B)\Big) \subseteq B$ and equality holds if f is onto.

Let $y \in f\Big(f^{-1}(B)\Big)$ so that by definition there is some $x \in f^{-1}(B)$ such that $f(x) = y$. We want to then show that $y \in B$. Since we have $x \in f^{-1}(B)$ we know that $f(x) \in B$ but then $y = f(x) \in B$. $\square$

For the second part, suppose f is onto. Then let $b \in B$ so that we'd like to prove $b \in f\Big(f^{-1}(B)\Big)$. Since f is onto we know there is some $x \in X$ such that $f(x) = b$. By definition $x \in f^{-1}(B)$. Then $f(x) = b \in f\Big(f^{-1}(B)\Big)$. $\square$

Problem 6. Suppose $f : X \to Y$ and $\{A_\alpha\}_{\alpha \in I}$ is an indexed collection of subsets of a set X. Prove $f\left(\bigcap_{\alpha \in I} A_\alpha\right) \subseteq \bigcap_{\alpha \in I} f(A_\alpha)$ with equality if f is one-to-one.

Let $y \in f\Big(\bigcap_{\alpha \in I} A_\alpha\Big)$ so that there must be some $x \in \bigcap_{\alpha \in I} A_\alpha$ such that $f(x) = y$. Then for all $\alpha \in I$ we have that $x \in A_\alpha$, and then $y \in f(A_\alpha)$. But then we have $y \in \bigcap_{\alpha \in I} f(A_\alpha)$. $\square$

Now for the second part assume that f is one-to-one. Also let $y \in \bigcap_{\alpha \in I} f(A_\alpha)$. Then for each $\alpha \in I$ we have $y \in f(A_\alpha)$. Because f is one-to-one it has an inverse, and then $f^{-1}(y) \in A_\alpha$. Thus we have $f^{-1}(y) \in \bigcap_{\alpha \in I} A_\alpha$ and since $f(f^{-1}(y)) = y$ we must have $y \in f(\bigcap_{\alpha \in I} A_\alpha)$. $\square$

Problem 7. Prove $|a-b| \leq |a-c| + |c-d| + |d-b|$ for all $a, b, c, d \in \mathbb{C}$.

This is the triangle inequality with two terms wiggled in.

$$\begin{aligned} |a-b| &= |a-c+c-d+d-b| \\ &\leq |a-c| + |c-d+d-b| \\ &\leq |a-c| + |c-d| + |d-b| \end{aligned}$$

Problem 8. Rudin page 23, problem 12. (Triangle inequality generalized to finite sums.)

Proof by induction: The case for two terms is the base-case and is the triangle inequality.

For the inductive case, suppose the theorem has been proved up to n terms. Then

$$|z_1 + z_2 + \cdots + z_{n+1}| \leq |z_1 + \cdots + z_n| + |z_{n+1}|$$

by a single application of the triangle inequality. Then by the inductive hypothesis

$$|z_1 + \cdots + z_n| + |z_{n+1}| \leq |z_1| + \cdots + |z_{n+1}|$$

Chaining these inequalities together gives

$$|z_1 + \cdots + z_n| \leq |z_1| + \ldots |z_n|$$

Problem 9. Rudin page 23, problem 13. (Reverse triangle inequality.)

We use the triangle inequality

$$|a+b| \le |a| + |b|$$

but with $a = x$ and $b = y - x$. Then we have

$$|x - (y - x)| \le |x| + |y - x|$$

which is

$$|y| \le |x| + |y - x|$$

which entails

$$|y| - |x| \le |y - x| = |x - y|$$

Now we repeat the whole argument but with $a = y$ and $b = x - y$

$$|y - (x - y)| \le |y| + |x - y|$$

which entails

$$|x| \le |y| + |x - y|$$

so

$$|x| - |y| \le |x - y|$$

Thus if $|x| - |y| \ge 0$ then

$$||x| - |y|| = |x| - |y| \le |x - y|$$

and if $|x| - |y| < 0$ then

$$||x| - |y|| = |y| - |x| \le |x - y|$$

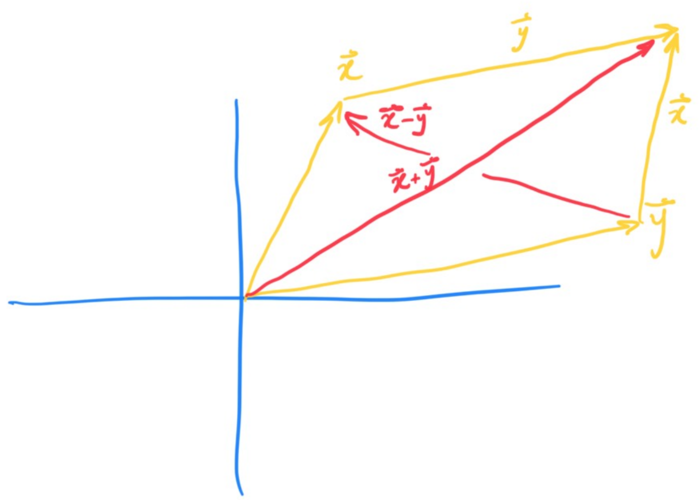
$\square$

Problem 10. Rudin page 23, problem 17. (Polarization identity.)

The proof merely expands everything in terms of the inner product, $|\vec{x}| = \vec{x} \cdot \vec{x}$.

$$
\begin{aligned}
|\vec{x}+\vec{y}|^2 + |\vec{x}-\vec{y}|^2 &= (\vec{x}+\vec{y})\cdot(\vec{x}+\vec{y}) + (\vec{x}-\vec{y})\cdot(\vec{x}-\vec{y}) \\
&= \vec{x}\cdot\vec{x} + 2(\vec{x}\cdot\vec{y}) + \vec{y}\cdot\vec{y} + \vec{x}\cdot\vec{x} - 2(\vec{x}\cdot\vec{y}) + \vec{y}\cdot\vec{y} \\
&= 2|\vec{x}|^2 + 2|\vec{y}|^2
\end{aligned}
$$

The geometric interpretation is that $\vec{x}$ and $\vec{y}$ form sides that you can close into a parallelogram–this is clearest in two or three dimensions, it's just the usual parallelogram addition law for vectors. The diagonal radiating from the origin is equal to $\vec{x}+\vec{y}$ and the remaining diagonal is $\vec{x}-\vec{y}$. The law then says that the sum of squared magnitudes of the diagonals is half the sum of the squared magnitudes of the sides.



Problem 11. Let $y_1 = 6$ and $y_{n+1} = \frac{2y_n - 6}{3}$ for all $n \in \mathbb{N}$. Prove:

(a.) $y_n > -6$ for all $n \in \mathbb{N}$

(b.) $\{y_n\}$ is decreasing.

(a.) We use induction where the base-case, $y_1 = 6 > -6$.

Now suppose the claim holds for all terms y_n with $n \leq N$. Then we show that $y_{N+1} > -6$.

$$y_{N+1} = \frac{2y_N - 6}{3} > \frac{2(-6) - 6}{3} = -6$$

(b.) We prove $y_{n+1} < y_n$ by induction on n. In the base-case, since $y_1 = 6$ and $y_2 = \frac{2(6)-6}{3} = 2$ then we clearly have $y_2 < y_1$.

For the inductive case we suppose this holds for all $n \leq N$ and now prove that $y_{N+2} < y_{N+1}$. The inequation

$$y_{N+2} = \frac{2y_{N+1} - 6}{3} < y_{N+1}$$

is equivalent to

$$\begin{aligned} 2y_{N+1} - 6 < 3y_{N+1} \quad &\Leftrightarrow \\ -6 < y_{N+1} \end{aligned}$$

and since this is guaranteed by part (a.) we're done. $\square$