Advanced Calculus, Homework 2

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Problem 1. Calculate, without proof, the suprema and infima for the following sets:

(a.) $A = \left\{\frac{m}{n} : m, n \in \mathbb{N} \text{ with } m < n\right\}$
(b.) $B = \left\{\frac{(-1)^m}{n} : m, n \in \mathbb{N}\right\}$
(c.) $C = \left\{\frac{n}{3n+1} : n \in \mathbb{N}\right\}$
(d.) $D = \left\{\frac{m}{m+n} : m, n \in \mathbb{N}\right\}$

(a.) $\sup A = 1$ and $\inf A = 0$.
(b.) $\sup B = 1$ and $\inf B = -1$.
(c.) $\sup C = 1/3$ and $\inf C = 1/4$.
(d.) $\sup D = 1, \inf D = 0$.

Problem 2. Rudin page 22, Problem 4. (Lower bounds are below upper bounds.)

Let E be a non-empty set bounded above and below by β and α respectively. We seek to show $\alpha \leq \beta$.

Take any element $x \in E$. Then $\alpha \leq x \leq \beta$.

Problem 3. Rudin page 22, Problem 5. ($\inf A = -\sup(-A)$)

Let A be a nonempty set of real numbers bounded below by α. We seek to show that $\inf A$ exists and is $-\sup(-A)$.

First we establish that $\sup(-A)$ exists, but clearly it must since $-A$ is also nonempty. This is because A is nonempty and if $x \in A$ then $-x \in -A$. Moreover, $-\alpha$ bounds it above. To see this let $x \in A$ so that $\alpha \leq x$ and then $-x \leq -\alpha$.

Hence $\sup(-A)$ exists and I claim that $-\sup(-A) = \inf A$. To see this we need to show both that $-\sup(-A)$ is a lower bound and then that it's the greatest such.

First let's see that it's a lower bound, so let's take any element $x \in A$. Then $-x \in -A$ and $-x \leq \sup(-A)$. Hence $-\sup(-A) \leq x$ and we're done with this part.

Next let's show that it's the greatest lower bound by considering any other lower bound β of A. Then as we've already seen this is enough to entail that $-\beta$ is an upper bound of $-A$. Then $\sup(-A) \leq -\beta$ so that $\beta \leq -\sup(-A)$. $\square$

Problem 4. Rudin page 22, problem 9. (The dictionary order gives $\mathbb{C}$ an order. Does it have the LUB property?)

We first prove commensurability, that is to say, precisely one of

$$w < z, \quad w = z, \quad z < w$$

holds for every $w, z \in \mathbb{C}$. For any two $z = a + ib, w = c + id$, if $a < c$ then $z < w$. If $a > c$ then $z > w$. So all that remains to show is the case where $a = c$.

If $a = c$ then we again have three cases. If $b < d$ then $z < w$ and if $b > d$ then $z > w$, otherwise $z = w$.

Next we prove transitivity. Suppose

$$a + ib < c + id$$
$$\text{and}$$
$$c + id < e + if$$

If $a < c$ then consider two cases: Either $c < e$ or $c = e$ and $d < f$. In either case, $a < e$ so $a + ib < e + if$. On the other hand suppose $a = c$ and $b < d$. Again consider two cases for the second inequality: Either $c < e$ or $c = e$ and $d < f$. If the former holds then $a < e$ and we're done. If the latter holds then $a = e$ and $b < d < f$ so again the first number is less than the last. $\square$

Does this set have the least upperbound property? No, because the set of all ix for $x \in \mathbb{R}$ is bounded above by 1. In fact it's bounded above by every number with real part greater than 0, and not bounded above by any other number. Since there is no least such element, this set has no least upper bound.

Problem 5. Let $A \subseteq \mathbb{R}$ be a nonempty set bounded above. Let $s \in \mathbb{R}$ have the property that for all $n \in \mathbb{N}$, we have $s + \frac{1}{n}$ an upper bound for A and also that $s - \frac{1}{n}$ is not an upper bound for A. Prove $s = \sup(A)$.

We may first want to prove that s is an upper bound. We argue by contradiction, supposing that there is some $x \in A$ such that $s < x$. By the Archimedean property there must be some $n \in \mathbb{N}$ such that $n(x - s) > 1$. So for this n we have $s + \frac{1}{n} < x$. But since $s + \frac{1}{n}$ is an upper bound then $x \leq s + \frac{1}{n}$. ↯

Next we show that it's the least upper bound, so assume $t < s$ and we show that t cannot be an upper bound for A. But $t < s$ implies again by the Archimedean property that there is some $1 < n(s - t)$. So $t < s - \frac{1}{n}$.

Since $s - \frac{1}{n}$ is not an upper-bound then there is some $x \in A$ such that $s - \frac{1}{n} \leq x$. Then $t < x$ and so t is not an upper-bound. $\square$

Problem 6. Prove that if X and Y are countable sets, then $X \times Y$ is.

We use the fact that countable unions of countable sets are countable. Then define $A_x = \{(x, y) | y \in Y\}$. I claim that this set is countable and prove this by exhibiting a bijection between it and Y, which we know to be countable. Take $\pi_2((x, y)) = y$, the projection mapping onto the second coordinate. With $\pi_2 : A_x \to Y$ this is one-to-one. This is because $\pi_2((x, y)) = \pi_2((x, y')) = y = y'$ and for all tuples the left coordinate is x. Hence $(x, y) = (x, y')$. And it is obviously onto since if $y \in Y$ then $\pi_2((x, y)) = y$.

Now

$$X \times Y = \bigcup_{x \in X} A_x$$

which makes this a countable union of countable sets, since the indexing set X is countable.

Problem 7. Give an example of a countable collection of disjoint open intervals. Explain.

Take the collection $\{I_n\}_{n\in\mathbb{N}}$ where

$$I_n = (n, n+1)$$

Obviously each $(n, n+1)$ is an open interval. Moreover each is disjoint from all the others, since they all lack the end-points. Finally the collection is countable because for each $n \in \mathbb{N}$ there is a unique $(n, n+1)$.

Problem 8. Prove that the set of odd integers is countable.

We use the map

$$\begin{aligned} 1 &\mapsto 1 \\ 2 &\mapsto -1 \\ 3 &\mapsto 3 \\ 4 &\mapsto -3 \\ &\vdots \end{aligned}$$

which is $f : \mathbb{N} \to \{z \in \mathbb{Z} : z \text{ is odd}\}$ and

$$f(n) = \begin{cases} n & \text{if } n \text{ is odd} \\ -n+1 & \text{if } n \text{ is even} \end{cases}$$

We now show this is injective. Suppose $f(n_1) = f(n_2)$. For the first case suppose $f(n_1) \geq 0$. We must then have that both n_1 and n_2 are odd, and $f(n_1) = f(n_2) = n_1 = n_2$. On the other hand if $f(n_1) < 0$ then both n_1 and n_2 are even. So $f(n_1) = f(n_2) = -n_1 + 1 = -n_2 + 1$ so $n_1 = n_2$.

And it is surjective. If z is any odd integer, first suppose it's positive. Then $f(z) = z$. Now suppose it's negative, then $f(-z+1) = -(-z+1) + 1 = z$.

I just realized that the definition of a bijection goes in the opposite direction. So let A be the set of odd integers. Since we have just shown $\mathbb{N} \sim A$, and since the book claims on page 25 that this is an equivalence relation, then we have $A \sim \mathbb{N}$.