MA 630 - Homework 3 (Module 2 - Section 1)

Solutions must be typeset in LaTeX and submitted to Canvas as a .pdf file. When applicable, write in complete sentences.

1. Let A and B be sets. For each of the following propositions, either prove it or give a counterexample.

   (a) If $A \subseteq B$, then $\mathscr{P}(A) \subseteq \mathscr{P}(B)$.

   (b) If $\mathscr{P}(A) \subseteq \mathscr{P}(B)$, then $A \subseteq B$.

   (a) Proof: Suppose $A \subseteq B$. We will show by element chasing that $\mathscr{P}(A) \subseteq \mathscr{P}(B)$. So let $X \in \mathscr{P}(A)$. (Note that some X must exist since the power set is never empty. This is because the empty set is a subset of every set.) By definition of the power set, $X \subseteq A$. Since $A \subseteq B$ therefore by the transitivity of the subset relation (Proposition 2.4), $X \subseteq B$. Then $X \in \mathscr{P}(B)$. $\square$

   (b) Proof: Suppose $\mathscr{P}(A) \subseteq \mathscr{P}(B)$. We will show that $A \subseteq B$ by element chasing, but we first have to handle the case when $A = \emptyset$. In this case trivially $A \subseteq B$. Now if $A \neq \emptyset$ then let $x \in A$. Then $\{x\} \subseteq A$ and therefore $\{x\} \in \mathscr{P}(A)$. Then $\{x\} \in \mathscr{P}(B)$ and so $\{x\} \subseteq B$. Therefore $x \in B$. $\square$

2. Prove that
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
   for all sets A, B, C.

   Proof: We show mutual inclusion, and fundamentally we break the proof into cases. The main first case is that neither $A \cup (B \cap C)$ nor $(A \cup B) \cap (A \cup C)$ are the empty set, and of course the remaining case is that one of them is empty.

   With the assumption that neither set is empty, to begin with we show $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$. For element-chasing, let $x \in A \cup (B \cap C)$. Then $x \in A$ or $x \in B \cap C$. We now further sub-divide the proof by cases.

   Case 1: $x \in A$. If $x \in A$ then $x \in A \cup B$ and $x \in A \cup C$. Therefore $x \in (A \cup B) \cap (A \cup C)$.

   Case 2: $x \in B \cap C$. If $x \in B \cap C$ then $x \in B$ and $x \in C$. Therefore $x \in A \cup B$ and $x \in A \cup C$ and so $x \in (A \cup B) \cap (A \cup C)$.

We have now finished the first inclusion, and proceed to show the reverse, $A \cup (B \cap C) \supseteq (A \cup B) \cap (A \cup C)$. Let $x \in (A \cup B) \cap (A \cup C)$ so $x \in A \cup B$ and $x \in B \cup C$. We again sub-divide the proof into two cases.

Case 1: $x \in A$. If $x \in A$ then $x \in A \cup (B \cap C)$.

Case 2: $x \notin A$. If $x \notin A$ then $x \in B$ and $x \in C$. Therefore $x \in B \cap C$ and so $x \in A \cup (B \cap C)$.

Hence we have finished the proof under the assumption that neither set is empty. We now consider the case when $A \cup (B \cap C) = \emptyset$. For contradiction suppose $(A \cup B) \cap (A \cup C) \neq \emptyset$ and in particular let $x \in (A \cup B) \cap (A \cup C)$. From the proof above we have already seen that this implies $x \in A \cup (B \cap C)$ which contradicts that this is the empty set.

Now suppose $(A \cup B) \cap (A \cup C) = \emptyset$ and for contradiction suppose $x \in A \cup (B \cap C)$. Again from the above proof, we know that this implies $x \in (A \cup B) \cap (A \cup C)$, contradicting that this is the empty set. $\square$

3. Prove that for all sets A and B,

$$(A \cup B) - (A \cap B) = (A - B) \cup (B - A).$$

Proof: We prove this by mutual inclusion, and each direction we prove by element chasing. We begin by showing $(A \cup B) - (A \cap B) \subseteq (A - B) \cup (B - A)$, and for now we assume the set on the left is not empty. Let $x \in (A \cup B) - (A \cap B)$. Then $x \in A \cup B$ and $x \notin A \cap B$. Without loss of generality suppose $x \in A$. Then $x \notin B$ and so $x \in A - B$. Thus $x \in (A - B) \cup (B - A)$.

For the reverse inclusion, again for now we assume that the set on the right is not empty. Let $x \in (A - B) \cup (B - A)$ and without loss of generality let $x \in A - B$. Then $x \in A$ and $x \notin B$. Hence $x \in A \cup B$ and $x \notin A \cap B$. So $x \in (A \cup B) - (A \cap B)$.

Finally we handle the case of the empty set. If $(A \cup B) - (A \cap B) = \emptyset$ then for contradiction suppose $x \in (A - B) \cup (B - A)$. The earlier parts of the proof have already shown that this entails $x \in (A \cup B) - (A \cap B)$, contradicting that the set is empty.

If $(A - B) \cup (B - A) = \emptyset$ then for contradiction suppose $x \in (A \cup B) - (A \cap B)$. Again by the previous parts of this proof, $x \in (A - B) \cup (B - A)$, contradicting that this set is empty. $\square$

4. Let A and B be sets.

(a) Prove that $\mathscr{P}(A \cap B) = \mathscr{P}(A) \cap \mathscr{P}(B)$.

(b) Is it true that $\mathscr{P}(A \cup B) = \mathscr{P}(A) \cup \mathscr{P}(B)$? Either prove or give a counterexample.

(a) Proof: We prove this by mutual inclusion, so to start with we show $\mathscr{P}(A \cap B) \subseteq \mathscr{P}(A) \cap \mathscr{P}(B)$. Let $X \in \mathscr{P}(A \cap B)$, which must always exist, as noted problem 1 part (a). Then $X \subseteq A \cap B$. Then $X \subseteq A$ and $X \subseteq B$. Hence $X \in \mathscr{P}(A)$ and $X \in \mathscr{P}(B)$. So $X \in \mathscr{P}(A) \cap \mathscr{P}(B)$.

For the reverse inclusion we begin by assuming $\mathscr{P}(A) \cap \mathscr{P}(B) \neq \emptyset$. So let $X \in \mathscr{P}(A) \cap \mathscr{P}(B)$. Then $X \in \mathscr{P}(A)$ and $X \in \mathscr{P}(B)$. So $X \subseteq A$ and $X \subseteq B$. So $X \subseteq A \cap B$ and so $X \in \mathscr{P}(A \cap B)$.

Finally if $\mathscr{P}(A) \cap \mathscr{P}(B) = \emptyset$ then suppose for contradiction that $X \in \mathscr{P}(A \cap B)$. From the earlier parts of this proof we have that $X \in \mathscr{P}(A) \cap \mathscr{P}(B)$ which contradicts that this is the empty set.

(b) Counter-example: Take $A = \{0\}$ and $B = \{1\}$. Then $\mathscr{P}(A \cup B) = \mathscr{P}(\{0,1\}) = \{\emptyset, \{0\}, \{1\}, \{0,1\}\}$. On the other hand $\mathscr{P}(A) = \{\emptyset, \{0\}\}$ and $\mathscr{P}(B) = \{\emptyset, \{1\}\}$. Hence $\mathscr{P}(A) \cup \mathscr{P}(B) = \{\emptyset, \{0\}, \{1\}\}$. Thus we see

$$\mathscr{P}(A \cup B) \neq \mathscr{P}(A) \cup \mathscr{P}(B).$$

5. For each natural number n, let

$$A_n = \left[\frac{1}{n^2}, 2n+1\right) = \left\{x \in \mathbb{R} : \frac{1}{n^2} \leq x < 2n+1\right\}.$$

Calculate (with proof) $\bigcup_{n \in \mathbb{N}} A_n$ and $\bigcap_{n \in \mathbb{N}} A_n$.

Solution: $\bigcup_{n \in \mathbb{N}} A_n = (0, \infty)$ and $\bigcap_{n \in \mathbb{N}} A_n = [1, 3)$.

Proof: First we show $\bigcup_{n \in \mathbb{N}} A_n = (0, \infty)$. Let $x \in \bigcup_{n \in \mathbb{N}} A_n$ so that there is some $n \in \mathbb{N}$ such that $x \in A_n$. Then $0 < \frac{1}{n^2} \leq x$, so $x \in (0, \infty)$.

Now let $x \in (0, \infty)$. There must exist some $n \in \mathbb{N}$ such that $\frac{1}{x} < n$. Then $\frac{1}{n} < x$ and therefore $\frac{1}{n^2} \leq \frac{1}{n} < x$. Moreover there must exist some m such that $\frac{x-1}{2} < m$, and therefore $x < 2m+1$. So now if we take $N = \max\{m, n\}$ then $\frac{1}{N^2} \leq \frac{1}{n^2} \leq x < 2m+1 \leq 2N+1$ so that $x \in A_N$ and hence $x \in \bigcup_{n \in \mathbb{N}} A_n$.

Next we show $\bigcap_{n\in\mathbb{N}} A_n = [1,3)$. We begin with $\bigcap_{n\in\mathbb{N}} A_n \subseteq [1,3)$. Let $x \in \bigcap_{n\in\mathbb{N}} A_n$ so that $x \in A_n$ for every $n \in \mathbb{N}$. We will prove by contradiction that $x \geq 1$, so suppose that $x < 1$. Then since $A_1 = [1,3)$ we have $x \notin A_1$ and therefore $x \notin \bigcap_{n\in\mathbb{N}} A_n$. Now we show by contradiction that $x < 3$, so suppose $x \geq 3$. Again $x \notin A_1$ so $x \notin \bigcap_{n\in\mathbb{N}} A_n$. So $x \in [1,3)$.

Finally we show $\bigcap_{n\in\mathbb{N}} A_n \supseteq [1,3)$. Let $x \in [1,3)$. Now for any $n \in \mathbb{N}$ we have $\frac{1}{n^2} \leq 1 \leq x$. Also we have $x < 3$ therefore $x - 1 < 2$ and so $\frac{x-1}{2} < 1$. From this we infer $\frac{x-1}{2} < n$ for any $n \in \mathbb{N}$. Therefore

$$\begin{aligned} x - 1 &< 2n \\ x &< 2n + 1 \end{aligned}$$

This then shows that for each $n \in \mathbb{N}$ we have $x \in A_n$ and hence $x \in \bigcap_{n\in\mathbb{N}} A_n$.