MA 630 - Homework 1 (Module 1 - Section 1)

Adam Frank

Solutions must be typeset in LaTeX and submitted to Canvas as a .pdf file. When applicable, write in complete sentences.

1. Let x and y be real numbers. Prove that if $2x^2 - 6x = 2y^2 - 6y$ and $x \neq y$, then $x + y = 3$. (Be sure to comment or note why it is important that $x \neq y$.)

   Solution: The following equations are equivalent

$$2x^2 - 6x = 2y^2 - 6y$$

$$x^2 - 3x = y^2 - 3y$$

$$x^2 - y^2 = 3(x - y)$$

$$(x + y)(x - y) = 3(x - y)$$

   In this last statement we know that $x \neq y$ and therefore $x - y \neq 0$. Hence we can divide both sides by $x - y$ to obtain

$$x + y = 3$$

2. Let a and b be integers. Prove that ab is odd if and only if a and b are both odd.

   Solution: If a, b are two odd numbers, and $a = 2m + 1$ and $b = 2n + 1$, then

$$ab = (2m + 1)(2n + 1) = 4mn + 2m + 2n + 1 = 2(2mn + m + n) + 1$$

   Hence there is an integer t such that $ab = 2t + 1$, namely $t = 2mn + m + n$.

   For the converse, we prove it by the contrapositive. So we will assume that ab is not odd and therefore even. We must show that not both a and b are odd. We will do this by contradiction, so we assume a and b are both odd and try to find a contradiction.

But we already know that if a and b are both odd then ab is odd. This contradicts the first assumption, that ab is even. $\lightning$

This shows that if ab is not odd then not both a and b are odd. Hence if a and b are both odd then ab is odd.

3. Let a and b be integers.

   (a) Prove that if ab is odd, then $a+b$ is even. As always, feel free to reference a previous homework problem.

   (b) Is the converse true? Either prove it or give a counterexample.

   Solution: (a) If ab is odd, from the previous problem (number 2) we already know both a and b are odd. Then there exist m, n such that $a = 2m+1$ and $b = 2n+1$ so that $a+b = 2m+2n+2 = 2(m+n+1)$. Thus $a+b = 2t$ where $t = m+n+1$ and so by definition $a+b$ is even.

   (b) The converse is false because there are some integers a, b such that $a+b$ is even while ab is not odd. For a counter-example take $a = b = 2$. Then $a+b = 4$ is even but $ab = 4$ is not odd.

4. Let m and n be integers which are greater than or equal to 2. Prove that $mn+1$ is not divisible by m.

   Solution: For contradiction suppose $m|(mn+1)$ and let $mn+1 = mk$, where k is an integer. Then $1 = m(k-n)$ so that $m|1$ and therefore $m = \pm 1$. But since $m \geq 2$ this is a contradiction. $\lightning$

5. Let n be an integer such that n^2 is even. Prove that n^2 is divisible by 4.

   Solution: Since n^2 is even, then by Theorem 1.12 we know that n is even. Then $n = 2m$ for some m. Then $n^2 = 4m^2$ which is divisible by 4.

6. Prove that for any natural number n, either n is a prime or a perfect square, or n divides $(n-1)!$.

   Solution: The claim has the form $P \vee Q \vee R$. We will prove the proposition by showing $\neg(P \vee Q) \to R$, which is equivalent to $(\neg P \wedge \neg Q) \to R$. That means we will assume n is not prime and not a perfect square, and use this to show that $n|(n-1)!$.

   Since n is not prime then n has a divisor, $1 < m < n$. Therefore there exists a $1 < k < n$ such that $mk = n$. Because n is not a perfect square then we must have that $m \neq k$.

   Now $(n-1)!$ has every factor from 1 to $n-1$. That is to say

$$(n-1)! = 1 \cdot 2 \cdots (n-1)$$

Because $1 < m < n$ then m must be one of these factors. Moreover, k must be some one of the factors other than m. Therefore $(n-1)! = mk \cdot t$ where t is the product of all the other factors which are not m and k. But since $n = mk$ we have $(n-1)! = n \cdot t$. Then by definition $n|(n-1)!$.