MA 630 - Homework 2 (Module 1 - Sections 1 and 2)

Solutions must be typeset in LaTeX and submitted to Canvas as a .pdf file. When applicable, write in complete sentences.

1. Let x be a positive real number. Prove that if $x - \dfrac{2}{x} > 1$, then $x > 2$ by

   (a) direct proof.

   (b) contrapositive proof.

   (c) proof by contradiction.

   Solution: (a) Direct proof: We assume $x > 0$ and $x - \frac{2}{x} > 1$, and try to show that $x > 2$. Each inequation below implies the next:

$$
\begin{aligned}
x - \frac{2}{x} &> 1 \\
x^2 - 2 &> x \\
x^2 - x - 2 &> 0 \\
(x-2)(x+1) &> 0
\end{aligned}
$$

   So $(x-2)(x+1)$ is a positive real. There are three cases: either both are positive, or both are negative, or one is positive and the other negative. This last case is impossible since the product of a positive and negative is negative. But also both cannot be negative because $x + 1 < 0$ entails $x < -1 < 0$ which contradicts that x is positive, as in the assumption. Hence both factors are positive, by disjunction elimination. So $x - 2 > 0$ and so $x > 2$. $\square$

   (b) Contrapositive: Still $x > 0$. We suppose $x \leq 2$ and show that $x - \frac{2}{x} \leq 1$. Since $x > 0$ then $x + 1 > 0$. Therefore each of the following inequations implies the next

$$
\begin{aligned}
x &\leq 2 \\
x - 2 &\leq 0 \\
(x-2)(x+1) &\leq 0 \\
x^2 - x - 2 &\leq 0 \\
x^2 - 2 &\leq x \\
x - \frac{2}{x} &\leq 1
\end{aligned}
$$

So $x - \frac{2}{x} \leq 1$ which is what we wanted. $\square$

(c) Contradiction: For contradiction suppose both $x - \frac{2}{x} > 1$ and that $x \leq 2$. By the argument in part (b) we already know that $x \leq 2$ entails $x - \frac{2}{x} \leq 1$. This contradicts our first assumption. $\lightning$ $\square$

2. Suppose x is an integer. Prove that $5x - 7$ is odd if and only if $9x + 2$ is even. *Hint: In both directions, first prove that x is even.*

   Solution: We first prove that if $5x-7$ is odd then $9x+2$ is even, so assume $5x-7 = 2k+1$ for some integer k. We will first prove that x is even by contradiction, so suppose x is odd and $x = 2m + 1$ for an integer m. Then

$$\begin{aligned} 5x - 7 &= 2k + 1 \\ 5(2m + 1) - 7 &= 2k + 1 \\ 2m - 2 &= 2k + 1 \\ 2(m - k) &= 3 \end{aligned}$$

   3 is an odd number and by the result above, it is also even, a contradiction. $\lightning$

   We have now shown that x is even, so let $x = 2n$. Then

$$\begin{aligned} 9x + 2 &= 9(2n) + 2 \\ &= 2(9n + 1) \end{aligned}$$

   Since $9n + 1$ is an integer then the above shows $9x + 2$ is even, as desired.

   Now we prove the converse, so assume that $9x + 2$ is even and let $9x + 2 = 2a$. We now prove x is even, again by contradiction, so suppose x is odd. Then let $x = 2b + 1$. Then

$$\begin{aligned} 9x + 2 &= 2a \\ 9(2b + 1) &= 2a \\ 18b + 9 &= 2a \\ 9 &= 2(a - 9b) \end{aligned}$$

   Since $a - 9b$ is an integer the above shows that 9 is even, but also 9 is odd, a contradiction. $\lightning$

   Hence we have that x is even so again suppose $x = 2n$. Then

$$\begin{aligned} 5x - 7 &= 5(2n) - 7 \\ &= 2(5n - 4) + 1 \end{aligned}$$

and since $5n - 4$ is an integer then $5x - 7$ is odd. $\square$

3. Prove that if k is an odd integer, then the equation $x^2 + x - k = 0$ has no integral solution.

   Solution: Suppose k is an odd integer, and we seek to prove $x^2 + x - k = 0$ has no integer solutions. For contradiction suppose $x \in \mathbb{Z}$ is any integer solution of $x^2 + x - k$. Then $x^2 - x = k$. Since k is odd therefore $x^2 + x$ is. But since $x^2 - x = x(x - 1)$ then it suffices to show that either x or $x - 1$ is even (because an even times any integer is even). Now to show that, we can show that if x is not even then $x - 1$ is. To that end let's suppose x is odd and $x = 2k + 1$. Then $x - 1 = 2k$ is even, as desired.

   At this point we have seen that $x(x - 1)$ is even, and therefore k is. But also k is odd, a contradiction. $\lightning$ $\square$

4. Let m and n be integers. Prove that $(m + 1)n^2$ is even if and only if m is odd or n is even.

   Solution: First we prove that if $(m + 1)n^2$ is even then either m is odd or n is even. So suppose $(m + 1)n^2$ is even and $(m + 1)n^2 = 2k$. To show the disjunction, we prove that if m is not odd then n is even. So suppose m is even and $m = 2a$. Then

$$\begin{aligned} (m + 1)n^2 &= 2k \\ (2a + 1)n^2 &= 2k \\ 2an^2 + n^2 &= 2k \\ n^2 &= 2(k - an^2) \end{aligned}$$

   The above shows that n^2 is even. From theorem 1.12 we know that therefore n is even, as desired.

   For the converse, we will show that if m is odd or n is even, then $(m + 1)n^2$ is even. So suppose that either m is odd or n is even. For a proof by cases first assume m is odd and so $m = 2b + 1$. Then

$$\begin{aligned} (m + 1)n^2 &= (2b + 2)n^2 \\ &= 2([b + 1]n^2) \end{aligned}$$

and since $(b+1)n^2$ is an integer the above then shows that $(m+1)n^2$ is even.

For the other case, suppose n is even. By theorem 1.12 then n^2 is even so let $n^2 = 2c$. Then

$$(m+1)n^2 = (m+1)2c = 2([m+1]c)$$

which shows $(m+1)n^2$ as desired. $\square$

5. (a) Let n be an integer. Prove that if n^2 is even, then n^2 is divisible by 4.
   (b) Prove that if k is an odd integer, then $2k$ is not divisible by 4.
   (c) Prove that the sum of the squares of two odd integers can not be equal to the square of an integer.

   Solution: (a) Suppose n^2 is even and we try to show that n^2 is divisible by 4. If n^2 is even then we've already seen that n must be even by theorem 1.12. So there is some t such that $n = 2t$. Then $n^2 = 4t^2$ and hence $n^2 = 4s$ where s is the integer t^2. By definition n^2 is divisible by 4.

   (b) We prove this by the contrapositive, so suppose $2k$ is divisible by 4 and we will try to show that k is not odd. There exists a t such that $2k = 4t$. But then $k = 2t$ and this directly shows k is even, and hence not odd.

   (c) Suppose a and b are odd integer, and we will show that there is no integer c such that $a^2+b^2 = c^2$. Since a and b are odd let $a = 2x+1$ and $b = 2y+1$. For contradiction suppose there is an integer c such that $a^2+b^2 = c^2$. Then

$$\begin{aligned}(2x+1)^2 + (2y+1)^2 &= c^2 \\ 4x^2 + 4x + 1 + 4y^2 + 4y + 1 &= c^2 \\ 2(2x^2 + 2x + 2y^2 + 2y + 2) &= c^2\end{aligned}$$

   The above shows c^2 is even. By part (a) c is divisible by 4. Moreover

$$2(2[x^2 + x + y^2 + y + 1]) = c^2$$

   and $2(x^2 + x + y^2 + y + 1)$ is odd. If we call $k = 2(x^2 + x + y^2 + y + 1)$ then we have $2k = c^2$ with k odd. Hence by part (b) c is not divisible by 4. But this contradicts the earlier finding that c^2 is divisible by 4. $\lightning$