Homework 8

A portion of the following problems will be graded according to the provided rubric.

1. Rudin pg 99 problem 6
2. Prove if f is continuous on $[a, b]$ with $f(x) > 0$ for all $a \leq x \leq b$, then $\frac{1}{f}$ is bounded on $[a, b]$.
3. Let $f(x) = x^3$
    a. Prove f is continuous on $\mathbb{R}$
    b. Prove f is not uniformly continuous on $\mathbb{R}$.
    c. Prove f is uniformly continuous on [1, 100].
    d. Prove f is uniformly continuous on (1, 3).
    e. Prove f is uniformly continuous on any bounded subset of $\mathbb{R}$
4. Prove that a uniformly continuous function preserves Cauchy sequences. That is, prove that if $f: X \rightarrow Y$ (where X and Y are metric spaces) is uniformly continuous and $\{x_n\}_{n=1}^{\infty} \subset X$ is a Cauchy sequence, then $\{f(x_n)\}_{n=1}^{\infty}$ is a Cauchy sequence.
5. Let $f, g: X \rightarrow X$ (where $X \subseteq \mathbb{R}$) be uniformly continuous.
    a. Prove $f + g$ is uniformly continuous
    b. Give an example to show fg is not always uniformly continuous
    c. Give an example to show $\frac{f}{g}$ is not always uniformly continuous
6. Rudin pg 99 problem 12
7. Rudin pg 102 problem 26
8. Rudin pg 100 problem 14
9. Prove any decreasing function that has the intermediate value property is continuous.
10. Give an example of each of the following or explain why such a request is impossible.
    a. A continuous function defined on an open interval with range equal to a closed interval.
    b. A continuous function defined on a closed interval with range equal to an open interval.
    c. A continuous function defined on all of $\mathbb{R}$ with range equal to $\mathbb{Q}$.