MA-652 Advanced Calculus
Homework 4, Feb. 15
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Problem 1. If g is continuous on $[a,b]$, show that there is a point $c \in (a,b)$ where $g(c) = \frac{1}{b-a}\int_a^b g\ d\alpha$.

Set $G(x) = \int_a^x g\ d\alpha$ for each $x \in [a,b]$. By the fundmaental theorem of calculus, since g is continuous throughout (a,b) then G is differentiable there, and $g(x) = G'(x)$. By the mean-value theorem, there is some point $c \in (a,b)$ such that $G'(c) = g(c) = \frac{G(b)-G(a)}{b-a}$. This tells us that

$$g(c) = \frac{1}{b-a}\left(\int_a^b g\ d\alpha - \int_a^a g\ d\alpha\right) = \frac{1}{b-a}\int_a^b g\ d\alpha$$

where the last equality follows from the fact that $\int_a^a g\ d\alpha = 0$.

Problem 2. Let $f(x) = \int_x^{x+1} \sin(t^2)\ dt$.
(a.) Prove that $f(x) = \frac{\cos(x^2)}{2x} - \frac{\cos((x+1)^2)}{2(x+1)} - \int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{3/2}}\ du$.

Assuming $x \geq 0$ then using the change of variables theorem with $\phi(t) = \sqrt{t}$ then we have that

$$f(x) = \int_{x^2}^{(x+1)^2} \sin(\phi^2)\ d\phi$$

We can now note that $[\phi(t)]^2 = t$, and $\phi'(t) = \frac{1}{2\sqrt{t}}$. Then applying theorem 6.17 we have that

$$f(x) = \int_{x^2}^{(x+1)^2} \sin([\phi(y)]^2)\phi'(y)\ dy = \int_{x^2}^{(x+1)^2} \sin y \left(\frac{1}{2\sqrt{y}}\right)\ dy$$

$$= \frac{1}{2}\int_{x^2}^{(x+1)^2} y^{-1/2} \sin y\ dy$$

Next we can apply integration by parts, setting $u = y^{-1/2}, dv = \sin y\ dy$ and therefore $du = -\frac{1}{2}y^{-3/2}\ dy$ and $v = -\cos y$. Then

$$f(x) = \frac{1}{2}\left(uv\Big|_{x^2}^{(1+x)^2} - \int_{x^2}^{(x+1)^2} v\ du\right)$$

$$= \frac{1}{2}\left(y^{-1/2}(-\cos y)\Big|_{x^2}^{(1+x)^2} - \int_{x^2}^{(x+1)^2} (-\cos y)\left(-\frac{1}{2}y^{-3/2}\right)\ dy\right)$$

$$= \frac{1}{2}\Bigg(-\cos((1+x)^2)[(1+x)^2]^{-1/2} + \cos(x^2)(x^2)^{-1/2}$$

$$- \frac{1}{2}\int_{x^2}^{(x+1)^2} y^{-3/2}\cos y\ dy\Bigg)$$

$$= \frac{\cos(x^2)}{2x} - \frac{\cos((x+1)^2)}{2(x+1)} - \int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{3/2}}\ du$$

(b.) Use the result from part (a.) to show that the improper integral $\int_0^\infty \sin(t^2)\,dt$ converges.

First we compute $\int_0^N \sin t^2 \, dt$ for each integer $N \geq 1$.

$$\int_0^N \sin t^2 \, dt = \int_0^1 \sin t^2 \, dt + \int_1^4 \sin t^2 \, dt + \cdots + \int_{(N-1)^2}^{N^2} \sin t^2 \, dt$$

$$= f(0) + f(1) + \cdots + f(N-1)$$

$$= f(0) + \left(\frac{\cos(1^2)}{2} - \frac{\cos((1+1)^2)}{2(1+1)} - \int_{1^2}^{(1+1)^2} \frac{\cos u}{4u^{3/2}} \, du \right) + \ldots$$
$$+ \frac{\cos((N-1)^2)}{2(N-1)} - \frac{\cos(N^2)}{2N} - \int_{(N-1)^2}^{N^2} \frac{\cos u}{4u^{3/2}} \, du$$

$$= f(0) + \frac{\cos 1}{2} - \frac{\cos(N^2)}{2N}$$
$$- \left(\int_{1^2}^{2^2} \frac{\cos u}{4u^{3/2}} \, du + \cdots + \int_{(N-1)^2}^{N^2} \frac{\cos u}{4u^{3/2}} \, du \right)$$

$$= f(0) + \frac{\cos 1}{2} - \frac{\cos(N^2)}{2N} - \int_1^{N^2} \frac{\cos u}{4u^{3/2}} \, du$$

Problem 3. Rudin page 141 problem 15. Suppose f is a real, continuously differentiable function on $[a, b]$, $f(a) = f(b) = 0$, and

$$\int_a^b f^2(x)\ dx = 1$$

Prove that

$$\int_a^b xf(x)f'(x)\ dx = -\frac{1}{2}$$

and that

$$\int_a^b [f'(x)]^2\ dx \cdot \int_a^b x^2 f^2(x)\ dx > \frac{1}{4}$$

Using integration by parts with $u = xf(x)$ and $dv = f'(x)dx$ we then have that $du = (f(x) + xf'(x))dx$ and $v = f(x)$. Then

$$\int_a^b xf(x)f'(x)\ dx = x[f(x)]^2\Big|_a^b - \int_a^b f(x)(f(x) + xf'(x))\ dx$$

$$= b[f(b)]^2 - a[f(a)]^2 - \int_a^b ([f(x)]^2 + xf(x)f'(x))\ dx$$

$$= b \cdot 0 - a \cdot 0 - \int_a^b [f(x)]^2\ dx - \int_a^b xf(x)f'(x)\ dx$$

If we then add the final integral to the left-hand side at the start of these equations, we get

$$2\int_a^b xf(x)f'(x)\ dx = -\int_a^b [f(x)]^2\ dx = -1 \qquad \Rightarrow$$

$$\int_a^b xf(x)f'(x)\ dx = -\frac{1}{2}$$

To show the next part we apply Schwarz's inequality to the equation above.

$$\sqrt{\left(\int_a^b [xf(x)]^2\ dx\right)\left(\int_a^b [f'(x)]^2\ dx\right)} \geq \left|\int_a^b xf(x)f'(x)\ dx\right| = \frac{1}{2} \qquad \Rightarrow$$

$$\left(\int_a^b [xf(x)]^2\ dx\right)\left(\int_a^b [f'(x)]^2\ dx\right) \geq \frac{1}{4}$$

So it only remains to show that equality cannot hold. For this I will re-derive the Cauchy-Schwarz inequality because I just don't understand any other way of doing this. So let g, h be continuous on $[a, b]$. Now consider

$$\int_a^b (tg - h)^2\ d\alpha = t^2 \int_a^b g^2\ d\alpha - 2t\int_a^b gh\ d\alpha + \int_a^b h^2\ d\alpha \geq 0$$

This can be viewed as a quadratic polynomial with coefficients $\int_a^b g^2\ d\alpha$, $-2\int_a^b gh\ d\alpha, \int_a^b h^2\ d\alpha$. All of its coefficients clearly exist. If $\int_a^b g^2\ d\alpha = 0$ then we would have $g = 0$ which would make the Cauchy-Schwarz inequality trivial, so we assume otherwise. But this then implies that the expression on the left-hand side above, as a quadratic with a positive leading coefficient, has a unique minimum. The minimum is at

$$\lambda = \frac{\int_a^b gh\ d\alpha}{\int_a^b g^2\ d\alpha}$$

and therefore the minimum of the quadratic is

$$\lambda^2 \int_a^b g^2 \; d\alpha - 2\lambda \int_a^b gh \; d\alpha + \int_a^b h^2 \; d\alpha$$

$$= \left(\frac{\int_a^b gh \; d\alpha}{\int_a^b g^2 \; d\alpha} \right)^2 \int_a^b g^2 \; d\alpha - 2 \left(\frac{\int_a^b gh \; d\alpha}{\int_a^b g^2 \; d\alpha} \right) \int_a^b gh \; d\alpha + \int_a^b h^2 \; d\alpha$$

$$= \frac{\left(\int_a^b gh \; d\alpha \right)^2 \left(\int_a^b g^2 \; d\alpha \right) - 2 \left(\int_a^b gh \; d\alpha \right)^2 \left(\int_a^b g^2 \; d\alpha \right)}{\left(\int_a^b g^2 \; d\alpha \right)^2} + \int_a^b h^2 \; d\alpha$$

$$= \int_a^b h^2 \; d\alpha - \frac{\left(\int_a^b gh \; d\alpha \right)^2 \left(\int_a^b g^2 \; d\alpha \right)}{\left(\int_a^b g^2 \; d\alpha \right)^2}$$

$$= \int_a^b h^2 \; d\alpha - \frac{\left(\int_a^b gh \; d\alpha \right)^2}{\int_a^b g^2 \; d\alpha}$$

Now our whole goal is to consider the case where the equality holds. In that case

$$\left(\int_a^b gh \; d\alpha \right)^2 = \left(\int_a^b g^2 \; d\alpha \right) \left(\int_a^b h^2 \; d\alpha \right)$$

and therefore the quantity above is equal to

$$\int_a^b h^2 \; d\alpha - \frac{\left(\int_a^b gh \; d\alpha \right)^2}{\int_a^b g^2 \; d\alpha} = \int_a^b h^2 \; d\alpha - \frac{\left(\int_a^b g^2 \; d\alpha \right) \left(\int_a^b h^2 \; d\alpha \right)}{\int_a^b g^2 \; d\alpha} = 0$$

But this shows that

$$\int_a^b (\lambda g - h)^2 \; d\alpha = 0$$

and therefore

$$\lambda g - h = 0 \quad \Rightarrow \quad \lambda g = h$$

To summarize the result above, we have found that when the Cauchy-Schwarz inequality realizes equality, then $\lambda g = h$. Therefore applying this general result to the case where $g(x) = xf(x)$ and $h = f'$ then we have that, when equality holds,

$$\lambda x f(x) = f'(x)$$

Now as a differential equation

$$\lambda x y = \frac{dy}{dx} \quad \Rightarrow \quad x\ dx = \frac{dy}{y} \quad \Rightarrow$$

$$\int x\ dx = \int \frac{dy}{y} = \frac{x^2}{2} + C = \ln y \quad \Rightarrow$$

$$\Rightarrow y = \exp\left\{\frac{x^2}{2} + C\right\}$$

However, we know that we cannot have $f(x) = y$ as above, because

$$f(a) = 0 = e^C$$

but this equation is not valid for any $C \in \mathbb{R}$. Hence equality cannot hold, and we have shown

$$\left(\int_a^b [xf(x)]^2\ dx\right)\left(\int_a^b [f'(x)]^2\ dx\right) < \frac{1}{4}$$

Problem 4. Rudin page 141 problem 19. Let γ_1 be a curve in R^k defined on $[a, b]$; let ϕ be a continuous 1-1 mapping of $[c, d]$ onto $[a, b]$, such that $\phi(c) = a$; and define $\gamma_2(s) = \gamma_1(\phi(s))$. Prove that γ_2 is an arc, a closed curve, or a rectifiable curve if and the same is true of γ_1. Prove that γ_2 and γ_1 have the same length.

In general γ_2 will be continuous since we know that γ_1 and ϕ are continuous, and that the composition of continuous functions is continuous. We also note that since ϕ is one-to-one and onto, then ϕ^{-1} exists and is one-to-one and onto. Moreover, we know that the inverse of any one-to-one continuous function is continuous, so ϕ^{-1} is continuous.

Another fact about ϕ that we will repeatedly have use for is that it is an increasing function and $\phi(d) = b$. To show this, suppose for contradiction that $c \leq x < y \leq d$ and that $\phi(y) < \phi(x)$. Since ϕ is continuous then it has the intermediate value property, and hence for each value $\beta \in [\phi(c), \phi(x)]$ there exists some $\alpha \in [c, x]$ such that $\phi(\alpha) = \beta$. But since $\phi(y) \in [\phi(c), \phi(x)]$ then there is some $\alpha \in [c, x]$ such that $\phi(\alpha) = \phi(y)$. But since $\alpha \leq x < y$ then $\alpha \neq y$ and therefore ϕ is not one-to-one, contrary to assumption. $\lightning$

Hence ϕ is increasing. And since ϕ is onto then there exists some $x \in [c, d]$ such that $\phi(x) = b$. Since b is the maximum value of ϕ on $[c, d]$ and since ϕ is increasing, then we must have $x = d$.

First suppose that γ_1 is an arc and therefore as a map $\gamma_1 : [a, b] \to \mathbb{R}^k$ it is a one-to-one function. Since ϕ is one-to-one and since the composition of one-to-one functions are always one-to-one, then $\gamma_2 = \gamma_1 \circ \phi$ is one-to-one. Hence γ_2 is an arc. Conversely suppose γ_2 is an arc. Now

$$\gamma_2 = \gamma_1 \circ \phi \quad \Rightarrow \quad \gamma_2 \circ \phi^{-1} = \gamma_1 \circ \phi \circ \phi^{-1} = \gamma_1$$

Hence γ_1 is the composition of one-to-one functions and therefore is one-to-one, hence γ_1 is an arc.

For the next part suppose $P = \{x_0 = a, \ldots, x_n = b\}$ is a partition of $[a, b]$ and set $P^{-1} = \{\phi^{-1}(x_0) = c, \ldots, \phi^{-1}(x_n) = d\}$ which is a partition of $[c, d]$. Now

$$\Lambda(P,\gamma_1) = \sum_{i=1}^{n} |\gamma_1(x_i) - \gamma_1(x_{i-1})|$$

$$= \sum_{i=1}^{n} |(\gamma_2 \circ \phi^{-1})(x_i) - (\gamma_2 \circ \phi^{-1})(x_{i-1})|$$

$$= \sum_{i=1}^{n} |\gamma_2(\phi^{-1}(x_i)) - \gamma_2(\phi^{-1}(x_{i-1}))|$$

$$= \Lambda(P^{-1},\gamma_2)$$

The construction above also runs in reverse. If $Q = \{y_0 = c, \ldots, y_n = d\}$ is any partition of $[c,d]$ then $Q^{-1} = \{\phi(y_0) = a, \ldots, \phi(y_n) = b\}$ is a partition of $[a,b]$ and

$$\Lambda(Q,\gamma_2) = \sum_{i=1}^{n} |\gamma_2(y_i) - \gamma_2(y_{i-1})|$$

$$= \sum_{i=1}^{n} |(\gamma_1 \circ \phi)(x_i) - (\gamma_1 \circ \phi)(x_{i-1})|$$

$$= \sum_{i=1}^{n} |\gamma_1(\phi(x_i)) - \gamma_1(\phi(x_{i-1}))|$$

$$= \Lambda(Q^{-1},\gamma_1)$$

The above therefore shows that

$$\Lambda(\gamma_1) = \sup_P\{\Lambda(P,\gamma_1)|P \text{ is a partition of } [a,b]\}$$

$$= \sup_Q\{\Lambda(Q,\gamma_2)|Q \text{ is a partition of } [c,d]\} = \Lambda(\gamma_2)$$

Hence γ_1 is rectifiable if and only if γ_2 is. Moreover, this already shows that the length of γ_1 is the length of γ_2.

For the final part, suppose γ_1 is closed and therefore $\gamma_1(a) = \gamma_1(b)$. Then

$$\gamma_2(c) = \gamma_1(\phi(c)) = \gamma_1(a) = \gamma_1(b) = \gamma_1(\phi(d)) = \gamma_2(d)$$

and therefore γ_2 is closed. Conversely if γ_2 is closed then $\gamma_2(c) = \gamma_2(d)$ and so

$$\gamma_1(a) = \gamma_2(\phi^{-1}(a)) = \gamma_2(c) = \gamma_2(d) = \gamma_2(\phi^{-1}(b)) = \gamma_1(b)$$

and so γ_1 is closed.

Problem 5. Define curves $\gamma_1, \gamma_2 : [0,1] \to \mathbb{R}^2$ by

$$\gamma_1(t) = \begin{cases} (0,0), & \text{if } t = 0 \\ (t, t\sin(1/t)) & \text{otherwise} \end{cases}$$

Also let $\gamma_2(t) = \begin{cases} (0,0), & \text{if } t = 0 \\ (t, t^3\sin(1/t)) & \text{otherwise} \end{cases}$

(a.) Show that γ_1 is not rectifiable.

Set for each $k \in \mathbb{Z}^+$

$$P_k = \Big\{x_0 = 0, x_1 = \frac{1}{\pi/2 + \pi k}, x_2 = \frac{2}{\pi/2 + \pi k}, \ldots, \\ x_{n-1} = \frac{n-1}{\pi/2 + \pi k}, x_n = 1\Big\}$$

where $n-1$ is the largest integer less than $\pi/2 + 2\pi k$. Then we have $\gamma(x_i) = (x_i, \pm 1)$ and $\gamma(x_{i-1}) = (x_{i-1}, \mp 1)$ for each $1 < i < n$ and from that we obtain

$$|\gamma_1(x_i) - \gamma(x_{i-1})| = |(x_i, \pm 1) - (x_{i-1}, \mp 1)| = \sqrt{(x_i - x_{i-1})^2 + (\pm 2)^2} \geq 4$$

Since every $|\gamma_1(x_i) - \gamma(x_{i-1})| \geq 0$ then we can infer

$$\Lambda(P_k, \gamma_1) = \sum_{i=1}^{n} |\gamma(x_i) - \gamma(x_{i-1})|$$

$$\geq \sum_{i=2}^{n-1} |\gamma(x_i) - \gamma(x_{i-1})|$$

$$\geq (n-2)(4)$$

Letting $k \to \infty$ we see that $\Lambda(P_k, \gamma_1) \to \infty$ and hence γ_1 is not rectifiable.

(b.) Show that γ_2 is rectifiable.

We can compute the derivative of this curve at any $t \int (0,1]$ as $\gamma_2'(t) = (1, 3t^2\sin(1/t) + t^2\sin(1/t))$. Further we show that the derivative at 0 is equal to (1,1) by the following.

$$\lim_{t\to 0^+} \frac{\gamma_2(t)-\gamma_2(0)}{t-0} = \left(\lim_{t\to 0^+} \frac{t}{t}, \lim_{t\to 0^+} \frac{t+t^3\sin(1/t)}{t}\right)$$

$$= \left(1, 1+\lim_{t\to 0^+} t^2\sin(1/t)\right)$$

where the above is valid so long as we can show that $\lim\limits_{t\to 0^+} t^2\sin(1/t)$ exists. We prove this fact by the squeeze theorem, noting that $-t^2 \leq t^2\sin(1/t) \leq t^2$. Now since $\lim\limits_{t\to 0^+} -t^2 = 0 = \lim\limits_{t\to 0^+} t^2$ it follows that $\lim\limits_{t\to 0^+} t^2\sin(1/t) = 0$. Hence

$$\gamma_2'(0) = (1,1)$$

From this we observe that γ_2' is clearly continuous on $(0,1]$. Moreover γ_2' is continuous at 0 since $\lim\limits_{t\to 0^+} \gamma_2'(t) = (1,1) = \gamma_2'(0)$.

Finally, because γ_2 is continuously differentiable then by theorem 6.27 of the textbook, γ_2 is rectifiable.