MA 638 - Section 8.2 Homework

1. Let R be an integral domain and suppose that every prime ideal in R is principal. This exercise proves that every ideal of R is principal, i.e., R is a PID.

(a) Assume that the set of ideals of R that are not principal is nonempty and prove that this set has a maximal element under inclusion (which, by hypothesis, is not prime). [Hint: Use Zorn's Lemma.]

(b) Let I be an ideal which is maximal with respect to being nonprincipal, and let $a, b \in R$ with $ab \in I$ but $a \notin I$ and $b \notin I$. Let $I_a = (I, a)$ be the ideal generated by I and a, let $I_b = (I, b)$ be the ideal generated by I and b, and define $J = \{r \in R | rI_a \subseteq I\}$. Prove that $I_a = (\alpha)$ and $J = (\beta)$ are principal ideals in R with $I \subsetneq I_b \subseteq J$ and $I_a J = (\alpha\beta) \subseteq I$.

(c) If $x \in I$ show that $x = s\alpha$ for some $s \in J$. Deduce that $I = I_a J$ is principal, a contradiction, and conclude that R is a PID.

Proof: (a) Let $\mathscr{I} \neq \emptyset$ be the set of all ideals of R which are not principal, and let $I_1 \in \mathscr{I}$. If I_1 is not maximal for $\mathscr{I}$ then there is some $I_2 \in \mathscr{I}$ such that $I_1 \subsetneq I_2$. Proceeding recursively, we develop an increasing chain of subsets

$$I_1 \subsetneq I_2 \subsetneq \dots$$

with $I_n \in \mathscr{I}$ for $n = 1, 2, \dots$ Sets are of course partially ordered by set containment. Now set $I = \bigcup_{i=1}^{\infty} I_i$. If $r \in R$ and $ri \in rI$ then for some n we have $i \in I_n$. Since I_n is an ideal therefore $ri \in I_n \subseteq I$ so that I is an ideal. Moreover, if I were prinicpal and $I = (\gamma)$ then we would have $\gamma \in I_n$ for some n, in which case $I = (\gamma) \subseteq I_n$. Since $I_n \subseteq I$ then this would make $I = I_n$ and then I_n is principal, contrary to assumption. Hence I is not principal.

(b) Since I is maximal and $I \subsetneq I_a$, then $I_a \notin \mathscr{I}$ and therefore I_a is principal. We let α be that element which generates I_a.

We want to also show that $I \subsetneq J$, and since we clearly have $I \subsetneq I_b$, then it suffices to show $I_b \subseteq J$. Now if $ci_a + db \in I_b$ then we seek to show that $(ci_a + db)I_a \subseteq I$. So also let $ei_b + fa \in I_a$, and therefore

$$(ci_a + db)(ei_b + fa) = cei_a i_b + cfai_a + dbei_b + dbfa.$$

Since I is an ideal then $cei_a i_b, cfai_a, dbei_b \in I$. We also know that $ab \in I$ and therefore $dfab \in I$. Hence $(ci + db)(ei + fa) = cei_a i_b + cfai_a + dbei_b + dbfa \in I$, hence $(ci + db) \in J$, hence $I_b \subseteq J$. From this it follows that

$$I \subsetneq I_b \subseteq J$$

and from that we infer that J is principal. Call the element which generates it β, so that $(\beta) = J$.

All that then remains is to show that $I_a J = (\alpha\beta) \subseteq I$. In general we have that $(\alpha)(\beta) = (\alpha\beta)$ so that this is trivial. To complete this part, let $j \in J$ so that we have $jI_a \subseteq I$. Therefore if $s \in I_a$ then we have $js \in I$. But this shows that for an arbitrary $j \in J$ and $s \in I_a$ we have $sj \in I$ and hence $I_a J \subseteq I$.

(c) Oh man, finally! Suppose $x \in I$ and therefore $x \in I_a$, so that we must have some $s \in R$ such that $x = s\alpha$. It's easy to see that $s \cdot (\alpha) = (s\alpha)$ since an arbitrary element from the left is $s \cdot (c \cdot \alpha)$ for some $c \in R$, but this is the same as an arbitrary element from the right, $c \cdot (s \cdot \alpha)$. Therefore

$$sI_a = s(\alpha) = (s\alpha) = (x) \subseteq I.$$

Then $sI_a \subseteq I$ and therefore $s \in J$. Since we now have $I \subseteq (\alpha\beta)$ and $(\alpha\beta) \subseteq I$ then we have $I \subseteq I_a J = (\alpha\beta)$ and therefore I is principal. But this contradicts our finding from part (a) that I is not principal.

Hence the assumption that the set of non-principal ideals is empty must be false. Therefore R is a PID.