MA 630 - Homework 4 (Module 2 - Section 2)

Solutions must be typeset in LaTeX and submitted to Canvas as a .pdf file. When applicable, write in complete sentences.

Use mathematical induction to solve each problem below.

1. Prove that
$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$
for all natural numbers n.

   Proof: Define $S = \left\{ n \in \mathbb{N} | \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} \right\}$. For the base-case, notice that since $1^3 = 1 = \frac{1^2(1+1)^2}{4}$ this is the condition which ensures $1 \in S$.

   For the inductive case, assume that $n \in \mathbb{N}$ and that $n \in S$. Then consider

$$\begin{aligned}
\sum_{i=1}^{n+1} i^3 &= \left(\sum_{i=1}^{n} i^3 \right) + (n+1)^3 \\
&= \frac{n^2(n+1)^2}{4} + \frac{4(n+1)^3}{4} \\
&= \frac{(n+1)^2(n^2 + 4(n+1))}{4} \\
&= \frac{(n+1)^2(n^2 + 4n + 4)}{4} \\
&= \frac{(n+1)^2(n+2)^2}{4}.
\end{aligned}$$

   The second equality follows from the inductive hypothesis. The above chain of equalities then shows that $n+1 \in S$. Hence by the Principle of Mathematical Induction $S = \mathbb{N}$, which is to say that for all $n \in \mathbb{N}$ we have $1^3 + \cdots + n^3 = \frac{n^2(n+1)^2)}{4}$. $\square$

2. Prove that $3^{4n+1} - 5^{2n-1}$ is divisible by 7 for all $n \in \mathbb{N}$. *Hint: It may be helpful to note that 81 = 77 + 4, and that 25 = 21 + 4.*

   Proof: Define $S = \{n \in \mathbb{N} | 3^{4n+1} - 5^{2n-1}\}$. For the base-case notice that

$$\begin{aligned} 3^{4(1)+1} - 5^{2(1)-1} &= 3^5 - 5 \\ &= 243 - 5 \\ &= 238 \\ &= 7 \cdot 34. \end{aligned}$$

   Hence this is divisible by 7 and so $1 \in S$.

   Now suppose for the inductive hypothesis that $n \in \mathbb{N}$ and $n \in S$. Since $n \in S$ we can infer that there is some $k \in \mathbb{Z}$ such that $3^{4n+1} - 5^{2n-1} = 7k$. Hence

$$\begin{aligned} 3^{4(n+1)+1} - 5^{2(n+1)-1} &= 3^{4n+5} - 5^{2n+1} \\ &= 3^4 \cdot 3^{4n+1} - 5^2 \cdot 5^{2n-1} \\ &= (77+4)3^{4n+1} + (21+4)5^{2n-1} \\ &= 77 \cdot 3^{4n+1} - 21 \cdot 5^{2n-1} + 4(3^{4n+1} - 5^{2n-1}) \\ &= 7(11 \cdot 3^{4n+1} - 3 \cdot 5^{2n-1}) + 4 \cdot 7k \\ &= 7(11 \cdot 3^{4n+1} - 3 \cdot 5^{2n-1} + 28). \end{aligned}$$

   Therefore $3^{4(n+1)+1} - 5^{2(n+1)-1} = 3^{4n+5} - 5^{2n+1}$ is divisible by 7, which is the condition for $n+1 \in S$. So by the Principle of Mathematical Induction we have $S = \mathbb{N}$, which is to say that for every $n \in \mathbb{N}$ we have that $3^{4n+1} - 5^{2n-1}$ is divisible by 7. $\square$

3. Prove that $\left(1 + \frac{1}{n}\right)^n < n$ for all natural numbers $n \geq 3$.

Proof: Define $S = \left\{n \in \mathbb{N} \middle| \left(1+\frac{1}{n}\right)^n < n\right\}$. Since $\left(1+\frac{1}{3}\right)^3 = \left(\frac{4}{3}\right)^3 = \frac{64}{27}$ and since $3 = \frac{81}{27}$, then we have $(1+\frac{1}{3})^3 < 3$. This establishes the base-case that $3 \in S$.

Now suppose for the inductive hypothesis that $n \in S$ and $n \geq 3$. Then

$$\begin{aligned}
\left(1+\frac{1}{n+1}\right)^{n+1} &= \left(1+\frac{1}{n+1}\right)^n \left(1+\frac{1}{n+1}\right) \\
&< \left(1+\frac{1}{n}\right)^n \left(1+\frac{1}{n+1}\right) \\
&< n\left(1+\frac{1}{n+1}\right) \\
&= n + \frac{n}{n+1} \\
&< n+1.
\end{aligned}$$

The third line above is justified by the inductive hypothesis, and the last line follows from $\frac{n}{n+1} < 1$, as this is equivalent to $n < n+1$. We then have that $n+1 \in S$ and therefore by the Principle of Mathematical Induction, $S = \mathbb{N} \setminus \{1, 2, 3\}$. $\square$

(Note: By a direct and tedious calculation we could confirm that indeed $1, 2, 3 \notin S$.)

4. (a) For which natural numbers n is $n^3 < 2^n$?

   (b) Prove your result. *Hint: It may be helpful to note that* $(n+1)^3 = n^3 + 3n^2 + 3n + 1$. *Then, to find a useful upper bound for this expression, note that* n^2 *is greater than both* n *and 1 whenever* $n > 1$.

(a) It is true for 1 but not for 2 through 9. It is again true for 10 and every number after that.

(b) *Proof:* Define $S = \{n \in \mathbb{N} | n^3 < 2^n\}$. For the base-case, notice that $10^3 = 1000 < 1024 = 2^{10}$. Hence $10 \in S$.

For the inductive hypothesis suppose that $n \in S$ and $n \geq 10$. Then

$$\begin{aligned}
(n+1)^3 &= n^3 + 3n^2 + 3n + 1 \\
&< 2^n + 3n^2 + 3n^2 + n^2 \\
&= 2^n + 7n^2 \\
&< 2^n + n \cdot n^2 \\
&= 2^n + n^3 \\
&< 2^n + 2^n \\
&= 2 \cdot 2^n \\
&= 2^{n+1}.
\end{aligned}$$

Note that because we assumed $n \geq 10 > 7$ that we were justified in stating that $7n^2 < n \cdot n^2$. Since we have now established the inductive case then, by the Principle of Mathematical Induction, $S \subseteq \{n \in \mathbb{N} | n \geq 10\}$. Because we directly checked smaller cases in part (a) we can further say that $S = \{n \in \mathbb{N} | n = 1 \text{ or } n \geq 10\}$. That is to say, for all natural numbers $n = 1$ or $n \geq 10$, we have that $n^3 < 2^n$. $\square$

5. (a) Let n be a natural number. Prove that $2\sqrt{n^2+n} + 1 \leq 2(n+1)$. You do not need to use induction. *Hint: First, consider* $4(n^2+n)$ *and* $(2n+1)^2$.

   (b) Let n be a natural number. Use induction to prove that

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \leq 2\sqrt{n} - 1.$$

(a) First note that $2\sqrt{n^2+n} + 1 \leq 2(n+1)$, if we distribute and subtract 1, is equivalent to

$$2\sqrt{n^2+n} \leq 2n + 1.$$

Since all quantities are positive, this is true if and only if $4(n^2+n) \leq (2n+1)^2$ which we get from squaring both sides. This is the same as

$4n^2 + 4n \leq 4n^2 + 4n + 1$. Since the right-hand side is one more than the left, this is clearly a true inequality. Hence we have shown that

$$2\sqrt{n^2 + n} + 1 \leq 2(n + 1).$$

(b) Define $S = \{n \in \mathbb{N} | \sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2\sqrt{n} - 1\}$. For the base case note that $\frac{1}{\sqrt{1}} = 1 = 2\sqrt{1} - 1$. Hence $1 \in S$. For the inductive hypothesis suppose that $n \in \mathbb{N}$ and $n \in S$. Then

$$\begin{aligned}
\sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} &= \left(\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \right) + \frac{1}{\sqrt{n+1}} \\
&\leq 2\sqrt{n} - 1 + \frac{1}{\sqrt{n+1}} \\
&= \frac{(2\sqrt{n} - 1)\sqrt{n+1} + 1}{\sqrt{n+1}} \\
&= \frac{2\sqrt{n^2 + n} - \sqrt{n+1} + 1}{\sqrt{n+1}} \\
&\leq \frac{2(n+1) - \sqrt{n+1}}{\sqrt{n+1}} \\
&= 2\sqrt{n+1} - 1.
\end{aligned}$$

This shows that the inductive case holds, and therefore by the Principle of Mathematical Induction $S = \mathbb{N}$. That is to say,

$$\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2\sqrt{n} - 1$$

for all natural numbers n. $\square$