MA-652 Advanced Calculus
Homework 1, Jan. 9
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## Problem 1. Let $f : [a, b] \to \mathbb{R}$ be differentiable at $x \in (a, b)$ and $k \in \mathbb{R}$. Prove that $(kf)'(x) = kf'(x)$.

If we define the difference quotient at $x \in (a, b)$,

$$\phi(x) = \frac{(kf)(t) - (kf)(x)}{t - x}$$

then our task is to find $\lim_{t \to x} \phi(x)$. But since

$$\lim_{t \to x} \phi = \lim_{t \to x} \frac{kf(t) - kf(x)}{t - x}$$

by definition of multiplying functions, then this is

$$\lim_{t \to x} k \frac{f(t) - f(x)}{t - x} = k \lim_{t \to x} \frac{f(t) - f(x)}{t - x} = kf'(x)$$

where this limit is guaranteed to exist by the differentiability of $f$ in this interval.

Problem 2. Let $f, g : [a, b] \to \mathbb{R}$ be differentiable at $x \in (a, b)$.

(a). Prove the quotient rule using the limit definition, wherever the denominator isn't 0.

For any $x \in (a, b)$ such that $g(x) \neq 0$ we will show that $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$. We define the difference quotient

$$\phi(t) = \frac{f(t)/g(t) - f(x)/g(x)}{t - x} = \frac{\frac{f(t)g(x)-f(x)g(t)}{g(t)g(x)}}{t - x}$$

Therefore

$$\left(\frac{f}{g}\right)'(t) = \lim_{t\to x} \frac{f(t)g(x) - f(x)g(t)}{[g(t)g(x)](t - x)}$$

$$= \frac{1}{g(x)} \lim_{t\to x} \frac{1}{g(t)} \left(\frac{f(t)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(t)}{t - x}\right)$$

$$= \frac{1}{g(x)} \lim_{t\to x} \frac{1}{g(t)} \cdot \lim_{t\to x} \left(\frac{f(t) - f(x)}{t - x} \cdot g(x) - f(x)\frac{g(t) - g(x)}{t - x}\right)$$

$$= \frac{1}{[g(x)]^2}(f'(x)g(x) - f(x)g'(x))$$

The final equation follows because we assumed that $g(x) \neq 0$ and therefore $\lim_{t\to x} \frac{1}{g(t)} = \frac{1}{g(x)}$. Also we assumed both functions are differentiable in the interval and hence $\lim_{t\to x} \frac{f(t) - f(x)}{t - x} = f'(x)$ and also $\lim_{t\to x} \frac{g(t) - g(x)}{t - x} = g'(x)$. The proof is then complete.

(b). Use the limit definition to find the derivative of $\frac{1}{x}$.

$$\lim_{t\to x}\frac{\frac{1}{t}-\frac{1}{x}}{t-x}=\lim_{t\to x}\frac{\frac{x-t}{tx}}{t-x}$$

$$=-\lim_{t\to x}\frac{1}{tx}$$

$$=-\frac{1}{x^2}$$

(c). Use (b) with the chain rule to derive the quotient rule.

$$\left(\frac{f}{g}\right)'(x)=\left(f(x)\cdot\frac{1}{g(x)}\right)'$$

$$=f'(x)\cdot\frac{1}{g(x)}+f(x)\left[\frac{1}{g(x)}\right]'$$

by the product rule. Now by the chain rule

$$\left[\frac{1}{g(x)}\right]'=-\frac{1}{[g(x)]^2}g'(x)$$

So we can infer from these two equations that

$$\left(\frac{f}{g}\right)'(x)=\frac{f'(x)}{g(x)}-f(x)\frac{g'(x)}{[g(x)]^2}$$

$$=\frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$$

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