Advanced Calculus, Homework 7

Adam Frank

August 2020

Problem 1. Prove the following limit statements.
a. $\lim_{x\to 2}(3x+4) = 10$
b. $\lim_{x\to 2}(x^2+x-1) = 5$
c. $\lim_{x\to 3}\frac{1}{x} = \frac{1}{3}$

(a) Let $\varepsilon \in \mathbb{R}^+$ and choose $\delta = \varepsilon/3$. Suppose x is any real number $|x-2| < \delta$. Then

$$|3x+4-10| = |3x-6| = 3|x-2| < 3(\varepsilon/3) = \varepsilon$$

(b) Let $\varepsilon \in \mathbb{R}^+$ and choose $\delta = \min\{\varepsilon/6, 1\}$. Suppose x is any real number $|x-2| < \delta$. First note that therefore $\delta \le 1$ and so $|x-2| < 1$. From that we infer $1 < x < 3$ and hence $|x+3| < 6$. Now we can see that

$$|x^2+x-1-5| = |x+3||x-2| < 6(\varepsilon/6) = \varepsilon$$

(c) Let $\varepsilon \in \mathbb{R}^+$ and choose $\delta = \min\{\varepsilon, 1\}$. Suppose x is any real number $|x-3| < \delta$. First note that therefore $\delta \le 1$ and so $|x-3| < 1$. From that we infer $2 < x < 4$ and hence $|3x| > 6$. Finally we infer $\frac{1}{|3x|} < \frac{1}{6}$. Now we can see that

$$\left|\frac{1}{x} - \frac{1}{3}\right| = \frac{|3-x|}{|3x|} < \varepsilon(1/6) < \varepsilon$$

Problem 2. Let $f : X \to \mathbb{R}$ be such that $\lim_{x\to p} f(x) = 0$.

a. Prove that if g is a bounded function on X, then $\lim_{x\to p} f(x)g(x) = 0$.

b. Give an example to show that $\lim_{x\to p} f(x)g(x)$ is not always equal to zero.

(a) Since g is bounded let M be the bound, so that $|g(x)| \leq M$ for all $x \in X$. Let $\varepsilon \in \mathbb{R}^+$ and let δ but such that $|x - p| < \delta$ entails $|f(x)| < \varepsilon/M$. Then for this same choice of δ, if $|x - p| < \delta$ then

$$|f(x)g(x)| = |f(x)||g(x)| < (\varepsilon/M)M = \varepsilon$$

(b) If $f(x) = x$ and

$$g(x) = \begin{cases} 1/x & \text{if } x \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

then we have

$$\lim_{x\to 0} f(x)g(x) = 1$$

Problem 3. Let $f(x) = \sqrt[3]{x}$.

a. Prove f is continuous at $p = 0$.

b. Prove f is continuous at any $p \neq 0$ (and thus is continuous everywhere).

(a.) Let $\varepsilon \in \mathbb{R}^+$ and set $\delta = \varepsilon^3$. Then if x is any number such that $|x| < \delta$ we have

$$|f(x) - f(0)| = \sqrt[3]{|x|} < \sqrt[3]{\varepsilon^3} = \varepsilon$$

(b) For a given ε choose $\delta = \min\{\varepsilon/p^{-2/3}, |p|\}$. First suppose $p > 0$. Note that because of this choice of δ the condition $|x - p| < \delta \leq p$ guarantees $p - p < x < p + p$. In particular we have $0 < x$. We can now infer

$$|f(x) - f(p)| = |\sqrt[3]{x} - \sqrt[3]{p}| \cdot \left(\frac{|x^{2/3} + x^{1/3}p^{1/3} + p^{2/3}|}{|x^{2/3} + x^{1/3}p^{1/3} + p^{2/3}|} \right)$$

$$= \frac{|x - p|}{|x^{2/3} + x^{1/3}p^{1/3} + p^{2/3}|}$$

$$< \frac{|x - p|}{p^{2/3}}$$

where the inequality holds because each quantity $x^{2/3}, x^{1/3}, p^{1/3}, p^{2/3}$ are positive. We can now infer

$$\frac{|x - p|}{p^{2/3}} < (\varepsilon/p^{-2/3})/p^{2/3} = \varepsilon$$

If on the other hand $p < 0$ then $|x - p| < \delta \leq |p| = -p$. This entails $p - (-p) < x < p + (-p)$ which ensures that $x < 0$. Then $x^{2/3}, x^{1/3}p^{1/3}$, and $p^{2/3}$ are all still positive numbers and hence we have

$$
\begin{aligned}
|f(x) - f(p)| &= |\sqrt[3]{x} - \sqrt[3]{p}| \cdot \left(\frac{|x^{2/3} + x^{1/3}p^{1/3} + p^{2/3}|}{|x^{2/3} + x^{1/3}p^{1/3} + p^{2/3}|} \right) \\
&= \frac{|x - p|}{|x^{2/3} + x^{1/3}p^{1/3} + p^{2/3}|} \\
&= \frac{|x - p|}{x^{2/3} + x^{1/3}p^{1/3} + p^{2/3}} \\
&< \frac{|x - p|}{p^{2/3}} \\
&< (\varepsilon / p^{-2/3})(p^{2/3}) = \varepsilon
\end{aligned}
$$

Problem 4. Let C be the Cantor set. Define $g : \mathbb{R} \to \{0, 1\}$ by

$$g(x) = \begin{cases} 1, & \text{if } x \in C \\ 0, & \text{if } x \notin C \end{cases}$$

(a) Prove that g is not continuous at any $x \in C$.
(b) Prove that g is continuous at every $x \notin C$.

(a) If $x \in C$ then let $\varepsilon = 1/2$ and we claim that for every $\delta > 0$ there exists some p such that both $|x - p| < \delta$ and yet $|g(x) - g(p)| \geq \varepsilon$. In particular we already know that $g(x) = 1$. If we can show that $p \notin C$ this suffices to show that $g(p) = 0$ and therefore $|g(x) - g(p)| = 1 \geq \varepsilon$.

So let $|x - p| < \delta$ and hence $x - \delta < p < x + \delta$. There must exist some k such that $\frac{1}{3^k} < \delta$. For this k, the set C_k must contain an interval end-point in $(x - \delta, x)$. Call the interval in C_k which contains an end-point in $(x - \delta, x)$ the interval I. Then there must be a point p not in I but with $p \in (x - \delta, x)$. Hence $p \notin C$, as desired.

(b) For any $x \notin C$ there is some C_k such that $x \notin C_k$. But that means that this point is in some interval $\left(\frac{m}{3^k}, \frac{m+1}{3^k}\right)$ where this interval is not in the Cantor set. If we then set $\delta = \min\left\{\left|x - \frac{m}{3^k}\right|, \left|\frac{m+1}{3^k} - x\right|\right\}$ then if x' is any number such that $|x - x'| < \delta$ this entails that $x' \in \left(\frac{m}{3^k}, \frac{m+1}{3^k}\right)$ and hence both $g(x) = 0 = g(x')$. Hence for any ε we select δ in this manner, and get that $|x - x'| < \delta$ entails $|g(x) - g(x')| = 0 < \varepsilon$.

Problem 5. Rudin page 98, Problem 2. If f is a continuous mapping of a metric space X into a metric space Y, prove that $f(\overline{E}) \subseteq \overline{f(E)}$ for every set $E \subseteq X$. Show by an example that this inclusion can be proper.

If $y \in f(\overline{E})$ then there exists an $x \in E \cup E'$ such that $f(x) = y$. If $x \in E$ then we are done since $f(x) = y \in f(E) \cup (f(E))' = \overline{f(E)}$.

On the other hand if $x \in E'$ then x is a limit point of E. We need to show that if $y \notin f(E)$ then y is a limit point of $f(E)$. So let $N(y)$ be any neighborhood of y.

Since $N(y)$ is open, then $f^{-1}(N(y))$ is open due to the continuity of f. Also $x \in f^{-1}(N(y))$ so there exists some neighborhood of x, call it $M(x)$, such that $M(x) \subseteq f^{-1}(N(y))$. Moreover x is a limit point of E so there exists some $x \neq p \in E$ such that $p \in f^{-1}(N(y))$. Then $f(p) \in N(y)$ and $f(p) \in f(E)$. Because $y \notin f(E)$ by assumption, then $f(p) \neq y$. Now we have that $y \in (f(E))'$.

For an example to show that the inclusion can be proper, take $X = (0, 1)$ and $Y = \mathbb{R}$ and $E = (0, 1)$ and $f(x) = x$. Then since $X = (0, 1)$ we have that in X the closure of E is just $(0, 1)$ and so

$$f(\overline{E}) = f((0, 1)) = (0, 1)$$

whereas

$$\overline{f(E)} = \overline{(0, 1)} = [0, 1]$$

Since 0 is in the latter but not the former, the inclusion is proper.

Problem 6. Let X, Y be metric spaces and $f : X \to Y$. Prove f is continuous if and only if $f^{-1}(C)$ is closed in X for every closed set C in Y.

We have already seen that the complement of any closed set is open. Since C is assumed to be closed in Y then its complement $Y \setminus C$ is open. We know that $f^{-1}(Y \setminus C)$ is open because f is continuous.

Moreover, I claim that $f^{-1}(Y \setminus C) = X \setminus f^{-1}(C)$. For if $x \in f^{-1}(Y \setminus C)$ then $f(x) \in Y \setminus C$ and so $f(x) \notin C$ so $x \in X \setminus f^{-1}(C)$. In the other direction, if $x \in X \setminus f^{-1}(C)$ then $f(x) \notin C$ so $f(x) \in (Y \setminus C)$ so $x \in f^{-1}(Y \setminus C)$.

Since we now have that $X \setminus f^{-1}(C)$ is open then $f^{-1}(C)$ is closed.

Problem 7. Rudin page 98, Problem 3. The zero set of a continuous real function on a metric space is closed.

The set $\{0\}$ is closed, and $f^{-1}(\{0\})$ is the zero set. By the previous problem, this set is closed.

Problem 8. Rudin page 98, Problem 4. Let f and g be continuous mappings of a metric space X into a metric space Y, and let E be a dense subset of X. Prove that $f(E)$ is dense in $f(X)$. If $g(p) = f(p)$ for all $p \in E$, prove that $g(p) = f(p)$ for all $p \in X$.

To show that $f(E)$ is dense in $f(X)$, we can show that $f(X) \subseteq \overline{f(E)}$. From the fact that $\overline{E} = X$ then we can infer $f(\overline{E}) = f(X)$. We already proved in this assignment that $f(\overline{E}) \subseteq \overline{f(E)}$. Hence $f(X) = f(\overline{E}) \subseteq \overline{f(E)}$ and so $f(E)$ is dense in $f(X)$.

Now suppose $g(p) = f(p)$ for all $p \in E$. We want to show also that if $p \notin E$ then $g(p) = f(p)$, so assume $p \notin E$. But since E is dense then $p \in E'$, that is to say p is a limit point of E.

Let $\{p_n\}$ be any sequence of points in E which approach p. Since $p_n \in E$ then $g(p_n) = f(p_n)$. Now because both functions are continuous

$$\lim_{n\to\infty} g(p_n) = g(p) = \lim_{n\to\infty} f(p_n) = f(p)$$