Module 1 Supplemental Notes 1

Recall

$\lim_{t \to x} f(t) = L$ means for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $0 < |x - t| < \delta$, then $|f(t) - L| < \varepsilon$.

Illustration

Definition

Let $f: [a, b] \to \mathbb{R}$. Fix $x \in [a, b]$ and define $(t) = \frac{f(t) - f(x)}{t - x}$ $(a < t < b, t \neq x)$. Let $f'(x) = \lim_{t \to x} \phi(t)$ (assuming the limit exists). If $f'(x)$ exists, we say f is **differentiable** at x. If $E \subset (a, b)$ and f' is defined at each point of E, we say f is **differentiable** on E.

Illustration

Example 1: $f(x) = |x|$

Theorem
Let $f: [a, b] \to \mathbb{R}$. If f is differentiable at $x \in (a, b)$, then f is continuous at x.

Proof:

Arithmetic of Differentiation
Suppose $f, g: [a, b] \to \mathbb{R}$ are differentiable at $x \in (a, b)$. Then $f + g$ and fg are differentiable at x and $\frac{f}{g}$ is differentiable at x provided $g(x) \neq 0$. Furthermore:

1. $(f + g)'(x) = f'(x) + g'(x)$
2. $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$
3. $\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$, for $g(x) \neq 0$

Proof:

Proposition

If $f(x) = x^n$ for $n \in \mathbb{Z}^+$, then f is differentiable everywhere and $f'(x) = nx^{n-1}$.

Proposition

Polynomials are differentiable everywhere. Furthermore, rational functions $\left(r(x) = \frac{p(x)}{q(x)}, \text{where } p \text{ and } q \text{ are polynomials}\right)$ are differentiable wherever $q(x) \neq 0$.

Chain Rule

Let $f: [a, b] \to \mathbb{R}$ be continuous and let $f'(x)$ exist at some point $x \in (a, b)$. Suppose $g: I \to \mathbb{R}$ for $I \supset Range(f)$ and g is differentiable at $f(x)$. If $h(t) = g\big(f(t)\big)$ $(a \leq t \leq b)$, then h is differentiable at x and $h'(x) = g'\big(f(x)\big)f'(x)$.

Illustration

Discussion

Proof:

Example 2: $f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$

Example 3: $f(x) = \begin{cases} x^2 sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$

Definition

Let $f: X \to \mathbb{R}$ where X is a metric space. We say f has a **local maximum** at $p \in X$ if there exists $r > 0$ such that $f(p) \geq f(q)$ for all $q \in N_r(p)$. We say f has a **local minimum** at $p \in X$ if there exists $r > 0$ such that $f(p) \leq f(q)$ for all $q \in N_r(p)$,

Theorem

Let $f: [a, b] \to \mathbb{R}$. If f has a local maximum at $p \in (a, b)$ and $f'(p)$ exists, then $f'(p) = 0$.

Proof:

Generalized Mean Value Theorem

If $f, g: [a, b] \to \mathbb{R}$ are continuous on $[a, b]$ and differentiable on (a, b). Then there exists $x \in (a, b)$ such that $[f(b) - f(a)]g'(x) = [g(b) - g(a)]f'(x)$.

Discussion

Proof:

Recall

A function $f: [a, b] \to \mathbb{R}$ is monotonically increasing if $a \leq x_1 < x_2 \leq b$, then $f(x_2) \geq f(x_1)$. A function $f: [a, b] \to \mathbb{R}$ is monotonically decreasing if $a \leq x_1 < x_2 \leq b$, then $f(x_2) \leq f(x_1)$.

Theorem

Suppose f is differentiable in $[a, b]$.

1. If $f'(x) \geq 0$ for all $x \in (a, b)$, then f is monotonically increasing
2. If $f'(x) \leq 0$ for all $x \in (a, b)$, then f is monotonically decreasing
3. If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant

Discussion

Example 4: $f(x) = \begin{cases} x^2 sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$

Theorem
Let $f: [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$ and $f'(a) < \lambda < f'(b)$. Then there exists $x \in (a, b)$ such that $f'(x) = \lambda$.

Proof:

Corollary
If $f: [a, b] \to \mathbb{R}$ is differentiable, then f' has no simple discontinuities.