Advanced Calculus, Homework 5

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Problem 1. Rudin page 78, Problem 1. (Convergence implies convergence in absolute value. Is the converse true?)

Suppose $\lim_{n\to\infty} s_n = L$ which is to say,

$$(\forall \varepsilon \in \mathbb{R}^+)(\exists N \in \mathbb{N})(\forall n \geq N, n \in \mathbb{N})\ (|s_n - L| < \varepsilon)$$

I claim that $\lim_{n\to\infty} |s_n| = |L|$. To prove this, let $\varepsilon \in \mathbb{R}^+$. We need to find some $N \in \mathbb{N}$ such that $||s_n| - |L|| < \varepsilon$ so long as $n \geq N$. First set N_0 to be such that $n \geq N_0$ entails

$$|s_n - L| < \varepsilon$$

as guaranteed by the assumption. But by the reverse triangle inequality $||s_n| - |L|| \leq |s_n - L|$. Hence N_0 is the value were seeking, since for all $n \geq N_0$

$$||s_n| - |L|| \leq |s_n - L| < \varepsilon$$

Hence by definition $\lim_{n\to\infty} |s_n| = |L|$.

The converse is false, as witnessed by the sequence $s_n = (-1)^n$. This converges in absolute value since $|s_n| = 1$, and yet s_n diverges.

Problem 2. Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be sequences in $\mathbb{C}^k$. Give an example of the following or explain why such a request is impossible.

a. Sequences $\{x_n\}$ and $\{y_n\}$ which both diverge, but whose sum $\{x_n + y_n\}$ converges.

b. Sequences $\{x_n\}$ and $\{y_n\}$ where $\{x_n\}$ converges and $\{y_n\}$ diverges, but whose sum $\{x_n + y_n\}$ converges.

(a.) $x_n = (-1)^n$ and $y_n = (-1)^{n+1}$. Each diverges but $x_n + y_n = 0$ which clearly converges (to 0).

(b.) This is impossible since $\{x_n + y_n\}$ must always diverge under these conditions. To prove this, suppose for contradiction that $\{x_n + y_n\}$ converges. Since the sum of convergent sequences converges, and we know that $\{-x_n\}$ converges, then we must have that $\{(x_n + y_n) + (-x_n)\}$ converges. But that is $\{y_n\}$, the convergence of which contradicts the divergence of $\{y_n\}$. $\lightning$

Hence we have seen that whenever $\{x_n\}$ converges and $\{y_n\}$ diverges, the sum $\{x_n + y_n\}$ diverges.

Problem 3. Let $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$. Let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence and assume $\lim_{n\to\infty} y_n = 0$. Show that $\lim_{n\to\infty} x_n y_n = 0$.

Let $\varepsilon \in \mathbb{R}^+$. Call M the bound for x_n so that that for all $n \in \mathbb{N}$ we have $|x_n| \leq M$. Choose $N \in \mathbb{N}$ such that for all $n \geq N$ we have $|y_n| < \varepsilon/M$. Now since $|y_n| < \varepsilon/M \leq \varepsilon/|x_n|$ we have

$$\begin{aligned} |x_n y_n - 0| < \varepsilon \quad &\Leftrightarrow \\ |y_n| < \varepsilon/|x_n| & \end{aligned}$$

and therefore $|y_n| < \varepsilon/M \leq \varepsilon/|x_n|$. Therefore this choice of N is that number which corresponds to ε in the definition of $\lim_{n\to\infty} x_n y_n = 0$.

Problem 4. Let $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty} \subseteq \mathbb{R}^k$. Give an example of each of the following or explain why such a request is impossible.

a. A sequence that has a subsequence that is bounded but contains no subsequence that converges.

b. A sequence that does not contain 0 or 1 but contains subsequences converging to each.

(a.) This is impossible, since the subsequence is itself a sequence which is bounded. And this sequence must have a convergent subsequence, by the theorem that all bounded sequences have a convergent subsequence.

(b.) A sequence where the odd terms converge to zero but the even terms converge to one:

$$x_n = \begin{cases} 1/(n+1) & \text{if } n \text{ is odd} \\ 1 + 1/n & \text{if } n \text{ is even} \end{cases}$$

No term of this sequence is 0 or 1 but the subsequence of even terms converges to 1. The subsequence of odd terms converges to 0.

Problem 5. Rudin page 78, Problem 3. If $s_1 = \sqrt{2}$ and

$$s_{n+1} = \sqrt{2 + \sqrt{s_n}} \quad (n = 1, 2, 3, \dots)$$

prove that $\{s_n\}$ converges and that $s_n < 2$.

I claim that this sequence is a bounded, monotonically increasing sequence. If true then the sequence must converge. If we prove that each $s_n < 2$ then since $s_n > 0$ we will know that $\{s_n\}$ is bounded by 2.

We prove both boundedness and monotonically increasing by means of induction. For boundedness, the base case is trivial since $\sqrt{2} < 2$.

Now assume that $s_n < 2$ for any $n \in \mathbb{N}$. Then

$$\begin{aligned} s_{n+1} = \sqrt{2 + \sqrt{s_n}} < 2 &\quad \Leftrightarrow \\ 2 + \sqrt{s_n} < 4 &\quad \Leftrightarrow \\ s_n < 4 & \end{aligned}$$

The mutual implications are valid because all quantities are positive, so squaring preserves order.

So $s_{n+1} < 2$.

Next we show that $s_n \leq s_{n+1}$ for every $n \in \mathbb{N}$, again by induction. In the base case,

$$\begin{aligned} s_1 = \sqrt{2} < s_2 = \sqrt{2 + \sqrt{2}} &\quad \Leftrightarrow \\ 2 < 2 + \sqrt{2} &\quad \Leftrightarrow \\ 0 < \sqrt{2} & \end{aligned}$$

Now that the base-case is done, we assume the inductive hypothesis that $s_n \leq s_{n+1}$ and try to show $s_{n+1} \leq s_{n+2}$.

$$\begin{aligned} s_{n+1} = \sqrt{2 + \sqrt{s_n}} \leq s_{n+2} = \sqrt{2 + \sqrt{s_{n+1}}} &\quad \Leftrightarrow \\ 2 + \sqrt{s_n} \leq 2 + \sqrt{s_{n+1}} &\quad \Leftrightarrow \\ s_n \leq s_{n+1} & \end{aligned}$$

Since we now have that the sequence is monotonically increasing and bounded, by the monotone convergence theorem, it must converge.

Problem 6. Prove the Monotone Convergence Theorem for a decreasing sequence.

We mimic the proof from Rudin for an increasing sequence. In particular, suppose $\{s_n\}_{n=1}^{\infty}$ is a sequence bounded below and monotonically decreasing. Because it's bounded below, the infimum exists, call this $s = \inf_{n \in \mathbb{N}} \{s_n\}$. We claim that $\lim_{n \to \infty} s_n = s$.

To prove this, let $\varepsilon \in \mathbb{R}^+$ be given. Now $s + \varepsilon$ is not an upper-bound and there must exist a element s_N such that $s \leq s_N < s + \varepsilon$. But since the sequence is decreasing and bounded below by s, then for all $n \geq N$ we must have $s_n \leq s_N$. Moreover since s is a lower bound by definition, $s \leq s_n \leq s_N$. Therefore

$$s_n - s = |s_n - s| < \varepsilon$$

for all $n \geq N$ which shows that $\lim_{n \to \infty} s_n = s$.

Problem 7. Prove the sequence defined by $x_1 = 1$ and $x_{n+1} = \frac{x_n+1}{4}$ converges and find its limit.

I claim that it is bounded below and monotonically decreasing. We can prove both by induction. In particular I start with the claim that $x_n \geq 0$ for all $n \in \mathbb{N}$. In the base-case this is obvious.

If we suppose that $x_n \geq 0$ then by multiplying and subtracting, and using $0 > -1$ we have

$$\begin{aligned} x_{n+1} = \frac{x_n + 1}{4} &\geq 0 \quad \Leftrightarrow \\ x_n &\geq 0 \end{aligned}$$

Next we show that the sequence is decreasing. Since $x_1 = 1 \geq x_2 = 1/2$ the base-case is trivial. Now suppose $x_n \geq x_{n+1}$ and consider

$$\begin{aligned} x_{n+1} = \frac{x_n + 1}{4} &\leq x_{n+2} = \frac{x_{n+1} + 1}{4} \quad \Leftrightarrow \\ x_n &\leq x_{n+1} \end{aligned}$$

Now that we see the sequence is decreasing and bounded, the limit must exist by the monotone convergence theorem. Now we must find the limit. Whatever the limit is, call it L, it must satisfy both $\lim\limits_{n\to\infty} x_n = L$ and $\lim\limits_{n\to\infty} x_{n+1} = L$. Then

$$\begin{aligned} \lim_{n\to\infty} x_{n+1} = L & \\ &= \lim_{n\to\infty} \frac{x_{n+1} + 1}{4} \\ &= \frac{\lim_{n\to\infty} x_n + 1}{4} \\ &= \frac{L+1}{4} \end{aligned}$$

which implies $4L = L + 1$ which implies $L = 1/3$.

Problem 8. Rudin page 78, Problem 5. For real sequences prove

$$\limsup_{n\to\infty}(a_n + b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

provided the sum on the right is not of the form $\infty - \infty$.

Say that $\lim_{n\to\infty} a_n = a^*$ and $\lim_{n\to\infty} b_n = b^*$. If one or the other of a^* or b^* is infinity, then the form of the sum must be, for some real number c, either $\infty + c$ or $c + \infty$ or $\infty + \infty$ or $-\infty - \infty$. In the first three cases, trivially these compute to just ∞. Since this is greater than or equal to any extended real number, the inequality must hold trivially. If they are of the form $-\infty - \infty$ then since the greatest subsequential limit of either goes to $-\infty$ then every subsequential limit does too. Therefore any sum of them does too. But now we have $\limsup_{n\to\infty}(a_n + b_n) = -\infty$. So again the result holds since $-\infty \le -\infty$. Therefore we only need to prove the case when both $a^*, b^* \in \mathbb{R}$.

We will accomplish this by definition of the supremum on the left-hand side. Since it is the supremum we can show a number is greater than or equal to it, by showing that the number is an upper bound on the underlying set. That is to say, we will show that $a^* + b^*$ is an upper bound on the set of subsequential limits of $\{a_n + b_n\}$. Further, we will do this by leaning heavily on the theorem given by Rudin which implies $a^* + \varepsilon$ has only finitely many points above it. (Theorem 3.17 part (b))

That is to say, more formally: For any $\varepsilon \in \mathbb{R}^+$ there is some N_1 such that for all $n \ge N_1$ we have

$$a_n < a^* + \varepsilon$$

Similarly there is an N_2 for the bs. That is to say, $\forall n \ge N_2$

$$b_n < b^* + \varepsilon \tag{1}$$

And moreover if we set $N = \max\{N_1, N_2\}$ then we have, for all $n \ge N$

$$\begin{aligned} a_n &< a^* + \varepsilon \\ b_n &< b^* + \varepsilon \end{aligned}$$

It follows that

$$a_n + b_n < a^* + b^* + 2\varepsilon$$

Since this holds for the whole sequence it also obviously holds for any subsequence. Therefore $a^* + b^* + 2\varepsilon$ is an upper-bound for any arbitrary subsequence. From this we infer that any subsequential limit of the left-hand side must satisfy

$$\lim_{i\to\infty} (a_{n_i} + b_{n_i}) \leq a^* + b^* + 2\varepsilon$$

Of course since this inequation holds for arbitrary positive ε then

$$\lim_{i\to\infty} (a_{n_i} + b_{n_i}) \leq a^* + b^*$$

Hence we have shown that $a^* + b^*$ is an upper bound on all subsequential limits. As described at the beginning this suffices to show that

$$\limsup_{n\to\infty} (a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

Problem 9. Show that if some sequence $\{p_n\}$ of a metric space X is Cauchy, and it has a subsequence $p_{n_i} \to p$ then we must have $p_n \to p$.

We prove this directly from the definition, so let $\varepsilon \in \mathbb{R}^+$. Since p_n is Cauchy then there must exist some $N_1 \in \mathbb{N}$ such that for all $m, n \geq N_1$ we have

$$d(p_m, p_n) < \varepsilon/2$$

Moreover, since the subsequence $p_{n_i} \to p$ then there must exist some N_2 such that for every index of the subsequence n_i such that $n_i \geq N_2$ we have

$$d(p_{n_i} - p) < \varepsilon/2$$

We now set $N = \max\{N_1, N_2\}$. Then for any $n \geq N$ we may pick any n_i in the subsequence $n_i \geq N$. Since this index in the subsequence is still an index in the original sequence, we can set $m = n_i$ and infer that

$$d(p_n, p_{n_i}) < \varepsilon/2$$

Now by the triangle inequality, which holds for any distance metric,

$$d(p_n, p) \leq d(p_n, p_{n_i}) + d(p_{n_i}, p)$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon$$

Hence $p_n \to p$.

Problem 10. Let $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty} \subseteq \mathbb{R}^k$. Give an example of each of the following or explain why such a request is impossible.

a. A Cauchy sequence that is not monotone.

b. A Cauchy sequence with an unbounded subsequence.

c. An unbounded sequence containing a Cauchy subsequence.

(a.) Here is a counter-example from $\mathbb{R}$:

$$a_n = \frac{(-1)^n}{n}$$

This is Cauchy because it goes to 0, and any convergent sequence is Cauchy. It is not monotone because it oscillates between negative and positive.

(b.) This is not possible since Cauchy sequences converge in $\mathbb{R}^k$, and convergent sequences are bounded.

(c.)

$$a_n = \begin{cases} n & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

The sequence is unbounded because, for any $M \in \mathbb{R}$ we there will be some $N \in \mathbb{N}$ such that $M < N$. If N is odd this already shows $M < a_N$ and if N is even then $M < a_{N+1}$. On the other hand the subsequence of even-indexed terms is Cauchy, trivially.

Problem 11. Prove that if $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty}$ are Cauchy sequences in $\mathbb{C}^k$, then $\{x_n + y_n\}_{n=1}^{\infty}$ is Cauchy.

If we could identify $\mathbb{C}^k$ with $\mathbb{R}^{2k}$, then the result would be immediate from the completeness of R^{2k}. However that's probably not kosher, so let's give a direct confirmation.

Let $\varepsilon \in \mathbb{R}^+$ be given. Then

$$|x_n + y_n - (x_m + y_m)| \leq |x_n - x_m| + |y_n + y_m|$$

so of course we choose N_1 such that for all $n, m \geq N_1$ we have

$$|x_n - x_m| < \varepsilon/2$$

and do likewise for N_2 corresponding to the ys. That is to say, $\forall m, n \geq N_2$

$$|y_n - y_m| < \varepsilon/2$$

Of course let $N = \max\{N_1, N_2\}$. Then

$$|x_n - x_m| + |y_n - y_m| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

Hence the sum of the sequences is Cauchy.

Problem 12. Prove that if $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty}$ are Cauchy sequences in $\mathbb{R}^k$, then $\{x_n y_n\}_{n=1}^{\infty}$ is Cauchy without using Theorem 3.3.

Since x_n and y_n are Cauchy sequences of $\mathbb{R}^k$, then they must converge. Since convergent sequences are bounded, then each of these is a bounded sequence.

We start by observing

$$\begin{aligned} |x_n y_n - x_m y_m| &= |x_n y_n - x_n y_m + x_n y_m - x_m y_m| \\ &\leq |x_n y_m - x_n y_m| + |x_n y_m - x_m y_m| \\ &= |x_n||y_n - y_m| + |y_m||x_n - x_m| \end{aligned}$$

Now since $\{x_n\}$ is Cauchy then it converges in $\mathbb{R}^k$ and therefore is bounded. Say $|x_n| \leq M_1$ for all $n \in \mathbb{N}$, and likewise $|y_n| \leq M_2$. Then

$$|x_n||y_n - y_m| + |y_m||x_n - x_m| \leq M_1|y_n - y_m| + M_2|x_n - x_m|$$

Now we choose N_1 such that for all $m, n \geq N_1$ we have

$$|x_n - x_m| < \frac{\varepsilon}{2M_2}$$

and similarly pick an N_2 such that

$$|y_n - y_m| < \frac{\varepsilon}{2M_1}$$

and of course we pick $N = \max\{N_1, N_2\}$. Then for all $m, n \geq N$

$$M_1|y_n - y_m| + M_2|x_n - x_m| < M_1\left(\frac{\varepsilon}{2M_2}\right) + M_2\left(\frac{\varepsilon}{2M_2}\right) = \varepsilon$$

Hence the product of the sequences is Cauchy.

Problem 13. Rudin page 82, Problem 23. Show that the distance between two Cauchy sequences converges in a metric space.

We are showing that $\{d(p_n, q_n)\}$ converges, so we show it's Cauchy. Intuitively we should have

$$d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n)$$

as a sort of "triangle-inequality like" fact. But we need to prove this before using it. From the triangle inequality we have both

$$d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_n)$$
$$d(p_m, q_n) \leq d(p_m, q_m) + d(q_m, q_n)$$

Using the second inequality to replace $d(p_m, q_n)$ in the first, we have the desired fact. Note that from $d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n)$ we can infer

$$d(p_n, q_n) - d(p_m, q_m) \leq d(p_n, p_m) + d(q_m, q_n)$$

Moreover none of the argument above depended on the order and choice of indices, so we can swap n for m and also obtain

$$d(p_m, q_m) - d(p_n, q_n) \leq d(p_n, p_m) + d(q_m, q_n)$$

so that when these two are put together we can say

$$|d(p_n, q_n) - d(p_m, q_m)| \leq d(p_n, p_m) + d(q_m, q_n)$$

Now for any $\varepsilon \in \mathbb{R}^+$ we choose N_1 such that $\forall m, n \geq N_1$ we have

$$d(p_m, p_n) < \varepsilon/2$$

and $\forall m, n \geq N_2$

$$d(q_m, q_n) < \varepsilon/2$$

and set $N = \max\{N_1, N_2\}$. Then

$$d(p_m, p_n) + d(q_m, q_n) < \varepsilon$$

This together with $|d(p_n, q_n) - d(p_m, q_m)| < d(p_m, p_n) + d(q_m, q_n)$ entails

$$|d(p_n, q_n) - d(p_m, q_m)| < \varepsilon$$

Hence $\{d(p_n, q_n)\}$ is Cauchy and is a sequence of real numbers. Since the real numbers are Cauchy-complete, then this sequence converges.