MA-652 Advanced Calculus
Homework 1, Jan. 9
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Problem 1. Let $f : [a, b] \to \mathbb{R}$ be differentiable at $x \in (a, b)$ and $k \in \mathbb{R}$. Prove that $(kf)'(x) = kf'(x)$.

If we define the difference quotient at $x \in (a, b)$,

$$\phi(x) = \frac{(kf)(t) - (kf)(x)}{t - x}$$

then our task is to find $\lim_{t \to x} \phi(x)$. But since

$$\lim_{t \to x} \phi = \lim_{t \to x} \frac{kf(t) - kf(x)}{t - x}$$

by definition of multiplying functions, then this is

$$\lim_{t \to x} k \frac{f(t) - f(x)}{t - x} = k \lim_{t \to x} \frac{f(t) - f(x)}{t - x} = kf'(x)$$

where this limit is guaranteed to exist by the differentiability of f in this interval.

Problem 2. Let $f, g : [a, b] \to \mathbb{R}$ be differentiable at $x \in (a, b)$.

(a). Prove the quotient rule using the limit definition, wherever the denominator isn't 0.

For any $x \in (a, b)$ such that $g(x) \neq 0$ we will show that $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$. We define the difference quotient

$$\phi(t) = \frac{f(t)/g(t) - f(x)/g(x)}{t - x} = \frac{\frac{f(t)g(x) - f(x)g(t)}{g(t)g(x)}}{t - x}$$

Therefore

$$\left(\frac{f}{g}\right)'(t) = \lim_{t \to x} \frac{f(t)g(x) - f(x)g(t)}{[g(t)g(x)](t - x)}$$

$$= \frac{1}{g(x)} \lim_{t \to x} \frac{1}{g(t)} \left(\frac{f(t)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(t)}{t - x} \right)$$

$$= \frac{1}{g(x)} \lim_{t \to x} \frac{1}{g(t)} \cdot \lim_{t \to x} \left(\frac{f(t) - f(x)}{t - x} \cdot g(x) - f(x) \frac{g(t) - g(x)}{t - x} \right)$$

$$= \frac{1}{[g(x)]^2} (f'(x)g(x) - f(x)g'(x))$$

The final equation follows because we assumed that $g(x) \neq 0$ and therefore $\lim_{t \to x} \frac{1}{g(t)} = \frac{1}{g(x)}$. Also we assumed both functions are differentiable in the interval and hence $\lim_{t \to x} \frac{f(t) - f(x)}{t - x} = f'(x)$ and also $\lim_{t \to x} \frac{g(t) - g(x)}{t - x} = g'(x)$. The proof is then complete.

(b). Use the limit definition to find the derivative of $\frac{1}{x}$.

$$\lim_{t\to x}\frac{\frac{1}{t}-\frac{1}{x}}{t-x}=\lim_{t\to x}\frac{\frac{x-t}{tx}}{t-x}$$

$$=-\lim_{t\to x}\frac{1}{tx}$$

$$=-\frac{1}{x^2}$$

(c). Use (b) with the chain rule to derive the quotient rule.

$$\left(\frac{f}{g}\right)'(x)=\left(f(x)\cdot\frac{1}{g(x)}\right)'$$

$$=f'(x)\cdot\frac{1}{g(x)}+f(x)\left[\frac{1}{g(x)}\right]'$$

by the product rule. Now by the chain rule

$$\left[\frac{1}{g(x)}\right]'=-\frac{1}{[g(x)]^2}g'(x)$$

So we can infer from these two equations that

$$\left(\frac{f}{g}\right)'(x)=\frac{f'(x)}{g(x)}-f(x)\frac{g'(x)}{[g(x)]^2}$$

$$=\frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$$

Problem 3. Rudin page 114 problem 1. If f is defined on $\mathbb{R}$ and $\forall x, y \in \mathbb{R}$ we have $|f(x) - f(x)| \leq (x - y)^2$, then prove that f is constant.

This feels like a complex analysis theorem—this sort of thing isn't supposed to be true for real functions! ☺

To show that f is constant we'll prove that the derivative is zero everywhere. That is to say, we'll show that at every $x \in \mathbb{R}$

$$\left| \frac{f(t) - f(x)}{t - x} \right| < \varepsilon$$

for each $\varepsilon \in \mathbb{R}^+$, whenever $|t - x| < \delta$ for some corresponding δ. Choose $\delta = \varepsilon$ in fact. Then if $|t - x| < \delta$ we have

$$\left| \frac{f(t) - f(x)}{t - x} \right| < \frac{(t - x)^2}{\delta} < \frac{\delta^2}{\delta} = \varepsilon$$

Problem 4. Rudin page 114 problem 2. Suppose $f'(x) > 0$ in (a, b). Prove that f is strictly increasing in (a, b). Let g be its inverse. Prove that g is differentiable and

$$g'(f(x)) = \frac{1}{f'(x)}$$

We already know that f is monotonically increasing due to theorem 5.11. Moreover, since $f'(x) > 0$ we have that there can be no two values $c, d \in (a, b)$ such that $c < d$ and $f(c) = f(d)$. If there were, by the mean-value theorem there would have to be a point $x_0 \in (c, d)$ such that $f'(x) = \frac{f(d)-f(c)}{d-c} = 0$.

Since f is strictly increasing it is one-to-one and has an inverse. By definition if g is the inverse then $f(g(y)) = y$ for all y in the co-domain of f. By the chain rule we have

$$f'(g(y))g'(y) = 1 \quad \Rightarrow$$

$$g'(y) = \frac{1}{f'(g(y))}$$

If we regard $y = f(x)$ and hence $g(y) = x$ then we have

$$g'(f(x)) = \frac{1}{f'(x)}$$

Problem 5. Rudin page 114 problem 4. If $C_0 + \frac{C_1}{2} + \cdots + \frac{C_n}{n+1} = 0$ then

$$C_0 + C_1 x + \cdots + C_n x^n = 0$$

has at least one solution in $[0, 1]$.

Define $F(x) = C_0 x + C_1 x^2/2 + \cdots + C_n x^n/(n+1)$ and observe that $F'(x) = C_0 + \cdots + C_n x^n$. Now $F(0) = 0$ and $F(1) = C_0 + \cdots + \frac{C_n}{n+1} = 0$. By the mean value theorem there is a point $c \in (0, 1)$ such that $F'(c) = \frac{F(1)-F(0)}{1-0} = 0$.

Problem 6. Rudin page 114 problem 5. Suppose f is defined and differentiable for every $x > 0$, and $\lim_{x\to\infty} f'(x) = 0$. Let $g(x) = f(x+1) - f(x)$. Prove that $\lim_{x\to\infty} g(x) = 0$.

Let $M \in \mathbb{R}$ be such that $|f'(a)| < \varepsilon$ for all $a > M$. We will show that, for this value of M, it follows that $|g(x)| < \varepsilon$ for all $x > M$.

Now for any such $x > M$ we have that $x+1 > M$. So on the interval $(x, x+1)$ we have that $\frac{f(x+1)-f(x)}{x+1-x} = f(x+1) - f(x) = f'(c)$ for some $c \in (x, x+1)$. Therefore $|f(x+1) - f(x)| = |f'(c)| < \varepsilon$. Since x was chosen arbitrarily in (M, ∞), we have shown

$$\lim_{x\to\infty} g(x) = 0$$

Problem 7. Rudin page 114 problem 6. Suppose (a) f is continuous for $x \geq 0$, (b) $f'(x)$ exists for $x > 0$, (c) $f(0) = 0$, (d) f' is monotonically increasing. Put $g(x) = \frac{f(x)}{x}$ for $x > 0$ and prove that g is monotonically increasing.

We show that the derivative is non-negative. Since

$$g'(x) = \frac{f'(x)x - f(x)}{x^2}$$

then it suffices to show that $f'(x)x - f(x) \geq 0$ for every $x \in \mathbb{R}^+$. To leverage condition (d) we apply the mean value theorem. There must exist a $0 < c < x$ such that

$$\frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x} = f'(c)$$

and hence $f(x) = f'(c)x$. Since $x > 0$, and since f' is increasing and $c < x$, we must have

$$f(x) = f'(c)x \leq f'(x)x$$

which implies

$$0 \leq f'(x)x - f(x)$$

as desired.

Problem 8. Rudin page 117 problem 22 abc.

First we need to show that based on these assumptions, $(kf)'(x)$ exists.

Problem 9. Rudin page 119 problem 26.

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Problem 10. Rudin page 119 problem 27.

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