MA 638 - Section 7.5 Homework

Throughout assume R is a ring with identity $1 \neq 0$.

1. Suppose that R, D, and Q are defined as in Theorem 7.15. Use the equivalence relation along with the defined addition and multiplication defined in the proof of Theorem 7.15 to prove the following:

   (a) Prove that Q is an abelian group under addition with additive identity $0/d$ for any $d \in D$.

   Proof: We first remark that R is assumed to be a commutative ring for this theorem and hence both addition and multiplication are commutative. Also, throughout the problem, we let $r, s, t \in R$ be arbitrary, and $d, e, f \in D$ be arbitrary.

   I next want to establish a few lemmas that I will use repeatedly in these proofs. First, since we saw in class that $\frac{d}{d}$ is the unitary element of Q regardless of the choice of $d \in D$, then it follows that

$$\frac{re}{de} = \frac{r}{d} \cdot \frac{e}{e} = \frac{r}{d}.$$

   The second lemma that I want to use throughout is that

$$\frac{r+s}{d} = \frac{r}{d} + \frac{s}{d}.$$

   The above follows from

$$\frac{r}{d} + \frac{s}{d} = \frac{rd+sd}{d^2} = \frac{(r+s)d}{d^2} = \frac{r+s}{d}.$$

   Now to begin the proof, we start with closure. To show closure, let $\frac{r}{d}, \frac{s}{e} \in Q$. Then $\frac{r}{d} + \frac{s}{e} = \frac{re+sd}{de}$. By construction, D is closed under multiplication and therefore $de \in D$. So $\frac{r}{d} + \frac{s}{e} \in Q$.

   We can see that $\frac{0}{d}$ is the additive identity since

$$\frac{r}{e} + \frac{0}{d} = \frac{rd+0e}{ed} = \frac{rd}{ed} = \frac{r}{e}.$$

   To see that Q is closed under inverses, observe that

$$\frac{r}{d} + \frac{-r}{d} = \frac{rd-rd}{d^2} = \frac{0}{d^2}.$$

   Since $d \in D$ then $d^2 \in D$. And from the above we already know that $\frac{0}{e}$ is the additive identity for any $e \in D$. Hence $\frac{-r}{d}$ is the additive inverse of $\frac{r}{d}$.

   To show that addition is associative

$$
\begin{aligned}
\frac{r}{d}+\left(\frac{s}{e}+\frac{t}{f}\right) &= \frac{r}{d}+\frac{sf+te}{ef}\\
&= \frac{ref+d(sf+te)}{def}\\
&= \frac{(re+ds)f+dte}{def}\\
&= \frac{(re+ds)}{de}+\frac{t}{f}\\
&= \left(\frac{r}{d}+\frac{s}{e}\right)+\frac{t}{f}.
\end{aligned}
$$

Finally to show commutativity

$$
\frac{r}{d}+\frac{s}{e}=\frac{re+sd}{de}=\frac{sd+re}{ed}=\frac{s}{e}+\frac{r}{d}.
$$

(b) Prove that multiplication is associative, distributive, and commutative.
Proof: Associative:

$$
\frac{r}{d}\cdot\left(\frac{s}{e}\cdot\frac{t}{f}\right)=\frac{r(st)}{d(ef)}=\frac{(rs)t}{(de)f}=\left(\frac{r}{d}\cdot\frac{s}{e}\right)\cdot\frac{t}{f}.
$$

Distributive:

$$
\begin{aligned}
\frac{r}{d}\cdot\left(\frac{s}{e}+\frac{t}{f}\right) &= \frac{r}{d}\cdot\frac{sf+te}{ef}\\
&= \frac{rsf+rte}{def}\\
&= \frac{rsf}{def}+\frac{rte}{def}\\
&= \frac{rs}{de}+\frac{rt}{df}\\
&= \frac{r}{d}\cdot\frac{s}{e}+\frac{r}{d}\cdot\frac{t}{f}.
\end{aligned}
$$

Commutative:

$$
\frac{r}{d}\cdot\frac{s}{e}=\frac{rs}{de}=\frac{sr}{ed}=\frac{s}{e}\cdot\frac{r}{d}.
$$

(c) Prove that the map $i : R \to Q$ where $i(r) = \frac{rd}{d}$ where $d \in D$ is an injective ring homomorphism.
Injective: Let $i(r) = i(s) = \frac{rd}{d} = \frac{sd}{d}$. Then by definition of equality $rd^2 = sd^2$. Since d is not zero or a zero divisor, by Proposition 2 of section 7.1, $rd = sd$. By the same principle, $r = s$ and therefore i is injective.
The additive homomorphism property:

$$i(r+s) = \frac{(r+s)d}{d} = \frac{rd+sd}{d} = \frac{rd}{d} + \frac{sd}{d} = i(r) + i(s).$$

The multiplicative homomorphism property:

$$i(r \cdot s) = \frac{rsd}{d} = \frac{rsd^2}{d^2} = \frac{rdsd}{d^2} = \frac{rd}{d} \cdot \frac{sd}{d} = i(r) \cdot i(s).$$

Hence i is an injective homomorphism.

2. Show by example that $R[x]$ does not have a field of fractions if R is not an integral domain.

Example: Let $R = \mathbb{Z}_4$ in which the number 2 is a zero-divisor since $2 \cdot 2 = 0$. Then the polynomial $2x^0 \in R[x]$ is a zero-divisor because $(2x^0)(2x^0) = 0x^0$. Hence if we consider the ring of fractions of $R[x]$, with $D = \{1x^0, 3x^0\}$, then $\frac{2x^0}{1x^0}$ is also a zero-divisor since

$$\frac{2x^0}{1x^0} \cdot \frac{2x^0}{1x^0} = \frac{0x^0}{1x^0}.$$

Because $\frac{2x^0}{1x^0}$ is a zero-divisor it cannot be a unit, and therefore the ring of fractions is not a field.