MA 638 - Section 7.4 Homework

INSTRUCTIONS: The solutions to all problems here should be typed up in LaTeX using correct mathematical notation and language. Proofs and explanations must be written in complete sentences using correct grammar and punctuation. Use mathematical displays and *implication* or *if and only if* arrows where appropriate to manipulate equations or to validate steps in a proof or argument. Your completed pdf will be uploaded in Canvas. Be sure to save your pdf file using the naming format "first initial last name underscore section number", i.e. Jane Doe would save her this homework as **jdoe_7-4.pdf**.

Throughout assume R is a ring with identity $1 \neq 0$.

1. Let R be a commutative ring. Prove that the principal ideal generated by x in $R[x]$ is a prime ideal if and only if R is an integral domain. Prove that (x) is a maximal ideal if and only if R is a field.

2. Assume R is commutative. Prove that if P is a prime ideal of R and P contains no zero divisors, then R is an integral domain.

3. Let $\phi : R \to S$ be a homomorphism of commutative rings. Prove that if P is a prime ideal of S, then either $\phi^{-1}(P) = R$ or $\phi^{-1}(P)$ is a prime ideal of R. Apply this to a special case when R is a subring of S and ϕ is the inclusion homomorphism to deduce that if P is a prime ideal of S, then $P \cap R$ is either R or a prime ideal of R. ($\phi^{-1}(P) = \{x \in R \mid \phi(x) \in P \subset S\}$)