MA 630 - Quiz 4 Group 3

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1. Let x be an integer, and set $S = \{n \in \mathbb{N} \; : \; x \text{ is odd if and only if } x^n \text{ is odd}\}$.

   (a) Prove that $1 \in S$.

   (b) Suppose that $n \in S$. Prove that if x is odd, then x^{n+1} is odd.

   (c) Suppose that $n \in S$. Prove that if x^{n+1} is odd, then x is odd.

   (d) Suppose that $n \in S$. The results of parts (b) and (c) imply that what other integer is an element of S? Explain.

   (e) Combining parts (a)-(d) with the Principle of Mathematical Induction, what result has been proved?

(a) *Proof:* To show that $1 \in S$ we need to show that x is odd if and only if x^1 is odd. But this is trivial since $x^1 = x$ for all integers x. So the statement is equivalent to "x is odd if and only if x is odd", and all sentences of the form "P if and only if P" are tautologically true. Hence $1 \in S$. □

(b) *Proof:* Assume that $n \in S$ and we will show that if x is odd then x^{n+1} is odd. So also assume x is odd and we'll show that x^{n+1} is odd.

Since we have assumed $n \in S$ then we have by definition of S that x is odd if and only if x^n is odd. Since we are also assuming x is odd, then we actually have that x^n is odd. Since x and x^n are both odd there must exist integers a, b such that $x = 2a + 1$ and $x^n = 2b + 1$. Hence,

$$\begin{aligned} x^{n+1} &= x \cdot x^n \\ &= (2a+1)(2b+1) \\ &= 4ab + 2a + 2b + 1 \\ &= 2(2ab + a + b) + 1. \end{aligned}$$

Now since $t = 2(2ab + a + b)$ is an integer, then $x^{n+1} = 2t + 1$ is odd. □

(c) Suppose $n \in S$ and we will show that if x^{n+1} is odd then x is odd. We will prove this by showing the converse, so suppose x is even and we will show that

x^{n+1} is even. Since x is even then there is some integer k such that $x = 2k$. Now we have

$$x^{n+1} = x \cdot x^n = 2kx^n.$$

Since $t = kx^n$ is an integer then $x^{n+1} = 2t$ is even. $\square$

(d) *Explanation:* Parts (b) and (c) together show that if $n \in S$ then x is odd if and only if x^{n+1} is odd. This consequent is the same thing as the condition for $n+1 \in S$. Hence parts (b) and (c) show that if $n \in S$ then $n+1 \in S$. $\square$

We may choose to set $n = 1$ since the above holds for every natural number. Then we have that if $1 \in S$ then $2 \in S$. From part (a) we know that $1 \in S$. Therefore we conclude that $2 \in S$.

(e) *E*xplanation: Part (a) establishes the base-case. Parts (b) and (c) together establish the inductive case. Hence by the Principle of Mathematical Induction, we have that $S = \mathbb{N}$. That is to say, for every natural number n, and for every integer x, we have that x is odd if and only if x^n is odd. $\square$

2. Let $S = \{n \in \mathbb{N} \ : \ 12 \text{ does not divide } n^4 - n^2\}$.

   (a) Explain why $1, 2, 3, 4, 5, 6 \notin S$.

   Proof/Explanation. Since $1^4 - 1^2 = 0$ and 12 divides 0, then $1 \notin S$. By the same logic, 12 divides each of the following

$$\begin{aligned}
2^4 - 2^2 &= 12, \\
3^4 - 3^2 &= 72 = 12 \cdot 6, \\
4^4 - 4^2 &= 240 = 12 \cdot 20, \\
5^4 - 5^2 &= 600 = 12 \cdot 50, \\
6^4 - 6^2 &= 1260 = 12 \cdot 105,
\end{aligned}$$

   and hence $2, 3, 4, 5, 6 \notin S$. $\square$

   (b) Suppose (for the sake of contradiction) that $S \neq \varnothing$. The Least Natural Number Principle implies that S has a smallest element, say k. Then, by part (a), $k \geq 7$ and thus, there exists a natural number m, $1 \leq m < k$

such that $k = m + 6$. What must be true about $m^4 - m^2$ and $k^4 - k^2$? Explain.

Proof/Explanation. If we suppose that $S \neq \varnothing$ and that $1 \leq m < k$ such that $k = m + 6$, then it must be true that there is an integer r such that

$$m^4 - m^2 = 12r.$$

Also by definition of k we have

$$k^4 - k^2 \neq 12t, \text{ for any integer } t.$$

$\square$

(c) Notice that

$$\begin{aligned} k^4 - k^2 &= (m+6)^4 - (m+6)^2 \\ &= m^4 + 24m^3 + 215m^2 + 852m + 1260 \\ &= m^4 + 24m^3 + (216m^2 - m^2) + 852m + 1260 \\ &= (m^4 - m^2) + (24m^3 + 216m^2 + 852m + 1260). \end{aligned}$$

Where is the contradiction?

Proof/Explanation.

Since

$$\begin{aligned} k^4 - k^2 &= (m^4 - m^2) + (24m^3 + 216m^2 + 852m + 1260) \\ &= 12r + 12(2m^3 + 18m^2 + 71m + 105) \\ &= 12(r + 2m^3 + 18m^2 + 71m + 105) \end{aligned}$$

hence $k^4 - k^2$ is divisible by 12. This contradicts the fact that $k^4 - k^2$ is not divisible by 12, from part (b). $\square$

(d) What result has been proved?

Proof/Explanation.
It has been proven that $S = \varnothing$. $\square$

3. Use induction to prove that

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

for all natural numbers n.
Proof: Let

$$S = \left\{ n \in \mathbb{N} : 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \right\}$$

.
We first show that $1 \in S$. Since

$$1^2 = \frac{1(1+1)(2 \cdot 1 + 1)}{6},$$

it follows that $1 \in S$.
Now, let $n \in S$. Thus,

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

We must show that $n+1 \in S$. That is, we will show that

$$1^2 + 2^2 + 3^2 + \cdots + n^2 + (n+1)^2 = \frac{(n+1)(n+2)[2(n+1)+1]}{6}.$$

We have

$$\begin{aligned}
1^2+2^2+3^2+\cdots+n^2+(n+1)^2 &= (1^2+2^2+3^2+\cdots+n^2)+(n+1)^2 \\
&= \frac{n(n+1)(2n+1)}{6}+(n+1)^2 \\
&= \frac{n(n+1)(2n+1)+6(n+1)^2}{6} \\
&= \frac{(n+1)[n(2n+1)+6(n+1)]}{6} \\
&= \frac{(n+1)(2n^2+n+6n+6)}{6} \\
&= \frac{(n+1)[2(n^2+3n+2)+(n+2)]}{6} \\
&= \frac{(n+1)[2(n+1)(n+2)+(n+2)]}{6} \\
&= \frac{(n+1)(n+2)[2(n+1)+1]}{6}.
\end{aligned}$$

Thus, $n+1 \in S$. By the Principle of Mathematical Induction, $S = \mathbb{N}$. Therefore,

$$1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

for all natural numbers n. □

4. Use induction to prove that $2^n + 1 < 2^{n+1} - 1$ for all natural numbers $n \geq 2$.
Proof: Let

$$S = \left\{ n \in \mathbb{N} : \forall n \geq 2, (2^n+1) < (2^{n+1}-1) \right\}$$

First, we shall show $2^n + 1 < 2^{n+1} - 1$ is valid for $n = 2$.

$$\begin{aligned}
2^2+1 &< 2^{2+1}-1 \\
4+1 &< 8-1 \\
5 &< 7.
\end{aligned}$$

Therefore we know that $2 \in S$. Now suppose that $n \in S$. We must show that

$n+1 \in S$. So considering the inequality for $n+1$, we get that

$$\begin{aligned}
2^{n+1}+1 &< 2^{(n+1)+1}-1 \\
2^{n+1}+1 &< 2^{n+2}-1 \\
2(2^n)+1 &< 4(2^n)-1 \\
2 &< 2(2^n) \\
1 &< 2^n.
\end{aligned}$$

Then, since n is a natural number, 2^n must be greater than 1. Thus, we have shown that the inequality holds for $n+1$, and therefore that $n+1 \in S$. Now, by the Principle of Mathematical Induction, we have shown that $2^n+1 < 2^{n+1}-1$ for all natural numbers $n \geq 2$. $\square$

5. Use induction to prove that $5^n - 2^n$ is divisible by 3 for all natural numbers n. *It may be helpful to note that* $5^{n+1} = 5 \cdot 5^n = (3+2)5^n$.

Proof.

$$\text{Let } S = \left\{ n \in \mathbb{N} : 3 \text{ divides } 5^n - 2^n \right\}.$$

First we show that $1 \in S$: If $n = 1$, then

$$5^n - 2^n = 5^1 - 2^1 = 5 - 2 = 3.$$

Since 3 divides 3, $1 \in S$.

Suppose that $n \in S$. Then 3 divides $5^n - 2^n$. We want to use this fact to prove that $n+1 \in S$. Note that $n+1 \in S$ if and only if 3 divides $5^{n+1} - 2^{n+1}$. And we have,

$$\begin{aligned}
5^{n+1} - 2^{n+1} &= 5(5^n) - 2(2^n) \\
&= (3+2)5^n - (2)2^n \\
&= (3)5^n + (2)5^n - (2)2^n \\
&= 2(5^n - 2^n) + (3)5^n.
\end{aligned}$$

Thus, $(3)5^n = 3k$ for some integer k, namely $k = 5^n$, proving that 3 divides $(3)5^n$. As we have previously shown, 3 also divides $5^2 - 2^n$, and so $n+1 \in S$. By the Principle of Mathematical Induction, $S = \mathbb{N}$. $\square$