MA 638 - Section 7.4 Homework

Throughout assume R is a ring with identity $1 \neq 0$.

1. Let R be a commutative ring. Prove that the principal ideal generated by x in $R[x]$ is a prime ideal if and only if R is an integral domain. Prove that (x) is a maximal ideal if and only if R is a field.

   For each part, we will use the following: Consider the evaluation map $\varphi : R[x] \to R$. If $p(x) \in R[x]$ then φ is given by $\varphi(p(x)) = p(0)$. Obviously we have both $\ker \varphi = (x)$ and $\mathrm{Im}\varphi = R$. Now $R[x]/(x) \cong R$ by the first isomorphism theorem.

   Proof part 1: Suppose (x) in $R[x]$ is prime. By theorem 13 of section 7.4, since (x) is prime therefore $R[x]/(x)$ is an integral domain. Since $R[x]/(x) \cong R$, then R is an integral domain.

   For the converse suppose that R is an integral domain. Therefore (x) is prime, again by theorem 13.

   Proof part 2: Suppose (x) is maximal and therefore by theorem 12 of section 7.4 $R[x]/(x)$ is a field. Since $R[x]/(x) \cong R$ then we have that R is a field. For the converse, suppose R is a field, then by theorem 12, (x) is maximal.

2. Assume R is commutative. Prove that if P is a prime ideal of R and P contains no zero divisors, then R is an integral domain.

   Proof: Let $r, s \in R$ such that $rs = 0$. Since P is an ideal, then $rs = 0 \in P$. Since P is prime either $r \in P$ or $s \in P$. Without loss of generality suppose $r \in P$. Therefore r is not a zero divisor, and since $rs = 0$ then either $r = 0$ or $s = 0$. Hence R is an integral domain.

3. Let $\phi : R \to S$ be a homomorphism of commutative rings. Prove that if P is a prime ideal of S then either $\phi^{-1}(P) = R$ or $\phi^{-1}(P)$ is a prime ideal of R. Apply this to a special case when R is a subring of S and ϕ is the inclusion homomorphism to deduce that if P is a prime ideal of S, then $P \cap R$ is either R or a prime ideal of R.

   Proof: First we establish that $\phi^{-1}(P)$ is an ideal, beginning with the fact that it is a commutative subgroup under addition. If $a, b \in \phi^{-1}(P)$ then $\phi(a), \phi(b) \in P$ and since P is an ideal, $\phi(a)+\phi(b) \in P$. Since ϕ is a homomorphism $\phi(a + b) = \phi(a) + \phi(b)$ and therefore $a + b \in \phi^{-1}(P)$. Moreover $-\phi(a) = \phi(-a) \in P$ and so $-a \in \phi^{-1}(P)$. Finally, associativity and commutativity follow from the fact that R is a ring.

   To next show closure under arbitrary products, let $r \in R$. Then $\phi(r)\phi(a) \in P$ since P is an ideal. But then $\phi(ra) = \phi(r)\phi(a)$ and so $ra \in \phi^{-1}(P)$. All that then remains is to show associativity and distributivity, but these are immediate from the fact that R is a ring.

   Now that we know $\phi^{-1}(P)$ is an ideal, let $a, b \in R$ and $ab \in \phi^{-1}(P)$. Therefore there exists some $p \in P$ such that $\phi(ab) = p$. Since ϕ was assumed to be a homomorphism then $\phi(ab) = \phi(a)\phi(b) \in P$ and therefore either $\phi(a) \in P$ or $\phi(b) \in P$. Without loss of generality suppose $\phi(a) \in P$ and therefore $a \in \phi^{-1}(P)$. This shows that if $\phi^{-1}(P) \neq R$ then $\phi^{-1}(P)$ is prime, which is equivalent to the statement that $\phi^{-1}(P) = R$ or $\phi^{-1}(P)$ is prime.

   If R is a subring of S and ϕ is the inclusion homomorphism, and if P is a prime ideal of S, then we have that $\phi^{-1}(P)$ is prime. But because ϕ is the inclusion homomorphism then $\phi^{-1}(P)$ just is those elements of R which are elements in P. That is to say $\phi^{-1}(P) = P \cap R$. Hence if $P \cap R \neq R$ then $P \cap R$ is prime.