MA-652 Advanced Calculus
Homework 1, Jan. 9
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Problem 1. Let $f : [a, b] \to \mathbb{R}$ be differentiable at $x \in (a, b)$ and $k \in \mathbb{R}$. Prove that $(kf)'(x) = kf'(x)$.

If we define the difference quotient at $x \in (a, b)$,

$$\phi(x) = \frac{(kf)(t) - (kf)(x)}{t - x}$$

then our task is to find $\lim\limits_{t \to x} \phi(x)$. But since

$$\lim_{t \to x} \phi = \lim_{t \to x} \frac{kf(t) - kf(x)}{t - x}$$

by definition of multiplying functions, then this is

$$\lim_{t \to x} k \frac{f(t) - f(x)}{t - x} = k \lim_{t \to x} \frac{f(t) - f(x)}{t - x} = kf'(x)$$

where this limit is guaranteed to exist by the differentiability of f in this interval.

Problem 2. Let $f, g : [a, b] \to \mathbb{R}$ be differentiable at $x \in (a, b)$.

(a). Prove the quotient rule using the limit definition, wherever the denominator isn't 0.

For any $x \in (a, b)$ such that $g(x) \neq 0$ we will show that $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$. We define the difference quotient

$$\phi(t) = \frac{f(t)/g(t) - f(x)/g(x)}{t - x} = \frac{\frac{f(t)g(x) - f(x)g(t)}{g(t)g(x)}}{t - x}$$

Therefore

$$\left(\frac{f}{g}\right)'(t) = \lim_{t \to x} \frac{f(t)g(x) - f(x)g(t)}{[g(t)g(x)](t - x)}$$

$$= \frac{1}{g(x)} \lim_{t \to x} \frac{1}{g(t)} \left(\frac{f(t)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(t)}{t - x} \right)$$

$$= \frac{1}{g(x)} \lim_{t \to x} \frac{1}{g(t)} \cdot \lim_{t \to x} \left(\frac{f(t) - f(x)}{t - x} \cdot g(x) - f(x) \frac{g(t) - g(x)}{t - x} \right)$$

$$= \frac{1}{[g(x)]^2} (f'(x)g(x) - f(x)g'(x))$$

The final equation follows because we assumed that $g(x) \neq 0$ and therefore $\lim_{t \to x} \frac{1}{g(t)} = \frac{1}{g(x)}$. Also we assumed both functions are differentiable in the interval and hence $\lim_{t \to x} \frac{f(t) - f(x)}{t - x} = f'(x)$ and also $\lim_{t \to x} \frac{g(t) - g(x)}{t - x} = g'(x)$. The proof is then complete.

(b). Use the limit definition to find the derivative of $\frac{1}{x}$.

$$\lim_{t\to x}\frac{\frac{1}{t}-\frac{1}{x}}{t-x} = \lim_{t\to x}\frac{\frac{x-t}{tx}}{t-x}$$

$$= -\lim_{t\to x}\frac{1}{tx}$$

$$= -\frac{1}{x^2}$$

(c). Use (b) with the chain rule to derive the quotient rule.

$$\left(\frac{f}{g}\right)'(x) = \left(f(x)\cdot\frac{1}{g(x)}\right)'$$

$$= f'(x)\cdot\frac{1}{g(x)} + f(x)\left[\frac{1}{g(x)}\right]'$$

by the product rule. Now by the chain rule

$$\left[\frac{1}{g(x)}\right]' = -\frac{1}{[g(x)]^2}g'(x)$$

So we can infer from these two equations that

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)}{g(x)} - f(x)\frac{g'(x)}{[g(x)]^2}$$

$$= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

Problem 3. Rudin page 114 problem 1. If f is defined on $\mathbb{R}$ and $\forall x, y \in \mathbb{R}$ we have $|f(x) - f(y)| \leq (x - y)^2$, then prove that f is constant.

This feels like a complex analysis theorem—this sort of thing isn't supposed to be true for real functions! ☺

To show that f is constant we'll prove that the derivative is zero everywhere. That is to say, we'll show that at every $x \in \mathbb{R}$

$$\left| \frac{f(t) - f(x)}{t - x} \right| < \varepsilon$$

for each $\varepsilon \in \mathbb{R}^+$, whenever $|t - x| < \delta$ for some corresponding δ. Choose $\delta = \varepsilon$ in fact. Then if $|t - x| < \delta$ we have

$$\left| \frac{f(t) - f(x)}{t - x} \right| \leq \frac{(t - x)^2}{|t - x|} = |t - x| < \delta = \varepsilon$$

Problem 4. Rudin page 114 problem 2. Suppose $f'(x) > 0$ in (a, b). Prove that f is strictly increasing in (a, b). Let g be its inverse. Prove that g is differentiable and

$$g'(f(x)) = \frac{1}{f'(x)}$$

We already know that f is monotonically increasing due to theorem 5.11. Moreover, since $f'(x) > 0$ we have that there can be no two values $c, d \in (a, b)$ such that $c < d$ and $f(c) = f(d)$. If there were, by the mean-value theorem there would have to be a point $x_0 \in (c, d)$ such that $f'(x) = \frac{f(d)-f(c)}{d-c} = 0$.

Since f is strictly increasing it is one-to-one and has an inverse. By definition if g is the inverse then $f(g(y)) = y$ for all y in the co-domain of f. By the chain rule we have

$$f'(g(y))g'(y) = 1 \quad \Rightarrow$$

$$g'(y) = \frac{1}{f'(g(y))}$$

If we regard $y = f(x)$ and hence $g(y) = x$ then we have

$$g'(f(x)) = \frac{1}{f'(x)}$$

Problem 5. Rudin page 114 problem 4. If $C_0 + \frac{C_1}{2} + \cdots + \frac{C_n}{n+1} = 0$ then

$$C_0 + C_1 x + \cdots + C_n x^n = 0$$

has at least one solution in $[0, 1]$.

Define $F(x) = C_0 x + C_1 x^2/2 + \cdots + C_n x^n/(n+1)$ and observe that $F'(x) = C_0 + \cdots + C_n x^n$. Now $F(0) = 0$ and $F(1) = C_0 + \cdots + \frac{C_n}{n+1} = 0$. By the mean value theorem there is a point $c \in (0, 1)$ such that $F'(c) = \frac{F(1)-F(0)}{1-0} = 0$.

Problem 6. Rudin page 114 problem 5. Suppose f is defined and differentiable for every $x > 0$, and $\lim_{x\to\infty} f'(x) = 0$. Let $g(x) = f(x+1) - f(x)$. Prove that $\lim_{x\to\infty} g(x) = 0$.

Set $\varepsilon \in \mathbb{R}^+$. Let $M \in \mathbb{R}$ be such that $|f'(a)| < \varepsilon$ for all $a > M$. We will show that, for this value of M, it follows that $|g(x)| < \varepsilon$ for all $x > M$.

Now for any such $x > M$ we have that $x + 1 > M$. So on the interval $(x, x+1)$ we have that $\frac{f(x+1)-f(x)}{x+1-x} = f(x+1) - f(x) = g(x) = f'(c)$ for some $c \in (x, x+1)$. Therefore $|g(x)| = |f(x+1) - f(x)| = |f'(c)| < \varepsilon$. Since x was chosen arbitrarily in (M, ∞), we have shown

$$\lim_{x\to\infty} g(x) = 0$$

Problem 7. Rudin page 114 problem 6. Suppose (a) f is continuous for $x \geq 0$, (b) $f'(x)$ exists for $x > 0$, (c) $f(0) = 0$, (d) f' is monotonically increasing. Put $g(x) = \frac{f(x)}{x}$ for $x > 0$ and prove that g is monotonically increasing.

We show that the derivative is non-negative. Since

$$g'(x) = \frac{f'(x)x - f(x)}{x^2}$$

then it suffices to show that $f'(x)x - f(x) \geq 0$ for every $x \in \mathbb{R}^+$. To leverage condition (d) we apply the mean value theorem. There must exist a $0 < c < x$ such that

$$\frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x} = f'(c)$$

and hence $f(x) = f'(c)x$. Since $x > 0$, and since f' is increasing and $c < x$, we must have

$$f(x) = f'(c)x \leq f'(x)x$$

which implies

$$0 \leq f'(x)x - f(x)$$

as desired.

Problem 8. Rudin page 117 problem 22 abc. Suppose f is a real function on $\mathbb{R}$. (a) If f is differentiable and $f'(t) \neq 1$ for all real t, prove that f has at most one fixed point.

Suppose $f(a) = a$ and $f(b) = b$ with $a < b$. Then by the mean value theorem there is a point $a < c < b$ such that $f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{b-a}{b-a} = 1$.

(b) Show that the function f defined by

$$f(t) = t + (1 + e^t)^{-1}$$

has no fixed point, although $0 < f'(t) < 1$ for all real t.

Suppose $f(a) = a = a + (1+e^a)^{-1}$ so that $(1+e^a)^{-1} = 0$, which is impossible. To see that $0 < f'(t) < 1$ note

$$f'(t) = 1 - (1 + e^t)^{-2} e^t$$

so that we need to show $0 < \frac{e^t}{(1+e^t)^2} < 1$. The first inequality is clear. The second is equivalent to

$$e^t < 1 + 2e^t + e^{2t} \quad \Leftrightarrow$$

$$0 < 1 + e^t + e^{2t}$$

where this last is clearly a sum of positive quantities for each real t.

(c) However, if there is a constant $A < 1$ such that $|f'(t)| \leq A$ for all real t, prove that a fixed point x of f exists, and that $x = \lim x_n$ where x_1 is arbitrary, and $x_{n+1} = f(x_n)$ for $n = 1, 2, \ldots$

First we show that the sequence is Cauchy so that the limit exists. Let $\varepsilon \in \mathbb{R}^+$. We first establish that the sequence values get closer by showing that for any $n \geq 1$ we have

$$|x_{n+1} - x_n| \leq A^{n-1}|x_2 - x_1|$$

The proof is by induction on n. The base-case is trivial since it's $|x_2 - x_1| \leq |x_2 - x_1|$. Now if the claim holds for $n \geq 1$ then apply the mean value theorem to x_n and x_{n+1}, so that there exists a c between them such that

$$\frac{f(x_{n+1}) - f(x_n)}{x_{n+1} - x_n} = f'(c)$$

Therefore

$$|f(x_{n+1}) - f(x_n)| = |x_{n+2} - x_{n+1}| = |f'(c)||x_{n+1} - x_n|$$

$$\leq A(A^{n-1})|x_2 - x_1| = A^n|x_2 - x_1|$$

Now we can consider that, if $m, n \in \mathbb{N}$ with $m \leq n$ then

$$|x_n - x_m| \leq |x_{m+1} - x_m| + \cdots + |x_n - x_{n-1}|$$

$$= (A^{m-1} + A^m + \cdots + A^{n-2})|x_2 - x_1|$$

$$= \left(\frac{1 - A^{n-1}}{1 - A} - \frac{1 - A^{m-1}}{1 - A}\right)|x_2 - x_1|$$

$$= (A^{m-1} - A^{n-1})\frac{|x_2 - x_1|}{1 - A}$$

So finally we can say, corresponding to ε, we choose $N \in \mathbb{N}$ such that $A^N < \left(\frac{1-A}{|x_2 - x_1|}\right)\varepsilon$. Then it follows from the above that

$$|x_n - x_m| \leq (A^{m-1} - A^{n-1})\frac{|x_2 - x_1|}{1 - A}$$

$$\leq A^N \frac{|x_2 - x_1|}{1 - A}$$

$$< \left(\frac{1 - A}{|x_2 - x_1|}\right)\varepsilon \cdot \left(\frac{|x_2 - x_1|}{1 - A}\right) = \varepsilon$$

Since we've now concluded that the sequence is Cauchy, we go on to compute its limit.

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} f(x_{n-1}) = f\left(\lim_{n\to\infty} x_{n-1}\right) = f(x)$$

The third equation is due to the assumption that f is differentiable and therefore continuous. This demonstrates that x is a fixed point of f.

Problem 9. Rudin page 119 problem 26. Suppose f is differentiable on $[a, b]$, $f(a) = 0$, and there is a real number A such that $|f'(x)| \leq A|f(x)|$ on $[a, b]$. Prove that $f(x) = 0$ for all $x \in [a, b]$. *Hint:* Fix $x_0 \in [a, b]$, let

$$M_0 = \sup |f(x)|, \quad M_1 = \sup |f'(x)|$$

for $a \leq x \leq x_0$. For any such x,

$$|f(x)| \leq M_1(x_0 - a) \leq A(x_0 - a)M_0$$

Hence $M_0 = 0$ if $A(x_0 - a) < 1$. That is, $f = 0$ on $[a, x_0]$.

From the mean value theorem, for any $x \in [a, x_0]$, we have some $c \in (a, x)$ such that

$$\frac{f(x) - f(a)}{x - a} = f'(x)$$

and so

$$\left|\frac{f(x)}{x - a}\right| = |f'(c)| \leq M_1$$

and so

$$|f(x)| \leq M_1(x - a) \leq M_1(x_0 - a)$$

since $M_1 \geq 0$ and $x_0 > x$. All that remains to show for the hint is that $M_1 \leq AM_0$ but this follows from the properties of suprema, in particular that $\sup cX = c \sup X$ where X is a set of real numbers and c is a non-negative constant. In this particular application we take X to be the range of f on $[a, x_0]$.

Now that we have proved the chain of inequalities in the hint, we have that $|f(x)| \leq A(x_0 - a)M_0$. We argue that $A(x_0 - a) < 1$ for some choice of x_0. But since A is constant then a choice of x_0 sufficiently close to a clearly exists.

Therefore on this interval $|f(x)| \leq A(x_0 - a)M_0 \leq M_0$. But also we have that $A(x_0 - a)M_0$ is an upper bound on $|f(x)|$ for all $a < x < x_0$ so that $M_0 \leq A(x_0 - a)M_0$ and hence $A(x_0 - a)M_0 = M_0$. Given that we have already established $A(x_0 - a) < 1$ this can only be true if $M_0 = 0$, and this immediately implies that $f = 0$ on $[a, x_0]$.

Now given that this holds, we repeat the argument above, this time on the interval $[x_0, b]$. We will again find that if $2x_0 \leq b$ then on $[x_0, 2x_0]$ we have that $f = 0$. Proceeding likewise, we continue until $nx_0 > b$ for some $n \in \mathbb{N}$. But in this case, we repeat the proof but with x_0 replaced by $b - nx_0$. Here again we find that $f = 0$ on $[(n-1)x_0, b]$.

The above then shows that on $[a, x_0] \cup [x_0, x_1] \cup \cdots \cup [x_{n-1}, b] = [a, b]$ we have $f = 0$.

Problem 10. Rudin page 119 problem 27. Let ϕ be a real function defined on a rectangle R in the plane, given by $a \leq x \leq b$, $\alpha \leq y \leq \beta$. A *solution* of the initial-value problem

$$y' = \phi(x, y), \quad y(a) = c, \quad (\alpha \leq c \leq \beta)$$

is, by definition, a differentiable function f on $[a, b]$ such that $f(a) = c$, $\alpha \leq f(x) \leq \beta$ and

$$f'(x) = \phi(x, f(x)) \quad (a \leq x \leq b)$$

Prove that such a problem has at most one solution if there is a constant A such that

$$|\phi(x, y_2) - \phi(x, y_1)| \leq A|y_2 - y_1|$$

whenever $(x, y_1) \in R$ and $(x, y_2) \in R$.

Hint: Apply exercise 26 to the difference of two solutions. Note that this uniqueness theorem does not hold for the initial-value problem

$$y' = y^{1/2}, \quad y(0) = 0$$

which has two solutions: $f(x) = 0$ and $f(x) = x^2/4$. Find all other solutions.

Suppose f_1, f_2 are two solutions, and define $h(x) = f_1(x) - f_2(x)$. So if I can show that $h(x)$ is the zero function, this implies that $f_1 = f_2$. We have that $h'(x) = (f_1(x) - f_2(x))' = \phi(x, f_1(x)) - \phi(x, f_2(x))$. Then

$$|h'(x)| = |(f_1 - f_2)'(x)| = |\phi(x, f_1(x)) - \phi(x, f_2(x))| \leq A|f_1(x) - f_2(x)| = A|h(x)|$$

and so by the previous exercise, $h = f_1 - f_2 = 0$ and so $f_1 = f_2$. This shows that there is at most one solution.

To find all solutions to the differential equaiton

$$y' = y^{1/2} \qquad y(0) = 0$$

we first set f to be some solution to this equation, and assume that it is not zero. Then $f' = f^{1/2}$ and also $f'' = \frac{1}{2}f^{-1/2}f' = \frac{1}{2}(f^{-1/2}f^{1/2}) = 1/2$. This then shows that $f' = \frac{x}{2} + C$ and therefore

$$f = \frac{x^2}{4} + Cx + D$$

And since $f(0) = 0 = \frac{0^2}{4} + C(0) + D = D$ then we can further state that every function of the form

$$f(x) = \frac{x^2}{4} + Cx$$

is a solution.