

#1

Adam Frank

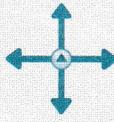
ϕ is not a ring homomorphism because it does not preserve addition. In particular

$$\phi\left(\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}\right) = \phi\left(\begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix}\right) = 25 - 25 = 0$$

whereas

$$\phi\left(\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\right) + \phi\left(\begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}\right) = 4 - 5 + 4 - 5 = -2.$$

Since these are unequal then ϕ does not preserve addition and so ϕ is not a ring homomorphism. \square



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#2

Adam Frank

(a) Let $C \subseteq M_2(\mathbb{R})$ be the subring isomorphic to \mathbb{C} that we discussed in a previous homework. We had seen that

$$C = \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \mid a, b \in \mathbb{R} \right\}.$$

(b) IF $a, b \in \mathbb{R}$ are not both 0 then

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}^{-1} = \frac{1}{a^2+b^2} \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

We can confirm this claim by computing

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \cdot \frac{1}{a^2+b^2} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \frac{1}{a^2+b^2} \begin{pmatrix} a^2+b^2 & 0 \\ 0 & a^2+b^2 \end{pmatrix}$$

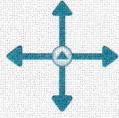
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(c) IF $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \begin{pmatrix} c & -d \\ d & c \end{pmatrix} \in C$ then these

commute since

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} c & -d \\ d & c \end{pmatrix} = \begin{pmatrix} ac-bd & -ad-bc \\ bc+ad & -bd+ac \end{pmatrix}$$

$$= \begin{pmatrix} c & -d \\ d & c \end{pmatrix} \begin{pmatrix} a & -b \\ b & a \end{pmatrix}. \quad \square$$



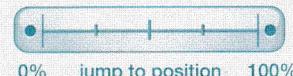
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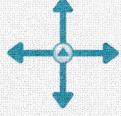


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#3

Adam Frank

This is not true, which we can see from the example $R = \mathbb{Z}$, $I = (2)$, $J = (3)$. Now $2 \in (2)$ and $3 \in (3)$ obviously, but $2+3=5$ and $5 \notin (2)$ and $5 \notin (3)$. Hence $5 \notin I \cup J$ so that $I \cup J$ isn't closed under addition. So $I \cup J$ isn't even a ring and therefore can't be an ideal. \square



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#4

Adam Frank

(a) Since $\bar{2} \neq \bar{0}$ but $\bar{2} \times \bar{2} = \bar{0}$ then $\bar{2}$ is a zero divisor.

Since $\bar{3} \times \bar{3} = \bar{9} = \bar{1}$ then $\bar{3}$ is a unit. $\bar{1}$ is always a unit in every ring.

So the set of zero divisors is $\{\bar{2}\}$. The set of units is $\{\bar{1}, \bar{3}\}$.

(b) It is necessary (but not sufficient) for $p(x) \in R[x]$ to be a unit, that the constant term is a unit.

(c) We can see that $2x^2 + 2x + 1$ is its own inverse by calculating

$$(\bar{2}x^2 + \bar{2}x + \bar{1})(\bar{2}x^2 + \bar{2}x + \bar{1}) =$$

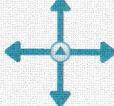
$$\bar{4}x^4 + \bar{4}x^3 + \bar{2}x^2 + \bar{4}x^3 + \bar{4}x^2 + \bar{2}x + \bar{2}x^2 + \bar{2}x + \bar{1}$$

$$= \bar{0}x^4 + \bar{0}x^3 + \bar{2}x^2 + \bar{0}x^3 + \bar{0}x^2 + \bar{2}x + \bar{2}x^2 + \bar{2}x + \bar{1}$$

$$= \bar{4}x^2 + \bar{4}x + \bar{1} = \bar{1}.$$

(d) A necessary condition for $p(x) = a_n x^n + \dots + a_0 \in R[x]$ to be a zero divisor is that both a_n and a_0 are zero divisors in R .

(e) Suppose for contradiction that $p(x) \in R[x]$ and $p(x) \neq \bar{0}$ and $f(x)p(x) = \bar{0}$. With $p(x) = \sum_{i=0}^n a_i x^i$ we would then have that the coefficient of x^{n+1} would be $\bar{1}a_n + \bar{2}a_{n-1} = \bar{0}$. Also the coefficient of x^n would be $\bar{2}a_n + \bar{1}a_{n-1} + \bar{2}a_{n-2} = \bar{0}$. Moreover it is clear that the coefficient of x^{n+2} is $\bar{2}a_n = \bar{0}$. Combining these equations,



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#4

(continued)

Adam Frank

$$\left. \begin{array}{l} a_{n-1} + \bar{2}a_{n-2} = \bar{0} \\ a_n + \bar{2}a_{n-1} = \bar{0} \end{array} \right\} \Rightarrow$$

$$a_n + \bar{2}(-\bar{2}a_{n-2}) = \bar{0} \Rightarrow$$

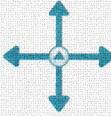
$$a_n - \bar{4}a_{n-2} = \bar{0}$$

$$\therefore a_n = \bar{0}.$$

But $a_n \neq \bar{0}$ since we assumed that the degree of $p(x)$ is n . Thus we have a contradiction.

Therefore $f(x)$ is not a zero divisor. Since the constant term of $f(x)$ is $\bar{2}$, and $\bar{2}$ is not a unit in R , then f is not a unit in $R[x]$. \square

(f) This is not a sufficient condition since the leading and constant terms of f are each zero divisors in R . However f is not a zero divisor. \square



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#5

Adam Frank

- (a) Prove that an element $r \in R$ cannot be both a unit and a zero divisor.

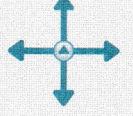
IF r is a unit and if $s \in R$ such that

$$rs = 0$$

then

$$r^{-1}rs = r^{-1}0 = 0 = 1 \cdot s = s.$$

Therefore r is not a zero divisor. \square



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#6

Adam Frank

(a) Suppose that I and J are comaximal ideals of R ,where R is a commutative ring with identity. So $I+J=R$. Let $r \in I \cap J$ so that $r \in I$ and $r \in J$.Let $x \in I$ and $y \in J$ be such that $xy=1$. Then

$$r = r \cdot 1 = r(xy) = rx + ry.$$

By definition of the products of rings, then

$$r = rx + ry \in IJ.$$

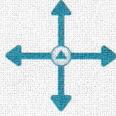
Conversely, let $i_1j_1 + i_2j_2 \in IJ$ be arbitrary.
 Then $i_1j_1, i_2j_2 \in I$ because I is an ideal. Also
 $i_1j_1, i_2j_2 \in J$ because R is commutative and therefore
 all ideals are 2-sided. Moreover ideals are closed
 under sums, so

$$i_1j_1 + i_2j_2 \in I$$

$$i_1j_1 + i_2j_2 \in J$$

$$\therefore i_1j_1 + i_2j_2 \in I \cap J.$$

Hence we have shown $I \cap J \subseteq IJ$ and
 $IJ \subseteq I \cap J$ so therefore $I \cap J = IJ$. \square



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#1

Adam Frank

Let $\phi: M_2(\mathbb{Z}) \rightarrow \mathbb{Z}$ be given by

$$\phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = ad - bc.$$

To see that ϕ is a ring homomorphism, we show first that it preserves addition. Let

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} e & f \\ g & h \end{pmatrix}\right) \in M_2(\mathbb{Z}).$$

Then

$$\begin{aligned} \phi\left(\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix}\right)\right) &= \phi\left(\begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}\right) \\ &= (a+e)(d+h) - (b+f)(c+g) \\ &= ad + ah + ed + eh - be - bg - fe - fg \\ &= \end{aligned}$$

$$\left(\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}\right) = \begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix} \mapsto 0$$

$$\begin{matrix} \downarrow & \downarrow \\ -1 & -1 \end{matrix}$$

$$(2x^2 + 2x + 1)(2x^2 + 2x + 1) = 4x^4 + 4x^3 + 2x^2 + 4x^4 + 4x^3 + 2x^2 + 2x^2 + 2x + 1$$

