#1

Adam Frank

ϕ is not a ring homomorphism because it does not preserve addition. In particular

$$\phi\left(\begin{pmatrix}1&2\\3&4\end{pmatrix}+\begin{pmatrix}4&3\\2&1\end{pmatrix}\right)=\phi\begin{pmatrix}5&5\\5&5\end{pmatrix}=25-25=0$$

whereas

$$\phi\begin{pmatrix}1&2\\3&4\end{pmatrix}+\phi\begin{pmatrix}4&3\\2&1\end{pmatrix}=4-5+4-5=-2.$$

Since these are unequal then ϕ does not preserve addition and so ϕ is not a ring homomorphism. $\square$

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#2 Adam Frank

(a) Let $\mathcal{C} \subseteq M_2(\mathbb{R})$ be the subring isomorphic to $\mathbb{C}$ that we discussed in a previous homework. We had seen that

$$\mathcal{C} = \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}.$$

(b) If $a, b \in \mathbb{R}$ are not both 0 then

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}^{-1} = \frac{1}{a^2+b^2} \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

We can confirm this claim by computing

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \cdot \frac{1}{a^2+b^2} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \frac{1}{a^2+b^2} \begin{pmatrix} a^2+b^2 & 0 \\ 0 & a^2+b^2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(c) If $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \begin{pmatrix} c & -d \\ d & c \end{pmatrix} \in \mathcal{C}$ then these commute since

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} c & -d \\ d & c \end{pmatrix} = \begin{pmatrix} ac-bd & -ad-bc \\ bc+ad & -bd+ac \end{pmatrix}$$

$$= \begin{pmatrix} c & -d \\ d & c \end{pmatrix} \begin{pmatrix} a & -b \\ b & a \end{pmatrix}. \qquad \square$$

#3

Adam Frank

This is not true, which we can see from the example $R = \mathbb{Z}$, $I = (2)$, $J = (3)$. Now $2 \in (2)$ and $3 \in (3)$ obviously, but $2+3=5$ and $5 \notin (2)$ and $5 \notin (3)$. Hence $5 \notin I \cup J$ so that $I \cup J$ isn't closed under addition. So $I \cup J$ isn't even a ring and therefore can't be an ideal. $\square$

#4 Adam Frank

(a) Since $\bar{2} \neq 0$ but $\bar{2} \times \bar{2} = \bar{0}$ then $\bar{2}$ is a zero divisor.

Since $\bar{3} \times \bar{3} = \bar{9} = \bar{1}$ then $\bar{3}$ is a unit. $\bar{1}$ is always a unit in every ring.

So the set of zero divisors is $\{\bar{2}\}$. The set of units is $\{\bar{1}, \bar{3}\}$.

(b) It is necessary (but not sufficient) for $p(x) \in R[x]$ to be a unit, that the constant term is a unit.

(c) We can see that $2x^2+2x+1$ is its own inverse by calculating

$$(\bar{2}x^2+\bar{2}x+\bar{1})(\bar{2}x^2+\bar{2}x+\bar{1}) =$$

$$\bar{4}x^4+\bar{4}x^3+\bar{2}x^2+\bar{4}x^3+\bar{4}x^2+\bar{2}x+\bar{2}x^2+\bar{2}x+\bar{1}$$

$$= \bar{0}x^4+\bar{0}x^3+\bar{2}x^2+\bar{0}x^3+\bar{0}x^2+\bar{2}x+\bar{2}x^2+\bar{2}x+\bar{1}$$

$$= \bar{4}x^2+\bar{4}x+\bar{1} = \bar{1}.$$

(d) A necessary condition for $p(x) = a_n x^n + \dots + a_0 \in R[x]$ to be a zero divisor is that both a_n and a_0 are zero divisors in R.

(e) Suppose for contradiction that $p(x) \in R[x]$ and $p(x) \neq \bar{0}$ and $f(x)p(x) = \bar{0}$. With $p(x) = \sum_{i=0}^{n} a_i x^i$ we would then have that the coefficient of x^{n+1} would be $\bar{1} \cdot a_n + \bar{2}a_{n-1} = \bar{0}$. Also the coefficient of x^n would be $\bar{2} \cdot a_n + \bar{1}a_{n-1} + \bar{2}a_{n-2} = \bar{0}$. Moreover it is clear that the coefficient of x^{n+2} is $\bar{2}a_n = \bar{0}$. Combining these equations,

#4 (continued)

Adam Frank

$$\left.\begin{array}{l} a_{n-1}+\bar{2}a_{n-2}=\bar{0} \\ a_n+\bar{2}a_{n-1}=\bar{0} \end{array}\right\} \Longrightarrow$$

$$a_n+\bar{2}(-\bar{2}a_{n-2})=\bar{0} \Longrightarrow$$

$$a_n-\bar{4}a_{n-2}=\bar{0}$$

$$\therefore \quad a_n=\bar{0}.$$

But $a_n \neq \bar{0}$ since we assumed that the degree of $p(x)$ is n. Thus we have a contradiction. Therefore $f(x)$ is not a zero divisor. Since the constant term of $f(x)$ is $\bar{2}$, and $\bar{2}$ is not a unit in R, then f is not a unit in $R[x]$. $\square$

(f) This is not a sufficient condition since the leading and constant terms of f are each zero divisors in R. However f is not a zero divisor. $\square$

#5

Adam Frank

(a) Prove that an element $r \in R$ cannot be both a unit and a zero divisor.

If r is a unit and if $s \in R$ such that

$$rs = 0$$

then

$$r^{-1}rs = r^{-1}0 = 0 = 1 \cdot s = s.$$

Therefore r is not a zero divisor. $\square$

#6

Adam Frank

(a) Suppose that I and J are comaximal ideals of R, where R is a commutative ring with identity. So $I+J=R$. Let $r \in I \cap J$ so that $r \in I$ and $r \in J$.

Let $x \in I$ and $y \in J$ be such that $x+y=1$. Then

$$r = r \cdot 1 = r(x+y) = rx+ry.$$

By definition of the products of rings, then

$$r = rx+ry \in IJ.$$

Conversely, let $i_1j_1 + i_2j_2 \in IJ$ be arbitrary. Then $i_1j_1, i_2j_2 \in I$ because I is an ideal. Also $i_1j_1, i_2j_2 \in J$ because R is commutative and therefore all ideals are 2-sided. Moreover ideals are closed under sums, so

$$i_1j_1 + i_2j_2 \in I$$

$$i_1j_1 + i_2j_2 \in J$$

$$\therefore \quad i_1j_1 + i_2j_2 \in I \cap J.$$

Hence we have shown $I \cap J \subseteq IJ$ and $IJ \subseteq I \cap J$ so therefore $I \cap J = IJ$. $\square$

1

Adam Frank

Let $\phi : M_2(\mathbb{Z}) \to \mathbb{Z}$ be given by

$$\phi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

To see that ϕ is a ring homomorphism, we show first that it preserves addition. Let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in M_2(\mathbb{Z}).$$

Then

$$\phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix}\right) = \phi\begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$$

$$= (a+e)(d+h) - (b+f)(c+g)$$

$$= ad + ah + ed + eh - be - bg - fe - fg$$

$$=$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix} \mapsto 0$$

$$\downarrow \qquad \downarrow$$

$$-1 \qquad -1$$

$$(2x^2+2x+1)(2x^2+2x+1) = \cancel{4x^2} + \cancel{4x^3} + 2x^2 + \cancel{4x^2} + \cancel{4x} + 2x + 2x^2 + 2x + 1$$