Advanced Calculus, Homework 3

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Problem 1. Rudin page 44, Problem 11. Determine which of the following are distance metrics for $x, y \in \mathbb{R}$.

$$d_1(x,y) = (x-y)^2$$

$$d_2(x,y) = \sqrt{|x-y|}$$

$$d_3(x,y) = |x^2 - y^2|$$

$$d_4(x,y) = |x-2y|$$

$$d_5(x,y) = \frac{|x-y|}{1+|x-y|}$$

d_1 is not a distance metric because it fails the triangle inequality: If $x = 1, y = 3$ and $r = 2$ then we have $d_1(1,3) = 4$ while on the other hand $d_1(1,2) + d_1(2,3) = 1 + 1 = 2$.

d_2 clearly satisfies "positivity": For all $x, y \in \mathbb{R}$ we have $\sqrt{|x-y|} \geq 0$. And $\sqrt{|x-y|} = 0$ implies $|x-y| = 0$ which in turn implies $x - y = 0$ and so $x = y$. Further if $x = y$ then clearly $\sqrt{|x-y|} = 0$.

It also clearly satisfies symmetry because of the symmetry of the absolute difference: $\sqrt{|x-y|} = \sqrt{|y-x|}$.

All that remains is to check the triangle inequality. Squaring both sides yields

$$
\begin{aligned}
d_2(x,y) = \sqrt{|x-y|} \leq \sqrt{|x-r|} + \sqrt{|r-y|} = d_2(x,r) + d_3(r,y) \quad &\iff \\
|x-y| \leq |x-r| + \sqrt{|x-r||y-r|} + |r-y|
\end{aligned}
$$

But from the triangle inequality applied to absolute value we already know

$$
\begin{aligned}
|x-y| &\leq |x-r| + |r-y| \\
&\leq |x-r| + \sqrt{|x-r||y-r|} + |r-y|
\end{aligned}
$$

The last inequality results from the fact that $\sqrt{|x-r||y-r|} \geq 0$. So d_2 satisfies the triangle inequality. Because it satisfies all three properties, d_2 is a distance metric.

d_3 fails the property that the distance is zero if and only if $x = y$. In particular, if $x = -1$ and $y = 1$ then $d_3(x,y) = |1-1| = 0$. Hence d_3 is not a distance metric.

d_4 fails the same property. In particular if $x = 2$ and $y = 1$ then $|x-2y| = 0$. Hence d_4 is not a distance metric.

d_5 clearly satisfies positivity, since both $|x-y|$ and $1+|x-y|$ are never negative. Also if $x = y = 0$ then clearly $d_5(x,y) = 0$. On the other hand if $d_5(x,y) = 0$ then we must have $|x-y| = 0$ and we already know that this entails $x = y$. Hence d_5 satisfies the first property.

d_5 also clearly satisfies symmetry since $|x-y| = |y-x|$.

So all that remains is to check the triangle inequality. Approaching it directly is a mess, or at least it was when I tried it. But clearly somehow it's related to $f(t) = \frac{t}{1+t}$ composed with d_5. For non-negative values, f acts like some kind of normalization, where it maps $|x-y|$ into the interval $[0,1)$. With that insight, perhaps proving that f is order-preserving is enough.

Certainly f is order preserving since

$$\frac{s}{1+s} \leq \frac{t}{1+t} \iff$$

$$s(1+t) \leq t(1+s) \iff$$

$$s \leq t$$

for $s, t > 0$.

With this in hand, we can now apply it where $s = |x-y|$ and $t = |x-r| + |r-y|$. Here we have that $s \leq t$ by the triangle inequality, and so

$$\begin{aligned}
d_5(x,y) = \frac{|x-y|}{1+|x-y|} \leq \frac{|x-r|+|r-y|}{1+|x-r|+|r-y|} \\
= \frac{|x-r|}{1+|x-r|+|y-r|} + \frac{|r-y|}{1+|x-r|+|y-r|} \\
\leq \frac{|x-r|}{1+|x-r|} + \frac{|r-y|}{1+|y-r|} = d_5(x,r) + d_5(r,y)
\end{aligned}$$

This shows $d_5(x,y) \leq d_5(x,r) + d_5(r,y)$ so d_5 satisfies the triangle inequality. Since d_5 satisfies all three properties, it's a distance metric.

Problem 2. Let $E = [0, 1] \cup (2, 3)$.

a. What are the limit points of E?

b. What are the isolated points of E?

c. What is the interior of E?

d. What is the closure of E?

e. Is E open, closed, clopen, or not open and not closed?

f. Find the boundary of E.

(a.) $E' = [0, 1] \cup [2, 3]$

Justification: For any $x \in [0, 1)$ and any neighborhood $N_r(x)$, let $y = \min\{1, x + r\}$ and so there is some $x < z < y$ with $z \in N_r(x) \cap E$. Hence $x \in E'$. For any $x \in [2, 3)$ and any neighborhood $N_r(x)$, let $y = \min\{2, x + r\}$ and so there is some $x < z < y$ with $z \in N_r(x) \cap [2, 3)$. Hence $x \in E'$. Next, if $N_r(1)$ is any neighborhood of 1, then let $y = \max\{0, 1 - r\}$. There is some $y < z < 1$ so that $z \in N_r(1) \cap E$. Hence $1 \in E'$. Finally, if $N_r(3)$ is any neighborhood of 3, then let $y = \max\{2, 3 - r\}$. There is some $y < z < 1$ so that $z \in N_r(3) \cap E$. Hence $3 \in E'$.

For any point outside of $[0, 1] \cup [2, 3]$ we can fit a neighborhood around it which separates it from E, so no such point is a limit point.

(b.) No isolated points.

Justification: Since every point inside E is a limit point, and limit points are never isolated points, then E contains no isolated points.

(c.) $E^\circ = (0, 1) \cup (2, 3)$

Justification: For any point $x \in (0, 1)$ the neighborhood $N_{\min\{x,1-x\}}(x) \subseteq E$ so these are interior points. For any point $x \in (2, 3)$ the neighborhood $N_{\min\{x-2,3-x\}}(x) \subseteq E$ so these are interior points. For any neighborhood of 0 or 1, there is a point in the neighborhood just to the left which is not in E, so these are not interior points. For any neighborhood of 2 or 3, there is a point just to the right that is not in E, so these are not interior points. Any other point is separated from E and not in E, so these cannot be interior points.

(d.) $\overline{E} = [0, 1] \cup [2, 3]$

Justification:

$$\begin{aligned} \overline{E} &= [0, 1] \cup (2, 3) \cup ([0, 1] \cup [2, 3]) \\ &= [0, 1] \cup [2, 3] \end{aligned}$$

(e.) Since we have seen that $E \neq E^\circ$, then E is not open.

(f.) {0,1,2,3}

Justification:

$$\begin{aligned}
\partial E &= \overline{E} \cap \overline{E^c} \\
&= ([0,1] \cup [2,3]) \cap \overline{(-\infty,0) \cup (1,2) \cup (3,\infty)} \\
&= ([0,1] \cup [2,3]) \cap (-\infty,0] \cup [1,2] \cup [3,\infty) \\
&= \{0,1,2,3\}
\end{aligned}$$

Problem 3. Prove that in any metric space (X, d) the sets X and $\emptyset$ are clopen.

First let's see that X is open: Take any point $p \in X$, and arbitrarily select the neighborhood of radius 1: $N_1(p)$. Trivially, this is a subset of X because X is the universal set. So X is open.

Next let's see that X is closed. If any point $p \in X$ is a limit point of X, then we need to show that $p \in X$. But this is is already assured from the assumption. So X is closed.

Next let's see that $\emptyset$ is open. Since there are no points in $\emptyset$ then it vacuously satisfies "for every point in $\emptyset$ there is a neighborhood of the point which is contained in $\emptyset$."

Finally let's see that $\emptyset$ is closed. For that we need to argue that there are no limit points, so that the set of limit points would be $\emptyset$. If $p \in X$ we show p is not a limit point of $\emptyset$. That follows immediately because if p were a limit point and N any neighborhood, we must have some other point inside of $\emptyset \cap N = \emptyset$, which cannot happen. Since we now know that the set of limit points is $\emptyset$ and since $\emptyset \subseteq \emptyset$, then by definition, $\emptyset$ is closed.

Problem 4. Rudin page 43, Problem 9 parts a, d, e, and f.

(a) Prove that E° is always open.

(d) Prove that $(E^\circ)^c = \overline{(E^c)}$.

(e) Do E and $\overline{E}$ always have the same interior?

(f) Do E and E° always have the same closure?

(a) Let $x \in E^\circ$, we want to find some neighborhood of x which is entirely inside of E°. But since x is an interior point of E then we have some neighborhood $N_r(x) \subseteq E$. I claim that every point in $N_r(x)$ is itself an interior point of E from which it would follow that $N_r(x) \subseteq E^\circ$.

To show this claim, let $p \in N_r(x)$ and we will show that p is an interior point of E. To do so we must find a neighborhood of p that is entirely contained in E. I claim that $N_{r-d(p,x)}(p)$ is such a neighborhood. To show this claim, we take any $q \in N_{r-d(x,p)}(p)$ and argue that $q \in E$. In particular I will show that $q \in N_r(x)$ and since we already have $N_r(x) \subseteq E$ then the desired result follows from it.

To obtain this part, we apply the triangle inequality to x, p and q. This tells us that $d(q,x) \leq d(q,p) + d(p,x)$. But since $q \in N_{r-d(p,x)}(p)$ then by definition $d(p,q) < r - d(p,x)$. Hence

$$d(q,x) < r - d(p,x) + d(p,x) = r$$

By definition it follows that $q \in N_r(x) \subseteq E$.

So as promised we now have that $N_{r-d(p,x)}(p) \subseteq E$ and from that, by definition, p is an interior point of E. From that, $N_r(x) \subseteq E^\circ$, which in turn entails that E° is open.

(d) We prove this by mutual containment. Let $x \in (E^\circ)^c$ so that x is not an interior point of E. To show that $x \in \overline{E^c} = E^c \cup (E^c)'$ we assume $x \notin E^c$ and prove that $x \in (E^c)'$. Note that this new assumption is the same as $x \in E$. Because x is not an interior point, we have that if N is any neighborhood of x then N must have some point $y \notin E$. Since $x \in E$ then $x \neq y$, and also notice $y \in E^c$. Then by definition, x is a limit point of E^c, so $x \in (E^c)'$.

Now let $x \in \overline{E^c}$. If $x \in E^c$ then for any neighborhood N of x, since $x \in N$ it must follow that $N \not\subseteq E$ and so $x \notin E^\circ$ and so $x \in (E^\circ)^c$. On the other hand if $x \in (E^c)'$ then every neighborhood N of x must intersect E^c at some point $y \neq x$. Since $y \in E^c$ then $y \notin E$ and so $N \not\subseteq E$. So again we have $x \notin E^\circ$ and therefore $x \in (E^\circ)^c$.

We have now shown containment in both directions. Hence $(E^\circ)^c = \overline{E^c}$.

(e) No.

Counter-example: $E = (0,1) \cup (1,2)$. Then we have $E^\circ = E$ and yet $\overline{E} = [0,2]$ which has interior $(\overline{E})^\circ = (0,2)$.

(f.) No.

Counter-example: $E = \{0\} \cup (1,2)$ so that $\overline{E} = \{0\} \cup [1,2]$. On the other hand $E^\circ = (1,2)$ and so $\overline{E^\circ} = [1,2]$.

Problem 5. Find two sets of real numbers A and B such that $A \cap B = \emptyset$ and $\overline{A} \cap \overline{B} \neq \emptyset$.

$A = (0, 1), \quad B = \{1\}$. Clearly their intersection is empty. Since $\overline{A} = [0, 1]$ and $\overline{B} = \{1\}$ then $\overline{A} \cap \overline{B} = \{1\}$.

Problem 6. Rudin page 43, Problem 7. (The closure of a finite union is the union of closures. In the infinite case, show that the closure of a union contains the union of the closure. Show by an example that the containment can be strict.)

First we prove that if C, D are two subsets of the universal set X then $\overline{C \cup D} = \overline{C} \cup \overline{D}$. I know this was proved in the lecture videos, but I didn't notice until after I wrote my own solution, and I wanted to see if my solution makes sense. Once we have this we will use induction on the number of sets in the collection.

First let $x \in \overline{C \cup D}$. If $x \in C \cup D$ then either $x \in \overline{C}$ or $x \in \overline{D}$ immediately, so that $x \in \overline{C} \cup \overline{D}$. If however $x \in (C \cup D)'$ then all we know is that a neighborhood of x always intersects either C or D, but we don't know which one. If it's in C that's not enough to know that x is a limit point of C. Because you might take another neighborhood of x and it has no intersection with C. So to motivate the proof, we imagine a sequence of neighborhoods generating a sequence of intersection points. Because the sequence is infinite, there must be an infinity of intersection points in C, or an infinity in D—and that will be enough for us to know which one x is a neighborhood of.

First let $N_r(x)$ be any neighborhood of x. Let $N_1(x), N_{1/2}(x), N_{1/3}(x), \ldots$ be neighborhoods of x each with radius of the form $1/m$ for some $m \in \mathbb{N}$. Since there must be some $M \in \mathbb{N}$ such that $\frac{1}{M} < r$ then for this M we have $N_{1/M}(x) \subseteq N_r(x)$.

Correspondingly denote $y_1 \in N_1(x)$, $y_2 \in N_{1/2}(x), \ldots$ where both $y_i \neq x$ and $y_i \in C \cup D$. The sequence of y_is either contains finitely many or infinitely many elements of C. If there are only finitely many elements in C then there are infinitely many elements in D. Hence there are infinitely many elements in one or the other. Without loss of generality, then, suppose there are infinitely many indices i such that $y_i \in C$.

Since there are infinitely many y_i in C then there must be some $y_m \in N_{1/m}(x)$ with $m \geq M$. Then $y_m \in N_{1/m}(x) \subseteq N_r(x) \subseteq N$. So we have shown that every neighborhood of x contains a point in C and therefore $x \in \overline{C} \subseteq \overline{C} \cup \overline{D}$. This concludes the argument for the containment $\overline{C \cup D} \subseteq \overline{C} \cup \overline{D}$.

Now let $x \in \overline{C} \cup \overline{D}$ and without loss of generality let $x \in \overline{C}$. If $x \in C$ then $x \in C \cup D \subseteq \overline{C \cup D}$. On the other hand if $x \in C'$ then every neighborhood of x contains a distinct point in C and therefore in $C \cup D$. That makes x a limit point of $C \cup D$ and so $x \in \overline{C \cup D}$.

Now that we've shown that the above, we can prove the full claim

$$\bigcup_{i=1}^{n} \overline{A}_i = \overline{\bigcup_{i=1}^{n} A_i}$$

by induction on n the number of sets. The base case $n = 1$ is trivial since

$$\bigcup_{i=1}^{1} \overline{A}_i = \overline{A}_1 = \overline{\bigcup_{i=1}^{1} A_i}$$

Now assume the inductive hypothesis, that the claim holds up to n, and consider

$$\begin{aligned}
\bigcup_{i=1}^{n+1} \overline{A}_i &= \left(\bigcup_{i=1}^{n} \overline{A}_i\right) \cup \overline{A}_{n+1} \\
&= \overline{\bigcup_{i=1}^{n} A_i} \cup \overline{A}_{n+1} \\
&= \overline{\left(\bigcup_{i=1}^{n} A_i\right) \cup A_{n+1}} \\
&= \overline{\bigcup_{i=1}^{n+1} A_i}
\end{aligned}$$

The second equality comes from the inductive hypothesis and the third from the proof for the case of two sets. Hence the claim is proved.

For the infinite case we need to show that

$$\bigcup_{i=1}^{\infty} \overline{A}_i \subseteq \overline{\bigcup_{i=1}^{\infty} A_i}$$

So let x be an element of the left-hand set. Then there must be some $1 \leq j$ such that $x \in \overline{A}_j$. If $x \in A_j$ then $x \in \bigcup_{i=1}^{\infty} A_i$ and therefore $x \in \overline{\bigcup_{i=1}^{\infty} A_i}$. On the other hand, if $x \in (A_j)'$ then we will show that $x \in \left(\bigcup_{i=1}^{\infty} A_i\right)'$. Let N be any neighborhood of x so that there must exist a $y \neq x$ such that $y \in N \cap A_j$. But then since $A_j \subseteq \bigcup_{i=1}^{\infty} A_i$ we must have $y \in N \cap \bigcup_{i=1}^{\infty} A_i$. Hence by definition $x \in \left(\bigcup_{i=1}^{\infty} A_i\right)' \subseteq \overline{\bigcup_{i=1}^{\infty} A_i}$. So we have that $\bigcup_{i=1}^{\infty} \overline{A}_i \subseteq \overline{\bigcup_{i=1}^{\infty} A_i}$.

Finally we need to show that the containment can be strict by exhibiting an example where the sets are unequal. Take the example $A_i = (-2+1/i,\ 2-1/i)$. Then $\overline{A}_i = [-2+1/i,\ 2-1/i]$. Also

$$\bigcup_{i=1}^{\infty} A_i = (-2, 2)$$

$$\overline{\bigcup_{i=1}^{\infty} A_i} = [-2, 2]$$

$$\bigcup_{i=1}^{\infty} \overline{A}_i = (-2, 2)$$

The last two equalities exhibit the fact that $\overline{\bigcup_{i=1}^{\infty} A_i} \neq \bigcup_{i=1}^{\infty} \overline{A}_i$.

Problem 7. Let $E \subset \mathbb{R}$ be a nonempty set bounded below. Show that $\inf(E) \in \overline{E}$.

We mimic the proof given in Rudin: If $\inf E \in E \subseteq \overline{E}$ then there's nothing to prove.

If $\inf E \notin E$ then we consider an arbitrary neighborhood $(\inf E - \varepsilon, \inf E + \varepsilon)$. In particular $\inf E + \varepsilon$ cannot be an lower bound on the set E and so there must be some $x \in E$ with $\inf E < x < \inf E + \varepsilon$. Then x is in this neighborhood, is also in E, and is not equal to $\inf E$. That makes $\inf E$ a limit point and so $\inf E \in \overline{E}$.

Problem 8. Let (X, d) be a metric space and $E \subset X$. Show that:

a. E° is the union of all open subsets of E.

b. $\overline{E}$ is the intersection of all the closed supersets of E.

(a.) We show this by mutual containment. Let $x \in E^\circ$. But we already have that $E^\circ \subseteq E$ and from the earlier exercise in this set, we know that E° is open. Hence x is contained in an open subset of E and therefore is in the union over all of these.

For the converse, let $x \in \bigcup\limits_{\substack{\mathcal{O} \text{ is open} \\ \mathcal{O} \subseteq E}} \mathcal{O}$ and in particular suppose $x \in O$ where O is an open set contained in E. Because O is open, every point in O is an interior point of O. So x is an interior point, and we can take some neighborhood $N \subseteq O$ with $x \in N$. Then $x \in O \subseteq E$ so $x \in E^\circ$.

(b.) Again by mutual containment, note that one direction is extremely easy. Since we know $\overline{E}$ is a closed set containing E, then $\bigcap\limits_{\substack{\mathcal{F} \text{ is closed} \\ E \subseteq \mathcal{F}}} \mathcal{F} \subseteq \overline{E}$.

For the converse, let $x \in \overline{E}$. If $x \in E$ and if F is any closed superset of E, then $x \in F$. Hence $x \in \bigcap\limits_{\substack{\mathcal{F} \text{ is closed} \\ E \subseteq \mathcal{F}}} \mathcal{F}$. On the other hand if $x \in E'$ then let F be any closed superset of E. If N is any neighborhood of x then there exists a $y \neq x$ such that $y \in N \cap E \subseteq N \cap F$. Hence by definition x is a limit point of F. Since F is closed then it contains all of its limit points, and therefore $x \in F$. Now since x is contained in every closed superset of E then x is in the intersection over all of these.

An interesting alternate proof "sketch" is to use $(E^\circ)^c = \overline{E^c}$ which we proved earlier. This doesn't quite work because, if we've already shown $E^\circ = \bigcup\limits_{\substack{\mathcal{O} \text{ is open} \\ \mathcal{O} \subseteq E}} \mathcal{O}$ then we'd like to take $\overline{E}$ and apply two complements, distributing the first one in, relating that to E° and using what we already know about this. So to flesh this avenue of proof out, we'd need to first prove $(\overline{E})^c = (E^c)^\circ$. That's probably possible, along the same lines at the earlier problem. But it's more work than I want to do for an alternate proof.

Problem 9. Prove that the complement of any $\mathcal{F}_\sigma$-set is a $\mathcal{G}_\delta$-set

De Morgan's does it all!

Let $F \in \mathcal{F}_\sigma$ and in particular suppose $F = \bigcup_{\alpha \in \mathbb{N}} F_\alpha$ where each F_α is a closed set. Then $F^c = \bigcap_{\alpha \in \mathbb{N}} F_\alpha^c$. Since the complement of any closed set is open, then by definition $\bigcap_{\alpha \in \mathbb{N}} F_\alpha^c \in \mathcal{G}_\delta$.