MA 630 - Homework 8 (Module 4 - Section 2)

Solutions must be typeset in LaTeX and submitted to Canvas as a .pdf file. When applicable, write in complete sentences.

1. Prove directly (i.e. using only the definitions of *bijection*, one-to-one, and onto) that if $f : A \to B$ and $g : B \to C$ are bijections, then $g \circ f$ is a bijection.

   Proof: (One-to-one:) Let $a, b \in X$ such that $g \circ f(a) = g \circ f(b)$. Then $g(f(a)) = g(f(b))$ and since g is one-to-one then $f(a) = f(b)$. Since f is one-to-one then $a = b$. Hence $g \circ f$ is one-to-one.

   (Onto:) Let $c \in C$, then since g is onto there must exist some $b \in B$ such that $g(b) = c$. Since f is onto B then there must exist some $a \in A$ such that $f(a) = b$. Hence

   $$g \circ f(a) = g(f(a)) = g(b) = c$$

   and therefore $g \circ f$ is onto.

   Hence $g \circ f$ is a bijection. $\square$

2. Let $f : A \to A$ and $g : A \to A$ be one-to-one functions from A onto A. Prove that
   $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

   Reminder: To prove that two functions, h_1 and h_2, are equal, it must be shown that h_1 and h_2 have the same domain, X, and that $h_1(x) = h_2(x)$ for all $x \in X$.

   Let $a \in A$, then $(f \circ g)^{-1}(a) = b$ is that element such that $(f \circ g)(b) = a$. This element is unique because $f \circ g$ is one-to-one as we demonstrated above. This is the same as $f(g(b)) = a$ and let's call $g(b) = c$. Then by g being one-to-one, we have that $b = g^{-1}(c)$. In this notation we also have $f(c) = a$ and $c = f^{-1}(a)$.

   On the other hand

   $$\begin{aligned}(g^{-1} \circ f^{-1})(a) &= g^{-1}(f^{-1}(a)) \\ &= g^{-1}(c) \\ &= b.\end{aligned}$$

This shows that for every $a \in A$, the values $(f \circ g)^{-1}(a)$ and $(g^{-1} \circ f^{-1})(a)$ are both b. Therefore $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$. $\square$

3. Prove that the function $f : \mathbb{R} - \{2\} \to \mathbb{R} - \{5\}$ given by $f(x) = \dfrac{5x+1}{x-2}$ is a bijection.

   Proof: Suppose $a, b \in \mathbb{R} - \{2\}$ are such that $f(a) = f(b)$. Then

$$
\begin{aligned}
\frac{5a+1}{a-2} &= \frac{5b+1}{b-2} \\
(5a+1)(b-2) &= (5b+1)(a-2) \\
5ab - 10a + b - 2 &= 5ab - 10b + a - 2 \\
-10a + b &= -10b + a \\
11b &= 11a \\
b &= a
\end{aligned}
$$

   which shows that f is one-to-one. To show f is onto, let $y \in \mathbb{R} - \{5\}$. Then the following are equivalent so long as $x \neq 2$.

$$
\begin{aligned}
\frac{5x+1}{x-2} &= y \\
5x + 1 &= xy - 2y \\
5x - xy &= -2y \\
x(5-y) &= -2y \\
x &= -\frac{2y}{5-y}.
\end{aligned}
$$

   Note that division by $5-y$ is valid since $y \neq 5$. Therefore we have $f\left(-\frac{2y}{5-y}\right) = y$ which shows that f is onto.

4. Let $f : X \to Y$, and let $A \subseteq X$.

   (a) Prove that $A \subseteq f^{-1}(f(A))$.

(b) Prove that if f is one-to-one, then $A = f^{-1}(f(A))$.

(a) *Proof:* Let $a \in A$ and let $y = f(a)$. Then $a \in f^{-1}(y)$. Since $y \in f(A)$ then $f^{-1}(y) \subseteq f^{-1}(f(A))$ by definition. But then $a \in f^{-1}(f(A))$ which shows $A \subseteq f^{-1}(f(A))$. $\square$

(b) *Proof:* We already have the inclusion $A \subseteq f^{-1}(f(A))$ from above. To prove the reverse, let $a \in f^{-1}(f(A))$ so that by definition $f(a) \in f(A)$. Hence also by definition there is some $a' \in A$ such that $f(a') = f(a)$. But since f is one-to-one, this requires $a = a'$ and then $a \in A$. This shows $f^{-1}(f(A)) \subseteq A$. $\square$

5. Let $f : X \to Y$, and let $A, B \subseteq X$.

   (a) Prove that $f(A \cup B) = f(A) \cup f(B)$.

   (b) Is it true that $f(A \cap B) = f(A) \cap f(B)$? Either prove or provide a counterexample.

(a) *Proof:* Let $y \in f(A \cup B)$ so that there is some $c \in A \cup B$ such that $f(c) = y$. Therefore $c \in A$ or $c \in B$, and therefore either $f(c) \in f(A)$ or $f(c) \in f(B)$. Then $y = f(c) \in f(A) \cup f(B)$, so $f(A \cup B) \subseteq f(A) \cup f(B)$.

Now let $y \in f(A) \cup f(B)$ so either $y \in f(A)$ or $y \in f(B)$. Without loss of generality let $y \in f(A)$ so that there is some $a \in A$ such that $y = f(a)$. Then $a \in A \cup B$ and therefore $y = f(a) \in f(A \cup B)$. Hence $f(A) \cup f(B) \subseteq f(A \cup B)$.

Since we have inclusion in both directions, $f(A \cup B) = f(A) \cup f(B)$. $\square$

(b) This is false, and a counterexample is $A = \{1\}$ and $B = \{2\}$ and $f(x) = 1$, the constant function 1. Then $A \cap B = \emptyset$ and $f(A \cap B) = \emptyset$. However, $f(A) = f(B) = \{1\}$ and so $f(A) \cap f(B) = \{1\}$.