MA-652 Advanced Calculus
Homework 6, Mar. 11

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Problem 1. Rudin page 165, Problem 1. Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.

Let $\{f_n\}$ be a sequence of bounded functions and $f_n \xrightarrow{u} f$. Let M_n be such that $|f_n(x)| \leq M_n$ for each $n \in \mathbb{N}$ and every $x \in E$. Then let $N \in \mathbb{N}$ be such that for each $n \geq N$ we have

$$|f_n(x) - f(x)| < 1$$

Let us then take the supremum over all $x \in E$:

$$A_n = \sup_{x \in E} |f_n(x) - f(x)| \leq 1$$

Notice that this in particular implies that f is bounded since $|f(x)| \leq |f_N(x)| + A_N \leq M_N + A_N$. Let us call the bound on f the number M_f.

Next we define

$$A = \sup_{n \geq N} A_n \leq 1$$

Then it follows that for all $x \in E$ and $n \geq N$, we have $|f_n(x) - f(x)| \leq A \leq 1$. In particular this implies that $|f_n(x)| \leq |f(x)| + 1 \leq M_f + 1$. Now set $B = \max\{M_1, \ldots, M_N, M_f + 1\}$ and then it follows that for all $x \in E$ and for all $n \in \mathbb{N}$ we have $|f_n(x)| \leq B$ and therefore the sequence is uniformly bounded.

Problem 2. Rudin page 165, Problem 2. If $\{f_n\}$ and $\{g_n\}$ converge uniformly on E, prove that $\{f_n + g_n\}$ converges uniformly on E. If, in addition, $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions, prove that $\{f_n g_n\}$ converges uniformly on E.

Let $\varepsilon \in \mathbb{R}^+$ and set N_1 such that for all $n \geq N_1$ we have $|f_n(x) - f(x)| < \varepsilon/2$ and set N_2 such that for all $n \geq N_2$ we have $|g_n(x) - g(x)| < \varepsilon/2$ for every $x \in E$. Then

$$|f_n(x) + g_n(x) - [f(x) + g(x)]| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)|$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon$$

Therefore $f_n + g_n \xrightarrow{u} f + g$.

Now suppose that both are sequences of bounded functions. From problem 1 we have that $\{f_n\}, \{g_n\}$ are each uniformly bounded, so let $|f_n(x)| \leq M_f$ and $|g_n(x)| \leq M_g$. We also saw in problem 1 that both f and g must themselves be bounded, so let us call their bounds M'_f and M'_g. Now set N_1 such that if $n \geq N_1$ we have

$$|f_n(x) - f(x)| < \varepsilon/M_g$$

and set N_2 such that if $n \geq N_2$ we have

$$|g_n(x) - g(x)| < \varepsilon/M'_f$$

and set $N = \max\{N_1, N_2\}$ and then if $n \geq N$ we have

$$|f_n(x)g_n(x) - f(x)g(x)| = |f_n(x)g_n(x) - f(x)g_n(x) + f(x)g_n(x) - f(x)g(x)|$$

$$\leq |f_n(x)g_n(x) - f(x)g_n(x)| + |f(x)g_n(x) - f(x)g(x)|$$

$$= |g_n(x)||f_n(x) - f(x)| + |f(x)||g_n(x) - g(x)|$$

$$< M_g(\varepsilon/M_g) + M'_f(\varepsilon/M'_f) = 2\varepsilon$$

Therefore $f_n + g_n \xrightarrow{u} f + g$.

Problem 3. Rudin page 165, Problem 3. Construct sequences $\{f_n, g_n\}$ which converge uniformly on some set E, but such that $\{f_n g_n\}$ converges uniformly on E.

On $E = (0,1)$ take $f_n(x) = \frac{1}{x}$, which trivially converges because every $|f_n(x) - f(x)| = 0 < \varepsilon$ for any $\varepsilon \in \mathbb{R}^+$. Next we take $g_n(x) = \frac{x}{xn+1}$ which pointwise converges to 0 for any $x \in E$. It also uniformly converges to 0, which we demonstrate by first maximizing the function.

$$g_n'(x) = \frac{(xn+1) - x(n)}{(xn+1)^2} = \frac{1}{(xn+1)^2} = 0$$

holds nowhere, and therefore this function has no local optima. Since this derivative is positive on $(0,1)$ then the function is increasing, and therefore $g_n(x) < g_n(1) = \frac{1}{(n+1)^2}$. Since we may choose N large enough that $\frac{1}{(n+1)^2} < \varepsilon$ then $g_n \xrightarrow{u} 0$.

On the other hand, $f_n g_n(x) = \frac{1}{x} \cdot \frac{x}{xn+1} = \frac{1}{xn+1}$. Clearly this point-wise converges to 0. But if we set $\varepsilon = \frac{1}{2}$ then there is no N such that $\frac{1}{xN+1} < \frac{1}{2}$. This is because the inequation is equivalent to

$$2 < xN + 1 \quad \Leftrightarrow \quad \frac{1}{N} < x$$

But there is always an x small enough that the above is not true. Therefore the convergence of $f_n g_n$ is not uniform.

Problem 4. Rudin page 167, Problem 11. Suppose $\{f_n\}, \{g_n\}$ are defined on E, and

(a) $\sum f_n$ has uniformly bounded partial sums;

(b) $g_n \to 0$ uniformly on E;

(c) $g_1(x) \geq g_2(x) \geq \cdots$ for every $x \in E$.

Prove that $\sum f_n g_n$ converges uniformly on E. *Hint:* Compare with theorem 3.42.

Set $A_n = \sum_{k=0}^{n} f_k(x)$ for each $n \geq 0$ and put $A_{-1} = 0$. Then if $0 \leq p \leq q$ we have

$$\sum_{n=p}^{q} f_n(x)g_n(x) = \sum_{n=p}^{q}(A_n - A_{n-1})g_n(x)$$

$$= \sum_{n=p}^{q} A_n g_n(x) - \sum_{n=p-1}^{q-1} A_n g_{n+1}(x)$$

$$= \sum_{n=p}^{q-1} A_n(g_n(x) - b_{n+1}(x)) + A_q g_q(x) - A_{p-1}g_p(x)$$

With this in hand we can now choose M such that for each partial sum $|A_n| \leq M$ for all n. If $\varepsilon \in \mathbb{R}^+$ then set N such that for all $n \geq N$ we have $|g_n(x)| < \frac{\varepsilon}{2M}$ for all $x \in E$. Then if $N \leq p \leq q$ we have

$$\left|\sum_{n=p}^{q} f_n(x)g_n(x)\right| = \left|\sum_{n=p}^{q-1} A_n(g_n(x) - b_{n+1}(x)) + A_q g_q(x) - A_{p-1}g_p(x)\right|$$

$$\leq \left|\sum_{n=p}^{q-1} M(g_n(x) - g_{n+1}(x)) + Mg_q(x) + Mg_p\right|$$

We note that the above inequality is true because $g_n(x) - g_{n+1}(x) \geq 0$ and because $g_q(x), g_p(x) \geq 0$. Then the above is equal to

$$= M \left| \sum_{n=p}^{q-1} (g_n(x) - g_{n+1}(x)) + g_q(x) + g_p \right|$$

$$= 2Mg_p(x) < 2M \left(\frac{\varepsilon}{2M} \right) = \varepsilon$$

Since the above is true for all $x \in E$ then $\sum f_n g_n$ converges uniformly on E.

Problem 5. Rudin page 168, Problem 15. Suppose f is a real continuous function on $\mathbb{R}$, $f_n(t) = f(nt)$ for $n = 1, 2, \ldots$ and $\{f_n\}$ is equicontinuous on $[0, 1]$. What conclusion can you draw about f?

We can show that f is in fact constant. For contradiction, suppose that f is not constant and in particular that $f(x) \neq f(y)$ for some $x \neq y$.

Problem 6. Rudin page 168, Problem 16. Suppose $\{f_n\}$ is an equicontinuous sequence of functions on a compact set K, and $\{f_n\}$ converges pointwise on K. Prove that $\{f_n\}$ converges uniformly on K.

Let $\varepsilon \in \mathbb{R}^+$ and set δ such that if $|x-y| < \delta$ for $x, y \in K$ then $|f_n(x) - f_n(y)| < \varepsilon$ for each $n \in \mathbb{N}$. For each $x \in K$ define $N_\delta(x)$ to be the neighborhood of x of radius δ. Since K is compact there must be a finite collection $x_1, \ldots, x_N$ such that $N_\delta(x_1), \ldots, N_\delta(x_N)$ is an open cover for K.

We now consider the Cauchy criterion for uniform convergence, so fix any $x \in K$ and find any x_i such that $x \in N_\delta(x_i)$ for $i = 1, \ldots, N$. Now since $\{f_n\}$ is converges at x_i then we use the Cauchy criterion. Let $N' \in \mathbb{N}$ such that if $p, q \geq N'$ then $|f_p(x_i) - f_q(x_i)| < \varepsilon$. Then for any such p, q we have

$$
\begin{aligned}
|f_p(x) - f_q(x)| &= |f_p(x) - f_p(x_i) + f_p(x_i) - f_q(x_i) + f_q(x_i) - f_q(x)| \\
&\leq |f_p(x) - f_p(x_i)| + |f_p(x_i) - f_q(x_i)| + |f_q(x_i) - f_q(x)| \\
&< 3\varepsilon
\end{aligned}
$$

Problem 7. Rudin page 168, Problem 18. Let $\{f_n\}$ be a uniformly bounded sequence of functions which are Riemann-integrable on $[a, b]$, and put

$$F_n(x) = \int_a^x f_n(t)\ dt \qquad (a \leq x \leq b)$$

Prove that there exists a subsequence $\{F_{n_k}\}$ which converges uniformly on $[a, b]$.

To apply theorem 7.25 we have from the Fundamental Theorem of Calculus that each F_n is continuous. If we let M be such that $|f_n(x)| \leq M$ then $|F_n(x)| \leq \left|\int_a^x M\ dt\right| = M(x-a)$. Therefore $F_n(x)$ is pointwise bounded. So it suffices to show that $\{F_n\}$ is equicontinuous.

Now for each $\varepsilon \in \mathbb{R}^+$ set $\delta = \varepsilon$. Then if $|x-y| < \delta$ we have

$$\begin{aligned} |F_n(x) - F_n(y)| &= \left|\int_a^x f_n(t)\ dt - \int_a^y f_n(t)\ dt\right| \\ &= \left|\int_x^y f_n(t)\ dt\right| \\ &\leq \left|\int_x^y M\ dt\right| \\ &= M|y-x| < M\varepsilon \end{aligned}$$

which proves equicontinuity.

Therefore by part (b) of theorem 7.25 we have that there is a subsequence of $\{F_n\}$ which converges uniformly on $[a, b]$.

Problem 8. Rudin page 169, Problem 20. If f is continuous on $[0,1]$ and if

$$\int_0^1 f(x)x^n \, dx = 0 \qquad (n = 0, 1, \dots)$$

prove that $f(x) = 0$ on $[0,1]$. *Hint:* The integral of the product of f with any polynomial is 0. Use the Weierstrass theorem to show that $\int_0^1 f^2(x)\, dx = 0$.

We first note that if $P = \sum_{i=0}^n a_i x^i$ is any polynomial, then

$$\int_0^1 P \cdot f \, dx = \sum_{i=0}^n a_i \int_0^1 x^i f \, dx = \sum 0 = 0$$

Now since f is continuous we let $P_n \xrightarrow{u} f$ as stated in the Stone-Weierstrass theorem. Then consider $\lim_{n\to\infty} \int_0^1 P_n f \, dx$. Due to theorem 7.16 we can infer

$$0 = \lim_{n\to\infty} \int_0^1 P_n f \, dx = \int_0^1 \lim_{n\to\infty} P_n f \, dx$$

The limit above is now the point-wise limit and therefore $\lim_{n\to\infty} P_n f = f^2$. Hence $\int_0^1 f^2 \, dx = 0$. By problem 9 of homework 3, this implies $f = 0$.

Problem 9. Rudin page 169, Problem 21. Let K be the unit circle in the complex plane (i.e. the set of all z with $|z| = 1$) and let $\mathscr{A}$ be the algebra of all functions of the form

$$f(e^{i\theta}) = \sum_{n=0}^{N} c_n e^{in\theta} \qquad (\theta \text{ real})$$

Then $\mathscr{A}$ separate points on K and $\mathscr{A}$ vanishes at no point of K, but nevertheless there are continuous functions on K which are not in the uniform closure of $\mathscr{A}$. *Hint:* For every $f \in \mathscr{A}$

$$\int_0^{2\pi} f(e^{i\theta}) e^{i\theta} \, d\theta = 0$$

and this is also true for every f in the closure of $\mathscr{A}$.

Since $f(z) = 0 + z + 0z^2 + \cdots \in \mathscr{A}$, then if $z_1 \neq z_2 \in K$ then we have that $f(z_1) = z_1 \neq z_2 = f(z_2)$. Hence $\mathscr{A}$ separates points.

Also note that $f(z) = 1 + 0z + \cdots \in \mathscr{A}$ and therefore for any point $z \in K$ we have $f(z) = 1 \neq 0$ and therefore $\mathscr{A}$ vanishes at no point in K.

Next we show that if $f = \sum_{n=0}^{N} c_n e^{in\theta} \in \mathscr{A}$ then $\int_0^{2\pi} f(e^{i\theta}) e^{i\theta} \, d\theta = 0$. This is because

$$\int_0^{2\pi} \left(\sum_{n=0}^{N} c_n e^{in\theta} \right) e^{i\theta} \ d\theta = \sum_{n=0}^{N} \left(c_n \int_0^{2\pi} e^{(n+1)i\theta} \ d\theta \right)$$

$$= \sum_{n=0}^{N} c_n \left(\left. \frac{e^{(n+1)i\theta}}{(n+1)i} \right|_0^{2\pi} \right)$$

$$= \sum_{n=0}^{N} \frac{c_n}{(n+1)i}(1-1)$$

$$= 0$$

Moreover if f is in the uniform closure of $\mathscr{A}$ then there is some $\{f_n\} \subseteq \mathscr{A}$ such that $f_n \xrightarrow{u} f$. Then due to theorem 7.16

$$\int_0^{2\pi} f \ d\theta = \int_0^{2\pi} \lim_{n\to\infty} f_n \ d\theta$$

$$= \lim_{n\to\infty} \int_0^{2\pi} f_n \ d\theta$$

$$= \lim_{n\to\infty} 0 = 0$$

Next consider the function $f(z) = \frac{1}{z} = e^{-i\theta}$ which is continuous on K. Yet

$$\int_0^{2\pi} e^{-i\theta} e^{i\theta} \ d\theta = \int_0^{2\pi} 1 \ d\theta = 2\pi$$

Hence $e^{-i\theta} \notin \mathscr{A}$.

Problem 10. Rudin page 169, Problem 22. Assume $f \in \mathscr{R}(\alpha)$ on $[a, b]$ and prove that there are polynomials P_n such that

$$\lim_{n\to\infty} \int_a^b |f - P_n|^2 \, d\alpha = 0$$

Let $\varepsilon \in \mathbb{R}^+$. From homework 3 problem 12 we know that there is a continuous function g such that $||f - g||_2 < \varepsilon$. Because g is continuous, due to the Stone-Weierstrass theorem, there is a sequence of polynomials P_n such that $P_n \xrightarrow{u} g$. So let N be such that for every $n \geq N$ we have $|g(x) - P_n(x)| < \varepsilon$ for every $x \in [a, b]$. Then if $n \geq N$, due to problem 12 from homework 3,

$$\begin{aligned}
\int_a^b |f - P_n|^2 \, d\alpha &= ||f - P_n||_2 \\
&\leq ||f - g||_2 + ||g - P_n||_2 \\
&< \varepsilon + \int_a^b |g - P_n|^2 \, d\alpha \\
&\leq \varepsilon + \int_a^b \varepsilon^2 \, d\alpha \\
&= \varepsilon + \varepsilon^2(\alpha(b) - \alpha(a))
\end{aligned}$$

Since ε may be chosen arbitrarily small, this shows $\lim_{n\to\infty} \int_a^b |f - P_n|^2 \, d\alpha = 0$.