Advanced Calculus, Homework 8

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Problem 1. Rudin page 99, Problem 6. Let $f : \mathbb{R} \to \mathbb{R}$ be defined on $E \subset \mathbb{R}$, and assume E is compact. Now prove that f is continuous if and only if the graph of f is compact.

Suppose f is continuous then since E is compact, by theorem 4.14 $f(E)$ is compact. Define the graph of f to be $\Gamma(f) \subseteq E \times f(E)$. Since E is compact in $\mathbb{R}$ then it's bounded, and since $f(E)$ is compact then $f(E)$ is bounded. Then $\Gamma(f)$ is bounded. It only remains to show that $\Gamma(f)$ is closed and then, since $\Gamma(f) \subseteq \mathbb{R}^2$ it will follow that $\Gamma(f)$ is compact.

Let $\{(x_n, y_n)\}$ be any sequence $(x_n, y_n) \in \Gamma(f)$ and let $(x_n, y_n) \to (x, y)$. We seek to show that $(x, y) \in \Gamma(f)$. First note that from chapter 2 we know that $x_n \to x$ and $y_n \to y$. Since E is compact it's therefore closed and therefore $x \in E$. Hence to show that $(x, y) \in \Gamma(f)$ we now only need to show that $y = f(x)$. That is to say we want $\lim_{n\to\infty} y_n = f(x)$. And since $(x_n, y_n) \in \Gamma(f)$ then $y_n = f(x_n)$ for each $n \in \mathbb{N}$. Now since f is continuous we know that

$$\lim_{n\to\infty} f(x_n) = f(x)$$

and hence $(x, y) \in \Gamma(f)$.

Now for the converse we suppose that $\Gamma(f)$ is compact and prove that f is continuous. We approach this by showing that f^{-1} maps closed sets to closed sets. So let $C \subseteq f(E)$ be a closed set, and we show $f^{-1}(C)$ is closed. To accomplish this we show that if $\{x_n\}$ is a sequence $x_n \in f^{-1}(C)$ and $x_n \to x$ then we must have $x \in f^{-1}(C)$.

To show this we will see that some subsequence $x_{n_k} \to t \in f^{-1}(C)$. From this it will follow that $x_{n_k} \to t = x$ since any subsequential limit is equal to the limit of the original sequence. This follows from the fact that $\Gamma(f)$ was assumed to be compact and therefore there exists some subsequence $\{(x_{n_k}, y_{n_k})\}$ which converges to a point $(x, y) \in \Gamma(f)$. Hence $x_{n_k} \to x$ and $f(x) = y$, which entails $x \in f^{-1}(y)$. Since C was assumed to be closed, and $y_{n_k} = f(x_{n_k}) \in C$ then $\lim_{n\to\infty} y_{n_k} = y \in C$. Therefore $x \in f^{-1}(y) \subseteq f^{-1}(C)$.

From all that was laid out above, it now follows that f is continuous.

Problem 2. Prove if f is continuous on $[a,b]$ with $f(x) > 0$ for all $a \leq x \leq b$ then $\frac{1}{f}$ is bounded on $[a,b]$.

Since f is continuous on the compact set $[a,b]$ then it attains its minimum at some point $c \in [a,b]$. That is to say, for all $x \in [a,b]$ we have $0 < f(c) \leq f(x)$. Therefore $\frac{1}{f(x)} \leq \frac{1}{f(c)}$ which makes $\frac{1}{f(c)}$ an upper bound on $\frac{1}{f}$. Of course also $0 < \frac{1}{f(x)}$ so that 0 is a lower bound.

Problem 3. Let $f(x) = x^3$

a. Prove f is continuous on $\mathbb{R}$

b. Prove f is not uniformly continuous on $\mathbb{R}$

c. Prove f is uniformly continuous on $[1, 100]$

d. Prove f is uniformly continuous on $(1, 3)$

e. Prove f is uniformly continuous on any bounded subset of $\mathbb{R}$

(a) Let $p \in \mathbb{R} \setminus \{0\}$ and we will show continuity at p. For any $\varepsilon \in \mathbb{R}^+$ let $\delta = \min\{\frac{\varepsilon}{9p^2}, |p|\}$. First note that if $|x - p| < \delta$ then $|x - p| \leq |p|$. I will demonstrate that $|x + p| \leq |3p|$.

First suppose $p > 0$ so that $|x-p| \leq p$ and therefore $p-p \leq x \leq p+p$. Since $0 \leq x$ then $0 \leq x + p = |x + p|$. And since $x \leq 2p$ then $x + p \leq 3p$. Therefore $|x + p| \leq 3p = |3p|$.

On the other hand if $p < 0$ then $|x-p| \leq -p$ and so $p-(-p) \leq x \leq p+(-p)$. Since $x \leq 0$ then $x+p \leq 0$ and so $|x+p| = -(x+p)$. Moreover, $2p \leq x$ implies $3p \leq x + p$. Hence $|x + p| = -(x + p) \leq -3p = |3p|$.

Using the fact that $|x + p| \leq 3p$ we can now argue that

$$
\begin{aligned}
|f(x) - f(p)| &= |x^3 - p^3| \\
&= |x - p||x^2 + 2xp + p^2| \\
&= |x - p||x + p|^2 \\
&< \frac{\varepsilon}{9p^2} \cdot 9p^2 = \varepsilon
\end{aligned}
$$

For the case where $p = 0$ we let $\delta = \varepsilon^{1/3}$. Then if $|x - p| = |x| < \delta$ we have

$$
\begin{aligned}
|f(x) - f(p)| &= |x^3| \\
&< (\varepsilon^{1/3})^3 = \varepsilon
\end{aligned}
$$

(b) We claim that with $\varepsilon = 1$ then for all $\delta \in \mathbb{R}^+$ there is always some pair

$x, y \in \mathbb{R}$ such that both $|x-y| < \delta$ and $|f(x) - f(y)| \geq \varepsilon$. To that end suppose δ is given. Then let $x = \frac{1}{\delta} + 1$ and $y = x + \delta/2$. From this we have

$$\begin{aligned}
|f(x) - f(y)| &= \left|y^3 - x^3\right| \\
&= |y - x||y^2 + 2xy + x^2| \\
&> |\delta/2||y^2| \\
&= \frac{\delta}{2} \cdot \left(\frac{1}{\delta^2} + \frac{1}{\delta} + \frac{1}{2} + \frac{1}{\delta} + 1 + \frac{\delta}{2} + \frac{1}{2} + \frac{\delta}{2} + \frac{\delta^2}{4}\right) \\
&> \frac{\delta}{2} \cdot \frac{2}{\delta} = 1
\end{aligned}$$

(c) Since f is continuous everywhere and $[1, 100]$ is a compact set, by theorem 4.19 then f is uniformly continuous on this set.

(d) Since f is continuous and $[1, 3]$ is compact, then f is uniformly continuous on $[1, 3]$. But if f is uniformly continuous on any set then it is uniformly continuous on any subset, hence f is uniformly continuous on $(1, 3)$.

Proof of my claim that if f is uniformly continuous on any set then f is uniformly continuous on a subset: Let f be uniformly continuous on E and let $A \subseteq E$. Let $\varepsilon \in \mathbb{R}^+$ and set δ such that $\forall x, y \in E$ we have $|x - y| < \delta$ entails $|f(x) - f(y)| < \varepsilon$. With this same δ, for any $x', y' \in A$ since $x', y' \in E$ then we must have $|x' - y'| < \delta$ implies $|f(x') - f(y')| < \varepsilon$.

(e) Let $X \subseteq \mathbb{R}$ be any bounded set. Then $\overline{X}$ is compact and therefore f is uniformly continuous on $\overline{X}$. Since $X \subseteq \overline{X}$ then f is uniformly continuous on X.

Problem 4. Prove that a uniformly continuous function preserves Cauchy sequences.

Let $\varepsilon \in \mathbb{R}^+$ be given, we will try to find an N such that $|f(a_p) - f(a_q)| < \varepsilon$ if $p, q \geq N$.

Since f is uniformly continuous then let δ be that value such that $|x - y| < \delta$ guarantees $|f(x) - f(y)| < \varepsilon$. Now choose N such that $|a_p - a_q| < \delta$ if $p, q \geq N$. Then it follows for any $p, q \geq N$ that $|a_p - a_q| < \delta$ and therefore $|f(a_p) - f(a_q)| < \varepsilon$, as we desired.

Problem 5. Let $f, g : X \to X$ where $X \subseteq \mathbb{R}$ be uniformly continuous.

a. Prove $f + g$ is uniformly continuous

b. Give an example to show fg is not always uniformly continuous

c. Give an example to show that f/g is not always uniformly continuous

(a) Let δ_1 be such that $|x-y| < \delta_1$ guarantees $|f(x) - f(y)| < \varepsilon/2$ and let δ_2 be such that $|x-y| < \delta_2$ guarantees $|g(x) - g(y)| < \varepsilon/2$, and let $\delta = \min\{\delta_1, \delta_2\}$. Then if $|x - y| < \delta$ we have

$$\begin{aligned} |f(x) + g(x) - [f(y) + g(y)]| &= |f(x) - f(y) + g(x) - g(y)| \\ &\leq |f(x) - f(y)| + |g(x) - g(y)| \\ &< 2\varepsilon/2 = \varepsilon \end{aligned}$$

(b) The function $f(x) = x$ is uniformly continuous on $X = \mathbb{R}$. With $g = f$ we see that $f(x)g(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.

(c) $f(x) = 1$ and $g(x) = x$ are each uniformly continuous on $X = (0, \infty)$. However, $f(x)/g(x) = 1/x$ is not.

Problem 6. Rudin page 99, Problem 12. A uniformly continuous function of a uniformly continuous function is uniformly continuous. (State this more precisely and prove it.)

The more precise statement is: Suppose $f : X \to Y$ and $g : Y \to Z$ are uniformly continuous functions and X, Y, Z are metric spaces. Prove that the composition function $h = g \circ f$ is uniformly continuous.

To prove this, let $\varepsilon \in \mathbb{R}^+$ be given and δ_1 such that $\forall x, y \in Y$ if $d_Y(x, y) < \delta_1$ then we have $d_Z(g(x), g(y)) < \varepsilon$. Now let $\delta \in \mathbb{R}^+$ be such that for all $a, b \in X$ if $d_X(a, b) < \delta$ then $d_Y(f(a), f(b)) < \delta_1$. From this it follows that if $a, b \in X$ are such that $d_X(a, b) < \delta$ then we have $d_Y(f(a), f(b)) < \delta_1$ and from that it follows, since $f(a), f(b) \in Y$, that

$$d_Z(g(f(a)), g(f(b))) = d_Z(h(a), h(b)) < \varepsilon$$

So h is uniformly continuous.

Problem 7. Rudin page 102, Problem 26. Suppose X, Y, Z are metric spaces, and Y compact. Let f map X into Y, let g be a continuous one-to-one mapping Y into Z, and put $h(x) = g(f(x))$ for $x \in X$. Prove that f is uniformly continuous if h is. Hint: g^{-1} has a compact domain $g(Y)$ and $f(x) = g^{-1}(h(x))$.

Prove also that f is continuous if h is continuous.

Show (by modifying Example 4.21, or another example) that the compactness of Y cannot be omitted from the hypothesis, even when X and Z are compact.

We assume h is uniformly continuous and try to show that f is. We first set, as an intermediate goal, to show that g^{-1} is uniformly continuous.

Now because Y is compact then $g(Y)$ is, since g was assumed to be continuous, and continuous functions map compact sets to compact sets. Moreover, g^{-1} is a function since g is one-to-one, and we have already seen (theorem 4.17) that the inverse of continuous functions is continuous. Hence g^{-1} is continuous. But then g^{-1} is a continuous function with domain $g(Y)$ which is compact. Hence g^{-1} is uniformly continuous.

Moreover, since $h = g \circ f$ then $f = g^{-1} \circ h$. As we've already proved in this exercise, the composition of uniformly continuous functions is uniformly continuous. So f is uniformly continuous.

Now rather than assume h is uniformly continuous we only assume that h is continuous, and try to show that f is continuous.

Still we know that g^{-1} exists, $g(Y)$ is compact, and $g(Y)$ is the domain of g^{-1}. Hence, still, g^{-1} is continuous by theorem 4.6. Also, still, $f = g^{-1} \circ h$. And the composition of continuous functions is continuous. Hence f is continuous.

We exhibit metric spaces X, Y, Z where X and Z are compact. And we exhibit functions $f : X \to Y$, $g : Y \to Z$, and $h = g \circ f$ where g is one-to-one and continuous, and h is uniformly continuous, but f is not continuous. Take

$$\begin{aligned} X &= [0, 2] \\ Y &= [0, 1) \cup [2, 3] \\ Z &= [0, 2] \end{aligned}$$

with

$$f(x) = \begin{cases} x & \text{if } x < 1 \\ x+1 & \text{if } x \geq 1 \end{cases}$$

and

$$g(x) = \begin{cases} x & \text{if } x < 1 \\ x-1 & \text{if } x \geq 2 \end{cases}$$

Notice that because the domain of g is $[0,1) \cup [1,2]$ then g is in fact continuous and one-to-one. And since $h(x) = x$ then h is uniformly continuous on $[0,2]$.

Problem 8. Rudin page 100, Problem 14. Let $I = [0, 1]$ be the closed unit interval. Suppose f is a continuous mapping of I into I. Prove that $f(x) = x$ for at least one $x \in I$.

If either $f(0) = 0$ or $f(1) = 1$ then there is nothing to prove. So assume both of these are not true and so $f(0) > 0$ and $f(1) < 1$. Now define the function $g(x) = f(x) - x$ which is a difference of continuous functions, and so therefore is continuous. Moreover $g(0) = f(0) > 0$ and $g(1) = f(1) - 1 < 0$.

Since $0 \in [g(1), g(0)]$ and g is continuous, then by the Intermediate Value Theorem, there must be a $c \in [0, 1]$ such that $g(c) = 0$. But this means $f(c) - c = 0$ so $f(c) = c$ which is what we hoped to prove.

Problem 9. Prove any decreasing function that has the intermediate value property is continuous.

Suppose $f : X \to \mathbb{R}$ where $X \subseteq \mathbb{R}$. Also suppose f is decreasing and has the intermediate value property. First we observe that $-f$ must be an increasing function, since if $a < b$ for any $a, b \in X$, we must have $f(a) > f(b)$ and therefore $-f(a) < -f(b)$.

We can also see that $-f$ must have the IVP. For if $-f(a) < -f(b)$ for any $a, b \in X$ and $a < b$ then we can let $-f(a) < y < -f(b)$. Then $f(a) > -y > f(b)$ and since f has the IVP then there exists some $a < c < b$ and $c \in X$ such that $f(c) = -y$. Then

$$f(a) > f(c) > f(b)$$

and therefore

$$-f(a) < -f(c) < -f(b)$$

$$-f(a) < y < -f(b)$$

which shows that $-f$ has the IVP.

Now since $-f$ is increasing and has the IVP, by the proof in lecture we know that $-f$ is continuous. Therefore $-(-f) = f$ is continuous by theorem 4.9.

Problem 10. Give an example of each of the following or explain why such a request is impossible.

a. A continuous function defined on an open interval with range equal to a closed interval.

b. A continuous function defined on a closed interval with range equal to an open interval.

c. A continuous function defined on all of $\mathbb{R}$ with range equal to $\mathbb{Q}$.

(a) Technically $\mathbb{R} = (-\infty, \infty)$ is an open interval and we could use the example of, say, $f(x) = \sin x$ or $f(x) = x$. But to take a bounded open interval, which might be the intention, then we can take $f(x) = \sin x$ on $(0, 2\pi)$. Then $f((0, 2\pi)) = [-1, 1]$.

(b) Technically we could take $\mathbb{R}$ for the domain again, and $\mathbb{R}$ for the range. Hence $f(x) = x$ would qualify. An even more interesting example might be $\tan x$ where the domain is $\mathbb{R}$ and the range is $(-\pi/2, \pi/2)$. These are closed and open intervals, respectively. If we require a bounded closed domain then we know the image of f will be compact and therefore not open.

(c) This is impossible because $\mathbb{R}$ is a connected set and $\mathbb{Q}$ is disconnected. Since continuous functions map connnected sets to connected sets, no continuous f can have $f(\mathbb{R}) = \mathbb{Q}$.