MA-652 Advanced Calculus
Homework 5, Feb. 24
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Problem 1. Let $f_n(x) = \frac{nx}{1+nx^2}$.
(a.) Find the pointwise limit of $\{f_n\}$ for all $x \in (0, \infty)$.

For any fixed $x \in (0, \infty)$ we can use L'Hospital's Rule to show that $\lim_{n\to\infty} f_n(x) = \dfrac{x}{x^2} = \dfrac{1}{x}$.

(b.) Is the convergence uniform on $(0, \infty)$?

No. Suppose for contradiction that the convergence were uniform, then we could apply theorem 7.11 and use the point 0 which is a limit point of $(0, \infty)$. Then for each $n = 1, 2, \dots$

$$\lim_{t \to 0^+} f_n(t) = A_n = \lim_{t \to 0^+} \frac{nt}{1 + nt^2} = 0$$

Then

$$\lim_{n \to \infty} A_n = 0$$

$$\lim_{t \to 0^+} \lim_{n \to \infty} f_n(t) = \lim_{t \to 0^+} \frac{1}{x} = \infty$$

The theorem implies that these two limits are equal, and yet we see that they are not, so we have a contradiction. ↯ Hence the convergence is not uniform.

(c.) Is the convergence uniform on $(0, 1)$?

No. Since the point 0 is a limit point of $(0, 1)$ then the exact same argument applies as above.

(d.) Is the convergence uniform on $(1, \infty)$?

Yes. Let $\varepsilon \in \mathbb{R}^+$ and consider

$$\left| f_n(x) - \frac{1}{x} \right| = \left| \frac{nx}{1+nx^2} - \frac{1}{x} \right|$$

$$= \left| \frac{nx^2 - (1+nx^2)}{x(1+nx^2)} \right|$$

$$= \left| \frac{1}{x(1+nx^2)} \right|$$

$$< \frac{1}{1+n}$$

where the final inequality is justified by the condition that $x > 1$. Of course there exists an $N \in \mathbb{N}$ such that

$$\frac{1}{\varepsilon} - 1 < N$$

and therefore

$$\frac{1}{1+N} < \varepsilon$$

Moreover for all $n \geq N$ we then have

$$\frac{1}{1+n} < \varepsilon$$

from which it follows that, for this N and all $n \geq N$ we have

$$\left| f_n(x) - \frac{1}{x} \right| < \varepsilon$$

Therefore $f_n \xrightarrow{u} \frac{1}{x}$ on $(1, \infty)$.

Problem 2. Rudin page 166 problem 5. Let

$$f_n(x) = \begin{cases} 0 & \left(x < \frac{1}{n+1}\right) \\ \sin^2\left(\frac{\pi}{x}\right) & \left(\frac{1}{n+1} \leq x \leq \frac{1}{n}\right) \\ 0 & \left(\frac{1}{n} < x\right) \end{cases}$$

Show that $\{f_n\}$ converges to a continuous function, but not uniformly. Use the series $\sum f_n$ to show that absolute convergence, even for all x, does not imply uniform convergence.

We define $f(x)$ to be the point-wise limit $\lim\limits_{n\to\infty} f_n(x)$. Clearly where $x \leq 0$ we have $f_n(x) = 0$ for all n and hence $f(x) = 0$ on the interval $(-\infty, 0]$. Next let $x > 0$ and pick any $N \in \mathbb{N}$ such that $\frac{1}{N} < x$. Then for all $n \geq N$ we have that $f_n(x) = 0$ and therefore for all such x we also have $f(x) = 0$. Therefore $\{f_n\} \xrightarrow{p.w.} 0$.

On the other hand the convergence is not uniform, since there is no $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon = 1$ for all $x \in \mathbb{R}$ and all $n \geq N$. This is because, for any N, we can always find a point $x \in (0, \frac{1}{1+N})$ such that $x = \frac{1}{1/2+2m}$ for some m. Such a point x must moreover lie within some interval $\left[\frac{1}{m}, \frac{1}{1+m}\right]$ for a natural number $m \geq N$. For these values we have

$$|f_m(x) - f(x)| = \left|\sin\left(\frac{\pi}{1/(1/2+2m)}\right)\right|$$

$$= \left|\sin\left(\frac{\pi}{2} + 2m\pi\right)\right| = 1$$

Hence the convergence is not uniform.

Next we show that $\sum f_n$ converges, and since each f_n is non-negative then this is the same as demonstrating absolute convergence. Again if $x \leq 0$ then each $f_n(x) = 0$ and then $\sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} 0 = 0$. Now if $x > 0$ then either x falls within precisely one interval of the form $\left[\frac{1}{1+N}, \frac{1}{N}\right]$, or it falls on an endpoint of one of these intervals. In the former case $\sum_{n=0}^{\infty} f_n(x) = f_N(x) = \sin^2(\pi/x)$. In the latter case, without loss of generality, suppose $x = \frac{1}{N}$ for some N. Then

$$\sum_{n=1}^{\infty} f_n(x) = f_N(x) + f_{N+1}(x) = 2\sin^2\left(\frac{\pi}{1/N}\right) = 2\sin^2(N\pi) = 0$$

Note that in such a case $\sum f_n(x) = \sin^2(\pi/x)$. Therefore

$$\sum f_n \xrightarrow{p.w.} g(x) = \begin{cases} 0 & \text{if } x \leq 0 \text{ or } x \geq 1 \\ \sin^2(\pi/x) & \text{otherwise} \end{cases}$$

We therefore can see that this is a case in which $\sum f_n$ pointwise converges absolutely. Finally we show that this convergence is not uniform. Set $\varepsilon = 1$ and let N be any natural number. Then set $x = \frac{1}{1/2+2m}$ such that $x < \frac{1}{1+N}$. Then $\sum f_n(x) = 0$ for every $n \geq N$, hence

$$\left|\sum f_n(x) - \sin^2(\pi/x)\right| = \sin^2\left(\frac{\pi}{1/(1/2+2m)}\right) = 1$$

and so the convergence is not uniform.

Problem 3. Rudin page 166 problem 6. Prove that the series

$$\sum_{n=1}^{\infty}(-1)^n\frac{x^2+n}{n^2}$$

converges uniformly in every bounded interval, but does not converge absolutely for any x.

Consider any $[a,b]$ and we will prove uniform convergence on this. First note that $M=\sup_{x\in[a,b]}x^2$ always exists since x^2 is continuous and has the extreme value property on any closed, bounded interval. Now notice that the sum is the same as

$$\sum_{n=1}^{\infty}(-1)^n\frac{x^2+n}{n^2}=\sum_{n=1}^{\infty}(-1)^n\frac{x^2+n}{n^2}+\sum_{n=1}^{\infty}(-1)^n\frac{n}{n^2}$$

$$=x^2\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^2}+\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$$

The two series at the end clearly converge, since they are both alternating series which satisfy the Alternating Series Test. That is to say both $\frac{1}{n}$ and $\frac{1}{n^2}$ each are nonnegative decreasing sequences and go to 0 as $n\to\infty$. Therefore they each Cauchy converge, so we let $N_1,N_2\in\mathbb{N}$ to be integers such that for all $p,q\geq N_1$ we have

$$\left|\sum_{n=p}^{q}(-1)^n\frac{1}{n^2}\right|<\varepsilon/M$$

and for all $p,q\geq N_2$ we have

$$\left|\sum_{n=p}^{q}(-1)^n\frac{1}{n}\right|<\varepsilon$$

Now set $N=\max\{N_1,N_2\}$ so that for any $p,q\geq N$ the above hold simultaneously. Then

$$
\begin{aligned}
\left|\sum_{n=p}^{q}(-1)^n \frac{x^2+n}{n^2}\right| &= \left|x^2\sum_{n=p}^{q}(-1)^n \frac{1}{n^2} + \sum_{n=p}^{q}(-1)^n \frac{1}{n}\right| \\
&\leq \left|x^2\sum_{n=p}^{q}(-1)^n \frac{1}{n^2}\right| + \left|\sum_{n=p}^{q}(-1)^n \frac{1}{n}\right| \\
&= x^2\left|\sum_{n=p}^{q}(-1)^n \frac{1}{n^2}\right| + \left|\sum_{n=p}^{q}(-1)^n \frac{1}{n}\right| \\
&< M(\varepsilon/M) + \varepsilon = 2\varepsilon
\end{aligned}
$$

Hence the series satisfies the Cauchy criterion for uniform convergence.

Next we show that the series does not converge absolutely anywhere on $[a, b]$. For this we merely observe that

$$
\begin{aligned}
\sum_{n=1}^{\infty}\left|(-1)^n \frac{x^2+n}{n^2}\right| &\geq \sum_{n=1}^{\infty}\left|\frac{n}{n^2}\right| \\
&= \sum_{n=1}^{\infty}\frac{1}{n}
\end{aligned}
$$

Since this last is the harmonic series, which diverges, then by the comparison test the series $\sum_{n=1}^{\infty}\left|(-1)^n \frac{x^2+n}{n^2}\right|$ diverges.

Problem 4. Let $f_n \to f$ pointwise and $f_n' \to g$ uniformly on $[a, b]$. Assume each f_n' is continuous, so that $\int_a^x f_n' \, d\alpha = f_n(x) - f_n(a)$ for all $x \in [a, b]$. Use this to prove $g(x) = f'(x)$.

First notice that

$$f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$$

$$= \lim_{t \to x} \frac{\lim_{n\to\infty} f_n(t) - \lim_{n\to\infty} f_n(x)}{t - x}$$

$$= \lim_{t \to x} \lim_{n \to \infty} \frac{f_n(t) - f_n(x)}{t - x}$$

$$= \lim_{t \to x} \lim_{n \to \infty} \frac{\int_x^t f_n' \, d\alpha}{t - x}$$

$$= \lim_{t \to x} \frac{\lim_{n\to\infty} \int_x^t f_n' \, d\alpha}{t - x}$$

Since each f_n is continuous, it is integrable. Because the convergence is uniform and each f_n' is continuous, by theorem 7.16 we have that

$$\lim_{t \to x} \frac{\lim_{n\to\infty} \int_x^t f_n' \, d\alpha}{t - x} = \lim_{t \to x} \frac{\int_x^t \lim_{n\to\infty} f_n' \, d\alpha}{t - x}$$

$$= \lim_{t \to x} \frac{\int_x^t g \, d\alpha}{t - x}$$

$$= \lim_{t \to x} \frac{\int_x^t g \, d\alpha - \int_x^x g \, d\alpha}{t - x}$$

Now since each f_n' is continuous and $f_n' \xrightarrow{u} g$, then g must be continuous. But now we notice that the above is in fact the definition of $\frac{d}{dt} \int_x^t g \, d\alpha$ and therefore by the Fundamental Theorem of Calculus, this is $g(x)$. Putting all of these together, $f' = g$.

Problem 5. Rudin page 166 problem 7. For $n = 1, 2, \ldots$ and x real, put

$$f_n(x) = \frac{x}{1 + nx^2}.$$

Show that $\{f_n\}$ converges uniformly to a function f, and that the equation

$$f'(x) = \lim_{n \to \infty} f_n'(x)$$

is correct if $x \neq 0$ but false if $x = 0$.

First note that for any fixed x we have $\lim_{n\to\infty} \frac{x}{1+nx^2} = 0$ pointwise. Hence we claim that $f_n \to 0$ uniformly. So let $\varepsilon \in \mathbb{R}^+$. Then set N' to be the next integer larger than $\frac{1}{4\varepsilon^2}$, and set N'' the next integer larger than $\frac{1}{2\varepsilon}$, and finally set $N = \max\{N', N''\}$. Then for any $n \geq N$ we note that the extrema of $\frac{x}{1+nx^2}$, for variable x, are found when we set to zero

$$f_n'(x) = \frac{1 \cdot (1 + nx^2) - x(2nx)}{(1 + nx^2)^2}$$

This then implies extrema where $1 - nx^2 = 0$ so that $x = \pm\sqrt{1/n}$ with extremal value

$$\left| \frac{\pm\sqrt{1/n}}{1 + n\sqrt{1/n}^2} \right| = \frac{\sqrt{1/n}}{2} = \frac{1}{2\sqrt{n}}$$

$$< \frac{1}{2\sqrt{\frac{1}{4\varepsilon^2}}}$$

$$= \varepsilon$$

As $x \to \pm\infty$ we have that $f_n(x) \to 1/n \leq 1/N < \varepsilon$. Since f_n is continous everywhere then $|f_n|$ does not take values beyond ε. Hence $f_n \xrightarrow{u} 0$.

For the final part, notice that

$$f'(x) = 0$$

$$f_n'(x) = \frac{1 - nx^2}{(1 + nx^2)^2}$$

Now if $x = 0$ then $f_n'(0) = \frac{1}{1} = 1$ and therefore $f'(x) \neq \lim_{n\to\infty} f_n'(x)$. On the other hand for any fixed $x \neq 0$ we have $\lim_{n\to\infty} \frac{1-nx^2}{(1+nx^2)} = 0$ and therefore $f'(x) = 0 = \lim_{n\to\infty} f_n'(x)$.

Problem 6. Let $g_n(x) = \frac{xn+x^2}{2n}$ and set $g(x) = \lim_{n\to\infty} g_n(x)$.

(a.) Compute $g(x)$ by algebraically taking the limit as $n \to \infty$ and then find $g'(x)$.

$$\lim_{n\to\infty} \frac{nx+x^2}{2n} = x/2 = g(x)$$

for each fixed x. Therefore $g'(x) = 1/2$.

(b.) Compute $g_n'(x)$ for each $n \in \mathbb{N}$ and show the sequence of derivatives converges uniformly on every interval $[-M, M]$. Conclude $g'(x) = \lim_{n\to\infty} g_n'(x)$.

$$g_n'(x) = \frac{1}{2n}(n+2x)$$

so that $\lim_{n\to\infty} g_n'(x) = 1/2$ for each x, pointwise. To show that the convergence is uniform on any interval $[-M, M]$, set N to be the next integer greater than M/ε. Then

$$\left| \frac{1}{2n}(n+2x) - \frac{1}{2} \right| = \left| \frac{x}{n} \right| < \frac{M}{N} < \varepsilon$$

So the convergence is uniform.

Problem 7. Rudin page 166 problem 8. If

$$I(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$$

and if $\{x_n\}$ is a sequence of distinct points of (a, b), and if $\sum |c_n|$ converges, prove the series

$$f(x) = \sum_{n=1}^{\infty} c_n I(x - x_n) \qquad (a \leq x \leq b)$$

converges uniformly and that f is continuous for every $x \neq x_n$.

We note that every for every n and for every x the term of the series $|c_n I(x - x_n)| \leq |c_n|$, and moreover we are told that $\sum |c_n|$ converges. Therefore by the Weierstrass theorem $f(x) = \sum c_n I(x - x_n)$ converges uniformly.

Now let $a \leq x' \leq b$ such that for every n we have $x' \neq x_n$. Since we have that the convergence is uniform, we can then use theorem 7.11 to infer

$$\lim_{t \to x'} f(x) = \lim_{t \to x'} \lim_{m \to \infty} \sum_{n=1}^{m} c_n I(x - x_n)$$

$$= \lim_{n \to \infty} \sum_{n=1}^{m} \lim_{t \to x'} c_n I(x - x_n)$$

$$= \lim_{n \to \infty} \sum_{n=1}^{m} c_n I(x' - x_n)$$

$$= f(x')$$

The third equation is due to the fact that $x \neq x_n$ for any n and therefore $I(x - x_n)$ is continuous for each n.

Problem 8. Rudin page 166 problem 9. Let $\{f_n\}$ be a sequence of continuous functions which converges uniformly to a function f on a set E. Prove that

$$\lim_{n\to\infty} f_n(x_n) = f(x)$$

for every sequence of points $x_n \in E$ such that $x_n \to x$ and $x \in E$. Is the converse of this true?

Since the convergence is uniform and each f_n is continuous then f is continuous on E. So let $\varepsilon \in \mathbb{R}^+$ and set $N_1 \in \mathbb{N}$ such that if $n \geq N_1$ then

$$|f_n(x) - f(x)| < \varepsilon$$

for all $x \in E$. In particular, for every choice of n

$$|f_n(x_n) - f(x_n)| < \varepsilon$$

where ε does not depend on n. Let $\{x_n\}$ be any sequence in E such that $x_n \to x'$. Also set $\delta \in \mathbb{R}^+$ such that if $|x - x'| < \delta$ then $|f(x) - f(x')| < \varepsilon$, and set N_2 such that if $n \geq N_2$ then

$$|x_n - x'| < \delta$$

Finally set $N = \max\{N_1, N_2\}$ and let $n \geq N$. Note that because $n \geq N$ then $|x_n - x'| < \delta$ and therefore $|f(x_n) - f(x')| < \varepsilon$. Then

$$\begin{aligned}
|f_n(x_n) - f(x')| &= |f_n(x_n) - f(x_n) + f(x_n) - f(x')| \\
&\leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x')| \\
&< \varepsilon + \varepsilon = 2\varepsilon
\end{aligned}$$

So $\lim_{n\to\infty} f_n(x_n) = f(x')$ as desired.

The converse is not true, which we can see in the example of problem 1. Here, if you pick any $x' \in (0, \infty)$ and then pick any sequence $\{x_n\}$ such that $x_n \to x'$, then

$$\lim_{n\to\infty} f_n(x_n) = \lim_{n\to\infty} \frac{nx_n}{1+(x_n)^2}$$

$$= \frac{nx'}{1+(x')^2}$$

We've already noted that each f_n is continuous on $E = (0, \infty)$ and yet $f_n \overset{u}{\not\to} f$.

Problem 9. Use the Weierstrass M-test to prove that if a powerseries $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ converges absolutely at the point x_0 then it converges uniformly on $[-c, c]$ where $c = |x_0|$.

Note that for each n we have $|a_n x^n| = |a_n||x|^n \leq |a_n|c^n = M_n$. Since we are given that $\sum_{n=0}^{\infty} |a_n x^n|$ converges, then by the comparison test $\sum_{n=0}^{\infty} M_n$ converges. So by the Weierstrass M-test, $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely on $[-c, c]$.