Name: **October 1, 2020**

Math 637: Exam 1

Directions: Work individually on the Test. Pick 5 out of the 8 listed problems to submit. Your solutions must be typed (in LaTeX) and your submission a single pdf. You will be graded on the correctness of your 5 solution as follows:

- 0 points - Nothing is correct or you left this problem blank.
- 1 - 3 points - The solution contains a major flaw or is not written in an acceptable mathematical format.
- 4 - 9 points - The solution contains only minor flaws and is written in an acceptable mathematical format.
- 10 points - The solution is flawless.

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1. Let G be a group. Prove at if $a \in G$ is the only element of order 2 then $a \in Z(G)$.
2. Let G be a group with $a \in G$ and suppose $|a| = 5$. Prove that $C_G(a) = C_G(a^3)$.
3. Prove that no group can have exactly two elements of order 2.
4. Let p be a prime number and define $Z = \{z \in \mathbb{C} | z^{p^n} = 1 \text{ for some } n \in \mathbb{Z}^+\}$. For each $k \in \mathbb{Z}^+$ define $H_k = \{z \in Z | z^{p^k} = 1\}$. Prove that $H_k \leq H_m$ if and only if $k \leq m$.
5. Let Z_n be a cyclic group of order n and for each $a \in \mathbb{Z}$ let
$$\sigma_a : Z_n \to Z_n \qquad \text{by} \qquad \sigma_a(x) = x^a \text{ for all } x \in Z_n.$$
Prove that σ_a is an automorphism of Z_n if and only if a and n are relatively prime.
6. Prove that no group is the union of two proper subgroups.
7. Let G be a finite group with more than one element. Show that G has an element of prime order.
8. Suppose that G is a finite abelian group and G has no element of order 2. Show that the mapping $x \mapsto x^2$ is an automorphism of G.