Advanced Calculus, Homework 6

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Problem 1. Rudin page 78, Problem 6abc. Investigate the behavior (convergence or divergence) of $\sum a_n$ if
(a) $a_n = \sqrt{n+1} - \sqrt{n}$
(b) $a_n = \frac{\sqrt{n+1}-\sqrt{n}}{n}$
(c) $a_n = (\sqrt[n]{n} - 1)^n$

(a) For any partial sum

$$\begin{aligned}\sum_{n=1}^{m} a_n &= (\sqrt{2}-\sqrt{1}) + (\sqrt{3}-\sqrt{2}) + \cdots + (\sqrt{m+1}-\sqrt{m}) \\ &= \sqrt{m+1} - \sqrt{1} = \sqrt{m+1} - 1\end{aligned}$$

Hence the limit as $m \to \infty$ is infinity and the series diverges.

(b)

$$\frac{\sqrt{n+1}-\sqrt{n}}{n} \cdot \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{n(\sqrt{n+1}+\sqrt{n})}$$

$$< \frac{1}{n\sqrt{n}} = \frac{1}{n^{1.5}}$$

The series $\sum \frac{1}{n^{1.5}}$ is a "p-series" with $p = 1.5 > 1$ and therefore this series converges. By the comparison test, the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n}$ must converge.

(c) We attempt the Root Test

$$\limsup_{n\to\infty} \sqrt[n]{(\sqrt[n]{n}-1)^n} = \limsup_{n\to\infty} \sqrt[n]{n} - 1$$

$$= 1 - 1 = 0 < 1$$

Note that $\limsup\limits_{n\to\infty} \sqrt[n]{n} = 1$ because we have proved in class that

$$\lim_{n\to\infty} \sqrt[n]{n} = \limsup_{n\to\infty} \sqrt[n]{n} = 1$$

Problem 2. Rudin page 78, Problem 7. Prove that the convergence of $\sum a_n$ implies the convergence of

$$\sum \frac{\sqrt{a_n}}{n}$$

if $a_n \geq 0$.

We find an M such that the partial sums are bounded, $\displaystyle\sum_{n=1}^{N} \frac{\sqrt{a_n}}{n} \leq M$. Then by theorem 3.24, we will have bounded partial sums, and every term is non-negative, therefore the series converges.

To find M, we consider

$$(\sqrt{a_1} + \sqrt{a_2}/2 + \cdots + \sqrt{a_N}/N) \leq M \iff$$

$$(\sqrt{a_1} + \sqrt{a_2}/2 + \cdots + \sqrt{a_N}/N)^2 \leq M^2$$

By the Cauchy-Schwarz inequality

$$\left(\sum_{n=1}^{N} \sqrt{a_n}\frac{1}{n}\right)^2 \leq \left(\sum_{n=1}^{N} \sqrt{a_n}^2\right)\left(\sum_{n=1}^{N} \left(\frac{1}{n}\right)^2\right) = \left(\sum_{n=1}^{N} a_n\right)\left(\sum_{n=1}^{N} \left(\frac{1}{n}\right)^2\right)$$

Because $\sum a_n$ converges, its partial sums are bounded by some M_1. Likewise we know $\sum \frac{1}{n^2}$ is a p-series with $p = 2$ and therefore converges, and therefore its partial sums are also bounded by some M_2. Therefore

$$\left(\sum_{n=1}^{N} a_n\right)\left(\sum_{n=1}^{N} \left(\frac{1}{n}\right)^2\right) \leq M_1 M_2, \quad \forall N \in \mathbb{N}$$

and so we define $M = \sqrt{M_1 M_2}$. Then we have that the partial sums are bounded, $\displaystyle\sum_{n=1}^{N} \sqrt{a_n}/n \leq M$. Because the terms of the series are all non-negative and the partial sums are bounded, then the series must converge.

Problem 3. Rudin page 79, Problem 9. Find the radius of convergence of each of the following power series:

(a) $\sum n^3 z^n$
(b) $\sum \frac{2^n}{n!} z^n$
(c) $\sum \frac{2^n}{n^2} z^n$
(d) $\sum \frac{n^3}{3^n} z^n$

(a)

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{n^3}}$$

but

$$\lim_{n\to\infty} \sqrt[n]{n^3} = (\lim_{n\to\infty} \sqrt[n]{n})^3$$

$$= 1^3 = 1$$

so this is therefore also the limsup. Hence

$$R = 1/1 = 1$$

(b)
We first note that the following limit exists for any z:

$$\lim_{n\to\infty} \left| \frac{\frac{2^{n+1}}{(n+1)!} z^{n+1}}{\frac{2^n}{n!} z^n} \right| = \lim_{n\to\infty} \left| \frac{2z}{n+1} \right|$$

$$= 0$$

Therefore the power series always converges at any z by the Ratio Test, since this limit is always less than 1. Hence the radius of convergence is infinity: $R = \infty$.

(c)

$$\lim_{n\to\infty} \sqrt[n]{\frac{2^n}{n^2}} = \lim_{n\to\infty} \frac{2}{(\sqrt[n]{n})^2} = \frac{2}{1} = 2$$

Thus

$$R = \frac{1}{\lim_{n\to\infty} \sqrt[n]{\frac{2^n}{n^2}}} = 1/2$$

(d)

$$\lim_{n\to\infty} \sqrt[n]{\frac{n^3}{3^n}} = \lim_{n\to\infty} \frac{(\sqrt[n]{n})^3}{3} = 1/3$$

Thus

$$R = \frac{1}{1/3} = 3$$

Problem 4. If $\sum_{n=1}^{\infty} a_n = A$ prove $\sum_{n=1}^{\infty} ca_n = cA$ for all $c \in \mathbb{R}$.

We can prove this entirely by the limit properties in a limit of partial sums.

$$\sum_{n=1}^{\infty} ca_n = \lim_{b\to\infty} \sum_{n=1}^{b} ca_n$$

$$= \lim_{b\to\infty} c \sum_{n=1}^{b} a_n$$

$$= c \lim_{b\to\infty} \sum_{n=1}^{b} a_n$$

$$= cA$$

Problem 5. Give a proof for the Comparison Test using the Monotone Convergence Theorem.

Suppose $\{a_n\}$ and $\{b_n\}$ are sequences of real numbers such that $|a_n| \leq b_n$ for every $n \geq N_0$ for some $N_0 \in \mathbb{N}$. Moreover suppose $\sum b_n$ converges to b. Then the sequence of partial sums $s_N = \sum_{n=1}^{N} b_n$ is bounded above by b. I claim that therefore b is also an upper bound on the sequence of partial sums of $|a_n|$. That is to say, for any choice of M I claim that

$$\sum_{n=1}^{M} |a_n| \leq b$$

This is obvious since for partial sums

$$\sum_{n=1}^{M} |a_n| \leq \sum_{n=1}^{M} b_n \leq b$$

But this shows by monotone convergence that the sequence of partial sums $\sum_{n=1}^{M} |a_n|$ is bounded above and monontonically increasing, so it too converges. But then this means that $\sum a_n$ converges absolutely, and therefore it converges.

Now for second part of the comparison test, suppose $\{a_n\}, \{d_n\}$ are sequence of real numbers such that $a_n \geq d_n \geq 0$ and suppose $\sum d_n$ diverges. Now for contradiction suppose that $\sum a_n$ converges. From the first part proved above, we must have that $\sum d_n$ converges, which is a contradiction. $\lightning$

Hence $\sum a_n$ must diverge.

Problem 6. Prove that if $\sum_{n=1}^{\infty} x_n$ converges absolutely and the sequence $\{y_n\}$ is bounded then $\sum_{n=1}^{\infty} x_n y_n$ converges.

Let $|y_n| \le M$. Then

$$\left|\sum_{n=p}^{q} x_n y_n\right| \le \sum_{n=p}^{q} |x_n||y_n| \le M \sum_{n=p}^{q} |x_n|$$

Now because $\sum x_n$ is absolutely convergent, there exists an N such that for all $p, q \ge N$

$$\sum_{n=p}^{q} |x_n| < \varepsilon/M$$

Thus for this N we have

$$\left|\sum_{n=p}^{q} x_n y_n\right| < M(\varepsilon/M) = \varepsilon$$

Hence $\sum x_n y_n$ satisfies the Cauchy criterion for convergence.

Problem 7. Give an example of each of the following or explain why such a request is impossible.

a. Two series $\sum x_n$ and $\sum y_n$ that both diverge but $\sum x_n y_n$ converges.

b. A convergent $\sum x_n$ and a bounded $\{y_n\}$ such that $\sum x_n y_n$ diverges.

c. Two sequences $\{x_n\}$ and $\{y_n\}$ where $\sum x_n$ and $\sum(x_n + y_n)$ both converge but $\sum y_n$ diverges.

(a) $\sum \frac{1}{n}$ and $\sum \frac{1}{n}$. They are each the harmonic series (or p-series with $p = 1$) and hence diverge. However $\sum \frac{1}{n^2}$ is a p-series with $p = 2$ and so it converges.

(b) $\sum \frac{(-1)^n}{n}$ and $y_n = (-1)^n$. The former is the alternating harmonic series. Because it is alternating, $\left|\frac{(-1)^n}{n}\right| = \frac{1}{n}$ is monotonically decreasing and goes to 0, this series converges. Clearly $|y_n| \leq 1$ so it is bounded.

However, $\sum \left(\frac{(-1)^n}{n} \cdot (-1)^n\right) = \sum \frac{1}{n}$ as we have said does not converge.

(c) This is impossible, since the convergence of the two series implies

$$\sum(x_n + y_n) - \sum x_n = \sum(x_n + y_n - x_n) = \sum y_n$$

must converge.