MA-652 Advanced Calculus
Homework 2, Jan. 13
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Problem 1. Rudin page 114 problem 7. Suppose $f'(x), g'(x)$ exist, $g'(x) \neq 0$, and $f(x) = g(x) = 0$. Prove that

$$\lim_{t \to x} \frac{f(t)}{g(t)} = \frac{f'(x)}{g'(x)}$$

$$\lim_{t \to x} \frac{f(t)}{g(t)} = \lim_{t \to x} \frac{(f(t) - f(x))/(t - x)}{(g(t) - g(x))/(t - x)} = \frac{\lim_{t \to x}(f(t) - f(x))/(t - x)}{\lim_{t \to x}(g(t) - g(x))/(t - x)} = \frac{f'(x)}{g'(x)}$$

Where the second equality is due to the fact that $g'(x) \neq 0$.

Problem 2. Rudin page 114 problem 8. Suppose f' is continuous on $[a, b]$ and $\varepsilon > 0$. Prove that there exists a $\delta > 0$ such that $\left|\frac{f(t)-f(x)}{t-x} - f'(x)\right| < \varepsilon$ whenever $0 < |t - x| < \delta$, for $a \leq x \leq b$, $a \leq t \leq b$. Does this hold for vector-valued functions?

Since f' is continuous on a compact set it is uniformly continuous. Hence

$$|f'(t) - f'(x)| < \varepsilon$$

for some δ_1 and all $|t - x| < \delta_1$. Hence

$$\left|f'(t) + \frac{f(x) - f(t)}{x - t} - \frac{f(x) - f(t)}{x - t} - f'(x)\right| =$$

Problem 3. Rudin page 115 problem 9. Let f be a continuous real function on $\mathbb{R}$, of which it is known that $f'(x)$ exists for all $x \neq 0$ and that $f'(x) \to 3$ as $x \to 0$. Does it follow that $f'(0)$ exists?

Problem 4. Rudin page 115 problem 11. Suppose f is defined in a neighborhood of x, and suppose $f'(x)$ exists. Show that

$$\lim_{h\to 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2} = f''(x)$$

Show by an example that the limit may exist even if $f''(x)$ does not. *Hint: Use Theorem 5.13.*

Problem 5. Rudin page 115 problem 12. If $f(x) = |x^3|$, compute $f'(x)$, $f''(x)$ for all real x, and show that $f^{(3)}(0)$ does not exist.

Problem 6. Let $a \in \mathbb{R}$ and $f : (a, \infty) \to \mathbb{R}$ be twice differentiable. (a) Use Taylor's Theorem to show that $f'(x) = \frac{1}{2h}[f(x+2h) - f(x)] - hf''(\xi)$ for some $\xi \in (x, x+2h)$.

(b) Use the result from part (a) to show that if M_0, M_1, M_2 are the LUBs of $|f(x)|, |f'(x)|$, and $|f''(x)|$ respectively on (a, ∞), then $|f'(x)| \leq hM_2 + \frac{M_0}{h}$.

(c) Use part (b) tos how $M_1^2 \leq 4M_0M_2$.

Problem 7. Rudin part 116 problem 16. Suppose f is twice differentiable on $(0, \infty)$, f'' is bounded on $(0, \infty)$, and $f(x) \to 0$ as $x \to \infty$. Prove that $f'(x) \to 0$ as $x \to \infty$. *Hint:* Let $a \to \infty$ in exercise 15.

Problem 8. Rudin part 116 problem 25. Suppose f is twice differentiable on $[a, b]$, $f(a) < 0$, $f(b) > 0$, $f'(x) \geq \delta > 0$, and $0 \leq f''(x) \leq M$ for all $x \in [a, b]$. Let ξ be the unique point in (a, b) at which $f(\xi) = 0$. Complete the details in the following outline of Newton's method for computing ξ.

(a) Choose $x_1 \in (\xi, b)$ and define $\{x_n\}$ by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Interpret this geometrically in terms of the tangent to the graph of f.

(b) Prove that $x_{n+1} < x_n$ and that $\lim_{n\to\infty} x_n = \xi$.

(c) Use Taylor's theorem to show that $x_{n+1} - \xi = \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2$ for some $t_n \in (\xi, x_n)$.

(d) If $A = M/2\delta$ deduce that

$$0 \leq x_{n+1} - \xi \leq \frac{1}{A}[A(x_1 - \xi)]^{2n}$$

(e) Show that Newton's method amounts to finding a fixed point of the function g defined by

$$g(x) = x - \frac{f(x)}{f'(x)}$$

How does $g'(x)$ behave for x near ξ?

(f) Put $f(x) = x^{1/3}$ on $(-\infty, \infty)$ and try Newton's method. What happens?