Advanced Calculus, Homework 4

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Problem 1. Rudin page 44, Problem 12. Prove that $K = \{1/n : n \in \mathbb{N}\} \cup \{0\}$ is compact directly from the definition.

Let $\mathcal{O}$ be any open cover of K, that is to say, $\mathcal{O}$ is a family of open sets such that $K \subseteq \bigcup_{O \in \mathcal{O}} O$. One open set $O_0 \in \mathcal{O}$ must contain 0. Then we may place a neighborhood around 0, call this $N_\varepsilon(0)$, such that $N_\varepsilon(0) \subseteq O_0$. All of the numbers within a radius of ε are inside $N_\varepsilon(0)$, leaving only finitely many numbers outside the neighborhood.

For each of the (finitely many) points outside of $N_\varepsilon(0)$, select one open set from $\mathcal{O}$ containing it. Call these $O_1, O_2, \cdots \in \mathcal{O}$ such that $1 \in O_1$ and $1/2 \in O_{1/2}, \ldots$, and so on. Let N be the smallest value such that $\frac{1}{N} < \varepsilon$. The finite subcollection $O_0, \ldots, O_{1/N}$ is a finite subcover. For if $n \in \mathbb{N}$ then either $\frac{1}{n} < \frac{1}{N}$ or $\frac{1}{n} \geq \frac{1}{N}$. In the former case $1/n < 1/N < \varepsilon$ and so $1/n \in O_0$. On the other hand if $\frac{1}{n} \geq \frac{1}{N}$ then $\frac{1}{N} \leq \frac{1}{n} \leq 1$ and so $1/n \in O_{1/n}$. And of course $0 \in O_0$.

Since every open cover has a finite subcover, then K is compact.

Problem 2. Rudin page 44, Problem 14. Give an example of an open cover of $(0,1)$ which has no finite subcover.

Take the open cover $\mathcal{O} = \{O_i | O_i = (1/i, 1-1/i) \text{ for } i \in \mathbb{N} \setminus \{1,2\}\}$. Any finite subcollection $\hat{\mathcal{O}} \subseteq \mathcal{O}$ has a largest indexed set O_N, and $\bigcup_{O_i \in \hat{\mathcal{O}}} O_i = O_N$. But there exists some $x \in \mathbb{R}$ such that $0 < x < 1/N$ so that therefore $x \notin O_N$. Hence $\hat{\mathcal{O}}$ cannot be a subcover.

This shows that $\mathcal{O}$ is an open cover $(0,1)$ such that there is no finite subcover.

Problem 3. Rudin page 44, Problem 15. Show that the finite/infinite intersection property of compact sets becomes false when "compact" is replaced by either "closed" or "bounded".

For the "closed" part we want to exhibit some family of closed sets such that every finite subfamily intersects. Further, we want the infinite intersection to be empty. Clearly $C_n = [n, \infty]$ is such a family.

For the "bounded" part we want to exhibit some family of bounded sets such that every finite subfamily intersects. Further we want that the infinite intersection is empty. Then $(0, 1/n)$ witnesses this property.

Problem 4. Let $K \subset \mathbb{R}$ be nonempty and compact. Show that $\inf(K) \in K$.

A compact set is always closed, so K is closed. Therefore $K = \overline{K}$. If we now show that $\inf(K) \in \overline{K}$ we are done. But this was proved in the previous homework, problem 7.

Problem 5. Let $K \subset \mathbb{R}^k$ be compact and $F \subset \mathbb{R}^k$ be closed. Determine if the following sets are always compact. If yes, prove it. If no, provide a counterexample.

a. $\overline{F^c \cup K^c}$

b. $K \setminus F$

c. $\overline{K \cap F^c}$

a. Not a compact set. Example: Let $F = [0,1] = K \subset \mathbb{R}^1$. Then $F^c \cup K^c = (-\infty, 0) \cup (1, \infty)$ and $\overline{F^c \cup K^c} = (-\infty, 0] \cup [1, \infty)$ which is unbounded and therefore not compact.

b. Not a compact set. Example: If $K = [0,1]$ and $F = [1,2]$ then $K \setminus F = [0,1)$ which is not closed and therefore not compact.

c. Must be compact. Proof: $K \cap F^c \subseteq K$ so that $\overline{K \cap F^c} \subseteq \overline{K} = K$ is therefore a closed subset of a compact set and therefore is closed (Theorem 2.35 of Rudin).

Problem 6. Let $P \subset \mathbb{R}^k$ be perfect and $K \subset \mathbb{R}^k$ be compact.

a. Is $P \cap K$ always compact? If yes, prove it. If no, provide a counterexample.

b. Is $P \cap K$ always perfect? If yes, prove it. If no, provide a counterexample.

a. Yes, this is compact. It is bounded because K is bounded and $P \cap K \subseteq K$. It is closed because finite intersections of closed sets are closed.

b. It is not always perfect. If $P = (-\infty, \infty)$ then P is perfect, and if $K = \{0\}$ then K is compact, but $P \cap K = \{0\}$ is not perfect because it contains an isolated point, 0.

Problem 7. Does there exist a perfect set consisting of only rational numbers? Why or why not?

No such set can exist. If a nonempty set in $\mathbb{R}$ is perfect, by theorem 2.43, it contains an uncountable number of points. Since any non-empty set of rational numbers is countable, then by the contrapositive of the above, any such set must not be perfect.

Problem 8. Rudin page 44, problem 19.

(a) If A and B are disjoint closed sets in some metric space X, prove they are separated.

(b) Prove the same for disjoint open sets.

(c) Fix $p \in X$, $\delta > 0$, define A to be the set of all $q \in X$ for which $d(p,q) < \delta$, define B similarly with $>$ in place of $<$. Prove that A and B are separated.

(d) Prove that every connected metric space with at least two points is uncountable. (Hint: use c)

(a.) We start by showing that, if A and B are disjoint closed sets in the metric space, then $A \cap \overline{B} = \emptyset$. But $A \cap \overline{B} = A \cap B = \emptyset$ by the equality of a closed set with its closure. By the same argument $\overline{A} \cap B = A \cap B = \emptyset$. Since both are empty, A and B are separated.

(b.) If A and B are now disjoint open sets in some metric space, then

$$A \cap \overline{B} = A \cap (B \cup B') = (A \cap B) \cup (A \cap B')$$

Since $A \cap B = \emptyset$ by assumption, we need only show $A \cap B' = \emptyset$. For contradiction, if $x \in A \cap B'$ then we find a neighborhood of x contained entirely in A. Call this neighborhood N so that $x \in N \subseteq A$. Because $x \in B'$ this neighborhood must intersect B at some point $y \in B \cap N$. But then $y \in N \subseteq A$ so $y \in A \cap B$. This contradicts the assumption that $A \cap B = \emptyset$. $\lightning$

(c.) A is a neighborhood of p and therefore open. We can prove that B is open by letting $x \in B$ and find a neighborhood of x contained entirely in B. In particular, a neighborhood of radius $d(p,x) - \delta$ around x will be entirely in B.

To show this, suppose for contradiction that there is some y in this neighborhood such that $d(p,y) \leq \delta$. Since y is assumed to be in the given neighborhood we have $d(x,y) < d(p,x) - \delta$ so that we can infer

$$d(p,x) \leq d(p,y) + d(y,x)$$

just from the triangle inequality and

$$d(p,y) + d(y,x) < \delta + (d(p,x) - \delta) = d(p,x)$$

from $d(x, y) < d(p, x) - \delta$. Chaining these inequalities together we have shown $d(p, x) < d(p, x)$. $\lightning$

Now that we know both A and B are open, then it follows from part (b.) that they are separated.

(d.) Let $x \neq y \in X$. Intuitively, the uncountability should come from $\mathbb{R}$ being uncountable, and for every $r \in \mathbb{R}$ there is some set of points $\{p \in X : d(p, x) = r\}$. For any two values of r we must have disjoint sets. So if every one of these sets is non-empty then there are uncountably many points in X. On the other hand, if one of these sets is empty then it can be leverages to split the space into two open sets *a la* part (c.). The only hitch is that when we split the space, we require that there is at least one element on the other side of the split—and that's where y comes in. We should be able to make x come out on one side and y on the other.

To formalize this intuition, suppose that the space is connected. To begin with, we show that it is impossible for some $C_r = \{p \in X : d(p, x) = r\}$ to be empty, where $r = t \cdot d(x, y)$ where $t \in (0, 1)$. This effectively has us looking at all of the radii in-between 0 and $d(x, y)$. If some C_r is empty, then define $A = \{p \in X : d(p, x) < r\}$ and $B = \{p \in X : d(p, x) > r\}$. Clearly $A \cup B = X$ since $A \cup B \cup C_r = X$ and $C_r = \emptyset$. Moreover, $x \in A$ and $y \in B$ so these sets are non-empty. And from part (c.) they are separated. This contradicts that X is connected. $\lightning$

Now since we know that every one of these C_r is non-empty and disjoint, then we can injectively map the uncountable set $(0, 1)$ to a subset of X. In particular, for each $t \in (0, 1)$ we map $f(t) = t_r$ where t_r is an arbitrarily chosen element from C_r.[1] Hence X is uncountable.

We have then seen that, for any connected space X containing at least two distinct elements, it must contain an uncountable infinity of elements.

[1] Because this is Axiom of Choice-y, it sounds like we're proving something is uncountable!

Problem 9. Give an example to show that if A is open and B is closed and A and B are disjoint, then they are not always separated.

$A = (0, 1), B = [1, 2]$
These are not separated because $\overline{A} \cap B = [0, 1] \cap [1, 2] = \{1\}$.

Problem 10. Find an example of a disconnected set whose closure is connected.

Take $A = (0, 1)$ and $B = (1, 2)$ and $C = A \cup B$. Then C is disconnected because it's the union of two disjoint open sets.

However, $\overline{C} = [0, 2]$ which is connected because it's an interval.