MA 638 - Section 7.1 Homework

INSTRUCTIONS: The solutions to all problems here should be typed up in LaTeX using correct mathematical notation and language. Proofs and explanations must be written in complete sentences using correct grammar and punctuation. Use mathematical displays and *implication* or *if and only if* arrows where appropriate to manipulate equations or to validate steps in a proof or argument. Your completed pdf will be uploaded in Canvas. Be sure to save your pdf file using the naming format "first initial last name underscore section number", i.e. Jane Doe would save her this homework as **jdoe_7-1.pdf**.

Throughout assume R is a ring with 1.

1. Prove (3) and (4) of Proposition 7.1.
2. The commutator of an element a in a ring R is defined as $C(a) = \{r \in R | ra = ar\}$ i.e the set of all elements in R that commute with a fixed element a. Prove that $C(a)$ is a subring of R containing a. Prove that the center of R is the intersection of the subrings $C(a)$ over all $a \in R$.
3. Prove that if R is an integral domain and $x^2 = 1$ for some $x \in R$, then $x = \pm 1$.
4. Suppose that x is a nilpotent element of a commutative ring R.
   (a) Prove that x is either zero or a zero divisor.
   (b) Prove that rx is nilpotent for every $r \in R$.
   (c) Prove that $1 + x$ is a unit in R.
5. A ring is called a boolean ring if $a^2 = a$ for all $a \in R$. Prove that every Boolean ring is commutative.
6. Let X be any nonempty set and let $\mathcal{P}(X)$ be the set of all subsets of X (the power set of X). Define addition and multiplication on $\mathcal{P}(X)$ by

$$A + B = (A - B) \cup (B - A) \quad \text{and} \quad A \times B = A \cap B$$

   i.e., addition is symmetric difference and multiplication is intersection.
   (a) Prove that $\mathcal{P}(X)$ is a ring under these operations ($\mathcal{P}(X)$ and its subrings are often referred to as *rings of sets*).
   (b) Prove that this ring is commutative, has identity, and is a Boolean ring.