Polynomials

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Introduction

A polynomial in one variable x is an expression of the form:

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where:

- n is a non-negative integer called the **degree** of the polynomial.
- $a_n, a_{n-1}, \ldots, a_1, a_0$ are **coefficients**, typically real or complex numbers.
- $a_n \neq 0$ is the **leading coefficient**.
- a_0 is the **constant term**.

Properties of Polynomials

1. Given polynomials $A(x)$ and $B(x) \neq 0$, there exist unique polynomials $q(x)$ and $r(x)$ such that:
$$A(x) = B(x)q(x) + r(x)$$
where $\deg(r) < \deg(B)$.
2. A polynomial $P(x)$ is divisible by the binomial $x - a$ if and only if $P(a) = 0$.
3. If a polynomial $P(x)$ is divisible by a polynomial $Q(x)$, then every zero of $Q(x)$ is also a zero of $P(x)$.
4. Any real polynomial $P(x)$ of degree n has a unique representation of the form:
$$P(x) = c(x - a_1)(x - a_2)\ldots(x - a_k)(x^2 - p_1 x + q_1)\ldots(x^2 - p_l x + q_l)$$
where $c \neq 0$, a_i, p_j, q_j are real numbers, and the quadratic terms satisfy $p_j^2 < 4q_j$. Additionally, $k + 2l = n$.
5. Let $P(x)$ be a polynomial with real coefficients, and let $a, b \in \mathbb{R}$ with $a < b$. If
$$P(a) \cdot P(b) < 0,$$
then there exists some $c \in (a, b)$ such that
$$P(c) = 0.$$

Elementary Symmetric Polynomials and Vieta's Formulas

Given n variables $x_1, x_2, \ldots, x_n$, the **elementary symmetric polynomials** are defined as follows:

- The first elementary symmetric polynomial:
$$e_1 = x_1 + x_2 + \cdots + x_n$$
- The second elementary symmetric polynomial:
$$e_2 = \sum_{1 \le i < j \le n} x_i x_j$$
- The general form:
$$e_k = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} x_{i_1} x_{i_2} \ldots x_{i_k}$$
for $1 \le k \le n$, where $e_n = x_1 x_2 \ldots x_n$.

Let $P(x)$ be a polynomial of degree n with roots $x_1, x_2, \ldots, x_n$:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

The relationships between the coefficients and the roots, known as **Vieta's formulas**, are:

$$\begin{aligned} e_1 &= x_1 + x_2 + \cdots + x_n = -\frac{a_{n-1}}{a_n}, \\ e_2 &= x_1 x_2 + x_1 x_3 + \cdots + x_{n-1} x_n = \frac{a_{n-2}}{a_n}, \\ &\vdots \\ e_n &= x_1 x_2 \ldots x_n = (-1)^n \frac{a_0}{a_n}. \end{aligned}$$

These formulas express the coefficients of a polynomial in terms of its roots.

Examples

Example 1: A Direct Application of Vieta

Problem. Let $a, b, c, x, y \in \mathbb{R}$ satisfy

$$a^3 + ax + y = 0, \qquad b^3 + bx + y = 0, \qquad c^3 + cx + y = 0,$$

with a, b, c pairwise distinct. Prove that $a + b + c = 0$.

Walk-through.

1. **Construct the cubic.** Each equality can be rewritten as $P(t) = t^3 + xt + y = 0$ with $t \in \{a, b, c\}$. Hence a, b, c are roots of P.
2. **Note the degree.** Since P is cubic and a, b, c are distinct, they comprise *all* the roots of P.

3. **Invoke Vieta's formulas.** For a monic cubic $t^3 + Bt^2 + Ct + D$ with roots r_1, r_2, r_3, Vieta states $r_1 + r_2 + r_3 = -B$. Here $B = 0$, so

$$a + b + c = -B = 0.$$

Therefore

$$\boxed{a + b + c = 0}$$

Example 2: Sign Information From Vieta's Formulas

Problem. Let $a, b, c \in \mathbb{R}$ satisfy

$$a + b + c > 0, \qquad ab + bc + ca > 0, \qquad abc > 0.$$

Show that $a, b, c > 0$.

Solution. Write the monic cubic with roots a, b, c:

$$P(x) = (x - a)(x - b)(x - c) = x^3 - (a + b + c)\, x^2 + (ab + bc + ca)\, x - abc.$$

By Vieta's formulas the given inequalities say that

$$\text{(i)} \; -(\text{coef. of } x^2) < 0, \qquad \text{(ii)} \; (\text{coef. of } x) > 0, \qquad \text{(iii)} \; -(\text{constant term}) < 0.$$

Note that 0 isn't a root of the polynomial given $abc > 0$. If at least one of a, b, c wasn't positive, the polynomial would have a negative root. But for $x < 0$ we have that:

$$\text{(i)}\; -x^3 < 0, \qquad \text{(ii)} \; -(a + b + c)x^2 < 0, \qquad \text{(iii)} \; (ab + bc + ca)x < 0, \qquad \text{(iv)} \; -abc < 0$$

and therefore $P(x) < 0$. Therefore $P(x)$ cannot have a negative root and:

$$\boxed{a > 0,\ b > 0,\ c > 0}$$

Interpolation Polynomial

Interpolation polynomials are used to construct a polynomial that passes through a given set of points. Given $n+1$ distinct points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$, there exists a unique polynomial $P(x)$ of degree at most n such that:

$$P(x_i) = y_i \quad \text{for each } i = 0, 1, \ldots, n.$$

A natural way to construct $P(x)$ is as a weighted sum of simpler polynomials:

$$P(x) = \sum_{i=0}^{n} y_i L_i(x).$$

Here, each **Lagrange basis polynomial** $L_i(x)$ is designed to satisfy the following two key properties:

1. **At $x = x_i$, we get** $L_i(x_i) = 1$ This ensures that when evaluating $P(x)$ at $x = x_i$, we get y_i, because all other terms $y_j L_j(x_i)$ vanish for $j \neq i$.

2. **At $x = x_j$ for $j \neq i$, we get** $L_i(x_j) = 0$ This ensures that the contribution from $y_j L_j(x)$ does not interfere with $P(x_i)$, keeping the interpolation intact.

To achieve these properties, we define:

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}.$$

Roots of Unity

Definition

The n**th roots of unity** are the complex solutions to the equation:

$$x^n = 1.$$

These roots can be expressed using Euler's formula:

$$\omega_k = e^{2\pi i k/n} = \cos\left(\frac{2\pi k}{n}\right) + i \sin\left(\frac{2\pi k}{n}\right),$$

for $k = 0, 1, \ldots, n-1$. Here, $\omega = e^{2\pi i/n}$ is a **primitive** nth root of unity, and the set $\{\omega^k\}_{k=0}^{n-1}$ comprises all nth roots of unity.

Algebraic and Geometric Properties

- **Closure under multiplication:** The set of nth roots of unity forms a cyclic group under multiplication. That is, $\omega^a \cdot \omega^b = \omega^{a+b \pmod{n}}$.
- **Sum of roots:** The sum of all nth roots of unity is zero:

$$\sum_{k=0}^{n-1} \omega_k = \sum_{k=0}^{n-1} \omega^k = 0.$$

- **Product of roots:** The product of all nth roots of unity is:

$$\prod_{k=0}^{n-1} \omega^k = \begin{cases} 1, & \text{if } n \text{ is odd,} \\ -1, & \text{if } n \text{ is even.} \end{cases}$$

- **Geometric interpretation:** When plotted on the complex plane, the nth roots of unity lie on the unit circle and are evenly spaced, forming the vertices of a regular n-gon centered at the origin.

Primitive Roots of Unity

A root ω^k is called a **primitive** nth root of unity if its order is n, i.e., $\omega^{k \cdot m} \neq 1$ for $1 \leq m < n$. There are $\phi(n)$ primitive nth roots of unity, where ϕ denotes Euler's totient function.

Applications in Problem Solving

1. Factoring Polynomials

Roots of unity are instrumental in factoring polynomials like $x^n - 1$. For example:

$$x^n - 1 = \prod_{k=0}^{n-1} (x - \omega^k).$$

This factorization is fundamental in understanding the structure of polynomials and in simplifying expressions.

2. Solving Equations Involving Sums of Powers

Consider the equation:

$$x^n + x^{n-1} + \cdots + x + 1 = 0.$$

This can be rewritten using the formula for the sum of a geometric series:

$$\frac{x^{n+1} - 1}{x - 1} = 0.$$

Solving $x^{n+1} = 1$ and $x \neq 1$ leads us to the $(n+1)$th roots of unity, excluding 1.

3. Cyclotomic Polynomials

The minimal polynomials over $\mathbb{Q}$ for primitive nth roots of unity are called **cyclotomic polynomials**, denoted $\Phi_n(x)$. They are defined as:

$$\Phi_n(x) = \prod_{\substack{1 \leq k \leq n \\ \gcd(k,n)=1}} (x - \omega^k).$$

These polynomials are irreducible over $\mathbb{Q}$ and have integer coefficients.

Examples

Example 1: Cube Roots of Unity

Find all cube roots of unity.

Solution: Solve $x^3 = 1$. The roots are:

$$\omega_0 = 1, \quad \omega_1 = e^{2\pi i/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad \omega_2 = e^{4\pi i/3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

Example 2: Factoring $x^4 + 1$

Factor $x^4 + 1$ over $\mathbb{C}$.

Solution: Note that $x^4 + 1 = x^4 - (-1)$. The 8th roots of unity satisfy $x^8 = 1$, so the roots of $x^4 + 1$ are the 8th roots of unity with odd indices:

$$x^4 + 1 = \prod_{k=1,3,5,7} (x - \omega^k).$$

Example 3: Sum of Powers of Roots of Unity

Compute $\sum_{k=0}^{n-1} \omega^{k\cdot m}$, where ω is a primitive nth root of unity.

Solution: This sum equals:

$$\sum_{k=0}^{n-1} \omega^{k\cdot m} = \begin{cases} n, & \text{if } n \mid m, \\ 0, & \text{otherwise.} \end{cases}$$

This property is useful in filtering specific coefficients in polynomial expressions.

Example 4: A Divisibility Criterion via Roots of Unity

Problem. For which positive integers n does

$$x^2 + x + 1 \mid x^{2n} + x^n + 1$$

hold?

Solution. We know that if a polynomial $Q(x)$ divides a polynomial $P(x)$ in $\mathbb{C}[x]$. then every root of Q is also a root of P. This motivates us to look at the roots of polynomial $x^2 + x + 1$. However these are the roots of unity!

Let $\zeta = \dfrac{-1 + i\sqrt{3}}{2}$; then ζ and ζ^2 are the primitive cube–roots of unity:

$$\zeta^3 = 1, \qquad \zeta \neq 1, \qquad \zeta^2 + \zeta + 1 = 0.$$

Because $\zeta^3 = 1$, every power ζ^n depends only on n mod 3. Write $n = 3q + k$ with $k \in \{0, 1, 2\}$. Then

$$\zeta^n = \zeta^k, \qquad \zeta^{2n} = \zeta^{2k}.$$

$$x^{2n} + x^n + 1 \Big|_{x=\zeta} = \zeta^{2k} + \zeta^k + 1 = \begin{cases} 3 & \text{if } k = 0, \\ 0 & \text{if } k = 1, \\ 0 & \text{if } k = 2. \end{cases}$$

Thus ζ (and, by the same calculation, ζ^2) is a root of $x^{2n} + x^n + 1$ precisely when $k \neq 0$, i.e. when $3 \nmid n$.

Therefore ζ and ζ^2 are roots of $x^{2n} + x^n + 1$ iff $3 \nmid n$. By the factor (or Remainder) Theorem, this happens exactly when the quadratic $x^2 + x + 1$ divides the trinomial.

$$\boxed{x^2 + x + 1 \mid x^{2n} + x^n + 1 \iff 3 \nmid n}$$

Checks.

- $n = 1, 2$: $\quad x^2 + x + 1$ divides $x^2 + x + 1$ and $x^4 + x^2 + 1$.
- $n = 3$: $\quad x^6 + x^3 + 1 = (x^2 + x + 1)(x^4 - x^2 + 1) + 3$; remainder $3 \neq 0$, so divisibility fails.

Problems

1. Let P be a nonconstant polynomial with integer coefficients. Prove that there is an integer x so that $P(x)$ is composite.
2. Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ be a polynomial with integer coefficients. Prove that $P(x)$ is irreducible over the integers if there exists a prime number p such that:
   - p divides a_i for all $i < n$,
   - p does not divide a_n, and
   - p^2 does not divide a_0.
3. Let real numbers a_1, a_2, a_3, a_4, a_5 satisfy
$$\frac{a_1}{k^2+1} + \frac{a_2}{k^2+2} + \frac{a_3}{k^2+3} + \frac{a_4}{k^2+4} + \frac{a_5}{k^2+5} = \frac{1}{k^2}, \qquad k = 1, 2, 3, 4, 5.$$
   Find
$$\frac{a_1}{37} + \frac{a_2}{38} + \frac{a_3}{39} + \frac{a_4}{40} + \frac{a_5}{41}.$$
4. The product of two of the four roots of the quartic equation $x^4 - 18x^3 + kx^2 + 200x - 1984 = 0$ is -32. Determine the value of k.
5. Given that the polynomials P and Q each have at least one real root, and the following equality holds:
$$P(1 + x + Q(x)^2) = Q(1 + x + P(x)^2),$$
   prove that P is identically equal to Q, denoted $P \equiv Q$.
6. Let P be a polynomial of degree n satisfying
$$P(k) = \binom{n+1}{k}^{-1} \qquad \text{for } k = 0, 1, ..., n.$$
   Determine $P(n+1)$.
7. Let $z_1, z_2, \ldots, z_n$ be the roots of the polynomial
$$P(x) = x^n + x^{n-1} + \cdots + x + 1.$$
   Determine the smallest natural number m such that the points $z_1^m, z_2^m, \ldots, z_n^m$ in the complex plane lie on the same line, if:
   a) $n = 2011$
   b) $n = 2010$
8. Let P be a polynomial of degree n with real coefficients. Assume that for all x in the interval $[0, 1]$, the absolute value of $P(x)$ satisfies the inequality $|P(x)| \leq 1$. Show that:
$$|P\left(-\frac{1}{n}\right)| \leq 2^{n+1} - 1$$
9. Determine all pairs of natural numbers (m, n) with $m, n \geq 3$ such that the expression
$$\frac{a^m + a - 1}{a^n + a^2 - 1}$$
   is an integer for infinitely many natural numbers a.