Combinatorial Games

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Introduction

Combinatorial games form a unique intersection of logic, strategy, and mathematical analysis. These games involve two players taking alternate turns under a strict set of rules, with no elements of chance, such as dice rolls or card draws. The absence of randomness ensures that the outcome depends purely on the players' strategies.

In this lecture, we explore the fundamental characteristics of combinatorial games, which include:

- Perfect information: Both players have full knowledge of the game's current state.
- Deterministic outcomes: No ties or randomness – one player will always have a winning strategy.
- Alternating moves: Players take turns, with each turn potentially shifting the game's state.

By analyzing these games, we employ ideas such as symmetry, positional analysis, and invariants to uncover optimal strategies. Our goal is not merely to determine who has a winning strategy but to understand the reasoning and mathematics underlying these strategies.

Symmetry

Symmetry plays a crucial role in many combinatorial games, allowing us to simplify the problem and predict outcomes. By identifying symmetrical positions or moves, we can often ensure that the opponent is forced to respond in a predictable manner, maintaining the symmetry of the game state. This idea is frequently seen in "copycat strategies," where one player mirrors the other's moves to maintain control of the game. For example, in games with symmetrical boards or configurations, the first player can leverage symmetry to ensure that they always have a move as long as their opponent does.

Positional Analysis

Positional analysis involves classifying game states as either winning or losing for the player whose turn it is. These states are typically divided into two categories:

- **N-positions (Next player wins):** The player whose turn it is has a winning strategy.
- **P-positions (Previous player wins):** The player who just made the last move has a winning strategy.

By understanding these positions, we can devise strategies that guide the game toward favorable outcomes. This method often involves working backward from the endgame to identify critical positions and optimal moves.

Invariants

Invariants are properties of the game that remain unchanged regardless of the moves made. These can include sums, parities, or geometric configurations that provide a framework for understanding the game's progression. By identifying and maintaining invariants, players can systematically constrain their opponent's options and ensure favorable outcomes. Invariants are particularly powerful in games where seemingly complex dynamics can be reduced to simple, unchanging principles.

Through these ideas, we not only solve specific problems but also develop a structured understanding of combinatorial games. This structured approach is essential for tackling more complex games and generalizing strategies to a wide range of problems.

Examples (Optional)

Example 1: When does the first player win?

Initially there are 12 stones on the table. Two players, Shef and Machine, move alternately. Shef always starts. On each turn a player removes either 1, 2, or 3 stones from the table. The winner is the one who takes the last stone.

Solution

Step 1. Play a few smaller games first

A principle we'll come to again and again in our problem solving journey is playing with small cases first. So here instead of playing with 12 stones, let's play with less than 12 stones. When there are $1, 2$, or 3 stone the Shef will win by taking all the stones. Let's now explore this further.

- We have $\underline{4}$ stones.
  - If Shef takes 1, 3 remain and Machine grabs all 3.
  - If Shef takes 2, 2 remain and Machine grabs both.
  - If Shef takes 3, 1 remains and Machine grabs it.

  Result: Shef, i.e., the **player that starts the game** with 4 stones **loses**.
- We have $\underline{5}$ stones.
  - If Shef removes **3** stones, 2 stones remain. Machine can then take the remaining 2 stones and win at once.
  - If Shef removes **2** stones, 3 stones remain. Machine can then take all 3 stones in one move and win the game immediately.
  - If Shef removes **1** stone, 4 stones remain. Now I have a question for you:

    What happens if there are 4 stones left for the Machine?

  This is the crucial insight! The Machine is then left in the same **position** the Shef was in when we had 4 stones. It's as though the game we analyzed with us having 4 stones appeared and the Machine became the first player. What does that mean for the Machine?

  Because we already know in that game that the player who starts loses, so because the Machine will be starter that means they will lose.

This means Shef has a winning strategy for 5 stones. The Shef just takes 1 stone and *knows* they will beat the Machine.

Result: Shef, i.e., the **player that starts the game** with 5 stones **wins**.

- We have 6 stones.
  - If Shef removes **3** stones, 3 stones remain. Machine can then take all 3 stones in one go and win.
  - If Shef removes **1** stone, 5 stones remain. From the previous step we know the starter with 5 stones *wins*. So Machine would win.
  - If Shef removes **2** stones, 4 stones remain.

    What happens if there are 4 stones left for the Machine?

    We already learned the starter with 4 stones *loses*. By leaving exactly 4 stones, Shef forces Machine into that losing position.

  Thus Shef has a winning strategy for 6 stones: remove **2** stones and hand the bad 4-stone pile to Machine. *Result:* Shef, i.e. the **player that starts the game** with 6 stones **wins**.

- We have 7 stones.
  - If Shef removes **3** stones, 4 stones remain. Machine is the starter on 4 stones and loses.
  - If Shef removes **2** stones, 5 stones remain. The starter with 5 stones wins, so Machine would win.
  - If Shef removes **1** stone, 6 stones remain. We just showed the starter with 6 stones wins, so Machine would win.

  Therefore Shef's winning move is to remove **3** stones and leave 4. *Result:* Shef, i.e. the **player that starts the game** with 7 stones **wins**.

- We have 8 stones.
  - If Shef removes **1** stone, 7 stones remain. The player who starts the game with with 7 stones has a winning strategy (see above), so Machine would win.
  - If Shef removes **2** stones, 6 stones remain. The player who starts the game with 6 stones has a winning strategy, so Machine would win.
  - If Shef removes **3** stones, 5 stones remain. The player who starts the game with with 5 stones wins, so Machine would win.

  No matter what Shef does, Machine will be the player who starts the game with on a winning pile. *Result:* Shef, i.e. the **player that starts the game** with 8 stones **loses**.

Now we've played a lot of games and we can basically build up to 12 and finish up.

Step 2. Play a the game with 12 stones

We have 12 stones. Using the pattern we just built (multiples of 4 are games when the first player lost, i. e. the second player had a winning strategy), whatever Shef leaves (11, 10, or 9), Machine can always remove enough stones to hand $\boxed{8}$ back to Shef:

$$11 \xrightarrow{\text{Machine takes 3}} 8 \qquad 10 \xrightarrow{\text{Machine takes 2}} 8 \qquad 9 \xrightarrow{\text{Machine takes 1}} 8$$

Now Shef is the player who starts the game with the number of stones being 8. This is like the mini-game we just analyzed with 8 stones. There we found that the Shef will lose, because the Machine has a winning strategy.

Result: Shef, i.e. the **player that starts the game** with 12 stones **loses**; Machine wins.

What we saw here was what we generally call Positional Analysis, which is just a fancy name for seeing who wins when we're playing a particular instance of a game.

You can also look at this game as having "sub-games" which are smaller instances of the game. Nothing really changes between these instances because the moves lead us to instances of a game with no "memory".

Example 2: Symmetry - yrtemmyS

There is a table with a square top. Two players, Shef and Machine, take turns putting a coin of radius 1 on the table. The player who cannot make a legal move loses the game. The Shef starts first. Who has the winning strategy?

Solution. The Shef places a coin at the center of the table. Thereafter, whenever the Machine places a coin at some position, the Shef places the next coin at the point symmetric to that position with respect to the center of the table. Since each move reduces the available area for future coins, the game must terminate, and by this pairing strategy the Shef always has a move whenever the Machine does. Hence the Shef wins.

Problems

1. Shef and Machine play a game with pebbles located in two piles on a table, one containing x pebbles and the other containing y pebbles. Each player chooses a pile and takes any number of pebbles from it. The player who takes the last pebble from the table wins. Shef plays first. Determine who has a winning strategy if:

   a) $x = 50$, $y = 50$;

   b) $x = 50$, $y = 100$.

2. Two players, Shef and Machine, play the following game. Starting with a positive integer n, they take turns replacing the current number m by $m - k^2$, where k is any positive integer satisfying $k^2 \leq m$. Shef makes the first move. The first player who reaches 0 wins. Who has the winning strategy for:

   (a) $n = 10$

   (b) $n = 45$?

3. Given a regular polygon with 2020 sides. Shef and Machine play a game where Shef plays first. In each move, they draw a diagonal of the polygon that does not intersect any previously drawn diagonal inside the polygon (diagonals may originate from the same vertex, but this is not considered an "inside" intersection). The player who cannot make a move loses. Who has a winning strategy?

4. On the board, the numbers 123 ... 9123 ... 912 ... .8912 are written, with a total of 2018 digits. Shef and Machine play a game, with Shef playing first. In each turn, one of them removes either the first two digits, the last two digits, or the first and last digit. The game ends when a two-digit number remains on the board. Machine wins if this number is divisible by 3, otherwise, Shef wins. Who has a winning strategy? Who has a winning strategy if there are 2022 digits written on the board (the last digit then being 6)?

5. On a circle, n points are given, initially with no two connected. Shef and Machine play a game, with Shef playing first. In each move, they connect two points that are not already directly connected. Determine for which n the first player has a winning strategy and for which the second, if:

   a) The player loses after whose move every point is connected to at least one other point;

   b) The player wins after whose move every point is connected to at least one other point.

6. 10 cards are placed on the table, with each bearing one of the digits 0,1,2, $\ldots$,9. Shef and Machine play a game, with Shef playing first. They take cards alternately, with the cards turned so they can see the numbers on them, until each has taken 5 cards. They arrange these cards from left to right to form a five-digit number (in the first move they must not take a zero).

   a) If the player whose number is divisible by 9 wins, who has a winning strategy? If both numbers are divisible by 9, or both are not, the game ends in a draw.

   b) If Shef wins if her number is divisible by 6, and otherwise Machine wins, who has a winning strategy?

7. The number 1000 is written on the board. Shef and Machine play a game, with Shef playing first and moves are made alternately. In each move, they erase the current number

and write a smaller natural number that does not divide it (i.e., if the current number is x, they erase x and write a natural number $t < x$ such that t does not divide x). The player who cannot make a move loses. Who has a winning strategy?