1 Antiderivatives

An antiderivate $F(x)$ of $f(x)$ is defined by

$$F'(x) = f(x)$$

Notation for $F(x)$:

$$F(x) = \int f(x)\,dx$$

Some basic examples of antiderivatives:

1. $\int e^x\,dx = e^x + c$
2. $\int 5x^2\,dx = \frac{5}{3}x^3 + c$
3. $\int \sec^2(x)\,dx = \tan(x) + c$

List of antiderivates:

1. $\int x\,dx = \frac{x^{n+1}}{n+1} + c$
2. $\int e^x\,dx = e^x + c$
3. $\int \frac{1}{x}\,dx = \ln x + c$
4. $\int n^x\,dx = \frac{n^x}{\ln n} + c$
5. $\int \cos(x)\,dx = \sin(x) + c$
6. $\int \sin(x)\,dx = -\cos(x) + c$
7. $\int \sec^2(x)\,dx = \tan(x) + c$
8. $\int \csc^2(x)\,dx = -\cot(x) + c$
9. $\int \tan(x)\sec(x)\,dx = \sec(x) + c$
10. $\int \cot(x)\csc(x)\,dx = -\csc(x) + c$
11. $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(x) + c$
12. $\int -\frac{1}{\sqrt{1-x^2}}\,dx = \cos^{-1}(x) + c$
13. $\int \frac{1}{1+x^2}\,dx = \tan^{-1}(x) + c$
14. $\int -\frac{1}{1+x^2}\,dx = \cot^{-1}(x) + c$
15. $\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}(x) + c$
16. $\int -\frac{1}{x\sqrt{x^2-1}}\,dx = \csc^{-1}(x) + c$

2 Definite Integrals

The definite integral of $f(x)$ from a to b is

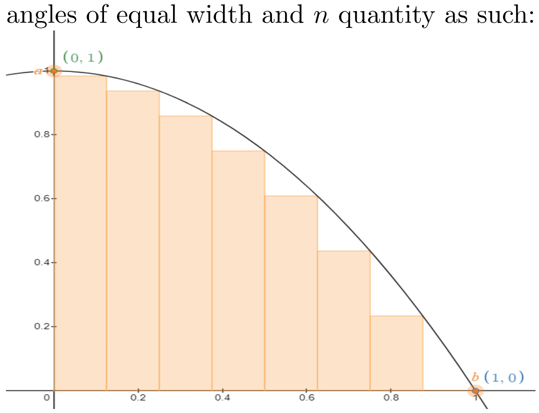
$$\int_a^b f(x)\,dx = \sum_{k=1}^{n} f(c_k)(x_k - x_{k-1})$$

where $\sum_{k=1}^{n} f(c_k)(x_k - x_{k-1})$ is known as the **Riemann Sum**.

Area and Riemann Sum

The Riemann Sum is an approximation of a region's area by adding up areas of multiple slices of the region.

For example, given the graph of $y = 1 - x^2$, we can imagine multiple rectangles of equal width and n quantity as such:



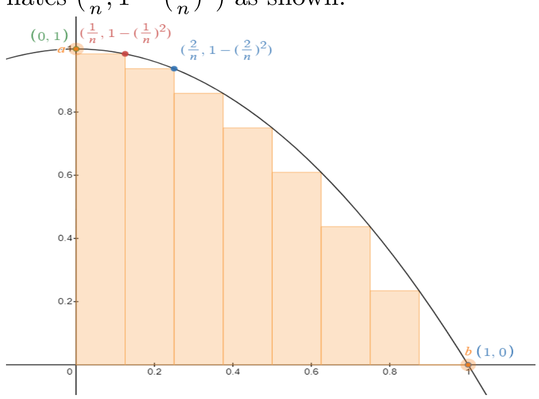
Let the area of the rectangle be represented by

$$\Delta_k A = \text{area of the rectangle} = \text{length} \times \text{width}$$

Since the rectangles are of equal width, we can say that each rectangle has a width of

$$\Delta_k x = \frac{1}{n}$$

We can then imagine each top-right corner of the rectangle to have the coordinates $(\frac{k}{n}, 1 - (\frac{k}{n})^2)$ as shown:



Because of this, the area of each rectangle (for this specific graph) is:

$$\Delta_k A = 1 - (\frac{k}{n})^2 \times \frac{1}{n}$$

And the approximate area of all these rectangles is:

$$\Delta_{\text{Total}} A = \Delta_1 A + \Delta_2 A + \Delta_3 A + ... + \Delta_n A$$

This can be summarized as:

$$\sum_{k=1}^{n} \Delta_k A = \sum_{k=1}^{n} [1 - (\frac{k}{n})^2](\frac{1}{n})$$

Since difference between approximate area and actual area gets smaller as more rectangles are added, we can find the actual area through:

$$\begin{aligned}
A &= \lim_{n\to\infty} \sum_{k=1}^{n} (1 - \frac{k^2}{n^2})(\frac{1}{n}) \\
&= \lim_{n\to\infty} (\frac{1}{n})(\sum_{k=1}^{n} 1 - \sum_{k=1}^{n} \frac{k^2}{n^2}) \\
&= \lim_{n\to\infty} (\frac{1}{n})(n - \frac{1}{n^2} \sum_{k=1}^{n} k^2) && \text{(Since for the summation we only care about } k\text{)} \\
&= \lim_{n\to\infty} (\frac{1}{n})(n - \frac{1}{n^2}(\frac{n(n+1)(2n+1)}{6})) && \text{(due to the sum of squares of natural numbers)} \\
&= \lim_{n\to\infty} (\frac{1}{n})(n - \frac{n(n+1)(2n+1)}{6n}) \\
&= \lim_{n\to\infty} 1 - \frac{(n+1)(2n+1)}{6n^2} \\
&= \lim_{n\to\infty} 1 - \frac{2n^2+3n+1}{6n^2} \\
&= \lim_{n\to\infty} 1 - \frac{2n^2}{6n^2} - \frac{3n}{6n^2} - \frac{1}{6n^2} \\
&= 1 - \frac{1}{3} \\
&= \frac{2}{3}
\end{aligned}$$

Note that using the power rule in integration yields the same result:

$$\int_0^1 1 - x^2\, dx = \left[x - \frac{x^2}{x}\right]_0^1 = 1 - \frac{1}{3} = \frac{2}{3}$$

Properties of Definite Integrals

$$1.\ \int_b^a f(x)\,dx = \lim_{||\Delta||\to 0} \sum_{k=1}^{n} f(c_k)(x_k - x_k) \tag{1}$$

$$2.\ \int_b^a f(x)\,dx = -\int_a^b f(x)\,dx \tag{2}$$

$$3.\ \int_a^a f(x)\,dx = 0 \tag{3}$$

$$4.\ \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \tag{4}$$

$$\tag{5}$$

$$5.\ \text{Let } m \le f(x) \le M \tag{6}$$

$$\forall x \text{ in } [a,b], \text{ then} \tag{7}$$

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a) \tag{8}$$

$$m \le \frac{1}{b-a}\int_a^b f(x)\,dx \le M \qquad \text{(average of } f \text{ over } [a,b]) \tag{9}$$

$$\tag{10}$$

$$6.\ \text{If } f(x) \le g(x) \text{ on } [a,b], \text{ then} \tag{11}$$

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx \tag{12}$$

$$\tag{13}$$

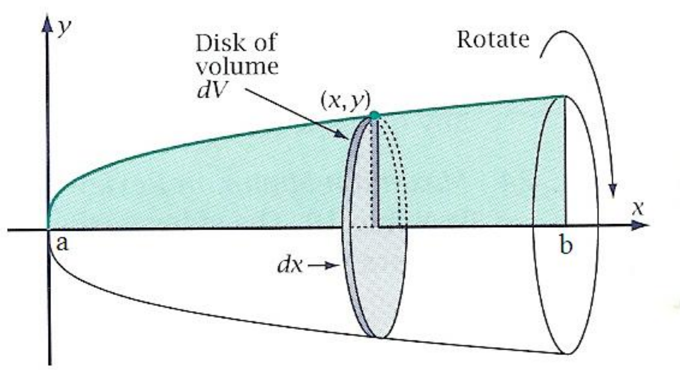
$$7.\ \int_a^b f(x-k)\,dx = \int_{a+k}^{b+k} f(x)\,dx \tag{14}$$

3 Fundamental Theorem of Calculus

4 Volume

Method of Slices

We can imagine an area element $A(x)$ or $A(y)$ that traverses through the solid vertically or horizontally. For example:



where volume is (with $A(x)$ being the area of the area element):

$$V = \int_c^d A(x)\, dx$$

Method of Disc/Washer

$$V = \pi \int_c^d f(x)^2\, dx$$

Method of Shells