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PHYS31.01: Analytical Physics 1

1. **Universal Gravitation Law**

$$|\vec{F}_g = \frac{Gm_1m_2}{r^2}|$$
$$|\vec{F}_e = \frac{k|q_1||q_2|}{r^2}|$$

One-to-one correspondence of Gravitation with Coulomb's Law

$$\textit{where } G = 6.67 * 10^{-11}[\frac{Nm^2}{kg^2}] \qquad \textit{(Gravitation Constant)}$$
$$\textit{where } k = 8.99 * 10^{9}[\frac{Nm^2}{C^2}] \qquad \textit{(Coulomb Constant)}$$

Acceleration due to gravity
↪ gravitation field
↪ Amount of gravitational force exerted by a particle/celestial body per unit mass.

$$|\vec{g}| = \frac{\vec{F}_g}{m_1} = \frac{Gm_2}{r^2}$$

So, on the Earth's surface,

$$|\vec{g}_e| = 9.8[m/s^2] = \frac{Gm_{earth}}{r_{earth}^2}$$

Our weight on the earth's surface is described by:

$$W = mg = m(\frac{Gm_{earth}}{R_e^2})$$

where W is the weight on the earth's surface.

2. **Examples:**

(a) Imagine a system of masses $m_1 = 4m$, $m_2 = 4m$, and $m_3 = m$ where m_1 is located at $(0, 0)$, m_2 is located at $(2a, 0)$, and m_3 is located at $(a, 3a)$.

i. What is the weight of m_3 due to m_1 & m_2? *Horizontal component of $\vec{F}_{1\to3}$ & $\vec{F}_{2\to3}$ vanish*
Distance between m_1 & m_3:

$$|\vec{r}_{1\to3}| = \sqrt{a^2 + 9a^2} = a\sqrt{10}$$

so:

$$|\vec{F}_{1\to3}| = |\vec{F}_{2\to3}| = \frac{G(4m)(m)}{(a\sqrt{10})^2} = (\frac{2}{5})(\frac{Gm^2}{a^2})$$
$$sin\theta = \frac{3}{\sqrt{10}}; cos\theta = \frac{1}{\sqrt{10}}$$
$$\vec{F}_{net,on3} = -\hat{j}(2)|\vec{F}_{1\to3}|sin\theta$$
$$\vec{F}_{net,on3} = -\hat{j}(\frac{4}{5})(\frac{3}{\sqrt{10}})((\frac{Gm^2}{a^2}))$$
$$= -\hat{j}(\frac{12}{5\sqrt{10}})((\frac{Gm^2}{a^2}))$$

ii. What is the gravitational field at point where m_3 is located?

$$\vec{g}_{on3} = \frac{\vec{F}_{net,on3}}{m_3}$$
$$= -\hat{j}\frac{12}{5\sqrt{10}}\frac{Gm}{a^2}$$

3. **Gravitational Potential Energy**
For a flat surface,

$$\vec{g} = -9.8[m/s^2]\hat{j}$$
$$where\ U_g = mgy$$
$$and\ |\vec{F}_g| = m|\vec{g}|.$$

For spherical body,

$$U_g = \frac{-Gm_1m_2}{r}$$
$$and\ |\vec{F}_g| = \frac{Gm_1m_2}{r^2}.$$

To show that the formula $U_g = mgy$ for the assumption of "flat earth surface" appears at the gravitational potential energy formula, we can use the assumption that on the Earth's surface, at some height h above the ground, we set a heigh $h << R_e$.

$$U_g = \frac{-GM_em}{R_e + h} = \frac{-GM_em}{R_e}(\frac{1}{1 + h/R_e})$$

By binomial expansion

$$U_g = \frac{-GM_em}{R_e}(1 - \frac{h}{R_e} + \frac{h^2}{R_e^2} - ...)$$
$$U_g = \frac{-GM_em}{R_e}(1 - \frac{h}{R_e}) \qquad (\textit{because the latter terms are cancellable.})$$
$$U_g = U_0 + mgh$$

By that, we have shown a **one-to-one correspondence** between the two formulas.

What, then is the essence of using the $-Gm_1m_2/r$ formula?
To calculate **escape velocity.**

What is the escape velocity of the Earth?
Use the *Energy Conservation* where the before case is before launch and

after case is when the rocket is at the outer space.

$$\begin{aligned} E_i &= U_i + K_i \\ &= -\frac{GM_e m}{R_e} + \frac{1}{2}mv^2 \end{aligned}$$

$$\begin{aligned} E_f &= U_f + K_f \\ &= 0 + 0 \end{aligned}$$

$$\begin{aligned} E_i &= E_f \\ -\frac{GM_e m}{R_e} + \frac{1}{2}mv^2 &= 0 \\ v &= \sqrt{\frac{2GM_e}{R_e}} \approx 11.2[km/s] \end{aligned}$$

MATH31.2: Mathematical Analysis IB

1. $r = 1 - 3sin\theta$

$$\begin{aligned} r_{max} = 4 &\Rightarrow 4 = 1 - 3sin\theta \\ &\Rightarrow 3 = -3sin\theta \\ &\Rightarrow -1 = sin\theta \\ &\Rightarrow \theta = (\frac{3\pi}{2}) && (or - \frac{\pi}{2}) \end{aligned}$$

$$\begin{aligned} r_{min} = 0 &\Rightarrow 0 = 1 - 3sin\theta \\ &\Rightarrow \frac{1}{3} = sin\theta \\ &\Rightarrow \theta = 0.34 \end{aligned}$$

$$(graph\ is\ a\ limacon\ with\ a\ loop)$$

BASIS:

$$\begin{aligned} 0 < |\frac{a}{b}| < 1 && (limacon\ with\ loop) \\ 1 < |\frac{a}{b}| < 2 && (limacon\ with\ dent) \\ |\frac{a}{b}| \geq 2 && (concave\ dimension) \end{aligned}$$

where: $r = a + bsin\theta$ *or* $r = a + bcos\theta$

2. **Area Under the Polar Curve**

   (a) Using the previous example:

$$A_{innerloop} = (2)(\frac{1}{2})\int_{sin^{-1}1/3}^{\pi/2}(1-3sin\theta)^2 d\theta$$
$$= \int[1-6sin\theta+9sin^2\theta]d\theta$$
$$= \theta+6cos\theta+\frac{9}{2}\int 1-cos2\theta d\theta$$
$$= [\frac{11}{2}\theta+6cos\theta-(\frac{9}{2})(\frac{sin2\theta}{2})]_{sin^{-1}1/3}^{\pi/2}$$

   (b) Set-up the integral for the area of the region inside both $r = 1$ and $r = 2sin\theta$.

$$A = (8)(\frac{1}{2})\int(r_{far}^2 - r_{near}^2)d\theta$$

$$r_{near} = 0$$

$$2sin2\theta = 1$$
$$sin2\theta = 1/2$$
$$\theta = \frac{\pi}{12}$$

$$A = 4\int_0^{\pi/12}((2sin2\theta)^2-(0^2))d\theta + 4\int_{\pi/12}^{\pi/4}((1)^2-(0^2))d\theta$$

3. **Lemniscate**

$$r^2 = acos2\theta \qquad\qquad r^2 = asin2\theta$$

$$
\begin{aligned}
\textit{Example: } r^2 &= 2cos2\theta \\
r &= \sqrt{2}, -\sqrt{2} \\
r^2 &= 2 & \Rightarrow cos2\theta = 1 \\
& & \Rightarrow 2\theta = 0
\end{aligned}
$$

$$
\begin{aligned}
r &= 0 \\
cos2\theta &= 0 \\
\theta &= \frac{\pi}{4}
\end{aligned}
$$

Area of lemniscate:

$$
\begin{aligned}
A &= (4)(\frac{1}{2})\int_0^{\pi/4} 2cos2\theta d\theta \\
&= 4[\frac{sin2\theta}{2}]_0^{\pi/4} \\
&= 2
\end{aligned}
$$