PHYS 112 PSet2

Capin, Mencias, Ong Ante, Ulpindo

October 9, 2024

Problem 1: Errm... What the Sigma?!

Problem:

Evaluate $e^{i\theta_y \hat{\sigma}_y}$ and $e^{i\theta_z \hat{\sigma}_z}$ where $\theta_y, \theta_z \in \mathbb{R}$.

Solution:
These expressions can be easily evaluated the same way $e^{i\theta_x \hat{\sigma}_x}$ was evaluated to be equal to $\begin{pmatrix} cos\theta_x & isin\theta_x \\ isin\theta_x & cos\theta_x \end{pmatrix}$.

First, we can try $e^{i\theta_y \hat{\sigma}_y}$.

$$\begin{aligned} e^{i\theta_y \hat{\sigma}_y} &= \sum_{n=0}^{\infty} \frac{(i\theta_y \hat{\sigma}_y)^n}{n!} \\ &= \sum_{k=0}^{\infty} \left[\frac{(i\theta_y \hat{\sigma}_y)^{2k}}{(2k)!} + \frac{(i\theta_y \hat{\sigma}_y)^{2k+1}}{(2k+1)!} \right] \end{aligned}$$

Where:

$$\begin{aligned} (i\theta_y \hat{\sigma}_y)^2 &= (-1)(\theta_y^2) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ &= -\theta_y^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= -\theta_y^2 1_{2x2}, \end{aligned}$$

and therefore:

$$\begin{aligned} [i\theta_y \hat{\sigma}_y]^{2k} &= (-1)^k \theta_y^{2k} 1_{2x2} \text{ and} \\ [i\theta_y \hat{\sigma}_y]^{2k+1} &= i(-1)^k \theta_y^{2k+1} \hat{\sigma}_y. \end{aligned}$$

Giving us:

$$\begin{aligned}
e^{i\theta_y\hat{\sigma}_y} &= \sum_{k=0}^{\infty}\left[\frac{(i\theta_y\hat{\sigma}_y)^{2k}}{(2k)!} + \frac{(i\theta_y\hat{\sigma}_y)^{2k+1}}{(2k+1)!}\right] \\
&= \sum_{k=0}^{\infty}\left[\frac{(-1)^k\theta_y^{2k}1_{2x2}}{(2k)!} + \frac{i(-1)^k\theta_y^{2k+1}\hat{\sigma}_y}{(2k+1)!}\right] \\
&= \sum_{k=0}^{\infty}\left[\frac{(-1)^k\theta_y^{2k}}{(2k)!}\right]1_{2x2} + i\sum_{k=0}^{\infty}\left[\frac{(-1)^k\theta_y^{2k+1}}{(2k+1)!}\right]\hat{\sigma}_y \\
&= (cos\theta_y)1_{2x2} + i(sin\theta_y)\hat{\sigma}_y \\
&= (cos\theta_y)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + i(sin\theta_y)\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \\
e^{i\theta_y\hat{\sigma}_y} &= \begin{pmatrix}cos\theta_y & sin\theta_y\\ -sin\theta_y & cos\theta_y\end{pmatrix}
\end{aligned}$$

Then, we can try $e^{i\theta_z\hat{\sigma}_z}$.

$$\begin{aligned}
e^{i\theta_z\hat{\sigma}_z} &= \sum_{n=0}^{\infty}\frac{(i\theta_z\hat{\sigma}_z)^n}{n!} \\
&= \sum_{k=0}^{\infty}\left[\frac{(i\theta_z\hat{\sigma}_z)^{2k}}{(2k)!} + \frac{(i\theta_z\hat{\sigma}_z)^{2k+1}}{(2k+1)!}\right]
\end{aligned}$$

Where:

$$\begin{aligned}
(i\theta_z\hat{\sigma}_z)^2 &= (-1)(\theta_z^2)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \\
&= -\theta_z^2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \\
&= -\theta_z^2 1_{2x2},
\end{aligned}$$

and therefore:

$$\begin{aligned}
[i\theta_z\hat{\sigma}_z]^{2k} &= (-1)^k\theta_z^{2k}1_{2x2} \text{ and} \\
[i\theta_z\hat{\sigma}_z]^{2k+1} &= i(-1)^k\theta_z^{2k+1}\hat{\sigma}_z.
\end{aligned}$$

Giving us:

$$
\begin{aligned}
e^{i\theta_z\hat{\sigma}_z} &= \sum_{k=0}^{\infty}\left[\frac{(i\theta_z\hat{\sigma}_z)^{2k}}{(2k)!} + \frac{(i\theta_z\hat{\sigma}_z)^{2k+1}}{(2k+1)!}\right] \\
&= \sum_{k=0}^{\infty}\left[\frac{(-1)^k\theta_z^{2k}1_{2x2}}{(2k)!} + \frac{i(-1)^k\theta_z^{2k+1}\hat{\sigma}_z}{(2k+1)!}\right] \\
&= \sum_{k=0}^{\infty}\left[\frac{(-1)^k\theta_z^{2k}}{(2k)!}\right]1_{2x2} + i\sum_{k=0}^{\infty}\left[\frac{(-1)^k\theta_z^{2k+1}}{(2k+1)!}\right]\hat{\sigma}_z \\
&= (cos\theta_z)1_{2x2} + i(sin\theta_z)\hat{\sigma}_z \\
&= (cos\theta_z)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + i(sin\theta_z)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \\
e^{i\theta_z\hat{\sigma}_z} &= \begin{pmatrix}cos\theta_z + isin\theta_z & 0\\ 0 & cos\theta_z - isin\theta_z\end{pmatrix}
\end{aligned}
$$

Problem 2: Expotentialist Therapy

Problem:

Reduce $e^{i\frac{\theta}{2}\hat{\sigma}_z}\hat{\sigma}_y e^{-i\frac{\theta}{2}\hat{\sigma}_z}$ in terms of superimposed Pauli matrices with no commutators.

Do this using two methods:

a using BCH formula

b using the result of Q1

Solution:
a using BCH formula, such that:

$$\begin{aligned} e^{-\hat{T}}Ae^{\hat{T}} &= \hat{A} + [\hat{A},\hat{T}] + \frac{1}{2!}[[\hat{A},\hat{T}],\hat{T}] + \frac{1}{3!}[[[\hat{A},\hat{T}],\hat{T}],\hat{T}] + ... \\ \textit{where: } \hat{T} &\equiv -i\frac{\theta}{2}\hat{\sigma}_z \textit{ and } \hat{A} \equiv \hat{\sigma}_y \end{aligned}$$

$$\begin{aligned} e^{i\frac{\theta}{2}\hat{\sigma}_z}\hat{\sigma}_y e^{-i\frac{\theta}{2}\hat{\sigma}_z} &= \hat{\sigma}_y + \left[\hat{\sigma}_y, -i\frac{\theta}{2}\hat{\sigma}_z\right] + \frac{1}{2!}\left[\left[\hat{\sigma}_y, -i\frac{\theta}{2}\hat{\sigma}_z\right], -i\frac{\theta}{2}\hat{\sigma}_z\right] + \frac{1}{3!}\left[\left[\left[\hat{\sigma}_y, -i\frac{\theta}{2}\hat{\sigma}_z\right], -i\frac{\theta}{2}\hat{\sigma}_z\right], -i\frac{\theta}{2}\hat{\sigma}_z\right] + ... \\ &= \hat{\sigma}_y + \left(-i\frac{\theta}{2}\right)[\hat{\sigma}_y,\hat{\sigma}_z] + \frac{1}{2!}\left(-i\frac{\theta}{2}\right)^2[[\hat{\sigma}_y,\hat{\sigma}_z],\hat{\sigma}_z] + \frac{1}{3!}\left(-i\frac{\theta}{2}\right)^3[[[\hat{\sigma}_y,\hat{\sigma}_z],\hat{\sigma}_z],\hat{\sigma}_z] + ... \end{aligned}$$

$$\begin{aligned} \textit{where: } [\hat{\sigma}_y,\hat{\sigma}_z] &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = 2i\hat{\sigma}_x \\ \textit{and: } [[\hat{\sigma}_y,\hat{\sigma}_z],\hat{\sigma}_z] &= 2i[\hat{\sigma}_x,\hat{\sigma}_z] = 2i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -2i\hat{\sigma}_y \end{aligned}$$

$$\begin{aligned} \therefore e^{i\frac{\theta}{2}\hat{\sigma}_z}\hat{\sigma}_y e^{-i\frac{\theta}{2}\hat{\sigma}_z} &= \hat{\sigma}_y + \left(-i\frac{\theta}{2}\right)[\hat{\sigma}_y,\hat{\sigma}_z] + \frac{1}{2!}\left(-i\frac{\theta}{2}\right)^2[[\hat{\sigma}_y,\hat{\sigma}_z],\hat{\sigma}_z] + \frac{1}{3!}\left(-i\frac{\theta}{2}\right)^3[[[\hat{\sigma}_y,\hat{\sigma}_z],\hat{\sigma}_z],\hat{\sigma}_z] + ... \\ &= \hat{\sigma}_y + \left(-i\frac{\theta}{2}\right)(2i)(\hat{\sigma}_x) + \left(-i\frac{\theta}{2}\right)^2(2i)(-2i)(\hat{\sigma}_y) + \left(-i\frac{\theta}{2}\right)^3(2i)(-2i)(2i)(\hat{\sigma}_x) + ... \end{aligned}$$

And this can be reduced to...

$$\begin{aligned} &= \sum_{k=0}^{\infty}\left[\left(-i\frac{\theta}{2}\right)^{2k}\left(\frac{2^{2k}}{(2k)!}\right)\right]\hat{\sigma}_y + \sum_{k=0}^{\infty}\left[\left(-i\frac{\theta}{2}\right)^{2k+1}\left(\frac{(-i)2^{2k+1}}{(2k+1)!}\right)\right]\hat{\sigma}_x \textit{ where } k \in \mathbb{Z} \\ &= \hat{\sigma}_y\sum_{k=0}^{\infty}\frac{(-1)^k\theta^{2k}}{(2k)!} - \hat{\sigma}_x\sum_{k=0}^{\infty}\frac{(-1)^k\theta^{2k+1}}{(2k+1)!} \end{aligned}$$

$$\therefore e^{i\frac{\theta}{2}\hat{\sigma}_z}\hat{\sigma}_y e^{-i\frac{\theta}{2}\hat{\sigma}_z} = \hat{\sigma}_y cos(\theta) - \hat{\sigma}_x sin(\theta)$$

(b) Using the result of Q1

$$e^{i\theta_z\hat{\sigma}_z} = \begin{pmatrix} cos\theta_z + isin\theta_z & 0 \\ 0 & cos\theta_z - isin\theta_z \end{pmatrix} \Rightarrow e^{i\frac{\theta}{2}\hat{\sigma}_z} = \begin{pmatrix} cos\left(\frac{\theta}{2}\right) + isin\left(\frac{\theta}{2}\right) & 0 \\ 0 & cos\left(\frac{\theta}{2}\right) - isin\left(\frac{\theta}{2}\right) \end{pmatrix}$$

$$\Rightarrow e^{-i\frac{\theta}{2}\hat{\sigma}_z} = \begin{pmatrix} cos\left(-\frac{\theta}{2}\right) + isin\left(-\frac{\theta}{2}\right) & 0 \\ 0 & cos\left(-\frac{\theta}{2}\right) - isin\left(-\frac{\theta}{2}\right) \end{pmatrix}$$

$$\begin{aligned}
\therefore e^{i\frac{\theta}{2}\hat{\sigma}_z}\hat{\sigma}_y e^{-i\frac{\theta}{2}\hat{\sigma}_z} &= \begin{pmatrix} cos\left(\frac{\theta}{2}\right) + isin\left(\frac{\theta}{2}\right) & 0 \\ 0 & cos\left(\frac{\theta}{2}\right) - isin\left(\frac{\theta}{2}\right) \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} cos\left(-\frac{\theta}{2}\right) + isin\left(-\frac{\theta}{2}\right) & 0 \\ 0 & cos\left(-\frac{\theta}{2}\right) - isin\left(-\frac{\theta}{2}\right) \end{pmatrix} \\
&= \begin{pmatrix} 0 & sin\left(\frac{\theta}{2}\right) - icos\left(\frac{\theta}{2}\right) \\ icos\left(\frac{\theta}{2}\right) + sin\left(\frac{\theta}{2}\right) & 0 \end{pmatrix} \begin{pmatrix} cos\left(-\frac{\theta}{2}\right) + isin\left(-\frac{\theta}{2}\right) & 0 \\ 0 & cos\left(-\frac{\theta}{2}\right) - isin\left(-\frac{\theta}{2}\right) \end{pmatrix} \\
&= \begin{pmatrix} 0 & \left(sin\left(\frac{\theta}{2}\right) - icos\left(\frac{\theta}{2}\right)\right)\left(cos\left(-\frac{\theta}{2}\right) - isin\left(-\frac{\theta}{2}\right)\right) \\ \left(cos\left(-\frac{\theta}{2}\right) + isin\left(-\frac{\theta}{2}\right)\right)\left(icos\left(\frac{\theta}{2}\right) + sin\left(\frac{\theta}{2}\right)\right) & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \left(sin\left(\frac{\theta}{2}\right) - icos\left(\frac{\theta}{2}\right)\right)\left(cos\left(\frac{\theta}{2}\right) + isin\left(\frac{\theta}{2}\right)\right) \\ \left(cos\left(\frac{\theta}{2}\right) - isin\left(\frac{\theta}{2}\right)\right)\left(icos\left(\frac{\theta}{2}\right) + sin\left(\frac{\theta}{2}\right)\right) & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 2sin\left(\frac{\theta}{2}\right)cos\left(\frac{\theta}{2}\right) + isin^2\left(\frac{\theta}{2}\right) - icos^2\left(\frac{\theta}{2}\right) \\ 2sin\left(\frac{\theta}{2}\right)cos\left(\frac{\theta}{2}\right) - isin^2\left(\frac{\theta}{2}\right) + icos^2\left(\frac{\theta}{2}\right) & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & sin(\theta) - icos(\theta) \\ sin(\theta) + icos(\theta) & 0 \end{pmatrix} \\
&= \hat{\sigma}_y cos(\theta) - \hat{\sigma}_x sin(\theta)
\end{aligned}$$

Problem 3: Bejeweled

Problem:

Using the properties of determinants, solve with a minimum of calculation the following equations for x.

a

$$\begin{vmatrix} x & a & a & 1 \\ a & x & b & 1 \\ a & b & x & 1 \\ a & b & c & 1 \end{vmatrix} = 0$$

b

$$\begin{vmatrix} x+2 & x+4 & x-3 \\ x+3 & x & x+5 \\ x-2 & x-1 & x+1 \end{vmatrix} = 0$$

Solution:

a Notice that in the following matrix, the following pairs of rows and columns are almost identical: 3rd and 4th rows; 2nd and 3rd rows; and the 1st and 4th columns. In this problem, we will use the property $det(\hat{(A)}) = 0$, if any pair of rows or columns are identical.

$$\begin{pmatrix} x & a & a & 1 \\ a & x & b & 1 \\ a & b & x & 1 \\ a & b & c & 1 \end{pmatrix}$$

Wherein the equality depends solely on whether the following cases exists: $(x = c)$, $(x = b)$, and $(x = a)$, respectively. All the rest of the pairs of rows and columns are identical with necessary extra conditions. Take for example, 1st and 2nd rows, where aside from $(x = a)$, $(a = b)$ must also be true.
However, given what we've obtained, we can form a cubic equation that represents the values of x where the determinant of the matrix is equal to zero, such that:

$$(x - c)(x - b)(x - a) = 0$$

b In this item, a technique called Gaussian elimination can be used, allowing for a triangular matrix to be obtained and used, such that:

$$\begin{vmatrix} x+2 & x+4 & x-3 \\ x+3 & x & x+5 \\ x-2 & x-1 & x+1 \end{vmatrix}$$

$$= \begin{vmatrix} x+2 & x+4 & x-3 \\ x+3-\left(\frac{x+3}{x+2}\right)x+2 & x-\left(\frac{x+3}{x+2}\right)x+4 & x+5-\left(\frac{x+3}{x+2}\right)x-3 \\ x-2 & x-1 & x+1 \end{vmatrix} \qquad \left(R_2 - \left(\frac{x+3}{x+2}\right)R_1 \rightarrow R_2\right)$$

$$= \begin{vmatrix} x+2 & x+4 & x-3 \\ 0 & \frac{-5x-12}{x+2} & \frac{7x+19}{x+2} \\ x-2 & x-1 & x+1 \end{vmatrix}$$

$$= \begin{vmatrix} x+2 & x+4 & x-3 \\ 0 & \frac{-5x-12}{x+2} & \frac{7x+19}{x+2} \\ x-2-\left(\frac{x-2}{x+2}\right)x+2 & x-1-\left(\frac{x-2}{x+2}\right)x+4 & x+1-\left(\frac{x-2}{x+2}\right)x-3 \end{vmatrix} \qquad \left(R_3 - \left(\frac{x+3}{x+2}\right)R_1 \rightarrow R_3\right)$$

$$= \begin{vmatrix} x+2 & x+4 & x-3 \\ 0 & \frac{-5x-12}{x+2} & \frac{7x+19}{x+2} \\ 0 & \frac{-x+6}{x+2} & \frac{8x-4}{x+2} \end{vmatrix}$$

$$= \begin{vmatrix} x+2 & x+4 & x-3 \\ 0 & \frac{-5x-12}{x+2} & \frac{7x+19}{x+2} \\ 0-\left(\frac{x-6}{5x+12}\right)0 & \frac{-x+6}{x+2}-\left(\frac{x-6}{5x+12}\right)\frac{-5x-12}{x+2} & \frac{8x-4}{x+2}-\left(\frac{x-6}{5x+12}\right)\frac{7x+19}{x+2} \end{vmatrix} \qquad \left(R_3 - \left(\frac{x-6}{5x+12}\right)R_2 \rightarrow R_3\right)$$

$$= \begin{vmatrix} x+2 & x+4 & x-3 \\ 0 & \frac{-5x-12}{x+2} & \frac{7x+19}{x+2} \\ 0 & 0 & \frac{33x+33}{5x+12} \end{vmatrix}$$

Where the determinant of the matrix is simply the product of the entries at the diagonal, such that:

$$\begin{vmatrix} x+2 & x+4 & x-3 \\ x+3 & x & x+5 \\ x-2 & x-1 & x+1 \end{vmatrix} = \begin{vmatrix} x+2 & x+4 & x-3 \\ 0 & \frac{-5x-12}{x+2} & \frac{7x+19}{x+2} \\ 0 & 0 & \frac{33x+33}{5x+12} \end{vmatrix} \qquad = (x+2)\left(\frac{-5x-12}{x+2}\right)\left(\frac{33x+33}{5x+12}\right)$$

$$= -33x - 33$$

Since, the above equation is also equal to zero, then:

$$-33x - 33 = 0$$
$$x = -1$$

Problem 4: "Shikai." - Shinji Hirako

Problem:
The elements of matrix $\hat{K}$ is given by

$$K_{ij} = a_i b_j \qquad \text{(where } a_i \text{ and } b_j \text{ are real numbers)}$$

(a) Construct a 2X2 matrix $\hat{S}$

$$\hat{S} = (\mathbb{1}_2 + i\hat{K})(\mathbb{1}_2 - i\hat{K})^{-1}$$

What are the elements of $\hat{S}$ in terms of a_1, a_2, b_1, and b_2?

(b) What is $\hat{S}^-1$ in terms of $\hat{K}$? Show that $\hat{S}^{-1}\hat{S} = \mathbb{1}_2$ explicitly.

Solution:
(a) We can first start by determining the elements of matrix $\hat{K}$ in terms of a_i and b_j:

$$\hat{K} = \begin{pmatrix} a_1b_1 & a_1b_2 \\ a_2b_1 & a_2b_2 \end{pmatrix}$$

$\hat{S}$ then becomes:

$$\begin{aligned}\hat{S} &= \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i\begin{pmatrix} a_1b_1 & a_1b_2 \\ a_2b_1 & a_2b_2 \end{pmatrix}\right)\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\begin{pmatrix} a_1b_1 & a_1b_2 \\ a_2b_1 & a_2b_2 \end{pmatrix}\right)^{-1} \\ &= \begin{pmatrix} 1 + ia_1b_1 & ia_1b_2 \\ ia_2b_1 & 1 + ia_2b_2 \end{pmatrix}\left(\begin{pmatrix} 1 - ia_1b_1 & -ia_1b_2 \\ -ia_2b_1 & 1 - ia_2b_2 \end{pmatrix}\right)^{-1}\end{aligned}$$

We can first try to find the inverse of the second factor.

$$\begin{pmatrix} 1 - ia_1b_1 & -ia_1b_2 \\ -ia_2b_1 & 1 - ia_2b_2 \end{pmatrix}^{-1} = \frac{\operatorname{adj}\begin{pmatrix} 1 - ia_1b_1 & -ia_1b_2 \\ -ia_2b_1 & 1 - ia_2b_2 \end{pmatrix}}{\det\begin{pmatrix} 1 - ia_1b_1 & -ia_1b_2 \\ -ia_2b_1 & 1 - ia_2b_2 \end{pmatrix}}$$

The determinant is

$$\begin{aligned}&(1 - ia_1b_1)(1 - ia_2b_2) - (-ia_1b_2)(-ia_2b_1) \\ &= 1 + (a_1b_1 - a_2b_2)i\end{aligned}$$

While the adjugate is

$$\begin{aligned}&\begin{pmatrix} 1 - ia_2b_2 & (-1) - ia_2b_1 \\ (-1) - ia_1b_2 & 1 - ia_1b_1 \end{pmatrix}^T \\ &= \begin{pmatrix} 1 - ia_2b_2 & ia_2b_1 \\ ia_1b_2 & 1 - ia_1b_1 \end{pmatrix}^T \\ &= \begin{pmatrix} 1 - ia_2b_2 & ia_1b_2 \\ ia_2b_1 & 1 - ia_1b_1 \end{pmatrix}\end{aligned}$$

Giving us

$$\hat{S} = \begin{pmatrix} 1 + ia_1b_1 & ia_1b_2 \\ ia_2b_1 & 1 + ia_2b_2 \end{pmatrix} \left(\frac{1}{1 + (a_1b_1 - a_2b_2)i} \begin{pmatrix} 1 - ia_2b_2 & ia_1b_2 \\ ia_2b_1 & 1 - ia_1b_1 \end{pmatrix} \right)$$

Moving the scalar quantity to the left side:

$$\hat{S} = \left(\frac{1}{1 + (a_1b_1 - a_2b_2)i} \right) \begin{pmatrix} 1 + ia_1b_1 & ia_1b_2 \\ ia_2b_1 & 1 + ia_2b_2 \end{pmatrix} \begin{pmatrix} 1 - ia_2b_2 & ia_1b_2 \\ ia_2b_1 & 1 - ia_1b_1 \end{pmatrix}$$

Multiplying the matrices gives us:

$$\begin{aligned}
\hat{S} &= \left(\frac{1}{1 + (a_1b_1 - a_2b_2)i} \right) \begin{pmatrix} (1 + ia_1b_1)(1 - ia_2b_2) + (ia_1b_2)(ia_2b_1) & (1 + ia_1b_1)(ia_1b_2) + (ia_1b_2)(1 - ia_1b_1) \\ (1 - ia_2b_2)(ia_2b_1) + (1 + ia_2b_2)(ia_2b_1) & (ia_2b_1)(ia_1b_2) + (1 + ia_2b_2)(1 - ia_1b_1) \end{pmatrix} \\
&= \left(\frac{1}{1 + (a_1b_1 - a_2b_2)i} \right) \begin{pmatrix} 1 + (a_1b_1 - a_2b_2)i & 2a_1b_2i \\ 2a_2b_1i & 1 + (a_2b_2 - a_1b_1)i \end{pmatrix} \\
\hat{S} &= \begin{pmatrix} 1 & \frac{2a_1b_2i}{1+(a_1b_1-a_2b_2)i} \\ \frac{2a_2b_1i}{1+(a_1b_1-a_2b_2)i} & \frac{1+(a_2b_2-a_1b_1)i}{1+(a_1b_1-a_2b_2)i} \end{pmatrix}
\end{aligned}$$

(b) We first express $\hat{S}$ in terms of K:

$$\hat{S} = \left(\frac{1}{1 + (K_{11} - K_{22})i} \right) \begin{pmatrix} 1 + (K_{11} - K_{22})i & 2K_{12}i \\ 2K_{21}i & 1 + (K_{22} - K_{11})i \end{pmatrix}$$

Meanwhile, for $\hat{S}^{-1}$ we can go back to our original expression for $\hat{S}$:

$$\hat{S} = (\mathbb{1}_2 + i\hat{K})(\mathbb{1}_2 - i\hat{K})^{-1}$$

Remember that $(\hat{A}\hat{B})^{-1} = \hat{B}^{-1}\hat{A}^{-1}$. Because of this, then

$$\hat{S}^{-1} = (\mathbb{1}_2 - i\hat{K})(\mathbb{1}_2 + i\hat{K})^{-1}$$

We can then bring out the components of $\hat{K}$:

$$\begin{aligned}
\hat{S}^{-1} &= \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \right) \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \right)^{-1} \\
&= \begin{pmatrix} 1 - iK_{11} & -iK_{12} \\ -iK_{21} & 1 - iK_{22} \end{pmatrix} \begin{pmatrix} 1 + iK_{11} & iK_{12} \\ iK_{21} & 1 + iK_{22} \end{pmatrix}^{-1}
\end{aligned}$$

Then we do the same steps to find the inverse of the second matrix. The determinant is

$$\begin{aligned}
&(1 + K_{11}i)(1 + K_{22}i) - (iK_{12})(iK_{21}) \\
&= 1 + (K_{11} + K_{22})i - K_{11}K_{22} + K_{12}K_{21} \\
&= 1 + (K_{11} + K_{22})i \qquad \text{(Because } K_{ij} = a_ib_j\text{)}
\end{aligned}$$

While the adjugate is

$$= \begin{pmatrix} 1 - K_{22}i & -K_{12}i \\ -K_{21}i & 1 - K_{11}i \end{pmatrix}$$

Giving us

$$\begin{aligned}
\hat{S}^{-1} &= \left(\frac{1}{1 + (K_{11} + K_{22})i}\right) \begin{pmatrix} 1 - iK_{11} & -iK_{12} \\ -iK_{21} & 1 - iK_{22} \end{pmatrix} \begin{pmatrix} 1 - K_{22}i & -K_{12}i \\ -K_{21}i & 1 - K_{11}i \end{pmatrix} \\
&= \left(\frac{1}{1 + (K_{11} + K_{22})i}\right) \begin{pmatrix} 1 + (K_{22} - K_{11})i & -2K_{12}i \\ -2K_{21}i & 1 + (K_{11} - K_{22})i \end{pmatrix}
\end{aligned}$$

Finally, we can multiply $\hat{S}^{-1}\hat{S}$:
(Note: Some steps are not shown here for brevity. Remember that $K_{ij} = a_i b_j$, so it means that $K_{12}K_{21} = K_{11}K_{22}$.)

$$\begin{aligned}
\hat{S}^{-1}\hat{S} &= \left(\frac{1}{1 + (K_{11} + K_{22})i}\right) \begin{pmatrix} 1 + (K_{22} - K_{11})i & -2K_{12}i \\ -2K_{21}i & 1 + (K_{11} - K_{22})i \end{pmatrix} \\
&\left(\frac{1}{1 + (K_{11} - K_{22})i}\right) \begin{pmatrix} 1 + (K_{11} - K_{22})i & 2K_{12}i \\ 2K_{21}i & 1 + (K_{22} - K_{11})i \end{pmatrix} \\
&= \left(\frac{1}{1 + (K_{11} + K_{22})^2}\right) \begin{pmatrix} 1 + K_{11}^2 + 2K_{11}K_{22} + K_{22}^2 & 0 \\ 0 & 1 + K_{11}^2 + 2K_{11}K_{22} + K_{22}^2 \end{pmatrix} \\
&= \left(\frac{1}{1 + K_{11}^2 + 2K_{11}K_{22} + K_{22}^2}\right) \begin{pmatrix} 1 + K_{11}^2 + 2K_{11}K_{22} + K_{22}^2 & 0 \\ 0 & 1 + K_{11}^2 + 2K_{11}K_{22} + K_{22}^2 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \mathbb{1}_2
\end{aligned}$$

Problem 5: Illuminati Confirmed

Problem:
Show that the eigenvalue equation for every triangular matrix has the form $(\lambda - A_{11})(\lambda - A_{22})...(\lambda - A_{NN})$

Solution:
First, we want to be able to show that the determinant of every triangular matrix is just the product of its diagonal entries. We can prove this by induction.
For our base case, we will use a $3x3$ triangular matrix:

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{pmatrix}$$

We can then use the Laplace expansion definition of a determinant. Note that we will be using the 1st column ($j = 1$) to make this easier:

$$\begin{aligned} \det(A) &= \sum_{i=1}^{3} (-1)^{i+j} A_{ij} M_{ij} \\ &= (-1)^{1+1} A_{11} M_{11} && \text{(The last two iterations are 0)} \\ &= (-1)^{1+1} A_{11} \cdot \det\left(\begin{pmatrix} A_{22} & A_{23} \\ 0 & A_{33} \end{pmatrix}\right) \\ &= A_{11}(A_{22}A_{33} - 0(A_23)) \\ &= A_{11}A_{22}A_{33} \end{aligned}$$

We can then assume that $\det A = A_{11}A_{22}\ldots A_{NN}$ where A is a NxN matrix.
Then let A' be a $N + 1$x$N + 1$ matrix, such that

$$A' = \begin{pmatrix} A_{11} & A_{12} & \ldots & A_{1(N+1)} \\ 0 & A_{22} & \ldots & A_{2(N+1)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & A_{(N+1)(N+1)} \end{pmatrix}$$

We can then take a look at its determinant:

$$\begin{aligned} \det(A') &= \sum_{i=1}^{N+1} (-1)^{i+j} A_{ij} M_{ij} \\ &= (-1)^{1+1} A_{11} M_{11} && \text{(The next iterations are 0)} \end{aligned}$$

We know $M_{11} = A_{22}A_{33}\ldots A_{(N+1)(N+1)}$ thanks to the induction hypothesis, so

$$\det(A') = A_{11}A_{22}A_{33}\ldots A_{(N+1)(N+1)}$$

Through induction, this proves the hypothesis.

This also works if the triangular matrix is flipped vertically, as we would just need to use the Laplace expansion on the first row instead.

We know that the eigenvalue equation of any matrix is given by

$$\det(A - \lambda \mathbb{1}) = 0$$

Given a triangular matrix A with dimension NxN, then we can use our previous proven statement:

$$(A_{11} - \lambda)(A_{22} - \lambda)(A_{33} - \lambda)\dots(A_{NN} - \lambda) = 0$$

We can then factor out (-1) to get our final form:

$$\begin{aligned}(-1)^N(\lambda - A_{11})(\lambda - A_{22})(\lambda - A_{33})\dots(\lambda - A_{NN}) &= 0 \\ (\lambda - A_{11})(\lambda - A_{22})(\lambda - A_{33})\dots(\lambda - A_{NN}) &= 0 && \text{(Transpose } (-1)^N\text{)}\end{aligned}$$

Problem 6: OKay...

Problem:

Given the matrix

$$\hat{A} = \begin{pmatrix} 1 & k & 3 \\ -k & 2 & -k \\ 1 & k & 3 \end{pmatrix}$$

(a) Find all the real possible values of k such that all the eigenvalues of $\hat{A}$ are real.

(b) With k being real, what are the corresponding eigenvectors?

Solution:
(a) We can first try to find the eigenvalue equation for $(\hat{A} - \lambda\mathbb{1})$ in terms of k. For this solution, we can use the Laplace expansion on the first row.

$$\begin{aligned} \det(\hat{A} - \lambda\mathbb{1}) &= (1-\lambda)[(2-\lambda)(3-\lambda) - (-k)(k)] \\ &+ (-1)(k)[(-k)(3-\lambda) - (-k)(1)] \\ &+ 3[(-k)(k) - (2-\lambda)(1)] = 0 \\ &\text{Through the magic powers of simplification...} \\ &= -\lambda^3 + 6\lambda^2 + (-8-2k^2)\lambda = 0 \\ &= (-\lambda)(\lambda^2 - 6\lambda + (8+2k^2)) = 0 \end{aligned}$$

For this section we only care about k, so we only take a look at the second factor as the first one does not have k.

$$\lambda^2 - 6\lambda + (8+2k^2) = 0$$

We can then find the value of λ in terms of k through the quadratic formula.

$$\lambda = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(8+2k^2)}}{2(1)}$$

Our eigenvalues only become real when $\sqrt{(-6)^2 - 4(1)(8+2k^2)}$ is real. For this to happen, we want the value inside the square root to be equal to or greater than 0.

$$\begin{aligned} (-6)^2 - 4(1)(8+2k^2) &\geq 0 \\ 36 - 4(8+2k^2) &\geq 0 \\ 36 - 32 - 8k^2 &\geq 0 \\ 8k^2 &\leq 4 \\ k^2 &\leq \frac{1}{2} \\ |k| &\leq \frac{1}{\sqrt{2}} \end{aligned}$$

From this, we can then say that for all the eigenvalues of $\hat{A}$ to be real, then it needs to be in the interval of $-\frac{1}{\sqrt{2}} \leq k \leq \frac{1}{\sqrt{2}}$.

(b) Going back to our eigenvalue equation:

$$(-\lambda)(\lambda^2 - 6\lambda + (8 + 2k^2)) = 0$$

One possible value for λ is 0 (since $-\lambda = 0 \Rightarrow \lambda = 0$), which we will exclude from the possible eigenvalues. Once again, taking a look at the other factor once again brings us back to the quadratic formula, which we can first simplify:

$$\begin{aligned} \lambda &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(8 + 2k^2)}}{2(1)} \\ &= \frac{6 \pm \sqrt{4 - 8k^2}}{2} \end{aligned}$$

To make things easier, we can just express the eigenvalue as λ first:

$$\begin{pmatrix} 1 & k & 3 \\ -k & 2 & -k \\ 1 & k & 3 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \lambda \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

Turning it into a system of equations:

$$\begin{aligned} 1\alpha + k\beta + 3\gamma &= \lambda\alpha \\ -k\alpha + 2\beta - k\gamma &= \lambda\beta \\ 1\alpha + k\beta + 3\gamma &= \lambda\gamma \end{aligned}$$

This implies that

$$\begin{aligned} \lambda\alpha &= \lambda\gamma \\ \alpha &= \gamma \end{aligned}$$

Leaving us with eigenvector $\begin{pmatrix} \alpha \\ \beta \\ \alpha \end{pmatrix}$ and

$$\begin{aligned} 1\alpha + k\beta + 3\alpha = \lambda\alpha \to 4\alpha + k\beta = \lambda\alpha \to (4 - \lambda)\alpha + k\beta = 0 \\ -k\alpha + 2\beta - k\alpha = \lambda\beta \to -2k\alpha + 2\beta = \lambda\beta \to -2k\alpha + (2 - \lambda)\beta = 0 \end{aligned}$$

Then:

$$\begin{aligned} (4 - \lambda)\alpha + k\beta &= -2k\alpha + (2 - \lambda)\beta \\ (4 - \lambda + 2k)\alpha &= (-k + 2 - \lambda)\beta \\ \beta &= \frac{4 - \lambda + 2k}{-k + 2 - \lambda}\alpha \end{aligned}$$

Which then gives us the eigenvector

$$|x\rangle = \alpha \begin{pmatrix} 1 \\ \frac{4-\lambda+2k}{-k+2-\lambda} \\ 1 \end{pmatrix}$$

Then by normalizing $|x\rangle$, we can then find α:

$$\alpha = \sqrt{\frac{-k+2-\lambda}{4-\lambda+2k}}$$

so that

$$|x\rangle = \sqrt{\frac{-k+2-\lambda}{4-\lambda+2k}} \begin{pmatrix} 1 \\ \frac{4-\lambda+2k}{-k+2-\lambda} \\ 1 \end{pmatrix}$$

Then finally, we can substitute λ to get the corresponding eigenvectors for any real value k:

$$|x\rangle = \sqrt{\frac{-k+2-(\frac{6\pm\sqrt{4-8k^2}}{2})}{4-(\frac{6\pm\sqrt{4-8k^2}}{2})+2k}} \begin{pmatrix} 1 \\ \frac{4-(\frac{6\pm\sqrt{4-8k^2}}{2})+2k}{-k+2-(\frac{6\pm\sqrt{4-8k^2}}{2})} \\ 1 \end{pmatrix}$$

Problem 7: Sisyphus, Part One

Problem:
Verify that the eigenvectors are orthogonal for the normal matrix

$$\hat{A} = \begin{pmatrix} 1 & 1+i & -1 \\ -1 & 1 & 1+i \\ 1+i & -1 & 1 \end{pmatrix}$$

Finding the determinant of $(\hat{A} - \lambda \mathbb{1}_2)$:

$$\begin{aligned} \det(\hat{A} - \lambda \mathbb{1}_2) &= (1-\lambda)[(1-\lambda)^2 - (-1)(1+i)] \\ &\quad - (1+i)[(-1)(1-\lambda) - (1+i)^2] \\ &\quad + (-1)[1 - (1+i)(1-\lambda)] = 0 \\ \\ &= -\lambda^3 + 3\lambda^2 + (-6-3i)\lambda + (1+5i) = 0 \\ &= \lambda^3 - 3\lambda^2 + (6+3i)\lambda - (1+5i) = 0 \end{aligned}$$

Through the magical powers of trial and error, we find that one of the factors is $\lambda_1 = 1+i$. Using factor theorem, we can then turn the expression into

$$(\lambda + (-1-i))(\lambda^2 + (-2+i)\lambda + (3+2i)) = 0$$

Using the quadratic formula:

$$\lambda = \frac{-(-2+i) \pm \sqrt{(-2+i)^2 - 4(3+2i)}}{2(1)}$$

$$\lambda = \frac{2 - i \pm \sqrt{3}\sqrt{-3-4i}}{2}$$

Through the expression $\sqrt{-3-4i} = x + iy$, we can determine that the square root is $(1-2i)$.

$$\lambda = \frac{2 - i \pm \sqrt{3}(1-2i)}{2}$$

We then get our other two eigenvalues:

$$\lambda_2 = \frac{2 - \sqrt{3} + (2\sqrt{3} - 1)i}{2}$$

$$\lambda_3 = \frac{2 + \sqrt{3} - (-1 - 2\sqrt{3})i}{2}$$

Now we can find their corresponding eigenvectors.
For $\lambda = 1+i$:

$$\begin{pmatrix} 1 & 1+i & -1 \\ -1 & 1 & 1+i \\ 1+i & -1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = (1+i) \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\begin{aligned}a+(1+i)b-c&=(1+i)a\\-a+b+(1+i)c&=(1+i)b\\(1+i)a-b+c&=(1+i)c\end{aligned}$$

We end up with

$$a=b=c$$

Which gives us an eigenvector of

$$a\begin{pmatrix}1\\1\\1\end{pmatrix}$$

For $\lambda=\frac{2-\sqrt{3}+(2\sqrt{3}-1)i}{2}$:

$$\begin{pmatrix}1&1+i&-1\\-1&1&1+i\\1+i&-1&1\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}=\frac{2-\sqrt{3}+(2\sqrt{3}-1)i}{2}\begin{pmatrix}a\\b\\c\end{pmatrix}$$

$$\begin{aligned}a+(1+i)b-c&=\frac{2-\sqrt{3}+(2\sqrt{3}-1)i}{2}a\\-a+b+(1+i)c&=\frac{2-\sqrt{3}+(2\sqrt{3}-1)i}{2}b\\(1+i)a-b+c&=\frac{2-\sqrt{3}+(2\sqrt{3}-1)i}{2}c\end{aligned}$$

We end up with

$$\begin{aligned}a&=\frac{-1+\sqrt{3}i}{2}c\\b&=\frac{-1-\sqrt{3}i}{2}c\end{aligned}$$

Which gives us an eigenvector of

$$c\begin{pmatrix}\frac{-1+\sqrt{3}i}{2}\\\frac{-1-\sqrt{3}i}{2}\\1\end{pmatrix}$$

For $\lambda=\frac{2+\sqrt{3}-(-1-2\sqrt{3})i}{2}$:

$$\begin{pmatrix}1&1+i&-1\\-1&1&1+i\\1+i&-1&1\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}=\frac{2+\sqrt{3}-(-1-2\sqrt{3})i}{2}\begin{pmatrix}a\\b\\c\end{pmatrix}$$

$$
\begin{aligned}
a + (1+i)b - c &= \frac{2+\sqrt{3}-(-1-2\sqrt{3})i}{2}a \\
-a + b + (1+i)c &= \frac{2+\sqrt{3}-(-1-2\sqrt{3})i}{2}b \\
(1+i)a - b + c &= \frac{2+\sqrt{3}-(-1-2\sqrt{3})i}{2}c
\end{aligned}
$$

We end up with

$$
\begin{aligned}
a &= \frac{-1-\sqrt{3}i}{2}c \\
b &= \frac{-1+\sqrt{3}i}{2}c
\end{aligned}
$$

Which gives us an eigenvector of

$$
c\begin{pmatrix} \frac{-1-\sqrt{3}i}{2} \\ \frac{-1+\sqrt{3}i}{2} \\ 1 \end{pmatrix}
$$

So we get the final eigenvectors

$$
\begin{aligned}
|x_1\rangle &= a\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\
|x_2\rangle &= c\begin{pmatrix} \frac{-1+\sqrt{3}i}{2} \\ \frac{-1-\sqrt{3}i}{2} \\ 1 \end{pmatrix} \\
|x_3\rangle &= c\begin{pmatrix} \frac{-1-\sqrt{3}i}{2} \\ \frac{-1+\sqrt{3}i}{2} \\ 1 \end{pmatrix}
\end{aligned}
$$

We can then verify they they are indeed orthogonal:

$$
\begin{aligned}
\langle (x_1)^T | x_2 \rangle &= (1)\frac{-1+\sqrt{3}i}{2} + (1)\frac{-1-\sqrt{3}i}{2} + (1)(1) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\langle (x_1)^T | x_3 \rangle &= (1)\frac{-1-\sqrt{3}i}{2} + (1)\frac{-1+\sqrt{3}i}{2} + (1)(1) \\
&= 0
\end{aligned}
$$

Problem 8: Sisyphus, Part Two

Problem:
Verify that $\hat{A}^\dagger$ in Q7 has the same set of eigenvectors but its eigenvalues are complex conjugates of that in $\hat{A}$.
Solution:
The hermitian conjugate is

$$\hat{A}^\dagger = \begin{pmatrix} 1 & -1 & 1-i \\ 1-i & 1 & -1 \\ -1 & 1-i & 1 \end{pmatrix}$$

Finding the eigenvalues:

$$\begin{aligned} \det(\hat{A}^\dagger - \lambda \mathbb{1}_2) &= (1-\lambda)[(1-\lambda)^2 - (-1)(1-i)] \\ &+ [(1-i)(1-\lambda) - (-1)^2] \\ &+ (1-i)[(1-i)^2 - (-1)(1-\lambda)] = 0 \\ &\lambda^3 - 3\lambda^2 + (6-3i)\lambda - (1-5i) = 0 \end{aligned}$$

Notice that the values of the non-λ numbers are the complex conjugates of their corresponding values in $\hat{A}$.

Our eigenvalues would be

$$\begin{aligned} \lambda_1 &= 1-i \\ \lambda_2 &= \frac{2-\sqrt{3}-(2\sqrt{3}-1)i}{2} \\ \lambda_3 &= \frac{2+\sqrt{3}+(-1-2\sqrt{3})i}{2} \end{aligned}$$

Then if we then express the eigenvector equation into a system of equations...
For λ_1:

$$\begin{aligned} a - b - (1-i)c &= (1-i)a \\ (1-i)a + b - c &= (1-i)b \\ -a - (1-i)b + c &= (1-i)c \end{aligned}$$

We get

$$a = b = c$$

For λ_2:

$$\begin{aligned} a - b - (1-i)c &= \frac{2-\sqrt{3}-(2\sqrt{3}-1)i}{2}a \\ (1-i)a + b - c &= \frac{2-\sqrt{3}-(2\sqrt{3}-1)i}{2}b \\ -a - (1-i)b + c &= \frac{2-\sqrt{3}-(2\sqrt{3}-1)i}{2}c \end{aligned}$$

We get

$$a = \frac{-1+\sqrt{3}i}{2}c$$
$$b = \frac{-1-\sqrt{3}i}{2}c$$

For λ_3:

$$a - b - (1-i)c = \frac{2+\sqrt{3}+(-1-2\sqrt{3})i}{2}a$$
$$(1-i)a + b - c = \frac{2+\sqrt{3}+(-1-2\sqrt{3})i}{2}b$$
$$-a - (1-i)b + c = \frac{2+\sqrt{3}+(-1-2\sqrt{3})i}{2}c$$

We get

$$a = \frac{-1-\sqrt{3}i}{2}c$$
$$b = \frac{-1+\sqrt{3}i}{2}c$$

We end up with the same relationships that we found in Q7, which essentially means that we get the same set of eigenvectors.

Problem 9: Pauli's Friends

Problem:
Revisit our old friends $\{\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z\}$
a) Check if they are Hermitian.

b) Obtain their eigenvalues and eigenvectors.
Solution:
(a) We take a look at the transpose and the hermitian conjugate.

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; \hat{\sigma}_x^T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; \hat{\sigma}_x^\dagger = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \; \hat{\sigma}_y^T = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \; \hat{\sigma}_y^\dagger = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
$$\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \; \hat{\sigma}_z^T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \; \hat{\sigma}_z^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

As the Hermitian conjugates are the same with the original matrices, we have shown that they are Hermitian.
(b) First, let us start with the eigenvalues.
For $\hat{\sigma}_x$:

$$\begin{aligned} \det(\hat{\sigma}_x) &= \det(\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix}) \\ \lambda^2 - 1 &= 0 \\ \lambda^2 &= 1 \\ \lambda &= \pm 1 \end{aligned}$$

For $\hat{\sigma}_y$:

$$\begin{aligned} \det(\hat{\sigma}_y) &= \det(\begin{pmatrix} -\lambda & -i \\ i & -\lambda \end{pmatrix}) \\ \lambda^2 - 1 &= 0 \\ \lambda^2 &= 1 \\ \lambda &= \pm 1 \end{aligned}$$

For $\hat{\sigma}_z$:

$$\begin{aligned} \det(\hat{\sigma}_z) &= \det(\begin{pmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{pmatrix}) \\ (1-\lambda)(-1-\lambda) &= 0 \\ \lambda &= \pm 1 \end{aligned}$$

Then we can find the eigenvectors.
For $\hat{\sigma}_x$ We start with $\lambda = 1$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\beta = \alpha$$
$$\alpha = \beta$$

This gives us

$$\alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Then with $\lambda = -1$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = - \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\beta = -\alpha$$
$$\alpha = -\beta$$

This gives us

$$\alpha \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

For $\hat{\sigma}_y$ We start with $\lambda = 1$

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$-\beta i = \alpha$$
$$\alpha i = \beta$$

This gives us

$$\alpha \begin{pmatrix} 1 \\ i \end{pmatrix}$$

Then with $\lambda = -1$

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = - \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$-\beta i = -\alpha$$
$$\alpha i = -\beta$$

This gives us

$$\alpha \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

Lastly $\hat{\sigma}_z$ We start with $\lambda = 1$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\begin{aligned} \alpha &= \alpha \\ -\beta &= \beta \end{aligned}$$

Since this implies $\beta = 0$, this gives us the eigenvector

$$\alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Then with $\lambda = -1$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = - \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\begin{aligned} \alpha &= -\alpha \\ -\beta &= -\beta \end{aligned}$$

Since this implies $\alpha = 0$, this gives us the eigenvector

$$\beta \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Problem 10: unitaream

Given the unitary operator

$$\hat{U} = \exp(-i\theta_y \hat{\sigma}_y)$$

1. Show that $\hat{U}$ is unitary.
2. Let:

$$|v_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

and consider the vector $|v_2\rangle = \hat{U}\,|v_1\rangle$. Demonstrate that:

a) $|v_1\rangle \neq |v_2\rangle$,

b) The norm of $|v_1\rangle$ is preserved under the unitary transformation.

Solution

1. Proving that $\hat{U}$ is unitary

The operator $\hat{U}$ is given by:

$$\hat{U} = \exp(-i\theta_y \hat{\sigma}_y)$$

We need to show that $\hat{U}$ satisfies the unitary condition:

$$\hat{U}^\dagger \hat{U} = I$$

First, recall that $\hat{\sigma}_y$ is Hermitian ($\hat{\sigma}_y^\dagger = \hat{\sigma}_y$) and satisfies the identity $\hat{\sigma}_y^2 = I$. Using the matrix exponential's series expansion:

$$\hat{U} = \exp(-i\theta_y \hat{\sigma}_y) = I\cos(\theta_y) - i\hat{\sigma}_y \sin(\theta_y)$$

The Hermitian conjugate of $\hat{U}$ is:

$$\hat{U}^\dagger = I\cos(\theta_y) + i\hat{\sigma}_y \sin(\theta_y)$$

Now, compute $\hat{U}^\dagger \hat{U}$:

$$\hat{U}^\dagger \hat{U} = (I\cos(\theta_y) + i\hat{\sigma}_y \sin(\theta_y))(I\cos(\theta_y) - i\hat{\sigma}_y \sin(\theta_y))$$

This expands to:

$$= I\cos^2(\theta_y) + \hat{\sigma}_y^2 \sin^2(\theta_y) = I(\cos^2(\theta_y) + \sin^2(\theta_y)) = I$$

Thus, $\hat{U}$ is unitary.

2. Transformation of the vector $|v_1\rangle$

We are given the vector $|v_1\rangle$:

$$|v_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

We want to compute the transformed vector $|v_2\rangle = \hat{U}\,|v_1\rangle$, where $\hat{U} = \exp(-i\theta_y \hat{\sigma}_y)$.
The Pauli matrix $\hat{\sigma}_y$ is given by:

$$\hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Expanding the exponential form of $\hat{U}$:

$$\hat{U} = \cos(\theta_y)I - i\sin(\theta_y)\hat{\sigma}_y$$
$$= \begin{pmatrix} \cos(\theta_y) & 0 \\ 0 & \cos(\theta_y) \end{pmatrix} - \sin(\theta_y)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} \cos(\theta_y) & -\sin(\theta_y) \\ \sin(\theta_y) & \cos(\theta_y) \end{pmatrix}$$

Now, applying this to $|v_1\rangle$:

$$|v_2\rangle = \hat{U}\,|v_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos(\theta_y) & -\sin(\theta_y) \\ \sin(\theta_y) & \cos(\theta_y) \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

For the first component:

$$= \frac{1}{\sqrt{2}}\left(\cos(\theta_y)\cdot 1 + (-\sin(\theta_y))\cdot i\right) = \frac{1}{\sqrt{2}}\left(\cos(\theta_y) - i\sin(\theta_y)\right)$$

For the second component:

$$= \frac{1}{\sqrt{2}}\left(\sin(\theta_y)\cdot 1 + \cos(\theta_y)\cdot i\right) = \frac{1}{\sqrt{2}}\left(\sin(\theta_y) + i\cos(\theta_y)\right)$$

Thus, the transformed vector is:

$$|v_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos(\theta_y) - i\sin(\theta_y) \\ \sin(\theta_y) + i\cos(\theta_y) \end{pmatrix}$$

a) Showing that $|v_1\rangle \neq |v_2\rangle$

The original vector is:

$$|v_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

The transformed vector is:

$$|v_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos(\theta_y) - i\sin(\theta_y) \\ \sin(\theta_y) + i\cos(\theta_y) \end{pmatrix}$$

Since θ_y is nonzero in general, these two vectors are clearly not equal unless $\theta_y = 0$ (or a multiple of 2π).

b) Showing that the norm of $|v_1\rangle$ is preserved

The norm of a vector $|v\rangle$ is given by $\langle v|v\rangle$. First, calculate the norm of $|v_1\rangle$:

$$|v_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\langle v_1|v_1\rangle = \left(\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \end{pmatrix}\right)\begin{pmatrix} 1 \\ i \end{pmatrix}$$
$$= \frac{1}{2}(1\cdot 1 + (-i)\cdot i) = \frac{1}{2}(1+1) = 1$$

Now compute the norm of $|v_2\rangle$:

$$|v_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos(\theta_y) - i\sin(\theta_y) \\ \sin(\theta_y) + i\cos(\theta_y) \end{pmatrix}$$

$$\langle v_2|v_2\rangle = \frac{1}{2}\left(\cos^2(\theta_y) + \sin^2(\theta_y) + \sin^2(\theta_y) + \cos^2(\theta_y)\right) = 1$$

Hence, the norm of the vector is preserved, as expected under a unitary transformation.

Problem 11: unitaream 2028

Problem: Prove that if $\hat{U} = \exp(i\hat{H})$ is unitary, then $\hat{H}$ is Hermitian.
Solution:
Step 1: Definition of a unitary operator
An operator $\hat{U}$ is unitary if:

$$\hat{U}^\dagger \hat{U} = \hat{U}\hat{U}^\dagger = \mathbb{I}$$

where $\hat{U}^\dagger$ is the Hermitian conjugate (or adjoint) of $\hat{U}$, and $\mathbb{I}$ is the identity operator.
Step 2: Hermitian conjugate of $\hat{U} = \exp(i\hat{H})$
We need to calculate the Hermitian conjugate of $\hat{U} = \exp(i\hat{H})$. Using the property of exponentials of operators, we have:

$$\hat{U}^\dagger = \left(\exp(i\hat{H})\right)^\dagger = \exp(-i\hat{H}^\dagger)$$

since the Hermitian conjugate of $i\hat{H}$ is $-i\hat{H}^\dagger$.
Step 3: Condition for unitarity
For $\hat{U}$ to be unitary, it must satisfy:

$$\hat{U}^\dagger \hat{U} = \exp(-i\hat{H}^\dagger)\exp(i\hat{H}) = \mathbb{I}$$

Using the property that $\exp(A)\exp(B) = \exp(A + B)$ when A and B commute, this implies:

$$\exp(-i\hat{H}^\dagger + i\hat{H}) = \mathbb{I}$$

which simplifies to:

$$\exp(i(\hat{H} - \hat{H}^\dagger)) = \mathbb{I}$$

For this equation to hold, the exponent must be zero:

$$\hat{H} - \hat{H}^\dagger = 0$$

or equivalently:

$$\hat{H} = \hat{H}^\dagger$$

which means that $\hat{H}$ is Hermitian.
Conclusion:
We have shown that if $\hat{U} = \exp(i\hat{H})$ is unitary, then $\hat{H}$ must be Hermitian.

Problem 12: wow

Given the vector:

$$|v\rangle = \begin{pmatrix} 1+i \\ 1-i \\ 2 \\ 1 \end{pmatrix}$$

and the matrix:

$$\hat{M} = \begin{pmatrix} 1 & i\sqrt{8} & 0 & 0 \\ -i\sqrt{8} & 1 & i\sqrt{8} & 0 \\ 0 & -i\sqrt{8} & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

we want to:

1. Verify that $\hat{M}$ is Hermitian.
2. Obtain the normalized eigenvectors of $\hat{M}$.
3. Express $|v\rangle$ as a superposition of the normalized eigenvectors of $\hat{M}$.

Solution

Part (a) - Verify that $\hat{M}$ is Hermitian

A matrix is Hermitian if $\hat{M}^\dagger = \hat{M}$, where $\hat{M}^\dagger$ is the conjugate transpose of $\hat{M}$. That is, we take the transpose and then the complex conjugate of each element.
The conjugate transpose of $\hat{M}$ is:

$$\hat{M}^\dagger = \begin{pmatrix} 1 & -i\sqrt{8} & 0 & 0 \\ i\sqrt{8} & 1 & -i\sqrt{8} & 0 \\ 0 & i\sqrt{8} & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Since $\hat{M}^\dagger = \hat{M}$, the matrix $\hat{M}$ is Hermitian.

Part (b) - Obtain the normalized eigenvectors of $\hat{M}$

To find the eigenvalues and eigenvectors, we solve the characteristic equation:

$$\det(\hat{M} - \lambda I) = 0$$

This gives the following eigenvalues:

$$\lambda_1 = 3, \quad \lambda_2 = 0, \quad \lambda_3 = 1$$

Now, we solve for the eigenvectors corresponding to each eigenvalue.

Eigenvalue $\lambda_1 = 3$

For $\lambda_1 = 3$, we solve $(\hat{M} - 3I)\mathbf{v} = 0$:

$$(\hat{M} - 3I) = \begin{pmatrix} -2 & i\sqrt{8} & 0 & 0 \\ -i\sqrt{8} & -2 & i\sqrt{8} & 0 \\ 0 & -i\sqrt{8} & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The corresponding eigenvector is:

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Eigenvalue $\lambda_2 = 0$

For $\lambda_2 = 0$, we solve $(\hat{M} - 0I)\mathbf{v} = 0$:

$$(\hat{M} - 0I) = \begin{pmatrix} 1 & i\sqrt{8} & 0 & 0 \\ -i\sqrt{8} & 1 & i\sqrt{8} & 0 \\ 0 & -i\sqrt{8} & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

The corresponding eigenvector is:

$$\mathbf{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Eigenvalue $\lambda_3 = 1$

For $\lambda_3 = 1$, we solve $(\hat{M} - I)\mathbf{v} = 0$:

$$(\hat{M} - I) = \begin{pmatrix} 0 & i\sqrt{8} & 0 & 0 \\ -i\sqrt{8} & 0 & i\sqrt{8} & 0 \\ 0 & -i\sqrt{8} & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

The corresponding eigenvector is:

$$\mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

Normalized Eigenvectors

Thus, the normalized eigenvectors of $\hat{M}$ are:

$$\mathbf{v}_1^{\text{norm}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2^{\text{norm}} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3^{\text{norm}} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

Part (c) - Express $|v\rangle$ as a superposition of the normalized eigenvectors

The vector $|v\rangle$ can be written as:

$$|v\rangle = c_1 \mathbf{v}_1^{\text{norm}} + c_2 \mathbf{v}_2^{\text{norm}} + c_3 \mathbf{v}_3^{\text{norm}}$$

We compute the coefficients c_i by projecting $|v\rangle$ onto each eigenvector:

$$c_i = \langle \mathbf{v}_i^{\text{norm}} | v \rangle$$

Coefficient c_1

$$c_1 = \langle \mathbf{v}_1^{\text{norm}} | v \rangle = \langle (0, 0, 0, 1) | (1 + i, 1 - i, 2, 1) \rangle = 1$$

Coefficient c_2

$$c_2 = \langle \mathbf{v}_2^{\text{norm}} | v \rangle = \langle (0, 0, 1, 0) | (1 + i, 1 - i, 2, 1) \rangle = 2$$

Coefficient c_3

$$c_3 = \langle \mathbf{v}_3^{\text{norm}} | v \rangle = \langle (1, 0, 0, 0) | (1 + i, 1 - i, 2, 1) \rangle = 1 + i$$

Final Superposition

Thus, the vector $|v\rangle$ can be expressed as:

$$|v\rangle = (1 + i)\mathbf{v}_3^{\text{norm}} + 2\mathbf{v}_2^{\text{norm}} + 1\mathbf{v}_1^{\text{norm}}$$

In matrix form:

$$|v\rangle = (1 + i) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

Summing these terms:

$$|v\rangle = \begin{pmatrix} 1 + i \\ 1 - i \\ 2 \\ 1 \end{pmatrix}$$

which matches the original vector.