Gamma Function

Thursday, September 5, 2024 13:10

For a non-negative integer n, the factorial $n!$ can be be obtained using the integral

$$n! = \int_0^\infty e^{-t} t^n \, dt$$

We can demonstrate this by repeated integration by parts (IBP)

Let $n \geq 1$, then

$$\int_0^\infty e^{-t} t^n \, dt = -\int_0^\infty t^n \, d(e^{-t}) \qquad d(e^{-t}) = -e^{-t} dt$$

$$= -\left[t^n e^{-t} \Big|_0^\infty - \int_0^\infty e^{-t} d(t^n) \right]$$

$$= -\left(\lim_{t \to \infty} t^n e^{-t} - 0 \cdot e^0 \right) + \int_0^\infty n t^{n-1} e^{-t} \, dt$$

$$= -0 + n \int_0^\infty e^{-t} t^{n-1} \, dt$$

$$\int_0^\infty e^{-t} t^n \, dt = n \int_0^\infty e^{-t} t^{n-1} \, dt$$

we let $n' = n-1$

$$\int_0^\infty e^{-t} t^{n'} \, dt = n' \int_0^\infty e^{-t} t^{n'-1} \, dt$$

go back to n

$$= (n-1) \int_0^\infty e^{-t} t^{n-2} \, dt$$

$$\int_0^\infty e^{-t} t^n \, dt = n(n-1) \underbrace{\int_0^\infty e^{-t} t^{n-2} \, dt}$$

you can continue doing this until you get the last two steps

E2. Show explicitly that $1! = 1$ and $0! = 1$

Gamma Function

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \, dt, \quad \mathrm{Re}(z) > 0$$

where z is a complex number

Observation

* $n! = \Gamma(n+1)$

By IBP

* $\Gamma(z+1) = z\Gamma(z)$
* $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty e^{-t} t^{-1/2} \, dt$$

let $x = t^{1/2}$, $dx = \frac{1}{2} t^{-1/2} dt$

$$\Gamma\left(\tfrac{1}{2}\right) = 2 \int_0^\infty e^{-x^2} dx$$

$$= \int_0^\infty e^{-x^2} dx + \underbrace{\int_0^\infty e^{-x^2} dx}$$

change variable $x \to -x$

$$\int_0^{-\infty} e^{-(-x)^2} d(-x) = -\int_0^{-\infty} e^{-x^2} dx$$

$$= \int_{-\infty}^0 e^{-x^2} dx$$

$$= \int_0^\infty e^{-x^2} dx + \int_{-\infty}^0 e^{-x^2} dx$$

$$= \int_{-\infty}^\infty e^{-x^2} dx \quad \text{(Gaussian integral)}$$

x is just a dummy variable

$$\Gamma\left(\tfrac{1}{2}\right) = \int_{-\infty}^\infty e^{-x^2} dx \quad \text{or} \quad \Gamma\left(\tfrac{1}{2}\right) = \int_{-\infty}^\infty e^{-y^2} dy$$

$$\left(\Gamma\left(\tfrac{1}{2}\right)\right)^2 = \int_{-\infty}^\infty dx \int_{-\infty}^\infty dy \, e^{-(x^2+y^2)}$$

Change the integration from rectangular (x, y) to polar (r, ϕ)

$$x = r\cos\phi$$
$$y = r\sin\phi$$

$$\int_{-\infty}^{+\infty} dx \int_{-\infty}^\infty dy \, e^{-(x^2+y^2)} = \underbrace{\int_0^{2\pi} d\phi}_{\text{finite}} \underbrace{\int_0^\infty dr \cdot r e^{-r^2}}$$

let $u = r^2$, $du = 2r\,dr$

$$\frac{1}{2}\int_0^\infty d(r^2) e^{-r^2} = \frac{1}{2}\int_0^\infty du \, e^{-u}$$

$$= 2\pi \times \frac{1}{2}$$

$$\left(\Gamma\left(\tfrac{1}{2}\right)\right)^2 = \pi \quad \text{or} \quad \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

$$\text{or} \quad \int_{-\infty}^\infty e^{-x^2} dx = \sqrt{\pi}$$

E3. What is $\int_{-\infty}^\infty \exp(-\alpha x^2) \, dx$ where $\alpha > 0$?

Q1 Consider the integral $\int_{-\infty}^\infty dx \cdot x^m \exp(-x^2)$

where m is a non-negative integer

a) argue why the integral is zero for odd m

b) show that for even m ($m = 2n$, n is a non-negative integer)

$$\int_{-\infty}^\infty x^{2n} \exp(-x^2) \, dx = \Gamma\left(n + \tfrac{1}{2}\right)$$

c) using the property $\Gamma(z+1) = z\Gamma(z)$, show that $\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n-1)!!}{2^n} = \sqrt{\pi}$.

Note: $k!! = k(k-2)(k-4)\ldots$

Apply proof of induction to confidently assert the relation

Ex:

$10!! = 8 \times 6 \times 4 \times 2$

$9!! = 9 \times 7 \times 5 \times 3 \times 1$

SIDE NOTES:

Jacobian

$$dx\,dy = J \, dr\,d\phi$$

$x = r\cos\phi$, $y = r\sin\phi$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \cos\phi & \sin\phi \\ -r\sin\phi & r\cos\phi \end{vmatrix}$$

$$= r\cos^2\phi - (-r\sin^2\phi)$$
$$= r(\cos^2\phi + \sin^2\phi)$$
$$= r$$

$$\boxed{dx\,dy = r\,dr\,d\phi}$$

or you can use your intuition (based on your familiarity with arc length relation