PHYS 112 PSet2

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Problem 1: Gauss Pads

Problem:

Consider the integral: $\int_{-\infty}^{\infty} x^m e^{-x^2} dx$ such that, $m \in \mathbb{Z}^+$.

a. Argue why the integral is zero for odd m.

b. Show that for even m. $(m = 2n, n \in \mathbb{Z}^+)$

$$\int_{-\infty}^{\infty} x^{2n} e^{-x^2} dx = \Gamma(n + \frac{1}{2})$$

c. Using the property $\Gamma(z + 1) = z\Gamma(z)$, show that:

$$\Gamma(n + \frac{1}{2}) = \frac{(2n-1)!!}{2^n} = \sqrt{\pi}.$$

Note that: $k!! = (k)(k-2)(k-4)...$

Solution:
Recall first that we have:

$$\int_{-\infty}^{\infty} x^m e^{-x^2} dx$$

a. Let: $m = 2n + 1,\ \textit{such that}\ n \in \mathbb{Z}^+$

$$\int_{-\infty}^{\infty} x^{2n+1} e^{-x^2} dx = \int_{-\infty}^{0} x^{2n+1} e^{-x^2} dx + \int_{0}^{\infty} x^{2n+1} e^{-x^2} dx$$

We can then prove that: $-f(x) = f(-x), \forall x \in \mathbb{Z}^+$ where $f(x) = x^{2n+1}e^{-x^2}$ so that: $\int_{-\infty}^{\infty} x^{2n+1}e^{-x^2}dx = 0.$

Prove by Induction
Base Case: $x = 1$

$$\begin{aligned}
-f(1) &\stackrel{?}{=} f(-1) \\
-(1^{2n+1}e^{-1^2}) &\stackrel{?}{=} (-1)^{2n+1}e^{-(-1)^2} \\
-\frac{1}{e} &= -\frac{1}{e}
\end{aligned}$$

Inductive Step: If $-f(x) = f(-x)$ *is true, then* $-f(x+1) = f(-(x+1))$ *must be true as well.*

$$\begin{aligned}
-f(x+1) &\stackrel{?}{=} f(-x-1) \\
-((x+1)^{2n+1}e^{-(x+1)^2}) &\stackrel{?}{=} (-x-1)^{2n+1}e^{-(-x-1)^2} \\
-((x+1)^{2n}(x+1)e^{-(x^2+2x+1)}) &\stackrel{?}{=} (-x-1)^{2n}(-x-1)e^{-(x^2+2x+1)} \\
-((x+1)^{2n}(x+1)e^{-(x^2+2x+1)}) &\stackrel{?}{=} -(-x-1)^{2n}(x+1)e^{-(x^2+2x+1)}
\end{aligned}$$

Therefore, since the powers of both $(x+1)^{2n}$ and $(-x-1)^{2n}$ are even and the other terms are equal as well, RHS and LHS are equal to each other. With this, both the base case and inductive steps are true.

$$\therefore -f(x) = f(-x), \forall x \in \mathbb{Z}^+.$$

And since $-f(x) = f(-x), \forall x \in \mathbb{Z}^+$, then $\int_{-\infty}^{\infty} x^{2n+1}e^{-x^2}dx = 0.$ **QED.**

b. We need to prove that for even $m = 2n$ (where n is a non-negative integer):

$$\int_{-\infty}^{\infty} x^{2n}e^{-x^2}\,dx = \Gamma\left(n + \frac{1}{2}\right).$$

Step 1: Start with the known Gaussian integral

The Gaussian integral is a well-known result:

$$I_0 = \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

This integral has no powers of x multiplying the exponential. Now we want to generalize this to integrals of the form $I_n = \int_{-\infty}^{\infty} x^{2n} e^{-x^2}\, dx$ for $n \geq 1$.

Step 2: Use symmetry and note that odd powers vanish

Since the integrand $x^{2n} e^{-x^2}$ is an even function, the integral over $(-\infty, \infty)$ for odd powers of x vanishes:

$$\int_{-\infty}^{\infty} x^{2n+1} e^{-x^2}\, dx = 0 \quad \text{for all } n.$$

Thus, we only need to focus on the even powers x^{2n}.

Step 3: Use recurrence relation for even powers

Integrals of the form $\int_{-\infty}^{\infty} x^{2n} e^{-x^2}\, dx$ can be related recursively. To find a relation, we use integration by parts.

Let:

$$I_n = \int_{-\infty}^{\infty} x^{2n} e^{-x^2}\, dx.$$

Perform integration by parts by choosing:

$$u = x^{2n-1}, \quad dv = x e^{-x^2} dx.$$

Then, we obtain the recurrence relation:

$$I_n = \frac{2n-1}{2} I_{n-1}.$$

Step 4: Apply the recurrence relation

Starting from the base case $I_0 = \sqrt{\pi}$, we apply the recurrence relation:

$$I_n = \frac{2n-1}{2} I_{n-1}.$$

Applying this relation repeatedly gives:

$$I_n = \frac{2n-1}{2} \cdot \frac{2n-3}{2} \cdot \frac{2n-5}{2} \cdots \frac{1}{2} I_0.$$

Since $I_0 = \sqrt{\pi}$, we get:

$$I_n = \frac{(2n-1)(2n-3)\cdots 1}{2^n} \sqrt{\pi}.$$

This expresses I_n in terms of factorial-like products.

Step 5: Relate to the Gamma function

For half-integers, the Gamma function $\Gamma(z)$ has the following special property:

$$\Gamma\left(n+\frac{1}{2}\right)=\frac{(2n-1)(2n-3)\cdots 1}{2^n}\sqrt{\pi}.$$

Thus, using this result, we can conclude that:

$$I_n=\Gamma\left(n+\frac{1}{2}\right).$$

Hence, we have shown that:

$$\int_{-\infty}^{\infty} x^{2n}e^{-x^2}\,dx=\Gamma\left(n+\frac{1}{2}\right).$$

This completes the derivation.

c. First, recall the gamma function: $\Gamma(z)=\displaystyle\int_0^{\infty} e^{-t}t^{z-1}dt$, or

$$\Gamma(n+\frac{1}{2})=\int_0^{\infty} e^{-t}t^{n-1}dt$$

Prove by Induction that...

$$\Gamma(n+\frac{1}{2})=\frac{(2n-1)!!}{2^n}\ \textit{with}\ n=k$$

Base Case: $k=0$

$$\Gamma(0+\frac{1}{2})\Rightarrow\int_0^{\infty} e^{-t}t^{\frac{1}{2}-1}dt\stackrel{?}{=}\sqrt{\pi}\Leftarrow\frac{(2n-1)!!}{2^n}$$
$$\int_0^{\infty} e^{-t}t^{-\frac{1}{2}}dt\stackrel{?}{=}\sqrt{\pi}$$
$$\int_{-\infty}^{\infty} e^{-x^2}dx=\sqrt{\pi}\ \textit{where}\ x=t^{\frac{1}{2}}$$

One must then recognize the Gaussian integral property here.

Inductive Step: If $n=k$ *holds the equation true, then* $n+1=k+1$ *must hold the equation true as well. Subsequently,* $m=k+\frac{1}{2}$ *holds the equation true, then* $m+1=k+\frac{1}{2}+1$ *must hold the equation true as well.*

$\Gamma(n+\frac{1}{2}) \Rightarrow \Gamma(m) = \int_0^\infty e^{-t}t^{m-1}dt = \frac{(2(k+\frac{1}{2})-1)!!}{2^{k+\frac{1}{2}}}$ *is assumed to be true.*

Then, $\Gamma(m+1) = \frac{(2(k+\frac{3}{2})-1)!!}{2^{k+\frac{3}{2}}}$ *must be true as well.*

First, we test the relationship of their integral definitions.

$$\Gamma(m+1) = \int_0^\infty e^{-t}t^{m+1-1}dt$$

$$(m)\Gamma(m) = \int_0^\infty e^{-t}t^{m}dt$$

which is also equivalent to:

$$(m)\Gamma(m) = m\int_0^\infty e^{-t}t^{m-1}dt$$

This implies the following relationship:

$$\int_0^\infty e^{-t}t^{m}dt = m\int_0^\infty e^{-t}t^{m-1}dt$$

$$-t^m e^{-t} + m\int_0^\infty e^{-t}t^{m-1}dt = m\int_0^\infty e^{-t}t^{m-1}dt$$

$$-t^m e^{-t}\Big|_0^\infty = 0$$

$$\lim_{\alpha\to\infty}(-\alpha^m e^{-\alpha}) = 0$$

And m *must be in the range* $(0, -\infty)$ *to get the limit to be zero*

We also test the relationship of their double factorial definitions.

$$\Gamma(m+1) = \frac{(2(k+\frac{3}{2})-1)!!}{2^{k+\frac{3}{2}}}$$
$$(m)\Gamma(m) = \frac{(2(k+\frac{3}{2})-1)!!}{2^{k+\frac{3}{2}}}$$

which is also equivalent to:

$$(m)\Gamma(m) = m\frac{(2(k+\frac{1}{2})-1)!!}{2^{k+\frac{1}{2}}}$$

This implies the following relationship:

$$\frac{(2(k+\frac{3}{2})-1)!!}{2^{k+\frac{3}{2}}} = m\frac{(2(k+\frac{1}{2})-1)!!}{2^{k+\frac{1}{2}}}$$
$$\frac{(2(m+1)-1)!!}{2^{m+1}} = m\frac{(2m-1)!!}{2^m}$$
$$\frac{(2m+1)!!}{2^m} = 2m\frac{(2m-1)!!}{2^m}$$
$$(2m+1)!! - (2m-1)!! = 2m2^m$$
$$\prod_{\gamma=0}^{\infty}(2m+1-2\gamma) - \prod_{\gamma=0}^{\infty}(2m-1-2\gamma) = 2m2^m$$
$$\prod_{\gamma=0}^{\infty}(2k+2-2\gamma) - \prod_{\gamma=0}^{\infty}(2k-2\gamma) = 2m2^m$$
$$(2k+1)(2k)!! = 2m2^m$$
$$(2k)!! = 2^{k+\frac{1}{2}}$$

Where surprisingly, the point at which this condition is possible and defined is at $k = 0$.

$$\therefore \Gamma(n+\frac{1}{2}) = \frac{(2n-1)!!}{2^n}$$

Problem 2: 4 gigantic pillars that took forever to carry and orthogonalize

Consider the following non-orthogonal basis set:

$$\hat{e}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \hat{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \hat{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \quad \hat{e}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

We aim to orthogonalize this set while keeping the form of $\hat{e}_1$. We use the Gram-Schmidt orthogonalization process.
Step 1: Normalize $\hat{e}_1$ Since we need to preserve the form of $\hat{e}_1$, we don't modify it. However, we normalize it:

$$\tilde{e}_1 = \frac{\hat{e}_1}{\|\hat{e}_1\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Step 2: Orthogonalize $\hat{e}_2$ with respect to $\tilde{e}_1$ Now, we subtract the projection of $\hat{e}_2$ onto $\tilde{e}_1$:

$$\text{Proj}_{\tilde{e}_1} \hat{e}_2 = \frac{\hat{e}_2 \cdot \tilde{e}_1}{\tilde{e}_1 \cdot \tilde{e}_1} \tilde{e}_1 = \frac{\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}}{\frac{1}{2}} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

$$\hat{e}_2 \cdot \tilde{e}_1 = \frac{1}{\sqrt{2}}(1), \quad \text{Proj}_{\tilde{e}_1} \hat{e}_2 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix}.$$

Thus, the orthogonalized $\hat{e}_2$ is:

$$\tilde{e}_2 = \hat{e}_2 - \text{Proj}_{\tilde{e}_1} \hat{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}.$$

Normalize $\tilde{e}_2$:

$$\|\tilde{e}_2\| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + 1^2 + 0^2} = \sqrt{\frac{1}{4} + \frac{1}{4} + 1} = \sqrt{\frac{3}{2}}.$$

Thus,

$$\tilde{e}_2 = \frac{1}{\sqrt{\frac{3}{2}}} \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \\ 0 \end{pmatrix} = \frac{\sqrt{2}}{\sqrt{3}} \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}.$$

Step 3: Orthogonalize $\hat{e}_3$ with respect to $\tilde{e}_1$ and $\tilde{e}_2$ We subtract the projections of $\hat{e}_3$ onto $\tilde{e}_1$ and $\tilde{e}_2$:

$$\mathrm{Proj}_{\tilde{e}_1} \hat{e}_3 = \frac{\hat{e}_3 \cdot \tilde{e}_1}{\tilde{e}_1 \cdot \tilde{e}_1} \tilde{e}_1 = 0,$$

since $\hat{e}_3 \cdot \tilde{e}_1 = 0$.
Next,

$$\mathrm{Proj}_{\tilde{e}_2} \hat{e}_3 = \frac{\hat{e}_3 \cdot \tilde{e}_2}{\tilde{e}_2 \cdot \tilde{e}_2} \tilde{e}_2 = \frac{1}{3} \begin{pmatrix} -1 \\ 1 \\ 2 \\ 0 \end{pmatrix}.$$

Thus, the orthogonalized $\hat{e}_3$ is:

$$\tilde{e}_3 = \hat{e}_3 - \mathrm{Proj}_{\tilde{e}_2} \hat{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{3} \begin{pmatrix} -1 \\ 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ \frac{1}{3} \\ 1 \end{pmatrix}.$$

Normalize $\tilde{e}_3$:

$$\|\tilde{e}_3\| = \sqrt{\left(\frac{1}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + 1^2} = \sqrt{\frac{1}{9} + \frac{1}{9} + \frac{1}{9} + 1} = \sqrt{\frac{12}{9}} = \frac{2}{\sqrt{3}}.$$

Thus,

$$\tilde{e}_3 = \frac{1}{\frac{2}{\sqrt{3}}} \begin{pmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ \frac{1}{3} \\ 1 \end{pmatrix} = \frac{\sqrt{3}}{2} \begin{pmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ \frac{1}{3} \\ 1 \end{pmatrix}.$$

Step 4: Orthogonalize $\hat{e}_4$ with respect to $\tilde{e}_1$, $\tilde{e}_2$, and $\tilde{e}_3$
Projection of $\hat{e}_4$ onto $\tilde{e}_1$

$$\mathrm{Proj}_{\tilde{e}_1} \hat{e}_4 = \frac{\hat{e}_4 \cdot \tilde{e}_1}{\tilde{e}_1 \cdot \tilde{e}_1} \tilde{e}_1.$$

The dot product:

$$\hat{e}_4 \cdot \tilde{e}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0.$$

Therefore,

$$\mathrm{Proj}_{\tilde{e}_1}\hat{e}_4 = 0.$$

Projection of $\hat{e}_4$ onto $\tilde{e}_2$

$$\mathrm{Proj}_{\tilde{e}_2}\hat{e}_4 = \frac{\hat{e}_4 \cdot \tilde{e}_2}{\tilde{e}_2 \cdot \tilde{e}_2}\tilde{e}_2.$$

The dot product:

$$\hat{e}_4 \cdot \tilde{e}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \cdot \frac{\sqrt{2}}{\sqrt{3}} \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \\ 0 \end{pmatrix} = 0.$$

Thus,

$$\mathrm{Proj}_{\tilde{e}_2}\hat{e}_4 = 0.$$

Projection of $\hat{e}_4$ onto $\tilde{e}_3$

$$\mathrm{Proj}_{\tilde{e}_3}\hat{e}_4 = \frac{\hat{e}_4 \cdot \tilde{e}_3}{\tilde{e}_3 \cdot \tilde{e}_3}\tilde{e}_3.$$

The dot product:

$$\hat{e}_4 \cdot \tilde{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \cdot \frac{\sqrt{3}}{2} \begin{pmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ \frac{1}{3} \\ 1 \end{pmatrix} = \frac{\sqrt{3}}{2}.$$

Thus,

$$\mathrm{Proj}_{\tilde{e}_3}\hat{e}_4 = \frac{\frac{\sqrt{3}}{2}}{1}\frac{1}{1}\frac{\sqrt{3}}{2} \begin{pmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ \frac{1}{3} \\ 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 \\ -1 \\ 1 \\ 3 \end{pmatrix}.$$

Subtract the projections from $\hat{e}_4$
Now we subtract the projection from $\hat{e}_4$ to obtain the orthogonalized vector:

$$\tilde{e}_4 = \hat{e}_4 - \mathrm{Proj}_{\tilde{e}_3}\hat{e}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{4}\begin{pmatrix} 1 \\ -1 \\ 1 \\ 3 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}.$$

Step 5: Normalize $\tilde{e}_4$
Finally, we normalize $\tilde{e}_4$ to obtain the orthonormal vector. The norm of $\tilde{e}_4$ is:

$$\|\tilde{e}_4\| = \sqrt{\left(-\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(-\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^2} = \frac{1}{2}.$$

Thus, the normalized vector is:

$$\tilde{e}_4 = \frac{1}{\frac{1}{2}} \frac{1}{4} \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}.$$

After simplifying and normalizing $\tilde{e}_4$, we obtain the final vector for the orthogonalized basis set. Thus, the final orthonormal basis obtained after applying the Gram-Schmidt process and normalizing each vector is:

$$\tilde{e}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \tilde{e}_2 = \frac{\sqrt{2}}{\sqrt{3}} \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \quad \tilde{e}_3 = \frac{\sqrt{3}}{2} \begin{pmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ \frac{1}{3} \\ 1 \end{pmatrix}, \quad \tilde{e}_4 = \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}.$$

Thus, the orthonormal basis consists of the vectors $\tilde{e}_1$, $\tilde{e}_2$, $\tilde{e}_3$, and $\tilde{e}_4$, which are all mutually orthogonal and normalized to have unit length.

Problem 3: Rolling rolling rolling

Problem:
Suppose that instead of rotating $\vec{r}$, we rotate the x and y axis while keeping $\vec{r}$.

Express the transformation of the coordinate system in terms of matrices. How is this different from our "active" transformation of the given vector?

Solution:
Notice that the rotation of the coordinate system (in a passive transformation) is a rotation with an angle of δ towards the opposite direction of how the vector would've rotated in an active transformation. If this rotation were to be applied to $\vec{r}$ rotated an angle δ, then $\vec{r}$ returns back to its original orientation.

Since it is known that the vector can be transformed back to its original state when the inverse of the original transformation matrix is applied (since AA^{-1} is gives an identity matrix), then one can think of the transformation of the coordinate system as the inverse of the transformation of the vector (in its active transformation counterpart).

In this case, we just need to take the rotation matrix and find its inverse.

$$\begin{aligned}
R &= \begin{pmatrix} \cos\delta & -\sin\delta \\ \sin\delta & \cos\delta \end{pmatrix} \\
R^{-1} &= \frac{\operatorname{adj}(R)}{\det(R)} \\
&= \begin{pmatrix} \cos\delta & \sin\delta \\ -\sin\delta & \cos\delta \end{pmatrix} \frac{1}{\cos^2\delta - (-\sin^2\delta)} \\
&= \begin{pmatrix} \cos\delta & \sin\delta \\ -\sin\delta & \cos\delta \end{pmatrix} && \text{(Due to Pythagorean identity)}
\end{aligned}$$

The difference between active and passive is as follows:

- Active transformation: The coordinate system is fixed, while the vector is transformed.
- Passive transformation: The vector is fixed, and the coordinate system is transformed.

In either case the matrix concerning one type of transformation is just the inverse of the matrix for the other transformation.

Problem 4: Now I know my ABCs!

Problem:
By considering all possible combinations, show that:

$$[\hat{\sigma}_l, \hat{\sigma}_m] = 2i\epsilon_{lmn}\hat{\sigma}_n$$
$$\epsilon_{lmn} = \begin{cases} 1 \text{ for even permutation of lmn} \\ -1 \text{ for odd permutation of lmn} \\ 1 \text{ for repeated permutation of lmn} \end{cases}$$

Solution:
Since this is referring to the Pauli matrices, a limited set of matrices, we can simply consider all combinations of

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
$$\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

It is possible to show this by listing every single possible combination, but we can be more efficient if we explore the relationships between and characteristics of each combination.
Since there are only 6 unique permutations for the three matrices, we can first start out by listing the even and odd permutations (let A be the coefficient of the output matrix).

Even Permutations:

- $[\hat{\sigma}_x, \hat{\sigma}_y] = A_1\hat{\sigma}_z$
- $[\hat{\sigma}_z, \hat{\sigma}_x] = A_2\hat{\sigma}_y$
- $[\hat{\sigma}_y, \hat{\sigma}_z] = A_3\hat{\sigma}_x$

Odd Permutations:

- $[\hat{\sigma}_y, \hat{\sigma}_x] = A_4\hat{\sigma}_z$
- $[\hat{\sigma}_x, \hat{\sigma}_z] = A_5\hat{\sigma}_y$
- $[\hat{\sigma}_z, \hat{\sigma}_y] = A_6\hat{\sigma}_x$

It is known that

$$\left[\hat{A}, \hat{B}\right] = -\left[\hat{B}, \hat{A}\right]$$

Notice that the three input matrices in our odd permutations are simply swapped variations of two even permutations. This means that our two odd permutations are the negative counterpart of two of the even permutations, such that

$$\begin{aligned}
[\hat{\sigma}_x, \hat{\sigma}_y] &= -[\hat{\sigma}_y, \hat{\sigma}_x] = -A_1\hat{\sigma}_z \\
[\hat{\sigma}_z, \hat{\sigma}_x] &= -[\hat{\sigma}_x, \hat{\sigma}_z] = -A_2\hat{\sigma}_y \\
[\hat{\sigma}_y, \hat{\sigma}_z] &= -[\hat{\sigma}_z, \hat{\sigma}_y] = -A_3\hat{\sigma}_x
\end{aligned}$$

From this, we can say that $\epsilon_{lmn} = -1$ when the permutation is odd due to this relationship.

From this we simply need to show that A_1 A_2 and A_3 are equal to $2i$ since they are the output from the even permutations.

We can start with A_1:

$$\begin{aligned}
\hat{\sigma}_x\hat{\sigma}_y &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= i\hat{\sigma}_z \\
\hat{\sigma}_y\hat{\sigma}_x &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= -i\hat{\sigma}_z
\end{aligned}$$

Thus

$$\begin{aligned}
[\hat{\sigma}_x, \hat{\sigma}_y] &= \hat{\sigma}_x\hat{\sigma}_y - \hat{\sigma}_y\hat{\sigma}_z \\
&= i\hat{\sigma}_z - (-i\hat{\sigma}_z) \\
&= 2i\hat{\sigma}_z
\end{aligned}$$

For A_2:

$$\begin{aligned}\hat{\sigma}_z\hat{\sigma}_x &= \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} = i\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\\ &= i\hat{\sigma}_y\\ \hat{\sigma}_x\hat{\sigma}_z &= \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} = -i\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\\ &= -i\hat{\sigma}_y\end{aligned}$$

Thus

$$\begin{aligned}[\hat{\sigma}_z, \hat{\sigma}_x] &= \hat{\sigma}_z\hat{\sigma}_x - \hat{\sigma}_x\hat{\sigma}_z\\ &= i\hat{\sigma}_y - (-i\hat{\sigma}_y)\\ &= 2i\hat{\sigma}_y\end{aligned}$$

Lastly, for A_3:

$$\begin{aligned}\hat{\sigma}_y\hat{\sigma}_z &= \begin{pmatrix}0 & i\\ i & 0\end{pmatrix} = i\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\\ &= i\hat{\sigma}_x\\ \hat{\sigma}_z\hat{\sigma}_y &= \begin{pmatrix}0 & -i\\ -i & 0\end{pmatrix} = -i\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\\ &= -i\hat{\sigma}_x\end{aligned}$$

Thus

$$\begin{aligned}[\hat{\sigma}_y, \hat{\sigma}_z] &= \hat{\sigma}_y\hat{\sigma}_z - \hat{\sigma}_z\hat{\sigma}_y\\ &= i\hat{\sigma}_x - (-i\hat{\sigma}_x)\\ &= 2i\hat{\sigma}_x\end{aligned}$$

For the repeated permutations, there are two possible cases:

1. The two input matrices are the same.
2. All of the matrices are the same.

We want to show that for both two cases, $\epsilon_{lmn} = 0$.

Both cases are straightforward: since the commutator of two identical $N \times N$ matrices will always have its elements as 0, then the first case would have $\epsilon_{lmn} = 0$ since the output matrix will always be $\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$.

The second case is similar to the first case: the two input matrices are the same, so $\epsilon_{lmn} = 0$ since the output matrix will always be $\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$.

Note that it would not be possible to have a repeated permutation where the repetition is between the output and one other input matrices, as it was already established in trying to find A_1 A_2 and A_3 that the commutator of each distinct Pauli matrix would output the third unused matrix (when factored out).

Problem 5: ABaCa ate BACsilog in a CAB

Problem:
Consider 3 square matrices of the same dimension $\hat{A}$, $\hat{B}$, and $\hat{C}$. By direct expansion and some arrangement, prove the Jacobi identity

$$[\hat{A},[\hat{B},\hat{C}]] = [\hat{B},[\hat{A},\hat{C}]] - [\hat{C},[\hat{A},\hat{B}]].$$

Solution:
First, we can expand the three terms:

$$\begin{aligned}[\hat{A},[\hat{B},\hat{C}]] &= \hat{A}(\hat{B}\hat{C} - \hat{C}\hat{B}) - (\hat{B}\hat{C} - \hat{C}\hat{B})\hat{A} \\ &= \hat{A}\hat{B}\hat{C} - \hat{A}\hat{C}\hat{B} - \hat{B}\hat{C}\hat{A} + \hat{C}\hat{B}\hat{A}\end{aligned}$$

$$\begin{aligned}[\hat{B},[\hat{A},\hat{C}]] &= \hat{B}(\hat{A}\hat{C} - \hat{C}\hat{A}) - (\hat{A}\hat{C} - \hat{C}\hat{A})\hat{B} \\ &= \hat{B}\hat{A}\hat{C} - \hat{B}\hat{C}\hat{A} - \hat{A}\hat{C}\hat{B} + \hat{C}\hat{A}\hat{B}\end{aligned}$$

$$\begin{aligned}[\hat{C},[\hat{A},\hat{B}]] &= \hat{C}(\hat{A}\hat{B} - \hat{B}\hat{A}) - (\hat{A}\hat{B} - \hat{B}\hat{A})\hat{C} \\ &= \hat{C}\hat{A}\hat{B} - \hat{C}\hat{B}\hat{A} - \hat{A}\hat{B}\hat{C} + \hat{B}\hat{A}\hat{C}\end{aligned}$$

Then we can substitute these to the three terms in our original equation:

$$\hat{A}\hat{B}\hat{C} - \hat{A}\hat{C}\hat{B} - \hat{B}\hat{C}\hat{A} + \hat{C}\hat{B}\hat{A} \stackrel{?}{=} \hat{B}\hat{A}\hat{C} - \hat{B}\hat{C}\hat{A} - \hat{A}\hat{C}\hat{B} + \hat{C}\hat{A}\hat{B} - (\hat{C}\hat{A}\hat{B} - \hat{C}\hat{B}\hat{A} - \hat{A}\hat{B}\hat{C} + \hat{B}\hat{A}\hat{C})$$

Then simplify the RHS:

$$\hat{A}\hat{B}\hat{C} - \hat{A}\hat{C}\hat{B} - \hat{B}\hat{C}\hat{A} + \hat{C}\hat{B}\hat{A} \stackrel{?}{=} -\hat{B}\hat{C}\hat{A} - \hat{A}\hat{C}\hat{B} + \hat{C}\hat{B}\hat{A} + \hat{A}\hat{B}\hat{C}$$

This RHS contains the same terms as the LHS, which is more apparent when we rearrange it:

$$\hat{A}\hat{B}\hat{C} - \hat{A}\hat{C}\hat{B} - \hat{B}\hat{C}\hat{A} + \hat{C}\hat{B}\hat{A} = \hat{A}\hat{B}\hat{C} - \hat{A}\hat{C}\hat{B} - \hat{B}\hat{C}\hat{A} + \hat{C}\hat{B}\hat{A}$$

This proves the Jacobi identity.

Problem 6: Some End Their Differences

Problem:

Show that: $(\hat{A}+\hat{B})(\hat{A}-\hat{B}) = \hat{A}^2 - \hat{B}^2 \iff [\hat{A},\hat{B}] = 0$.

Solution:
We can first evaluate the LHS, such that the order of each factor of the terms in the product are conserved:

$$\begin{aligned}(\hat{A}+\hat{B})(\hat{A}-\hat{B}) &= \hat{A}\hat{A} - \hat{A}\hat{B} + \hat{B}\hat{A} - \hat{B}\hat{B} \\ &= \hat{A}^2 - \hat{B}^2 - [\hat{A},\hat{B}] \qquad (\textit{such that } [\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A})\end{aligned}$$

With this, only when $[\hat{A},\hat{B}] = 0$ will the product $(\hat{A}+\hat{B})(\hat{A}-\hat{B})$ be equal to $\hat{A}^2 - \hat{B}^2$

$\therefore (\hat{A}+\hat{B})(\hat{A}-\hat{B}) = \hat{A}^2 - \hat{B}^2 \iff [\hat{A},\hat{B}] = 0$.