PHYS112 - ProbSet1

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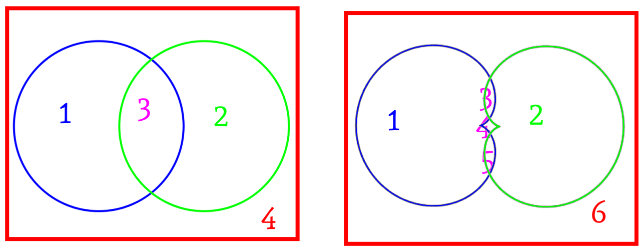
Problem 1: Enclaves
Are these two Venn diagrams equivalent?



Explain your answer.

Solution. The two Venn diagrams are **equivalent**.

To determine whether the two Venn diagrams are equivalent or not, we can simply try to reason whether or not the *amount* of regions and events in the sample spaces of both venn diagrams show correspondence. In particular, using the definition of the equivalence of sets, we can assume that the cardinality of each Venn diagram will concern both the events and regions, independently.

First, one can count that the amount of events that correspond to each Venn diagram is two(2). Following this, we can use the fact that each Venn diagram contains 2^n regions, where n is the amount of events. This leaves us with the fact that there are four(4) regions corresponding to each Venn diagram.

Given this, we can at least assume that the two Venn diagrams are equivalent.

However, the unequal amount of labeled 'regions' between the two Venn diagrams should be also addressed. One may assume that these excess labels indicate unequal cardinality across the two Venn diagrams.

Perhaps, it should be known that the Venn diagram is but a representation of elements, contained in the set itself. Thus, any error caused by a bad graphical interpretation of the set (in this case, the redundance of regions 3, 4, and 5) should not amount to any complications that strays the set from its analytical properties.

Problem 2: Prove It!
Given the union of n general events, prove by induction upon n that:

$$\begin{aligned}\Pr(A_1 \cup A_2 \cup ... \cup A_n) &= \sum_i \Pr(A_i) - \sum_{i,j} \Pr(A_i \cap A_j) \\ &+ \sum_{i,j,k} \Pr(A_i \cap A_j \cap A_k) \\ &- ... + (-1)^{(n+1)} \Pr(A_1 \cap A_2 \cap ... \cap A_n)\end{aligned}$$

*summation with different indices are understood to be not equal.

$$(i \neq j, i \neq j \neq k, ...)$$

Proof. We use mathematical induction.

Base Case: Setting $n = 2$, we get:

$$\begin{aligned}\Pr(A_1 \cup A_2) &= \sum_i \Pr(A_i) - \sum_{i,j} \Pr(A_i \cap A_j) \\ &= \Pr(A_1) + \Pr(A_2) - \Pr(A_1 \cap A_2)\end{aligned}$$

where the last statement is one of the Probability Axioms and Theorems.
Induction Step: Let $n = z \in \mathbb{N}$ and:

$$\begin{aligned}\Pr(A_1 \cup A_2 \cup ... \cup A_z) &= \sum_i \Pr(A_i) - \sum_{i,j} \Pr(A_i \cap A_j) + \sum_{i,j,k} \Pr(A_i \cap A_j \cap A_k) \\ &- ... + (-1)^{(z+1)} \Pr(A_1 \cap A_2 \cap ... \cap A_z) \\ &= \sum_i \Pr(A_i) - \sum_{i,j} \Pr(A_i \cap A_j) + ... + (-1)^{z+1} \Pr(A_i \cap ... \cap A_z)\end{aligned}$$

Then, we use one of the Probability Axioms and Theorems, such that:

$$\Pr((A_1 \cup A_2 \cup ... \cup A_z) \cup A_{z+1}) = \Pr(A_1 \cup A_2 \cup ... \cup A_z) + \Pr(A_{z+1}) - \Pr((A_1 \cup A_2 \cup ... \cup A_z) \cap A_{z+1})$$

Where, one can notice the following observations:
1. The first term is the unity of all sets but A_{z+1}.
2. The second terms is the entire set A_{z+1}.
3. The last term is simply equivalent to: $(-1)^{z+1} \Pr(\cap_z A_z)$, or the intersection of all sets concerned.

We can then apply the induction hypothesis:

$$\begin{aligned}&\Pr(A_1 \cup A_2 \cup ... \cup A_n) + \Pr(A_{z+1}) - \Pr((A_1 \cup A_2 \cup ... \cup A_z) \cap A_{z+1}) \\ &= \sum_i^z \Pr(A_i) - \sum_{i,j}^z \Pr(A_i \cap A_j) + ... + (-1)^{z+1} \sum_{i,j,...,z}^z \Pr(A_i \cap ... \cap A_z) + \Pr(A_{z+1}) \\ &- \Pr((A_1 \cap A_{z+1}) \cup ... \cup (A_z \cap A_{z+1}))\end{aligned}$$

Then we can once again use the induction hypothesis on the last term:

$$\Pr((A_1 \cap A_{z+1}) \cup ... \cup (A_z \cap A_{z+1}))$$
$$= \sum_i^z \Pr(A_i) + ... + (-1)^{z+1} \sum_{i,j...z}^z \Pr(A_i \cap ... \cap A_{z+1})$$

Then applying that to the entire expression:

$$\sum_i^z \Pr(A_i) - \sum_{i,j}^z \Pr(A_i \cap A_j) + ... + (-1)^{z+1} \sum_{i,j,...,z}^z \Pr(A_i \cap ... \cap A_z) + \Pr(A_{z+1})$$
$$- \sum_i^z \Pr(A_i) - ... - (-1)^{z+1} \sum_{i,j...z}^z \Pr(A_i \cap ... \cap A_{z+1})$$

We can then notice a couple of things:

1. $\sum_i^z \Pr(A_i) + \Pr(A_{z+1})$ is equivalent to $\sum_i^{z+1} \Pr(A_i)$.

2. $-\sum_{i,j}^z \Pr(A_i \cap A_j) - \sum_i^z \Pr(A_i \cap A_{z+1})$ is equivalent to $-\sum_{i,j}^{z+1} \Pr(A_i \cap A_j)$, and the subsequent pattern that arises from it applies to the terms expressed through "...".

3. Due to the aforementioned pattern, $(-1)^{z+1} \sum_{i,j,...,z}^z \Pr(A_i \cap ... \cap A_z)$ should become $(-1)^{z+1} \sum_{i,j,...,z}^{z+1} \Pr(A_i \cap ... \cap A_z)$.

4. $-(-1)^{z+1}$ is equivalent to $+(-1)^{(z+1)+1}$.

5. Since the sample set is already exhausted at $\sum_{i,j...z}^z \Pr(A_i \cap ... \cap A_{z+1})$, it can also be said to be equivalent to $\sum_{i,j...z+1}^z \Pr(A_i \cap ... \cap A_{z+1})$.

We then get the following expression:

$$\sum_i^{z+1} \Pr(A_i) - \sum_{i,j}^{z+1} \Pr(A_i \cap A_j) + ... + (-1)^{z+1} \sum_{i,j,...,z}^{z+1} \Pr(A_i \cap ... \cap A_z)$$
$$+ (-1)^{(z+1)+1} \sum_{i,j...z+1}^{z+1} \Pr(A_i \cap ... \cap A_{z+1})$$

As it follows the induction hypothesis when $n = z + 1$, this proves the inclusion-exclusion principle.

Problem 3: Anti Target Practice Target Practice Club
Two duelists, A and B, take alternate shots at each other, and the duel isover when a shot (fatal or otherwise!) hits its target. Each shot fired by A has a probability α of hitting B, and each shot fired by B has a probability β of hitting A. Calculate the probabilities P_1 and P_2, defined as follows, that A will win such a duel: P_1, A fires the first shot; P_2, B fires the first shot. If they agree to fire simultaneously, rather than alternately, what is the probability P_3 that A will win, i.e. hit B without being hit himself?

Solution. Instead of approaching the problem using the solution in the book, we simply took the probability of B winning instead first, to use that in finding the probability of A winning, where we get:

$$\bar{P}_1 \rightarrow \textit{ probability that } B \textit{ wins if } A \textit{ shoots first}$$
$$\bar{P}_2 \rightarrow \textit{ probability that } B \textit{ wins if } B \textit{ shoots first}$$

And using the following to find for $\bar{P}_1$ and $\bar{P}_2$.

$$\bar{P}_1 : C_1 \rightarrow \textit{ B hits },\qquad C_2 \rightarrow \textit{ B misses, A hits },\qquad C_3 \rightarrow \textit{ Both miss}$$
$$\bar{P}_2 : D_1 \rightarrow \textit{ A hits },\qquad D_2 \rightarrow \textit{ A misses.}$$

We also assume that an event Z indicates the event where B wins, such that:

$$\begin{aligned}
\bar{P}_1 &= \sum_i Pr(D_i)Pr(Z|D_i) \\
&= ((\alpha) \times 0) + ((1-\alpha) \times \bar{P}_2) \\
\bar{P}_2 &= \sum_i Pr(C_i)Pr(Z|C_i) \\
&= (\beta \times 1) + ((1-\beta)(\alpha) \times 0) + ((1-\beta)(1-\alpha) \times \bar{P}_2)
\end{aligned}$$

Since, the $Pr(Z|C_i)$ makes the function recursive:

$$Pr(Z|C_1) = 1,\qquad Pr(Z|C_2) = 0,\qquad Pr(Z|C_3) = \bar{P}_2$$
$$Pr(Z|C_4) = 0,\qquad Pr(Z|C_5) = \bar{P}_2$$

Then, we rearrange $\bar{P}_1$ and get P_1 from $P_1 = 1 - \bar{P}_1$:

$$\begin{aligned}
\bar{P}_1 &= (1-\alpha)(\frac{\beta}{\alpha+\beta-\alpha\beta}) \\
&= \frac{\beta-\alpha\beta}{\alpha+\beta-\alpha\beta} \\
P_1 &= 1 - \frac{\beta-\alpha\beta}{\alpha+\beta-\alpha\beta} \\
&= \frac{\alpha+\beta-\alpha\beta-\beta+\alpha\beta}{\alpha+\beta-\alpha\beta} \\
&= \frac{\alpha}{\alpha+\beta-\alpha\beta}
\end{aligned}$$

To get P_2, we use the same procedure.

$$\begin{aligned}
\bar{P}_2 &= (\frac{\beta}{\alpha+\beta-\alpha\beta}) \\
P_2 &= 1-(\frac{\beta}{\alpha+\beta-\alpha\beta}) \\
&= (\frac{\alpha+\beta-\alpha\beta-\beta}{\alpha+\beta-\alpha\beta}) \\
&= \frac{\alpha(1-\beta)}{\alpha+\beta-\alpha\beta}
\end{aligned}$$

P_3 can also be found using a similar method. By finding the probability of Z and $\bar{P}_1$ as follows:

$$\begin{aligned}
\bar{P}_3 : E_1 &\rightarrow \textit{ B hits, A misses }, & E_2 &\rightarrow \textit{ A hits, B misses} \\
E_3 &\rightarrow \textit{ Both hit,} & E_4 &\rightarrow \textit{ Both miss}
\end{aligned}$$

Then, $Pr(Z|E_i)$ makes the function recursive:

$$\begin{aligned}
Pr(Z|E_1) &= 1, & Pr(Z|E_2) &= 0 \\
Pr(Z|E_3) &= 0, & Pr(Z|E_4) &= \bar{P}_3
\end{aligned}$$

Thus:

$$\begin{aligned}
\bar{P}_3 &= \sum_i Pr(E_i)Pr(Z|E_i) \\
&= (\beta(1-\alpha)\times 1) + (\beta(1-\alpha)\times 0) + (\alpha\beta\times 0) + ((1-\alpha)(1-\beta)\times \bar{P}_3) \\
&= \frac{\beta-\alpha\beta+\alpha\beta}{\alpha+\beta-\alpha\beta} \\
&= \frac{\beta}{\alpha+\beta-\alpha\beta}
\end{aligned}$$

Then, we rearrange $\bar{P}_3$ and get P_3 from $P_3 = 1-\bar{P}_3$:

$$\begin{aligned}
P_3 &= 1-\frac{\beta}{\alpha+\beta-\alpha\beta} \\
&= \frac{\alpha+\beta-\alpha\beta-\beta}{\alpha+\beta-\alpha\beta} \\
&= \frac{\alpha(1-\beta)}{\alpha+\beta-\alpha\beta}
\end{aligned}$$

Therefore, we get the following values:

$$\begin{aligned}
P_1 &= \frac{\alpha}{\alpha+\beta-\alpha\beta} \\
P_2 &= \frac{\alpha(1-\beta)}{\alpha+\beta-\alpha\beta} \\
P_3 &= \frac{\alpha(1-\beta)}{\alpha+\beta-\alpha\beta}
\end{aligned}$$

Problem 4: XxXDefectiveXxX
An electronics assembly firm buys its microchips from three different suppliers; half of them are bought from firm X, whilst firms Y and Z supply 30% and 20%, respectively. The suppliers use different quality-control procedures and the percentages of defective chips are 2%, 4%, and 4% for X, Y, and Z, respectively. The probabilities that a defective chip will fail two or more assembly-line tests are 40%, 60%, and 80%, respectively, whilst all defective chips have a 10% chance of escaping detection. An assembler finds a chip that fails only one test. What is the probability that it came from supplier X?

Solution. We are asked to find the probability that the chip came from supplier X given that it failed exactly one assembly-line test. Let $F = 1$ represent the event that the chip failed exactly one test, and let D represent the event that the chip is defective.
Given data:

$$\begin{aligned} P(X) &= 0.5, & P(Y) &= 0.3, & P(Z) &= 0.2, \\ P(D|X) &= 0.02, & P(D|Y) &= 0.04, & P(D|Z) &= 0.04. \end{aligned}$$

The probabilities of failing exactly one test given that the chip is defective are:

$$\begin{aligned} P(F = 1|D, X) &= 0.5 \quad (\text{since } 1 - 0.4 - 0.1 = 0.5), \\ P(F = 1|D, Y) &= 0.3 \quad (\text{since } 1 - 0.6 - 0.1 = 0.3), \\ P(F = 1|D, Z) &= 0.1 \quad (\text{since } 1 - 0.8 - 0.1 = 0.1). \end{aligned}$$

We will use Bayes' Theorem to calculate the probability:

$$P(X|F = 1) = \frac{P(F = 1|D, X) \cdot P(D|X) \cdot P(X)}{P(F = 1)},$$

where $P(F = 1)$ is the total probability of a chip failing exactly one test:

$$P(F = 1) = P(F = 1|D, X) \cdot P(D|X) \cdot P(X) + P(F = 1|D, Y) \cdot P(D|Y) \cdot P(Y) + P(F = 1|D, Z) \cdot P(D|Z) \cdot P(Z).$$

Calculating the individual terms:

$$\begin{aligned} P(F = 1|D, X) \cdot P(D|X) \cdot P(X) &= (0.5)(0.02)(0.5) = 0.005, \\ P(F = 1|D, Y) \cdot P(D|Y) \cdot P(Y) &= (0.3)(0.04)(0.3) = 0.0036, \\ P(F = 1|D, Z) \cdot P(D|Z) \cdot P(Z) &= (0.1)(0.04)(0.2) = 0.0008. \end{aligned}$$

Summing these values gives:

$$P(F = 1) = 0.005 + 0.0036 + 0.0008 = 0.0094.$$

Finally, applying Bayes' Theorem:

$$P(X|F = 1) = \frac{0.005}{0.0094} \approx 0.5319.$$

Thus, the probability that the chip came from supplier X, given that it failed exactly one assembly-line test, is approximately 0.5319 or 53.19%.

Problem 5: Little John
A boy is selected at random amongst the children belonging to families with n children. It is known that he has at least two sisters. Show that the probability he has $k-1$ brothers is

$$\frac{(n-1)!}{(2^{n-1}-n)(k-1)!(n-k)!}$$

for $1 \leq k \leq n-2$ and zero for other values of k. Assume that boys and girls are equally likely.

Solution. We first let A_j be the event where the boy's family contains j boys and $n-j$ girls amongst n siblings. Then:

$$Pr(A_j) = \frac{{}^{n-1}C_{j-1}(\frac{1}{2})^{n-1}}{\sum_{j=1}^{n} {}^{n-1}C_{j-1}(\frac{1}{2})^{n-1}}$$

Which is simply the combination operation in the numerator but multiplied to a factor of $\frac{1}{2^{n-1}}$ from the geometric series factor.

$$Pr(A_j) = \frac{(n-1)!}{2^{n-1}(j-1)!(n-j)!}$$

Then, if T is the event that the boy has at least two sisters, we apply Baye's Theorem on the following:

$$Pr(A_k|T) = \frac{Pr(A_k)}{\sum_{j=1}^{n-2} Pr(A_j)}$$
$$for, 1 \leq k \leq n-2.$$

Notice that this denominator is similar to the denominator in the case where the known minimum amount of sisters is not known, only omitting the $j = n-1$ and $j = n$ terms. So, the denominator of $Pr(A_k|B)$ should simply be:

$$\begin{aligned}
&1 - \frac{(n-1)!}{2^{n-1}(n-2!)1!} - \frac{(n-1)!}{2^{n-1}(n-1!)0!} \\
&= \frac{2^{n-1}-(n-1)-1}{2^{n-1}} \\
&= \frac{2^{n-1}-n}{2^{n-1}}
\end{aligned}$$

Thus, we can substitute this back to $Pr(A_k|T)$, where $Pr(A_k)$ is simply equivalent to $Pr(A_j)$ when $k = j$. This leaves us with,

$$Pr(A_j) = \frac{(n-1)!}{2^{n-1}(k-1)!(n-k)!}\frac{2^{n-1}}{2^{n-1}-n} \quad = \frac{(n-1)!}{(2^{n-1}-n)(k-1)!(n-k)!}$$

Leaving us with the same formula assumed to be the correct one in the problem.