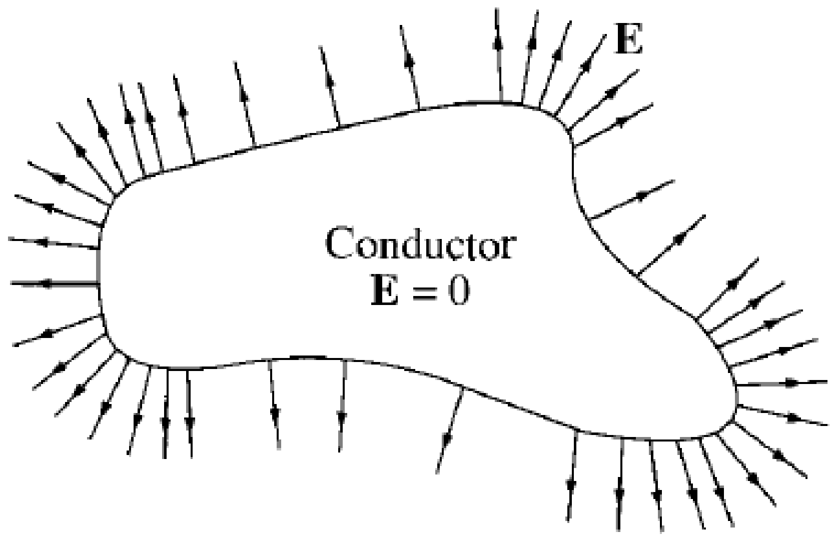
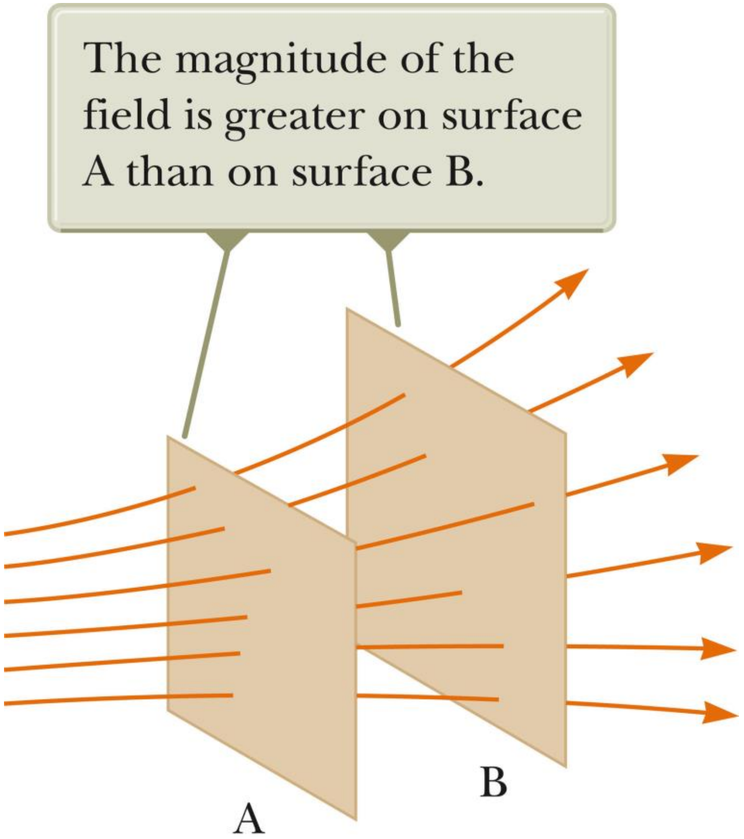
Gauss' Law



Charge and Electric Flux

Electric field lines

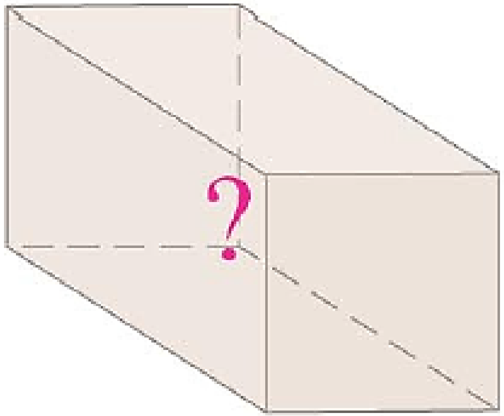
- *Electric field lines* give us a means of representing the electric field pictorially.
- The electric field vector is tangent to the electric field line at each point.
- The number of lines per unit area through a surface perpendicular to the lines is proportional to the magnitude of the electric field in that region.



Charge and Electric Flux

Electric field lines

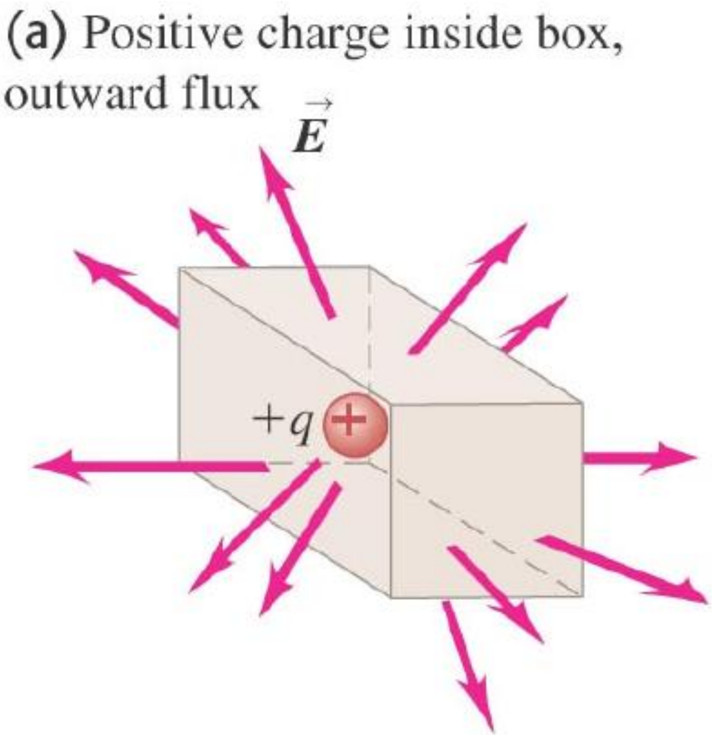
- If the electric field pattern is known in a given region, what can we determine about the charge distribution in that region?

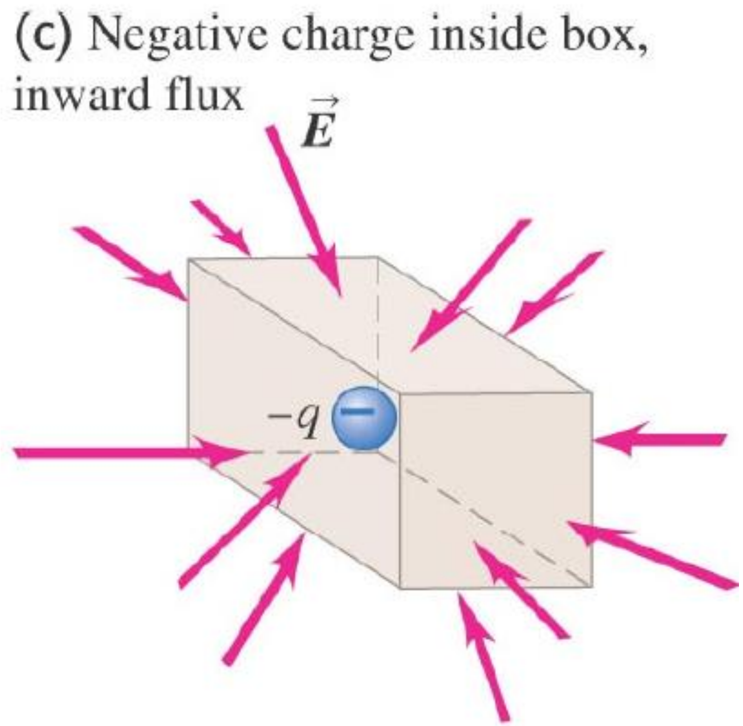


A box containing an unknown amount of charge

Charge and Electric Flux

Flux of the electric field lines

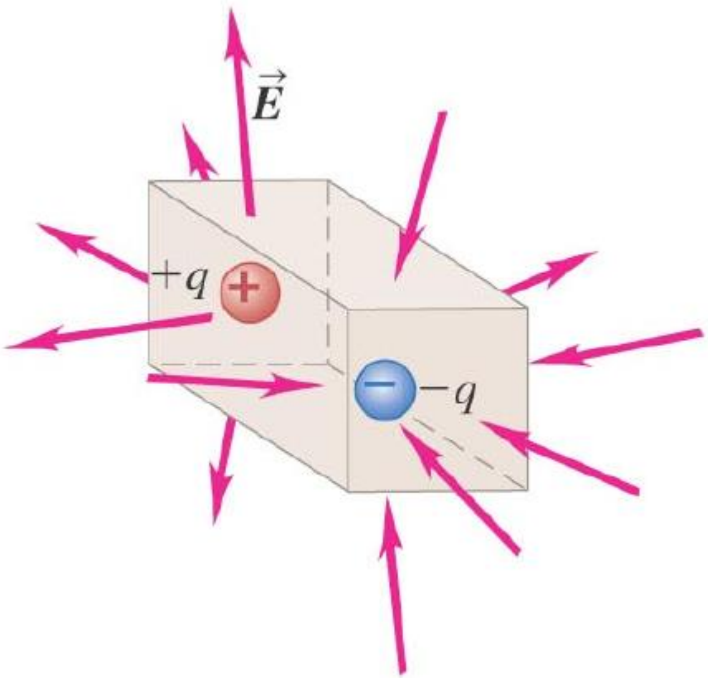




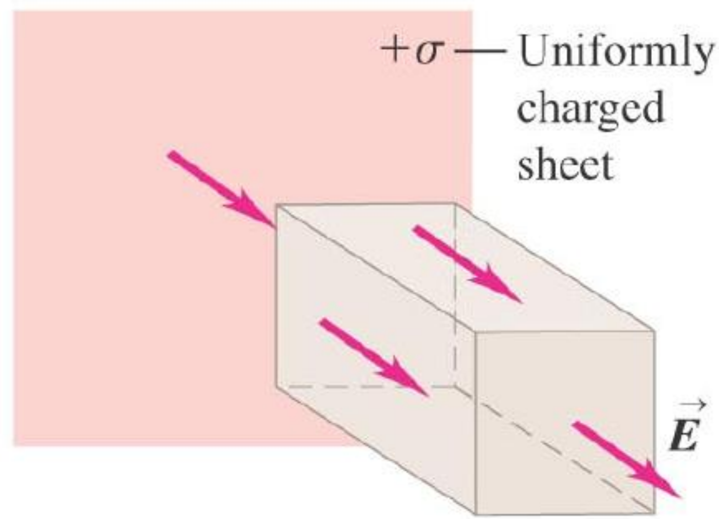
Charge and Electric Flux

Flux of the electric field lines

(b) Zero *net* charge inside box, inward flux cancels outward flux.



(c) No charge inside box, inward flux cancels outward flux.



Charge and Electric Flux

Electric flux Φ_E

- For a uniform electric field:

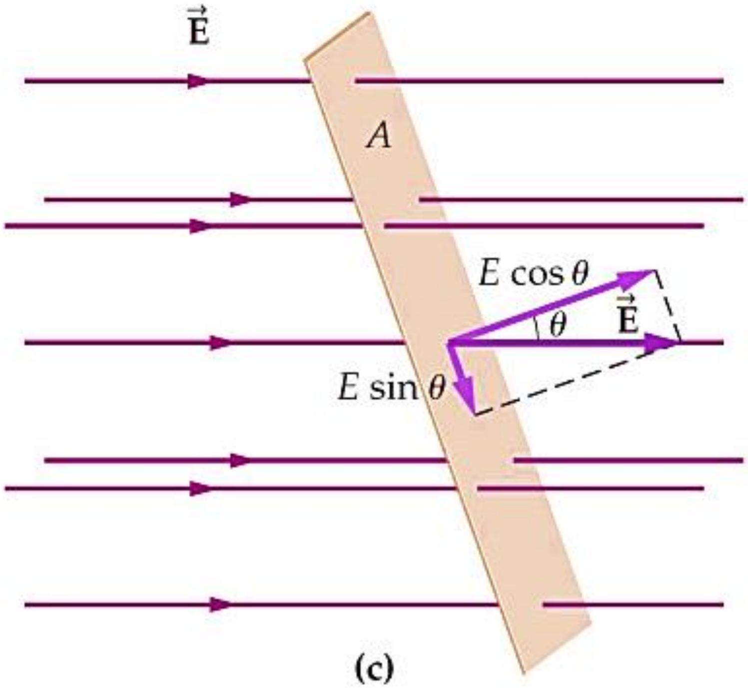
$$\Phi_E = E_{\perp} A$$

- Define the vector area $\vec{\boldsymbol{A}}$

$$\vec{\boldsymbol{A}} = A\,\hat{\boldsymbol{n}}$$

$$\Phi_E = \vec{\boldsymbol{E}} \cdot \vec{\boldsymbol{A}}$$

- Units: $(\mathrm{N} \cdot \mathrm{m}^2/\mathrm{C})$

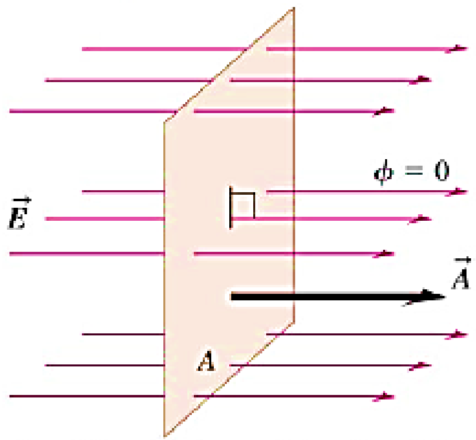


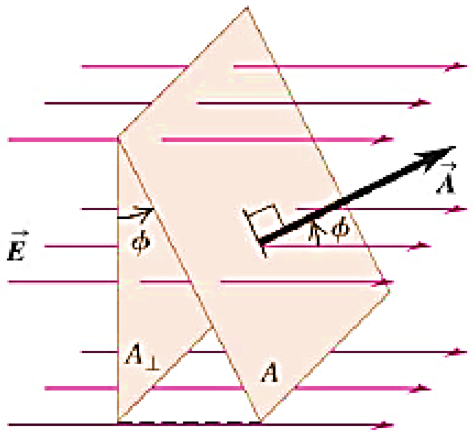
(c)

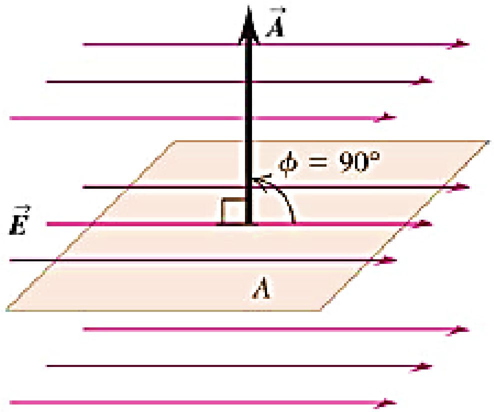
Charge and Electric Flux

Electric flux Φ_E

$$\Phi_E = \vec{E} \cdot \vec{A}$$







$$\Phi_E = EA$$

$$\Phi_E = EA\cos\phi$$

$$\Phi_E = 0$$

Charge and Electric Flux

Electric flux Φ_E

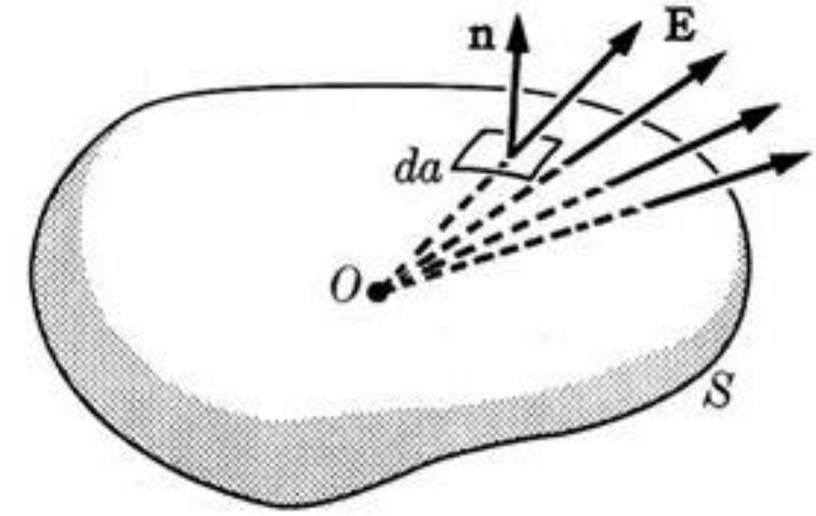
For a *non-uniform* electric field, irregular surface:

- On a surface element dA :

$$d\Phi_E = \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}}$$

- For the complete surface

$$\Phi_E = \int_S \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}}$$



Charge and Electric Flux

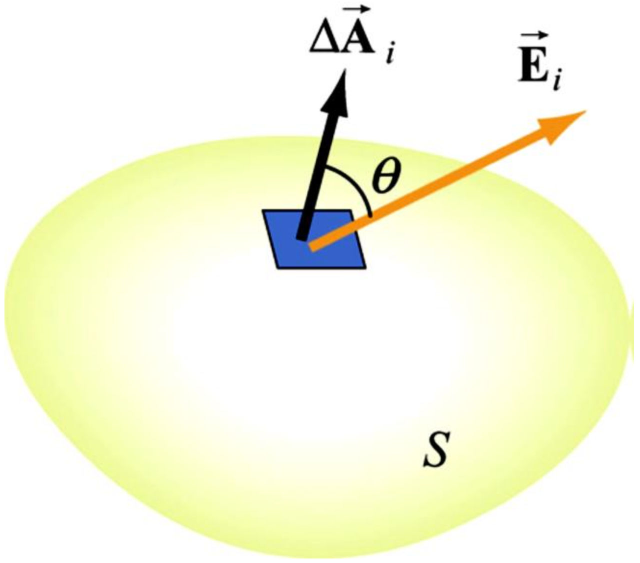
Net electric flux Φ_{Net}

For closed (containing a *volume*) surface:

- The *net electric flux*

$$\Phi_{Net} = \oint \vec{E} \cdot d\vec{A}$$

($d\vec{A}$ points outward)



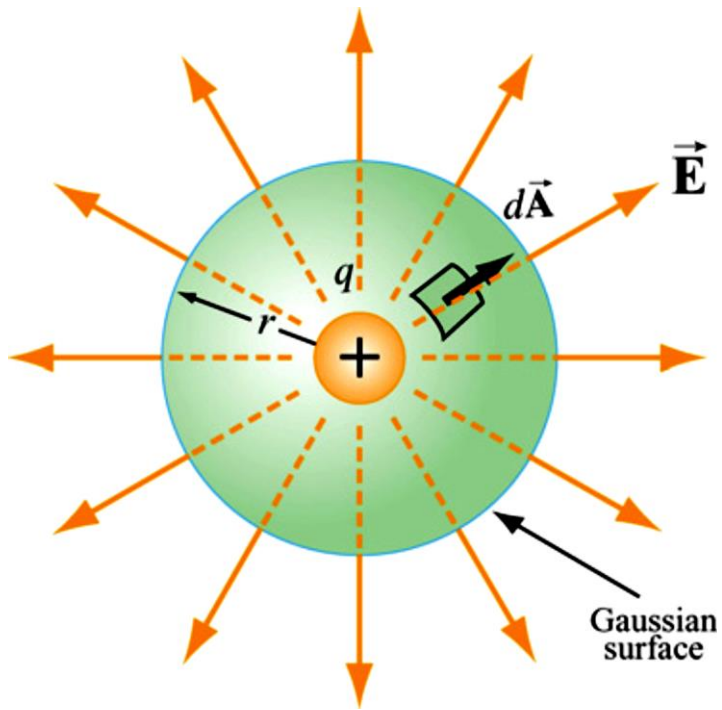
Gauss' Law

Net electric flux Φ_{Net}

For point charge *q* enclosed by a spherical surface

$$\vec{\mathbf{E}} = k\frac{q}{r^2}\hat{\mathbf{r}}$$

$$d\vec{A} = dA\,\hat{\mathbf{r}}$$



- The *net electric flux*

$$\Phi_{Net} = \left(k\frac{q}{r^2}\right)\oint dA$$

$$= \left(k\frac{q}{r^2}\right)(4\pi r^2)$$

$$\Phi_{Net} = \frac{q}{\epsilon_0}$$

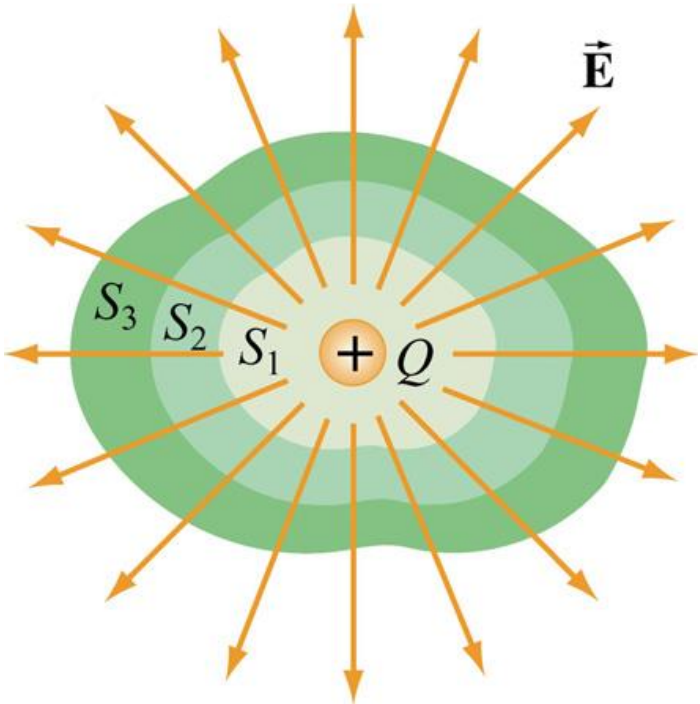
Gauss' Law

Net electric flux Φ_{Net}

For point charge *q* enclosed by any closed surface

$$\Phi_{Net} = \frac{q}{\epsilon_0}$$

- The shape of the closed surface can be arbitrarily chosen — the same result is obtained.

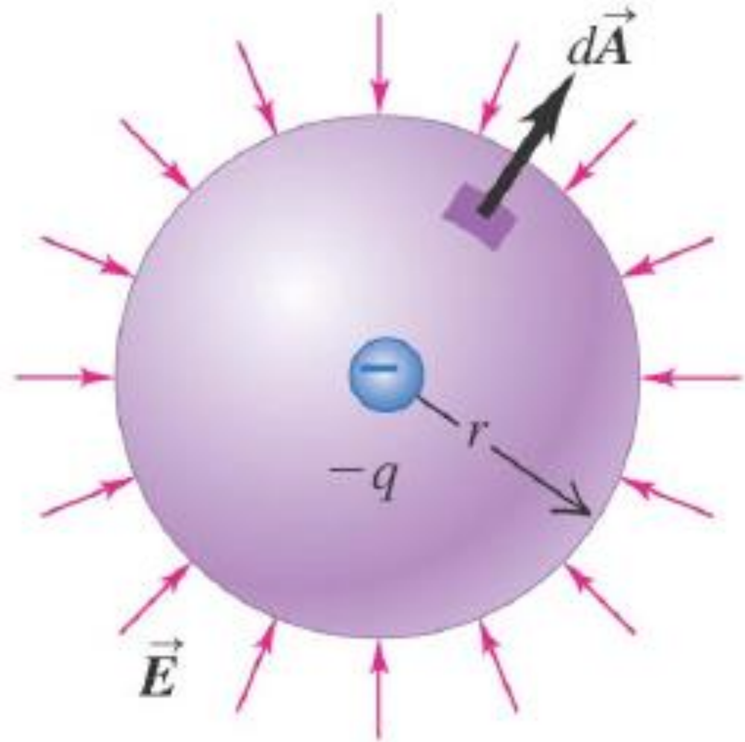


Gauss' Law

Net electric flux Φ_{Net}

For negative point charge –*q*

$$\Phi_{Net} = -\frac{q}{\epsilon_0}$$

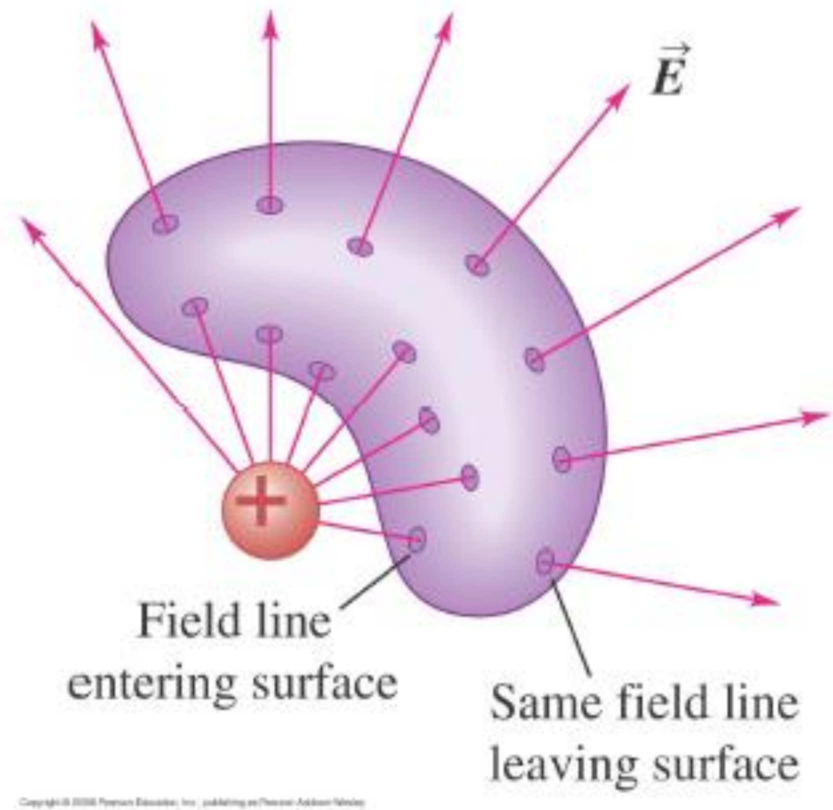


Gauss' Law

Net electric flux Φ_{Net}

When the point charge q is *not* enclosed by the surface

$$\Phi_{Net} = 0$$



Gauss' Law

Net electric flux Φ_{Net}

For N point charges q_i enclosed by the surface

$$\vec{\mathbf{E}} = k\sum_{j=1}^{N}\frac{q_j}{{r_j}^2}\hat{\mathbf{r}}_j$$

- The *net electric flux*

$$\Phi_{Net} = \frac{1}{\epsilon_0}\sum_{j=1}^{N} q_j$$

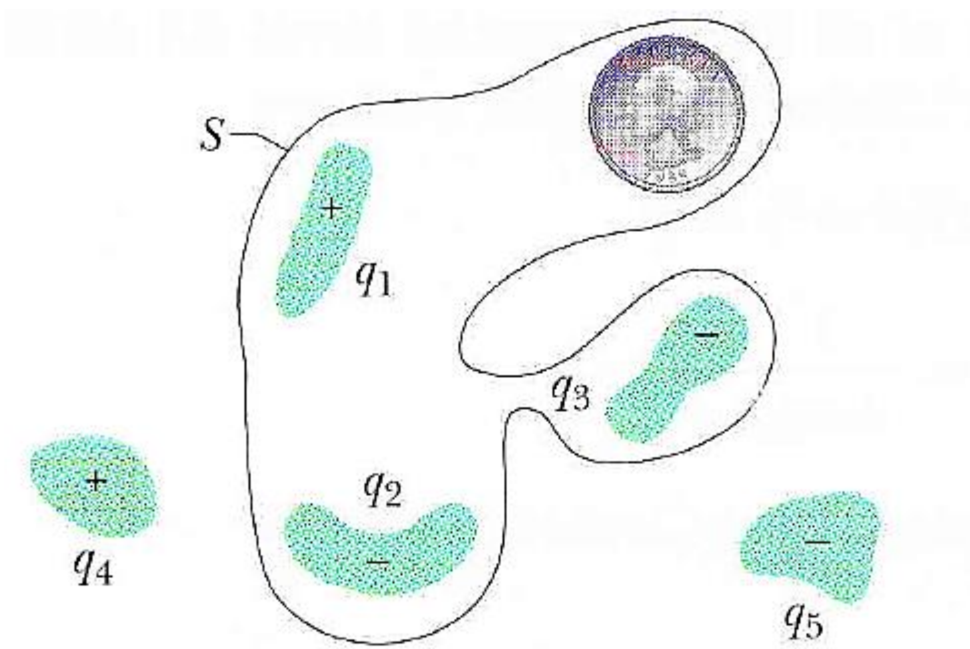
$$\Phi_{Net} = \frac{1}{\epsilon_0}\,Q_{Enclosed}$$

Gauss' Law

Gauss' Law

For N point charges q_i enclosed by the surface

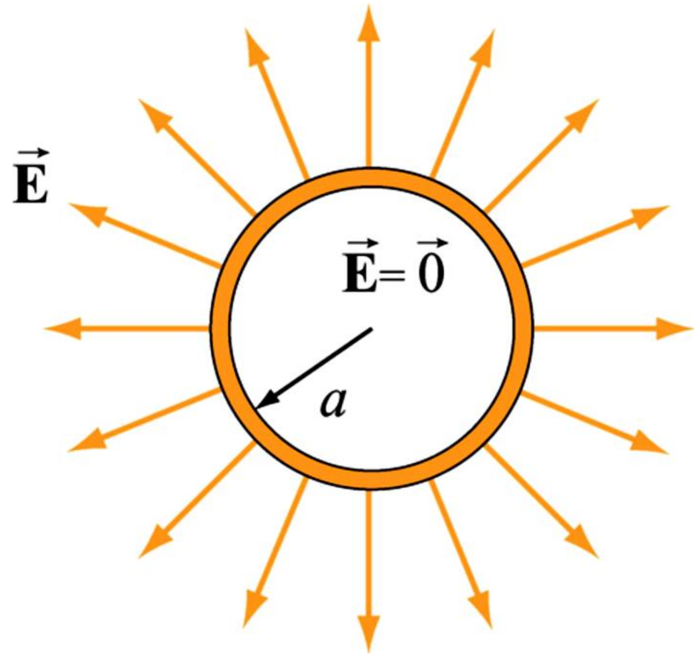
$$\Phi_{Net} = \frac{1}{\epsilon_0} Q_{Enclosed}$$



Applications of Gauss' Law

Example: Spherical shell

A thin spherical shell of radius *a* has a charge +*Q* evenly distributed over its surface. Find the electric field both inside and outside the shell.



Applications of Gauss' Law

Example: *Spherical shell*

Solution: The charge distribution is spherically symmetric. The electric field must be radially symmetric and directed outward. Use spherical *Gaussian surface*.

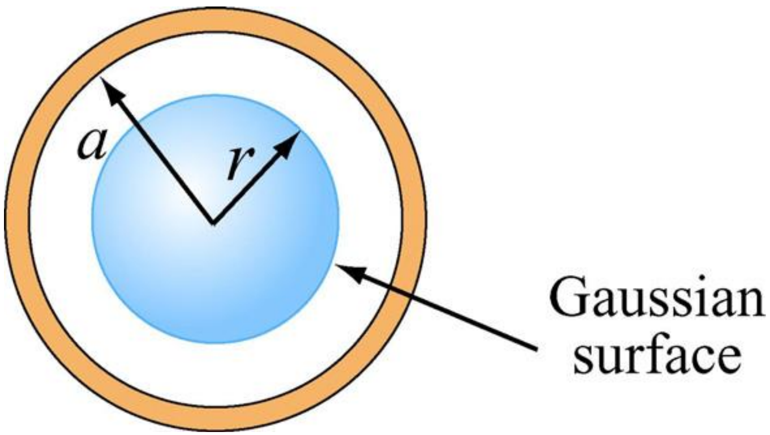
- Case 1: $r < a$

$$\Phi_{Net} = E_{in}\, A = \frac{Q_{Enc}}{\epsilon_0}$$

$$= 0$$

- Therefore:

$$E_{in} = 0$$



Applications of Gauss' Law

Example: *Spherical shell*

- Case 2: $r > a$

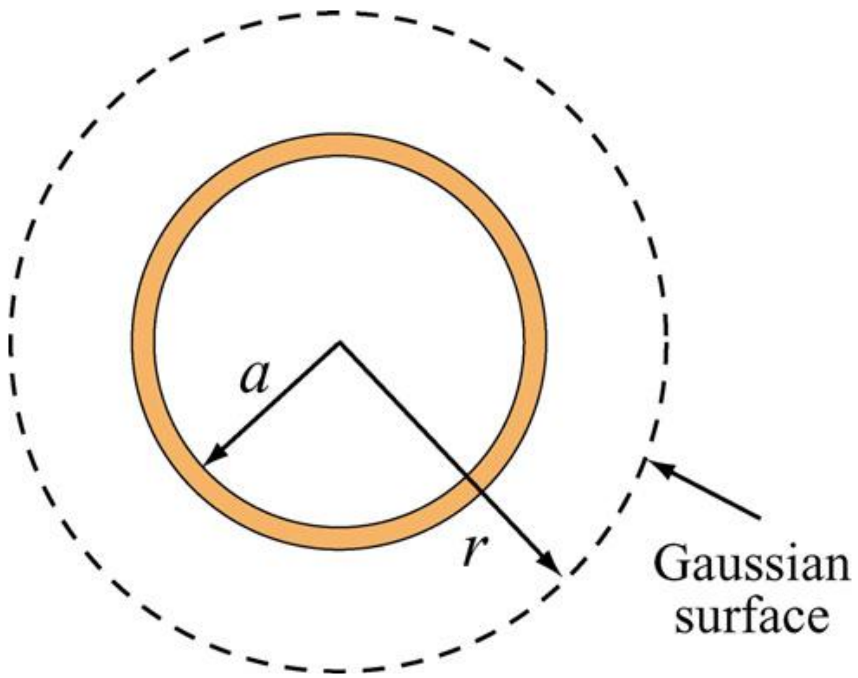
$$\Phi_{Net} = E_{out}A = E_{out}(4\pi r^2)$$

$$\frac{Q_{Enc}}{\epsilon_0} = \frac{Q}{\epsilon_0}$$

- Therefore:

$$E_{out} = \frac{Q}{\epsilon_0(4\pi r^2)}$$

$$= k\frac{Q}{r^2}$$

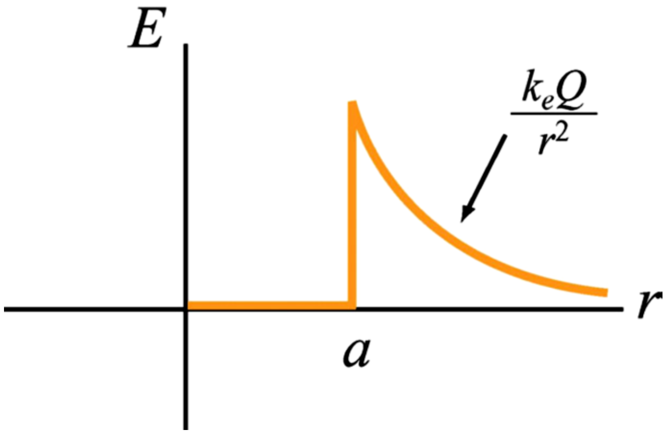


Applications of Gauss' Law

Example: *Spherical shell*

- Solution:

$$E(r) = \begin{cases} 0 & r \le a \\ k\dfrac{Q}{r^2} & r \ge a \end{cases}$$



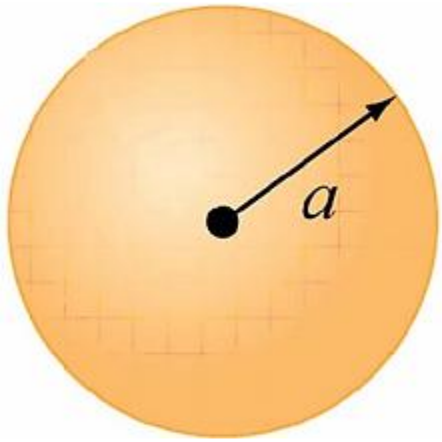
Applications of Gauss' Law

Example: Non-Conducting Solid Sphere

An electric charge $+Q$ is *uniformly distributed* throughout a non-conducting solid sphere of radius a. Determine the electric field everywhere inside and outside the sphere.

- Charge density:

$$\rho = \frac{Q}{(4\pi a^3/3)}$$



Applications of Gauss' Law

Example: Non-Conducting Solid Sphere

Solution: The electric field must be radially symmetric and directed outward. The magnitude of the electric field is constant on spherical surfaces of radius *r*.

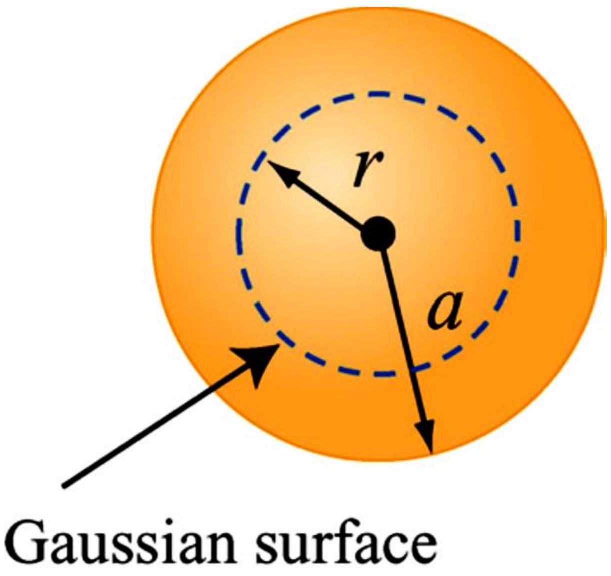
- Case 1: $r < a$

$$\Phi_{Net} = E_{in}\, A = E_{in}(4\pi r^2)$$

$$\frac{Q_{Enc}}{\epsilon_0} = \frac{\rho(4\pi a^3/3)}{\epsilon_0}$$

- Therefore:

$$E_{in} = \frac{\rho r}{3\epsilon_0} = kQ\left(\frac{r}{a^3}\right)$$



Applications of Gauss' Law

Example: Non-Conducting Solid Sphere

- Case 2: $r > a$

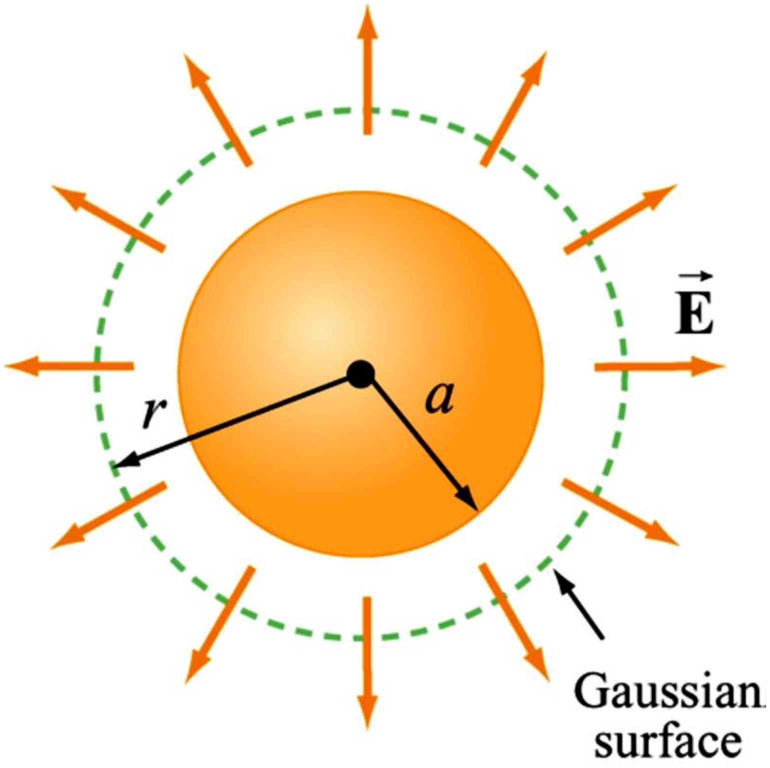
$$\Phi_{Net} = E_{out}\,A = E_{out}(4\pi r^2)$$

$$\frac{Q_{Enc}}{\epsilon_0} = \frac{Q}{\epsilon_0}$$

- Therefore:

$$E_{out} = \frac{Q}{\epsilon_0(4\pi r^2)}$$

$$= k\frac{Q}{r^2}$$

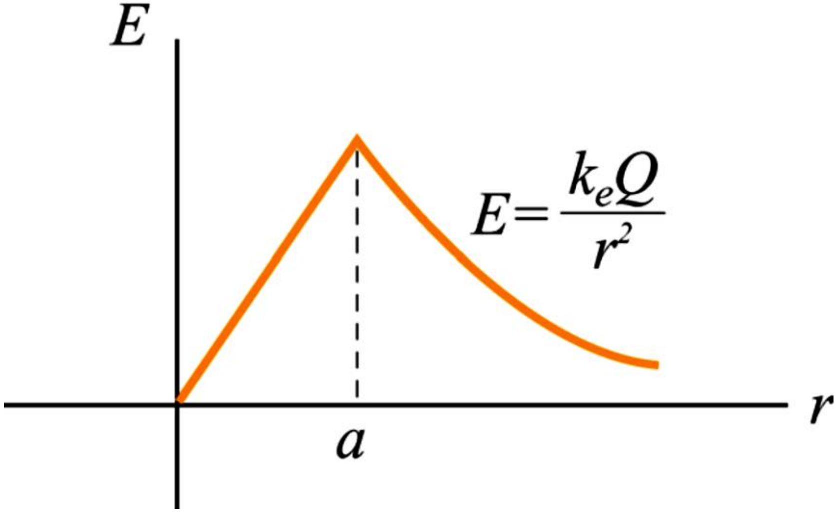


Applications of Gauss' Law

Example: Non-Conducting Solid Sphere

- Solution:

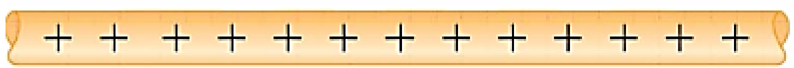
$$E(r) = \begin{cases} kQ\left(\dfrac{r}{a^3}\right) & r \le a \\ k\dfrac{Q}{r^2} & r \ge a \end{cases}$$

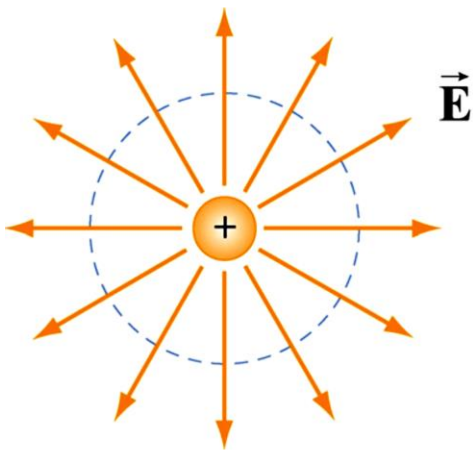


Applications of Gauss' Law

Example: Infinitely Long Rod of Uniform Charge Density

An infinitely long rod of negligible radius has a uniform charge density λ. Calculate the electric field at a distance from the wire.





Solution: The electric field must point *radially away* from the symmetry axis of the rod. The magnitude of the electric field is constant on cylindrical surfaces of radius *r*.

Applications of Gauss' Law

Example: Infinitely Long Rod of Uniform Charge Density

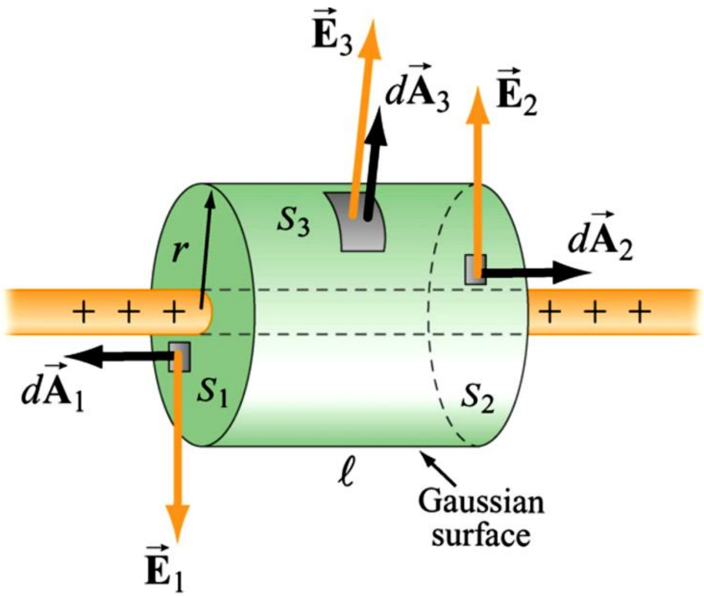
Solution: We choose a coaxial cylinder as our Gaussian surface.

$$\Phi_{Net} = \oint \vec{E} \cdot d\vec{A} = \int \vec{E}_1 \cdot d\vec{A}_1 + \int \vec{E}_2 \cdot d\vec{A}_2 + \int \vec{E}_3 \cdot d\vec{A}_3$$

- The ends S_1 and S_2 yield $\Phi = 0$

$$\Phi_{Net} = \int \vec{E}_3 \cdot d\vec{A}_3 = E_3(2\pi\, r\ell)$$

$$\frac{Q_{Enc}}{\epsilon_0} = \frac{(\lambda\, \ell)}{\epsilon_0}$$



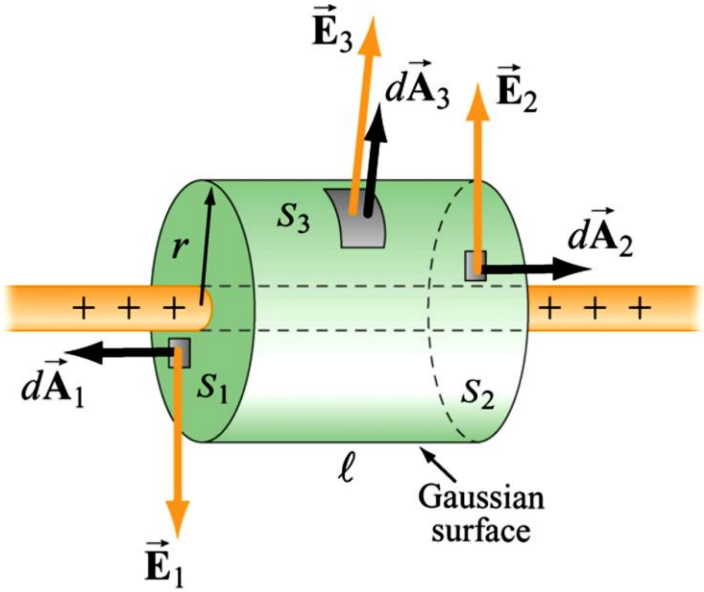
Applications of Gauss' Law

Example: Infinitely Long Rod of Uniform Charge Density

Solution:

- Therefore:

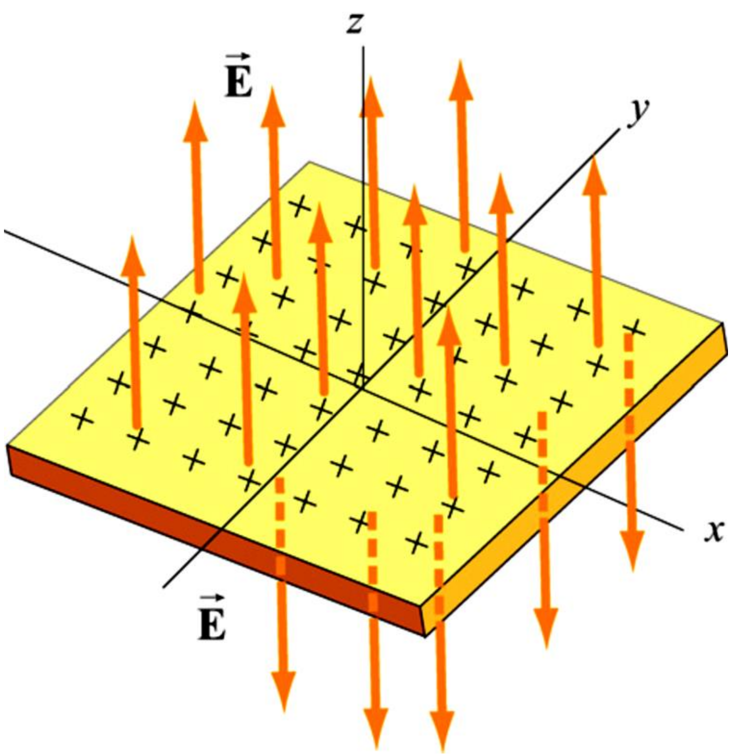
$$E(r) = \frac{\lambda}{2\pi\epsilon_0 r} = 2k\left(\frac{\lambda}{r}\right)$$



Applications of Gauss' Law

Example: Infinitely Plane of Charge

Consider an infinitely wide non-conducting sheet in the *xy*-plane with uniform surface charge density σ. Determine the electric field everywhere in space.

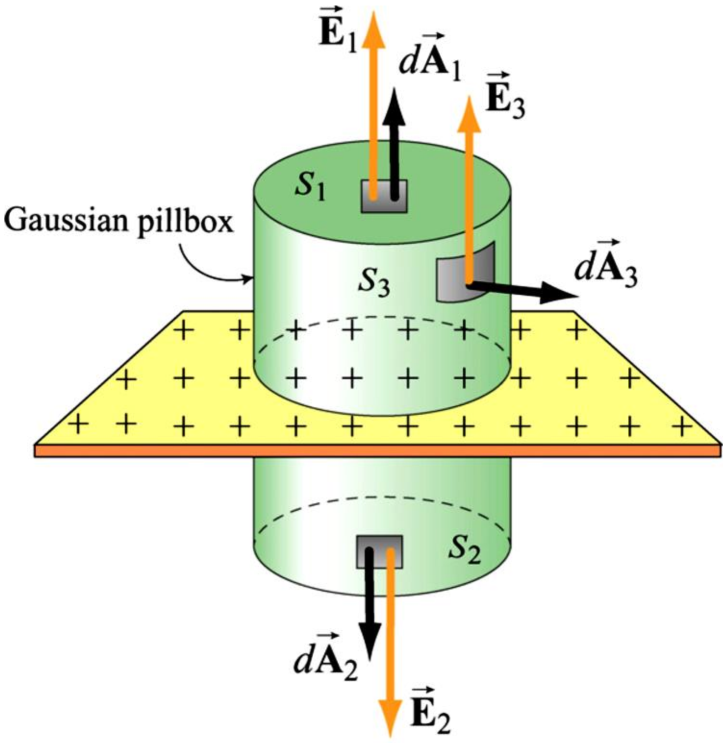


Applications of Gauss' Law

Example: Infinitely Plane of Charge

Solution: By planar symmetry, the electric field must point perpendicularly away from the plane. The magnitude of the electric field is constant on planes parallel to the non-conducting plane.

- We choose our Gaussian surface to be a cylinder ("pillbox").
    - The pillbox consists of three parts: two end-caps and the cylindrical side.



Applications of Gauss' Law

Example: Infinitely Plane of Charge

Solution:

$$\Phi_{Net} = \oint \vec{E} \cdot d\vec{A} = \int \vec{E}_1 \cdot d\vec{A}_1 + \int \vec{E}_2 \cdot d\vec{A}_2 + \int \vec{E}_3 \cdot d\vec{A}_3$$

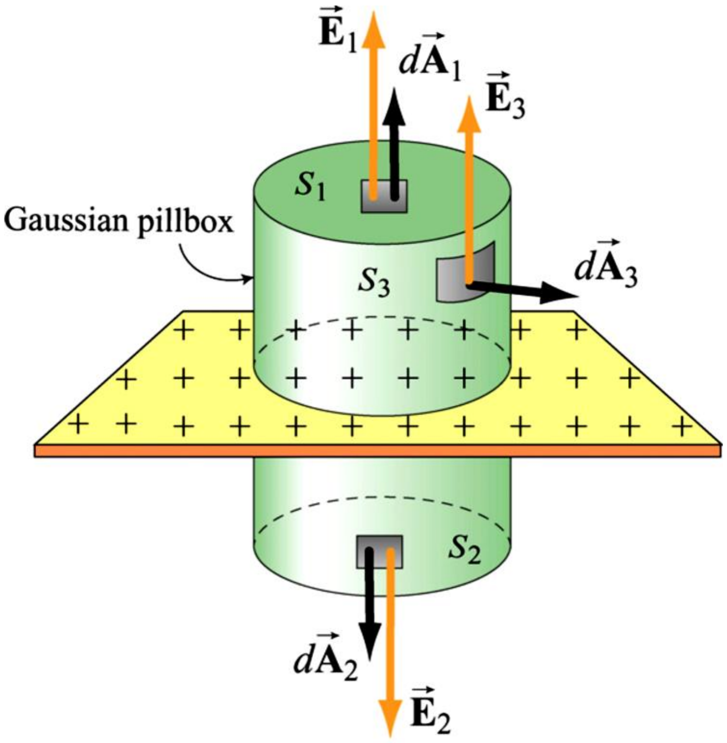
- The side S_3 yield $\Phi = 0$

$$\Phi_{Net} = E_1 A + E_2 A = 2EA$$

$$\frac{Q_{Enc}}{\epsilon_0} = \frac{(\sigma\, A)}{\epsilon_0}$$

- Therefore:

$$E = \frac{\sigma}{2\epsilon_0} = 2\pi k\, \sigma$$

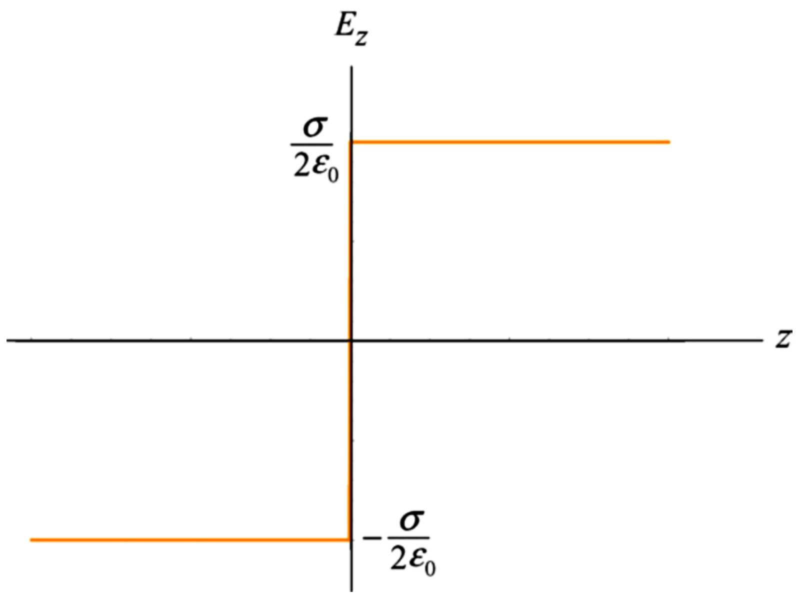


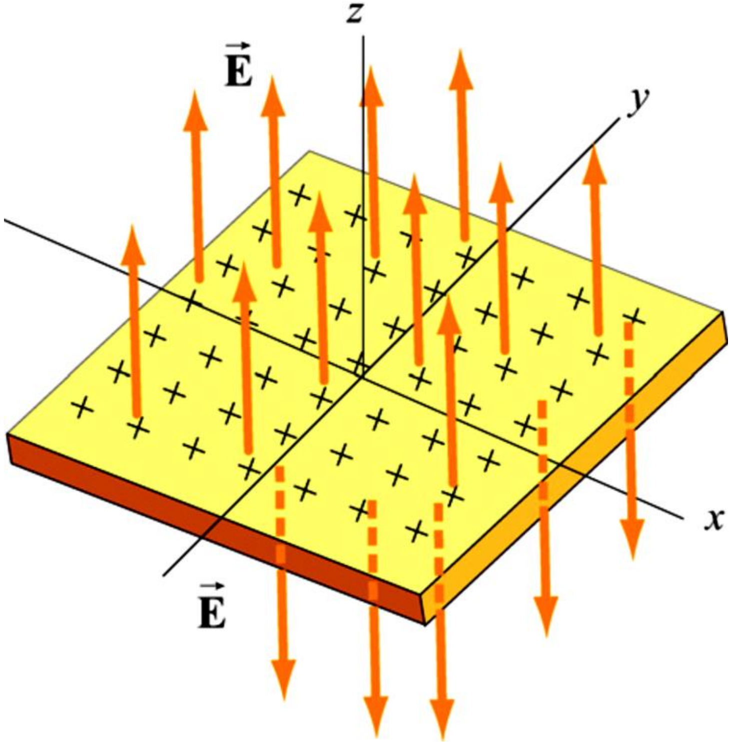
Applications of Gauss' Law

Example: Infinitely Plane of Charge

Solution:

$$E(x,y,z) = \begin{cases} -\dfrac{\sigma}{2\epsilon_0} & z < a \\ +\dfrac{\sigma}{2\epsilon_0} & z > a \end{cases}$$

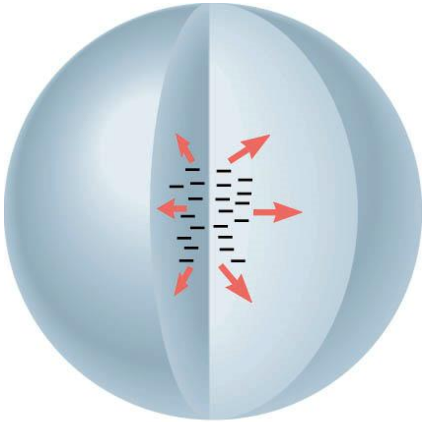




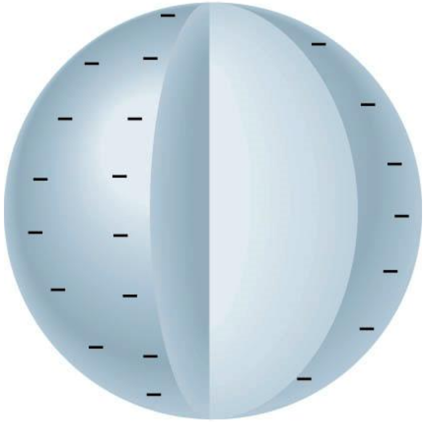
Charges on Conductors

Electric field in a conductor

- At equilibrium under electrostatic conditions, the *electric field* is zero at any point within a conducting material.
- At equilibrium under electrostatic conditions, any *excess charge* resides on the surface of a conductor.



(*a*)

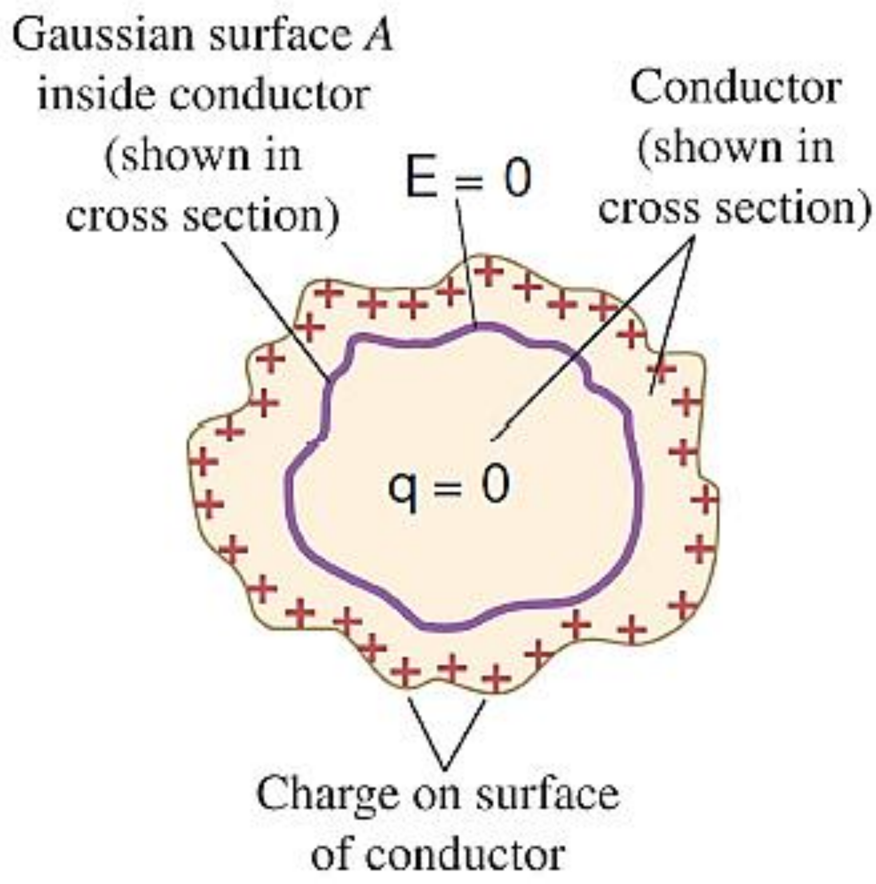


(*b*)

Charges on Conductors

Electric field in a conductor

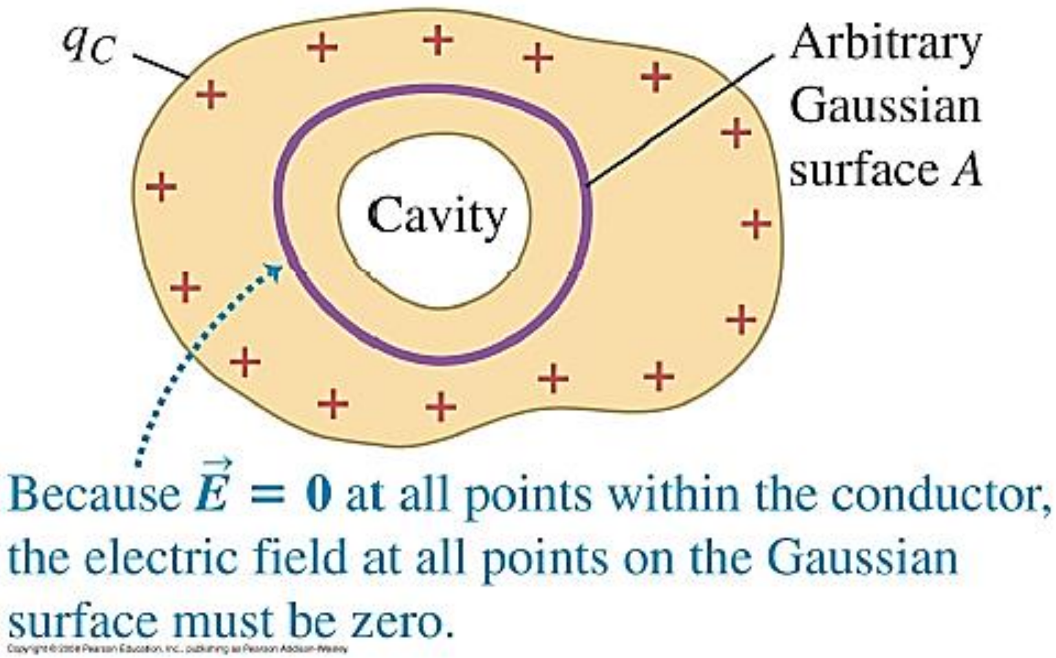
- Excess charge on a solid conductor.



Charges on Conductors

Electric field in a conductor

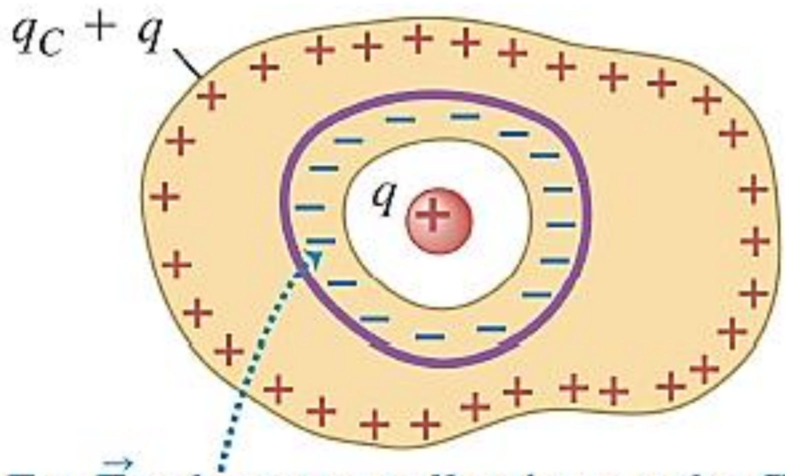
- Excess charge on solid conductor with internal cavity.



Charges on Conductors

Electric field in a conductor

- An isolated point charge suspended in the cavity of charged solid conductor with internal cavity.

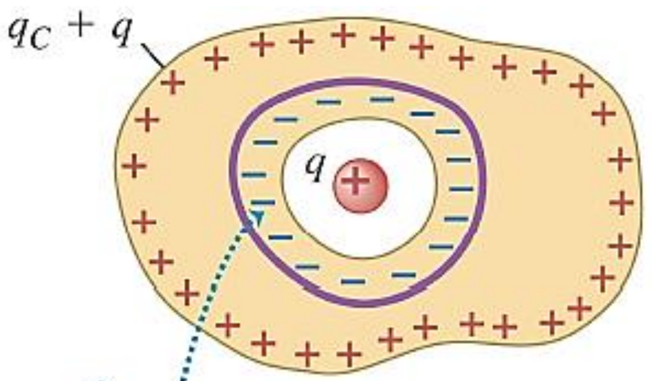


For $\vec{E}$ to be zero at all points on the Gaussian surface, the surface of the cavity must have a total charge $-q$.

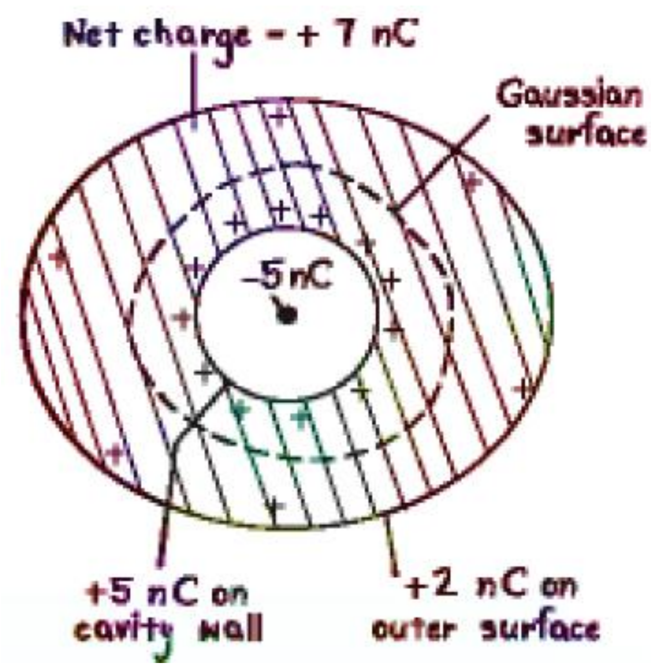
Charges on Conductors

Example: Conductor with internal cavity

A solid conductor carries a total charge of $+7\ \mathrm{nC}$. Within the cavity, insulated from the conductor, is a point charge of $-5\ \mathrm{nC}$. How much charge is on each surface of the conductor?



For $\vec{E}$ to be zero at all points on the Gaussian surface, the surface of the cavity must have a total charge $-q$.



Charges on Conductors

Electric field in a conductor

Electric field on the surface of a charged solid conductor.

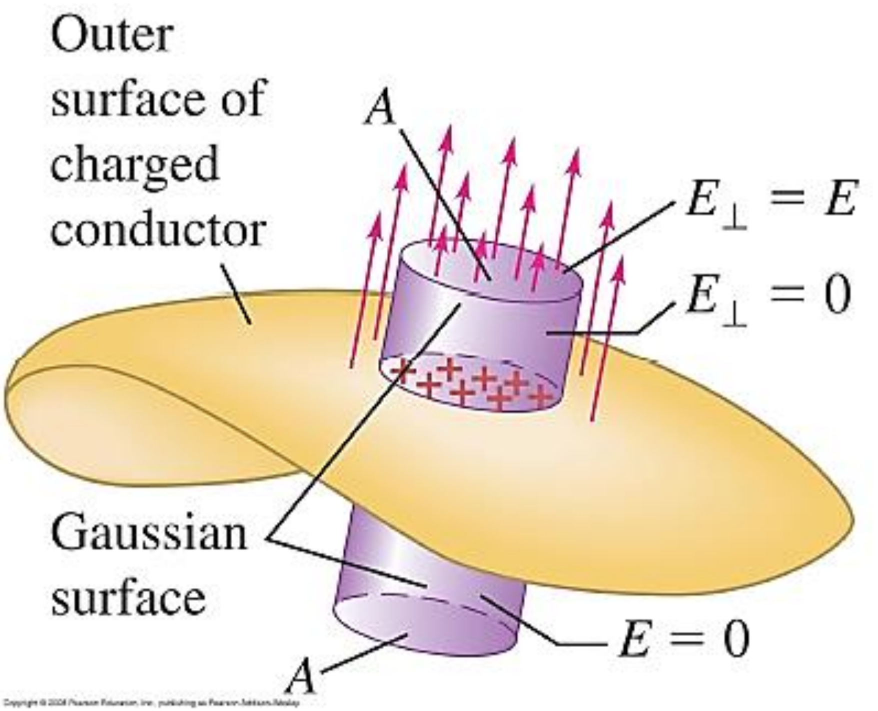
- Static equilibrium:

$$E_{\parallel} = 0$$

(No tangential field)

- Perpendicular field only:

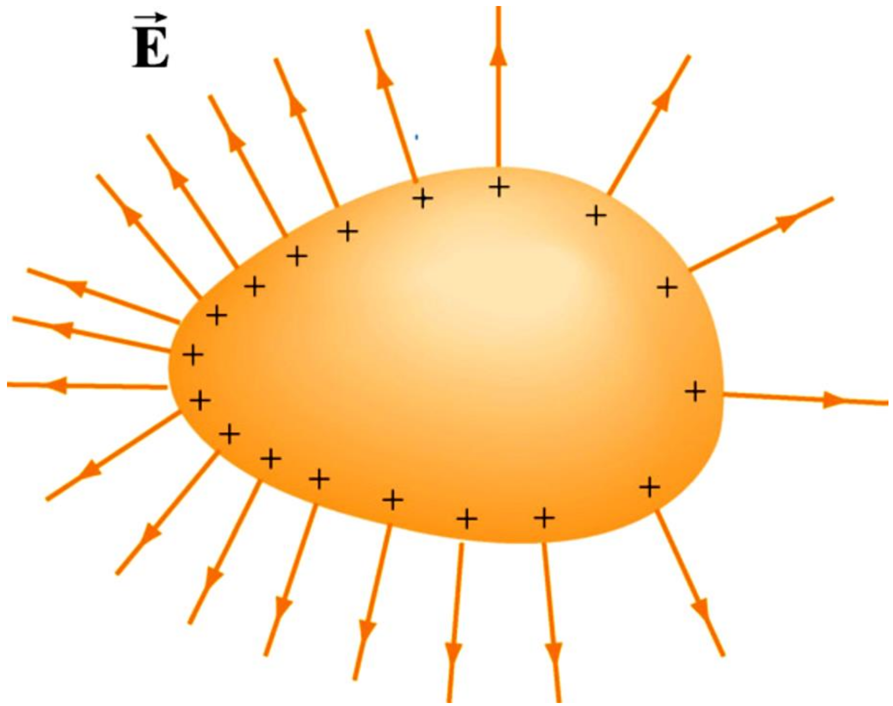
$$E_{\perp} = \frac{\sigma}{\epsilon_0}$$



Charges on Conductors

Electric field in a conductor

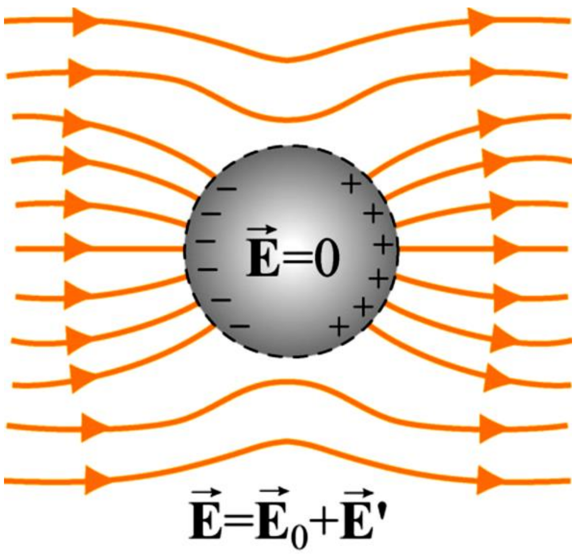
- Electric field on the surface of a charged solid conductor.



Charges on Conductors

Electric field in a conductor

- The conductor shields any charge within it from electric fields created outside the conductor.

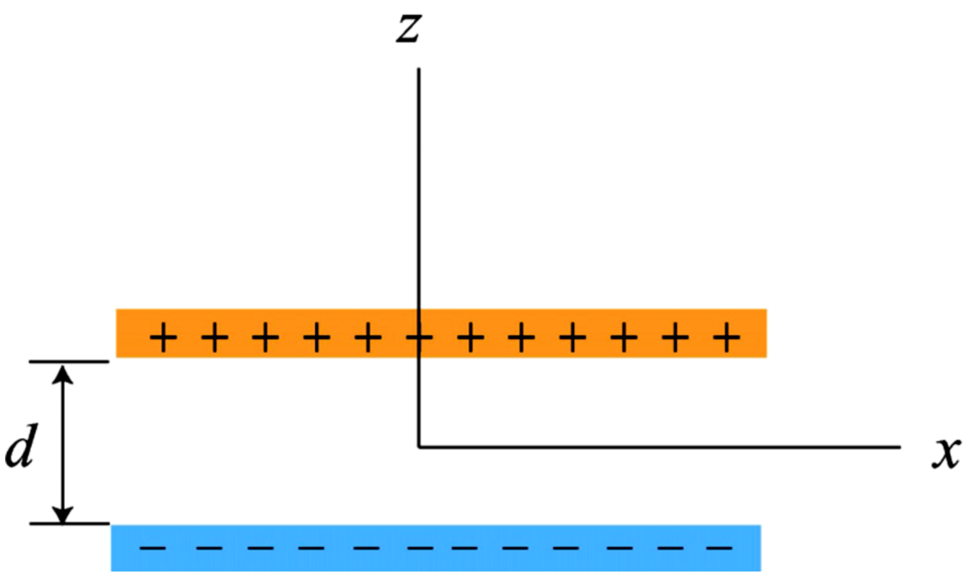


  - At electrostatic equilibrium, *electric field* must vanish inside a conductor. The conductor is *polarized*.

Charges on Conductors

Example: Parallel-plate capacitor

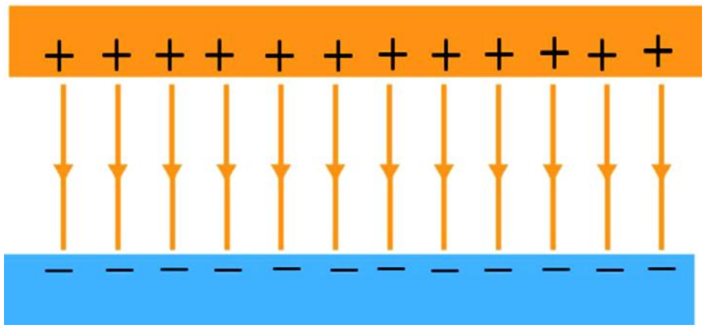
Two parallel infinite conducting planes lying in the *xy*-plane are separated by a distance *d*. Each plane is uniformly charged with equal but opposite surface charge densities. Find the electric field everywhere in space.



Charges on Conductors

Example: Parallel-plate capacitor

- Solution:
  a) By superposition of two infinite planes
  b) Using Gauss' law



$$E(x,y,z) = \begin{cases} 0 & z > d/2 \\ -\dfrac{\sigma}{\epsilon_0} & d/2 > z > -d/2 \\ 0 & z < -d/2 \end{cases}$$

END