Limit Comparison Test

Consider

$$\sum_{n=1}^{\infty} \frac{2n^3-1}{n^3}; \;\; a_n = \frac{2n^2-1}{n^3} > 0$$

If we try to do the direct comparison test:

$$\begin{aligned} 2n^2 - 1 &< 2n^2 \\ \frac{2n^2-1}{n^3} &< \frac{2n^2}{n^3} = \frac{2}{n} \end{aligned}$$

so

$$\sum_{n=1}^{\infty} \frac{2n^3-1}{n^3} < \sum_{n=1}^{\infty} \frac{2}{n} = 2\sum_{n=1}^{\infty} \frac{1}{n}$$

The latter summations are divergent, so direct comparison test **does not work** here.

Going back:

$$\sum_{n=1}^{\infty} \frac{2n^3-1}{n^3}; \;\; a_n = \frac{2n^2-1}{n^3} \sim \frac{2n^2}{n^3} = \frac{2}{n}$$

(Note that $\sim$ means "related to")
Then:

$$\begin{aligned} \lim_{n\to\infty} \frac{\frac{2n^2-1}{n^3}}{\frac{2}{n}} &= \lim_{n\to\infty} \frac{2n^2-1}{n^3} \cdot \frac{n}{2} \\ &= \lim_{n\to\infty} \frac{2n^3-1}{2n^3} \\ &= \lim_{n\to\infty} \frac{2-\frac{1}{n^2}}{2} && \text{(multiply by } \frac{1/n^3}{1/n^3}\text{)} \\ &= 1 \end{aligned}$$

(alternatively, L'Hospital's rule also works here)

As $n \to \infty$, $\frac{2n^2-1}{n^3} = \frac{2}{n}$.

Since $\sum_{n=1}^{\infty} \frac{2}{n}$ is divergent, then $\sum_{n=1}^{\infty} \frac{2n^3-1}{n^3}$ is divergent.

This is called the **limit comparison test**.

Summary of Limit Comparison Test

In summary, given

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is positive and finite:

1. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} b_n$ converges.
2. If $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

Example 1

$$\sum_{n=1}^{\infty} \frac{\sqrt[3]{4n^7 - 3n^2 + 4}}{\sqrt{4n^9 + n + 2}};$$

Then

$$a_n = \frac{\sqrt[3]{4n^7 - 3n^2 + 4}}{\sqrt{4n^9 + n + 2}} \sim \frac{\sqrt[3]{n^7}}{\sqrt{n^9}} = \frac{1}{n^{13/6}}$$

and $a_n > 0$

So

$$\begin{aligned}
\lim_{n\to\infty} \frac{\frac{\sqrt[3]{4n^7-3n^2+4}}{\sqrt{4n^9+n+2}}}{\frac{\sqrt[3]{n^7}}{\sqrt{n^9}}} &= \lim_{n\to\infty} \frac{\frac{\sqrt[3]{4n^7-3n^2+4}}{n^7}}{\frac{\sqrt{4n^9+n+2}}{n^9}} \\
&= \lim_{n\to\infty} \frac{\sqrt[3]{4 - 3/n^5 + 4/n^7}}{\sqrt{4 + 1/n^8 + 2/n^9}} \\
&= \frac{\sqrt[3]{4}}{\sqrt{4}}
\end{aligned}$$

This is a positive and finite. We can then use the LCT.

$$\sum_{n=1}^{\infty} \frac{1}{n^{13/6}} \text{convergent} \Longrightarrow_{\text{limit comparison test}} \sum_{n=1}^{\infty} \frac{\sqrt[3]{4n^7 - 3n^2 + 4}}{\sqrt{4n^9 + n + 2}} \text{convergent}$$

($\frac{1}{n^{13/6}}$ is a p-series with $p = \frac{13}{6} > 1$)

Example 2

$$\sum_{n=1}^{\infty} \sin^4 \frac{1}{n}; \quad a_n = \sin^4 \frac{1}{n} > 0$$

Remember that $\sin x \approx x$ for very small x since

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

Then

$$a_n = \sin^4 \frac{1}{n} \sim \frac{1}{n^4}$$

$$\begin{aligned}
\lim_{n\to\infty} \frac{\sin^4 \frac{1}{n}}{\frac{1}{n^4}} &= \lim_{n\to\infty} \left(\frac{\sin \frac{1}{n}}{\frac{1}{n}} \right)^4 \\
&= 1^4 \\
&= 1
\end{aligned}$$

This is positive and finite. We can then use the LCT.

$$\sum_{n=1}^{\infty} \frac{1}{n^4} \text{convergent} \Longrightarrow_{\text{limit comparison test}} \sum_{n=1}^{\infty} \sin^4 \frac{1}{n} \text{convergent}$$