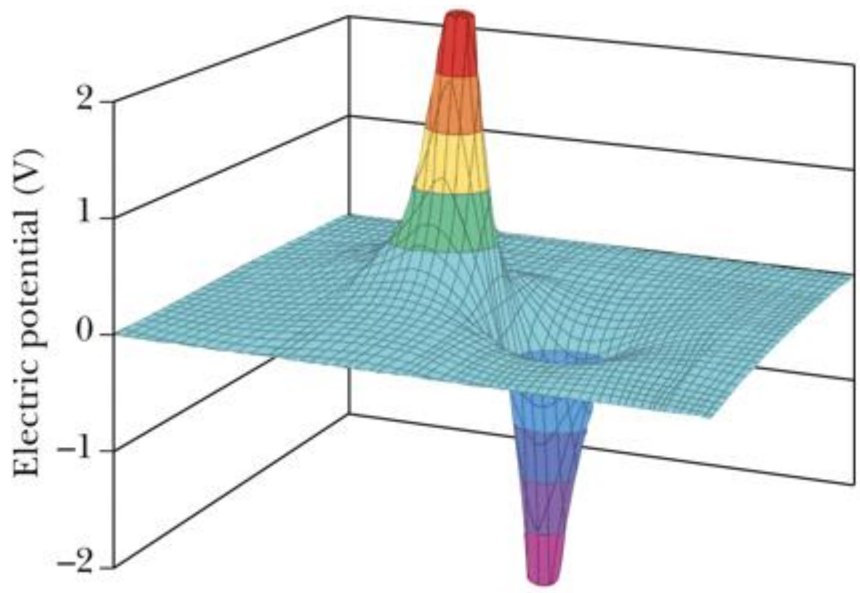
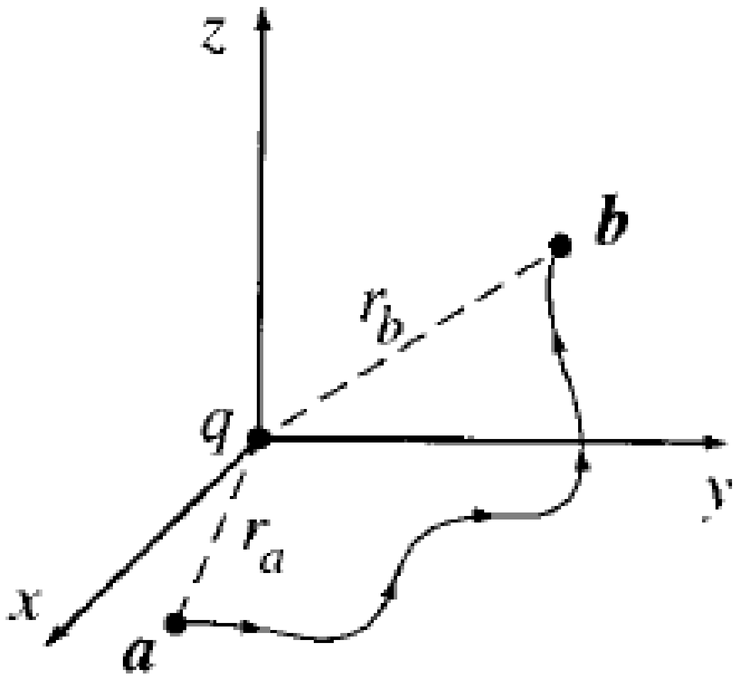
Electric Potential



Electric Potential Energy

Work done by the electric force

- When a charged particle moves in an electric field, the field exerts a force that can do work on the particle.
- The work can be expressed in terms of electric potential energy.



Electric Potential Energy

Work done by the field of a point charge

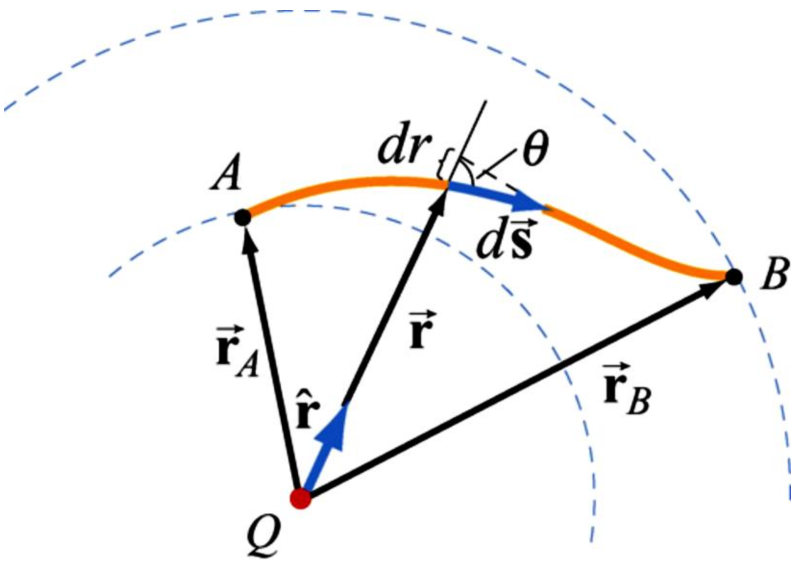
Moving a test charge q_0 from *a* to *b*.

$$\vec{\mathbf{F}} = k\frac{q_0 Q}{r^2}\hat{\mathbf{r}}$$

- The work done.

$$W_{a\to b} = \int_a^b \vec{\mathbf{F}}\cdot d\vec{\boldsymbol{s}}$$

$$= kq_0 Q\int_a^b \frac{1}{r^2}\hat{\mathbf{r}}\cdot d\vec{\boldsymbol{s}}$$



$$\hat{\mathbf{r}}\cdot d\vec{\boldsymbol{s}} = ds(\cos\theta) \ = dr$$

Electric Potential Energy

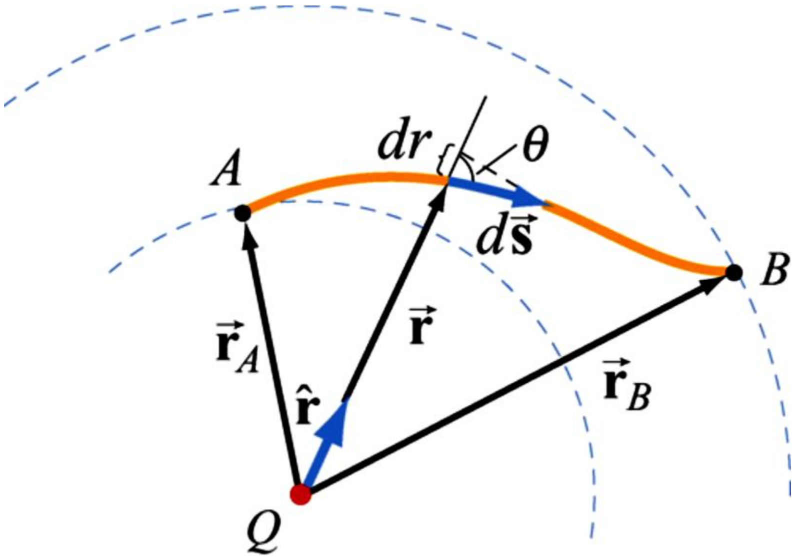
Work done by the field of a point charge

Moving a test charge q_0 from *a* to *b*.

- The work done.

$$W_{a\to b} = kq_0Q\int_a^b \frac{1}{r^2}dr$$

$$= -kq_0Q\left(\frac{1}{r_b} - \frac{1}{r_a}\right)$$



Work done is independent of actual path!

Electric Potential Energy

Work done by uniform electric field

Moving a test charge q_0 from *a* to *b*.

$$\vec{\mathbf{F}} = q_0\vec{\mathbf{E}}$$

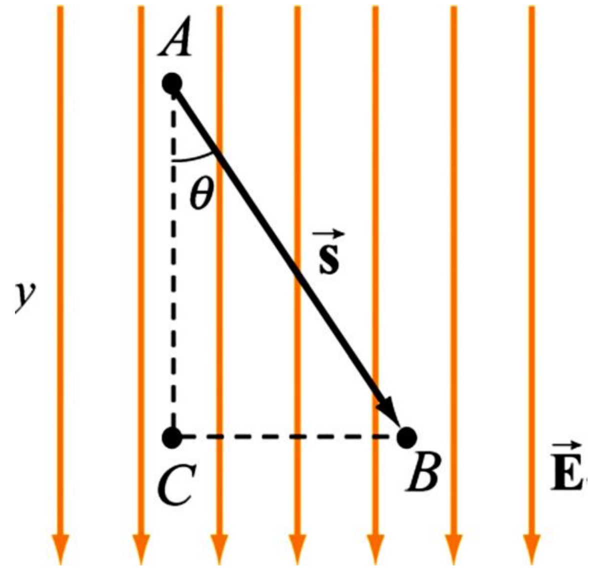
- The work done.

$$W_{a\to b} = \int_a^b \vec{\mathbf{F}} \cdot d\vec{\boldsymbol{s}}$$

$$= q_0 \int_a^b E\cos\theta\, ds$$

$$= q_0 E \int_a^b dy$$

$$W_{a\to b} = q_0 E(y_b - y_a)$$



Work done is independent of actual path!

Electric Potential Energy

The electric potential energy

The work done.

$$W_{a\to b} = \int_a^b \vec{\mathbf{F}} \cdot d\vec{\boldsymbol{s}} = -\Delta U$$
$$= -(U_b - U_a)$$

- For the field of a ***point charge***:

$$W_{a\to b} = -\left(\frac{kq_0Q}{r_b} - \frac{kq_0Q}{r_a}\right) \; ; \quad U = \frac{kq_0Q}{r}$$

- For a ***uniform field***:

$$W_{a\to b} = \; (q_0Ey_b - q_0Ey_a) \; ; \quad U = -q_0E\, y$$

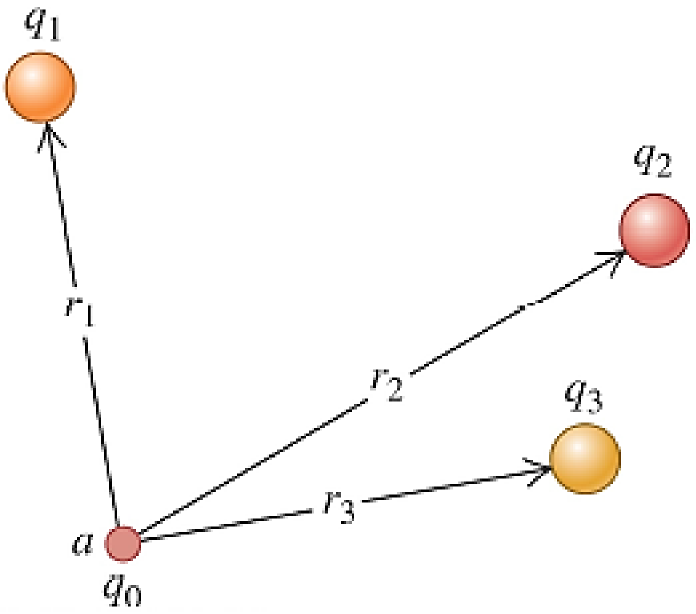
Electric Potential Energy

The electric potential energy with several point charges

- By *superposition*:

$$U = kq_0\left(\frac{q_1}{r_1} + \frac{q_2}{r_2} + \frac{q_3}{r_3} + \cdots\right)$$

$$= kq_0\sum_i\left(\frac{q_i}{r_i}\right)$$



Electric Potential

Potential energy per unit charge

- Electric field *E* is *electric force per unit charge*. The potential *V* is the *electric potential energy per unit charge*.

$$\frac{U}{q_0} \equiv V \; ; \qquad U = q_0 V$$

- Unit: $\text{Joule/Coulomb} = \text{Volt (V)}$

- Work done per unit charge:

$$\frac{W_{a\to b}}{q_0} = -\frac{\Delta U}{q_0} = -(V_b - V_a)$$

$$= (V_a - V_b) \equiv V_{ab}$$

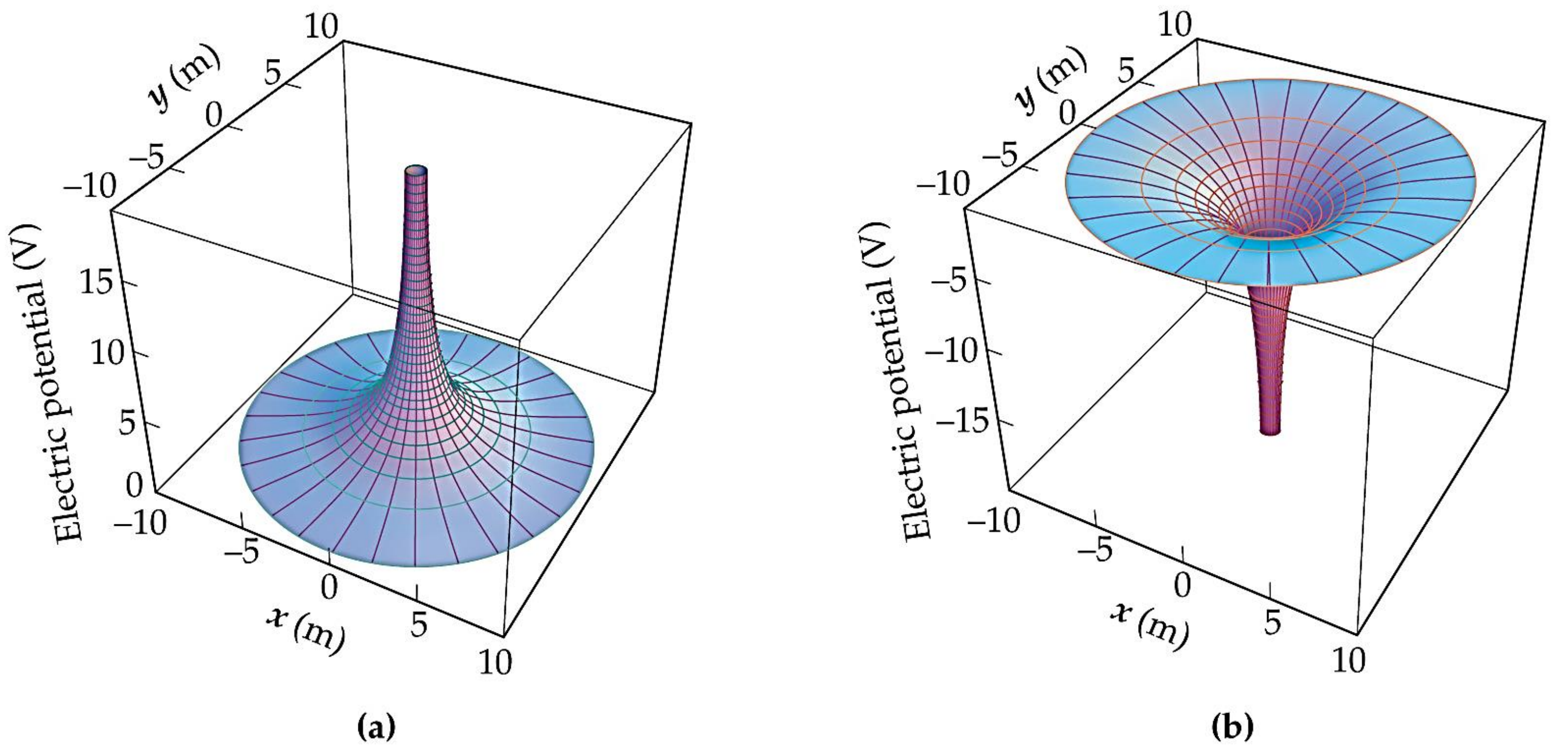
Also called the *voltage*.

Electric Potential

Electric potential

- Electric potential for a point charge

$$V(r) = \frac{kQ}{r}$$

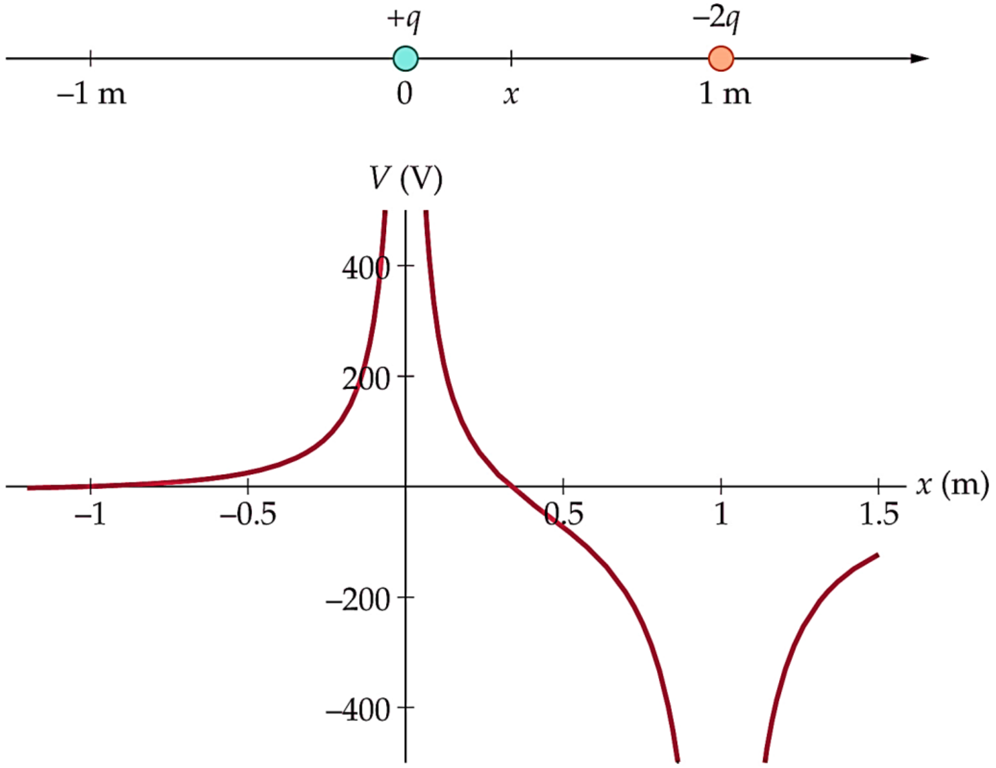


Electric Potential

Electric potential

- Electric potential for two point charges

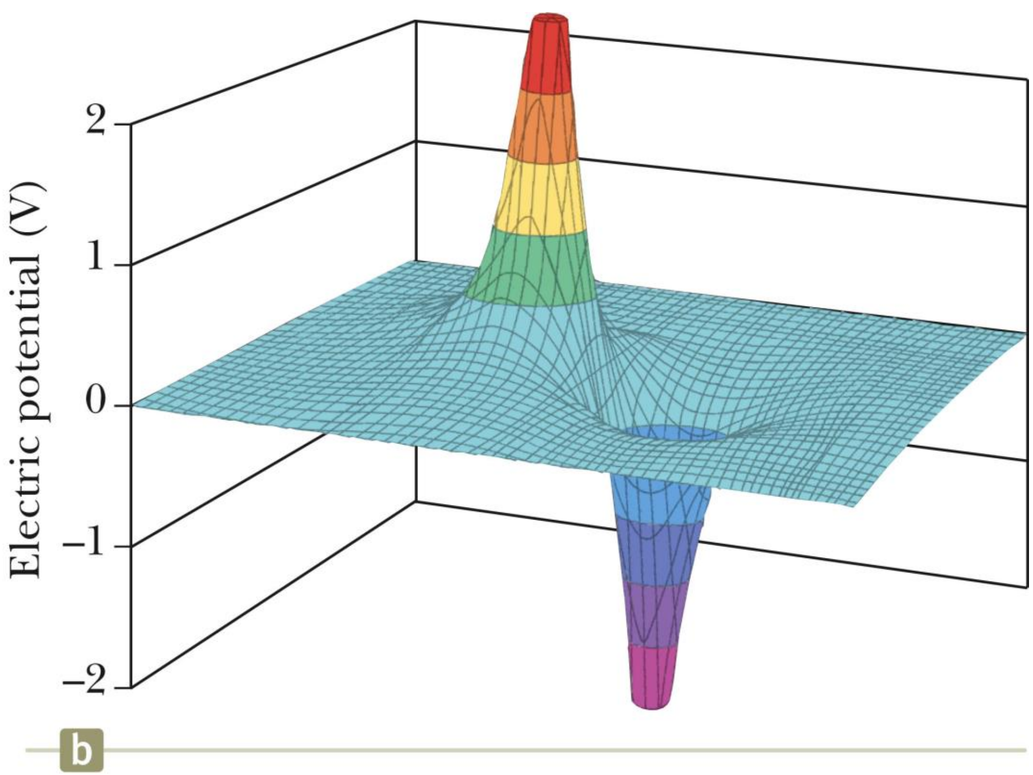
$$V(r) = \frac{kq_1}{r_1} + \frac{kq_2}{r_2}$$



Electric Potential

Electric potential

- Electric potential of a dipole



Electric Potential

Electric potential

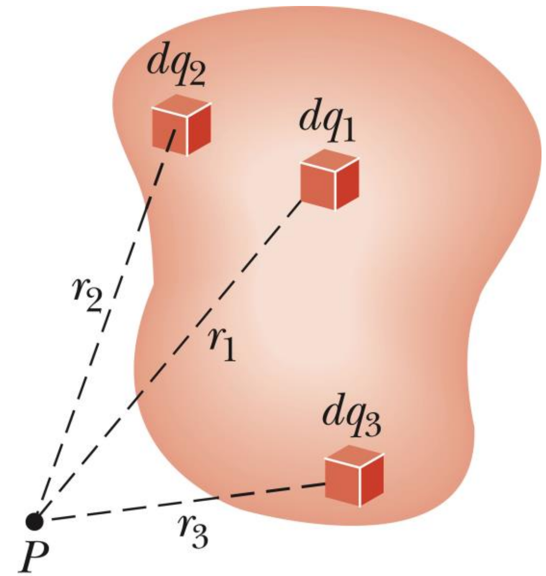
For continuous charge distribution

- Potential at *P* due to element dq

$$dV = k\frac{dq}{r}$$

- The total potential

$$V = \int_{body} dV = k\int_{body} \frac{dq}{r}$$

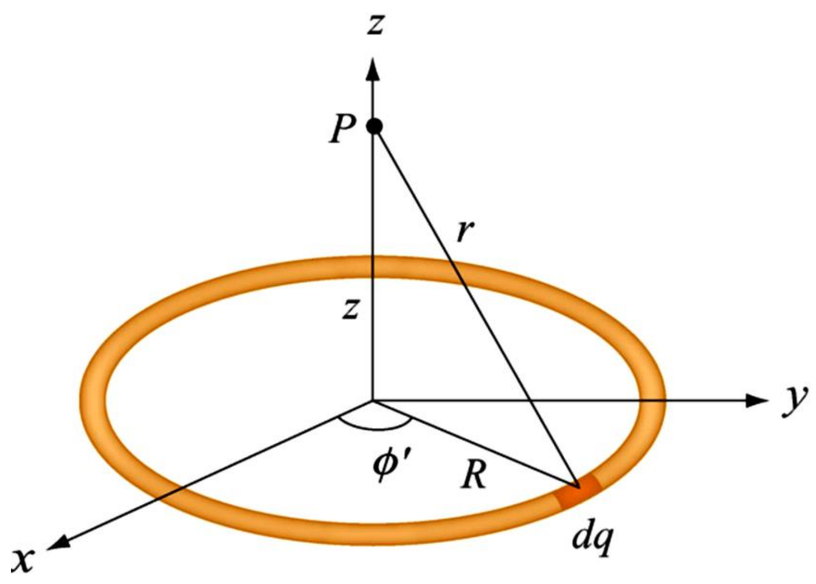


$$dq = \begin{cases} \lambda dl & \text{(Length)} \\ \sigma dA & \text{(Surface)} \\ \rho d\tau & \text{(Volume)} \end{cases}$$

Calculating Electric Potential

Example: Uniformly charged ring

Consider a uniformly charged ring of radius *R* and charge density λ. What is the electric potential at a distance *z* from the central axis?



- Divide the ring into differential elements $dl = R\,d\phi'$.

Calculating Electric Potential

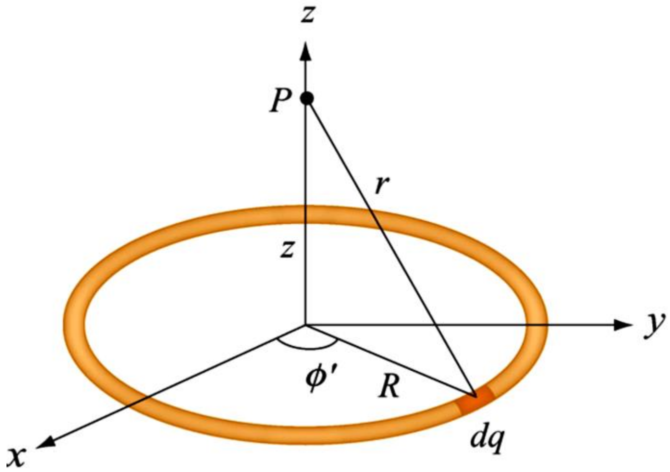
Example: Uniformly charged ring

Solution:

- For the differential element dl :

$$dq = \lambda dl = \lambda R\, d\phi'$$

$$dV = k\frac{dq}{r} = k\lambda R \frac{d\phi'}{\sqrt{R^2 + z^2}}$$



- The total potential:

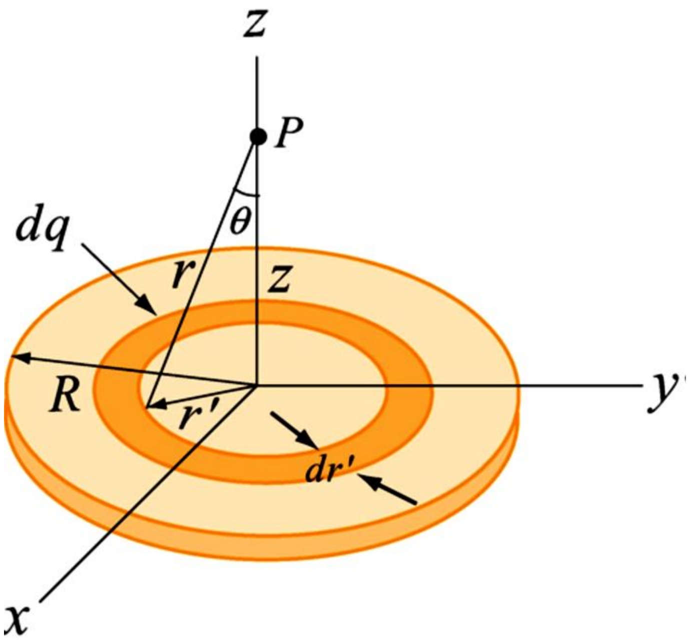
$$V = \frac{k\lambda R}{\sqrt{R^2 + z^2}} \int d\phi' = \frac{k\lambda R}{\sqrt{R^2 + z^2}} (2\pi)$$

$$V(z) = \frac{kQ}{\sqrt{R^2 + z^2}}$$

Calculating Electric Potential

Example: Uniformly charged disk

Consider a uniformly charged disk of radius *R* and charge density σ lying in the *xy*-plane. What is the electric potential at a distance *z* from the central axis?



- Divide the disk into thin ring elements $dA = 2\pi r' dr'$

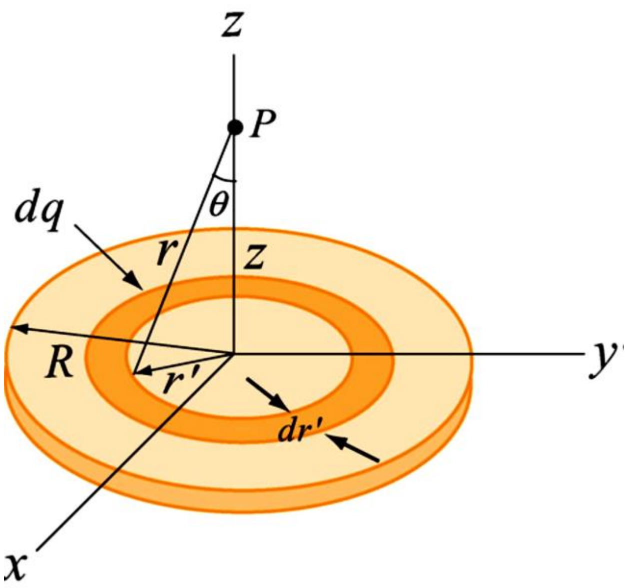
Calculating Electric Potential

Example: Uniformly charged disk

Solution:

- For the differential element dl :

$$dq = \sigma dA = 2\pi\sigma\, r'\, dr'$$



- From previous result (ring) :

$$dV = \frac{k\, dq}{\sqrt{r'^2 + z^2}} = \frac{2\pi k\sigma\, r'\, dr'}{\sqrt{r'^2 + z^2}}$$

- The total potential:

$$V = 2\pi k\sigma \int_0^R \frac{r'\, dr'}{\sqrt{r'^2 + z^2}} == 2\pi k\sigma \left[\sqrt{r'^2 + z^2} - z\right]_0^R$$

Calculating Electric Potential

Example: Uniformly charged disk

Solution:

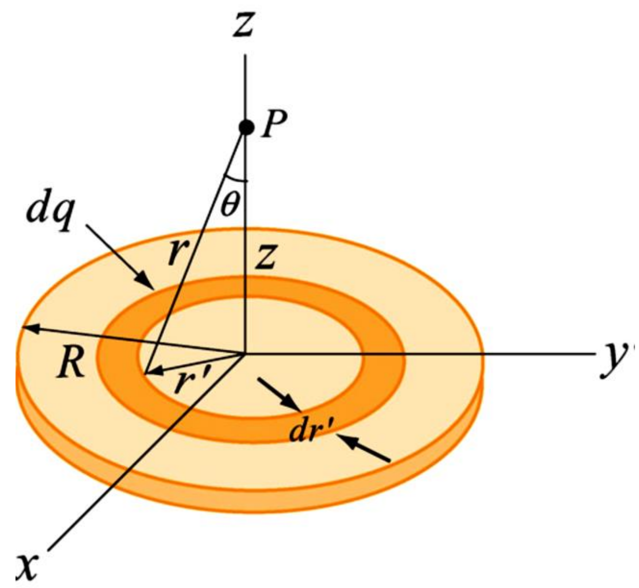
- The total potential:

$$V(z) = 2\pi k\sigma\left[\sqrt{R^2+z^2} - z\right]$$

- Limiting case 1: $z \gg R$

$$\sqrt{R^2+z^2} \approx z\left(1+\frac{R^2}{2z^2}+\cdots\right)$$

$$V(z) \approx 2\pi k\sigma\left[\frac{R^2}{2z}\right] = k\frac{\pi\sigma R^2}{z} = k\frac{Q}{z}$$



Calculating Electric Potential

Example: Uniformly charged disk

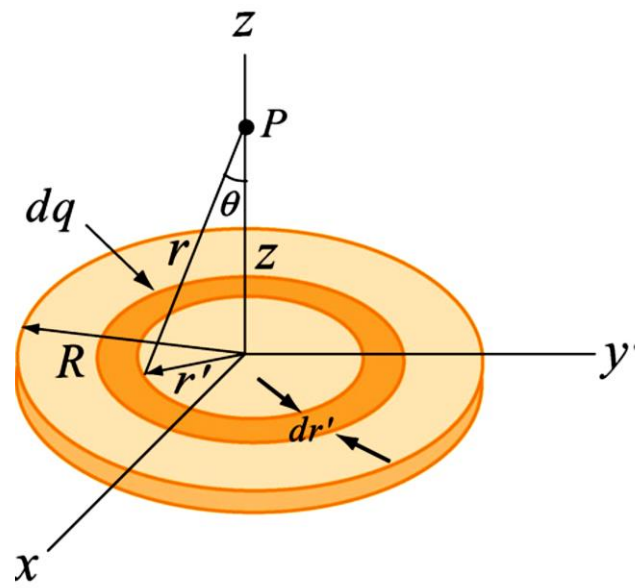
Solution:

- The total potential:

$$V(z) = 2\pi k\sigma\left[\sqrt{R^2 + z^2} - z\right]$$

- Limiting case 2: $z \ll R$

$$V(z) \approx 2\pi k\sigma[R - z]$$

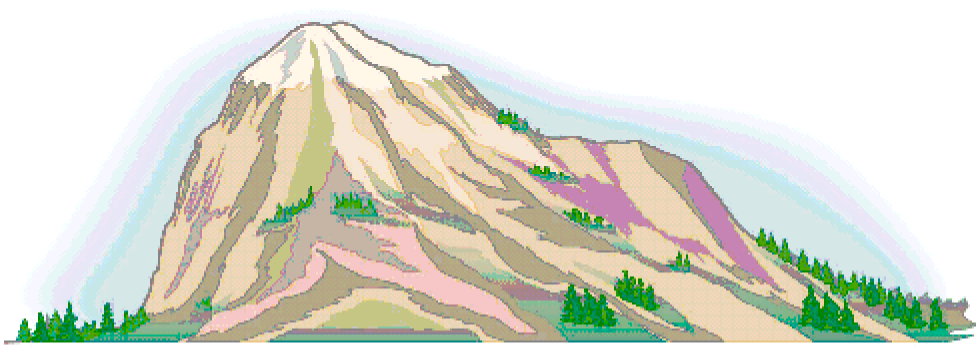


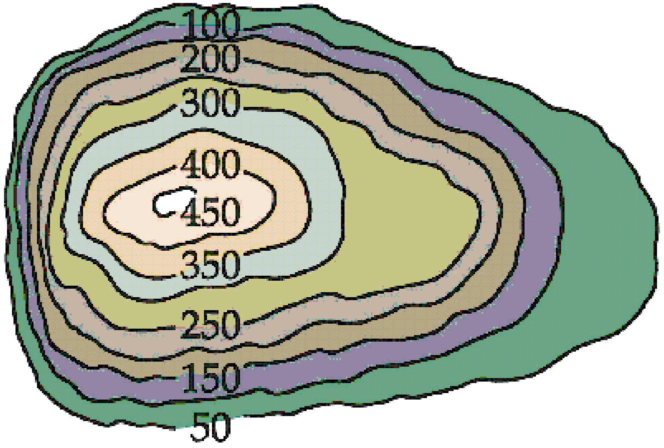
Equipotential Surfaces

Equipotential curves

On a contour map, the curves mark constant elevation

- The steepest slope is perpendicular to the curves.
- The closer together the curves, the steeper the slope.



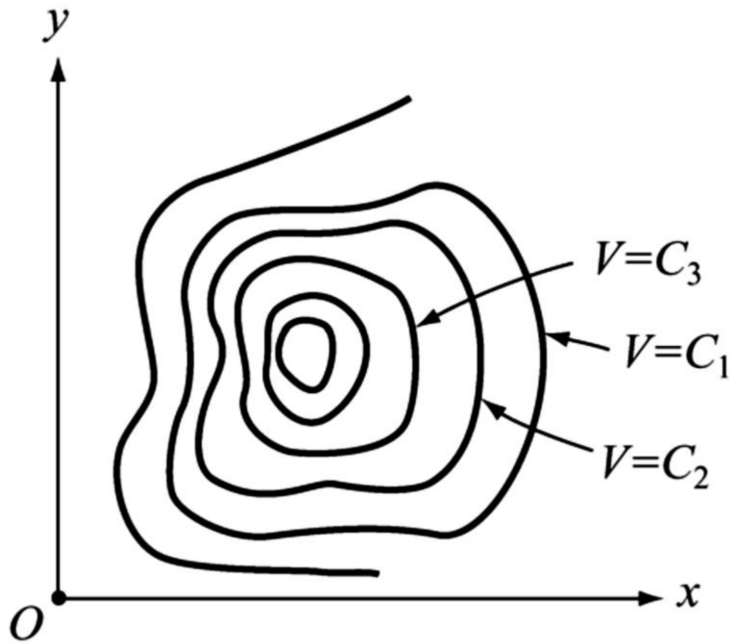


Equipotential Surfaces

Equipotential curves

A system in two dimensions has an electric potential $V(x, y)$.

- The curves characterized by constant $V(x, y) =$ constant are called ***equipotential*** curves.



Equipotential Surfaces

Equipotential surfaces

In three dimensions the electric potential is $V(x, y, z)$.

- $V(x, y, z) =$ constant are equipotential surfaces.

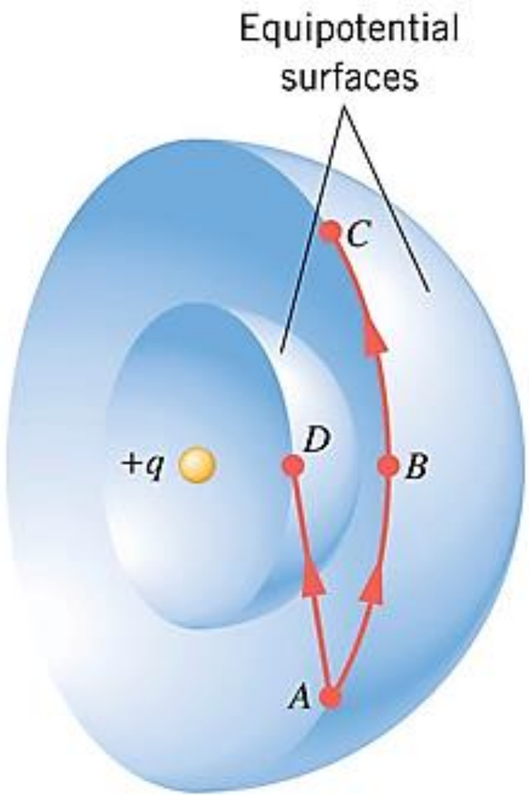
- The total potential:

$$V(x, y, z) = \frac{kq}{r} = \frac{kq}{\sqrt{x^2 + y^2 + z^2}}$$

- $V(x, y, z) =$ constant :

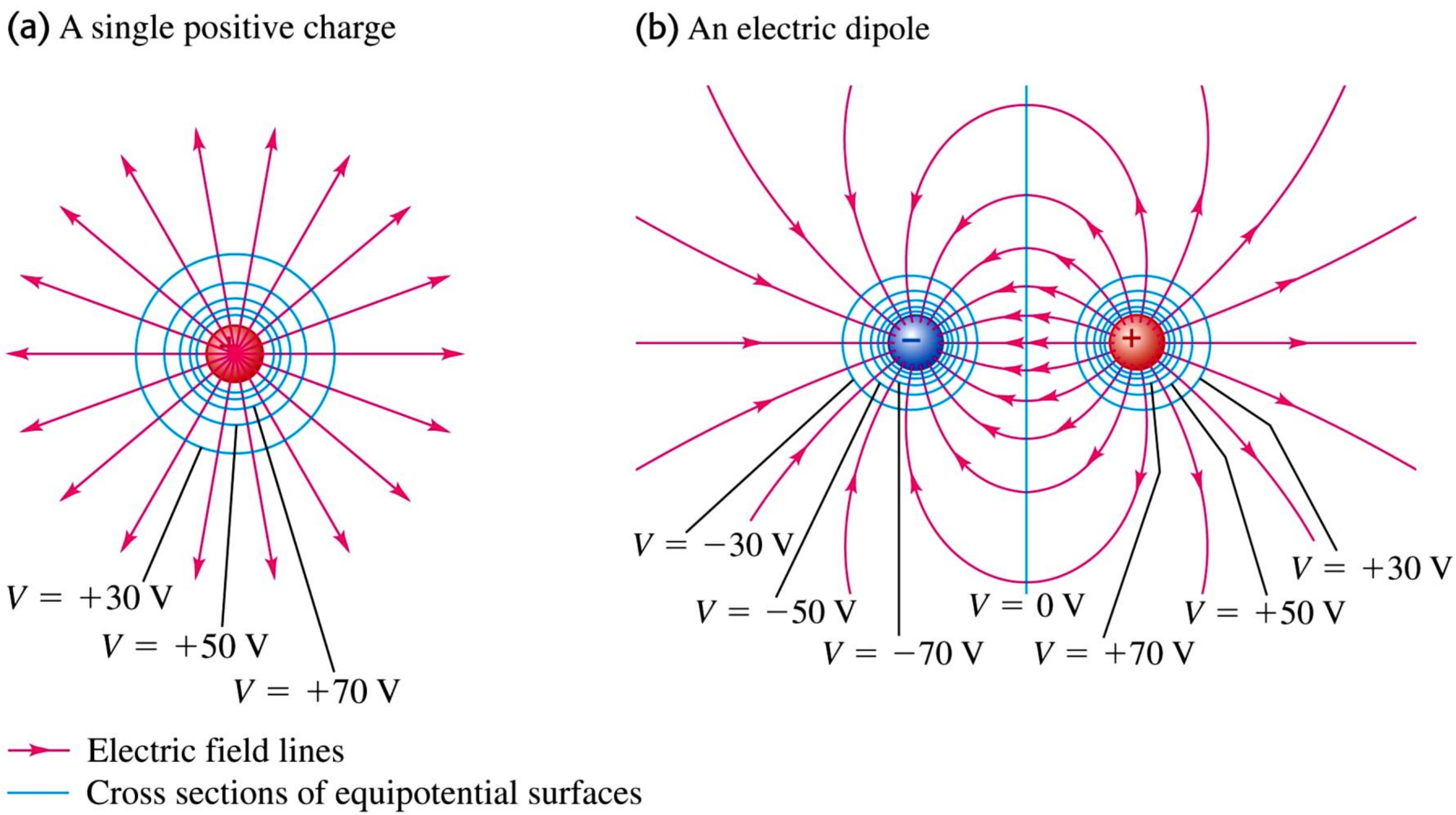
$$x^2 + y^2 + z^2 = \text{constant}$$

(Spherical shells)



Equipotential Surfaces

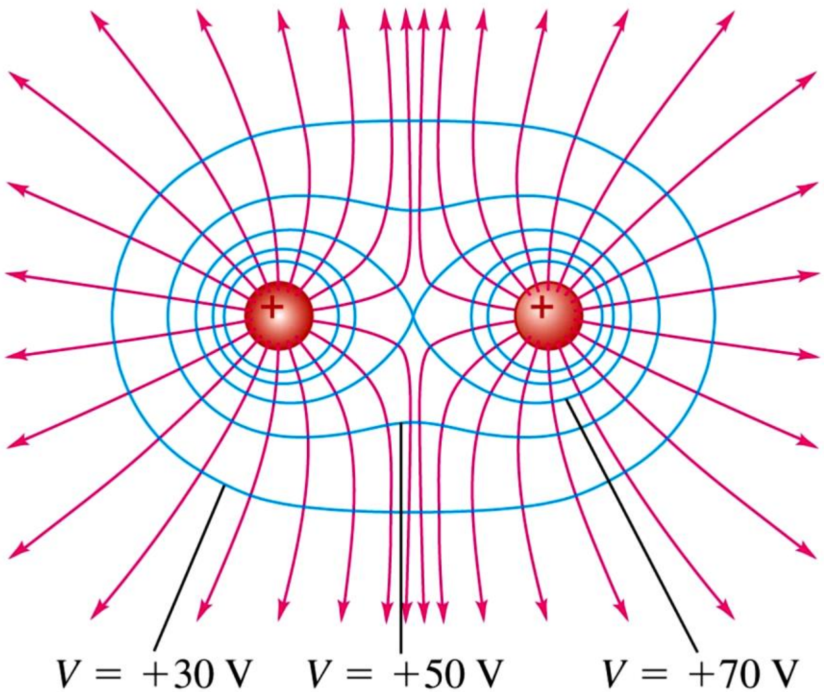
Equipotential surfaces



Equipotential Surfaces

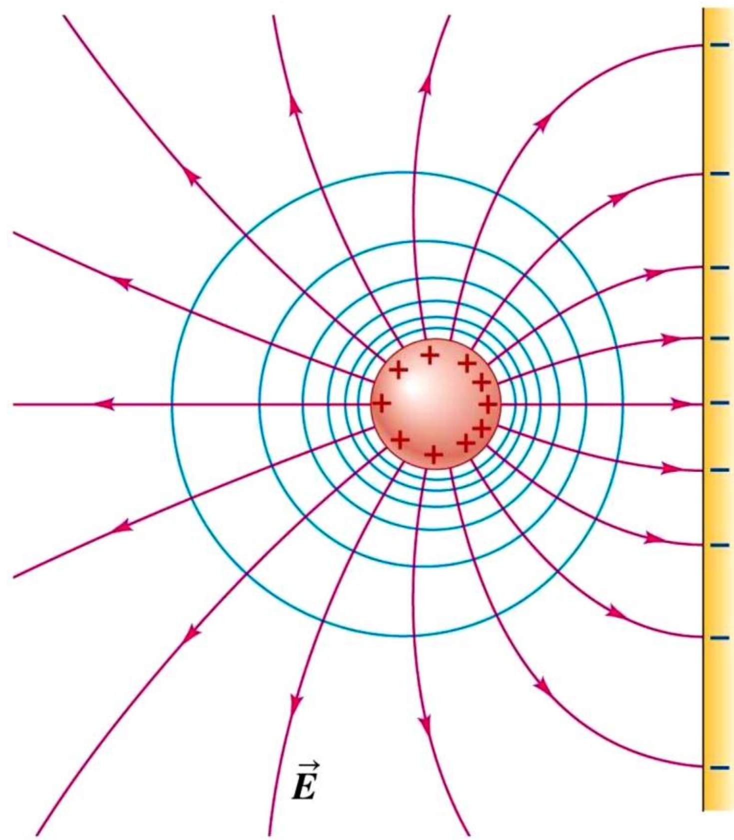
Equipotential surfaces

(c) Two equal positive charges



Electric field lines

Cross sections of equipotential surfaces



Cross sections of equipotential surfaces

Electric field lines

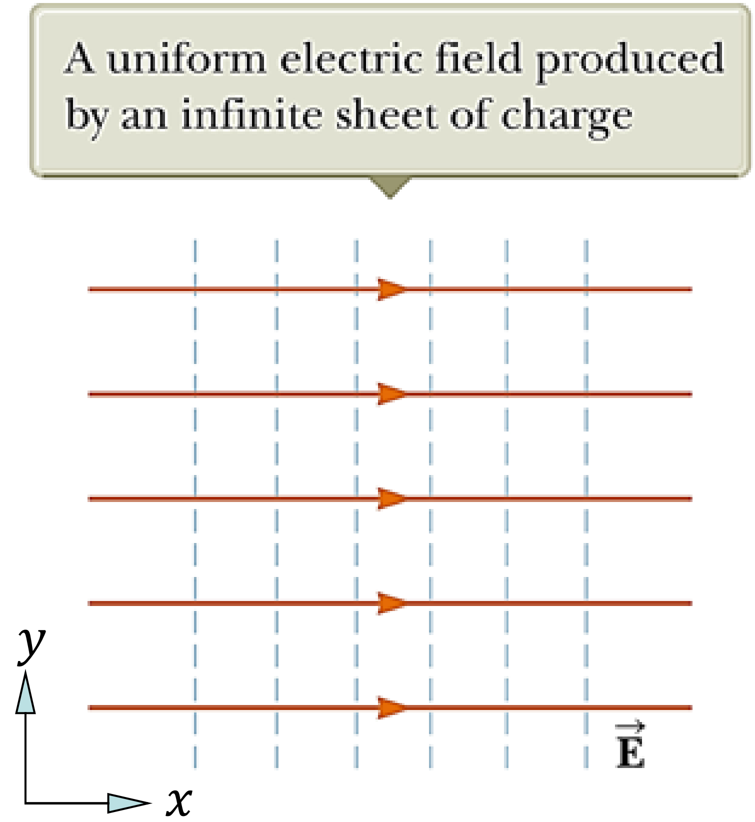
Equipotential Surfaces

Equipotential surfaces

Uniform electric field $\boldsymbol{E}$

$$V = -E\,x$$

- The equipotential surfaces are parallel planes $x =$ constant (dashed blue lines).
- The equipotentials are everywhere perpendicular to the electric field lines.



Equipotential and Conductors

Conductors

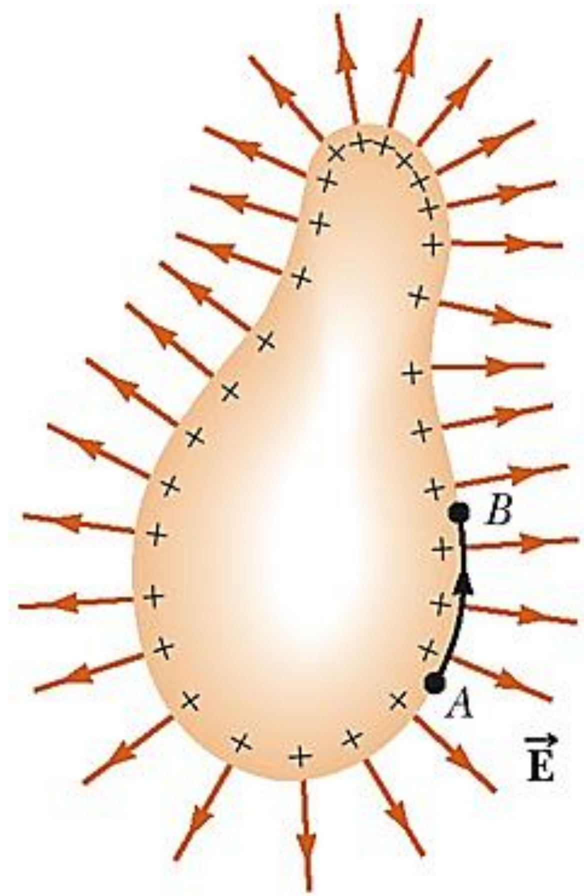
Moving a test charge on the surface of a conductor

$$W = -q_0(V_B - V_A)$$

- Since $\boldsymbol{E}$ is perpendicular to the surface, no work is required to move the charge ($W = 0$).

$$V_B = V_A$$

- All points *on the surface* of a charged conductor in electrostatic equilibrium are at the same potential.



Equipotential and Conductors

Conductors

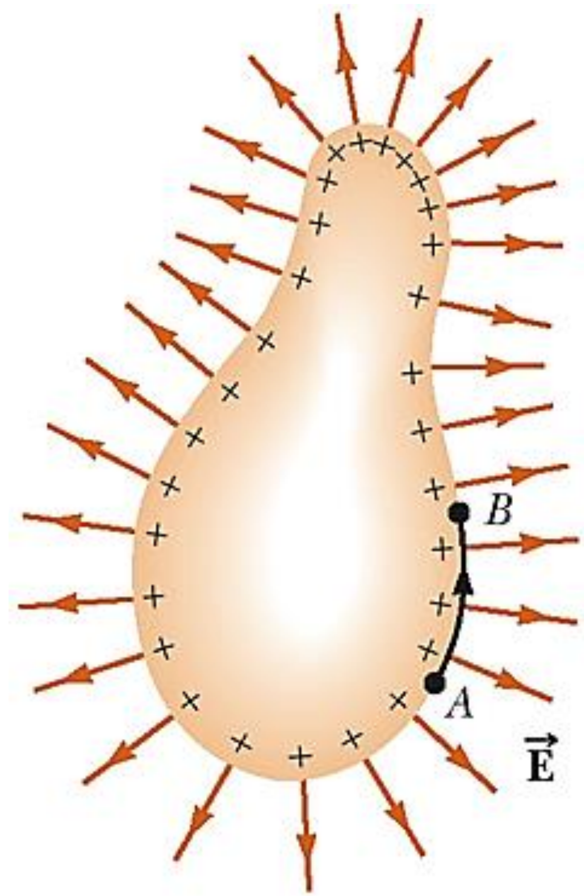
Moving a test charge from the surface to the interior of a conductor

$$W = -q_0(V_{in} - V_A)$$

- Since $E = 0$ inside, no work is required ($W = 0$).

$$V_{in} = V_A$$

- The potential everywhere *inside* the conductor is constant and equal to its value at the surface.



Potential Gradient

Electric field from potential

For the field of a point charge

$$\Delta U = -\int_a^b F \cdot dr$$

- The potential difference.

$$\Delta V = -\int_a^b E \cdot dr = \int_a^b dV$$

$$dV = -E \cdot dr$$

$$E = -\frac{dV}{dr}$$

Potential Gradient

Electric field from potential

In general, for $V = V(x, y, z)$

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$$

- We identify

$$\int_a^b dV = -\int_a^b \vec{E} \cdot d\vec{r}$$

$$d\vec{r} = dx\ \hat{\imath} + dy\ \hat{\jmath} + dz\ \hat{k}$$

$$\vec{E} \cdot d\vec{r} = E_x dx + E_y dy + E_z dz$$

$$E_x = -\frac{\partial V}{\partial x}\ ; \quad E_y = -\frac{\partial V}{\partial y}; \quad E_z = -\frac{\partial V}{\partial z}$$

Potential Gradient

Example: Point charge Q

$$V(x,y,z) = \frac{kQ}{r} = \frac{kQ}{\sqrt{x^2+y^2+z^2}}$$

- We identify

$$E_x = -\frac{\partial V}{\partial x} = kQ\frac{x}{(x^2+y^2+z^2)^{3/2}}$$

$$E_y = kQ\frac{y}{(x^2+y^2+z^2)^{3/2}}\ ;\ E_z = kQ\frac{z}{(x^2+y^2+z^2)^{3/2}}$$

$$E = \sqrt{E_x^2+E_y^2+E_z^2} = kQ\frac{1}{(x^2+y^2+z^2)} = \frac{kQ}{r^2}$$

END