4-5 Summary

Methods on Solving:

1. Homogeneous Equations

Condition: Given an inseparable FOODE in the form

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

where $M(x,y)$ and $N(x,y)$ are homogeneous functions of the same degree.

Steps:

1. Use the substitution $y = vx$ or $v = \frac{y}{x}$.
2. Using the substitution in Step 1, find that $dy = v\,dx + x\,dv$ (through the product rule).
3. Substitute dy and y with the equations that we found in Steps 1 and 2.
4. If the condition is met, certain terms should cancel out such that we get a separable equation.

2. Linear Equation

Condition: Given a linear FOODE that we can write in the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where $P(x)$ and $Q(x)$ are functions solely in terms of x.

Steps:

1. Find the integrating factor, $\mu(x) = e^{\int P(x)\,dx}$.
2. Multiply both sides by $\mu(x)$.
3. The LHS can then be written as an exact derivative $\frac{d}{dx}(\mu(x)y)$ (thanks to the product rule).
4. Integrate both sides.

3. Functions of 2 Variables

Condition: Given a FOODE in the form

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

there should exist a $f(x,y)$ such that when

$$\frac{\delta f}{\delta x} = M(x,y)\,;\,\frac{\delta f}{\delta y} = N(x,y)$$

then

$$\frac{\delta M}{\delta y} = \frac{\delta N}{\delta x}$$

Steps (Method 1):

1. Integrate $\frac{\delta f}{\delta x} = M(x,y)$ (with respect to x) and $\frac{\delta f}{\delta y} = N(x,y)$ (with respect to y) to get $f(x,y) = \int M(x,y) + C(x)$ and $f(x,y) = \int M(x,y)\,dx + C(x)$ and $f(x,y) = \int N(x,y)\,dy + D(y)$.
2. There should be common terms and uncommon terms. The uncommon terms should be the values of $C(x)$ and $D(y)$.

3. Construct the final expression using these terms.
4. Since $M(x, y)\, dx + N(x, y)\, dy = 0$ should be interpreted as $df = 0$, then simply equate the final expression to C.

Steps (Method 2):

1. Integrate $\frac{\delta f}{\delta x} = M(x, y)$ (with respect to x) to get $f(x, y) = \int M(x, y)\, dx + C(x)$.
2. We can then differentiate both sides to get $\frac{\delta f}{\delta x} = M(x, y) + \frac{dC}{dy}$.
3. We can then equate the equation found in Step 2 with $\frac{\delta f}{\delta y} = N(x, y)$.
4. Find $C(y)$ and then construct the final equation found from Step 1.
5. Since $M(x, y)\, dx + N(x, y)\, dy = 0$ should be interpreted as $df = 0$, then simply equate the final expression to C.