Summary for Sequences and Series (Part 1)

For Sequences

Given any sequence a_n, if there exists a $\lim_{n\to\infty} a_n = L$, then the sequence is convergent. If not, then the sequence is divergent.

Monotonic Sequence Theorem

A sequence $\{a_n\}$ is increasing if $a_n < a_{n+1}$ for all $n \geq 1$. It is decreasing if $a_n > a_{n+1}$ for all $n \geq 1$.
In either case, it is monotonic. To find if it is monotonic, you can:

1. Compare a_n and a_{n+1} by checking $\frac{a_{n+1}}{a_n}$.
2. Check its first derivative.
3. (For recursive sequences) Find if $a_{n+1} - a_n$ is positive or negative. If positive, it is increasing. If negative, it is decreasing.

A sequence $\{a_n\}$ is **bounded above** if there is a number M such that $a_n \leq M$ for all $n \geq 1$.
It is **bounded below** if there is a number m such that $a_n \geq m$ for all $n \geq 1$.
It is **bounded** if it is both bounded above and bounded below.

If a sequence is both **bounded** and **monotonic**, then it is convergent.

For Series

Divergence Test

Given

$$\sum_{n=1}^{\infty} a_n$$

Then if $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ will diverge.

Note that this does NOT mean that $\sum_{n=1}^{\infty} a_n$ will converge if $\lim_{n\to\infty} a_n = 0$.

Integral Test

Given

$$\sum_{n=1}^{\infty} a_n$$

Let $f(x)$ be a function satisfying:

1. $a_n = f(n)$
2. f should be positive, continuous, and decreasing on $[1, \infty)$.
   (If there is no $f(x)$ that satisfies both conditions, then we cannot use the integral test.)

Then:

1. If $\int_1^\infty f(x)dx$ is convergent, then $\sum_{n=1}^\infty a_n$ is convergent.
2. If $\int_1^\infty f(x)dx$ is divergent, then $\sum_{n=1}^\infty a_n$ is divergent.

Direct Comparison Test

Given $\sum a_n$ and $\sum b_n$, with $a_n, b_n \geq 0$ for all n and $a_n \leq b_n$ for all $n,
Then

1. If $\sum b_n$ is convergent, then $\sum a_n$ is convergent.
2. If $\sum a_n$ is divergent, then $\sum b_n$ is divergent.

Limit Comparison Test

Given

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is positive and finite:

1. If $\sum_{n=1}^\infty b_n$ converges, then $\sum_{n=1}^\infty a_n$ converges.
2. If $\sum_{n=1}^\infty b_n$ diverges, then $\sum_{n=1}^\infty a_n$ diverges.

Alternating Series Test

Given

$$\sum_{n=1}^\infty (-1)^{n-1} a_n$$

If $\lim_{n\to\infty} a_n = 0$ and $\{a_n\}$ is decreasing, then the sequence is convergent.

Note that this can ONLY conclude convergence.

An Extension of the Test

There are two types of convergent series:

1. Absolutely Convergent Series: convergent regardless if absolute values are taken
2. Conditionally Convergent: series becomes divergent if absolute values are taken.

   Then, if $\sum_{n=1}^\infty |a_n|$ is convergent, then $\sum_{n=1}^\infty a_n$ is convergent. This is helpful when it is easier to find the former than the latter.

Common Series & Sequences (for Comparison Tests)

P-series

The series in the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

where p is any real value.
For this series:

1. It is convergent for $p > 1$.
2. It is divergent for $p \leq 1$. ($p = 1$ is known as the harmonic series).

Geometric Sequence & Series

The geometric sequence is in the form

$$a_n = ar^{n-1}$$

where r is any value.
When turned into the series

$$\sum_{n=1}^{\infty} ar^n$$

It is:

1. Convergent when $r < 1$ (where its sum becomes $\frac{a}{1-r}$)
2. Divergent when $r \geq 1$

What Tests to Use?

1. Before everything, try the divergence test first.
2. If the form looks something like $\frac{f(n)}{g(n)}$ where $f(n)$ and $g(n)$ are polynomials of n, the Limit Comparison Test may work. With this, it will most likely be similar to a p-series.
3. If you see something that involves $(-1)^n$, then you are likely to use the Alternating Series Test. If $\lim_{n \to \infty} a_n = 0$ is difficult to find, then you can probably try to show that $\sum_{n=1}^{\infty} |b_n|$ (where b_n is the full expression of the series) is convergent. If you cannot determine if b_n is convergent, then good luck lol
4. If you know of a known series (e.g. p-series and geometric series) that is greater than or less than the given series (whether the one or the other applies kinda is just based on intuition), then you can use the Direct Comparison Test.
5. If all the values of the series are positive and you cannot use either comparison tests, then you may need to use the Integral Test.