APPENDIX 3

Auxiliary Material

A3.1 Formulas for Special Functions

For tables of numeric values, see Appendix 5.

Exponential function e^x (Fig. 545)

$$e = 2.71828\ 18284\ 59045\ 23536\ 02874\ 71353$$

(1) $$e^x e^y = e^{x+y}, \qquad e^x/e^y = e^{x-y}, \qquad (e^x)^y = e^{xy}$$

Natural logarithm (Fig. 546)

(2) $$\ln(xy) = \ln x + \ln y, \qquad \ln(x/y) = \ln x - \ln y, \qquad \ln(x^a) = a \ln x$$

$\ln x$ is the inverse of e^x, and $e^{\ln x} = x$, $e^{-\ln x} = e^{\ln(1/x)} = 1/x$.

Logarithm of base ten $\log_{10} x$ or simply $\log x$

(3) $$\log x = M \ln x, \qquad M = \log e = 0.43429\ 44819\ 03251\ 82765\ 11289\ 18917$$

(4) $$\ln x = \frac{1}{M} \log x, \qquad \frac{1}{M} = \ln 10 = 2.30258\ 50929\ 94045\ 68401\ 79914\ 54684$$

$\log x$ is the inverse of 10^x, and $10^{\log x} = x$, $10^{-\log x} = 1/x$.

Sine and cosine functions (Figs. 547, 548). In calculus, angles are measured in radians, so that $\sin x$ and $\cos x$ have period 2π.

$\sin x$ is odd, $\sin(-x) = -\sin x$, and $\cos x$ is even, $\cos(-x) = \cos x$.

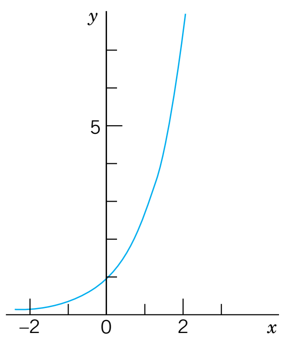

Fig. 545. Exponential function e^x

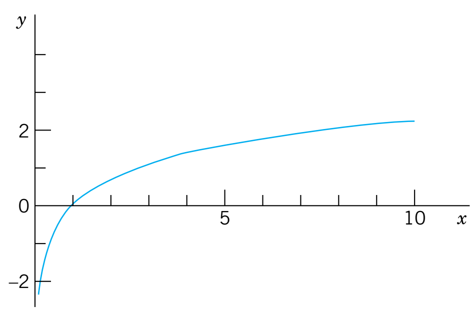

Fig. 546. Natural logarithm $\ln x$

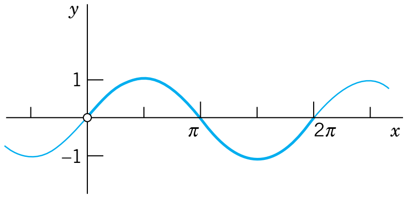

Fig. 547. sin x

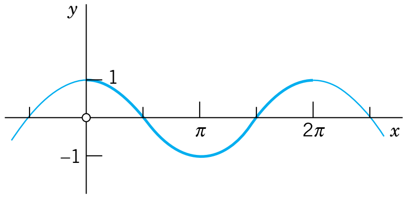

Fig. 548. cos x

$$1° = 0.01745\,32925\,19943 \text{ radian}$$

$$1 \text{ radian} = 57°\,17'\,44.80625''$$

$$= 57.29577\,95131°$$

$$\sin^2 x + \cos^2 x = 1 \tag{5}$$

$$\begin{cases} \sin(x+y) = \sin x \cos y + \cos x \sin y \\ \sin(x-y) = \sin x \cos y - \cos x \sin y \\ \cos(x+y) = \cos x \cos y - \sin x \sin y \\ \cos(x-y) = \cos x \cos y + \sin x \sin y \end{cases} \tag{6}$$

$$\sin 2x = 2 \sin x \cos x, \qquad \cos 2x = \cos^2 x - \sin^2 x \tag{7}$$

$$\begin{cases} \sin x = \cos\left(x - \dfrac{\pi}{2}\right) = \cos\left(\dfrac{\pi}{2} - x\right) \\ \cos x = \sin\left(x + \dfrac{\pi}{2}\right) = \sin\left(\dfrac{\pi}{2} - x\right) \end{cases} \tag{8}$$

$$\sin(\pi - x) = \sin x, \qquad \cos(\pi - x) = -\cos x \tag{9}$$

$$\cos^2 x = \tfrac{1}{2}(1 + \cos 2x), \qquad \sin^2 x = \tfrac{1}{2}(1 - \cos 2x) \tag{10}$$

$$\begin{cases} \sin x \sin y = \tfrac{1}{2}[-\cos(x+y) + \cos(x-y)] \\ \cos x \cos y = \tfrac{1}{2}[\cos(x+y) + \cos(x-y)] \\ \sin x \cos y = \tfrac{1}{2}[\sin(x+y) + \sin(x-y)] \end{cases} \tag{11}$$

$$\begin{cases} \sin u + \sin v = 2 \sin \dfrac{u+v}{2} \cos \dfrac{u-v}{2} \\ \cos u + \cos v = 2 \cos \dfrac{u+v}{2} \cos \dfrac{u-v}{2} \\ \cos v - \cos u = 2 \sin \dfrac{u+v}{2} \sin \dfrac{u-v}{2} \end{cases} \tag{12}$$

$$A \cos x + B \sin x = \sqrt{A^2 + B^2} \cos(x \pm \delta), \qquad \tan \delta = \frac{\sin \delta}{\cos \delta} = \mp \frac{B}{A} \tag{13}$$

$$A \cos x + B \sin x = \sqrt{A^2 + B^2} \sin(x \pm \delta), \qquad \tan \delta = \frac{\sin \delta}{\cos \delta} = \pm \frac{A}{B} \tag{14}$$

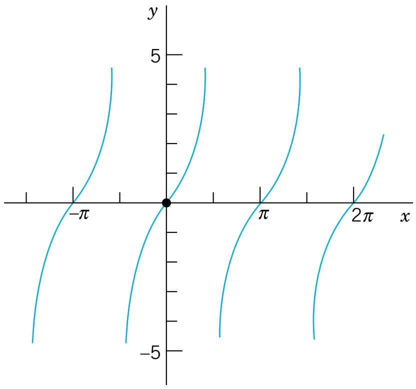

Fig. 549. tan *x*

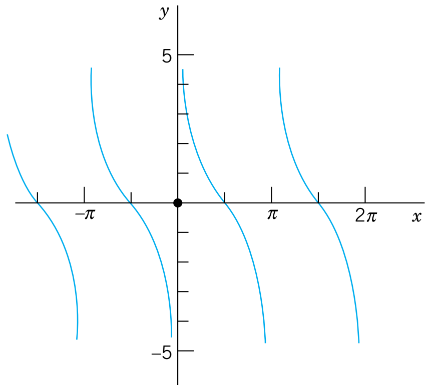

Fig. 550. cot *x*

Tangent, cotangent, secant, cosecant (Figs. 549, 550)

$$\tan x = \frac{\sin x}{\cos x}, \qquad \cot x = \frac{\cos x}{\sin x}, \qquad \sec x = \frac{1}{\cos x}, \qquad \csc x = \frac{1}{\sin x} \tag{15}$$

$$\tan (x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}, \qquad \tan (x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y} \tag{16}$$

Hyperbolic functions (hyperbolic sine sinh x, etc.; Figs. 551, 552)

$$\sinh x = \tfrac{1}{2}(e^x - e^{-x}), \qquad \cosh x = \tfrac{1}{2}(e^x + e^{-x}) \tag{17}$$

$$\tanh x = \frac{\sinh x}{\cosh x}, \qquad \coth x = \frac{\cosh x}{\sinh x} \tag{18}$$

$$\cosh x + \sinh x = e^x, \qquad \cosh x - \sinh x = e^{-x} \tag{19}$$

$$\cosh^2 x - \sinh^2 x = 1 \tag{20}$$

$$\sinh^2 x = \tfrac{1}{2}(\cosh 2x - 1), \qquad \cosh^2 x = \tfrac{1}{2}(\cosh 2x + 1) \tag{21}$$

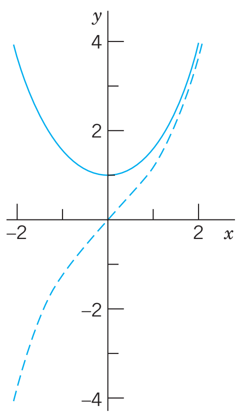

Fig. 551. sinh *x* (dashed) and cosh *x*

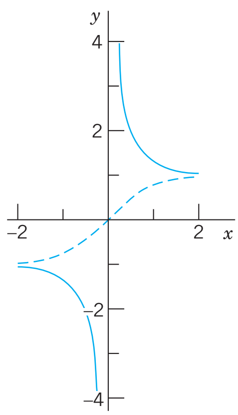

Fig. 552. tanh *x* (dashed) and coth *x*

$$\text{(22)} \qquad \begin{cases} \sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y \\ \cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y \end{cases}$$

$$\text{(23)} \qquad \tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$$

Gamma function (Fig. 553 and Table A2 in App. 5). The gamma function $\Gamma(\alpha)$ is defined by the integral

$$\text{(24)} \qquad \Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1}\, dt \qquad (\alpha > 0),$$

which is meaningful only if $\alpha > 0$ (or, if we consider complex α, for those α whose real part is positive). Integration by parts gives the important *functional relation of the gamma function,*

$$\text{(25)} \qquad \Gamma(\alpha + 1) = \alpha\Gamma(\alpha).$$

From (24) we readily have $\Gamma(1) = 1$; hence if α is a positive integer, say k, then by repeated application of (25) we obtain

$$\text{(26)} \qquad \Gamma(k + 1) = k! \qquad (k = 0, 1, \cdots).$$

This shows that *the gamma function can be regarded as a generalization of the elementary factorial function.* [Sometimes the notation $(\alpha - 1)!$ is used for $\Gamma(\alpha)$, even for noninteger values of α, and the gamma function is also known as the **factorial function**.]

By repeated application of (25) we obtain

$$\Gamma(\alpha) = \frac{\Gamma(\alpha+1)}{\alpha} = \frac{\Gamma(\alpha+2)}{\alpha(\alpha+1)} = \cdots = \frac{\Gamma(\alpha+k+1)}{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+k)}$$

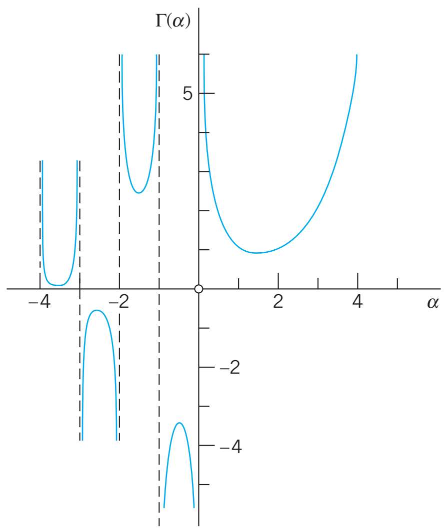

Fig. 553. Gamma function