

# Linear Programming

# What is Linear Programming?

- A method for solving certain types of minimization/maximization problems.
- The problems must be expressible by a system of inequalities
- The inequalities must have very specific constraints.

Our first example:

maximize  $x_1 + x_2$   
subject to

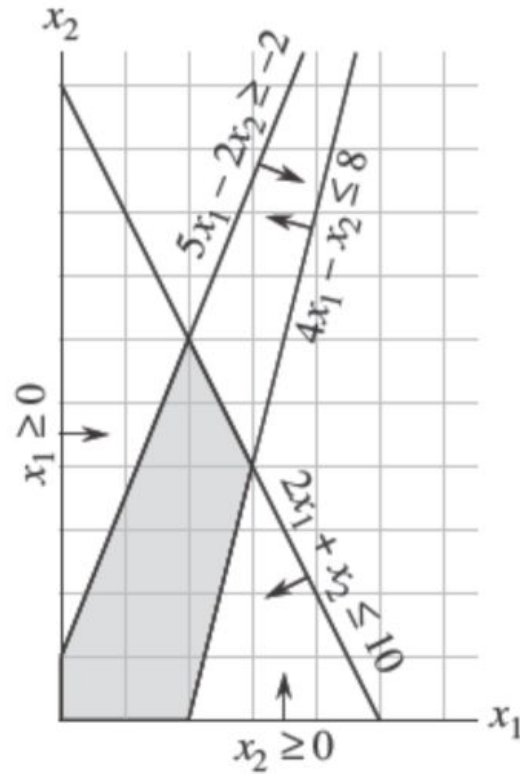
$$4x_1 - x_2 \leq 8$$

$$2x_1 + x_2 \leq 10$$

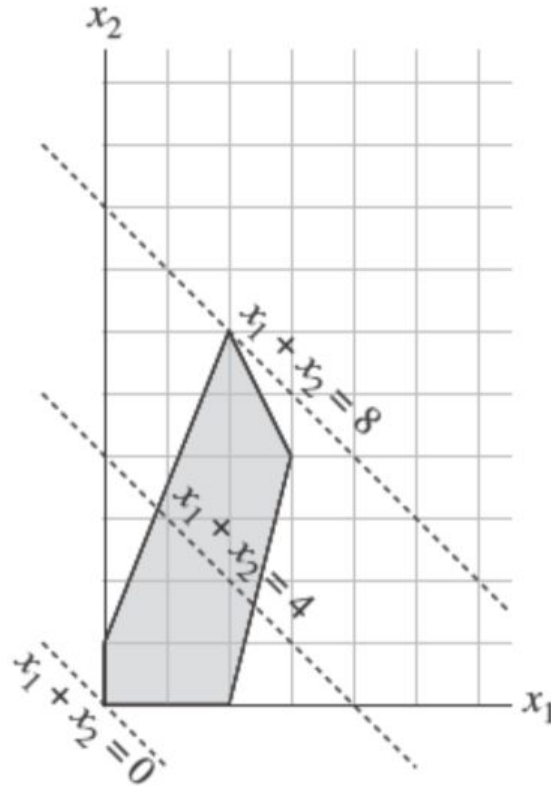
$$5x_1 - 2x_2 \geq -2$$

$$x_1, x_2 \geq 0$$

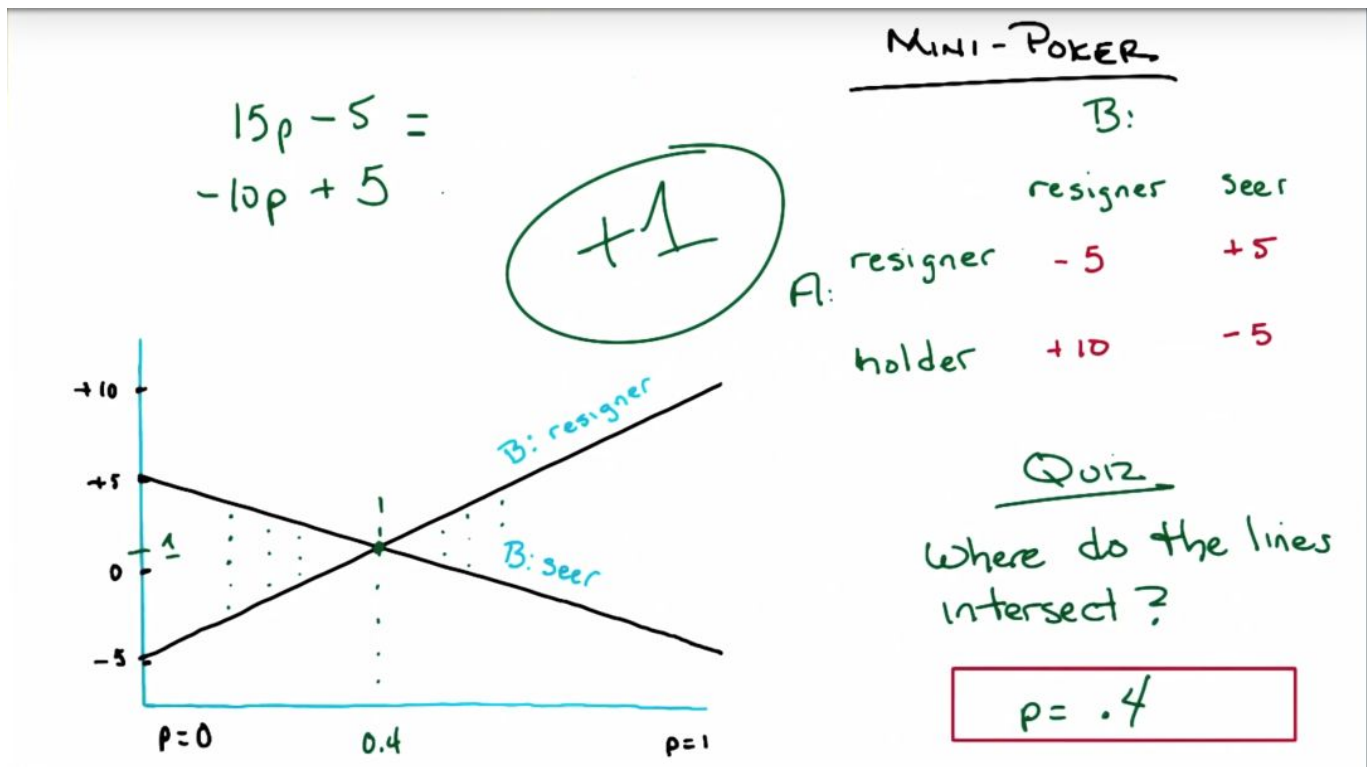
# Let's graph it!



And, we are done...

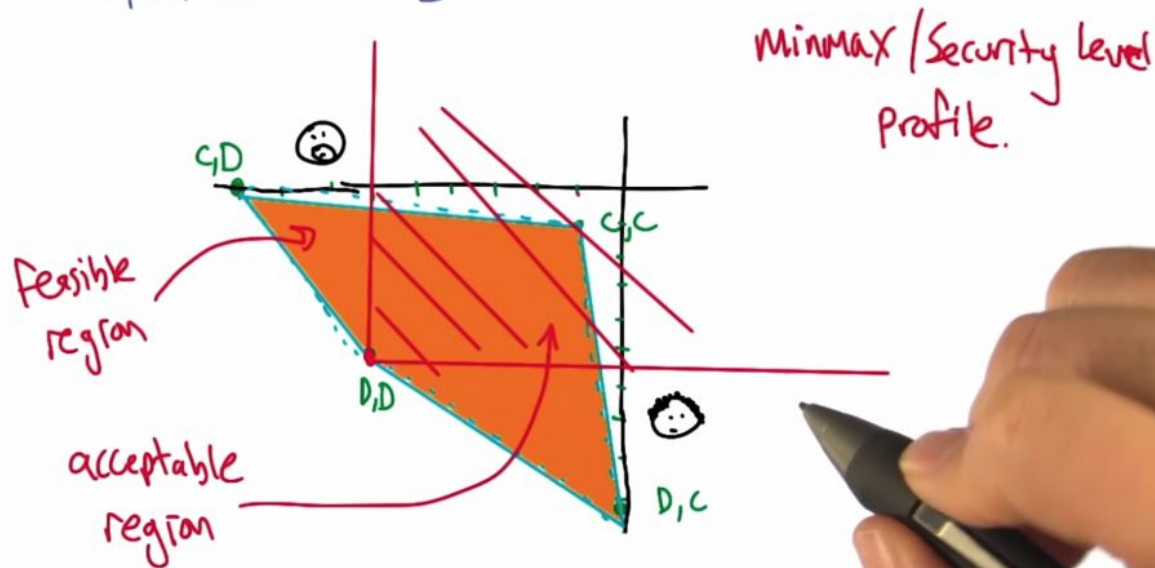


This should look familiar

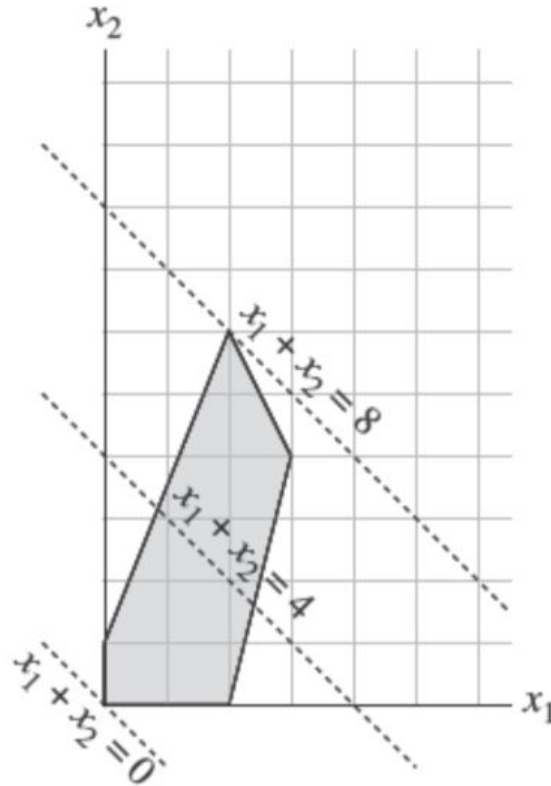


Seriously, really familiar...

## Repeated Games and the Folk Theorem



So, how do we use a computer to solve this?





# Linear Programming!

- Get yourself a LP solver: glpk, CVXOPT, SCPSolver, etc...
- Express your problem as a system of inequalities.
- Run your solver on the resulting matrix.

## Back to our example

maximize  $x_1 + x_2$   
subject to

$$4x_1 - x_2 \leq 8$$

$$2x_1 + x_2 \leq 10$$

$$5x_1 - 2x_2 \geq -2$$

$$x_1, x_2 \geq 0$$

Almost there

minimize  $-x_1 - x_2$

subject to

$$4x_1 - x_2 \leq 8$$

$$2x_1 + x_2 \leq 10$$

$$5x_1 - 2x_2 \geq -2$$

$$x_1, x_2 \geq 0$$

## "Standard Form"

minimize  $-x_1 - x_2$

subject to

$$4x_1 - x_2 \leq 8$$

$$2x_1 + x_2 \leq 10$$

$$-5x_1 + 2x_2 \leq 2$$

$$x_1, x_2 \geq 0$$

# CVXOPT LP Solver

<http://cvxopt.org/userguide/coneprog.html#cvxopt.solvers.lp>

```
cvxopt.solvers.lp(c, G, h[, A, b[, solver[, primalstart[, dualstart]]]])
```

## Explicit Multipliers

minimize  $-1x_1 - 1x_2$

subject to

$$4x_1 - 1x_2 \leq 8$$

$$2x_1 + 1x_2 \leq 10$$

$$-5x_1 + 2x_2 \leq 2$$

$$x_1, x_2 \geq 0$$

# CVXOPT LP Parameters

minimize  $-1x_1 - 1x_2$   $\leftarrow c$   
subject to

$$\begin{array}{rcl} 4x_1 - 1x_2 & \leq & 8 \\ 2x_1 + 1x_2 & \leq & 10 \\ -5x_1 + 2x_2 & \leq & 2 \\ x_1, x_2 & \geq & 0 \end{array}$$

$\mathbf{G}$   $\mathbf{h}$

# CVXOPT LP Parameters

**c** [-1, -1]

**G** [4, 2, -5], [-1, 1, 2]

**h** [8, 10, 2]

`cvxopt.solvers.lp(c, G, h[, A, b[, solver[, primalstart[, dualstart]]]])`



# Code

```
from cvxopt import matrix, solvers

c = matrix([-1., -1.])
G = matrix([[4., 2., -5.], [-1., 1., 2.]])
h = matrix([8., 10., 2.])

solution = solvers.lp(c, G, h)
```

## Just a small bit harder... The game of "chicken"

	$L$	$R$
$T$	6,6	2,7
$B$	7,2	0,0

The correlated equilibria in this game are described by the probability constraints  $\pi_{TL} + \pi_{TR} + \pi_{BL} + \pi_{BR} = 1$  and  $\pi_{TL}, \pi_{TR}, \pi_{BL}, \pi_{BR} \geq 0$  together with the following rationality constraints:

$$-1\pi_{TL} + 2\pi_{TR} \geq 0$$

$$1\pi_{BL} - 2\pi_{BR} \geq 0$$

$$-1\pi_{TL} + 2\pi_{BL} \geq 0$$

$$1\pi_{TR} - 2\pi_{BR} \geq 0$$

# Four Inequalities

		3 ↓	4 ↓
		<i>L</i>	<i>R</i>
1 →	<i>T</i>	6,6	2,7
2 →	<i>B</i>	7,2	0,0

The correlated equilibria in this game are described by the probability constraints  $\pi_{TL} + \pi_{TR} + \pi_{BL} + \pi_{BR} = 1$  and  $\pi_{TL}, \pi_{TR}, \pi_{BL}, \pi_{BR} \geq 0$  together with the following rationality constraints:

$$-1\pi_{TL} + 2\pi_{TR} \geq 0$$

$$1\pi_{BL} - 2\pi_{BR} \geq 0$$

$$-1\pi_{TL} + 2\pi_{BL} \geq 0$$

$$1\pi_{TR} - 2\pi_{BR} \geq 0$$

# Four Inequalities

Formula 1

$$\begin{aligned}6\pi_{L|T} + 2\pi_{R|T} &\geq 7\pi_{L|T} + 0\pi_{R|T} \\-1\pi_{L|T} + 2\pi_{R|T} &\geq 0 \\-1\pi_{TL} + 2\pi_{TR} &\geq 0\end{aligned}$$

Formula 2

$$\begin{aligned}7\pi_{L|B} + 0\pi_{R|B} &\geq 6\pi_{L|B} + 2\pi_{R|B} \\1\pi_{L|B} - 2\pi_{R|B} &\geq 0 \\1\pi_{BL} - 2\pi_{BR} &\geq 0\end{aligned}$$

Formula 3

$$\begin{aligned}6\pi_{T|L} + 2\pi_{B|L} &\geq 7\pi_{T|L} + 0\pi_{B|L} \\-1\pi_{T|L} + 2\pi_{B|L} &\geq 0 \\-1\pi_{TL} + 2\pi_{BL} &\geq 0\end{aligned}$$

Formula 4

$$\begin{aligned}7\pi_{T|R} + 0\pi_{B|R} &\geq 6\pi_{T|R} + 2\pi_{B|R} \\1\pi_{T|R} - 2\pi_{B|R} &\geq 0 \\1\pi_{TR} - 2\pi_{BR} &\geq 0\end{aligned}$$

note:  $\pi_{xy} = \pi(x|y)\pi(y) = \pi(x \cap y) = \pi(y|x)\pi(x) = \pi_{yx}$

Four Inequalities, four variables: it is party time.

$$\pi_{TL} + \pi_{TR} + \pi_{BL} + \pi_{BR} = 1$$

s.t.

$$-1\pi_{TL} + 2\pi_{TR} \geq 0$$

$$1\pi_{BL} - 2\pi_{BR} \geq 0$$

$$-1\pi_{TL} + 2\pi_{BL} \geq 0$$

$$1\pi_{TR} - 2\pi_{BR} \geq 0$$

$$x_1 = \pi_{TL}, \ x_2 = \pi_{TR}, \ x_3 = \pi_{BL}, \ x_4 = \pi_{BR}$$

"Standard Form"?

$$1x_1 + 1x_2 + 1x_3 + 1x_4 = 1$$

s.t.

$$-1x_1 + 2x_2 \geq 0$$

$$1x_3 - 2x_4 \geq 0$$

$$-1x_1 + 2x_3 \geq 0$$

$$1x_2 - 2x_4 \geq 0$$

"Standard Form"?

$$1x_1 + 1x_2 + 1x_3 + 1x_4 = 1$$

s.t.

$$1x_1 - 2x_2 \leq 0$$

$$-1x_3 + 2x_4 \leq 0$$

$$1x_1 - 2x_3 \leq 0$$

$$-1x_2 + 2x_4 \leq 0$$

## New parameters

$$1x_1 + 1x_2 + 1x_3 + 1x_4 = 1$$

s.t.



A



b

$$1x_1 - 2x_2 \leq 0$$

$$-1x_3 + 2x_4 \leq 0$$

$$1x_1 - 2x_3 \leq 0$$

$$-1x_2 + 2x_4 \leq 0$$



## Old parameters

$$1x_1 + 1x_2 + 1x_3 + 1x_4 = 1$$

s.t.

$$\begin{array}{l} 1x_1 - 2x_2 + 0x_3 + 0x_4 \leq 0 \\ 0x_1 + 0x_2 - 1x_3 + 2x_4 \leq 0 \\ 1x_1 + 0x_2 - 2x_3 + 0x_4 \leq 0 \\ 0x_1 - 1x_2 + 0x_3 + 2x_4 < 0 \end{array}$$

# Code

```
from cvxopt import matrix, solvers

c = matrix([1., 1., 1., 1.])
G = matrix([[1., 1., 0., 0.], [-2., 0., 0., -1.],
            [0., -2., -1., 0.], [0., 0., -2., -2.]])
h = matrix([0., 0., 0., 0.])

A = matrix([[1.], [1.], [1.], [1.]])
b = matrix(1.)

solution = solvers.lp(c, G, h, A, b)
```

Presentation and Code samples can be found at:  
<https://github.com/axonal/cvxopt-tutorial>