

# **Microscopic Simulation of Solid State Sintering Regarding Irregularly Shaped Powder Particles**

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November 20, 2023

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# Preface

# 1 Introduction

## 1.1 The RefraBund Project

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### 1.1.3 Process Routes

## 1.2 State of the Art

### 1.2.1 Basic Theory of Sintering

### 1.2.2 Sintering Models with Sharp Interfaces

### 1.2.3 Sintering Models with Diffuse Interfaces

### 1.2.4 Monte-Carlo-Methods (MCM)

#### Classic Monte Carlo Methods

#### Kinetic Monte Carlo Methods

### 1.2.5 The Thermodynamic Extremal Principle (TEP)

General things about internal state and non-equilibrium  
Applications

#### Classic Formulation

The classic formulation of the principle was basically formulated by Svoboda and Turek [4], however dependent on the works of Ziegler [5, 6] and Onsager [3]. It is based on the assumption, that the dissipation in the system is always maximized with respect to thermodynamic and kinetic constraints. This is equivalent to the statement, that the system always tries to approach the equilibrium as fast as possible.

The dissipation  $\mathcal{D}$  can be formulated as in Equation 1.1a in dependence on the vector of external state variables  $\mathbf{X}$ , the vector of internal state variables  $\mathbf{x}$  and the velocity vector of internal state variables  $\dot{\mathbf{x}}$ . It is a bilinear form in the velocity vector of internal state variables and the vector of thermodynamic forces. The latter can be expressed as  $\partial G(\mathbf{X}, \mathbf{x})/\partial \mathbf{x}$ , where  $G$  is the Gibbs energy of the system. The Gibbs energy is as state function independent of  $\dot{\mathbf{x}}$ , thus the thermodynamic forces do not include any kinetic constraints. Since  $\mathcal{D}$  is linear in  $\dot{\mathbf{x}}$ , maximizing without further constraints is not meaningful. Therefore, an additional dissipation function  $\mathcal{Q}$  is introduced, which must be always equal to  $\mathcal{D}$ , since both describe the dissipation of the process (compare Equation 1.1b). However,  $\mathcal{Q}$  is formulated in terms of the kinetic conditions of the process and is generally of a non-linear form in  $\dot{\mathbf{x}}$ , often a quadratic form. The actual form

of  $\mathcal{Q}$  heavily depends on the regarded process, but it commonly includes empirical kinetic material parameters such as diffusion coefficients or mobilities.

$$\mathcal{D}(\mathbf{X}, \mathbf{x}, \dot{\mathbf{x}}) = \frac{\partial G(\mathbf{X}, \mathbf{x})}{\partial \mathbf{x}} \cdot \dot{\mathbf{x}} \rightarrow \max_{\dot{\mathbf{x}}} \quad (1.1a)$$

$$\mathcal{D}(\mathbf{X}, \mathbf{x}, \dot{\mathbf{x}}) - \mathcal{Q}(\mathbf{X}, \mathbf{x}, \dot{\mathbf{x}}) = 0 \quad (1.1b)$$

The constrained optimization problem in Equation 1.1 can be solved using the Lagrange formalism. The problem is reformulated as the Lagrange functional  $\mathcal{L}$  in Equation 1.2 with the additional parameter  $\lambda$ .

$$\mathcal{L} = \mathcal{D}(\mathbf{X}, \mathbf{x}, \dot{\mathbf{x}}) + \lambda (\mathcal{D}(\mathbf{X}, \mathbf{x}, \dot{\mathbf{x}}) - \mathcal{Q}(\mathbf{X}, \mathbf{x}, \dot{\mathbf{x}})) \quad (1.2)$$

Setting the gradient of  $\mathcal{L}$  with respect to  $\dot{\mathbf{x}}$  and  $\lambda$  equal zero gives the optimum of the dissipation  $\mathcal{D}$  under the given constraints. Note that the derivative with respect to  $\lambda$  is always identical to the constraint equation.

$$\mathcal{L}_{\dot{\mathbf{x}}} = (1 + \lambda) \frac{\partial G(\mathbf{X}, \mathbf{x})}{\partial \mathbf{x}} - \lambda \frac{\partial \mathcal{Q}(\mathbf{X}, \mathbf{x}, \dot{\mathbf{x}})}{\partial \dot{\mathbf{x}}} \stackrel{!}{=} 0 \quad (1.3a)$$

$$\mathcal{L}_{\lambda} = \mathcal{D}(\mathbf{X}, \mathbf{x}, \dot{\mathbf{x}}) - \mathcal{Q}(\mathbf{X}, \mathbf{x}, \dot{\mathbf{x}}) \stackrel{!}{=} 0 \quad (1.3b)$$

For the simple case of  $\mathcal{Q}$  being quadratic in  $\dot{\mathbf{x}}$  the solution of the system is given in Equation 1.4, which is a linear system of equations of the same size as  $\dot{\mathbf{x}}$ . Compare Svoboda and Turek [4] and Fischer, Svoboda, and Petryk [1] on this.

$$-\frac{\partial G(\mathbf{X}, \mathbf{x})}{\partial \mathbf{x}} = \frac{1}{2} \frac{\partial \mathcal{Q}(\mathbf{X}, \mathbf{x}, \dot{\mathbf{x}})}{\partial \dot{\mathbf{x}}} \quad (1.4)$$

Note, that for this formulation it is required, that the dissipation  $\mathcal{D}$  and the dissipation function  $\mathcal{Q}$  must depend on the same kinetic variables  $\dot{\mathbf{x}}$ . Often the fluxes  $\mathbf{j}$  are used therein.

## Generalized Formulation

Recently, Hackl, Fischer, and Svoboda [2] published a generalized formulation of the principle breaking up the need to have the same kinetic variables in the dissipation  $\mathcal{D}$  and the dissipation function  $\mathcal{Q}$ . The following elaborations use a different notation than in the reference, which fits better to the needs of the application in section 4.6, but the meaning is generally equivalent. The dissipation  $\mathcal{D}$  is defined in the same way as before, but the dissipation function  $\mathcal{Q}$  does not include the velocities of internal state  $\dot{\mathbf{x}}$ , but instead the fluxes  $\mathbf{j}$  as in Equation 1.5b. Note, that the velocities  $\dot{\mathbf{x}}$  are *not* required to be identical to the fluxes  $\mathbf{j}$  here, neither they must have the same size. The connections between the velocities  $\dot{\mathbf{x}}$  and the fluxes  $\mathbf{j}$  are introduced by the constraints  $\mathbf{f}$  as in Equation 1.5c. These are required to be able to solve the problem, so they shall be called *required* constraints hereinafter.  $\mathbf{f}$  and  $\mathbf{j}$  must be of same size. But it is also possible to include several *additional* constraints  $\mathbf{g}$ . These may be used for other requirements on the validity of the model, for example geometric constraints. The constraints may include several *auxiliary variables*  $\boldsymbol{\xi}$  that are not part of  $\mathbf{x}$  or  $\mathbf{j}$  and occur therefore neither in  $\mathcal{D}$  nor  $\mathcal{Q}$ . See section 4.6 for details on their application.

$$\mathcal{D}(\mathbf{X}, \mathbf{x}, \dot{\mathbf{x}}) = \frac{\partial G(\mathbf{X}, \mathbf{x})}{\partial \mathbf{x}} \cdot \dot{\mathbf{x}} \rightarrow \max_{\dot{\mathbf{x}}} \quad (1.5a)$$

$$\mathcal{D}(\mathbf{X}, \mathbf{x}, \dot{\mathbf{x}}) - \mathcal{Q}(\mathbf{X}, \mathbf{x}, \mathbf{j}) = 0 \quad (1.5b)$$

$$\mathbf{f}(\mathbf{X}, \mathbf{x}, \dot{\mathbf{x}}, \mathbf{j}, \boldsymbol{\xi}) = 0 \quad (1.5c)$$

$$\mathbf{g}(\mathbf{X}, \mathbf{x}, \dot{\mathbf{x}}, \mathbf{j}, \boldsymbol{\xi}) = 0 \quad (1.5d)$$

With this generalized formulation the Lagrange functional writes as in Equation 1.6. Here we have more Lagrange parameters.  $\lambda_1$  is equivalent to the classic formulation. The vector parameters  $\boldsymbol{\lambda}_2$  and  $\boldsymbol{\lambda}_3$  for the required and additional constraints are of the same size as  $\mathbf{f}$  resp.  $\mathbf{g}$ .

$$\mathcal{L} = \mathcal{D}(\mathbf{X}, \mathbf{x}, \dot{\mathbf{x}}) + (\mathcal{D}(\mathbf{X}, \mathbf{x}, \dot{\mathbf{x}}) - \mathcal{Q}(\mathbf{X}, \mathbf{x}, \mathbf{j})) \lambda_1 + \mathbf{f}^T(\mathbf{X}, \mathbf{x}, \dot{\mathbf{x}}, \mathbf{j}, \boldsymbol{\xi}) \cdot \boldsymbol{\lambda}_2 + \mathbf{g}^T(\mathbf{X}, \mathbf{x}, \dot{\mathbf{x}}, \mathbf{j}, \boldsymbol{\xi}) \cdot \boldsymbol{\lambda}_3 \quad (1.6)$$

As before, the gradient of  $\mathcal{L}$  is set equal to zero to obtain the constrained optimum. Finding a simplified equation system as done above (Equation 1.4) is here not possible due to the constraints.

$$\mathcal{L}_{\dot{\mathbf{x}}} = \frac{\partial G(\mathbf{X}, \mathbf{x})}{\partial \mathbf{x}} (1 + \lambda_1) + \frac{\partial \mathbf{f}^T(\mathbf{X}, \mathbf{x}, \dot{\mathbf{x}}, \mathbf{j}, \boldsymbol{\xi})}{\partial \dot{\mathbf{x}}} \cdot \boldsymbol{\lambda}_2 + \frac{\partial \mathbf{g}^T(\mathbf{X}, \mathbf{x}, \dot{\mathbf{x}}, \mathbf{j}, \boldsymbol{\xi})}{\partial \dot{\mathbf{x}}} \cdot \boldsymbol{\lambda}_3 \stackrel{!}{=} 0 \quad (1.7a)$$

$$\mathcal{L}_{\mathbf{j}} = -\frac{\partial \mathcal{Q}(\mathbf{X}, \mathbf{x}, \mathbf{j})}{\partial \mathbf{j}} \lambda_1 + \frac{\partial \mathbf{f}^T(\mathbf{X}, \mathbf{x}, \dot{\mathbf{x}}, \mathbf{j}, \boldsymbol{\xi})}{\partial \mathbf{j}} \cdot \boldsymbol{\lambda}_2 + \frac{\partial \mathbf{g}^T(\mathbf{X}, \mathbf{x}, \dot{\mathbf{x}}, \mathbf{j}, \boldsymbol{\xi})}{\partial \mathbf{j}} \cdot \boldsymbol{\lambda}_3 \stackrel{!}{=} 0 \quad (1.7b)$$

$$\mathcal{L}_{\boldsymbol{\xi}} = \frac{\partial \mathbf{f}^T(\mathbf{X}, \mathbf{x}, \dot{\mathbf{x}}, \mathbf{j}, \boldsymbol{\xi})}{\partial \boldsymbol{\xi}} \cdot \boldsymbol{\lambda}_2 + \frac{\partial \mathbf{g}^T(\mathbf{X}, \mathbf{x}, \dot{\mathbf{x}}, \mathbf{j}, \boldsymbol{\xi})}{\partial \boldsymbol{\xi}} \cdot \boldsymbol{\lambda}_3 \stackrel{!}{=} 0 \quad (1.7c)$$

$$\mathcal{L}_{\lambda_1} = \mathcal{D}(\mathbf{X}, \mathbf{x}, \dot{\mathbf{x}}) - \mathcal{Q}(\mathbf{X}, \mathbf{x}, \mathbf{j}) \stackrel{!}{=} 0 \quad (1.7d)$$

$$\mathcal{L}_{\boldsymbol{\lambda}_2} = \mathbf{f}(\mathbf{X}, \mathbf{x}, \dot{\mathbf{x}}, \mathbf{j}, \boldsymbol{\xi}) \stackrel{!}{=} 0 \quad (1.7e)$$

$$\mathcal{L}_{\boldsymbol{\lambda}_3} = \mathbf{g}(\mathbf{X}, \mathbf{x}, \dot{\mathbf{x}}, \mathbf{j}, \boldsymbol{\xi}) \stackrel{!}{=} 0 \quad (1.7f)$$

### 1.2.6 Miscellaneous

## 2 Aim and Scope

This work is intended to create and evaluate a novel model approach for sintering of irregularly shaped particles. The model is based on a *finite differences* (FD) approach of particle description, thus the interfaces are described as sharp lines. For the solution of the complicated partial differential equation system describing the diffusional flows and geometry evolution, the *thermodynamic extremal principle* (TEP) shall be applied. The model shall be restricted to 2D-space as a proof of concept, but may be extended to 3D-space as well. The latter is out of scope of this work.

The work resides under the circumstances of the RefraBund project as described in section 1.1. Therefore, the following influences shall be included in the model and investigated by simulation as well experiment.

**Multi-Material Powder** The investigated material consists of aluminium oxide powder as well as metallic niobium resp. tantalum powder. The model must therefore be able to respect different material properties on particles in contact to each other with their implication of grain boundary shape and kinetic behavior.

**Multi-Scale Powder** For achieving good thermal shock properties a component shall consist of coarse-grained material with fine-grained additions as filler and bonder. The model must be able to deal with particle sizes that differ by several orders of magnitude, which is mainly a topic of computational effort and numerical stability.

**Aggregate Powders** The coarse grained powder is obtained by sintering of fine-grained composite powders to produce aggregate particles. The aggregate particles are not homogeneous in their properties and include a certain porosity. The model must include contacts of aggregate particles by average approaches to circumvent the need of local structure description and modelling.

**Powder Mixture** The production process involves mixing of different powder fractions to obtain optimal properties in processing and application. The mixing shall be modelled by statistical approaches to enable the investigation of different powder mixture to support product development.

The predictions of the model are characterized in this work in comparison to other models. The model is validated against sintering experiments under consideration of microstructure and physical material properties.



## **3 Powder Analysis and Representation**

### **3.1 Classic Methods of Powder Characterization**

### **3.2 Particle Description by Parametrized Shape Functions**

### **3.3 Characterization of Powders and Powder Mixtures**

# 4 Model Development

## 4.1 A Discrete Model of Powder Particles

- Particles, Coordinate System
- Node Types
- Multi-Scale Considerations, Matrix

Continuous description of the particle surface geometry is only possible for nearly ideal geometries. For complicated geometries a discretized approach is feasible. For the current work, the concept of a node shall be introduced. A node is here considered as a discrete point of the particle surface connected with its neighbors by straight lines. The spline of those lines is defining the surface of the particle.

The location of each node in space is defined by a tuple of polar coordinates  $(\varphi, r)$ , where  $\varphi$  is the angle coordinate and  $r$  the radius reps. distance from origin. The origin of the polar coordinates is distinct for each particle and is considered as the center of the particle, although it is generally not identical with the barycenter of the particle. With this concept the particle can be moved in space without translating the surface node coordinates and the description of node evolution is simplified, since only the local geometry must be regarded.

In regard of sintering processes, the contact properties of multiple particles are investigated. The common interface of two particles in contact is commonly called a sinter neck. It consists of a grain boundary bounded by triple points of the grain boundary and the two adjacent free surfaces. Until here, three types of nodes can be identified:

**Surface Nodes** forming the free surface of a particle in contact to atmosphere or vacuum.

**Grain Boundary Nodes** forming the grain boundary in a particle contact.

**Neck Nodes** representing the triple point between grain boundary and two surfaces.

Details on the conditions at those nodes are given in the following sections.

To locate particles in space a tree-like structure shall be used. In the center of the absolute space a single particle is located, which will not change its position and all other particles' locations are defined relative to that. This particle shall be called the *root particle*. It's nearest neighbors in contact define their coordinates as polar coordinates  $(\varphi_P, r_P)$  with origin at the center of the root particle. In accordance to common nomenclature of such data structures, the involved particles shall be called *parents* resp. *children*. Each particle may have further children in this structure, which define their coordinates with respect to their parents center. The coordinate system of the children is identical to the coordinate system of the parent's nodes. Aside it's center location, a particle may be rotated around it's center point. This rotation angle is given as  $\omega$ .

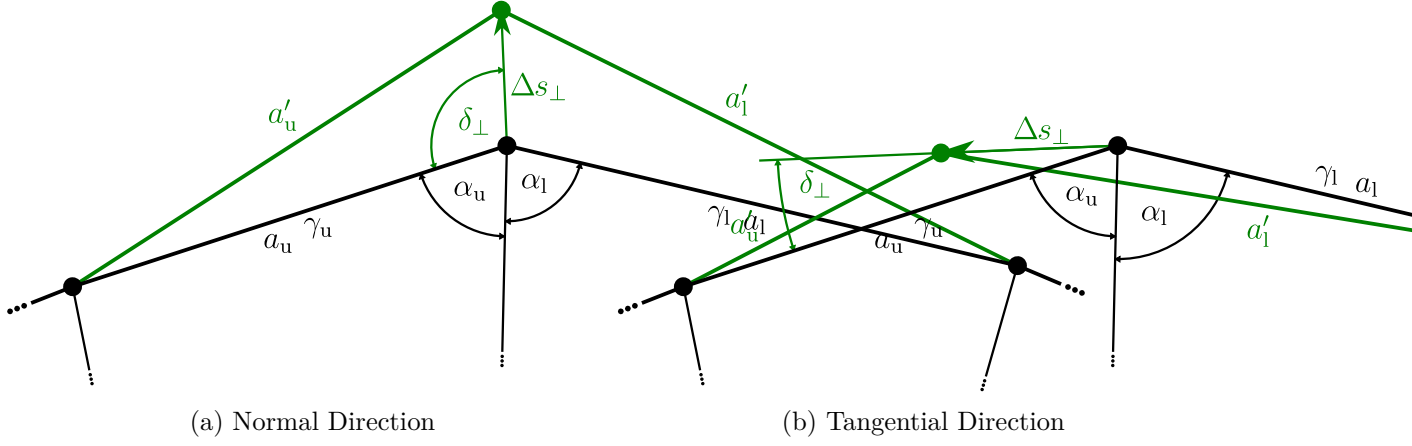


Figure 4.1: Shifting of Surface Nodes

## 4.2 Considerations on Free Surfaces

If a node is displaced (shifted) in space, a change in Gibbs energy occurs due to the change in surface resp. interface area. The amount of energy bound in a surface or interface is described by the interface energy  $\gamma$ . Since a 2D problem is regarded, the length of the surface line  $a$  is a measure of present surface area. The change of Gibbs energy due to node shifting is described by Equation 4.1 with the prime values as measures after shifting.

$$\delta G_{\perp} = (a_u' - a_u + a_l' - a_l) \gamma_S \quad (4.1)$$

The shifting of nodes is separated into two components as shown in Figure 4.1. The normal vector points under an angle of  $\delta_{\perp}$  acc. to Equation 4.2 to both surface lines.

$$\delta_{\perp} = \pi - \frac{1}{2} (\alpha_u + \alpha_l) \quad (4.2)$$

With a certain normal shift  $\delta s_{\perp}$ , the surface lengths after shifting are calculated by Equation 4.3 and Equation 4.4.

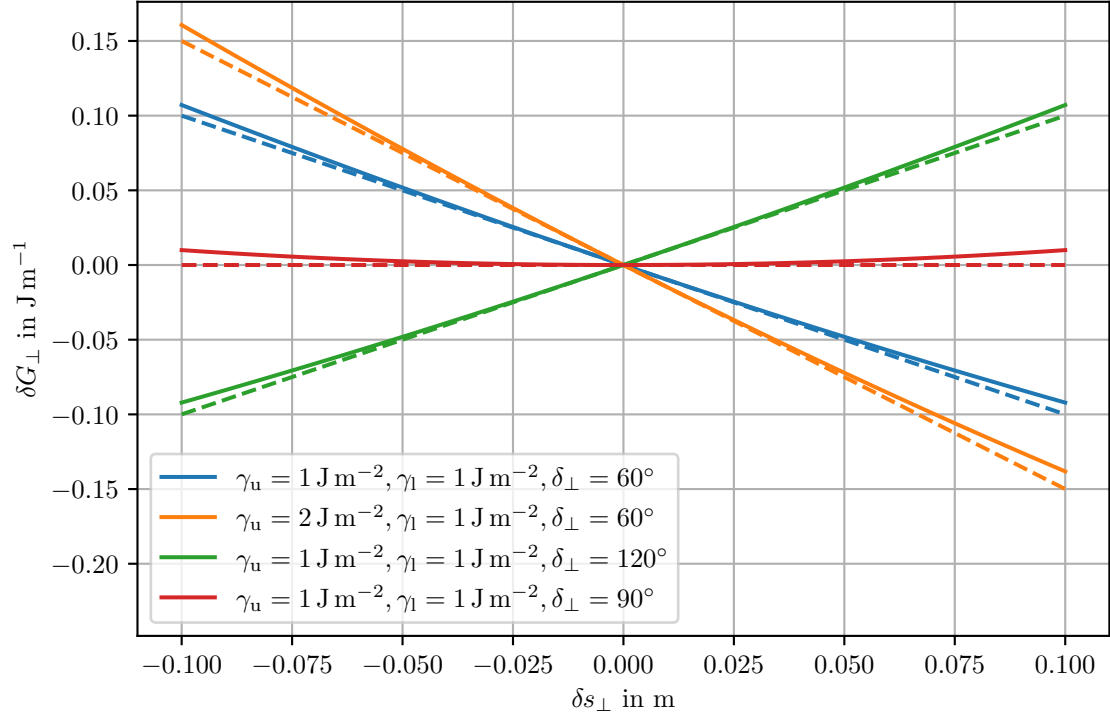
$$a_u' = \sqrt{a_u^2 + \delta s_{\perp}^2 - 2a_u \delta s_{\perp} \cos \delta_{\perp}} \quad (4.3)$$

$$a_l' = \sqrt{a_l^2 + \delta s_{\perp}^2 - 2a_l \delta s_{\perp} \cos \delta_{\perp}} \quad (4.4)$$

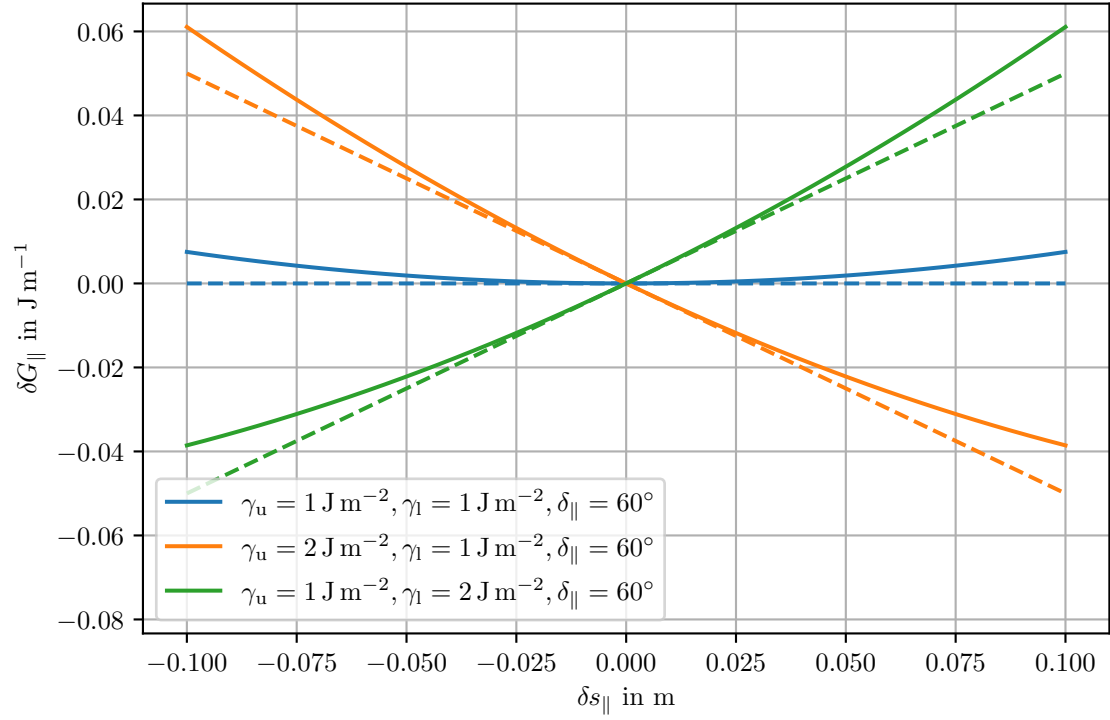
The slope of the tangent in  $\delta s_{\perp} = 0$  is calculated as in Equation 4.5. Figure 4.2a shows the change in Gibbs energy due to normal shifting with different values of  $\delta_{\perp}$  and  $\gamma$ . A  $\delta_{\perp} > 90^\circ$  means a convex surface, thus energy gain when inward shifting (negative  $\delta s_{\perp}$ ). A  $\delta_{\perp} < 90^\circ$  means a concave surface, thus energy gain when outward shifting (positive  $\delta s_{\perp}$ ). A  $\delta_{\perp} = 90^\circ$  means a even surface, thus energy loss in both shifting directions. Note, that the slope is dependent on the sum of  $\gamma_u$  and  $\gamma_l$ , whereas the monotonicity depends on  $\delta_{\perp}$ .

$$\frac{\partial G}{\partial s_{\perp}} = \lim_{\delta s_{\perp} \rightarrow 0} \frac{\delta G_{\perp}}{\delta s_{\perp}} = -(\gamma_u + \gamma_l) \cos \delta_{\perp} \quad (4.5)$$

$$\frac{\partial V}{\partial s_{\perp}} = \frac{1}{2} (a_u + a_l) \sin \delta_{\perp} \quad (4.6)$$



(a) Normal Direction



(b) Tangential Direction

Figure 4.2: Change in Gibbs Energy Due to Node Shifting (tangents dashed)

Regarding the normal shifting, the direction vector is under an angle of  $\delta_{\parallel}$  acc. to. Equation 4.7 to the upper surface line.

$$\delta_{\parallel} = \delta_{\perp} - \frac{\pi}{2} \quad (4.7)$$

The change in Gibbs energy is calculated in a similar way acc. to Equation 4.8, but the shifted surface lengths calculate as in Equation 4.9 and Equation 4.10 in dependence on the tangential shift  $\delta s_{\parallel}$ . Note the signs before the cosine terms.

$$\delta G_{\perp} = (a_u' - a_u + a_l' - a_l) \gamma_S \quad (4.8)$$

$$a_u' = \sqrt{a_u^2 + \delta s_{\parallel}^2 - 2a_u \delta s_{\parallel} \cos \delta_{\parallel}} \quad (4.9)$$

$$a_l' = \sqrt{a_l^2 + \delta s_{\parallel}^2 + 2a_l \delta s_{\parallel} \cos \delta_{\parallel}} \quad (4.10)$$

The slope of the tangent in  $\delta s_{\parallel} = 0$  is calculated as in Equation 4.11. Figure 4.2b shows the change in Gibbs energy due to normal shifting with different values of  $\gamma$ . The slope and monotonicity of these curves is here dependent on the *difference* of  $\gamma_u$  and  $\gamma_l$ . The convexity or concavity of the surface is here only of minor importance. Whether the interface with higher  $\gamma$  is located on the upper or lower side determines the monotonicity of the curves. In the special case when  $\gamma_u = \gamma_l$ , the curve has a minimum in  $\delta s_{\parallel} = 0$ , meaning that any shift will produce an energy loss. This is the case on all nodes except neck nodes.

$$\frac{\partial G}{\partial s_{\parallel}} = \lim_{\delta s_{\parallel} \rightarrow 0} \frac{\delta G_{\parallel}}{\delta s_{\parallel}} = -(\gamma_u - \gamma_l) \cos \delta_{\parallel} \quad (4.11)$$

$$\frac{\partial V}{\partial s_{\parallel}} = \frac{1}{2} (a_l - a_u) \sin \delta_{\parallel} \quad (4.12)$$

### 4.3 Considerations on Grain Boundaries

In contrast to free surfaces, where the geometric evolution of a node is not constrained by the surrounding space, in grain boundaries the node shifting acts counter the solid material of the other particle. This introduces additional constraints, since the grain boundary must not form holes and must not overlap. The geometric constraints are dependent on the local evolution of the grain boundary nodes, as well as the relative position change of the particles.

Starting from a compact grain boundary, meaning that there are no holes and/or overlap present within it, the condition of maintaining the compactness of the grain boundary can be formulated as follows, if the postions of the particles are fixed.

$$\delta s_{\perp L} = -\delta s_{\perp R} \quad (4.13)$$

$$\delta s_{\parallel L} = -\delta s_{\parallel R} \quad (4.14)$$

But in reality the positions of particles to each other change, which can be observed macroscopically as shrinkage. However, the relation between node shifting and particle movement cannot be easily formulated, since the particle movement is effected by the ensemble of all nodes involved





analytically, so numerical time integration is required. Usually, the transition from infinitesimal description to finite time steps would be performed as a last step, but doing this step at the beginning opens possibilities of simplifying the equations and clarifies the following elaborations. At this point, it does not matter if a simple Eulerian integration is used or more complex methods like Runge-Kutta. The key feature is, that we use a finite time step width  $\delta t$ , which is fixed in each time step calculation, but may change from step to step. So, the internal state velocities  $\dot{x}$  are directly linked linearly to finite changes using the time step as follows:

$$\delta x = \dot{x} \cdot \delta t \quad (4.20)$$

#### 4.6.2 Choice of Variables

First, the internal state variables  $\mathbf{x}$ , the internal state velocities  $\dot{\mathbf{x}}$  and the fluxes  $\mathbf{j}$  must be chosen. The feasibility of the approach heavily depends on this choice.

The choice of the fluxes is straightforward with the diffusional fluxes. One has two diffusional fluxes on each node  $j_u$  and  $j_l$ , however the flux of the upper node to the lower and the flux from the lower to the upper are always equal due to constance of mass. Additional fluxes are to be found in the fluxes from the particle to the matrix  $j_M$ , if a matrix is present in the inter-particle spaces.

The internal state variables are the polar coordinates of the nodes  $[r, \varphi]$  in with pole in their particle's center, since they determine the thermodynamic forces and the main aim of the simulation is to follow the time evolution of the particles and nodes. The coordinates of the particles  $[r_P, \varphi_P, \omega_P]$  are not used as internal state variables, since they are unambiguously defined, if the coordinates of all nodes relative to their particle are known and also which nodes are in contact to each other. Moreover, the particle coordinates are auxiliary variables (included in  $\xi$ ) to ease formulation of the geometric constraints.

The straightforward choice of the internal state velocities are the finite changes of internal state variables (e.g.  $\Delta r$  and  $\Delta \varphi$ ). However, for the changes in node coordinates this is not feasible, since they do not depend linearly on the fluxes due to triangular function relations. A more appropriate way is to define the node shifts  $\delta s_\perp$  and  $\delta s_\parallel$  as internal state velocities, so that the required constraints  $\mathbf{f}$  become linear (see subsection 4.6.5). As they are determined, translation into new coordinates is directly possible using the relations obtained in the previous sections.

The external state variables (such as temperature  $T$  and pressure  $p$ ) are not considered in the following, since they are assumed constant.

#### 4.6.3 Dissipation $\mathcal{D}$

The dissipation at one node is given by the product node shifts and the respective Gibbs energy derivatives as in Equation 4.21. The Gibbs energy derivatives were determined in previous sections for the different node types. The dissipation of the whole system is the sum of all node dissipations, since all thermodynamic forces occur at nodes. Shifting of particles alone does not follow a thermodynamic force, but is determined by the ensemble of the involved nodes.

$$\mathcal{D} = \sum^N \left[ \frac{\partial G}{\partial s_\perp} \delta s_\perp + \frac{\partial G}{\partial s_\parallel} \delta s_\parallel \right] \quad (4.21)$$



#### 4.6.4 Dissipation Function $\mathcal{Q}$

$$\mathcal{Q} = \sum^{\mathbb{N}} \frac{1}{2} \frac{RT}{V_m \xi^\circ} \left( \frac{a_u j_u^2}{D_u} + \frac{a_l j_l^2}{D_l} \right) \delta t \quad (4.22)$$

#### 4.6.5 Required Constraints $f$

Relation between fluxes resp. volume change and shift.

$$\frac{\partial V}{\partial s_\perp} \delta s_\perp + \frac{\partial V}{\partial s_\parallel} \delta s_\parallel = \delta V = (j_u + j_l) \delta t \quad (4.23)$$

#### 4.6.6 Additional Constraints $g$

For each contact node, where:

$$\delta r_P = \frac{\partial r_P}{\partial s_\perp} \delta s_\perp + \frac{\partial r_P}{\partial s_\parallel} \delta s_\parallel \quad (4.24a)$$

$$\delta \varphi_P = \frac{\partial \varphi_P}{\partial s_\perp} \delta s_\perp + \frac{\partial \varphi_P}{\partial s_\parallel} \delta s_\parallel \quad (4.24b)$$

$$\delta \omega_P = \frac{\partial \omega_P}{\partial s_\perp} \delta s_\perp + \frac{\partial \omega_P}{\partial s_\parallel} \delta s_\parallel \quad (4.24c)$$

# 5 Software Implementation of the Model

Reference to open source code

## 5.1 Representation of Particles and Nodes

- Classes
- Tree Structure
- Coordinate Systems

## 5.2 Numerical Solution Procedure

- TEP Solution
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- Monte-Carlo Drawing

## 5.3 Calculation and Extraction of Key Features

- Volume Cell, Shrinkage
- Neck Measures

## **6 Model Validation**

### **6.1 Investigations on Simple Test Cases**

**6.1.1 A Single Particle Free in Space**

**6.1.2 A Particle Pair Free in Space**

**6.1.3 A Particle Pair at Different Contact Angles**

**6.1.4 A Particle Pair with Asymmetric Material Properties**

**6.1.5 A Single Particle Embedded in a Matrix**

**6.1.6 A Particle Pair Embedded in a Matrix**

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