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Machine Learning Hw 3

1)

a. $\bar{g}(x) = E_D[g(x)]$

$$E_D[g(x)] = E_D[Ax + B] \text{ where } A = \frac{X_1^2 - X_2^2}{X_1 - X_2} = sign(X_1 + X_2)$$
$$E_D[g(x)] = E_D[(X_1 + X_2)(x - X_1) + X_1^2]$$
$$E_D[g(x)] = E_D[X_1 x - X_1^2 + X_2 x - X_1 X_2 + X_1^2]$$
$$E_D[g(x)] = E_D[(X_1 + X_2)x - X_1 X_2]$$
$$since\ E_D[(X_i)] = 0, we\ can\ then\ conclude\ that:$$
$$\bar{g}(x) = 0$$

In [1]:
```
import matplotlib.pyplot as plt
import sklearn
import numpy as np
import pandas as pd
import random as rd
from sklearn.linear_model import Perceptron
from numpy.linalg import inv
from scipy import integrate
from scipy.integrate import quad
from scipy.integrate import tplquad
```

b) and c)

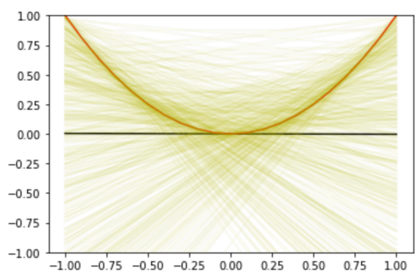
In [2]:
```

#returns the average slope and intercept over n simulations
def g_bar(n):
    x = 0
    y = 0
    for i in range(n):
        a = np.random.uniform(-1, 1)
        b = np.random.uniform(-1, 1)
        x = x + a + b
        y = y + (- a * b)
    x = x/n
    y = y/n
    return (x , y)

answer = []
coordinate = g_bar(100000)
answer.append(coordinate)
```

In [3]:
```
answer
```

Out[3]:
```
[(-0.0035985163725865653, 3.638724189882314e-05)]
```

In [4]:

```python
#graphing g bar - Denoted by blue
for x in range(len(answer)):
    slope, intercept = answer[x]

    gx = np.array([-1, 1])
    gy = slope * gx + intercept
    plt.plot(gx,gy,c = 'black',alpha=1.0)



#graphing f(x) - Denoted by red
x1 = []
for x in range(21):
    x1.append(-1 + x *0.1)
y1 = [x1[x]**2for x in range(len(x1))]

plt.plot(x1,y1, c = 'r', alpha = 1.0)

#graphing all possible g(x) - Denoted by the yellow
for x in range(500):
    x_r = np.array([-1, 1])
    a =rd.uniform(-1,1)
    b =rd.uniform(-1,1)
    slope = a+b
    b = -a *b

    y_r = slope*x_r+b
    plt.plot(x_r, y_r, c = 'y', alpha =0.05)

plt.ylim([-1, 1])
plt.show()
```



In [5]:

```python
#generating variance
def integrand_variance(x, x1, x2):
    slope, intercept = answer[0]
    g_bar = slope * x + intercept
    a1 = (x1 + x2)
    b1 = -x1 * x2
    g_data = a1 * x + b1
    return 1 / 8 * (g_data - g_bar) ** 2

#generating bias
def integrand_bias(x):
    slope, intercept = answer[0]
    g_bar = slope * x + intercept
    target = x**2
    return (g_bar - target) ** 2 / 2



ans, err = quad(integrand_bias, -1, 1)
ans_v, err_v = integrate.tplquad(integrand_variance, -1, 1, lambda x: -1, lambda x: 1,lambda x, y: -1, lambda x, y
print("Variance is {}".format(ans_v))
print("Bias is {}".format(ans))
#error out is = bias + variance
print("Error out is {}".format(ans + ans_v))
```

```
Variance is 0.3333376510973926
Bias is 0.19998005960279341
Error out is 0.5333177107001861
```

2)

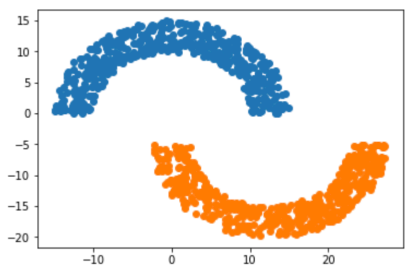
LFD 3.1

In [6]:

```
rad = 10
thk = 5
sep = 5
N = 2000
n = 1000

#Seperating pic into top half vs bottom half
#generate outputs (-1, 1) in array y
#for all y>1,
#tags all points in bottom half as -1, top as +1
#saves everything into one array
#return array of coordinates X (X,y) and corresponding output in another array y (-1, 1)

#upper half center coordinates
upper_x = 0
upper_y = 0

#bottom half center coordinates
lower_x = upper_x + rad + thk / 2
lower_y = upper_y - sep
#creating the theta needed to generate semicircles
Theta = np.random.uniform(0, 2*np.pi, n)
#uniformly drawing 'n' samples between bounds of (rad, rad + thickness)
converter = np.random.uniform(rad, rad+thk, n)
#taking array Theta and changing all values < pi to 1, > to 0. Making a binary array
#this allows us to know which data points are on top half, which on bottom
y = 2 * (Theta < np.pi) - 1
X = np.zeros((n, 2))
#for all values out put is +1, input the coordinates of the center for top half
#for all values out put is -1, input the coordinates of the center for bottom half
X[y > 0] = np.array([upper_x, upper_y])
X[y < 0] = np.array([lower_x, lower_y])
#for each element in X, convert the x coordinate [0] and y coordinate [2]
#by their corresponding angle theta
X[:, 0] += np.cos(Theta) * converter
X[:, 1] += np.sin(Theta) * converter

```

In [7]:

```
#plotting the first half and second half seperately
plt.scatter(X[y>0][:, 0], X[y>0][:, 1])
plt.scatter(X[y<0][:, 0], X[y<0][:, 1])
plt.show()
```



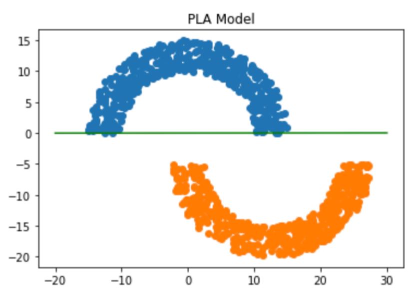
a)

In [8]:

```python
def PLA(X, y):
    input_size, d = X.shape
    #making default 0 weights
    w = np.zeros(d)
    iterator = 0
    #used to check for convergence
    n = X.shape[0]
    check = 0
    #PLA application - checking if converged
    while not(check == n):
        #checking if missclassified
        if np.sign(X[iterator, :].dot(w) * y[iterator]) <= 0:
            #PLA weight adjustment
            w += y[iterator] * X[iterator, :]
        #moving on to next weight, input, corresponding output, etc.
        iterator += 1
        #restarting the loop when iterator == size of inputs
        if iterator == input_size:
            iterator = 0
        #computing the sum input times weight
        check = np.sum(X.dot(w) * y > 0)
    return w
```

In [9]:

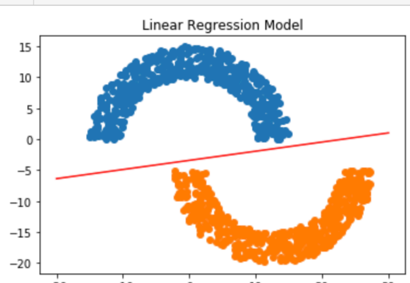
```python
X_treat = np.c_[np.ones(int(N/2)), X]
weights = PLA(X_treat, y)

#boundaries for the PLA
a_1 = np.array([-20,30])
#corresponding output
b_1 = - (weights[0] + weights[1] * a_1) / weights[2]
#plotting the top half
plt.scatter(X[y>0][:, 0], X[y>0][:, 1])
#plotting the bottom half
plt.scatter(X[y<0][:, 0], X[y<0][:, 1])
#plotting the PLA result
plt.title('PLA Model')
plt.plot(a_1, b_1,c='g')
plt.show()

```



b)

In [10]:

```python
X_treat = np.c_[np.ones(int(N/2)), X]
w1 = inv(X_treat.T.dot(X_treat)).dot(X_treat.T).dot(y)
b2 = - (w1[0] + w1[1] * a_1) / w1[2]
plt.scatter(X[y>0][:, 0], X[y>0][:, 1])
plt.scatter(X[y<0][:, 0], X[y<0][:, 1])
plt.title('Linear Regression Model')
plt.plot(a_1, b2, c="r")
plt.show()
```



3)

a. $f(c) = (Xc - y)^T(Xc - y)$
$f(c) = (Xc)^T Xc - (Xc)^T y - Xc(y)^T + yy^T$
$This\ can\ be\ derived\ by\ multiplying\ out\ the\ terms$

b. $e(c) = -(y)^T Xc = Ac$
$A = -(y)^T X$
$From\ the\ textbook$:
$e(c+s) = e(c) + Dc + o(s)$
$e(c) + e(s) = e(c) + Dc + o(s)$
$Ac + -(y)^T Xs = Ac + Dc + o(s)$
$Extracting\ the\ derivative$
$-(y)^T Xs = Dc + o(s)$
$Divide\ both\ sides\ by\ s$
$$\frac{-(y)^T Xs}{s} = \frac{Dc}{s} + \frac{o(s)}{s}$$
$As\ the\ value\ of$ s $approaches\ 0, \frac{||o(s)||}{||s||} \to 0\ from\ the\ textbook$
$$\frac{-(y)^T Xs}{s} = \frac{Dc}{s} \to \boldsymbol{-(y)^T X = Dc}$$

c. $q(c) = c^T Qc\ where\ c^T Qc = (Xc)^T Xc$
$q(c+s) = (c+s)^T Q(c+s)$
$q(c+s) = (c+s)(Qc+Qs)$
$Using\ the\ part\ from\ above: q(c+s) = q(c) + Dc + o(s)$
$(c+s)(Qc+Qs) = c^T Qc + Dc + o(s)$
$(c+s)Qc + (c+s)Qs = c^T Qc + Dc + o(s)$
$cQc + sQc + cQs + sQs = c^T Qc + Dc + o(s)$
$sQc + cQs + sQs = Dc + o(s)$
$Divide\ both\ sides\ by\ s$
$$Qc = \frac{Dc}{s} + 0,$$
$As\ the\ value\ of$ s $approaches\ 0, \frac{||o(s)||}{||s||} \to 0\ from\ the\ textbook,$
$hence\ all\ terms\ with\ s * Qc\ are\ reduced\ to\ 0$
$Dc = Qc$

d. $f(c) = (Xc)^T Xc - (Xc)^T y - Xc(y)^T + yy^T$
$f(c) = (Xc)^T Xc - X^T c(y) - Xc(y)^T + yy^T$
$X^T c(y)\ and\ Xc(y)^T\ follow\ the\ same\ derivative\ from\ e(c)$
$yy^T\ goes\ to\ zero$
$\boldsymbol{f'(c) = X^T Xc + (y)^T X + yX}$

e. $since\ \nabla f(x) = 0,\ then\ we\ can\ set\ X^T Xc + (y)^T X + yX = 0 \to \boldsymbol{c} = \frac{\boldsymbol{(y)^T X + yX}}{\boldsymbol{X^T X}}$

4)

a)

In [11]:

```
count = 0
for x in range (100000):
    #generating the rows and columns
    row = rd.randint(3,9)
    col = rd.randint(1, row-1)
    #making the array
    array = np.random.rand(row,col)
    #transposing the array
    array_t = array.transpose()
    #multiplying
    mult = np.dot(array_t,array)
    #counting number of times that it is invertible
    if abs(np.linalg.det(mult)) > 10**(-10):
        count+=1
count = count/100000
count
```

Out[11]: 1.0

b)

It looks like a matrix is always invertible when it is the dot product of X^T and X. It's important to note that this is what allows us to use matrices for linear regression.

5)

a.

$$9 = (x_1 + 5)^2 + (x_2 - 2)^2$$
$$0 = x_1^2 + 10x_1 + 25 + x_2^2 - 4x_2 + 4 - 9$$
$$0 = 20 + 10x_1 - 4x_2 + x_1^2 + x_2^2$$
$$0 = 20z_1 + 10z_2 - 4z_3 + z_4 + z_5 + 0z_6$$

b.

5)

b)

In [ ]:
```
from sklearn.linear_model import Perceptron
```

In [11]:
```
dataset_1 = np.loadtxt('sampleQuadData2.txt')
(numSamples_1, numFeatures_1) = dataset_1.shape
feat_1 = dataset_1[:,range(numFeatures_1-1)].reshape((numSamples_1, numFeatures_1-1))
output_1 = dataset_1[:, numFeatures_1-1].reshape((numSamples_1,))

(numSamples_1, numFeatures_1) = feat_1.shape

perceptron_1 = Perceptron(fit_intercept=False)
perceptron_1.fit(feat_1,output_1)
perceptron_1.score(feat_1,output_1)


```

```
/Users/alexanderxu/anaconda3/lib/python3.7/site-packages/sklearn/linear_model/stochastic_gradient.py:166: FutureWarning: max_iter and tol parameters have been added in Perceptron in 0.19. If both are left unset, they default to max_iter=5 and tol=None. If tol is not None, max_iter defaults to max_iter=1000. From 0.21, default max_iter will be 1000, and default tol will be 1e-3.
  FutureWarning)
```

Out[11]: 0.52

In [12]:
```
dataset_2 = np.loadtxt('sampleQuadData2Transformed.txt')
(numSamples_2, numFeatures_2) = dataset_2.shape
feat_2 = dataset_2[:,range(numFeatures_2-1)].reshape((numSamples_2, numFeatures_2-1))
output_2 = dataset_2[:, numFeatures_2-1].reshape((numSamples_2,))

perceptron_2 = Perceptron(fit_intercept=False)
perceptron_2.fit(feat_2,output_2)
perceptron_2.score(feat_2,output_2)
```

```
/Users/alexanderxu/anaconda3/lib/python3.7/site-packages/sklearn/linear_model/stochastic_gradient.py:166: FutureWarning: max_iter and tol parameters have been added in Perceptron in 0.19. If both are left unset, they default to max_iter=5 and tol=None. If tol is not None, max_iter defaults to max_iter=1000. From 0.21, default max_iter will be 1000, and default tol will be 1e-3.
  FutureWarning)
```

Out[12]: 0.89

In [13]:
```
print(perceptron_1.coef_)
print('Error Rate for 1 is:')
print(1-perceptron_1.score(feat_1,output_1))
print(perceptron_2.coef_)
print('Error Rate for 2 is:')
print(1-perceptron_2.score(feat_2,output_2))
```

```
[[-2.02428985  3.75626857]]
Error Rate for 1 is:
0.48
[[ 29.        -81.788771  10.390278   1.120849 157.443083   3.924539]]
Error Rate for 2 is:
0.10999999999999999
```

The error rates for the non transformed data set is 0.48, whereas the error rate for 2 is 0.10999, implying 89% accuracy. This makes sense in that there are more variables in the transformed data's boundary space, allowing a better fitting fencing between the positive and negative data outputs

c.

The weights for the untransformed data is: [-2.02, 3.756]

The weights for the transformed data is: [29, -81.788, 10.39, 1.12, 157.44, 3.92]

Boundary in X-Space: $29 - 81.788x_1 + 10.39x_2 + 1.12x_1^2 + 157.44x_2^2 + 3.92x_1x_2$

Boundary in Z-Space: $29z_1 - 81.788z_2 + 10.39z_3 + 1.12z_4 + 157.44z_5 + 3.92z_6$

d. Since $1.12x_1^2 + 157.44x_2^2$ coefficients are positive, by analytical geometry the shape is an ellipse. This shape makes sense since if you plot the data set, the positive data points are bunched up in the center of the graph in a circular shape. We know it's not a circle since there exists a Z6.