EXERCISES 2.11-2.12

POLYNOMIALS – SUM AND PRODUCT OF ROOTS

Compiled by Christos Nikolaidis

POLYNOMIALS – FACTOR AND REMAINDER THEOREMS

A. Practice Questions

1. Consider the cubic function $f(x) = ax^3 + 2x^2 + 3x + 4$. Find the value of a in each of the following cases

(a) the graph of the function passes through the point (1,10)

(b) $f(x)$ is divisible by $(x-1)$

(c) when $f(x)$ is divided by $(x-1)$, the remainder is 10.

(d) Confirm the result in (c) by using long division.

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

2. Consider the cubic function $f(x) = ax^3 + bx^2 + 3x + 4$. Find the values of a and b in each of the following cases

(a) $f(x)$ is divisible by $(x-1)$ and leaves a remainder 6 when divided by $(x+1)$.

(b) $f(x)$ is divisible by $(x^2 - 1)$.

(c) $f(x)$ leaves a remainder $-3x + 3$ when divided by $(x^2 - 1)$.

3. Consider the cubic function $f(x) = 2x^3 + ax^2 + bx + c$. Find the values of a,b,c

(a) if the graph of the function passes through the points (1,0), (–1 ,2), and (0,3).

(b) if the graph of the function passes through the points (1,0), (–1,0), and (3,0).

4. Consider the polynomial $f(x) = 4x^3 - 4x^2 - 5x + 3$.

(a) Show that $x = -1$ is a root and find the other two roots by using long division.

(b) Express the polynomial in the form $f(x) = (x - a)(bx - c)(dx - e)$, where $a, b, c, d, e \in Z$.

5. Consider the following cubic polynomial

$$f(x) = (x-1)(x^2 + (2-k)x + k^2)$$

(a) Show that $x = 1$ cannot be a repeated root.

(b) Find the values of k so that the polynomial has

(i) Exactly one real root. (ii) Exactly two real roots (iii) Exactly three real roots

(c) Find the roots in each case.

B. Past Paper questions (*SHORT*)

6. Consider $f(x) = x^3 - 2x^2 - 5x + k$. Find the value of k if $(x + 2)$ is a factor of $f(x)$.

.....

.....

.....

.....

.....

.....

.....

(Total 6 marks)

7. When the function $f(x) = 6x^4 + 11x^3 - 22x^2 + ax + 6$ is divided by $(x + 1)$ the remainder is -20. Find the value of a.

.....

.....

.....

.....

.....

.....

.....

(Total 4 marks)

8. When the polynomial $x^4 + ax + 3$ is divided by $(x - 1)$, the remainder is 8. Find the value of a.

.....

.....

.....

.....

.....

.....

.....

(Total 6 marks)

9. The polynomial $p(x) = x^3 + ax^2 - 3x + b$ is divisible by $(x - 2)$ and has a remainder 6 when divided by $(x + 1)$. Find the value of a and of b.

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

(Total 6 marks)

Extra question

Factorise $p(x)$ and state its roots

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

10. The polynomial $p(x) = (ax + b)^3$ leaves a remainder of –1 when divided by $(x + 1)$, and a remainder of 27 when divided by $(x – 2)$. Find the values of the real numbers a and b.

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

(Total 4 marks)

11. The polynomial $f(x) = x^3 + 3x^2 + ax + b$ leaves the same remainder when divided by $(x – 2)$ as when divided by $(x + 1)$. Find the value of a.

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

(Total 5 marks)

Extra question
State the possible values of b.

.....

12. Given that $(x - 2)$ and $(x + 2)$ are factors of $f(x) = x^3 + px^2 + qx + 4$, find the value of p and of q.

.....

.....

.....

.....

.....

.....

.....

.....

(Total 6 marks)

Extra question

Solve the equation $f(x) = 0$.

.....

.....

.....

13. The polynomial $P(x) = 2x^3 + ax^2 - 4x + b$ is divisible by $(x - 1)$ and by $(x + 3)$. Find the value of a and of b.

.....

.....

.....

.....

.....

.....

.....

.....

(Total 6 marks)

Extra question:

Factorise $f(x)$

.....

.....

.....

14. The polynomial $x^2 - 4x + 3$ is a factor of $x^3 + (a - 4)x^2 + (3 - 4a)\,x + 3$. Calculate the value of the constant a.

.....

.....

.....

.....

.....

.....

.....

.....

(Total 6 marks)

Extra question:

Factorise the cubic polynomial

.....

.....

.....

.....

Extra question (a slight modification of the question above)

The polynomial $x^2 - 4x + 3$ is a factor of $x^3 + (a - 4)x^2 + (3 - 4a)\,x + b$. Calculate the value of the constant a and of the constant b.

.....

.....

.....

.....

.....

.....

.....

.....

.....

15. When the polynomial $P(x) = 4x^3 + px^2 + qx + 1$ is divided by $(x - 1)$ the remainder is -2.
When $P(x)$ is divided by $(2x - 1)$ the remainder is $\frac{13}{4}$. Find the value of p and of q.

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

(Total 6 marks)

Extra question

When $P(x) = 4x^3 + px^2 + qx + 1$ is divided by $(x - 1)(2x - 1)$ the remainder is $\frac{-21x + 17}{2}$
Find the value of p and of q.

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

INEQUALITIES

A. Practice Questions

16. (a) Sketch the graph of the function $f(x) = (x-1)^2(x+1)(x-2)$
Indicate only the x-intercepts, and the y-intercept.

(b) Hence solve the following inequalities

(i) $(x-1)^2(x+1)(x-2) \geq 0$

(ii) $\dfrac{(x-1)^2(x+1)}{x-2} > 0$ (iii) $\dfrac{(x-1)^2(x+1)}{x-2} \geq 0$

17. Solve the inequalities

(a) $\dfrac{5(x-1)(x-2)^2}{(x-3)^3} \geq 0$ (b) $\dfrac{(x-1)(x-3)^3}{5(x-2)^2} \geq 0$ (c) $\dfrac{(x-2)^2(x-3)^3}{5(x-1)} \geq 0$

18. The polynomial $f(x) = x^3 - 4x^2 + 3x + a$ is divisible by $(x - 1)$.

(a) find the value of a

(b) give full factorization of $f(x)$.

(c) sketch the corresponding graph by indicating only the x-intercepts.

(d) solve the inequality $f(x) \leq 0$.

19. The polynomial $f(x) = x^3 - 7x^2 + ax - 9$ is divisible by $(x - 1)$.

(a) find the value of a

(b) give full factorization of $f(x)$.

(c) sketch the corresponding graph by indicating only the x- intercepts.

(d) solve the inequality $f(x) \leq 0$.

20. The polynomial $f(x) = 3x^3 - a$ is divisible by $(x - 1)$.

(a) find the value of a

(b) give full factorization of $f(x)$.

(c) sketch the corresponding graph by indicating only the x- intercepts.

(d) solve the inequality $f(x) \leq 0$.

B. Past Paper questions (*SHORT*)

21. Solve the inequality $x^2 - 4 + \frac{3}{x} < 0$.

Solve with a GDC (sketch a graph and give the answer)

.....

.....

.....

.....

.....

.....

.....

.....

.....

(Total 6 marks)

Solve without a GDC

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

(Total 6 marks)

22. Let $f(x) = \dfrac{x+4}{x+1}$, $x \neq -1$ and $g(x) = \dfrac{x-2}{x-4}$, $x \neq 4$. Find the set of values of x such that $f(x) \leq g(x)$.

Solve with a GDC (sketch a graph and give the answer)

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

(Total 6 marks)

Solve without a GDC

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

(Total 6 marks)

For exercises 23-26 do not use a GDC. (Then confirm your final answer by using a GDC)

23. Find the values of x for which $|5 - 3x| \leq |x + 1|$.

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

(Total 3 marks)

24. Solve the inequality $|x - 2| \geq |2x + 1|$.

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

(Total 6 marks)

25. Solve the inequality $\left|\frac{x+9}{x-9}\right| \leq 2$

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

(Total 6 marks)

26. Find the largest set of values of x such that the function f given by $f(x)=\sqrt{\frac{8x-4}{x-3}}$ takes real values.

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

(Total 6 marks)

27. The functions f and g are defined by

$$f(x) = 2x - 1, \qquad g(x) = \frac{x}{x+1},\ x \neq -1$$

Find the values of x for which $(f \circ g)(x) \leq (g \circ f)(x)$

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

(Total 6 marks)

SUM AND PRODUCT OF ROOTS

A. Practice Questions

28. Complete the following table: for each polynomial find the sum and the product of the roots (allowing non-real roots and repetition of roots) as well as the remainder when the polynomial is divided by (x-1).

Polynomial	Sum of roots	Product of roots	Remainder by (x-1)
$f(x) = 2x^4 + 6x^3 + 5x^2 - 7x + 8$			
$f(x) = 2x^5 + 6x^3 + 5x^2 - 7x + 8$			
$f(x) = x^{10} - x^9 - 1$			

.....

.....

.....

.....

29. Consider the cubic function $f(x) = ax^3 + 2x^2 + 3x + 4$. Find the value of a

(a) if the sum of the roots of the cubic polynomial is 10

(b) if the product of the roots of the cubic polynomial is 10

.....

.....

.....

.....

.....

30. Consider the cubic function $f(x) = ax^3 + bx^2 + 3x + 4$. Find the values of a and b given that the sum of the roots is 10 and the product of the roots is 12

.....

.....

.....

.....

.....

.....

31. The polynomial

$$f(x) = x^4 - 2x^3 + ax^2 + bx + 3$$

is divisible by $(x-1)$ and the quotient polynomial is $q(x)$. Find

(a) The sum and the product of the roots of $f(x)$

(b) The degree of $q(x)$

(c) The sum of a and b.

(d) The sum and the product of the roots of $q(x)$.

32. Consider the polynomial

$$f(x) = ax^4 + bx^3 + cx^2 + dx + 16$$

When $f(x)$ is divided by $(x+1)$ the remainder is 120, while $(x-1)$ is a factor.
The sum and the product of its roots are both 8.

(a) Find the values of a, b, c and d.

When $f(x)$ is divided by the factor $(x-1)$ the quotient $q(x)$ is a cubic function.

(b) Find the sum and the product of the roots of $q(x)$.

33. The polynomial

$$f(x) = x^4 + ax^3 + bx^2 + cx + d$$

is divisible by $(x-1)$ and $(x-2)$. The sum of its roots is 7, the product of its roots is 0.

(a) Find the values of a, b, c and d.

(b) Factorize $f(x)$.

34. Let α, β be the roots of the quadratic $f(x) = 5x^2 - 2x - 4$

Without estimating the roots α, β , find

(a) $\alpha + \beta$ and $\alpha\beta$

(b) $\alpha^2 + \beta^2$ and $\alpha^3 + \beta^3$

(c) a quadratic with integer coefficients which has roots α^2, β^2.

(d) a quadratic with integer coefficients which has roots $\frac{1}{\alpha}, \frac{1}{\beta}$.

Remember: in a cubic function $ax^3 + bx^2 + cx + d$ with roots r_1, r_2, r_3

$$r_1 + r_2 + r_3 = -\frac{b}{a} \qquad r_1 r_2 r_3 = -\frac{d}{a} \qquad \text{and} \qquad r_1 r_2 + r_2 r_3 + r_3 r_1 = \frac{c}{a}$$

35. Let α, β, γ be the roots of the cubic function

$$f(x) = x^3 - 5x^2 - 7x + 3$$

Find, without estimating the roots α, β, γ

(a) $\alpha + \beta + \gamma$, $\alpha\beta\gamma$ and $\alpha\beta + \beta\gamma + \gamma\alpha$

(b) $\alpha^2 + \beta^2 + \gamma^2$

(c) $(\alpha\beta)^2 + (\beta\gamma)^2 + (\gamma\alpha)^2$

(d) a cubic polynomial which has roots $\alpha^2, \beta^2, \gamma^2$