# MATH HL

## EXERCISES 2.11-2.12 SOLUTIONS

## POLYNOMIALS – SUM AND PRODUCT OF ROOTS

**Compiled by: Christos Nikolaidis**

## POLYNOMIALS – FACTOR AND REMAINDER THEOREMS

### A. Practice Questions

**1.** (a) $a = 1$ (b) $a = -9$ (c) $a = 1$

**2.** (a) $a = -6 \quad b = -1$ (b) $a = -3$, $b = -4$ (c) $a = -6 \quad b = -1$

**3.** (a) $a = -2 \quad b = -3 \quad c = 3$ (b) $a = -6, b = -2, c = 6$

**4.** (a) $x = -1$, $x = \dfrac{1}{2}$, $x = \dfrac{3}{2}$ (b) $f(x) = (x+1)(2x-1)(2x-3)$

**5.** b) $k < -2 \ \ or \ \ k > \dfrac{2}{3}$ b) $k = -2 \ \ or \ \ k = \dfrac{2}{3}$ c) $-2 \le k \le \dfrac{2}{3}$

### B. Past Paper questions (*SHORT*)

**6.** If $x + 2$ is a factor of $f(x)$ then $f(-2) = 0 \Rightarrow \ k = 6$ (C6)

**[6]**

**7.** By the remainder theorem, $f(-1) = 6 - 11 - 22 - a + 6 = -20 \Leftrightarrow a = -1$ (M1)(M1)(A2)

**[4]**

**8.** $f(x) = x^4 + ax + 3$, $f(1) = 8 \Rightarrow 1 + a + 3 = 8 \ \Rightarrow \ a = 4$ (C6)

**[6]**

**9.** Using the remainder and factor theorems, or long division,
$8 + 4a - 6 + b = 0$ and $-1 + a + 3 + b = 6$ (M1)(A1)(M1)(A1)
$a = -2, b = 6$ (A1)(A1) (C6)

**[6]**

**Extra question**
$p(x) = (x-2)(x^2-3) = (x-2)(x+\sqrt{3})(x-\sqrt{3})$ roots: $x = 2,\ x = -\sqrt{3},\ x = \sqrt{3}$

**10.** $p(x) = (ax + b)^3$

$p(-1) = -1 \Rightarrow (b - a)^3 = -1 \Rightarrow b - a = -1$ (M1) (A1)

$p(2) = 27 \ \Rightarrow (2a + b)^3 = 27 \Rightarrow 2a + b = 3$ (A1)

Thus, $a = \dfrac{4}{3}$, $b = \dfrac{1}{3}$. (A1)

**[4]**

**11.** $f(2) = 8 + 12 + 2a + b = 2a + b + 20$ (M1)
$f(-1) = -1 + 3 - a + b = 2 - a + b.$ (M1)
These remainders are equal when $2a + 20 = 2 - a$ giving $a = -6$. (A1) (C3)

**[3]**

**Extra question:**
$b$ can be any real number.

**12.** $f(2) = 8 + 4p + 2q + 4 = 0$

$\Rightarrow 4p + 2q = -12$ *M1A1*

$f(-2) = -8 + 4p - 2q + 4 = 0$

$\Rightarrow 4p - 2q = 4$ *M1A1*

$\Rightarrow 8p = -8$

$\Rightarrow p = -1$ *A1*

$\Rightarrow -4 + 2q = -12$

$\Rightarrow q = -4$ *A1* *N4*

**OR**

$f(x) = x^3 + px + qx + 4 \equiv (x-2)(x+2)(x+a)$ *M1*

Equate co-efficients of $x^0$ : $4 = -4a \Rightarrow a = -1$ *M1*

$\Rightarrow f(x) = (x^2 - 4)(x-1) = x^3 - x^2 - 4x + 4$ *M1A1*

$\Rightarrow p = -1$ and $q = -4$ *A1A1* *N4*

**[6]**

**Extra question**

$x = 2,\ x = -2,\ x = 1$

**13.**

$(x-1)$ is a factor of $P(x) \Rightarrow P(1) = 0$ *(M1)*

$\Rightarrow a + b = 2$ *A1*

$(x+3)$ is a factor of $P(x) \Rightarrow P(-3) = 0$ *(M1)*

$\Rightarrow 9a + b = 42$ *A1*

Solving $\Rightarrow a = 5$ , $b = -3$ *(M1)A1* *N4*

**[6]**

**Extra question:**

$P(x) = (x-1)(x+3)(2x+1)$

**14.** **METHOD 1**

$x^2 - 4x + 3 = (x-3)(x-1)$ (M1)(A1)

**EITHER** $1 + (a-4) + (3-4a) + 3 = 0$ Solving, $a = 1$ (M1)(A1)(M1)(A1)

**OR** $27 + 9(a-4) + 3(3-4a) + 3 = 0$ Solving, $a = 1$ (M1)(A1) (M1)(A1) (C6)

**METHOD 2**

Using the information given it follows that

$x^3 + (a-4)x^2 + (3-4a)x + 3 \equiv (x^2 - 4x + 3)(x+1)$ (M1)(A1)

Comparing coefficients of $x^2$ (or $x$) (M1)

$a - 4 = -3$ (or $3 - 4a = -1$) (A1)(A1)

giving $a = 1$ (A1) (C6)

**[6]**

**Extra question:**

$(x-3)(x-1)(x+1)$

**15.**

$P(x) = 4x^3 + px^2 + qx + 1$

$P(1) = 4(1)^3 + p(1)^2 + q(1) + 1 = -2$ *(M1)*

$\Rightarrow p + q = -7$ *(A1)*

$P\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)^3 + p\left(\frac{1}{2}\right)^2 + q\left(\frac{1}{2}\right) + 1 = \frac{13}{4}$ *(M1)*

$\Rightarrow p + 2q = 7$ *(A1)*

Solving gives $p = -21, q = 14$ *(A1)(A1)* *(C6)*

**[6]**

**Extra question**

The results for p and q are exactly the same!

**INEQUALITITES**

## A. Practice Questions

**16.** (a) According to the graph

| $f(x)$ | $-1$ | $1$ | $2$ | |
|---|---|---|---|---|
| sign | $+$ | $-$ | $-$ | $+$ |

(b) (i) $x \in (-\infty,-1] \cup \{1\} \cup [2,+\infty)$

(ii) $x \in (-\infty,-1) \cup (2,+\infty)$

(iii) we exclude the roots of the denominator $x \in (-\infty,-1] \cup \{1\} \cup (2,+\infty)$

**17.** For all of them we consider the polynomial $5(x-1)(x-2)^2(x-3)^3$

| $f(x)$ | $1$ | $2$ | $3$ | |
|---|---|---|---|---|
| sign | $+$ | $-$ | $-$ | $+$ |

(the solution for $f(x) \geq 0$ is $x \in (-\infty,1] \cup \{2\} \cup [3,+\infty)$

For each inequality we exclude the roots of the denominator. The corresponding solutions are

$x \in (-\infty,1] \cup \{2\} \cup (3,+\infty)$ $\quad$ $x \in (-\infty,1] \cup [3,+\infty)$ $\quad$ $x \in (-\infty,1) \cup \{2\} \cup [3,+\infty)$

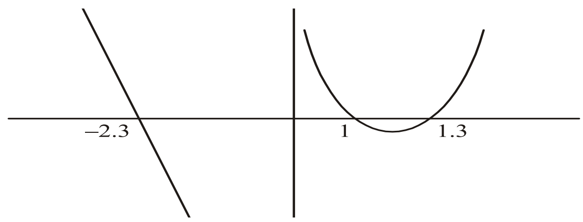
**18.** a) $a = 0$ b) $f(x) = x(x-1)(x-3)$ d) $x \leq 0$ *or* $1 \leq x \leq 3$

**19.** a) $a = 15$ b) $f(x) = (x-1)(x-3)^2$ d) $x \leq 1$ or $x = 3$

**20.** a) $a = 3$ b) $f(x) = 3(x-1)(x^2 + x + 1)$ d) $x \leq 1$

## B. Past Paper questions (*SHORT*)

**21.** **METHOD 1**



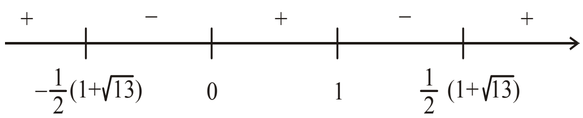

(M2)

$x^2 - 4 + \dfrac{3}{x} < 0 \Rightarrow -2.30 < x < 0$ or $1 < x < 1.30$ (G2)(G2) (C6)

**METHOD 2**

$x^2 - 4 + \dfrac{3}{x} < 0 \Rightarrow \dfrac{x^3 - 4x + 3}{x} < 0 \Rightarrow \dfrac{(x-1)(x^2 + x - 3)}{x} < 0$ (M1) (M1)

Critical values: $1, \dfrac{1}{2}(-1 \pm \sqrt{13}), 0$ (A2)

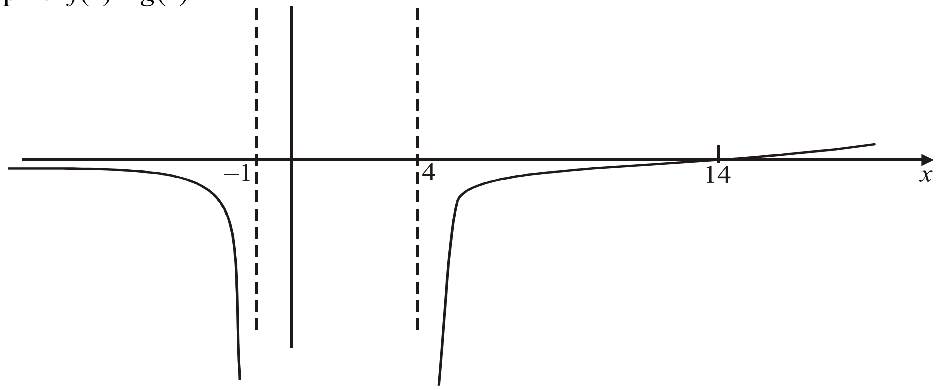



$\Rightarrow -\dfrac{1}{2}(\sqrt{13} + 1) < x < 0$ or $1 < x < \dfrac{1}{2}(\sqrt{13} - 1)$ (A1)(A1) (C6)

**[6]**

**22.** **METHOD 1**

Graph of $f(x) - g(x)$




M1

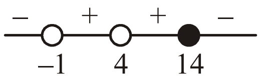
$x < -1$ or $4 < x \leq 14$ A1A1 6

**METHOD 2**

$\frac{x+4}{x+1} - \frac{x-2}{x-4} \leq 0 \Rightarrow \frac{x^2 - 16 - x^2 + x + 2}{(x+1)(x-4)} \leq 0 \Rightarrow \frac{x-14}{(x+1)(x-4)} \leq 0$ M1 A1

Critical value of $x = 14$ A1

Other critical values $x = -1$ and $x = 4$ A1




$x < -1$ or $4 < x \leq 14$ A1A1 6

***Note**: Each value and inequality sign must be correct.*

**[6]**

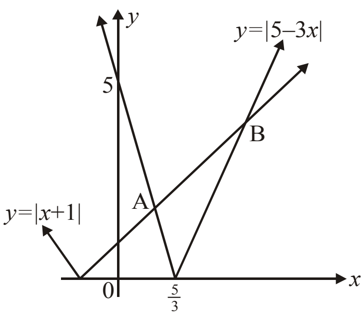
**23.** **METHOD 1**

$|5 - 3x| \leq |x + 1| \Rightarrow 25 - 30x + 9x^2 \leq x^2 + 2x + 1$ (M1)

$\Rightarrow 8x^2 - 32x + 24 \leq 0 \Rightarrow 8(x - 1)(x - 3) \leq 0$ (M1)

$\Rightarrow 1 \leq x \leq 3$ (A1) (C3)

**METHOD 2**



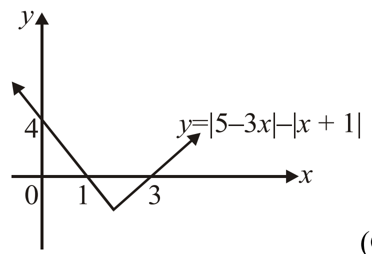

(G1)

We obtain A = (1, 2) and B = (3, 4). (G1)

Therefore, $1 \leq x \leq 3$. (A1)(C3)

**METHOD 3**

Sketch the graph of $y = |5 - 3x| - |x + 1|$.




(G2)

From this graph we see that

$y \leq 0$ for $1 \leq x \leq 3$. (A1)(C3)

**[3]**

**24.** **METHOD 1**

The graphs of $y = |x-2|$ and $y = |2x+1|$ meet where

$(x-2) = (2x+1) \Rightarrow x = -3$ (M1)(A1)

$(x-2) = -(2x+1) \Rightarrow x = \frac{1}{3}$ (M1)(A1)

Test any value, *e.g.* $x = 0$ satisfies inequality (M1)

so $x \in \left[-3, \frac{1}{3}\right]$. (A1) (C6)

**METHOD 2**

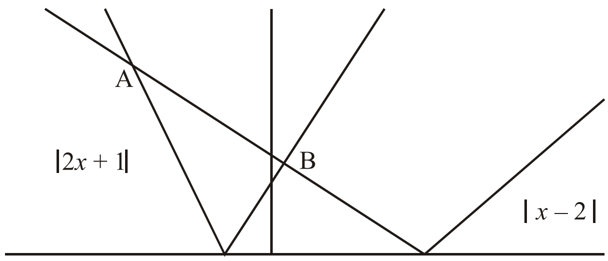
$(x-2)^2 \geq (2x+1)^2 \Rightarrow x^2 - 4x + 4 \geq 4x^2 + 4x + 1$ (M1) (A1)

$3x^2 + 8x - 3 \leq 0 \Rightarrow (3x-1)(x+3) \leq 0$ (or find roots of equation) (A1) (A1)

Test any value, *e.g.* $x = 0$ satisfies inequality. (M1)

So $x \in \left[-3, \frac{1}{3}\right]$. (A1) (C6)

**METHOD 3**




(G1) (G1)

We obtain for A, $x = -3$ and for B, $x = \frac{1}{3}$ (G1) (G1)

From the graph, $x \in \left[-3, \frac{1}{3}\right]$. (M1)(A1) (C6)

***Note:*** *Award (C5) for an open interval.*

**[6]**

**25.** ***Note:*** *If no working shown or if working is incorrect, award (C3) for one correct interval.*

**METHOD 1**

The critical values occur when $\frac{x+9}{x-9} = \pm 2 \to x = 3, 27$ (M1)(A1)

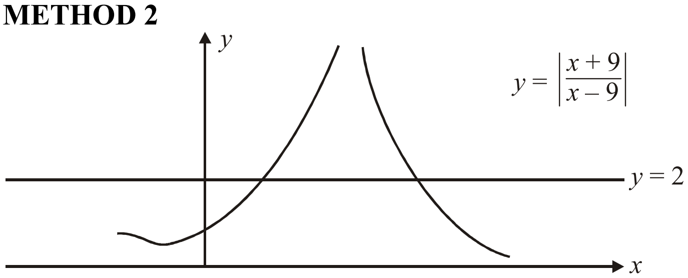
Consider $[-\infty, 3]$: value of function at 0 is 1 which is $\leq 2$. (A1)

Consider $[3, 27]$: value of function at 12 is 7 which is not $\leq 2$. (A1)

Consider $[27, \infty[$: value of function at 36 is $\frac{5}{3}$ which is $\leq 2$. (A1)

The required solution set is therefore $]-\infty, 3] \cup [27, \infty[$ (A1) (C6)

**METHOD 2**




(M1)(A1)(A1)

$]-\infty, 3] \cup [27, \infty[$ (A1)(A1)(A1) (C6)

***Notes:*** *Penalize* ***[1 mark]*** *for open end at 3 and /or 27.*
*Award the final (A1) for the symbol $\cup$ or the word 'or'.*

**[6]**

**26.** either algebraically or graphically: $x \le 0.5$ or $x > 3$

**[6]**

**27.** $(f \circ g)(x) = 2\left(\frac{x}{x+1}\right) - 1$ *(A1)*

$(g \circ f)(x) = \frac{2x-1}{2x}$ *(A1)*

**EITHER**

$\frac{2x}{x+1} - 1 \le \frac{2x-1}{2x}$ ***M1***

Getting 0 on one side ***(M1)***

$\frac{2x}{x+1} - 1 - \frac{1-3x}{2x} \le 0 \quad \left(\frac{1-3x}{2x(x+1)} \le 0\right)$

$-1 < x < 0$ or $x \ge \frac{1}{3}$. ***A1A1***

**OR**

For attempting to graph $(f \circ g)(x)$ and $(g \circ f)(x)$ or $(f \circ g)(x) - (g \circ f)(x)$ ***M1***

For an accurate graph. ***A1***

$-1 < x < 0$ or $x \ge \frac{1}{3}$. ***A1A1***

**Notes:** The inequality sign **and** the values of $x$ must both be correct.
Accept ($-1 < x < 0$ **and** $x \ge \frac{1}{3}$).

**[6]**

**SUM AND PRODUCT OF ROOTS**

**A. Practice Questions**

**28.**

| Polynomial | Sum of roots | Product of roots | Remainder |
|---|---|---|---|
| $f(x) = 2x^4 + 6x^3 + 5x^2 - 7x + 8$ | -3 | 4 | 14 |
| $f(x) = 2x^5 + 6x^3 + 5x^2 - 7x + 8$ | 0 | -4 | 14 |
| $f(x) = x^{10} - x^9 - 1$ | 1 | -1 | -1 |

**29.** (a) $a = -\frac{1}{5}$ (b) $a = -\frac{2}{5}$

**30.** (a) $a = -\frac{1}{3}$ (b) $b = \frac{10}{3}$

**31.** (a) Sum = 2 , Product = 3
(b) deg = 3 (cubic)
(c) $f(1) = 0 \Rightarrow a + b = -2$
(d) Sum = 1, Product = 3

**32.** (a) $a = 2$ , $b = -16$, $c = 42$, $d = -44$

Indeed,

Product = $\dfrac{16}{a} = 8 \Rightarrow a = 2$  Sum = $-\dfrac{b}{a} = 8 \Rightarrow b = -16$

$f(1) = 0 \Rightarrow a + b + c + d + 16 = 0 \Rightarrow c + d = -2$

$f(-1) = 120 \Rightarrow a - b + c - d + 16 = 0 \Rightarrow c - d = 86$

Therefore, $c = 42$ , $d = -44$

(b) Sum = 7, Product = 8

**33.** (a) Sum = $-\dfrac{a}{1} = 7 \Rightarrow a = -7$ ,  Product = $\dfrac{d}{1} = 0 \Rightarrow d = 0$

$f(1) = 0 \Rightarrow 1 + a + b + c + d = 0 \Rightarrow b + c = 6$

$f(2) = 0 \Rightarrow 16 + 8a + 4b + 2c + d = 0 \Rightarrow 2b + c = 20$

Therefore, $b = 14$  $c = -8$

(b) We know three roots, 0,1 and 2. Since the sum is -7 the third root is 4.

Hence $f(x) = x(x-1)(x-2)(x-4)$

**34.** (a) $\alpha + \beta = \dfrac{2}{5}$, $\alpha\beta = -\dfrac{4}{5}$

(b) We use $(\alpha+\beta)^2 = \alpha^2 + 2\alpha\beta + \beta^2$

$\Rightarrow \alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

Thus $\alpha^2 + \beta^2 = \dfrac{4}{25} + \dfrac{8}{5} = \dfrac{44}{25}$

We use $(\alpha+\beta)^3 = \alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3$

$\Rightarrow \alpha^3 + \beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$

Thus $\alpha^3 + \beta^3 = \dfrac{8}{125} + 3 \cdot \dfrac{4}{5} \cdot \dfrac{2}{5} = \dfrac{128}{125}$

(c) Since $\alpha^2 + \beta^2 = \dfrac{44}{25}$ and $\alpha^2\beta^2 = \dfrac{16}{25}$

the polynomial is $25x^2 - 44x + 16$

(d) Since $\dfrac{1}{\alpha} + \dfrac{1}{\beta} = \dfrac{\alpha+\beta}{\alpha\beta} = -\dfrac{2}{4}$ and $\dfrac{1}{\alpha} \cdot \dfrac{1}{\beta} = -\dfrac{5}{4}$

the polynomial is $4x^2 + 2x - 5$ (or any multiple).

**35.** (a) $\alpha + \beta + \gamma = 5$ , $\alpha\beta\gamma = -3$ and $\alpha\beta + \beta\gamma + \gamma\alpha = -7$

(b) We use $(\alpha+\beta+\gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2\alpha\beta + 2\beta\gamma + 2\gamma\alpha$

$\Rightarrow \alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$

Thus $\alpha^2 + \beta^2 + \gamma^2 = 25 + 2(-7) = 39$

(c) We use $(\alpha\beta + \beta\gamma + \gamma\alpha)^2 = (\alpha\beta)^2 + (\beta\gamma)^2 + (\gamma\alpha)^2 + 2\alpha\beta^2\gamma + 2\alpha\beta\gamma^2 + 2\alpha^2\beta\gamma$

$\Rightarrow (\alpha\beta)^2 + (\beta\gamma)^2 + (\gamma\alpha)^2 = (\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2\alpha\beta\gamma(\alpha+\beta+\gamma)$

Thus $(\alpha\beta)^2 + (\beta\gamma)^2 + (\gamma\alpha)^2 = (-7)^2 - 2(-3)(5) = 79$

(d) Since $\alpha^2 + \beta^2 + \gamma^2 = 39$, $\alpha^2\beta^2\gamma^2 = (\alpha\beta\gamma)^2 = 9$ , $\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 = 79$

the polynomial is $x^3 - 39x^2 + 79x - 9$ (or any multiple).