CHAPTER 3

THE VEHICLE ROUTING AND SCHEDULING PROBLEM WITH TIME WINDOW CONSTRAINTS; FORMULATION AND HEURISTICS

3.1 Introduction

In this chapter we introduce a mixed integer programming formulation for the (VRSPTW) and describe various approximate methods for its solution. To begin with, let us define a time window for a customer, say i, by giving an earliest (e_i) and latest (d_i) service time.

We distinguish several situations with respect to the constraints imposed on the time of delivery to customers:

i) $e_i = 0$, $d_i = T$, for all i

ii) $e_i = d_i = t_i$, for all i

iii) $e_i = 0$, $d_i < T$, for some i

iv) $e_i > 0$, $d_i = T$ for some i

v) $e_i \geq 0$, $d_i \leq T$, for some i, where T is a scalar chosen to be larger than the total schedule time of any feasible route, and t_i is the arrival time at customer i. Case i) is the vehicle routing problem, while case ii) is the vehicle scheduling problem. In cases iii) and iv), one-sided time windows are present in the form of due dates (case iii) or earliest delivery times (case iv). Such time-windows are encountered, for example, in some dial-a-ride problems where customers specify desired pick-up times or latest delivery times. Case v) is the general (or "mixed") problem with time windows, where two-sided time windows are imposed on possible service times. When time windows are present, both the routing and scheduling of vehicles needs to be performed.

We now introduce the (VRSPTW) formally. Our formulation is based on a model for the basic routing problem given by Fisher and Jaikumar(1981). The following notation is needed:

<u>Constants</u>:

K = number of vehicles.

n = number of demand points; 0 denotes the central depot.

b_k = capacity of vehicle k.

q_i = demand of customer i.

c_{ijk} = cost of direct travel for vehicle k, from customer i to customer j.

t_{ij} = direct travel time from customer i to customer j.

s_i = service time at customer i.

e_i = earliest time allowed for delivery to customer i.

d_i = latest time for delivery to customer i.

T = a scalar chosen to be larger than the travel time of any feasible route.

Variables:

$$y_{ik} = \begin{cases} 1, & \text{if customer i's order is delivered by vehicle k.} \\ 0, & \text{otherwise.} \end{cases}$$

$$x_{ijk} = \begin{cases} 1, & \text{if vehicle k travels directly from customer i to customer j.} \\ 0, & \text{otherwise.} \end{cases}$$

t_i = arrival time at customer i.

s=start $t_{0,s}^{k}$ = departure time of vehicle k from the depot.

f=finish $t_{0,f}^{k}$ = arrival time of vehicle k at the depot.

We now state an MIP formulation for the problem of routing and scheduling capacitated vehicles subject to time windows constraints.

MIP Formulation f

Problem

min Σ_{ijk} c

S.t Σ_i q_i

Σ_k

Σ_i

Σ_j

$t_j \geq t_i$ +

$t_{0,f}^{k} \geq t_i + s$

$t_j \geq t_{0,s}^{k}$ +

$e_i \leq t_i \leq d$

$e_0 \leq t_{0,p}^{k} \leq$

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MIP Formulation for the Vehicle Routing and Scheduling Problem with Time Windows (VRSPTW).

min $\quad \Sigma_{ijk} \; c_{ijk}x_{ijk}$, subject to: $\qquad$ (3.1)

s.t. $\quad \Sigma_i \; q_i y_{ik} \le b_k, \; k=1,\dots,K$ $\qquad$ (3.2)

$$\Sigma_k \; y_{ik} = \begin{cases} K, & i=0 \\ 1, & i=1,\dots,n \end{cases} \qquad (3.3)$$

$$\Sigma_i \; x_{ijk} = y_{jk}, \; j=0,\dots,n; \; k=1,\dots,K \qquad (3.4)$$

$$\Sigma_j \; x_{ijk} = y_{ik}, \; i=0,\dots,n; \; k=1,\dots,K \qquad (3.5)$$

$$t_j \ge t_i + s_i + t_{ij} - (1-x_{ijk})T, \; \begin{matrix} i,j=1,\dots,n \\ k=1,\dots,K \end{matrix} \qquad (3.6)$$

$$t_{0,f}^{k} \ge t_i + s_i + t_{i0} - (1-x_{i0k})T, \; \begin{matrix} i=1,\dots,n \\ k=1,\dots,K \end{matrix} \qquad (3.7)$$

$$t_j \ge t_{0,s}^{k} + s_i + t_{0j} - (1-x_{0jk})T, \; \begin{matrix} j=1,\dots,n \\ k=1,\dots,K \end{matrix} \qquad (3.8)$$

$$e_i \le t_i \le d_i \quad , \; i=1,\dots,n \qquad (3.9)$$

$$e_0 \le t_{0,p}^{k} \le d_0 \; , \; p=s,f, \; k=1,\dots,K \qquad (3.10)$$

$$y_{ik} = 0 \text{ or } 1, \; i=0,\dots,n; \; k=1,\dots,K \qquad (3.11)$$

$$x_{ijk} = 0 \text{ or } 1, \; i,j=0,\dots,n; \; k=1,\dots,K \qquad (3.12)$$

$$t_i \ge 0, \; i=0,\dots,n \qquad (3.13)$$

Constraints (3.2) ensure that each route's demand is within the capacity limit of the vehicle servicing that route. (3.3) state that each route originates and terminates at the depot, and that every customer is on exactly one route. For every k, and y_{ik} fixed to satisfy (3.2)-(3.3) and (3.11), constraints (3.4)-(3.10), (3.12) are imposed on the customers assigned to vehicle k. (3.4)-(3.5) constrain a route to enter and leave each customer location exactly once. Constraints (3.6)-(3.8) ensure compatible arrival times. Moreover, (3.6) eliminate all oriented subtours. A proof of their validity is given below. (3.9) restrict the arrival time at a customer to its time window. (3.10) are depot time window constraints on vehicle departure and arrival times. We will show now that constraints (3.6) eliminate all the oriented subtours. For this, let $T(y_{ik}) = \{ i : y_{ik} = 1 \}$ and $T'(y_{ik}) = T - \{0\}$.

<u>Lemma 3.1</u>. For given k and y_{ik} fixed to satisfy (3.2)-(3.3) and (3.11), the constraints:

$$t_j \geq t_i + t_{ij} + s_i - (1-x_{ijk})T, \quad i,j \in T'(y_{ik})$$

are satisfied if and only if there are no oriented subtours.

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Proof: Clearly these constraints are satisfied when there are no oriented subtours. We prove now the second half of the lemma by contradiction. Let $T'(y_k) = \{i_1,\ldots,i_{n(k)}\}$, and let the variables $\{ x_{ijk} : i,j=i_l, l=1,\ldots,n(k), i\neq j\}$ and $\{ t_i : i=i_l, l=1,\ldots,n(k)\}$ satisfy the above constraints. Assume now that there exists an oriented subtour of order, say p. Without loss of generality, assume that the subtour is $\{i_1,\ldots,i_p\}$. We must have:

$$x_{i_1 i_2 k} = x_{i_2 i_3 k} = \cdots = x_{i_p i_1 k} = 1.$$

Since

$$t_{i_m} \geq t_{i_{m-1}} + s_{i_{m-1}} + t_{i_{m-1} i_m}, \quad m=2,\ldots,p,$$

trivially,

$$t_{i_p} \geq t_{i_1} + \Sigma_{m=2}^{p} t_{i_{m-1} i_m} \qquad (3.14)$$

Also,

$$t_{i_1} \geq t_{i_p} + s_p + t_{i_p i_1} \qquad (3.15)$$

From (3.14) and (3.15) it follows that $t_{i_{m-1} i_m} = t_{i_p i_1} = 0$ and $s_m = 0$ for all m. We have reached a contradiction; therefore there cannot be any oriented subtours.

It follows from the above lemma that the usual subtour elimination constraints, found in (VRP) formulations are superfluous in our formulation of the (VRSPTW). Concerning the scalar T used in constraints (3.6), note that it is sufficient to take $T=\max\{d_i+s_i+t_{ij}-e_j\}$. Alternatively, for each arc (i,j), we can define its associated constant T as the quantity in the above parantheses.

Miller et al.(1960) have introduced the following subtour elimination constraints for the (TSP):

$$a_i - a_j + (n+1)\, x_{ij} \le n \qquad (3.16)$$
$$i=0,1,\ldots,n;\ j=1,\ldots,n+1,\ i\ne j$$

where the home city is labelled as city 0 and city n+1, and a_i is a real number associated with city i, $1\le a_i\le n+1$, $i=1,\ldots,n+1$, and $x_{ij}=x_{ijk}$ with $k=1$. If we let $t_i=a_i$, $t_{ij}=x_{ij}$, $T=n$, $s_i=0$, all i,j, then one can see that (3.16) is a particular case of the constraints (3.6). In other words, in the classical (TSP), where only the sequencing and not the scheduling of the customers is required, one can assume unit travel times, which leads to the constraints (3.16). This generalization of the classical (TSP) subtour elimination constraints has been derived independently by Desrosiers et al.(1983A), who use it in the context of school bus scheduling.

A solution for the (VRSPTW) provides an allocation of customers to vehicles (variables y_{ik}), the routes (variables x_{ijk}), i.e. the sequence in which the customers are going to be visited, and the vehicle schedules (variables t_i). We

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The above formulation is very flexible, and, as such, it can accomodate a number of important problem complexities. To start with, we should remark that in the vehicle routing field, one usually assumes a fleet of identical vehicles i.e. $b_k=b$, for all k; hence $c_{ijk}=c_{ij}$, for all k. The fleet size is assumed to be large enough to permit the existence of a feasible solution. Moreover, the cost of travel c_{ij} is considered to be a function of distance (usually $c_{ij}=d_{ij}$, where d_{ij} is the distance between i and j) and sometimes of travel time ($c_{ij}=t_{ij}$). A more general criterion is provided by the weighted combination of the above criteria, i.e. $c_{ij}=a_1 d_{ij}+a_2 t_{ij}$.

In the presence of time windows the primary objective becomes the minimization of the total routing and scheduling costs, which include not only the total travel distance and time costs considered for routing problems, but also the cost of waiting time which is inccured when a vehicle arrives too early at a customer location or when the vehicle is loaded or unloaded. For example, in many realistic environments, the unit cost of travel involves a cost per unit distance (ex: gasoline) and a cost per unit time (ex: driver's wages). Then, $c_{ij} = a_1 d_{ij} + a_2(t_j - t_i)$, $a_1+a_2=1$.