Harmonic Measure TD1

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Proposition 1: Polar coordinates There exists a Borel measure σ_{n-1} on S_{n-1} such that

$$\int_{\mathbb{R}^n} f \mathrm{d}\mathcal{L}_n = \iint_{(r,x)\in(0,+\infty)\times S_{n-1}} f(rx) r^{n-1} \mathrm{d}r \mathrm{d}\sigma_{n-1}(x)$$

if f is measurable and nonnegative, or if f is integrable.

In particular, if f is radial, i.e., $f(x) = \varphi(|x|)$,

$$\int_{\mathbb{R}^n} \varphi(|x|) \mathrm{d}\mathcal{L}_n = s_{n-1} \int_0^{+\infty} r^{n-1} \varphi(r) \mathrm{d}r,$$

where

$$s_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

is the area of the unit sphere in $\mathbb{R}^n$.

Definition 1 (The Hardy–Littlewood Operator): If f is a locally integrable function on $\mathbb{R}^n$, one sets

$$\mathrm{M}f(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| \mathrm{d}y.$$

$\mathrm{M}f$ is called the **Hardy–Littlewood maximal function** and the sublinear operation $f \mapsto \mathrm{M}f$ the **Hardy–Littlewood maximal operator**.

Proposition 2:
1. There exists a constant $C > 0$ (depending on the dimension n) such that, for all f and all $t > 0$, one has
$$|\{\mathrm{M}f > t\}| \leqslant \frac{C}{t} \|f\|_1 \ .$$
2. For any $p \in (1, +\infty)$, there exists $C_p > 0$ (depending on n) such that, for any f, one has
$$\|\mathrm{M}f\|_p \leqslant C_p \|f\|_p.$$

Exercise 1: A function from a topological space X to $\mathbb{R}$ is said to be lower semicontinuous (lsc, for short) if, for all $t \in \mathbb{R}$, the set $\{f > t\}$ is open. Show that the function $\mathrm{M}f$ is lsc, hence measurable.

Proof：记

$$\mathrm{M}f(x) = \sup_{r>0} f_r(x) = \sup_{r>0} ⨍_{B(x,r)} |f(y)| \,\mathrm{d}y,$$

那么有 $\{f > t\} = \bigcup_{r>0} \{f_r > t\}$，只要证明每个 $\{f_r > t\}$ 是开集即可，那么只要证明 $\forall r > 0, f_r(x)$ 对 x 连续。对 f_r 换元得到

$$\begin{aligned} f_r(x) &= ⨍_{B(x,r)} |f(y)| \,\mathrm{d}y \\ &= ⨍_{B(0,r)} |f(x+y)| \,\mathrm{d}y \\ &= \frac{1}{|B(0,r)|} \int_{\mathbb{R}} 1_{B(0,r)} |f(x+y)| \,\mathrm{d}y \,. \end{aligned}$$

取 $x_n \to x$，首先有

$$1_{B(0,r)} |f(x_n + y)| \xrightarrow{\text{a.e.}} 1_{B(0,r)} |f(x+y)|,$$

且在 $|x_n - x| < r$ 时有

$$1_{B(0,r)} |f(x_n + y)| \leqslant 1_{B(0,2r)} |f(x+y)|$$

而后者可积，那么由 *Lebesgue* 控制收敛定理知道 $f_r(x_0) \to f_r(x)$，从而 f_r 连续。 □

Exercise 2：Let a and b be two numbers such that $a < b$. Compute $\mathrm{M}\mathbb{1}_{[a,b]}$.

Proof：对自变量分类易得

$$\mathrm{M}1_{[a,b]} = \begin{cases} \dfrac{b-a}{2(x-a)}, x > b \\ \dfrac{1}{2}, x \in \{a, b\} \\ 1, x \in (a,b) \\ \dfrac{b-a}{2(b-x)}, x < a. \end{cases}$$

□

Exercise 3：Let $f = \mathbb{1}_{B(0,1)}$ be the indicator function of the unit ball in $\mathbb{R}^n$. Show that, for $|x| > 1$, $\mathrm{M}f(x) \leqslant C/(|x|-1)^n$, where $C > 0$ is a constant. Conclude that, for $p > 1$, $\mathrm{M}f \in L^p(\mathbb{R}^n)$.

Proof：沿用之前对 $f_r(x)$ 的记号，题目也即证明

$$\mathrm{M}f(x) = \sup_{r>0} f_r(x) \leqslant \frac{C}{(|x|-1)^n}.$$

对 $\forall r > |x| + 1$ 有

$$f_r(x) = \frac{\displaystyle\int_{\mathbb{R}} f(y)\,\mathrm{d}y}{|B(x,r)|} = \frac{|B(0,1)|}{|B(x,r)|} = \frac{1}{r^n} \leqslant \frac{1}{(|x|-1)^n}.$$

对 $\forall r \in [|x|-1, |x|+1]$ 有

$$f_r(x) = \frac{\int_{B(x,r)} f(y)\,\mathrm{d}y}{|B(x,r)|} \leqslant \frac{|B(0,1)|}{|B(x,|x|-1)|} = \frac{1}{(|x|-1)^n}$$

对 $\forall r < |x|-1$, $f_r(x)$ 无意义。 □

Exercise 4: Prove the second assertion (L^p-boundedness of M) of the theorem above.

Proof: 为去掉绝对值，不妨设 $f \geqslant 0$。首先断言算子 M 满足 $\mathrm{M}(f+g) \leqslant \mathrm{M}f + \mathrm{M}g$。令

$$g(x) = \begin{cases} f(x), f(x) > t/2 \\ 0, f(x) \leqslant t/2 \end{cases}, \quad h(x) = f(x) - g(x),$$

那么 $f = g + h$，$\mathrm{M}f \leqslant \mathrm{M}g + \mathrm{M}h$，而 $\mathrm{M}h \leqslant \sup h = t/2$，那么就知道 $\{Mf > t\} \subset \{Mg > t/2\}$，再结合第一条命题就知道

$$|\{Mf > t\}| \leqslant |\{Mg > t/2\}| \leqslant \frac{2C}{t} \int_{\mathbb{R}^n} g\,\mathrm{d}x = \frac{2C}{t} \int_{\{f > t/2\}} f\,\mathrm{d}x,$$

那么直接应用积分变换

$$\int_{\mathbb{R}^n} f^p\,\mathrm{d}x = \int_0^\infty p t^{p-1}\ |\{f > t\}|\,\mathrm{d}t$$

就知道

$$\begin{aligned}
\|Mf\|^p &= \int_{\mathbb{R}} \mathrm{M}f^p\,\mathrm{d}x \\
&= \int_0^\infty p t^{p-1}\ |\{\mathrm{M}f > t\}|\,\mathrm{d}t \\
&\leqslant \int_0^\infty p t^{p-1} \frac{2C}{t} \int_{\{f(x) > t/2\}} f\,\mathrm{d}x\,\mathrm{d}t \\
&= 2Cp \int_{\mathbb{R}^n} f(y) \int_0^{2f(y)} t^{p-2}\,\mathrm{d}t\,\mathrm{d}y \\
&= \frac{2^p C p}{p-1}\ \|f\|_p.
\end{aligned}$$

其中用了 *Fubini* 定理进行积分换序。 □

Exercise 5: If $f \in L^1(\mathbb{R}^n)$ such that $f \neq 0$, then $\mathrm{M}f \notin L^1(\mathbb{R}^n)$ (prove that

$$\mathrm{M}f(x) \geqslant \frac{C}{|x|^n}$$

for $|x|$ large enough, where $C > 0$ is a constant).

Proof: 这里的 $f \neq 0$ 应当是 $f \neq 0$ *a.e.*，所以存在球 $B(0, r)$ 使得

$$\int_{B(0,r)} |f(x)| \, \mathrm{d}x = A > 0.$$

对 $|x| > r$，考虑 $B(x, 2|x|)$ 上的积分，得到

$$\begin{aligned} \mathrm{M}f(x) &\geqslant \frac{1}{|B(x, 2|x|)|} \int_{B(x,2|x|)} |f(x)| \, \mathrm{d}x \\ &= \frac{C}{2^n |x|^n} \cdot A. \end{aligned}$$

□

Exercise 6:

1. Let $\rho > 3$. Let $\mathcal{B}$ be a collection of balls of bounded radii. Prove that one can extract from $\mathcal{B}$ a sequence of disjoint balls $(\{B_j\})$ such that

$$\mathcal{L}\left(\bigcup_{B \in \mathcal{B}} B\right) \leqslant \sum_{j \geqslant 1} \mathcal{L}(B_j^*) = \rho^n \sum_{j \geqslant 1} \mathcal{L}(B_j),$$

   where B_j^* is the ball with the same center as B_j and whose radius is ρ times the radius of B_j.
2. Give a proof of the first assertion of Theorem above which does not use the regularity of the Lebesgue measure.

Proof: *Set* $\varepsilon = (\rho - 3)/\rho$ *and* $\mathscr{B}_0 = \{B_j\}$. *Choose* $B_1' \in \mathscr{B}_0$ *such that*

$$r_{B_1'} > (1 - \varepsilon) \sup_{B \in \mathscr{B}_0} \{r_B\}.$$

Set $\mathscr{B}_1 = \{B \in \mathscr{B}_0 : B \cap B_{1'} = \varnothing\}$. *Choose* ${B_2}' \in \mathscr{B}_1$ *similarly, i.e. such that*

$$r_{B_2'} > (1 - \varepsilon) \sup_{B \in \mathscr{B}_1} \{r_B\}.$$

So on we can obtain a sequence of balls $\{{B_n}'\}$ *and* $\{\mathscr{B}_n\}$. *We assert that* $\{{B_n}'\}$ *is the ball required. From the process of construction we can easily observe that* $\{{B_n}'\}$ *are disjoint. First we prove that for* $B \in \mathscr{B}_i \setminus \mathscr{B}_{i+1}$ *we have* $B \subset \rho {B_i}'$. *Since* $B \cap {B_i}' \neq \varnothing$, $r_B \leqslant \sup_{B^* \in \mathscr{B}_i} \{r_{B^*}\}$ *then*

$$d\left(c_{{B_i}'}, c_B\right) + r_B \leqslant r_{{B_i}'} + 2r_B \leqslant 3 \sup_{B \in \mathscr{B}_i} \{r_B^*\} = \rho r_{{B_i}'}$$

where c_B *stands for the center point of* B. *Hence* $B \subset \rho {B_i}'$. *Second we need to show that* $\bigcap_i \mathscr{B}_i = \varnothing$. *Without lost of generality, we can assume all center of balls are bounded in* $[0, 1]^n$. *Then it is easy to determine the result.* □

Exercise 7: Let $f_1, f_2, \cdots, f_m, \cdots$ be a nondecreasing sequence of nonnegative functions in $L^1(\mathbb{R}^n)$. Let f be the pointwise limit of f_m. Show that, for all $x \in \mathbb{R}^n$

$$\mathrm{M}f(x) = \lim_{m\to\infty} \mathrm{M}f_m(x).$$

Proof: 用 *Lebesgue* 控制定理。记

$$f_r(x) = \fint_{B(x,r)} f(x)\,\mathrm{d}x, f_r^{(n)}(x) = \fint_{B(x,r)} f_r^{(n)}(x)\,\mathrm{d}x,$$

那么只要证明

$$\sup_r f_r(x) = \lim_{n\to\infty} \sup_r f_r^{(n)}(x).$$

对 $\forall r > 0$ 有

$$f_r^{(n)}(x) = \fint_{B(x,r)} f_r^{(n)}(x)\,\mathrm{d}x \leqslant \fint_{B(x,r)} f_r(x)\,\mathrm{d}x < \infty,$$

用 *Lebesgue* 控制收敛定理知道 $f_r(x) \uparrow f_r^{(n)}(x)$ 对 $\forall r > 0$ 成立。

接下来证明 $\sup_r f_r^{(n)}(x) \uparrow \sup_r f_r(x)$。

1. 对 $\forall x_0 \in \mathbb{R}^n$ 有

$$f_r^{(n)}(x) \uparrow f_r(x) \leqslant \sup_r f_r(x),$$

那么

$$\lim_n \sup \sup_r f_r^{(n)}(x) \leqslant \sup_r f_r(x);$$

2. 取 $\{x_n \in \mathbb{R}^n\}$ 使得 $f_r(x_n) \uparrow \sup_r f_r(x)$，又知道 $f_r^{(m)}(x_n) \uparrow f_r(x_n)$，取 $m = n$ 直接得到

$$f_r^{(n)}(x_n) \uparrow \sup_r f_r(x),$$

那么自然有

$$\lim_n \inf \sup_r f_r^{(n)}(x) \geqslant \sup_r f_r(x).$$

□

Exercise 8: If $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, define

$$\widetilde{M}f(x) = \sup\left\{ \frac{1}{|B|} \int_B |f(y)| \mathrm{d}y : B \text{ an open ball}, x \in B \right\}.$$

1. Show that the set $\left\{ \widetilde{M}f > t \right\}$ is open.
2. Show that we have

$$\left|\left\{\widetilde{M} f>t\right\}\right| \leqslant \frac{3^n}{t} \int_{\left\{\widetilde{M} f>t\right\}}|f| \mathrm{d} \lambda .$$

3. Let $p \in(1,+\infty)$. Show that

$$\int\left(\widetilde{M} f\right)^p \mathrm{d} \lambda \leqslant \frac{3^n p}{p-1} \int|f|\left(\widetilde{M} f\right)^{p-1} \mathrm{d} \lambda \leqslant \frac{3^n p}{p-1}\|f\|_p\left(\int\left(\widetilde{M} f\right)^p \mathrm{d} \lambda\right)^{\frac{p-1}{p}} .$$

4. Let $p \in(1,+\infty)$. Show that $\|\widetilde{M} f\|_p \leqslant \frac{3^n p}{p-1}\|f\|_p$. Hint: Use Exercises 1.3 and 1.7 and the preceding inequality.
5. Show that

$$\|\mathrm{M} f\|_p \leqslant \frac{3^n p}{p-1}\|f\|_p \text { . }$$

Proof: 不妨假设 $f(x) \geqslant 0$，否则可以加上绝对值从而变成非负函数。

□

Exercise 9: A positive Borel measure μ on $\mathbb{R}^n$ is said to be doubling if there exists a constant C such that, for all $x \in \mathbb{R}^n$ and all $r>0$, one has

$$\mu(B(x, 2 r)) \leqslant C \mu(B(x, r)) .$$

For such a measure, prove that, for all $\gamma>1$, there exists C_γ such that, for all $x \in \mathbb{R}^n$ and all $r>0$, one has

$$\mu(B(x, \gamma r)) \leqslant C_\gamma \mu(B(x, r)) .$$

Exercise 10: Prove that, for $\alpha>0$, the measure (on $\mathbb{R}^n$) $\mathrm{d} \mu(x)=|x|^\alpha \mathrm{d} x$ is doubling.

Exercise 11: Let μ be a doubling measure on $\mathbb{R}^n$. If f is locally integrable with respect to μ, one sets

$$M_\mu f(x)=\sup _{r>0} \frac{1}{\mu(B(x, r))} \int_{B(x, r)}|f| \mathrm{d} \mu .$$

Prove that there exists C such that, for all $f \in L^1(\mu)$ and all $t>0$, one has

$$\mu\left(M_\mu f>t\right) \leqslant \frac{C}{t}\|f\|_{L^1(\mu)} \text { . }$$

Exercise 12:
1. Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space, and let f be a nonnegative measurable function on X. Show that, for $0<p<+\infty$ and $0<u<v<+\infty$, we have

$$\int_{f>u} f^p \mathrm{d}\mu = p \int_u^{+\infty} t^{p-1} \mu(\{f > t\}) \mathrm{d}t + u^p \mu(\{f > u\}),$$

$$\int_{f \leqslant v} f^p \mathrm{d}\mu = p \int_0^v t^{p-1} \mu(\{f > t\}) \mathrm{d}t - v^p \mu(\{f > v\}),$$

$$\int_{u<f \leqslant v} f^p \mathrm{d}\mu = p \int_u^v t^{p-1} \mu(\{f > t\}) \mathrm{d}t - v^p \mu(\{f > v\}) + u^p \mu(\{f > v\}).$$

2. Let f be an integrable function on $\mathbb{R}^n$. Show that if

$$\int_{\mathbb{R}^n} |f(x)| \log^+(|f(x)|) \mathrm{d}x$$

is finite, then $\mathrm{M}f$ is locally integrable. Hint: If B is a ball, write

$$\int_B \mathrm{M}f \mathrm{d}\lambda \leqslant 2\lambda(B) + \int_{\mathrm{M}f>2} \mathrm{M}f \mathrm{d}\lambda,$$

and use 1. and the inequality

$$t\lambda(\{\mathrm{M}f > t\}) \leqslant C \int_{|f|>t/2} |f| \mathrm{d}\lambda.$$