Mathematics Courses
Tasks Answers Series

Real Analysis: Tasks Answers

Fourth Edition,

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Preliminaries on Sets, Mappings, and Relations

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4 | Lebesgue Integration

4.1 | The Rimemann Integral

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4.2 | The Lebesgue Integral of a Bounded Measurable Function over a Set of Finite Measure

4.3 | The Lebesgue Integral of a Measurable Nonnegatice Function

4.4 | The General Lebesgue Integral

4.5 | Contable Addativity and Continuity of Integration

4.6 | Uniform Integrability: The Vitali Convergence Theorem

Task 4.40: Let f be integrable over $\mathbb{R}$. Show that the function F defined by

$$F(x) = \int_{-\infty}^{x} f \text{ for all } x \in \mathbb{R}$$

is properly defined and continuous. Is it necessarily Lipschitz?

Task 4.41: Show that Proposition 25 is false if $E = \mathbb{R}$.

Task 4.42: Show that Theorem 26 is false without the assumption that the h_n's are nonnegative.

Proof: 考虑函数列

$$h_{n(x)} = \begin{cases} n, 0 \leq x \leq \frac{1}{2^n} \\ -n, \frac{1}{2^n} < x \leq \frac{1}{2^{n-1}} \\ 0, \text{其他情况} \end{cases}$$

此时 $\lim \int_{[0,1]} h_n = 0$，但是在 $[0,1]$ 上不一致可积.

考虑函数列

Task 4.43: Let the sequences of functions $\{h_n\}$ and $\{g_n\}$ be uniformly integrable over E. Show that for any α and β, the sequence of linear combinations $\alpha f_n + \beta g_n$ also is uniformly integrable over E.

Proof: 任取 $\varepsilon > 0$. 设 $\delta_1, \delta_2 > 0$ 满足 $\{h_n\}, \{g_n\}$ 在 $\frac{\varepsilon}{2\,|\alpha|}, \frac{\varepsilon}{2\,|\beta|}$ 下对应的一致可积条件. 那么令 $\delta = \min\{\delta_1, \delta_2\}$. 对任意 E 的可测子集 A，且 $m(A) < \delta$，则有

$$\begin{aligned}\int_A |\alpha f_n + \beta g_n| &\leq |\alpha| \int_A |f_n| + |\beta| \int_A |g_n| \\ &= |\alpha| \cdot \frac{\varepsilon}{2\ |\alpha|} + |\beta| \cdot \frac{\varepsilon}{2\ |\beta|} \\ &< \varepsilon.\end{aligned}$$

Task 4.44: Let f be integrable over $\mathbb{R}$ and $\varepsilon > 0$. Establish the following three approximation properties.

1. There is a simple function η on $\mathbb{R}$ which has finite support and $\int_{\mathbb{R}} |f - \eta| < \varepsilon$.
2. There is a step function s on $\mathbb{R}$ which vanished outside a closed, bounded interval and $\int_{\mathbb{R}} |f - s| < \varepsilon$.
3. There is a continuous function g on $\mathbb{R}$ which vanished outside a bounded set and $\int_{\mathbb{R}} |f - g| < \varepsilon$.

Proof: 不妨假设 f 在 $\mathbb{R}$ 上非负，否则只要对 f 非负和 f 为负的部分分别应用结论即可.

对于第一问，由定义知道

$$\int_{\mathbb{R}} f = \sup \left\{ \int_{\mathbb{R}} h \mid h \text{ 是有界、可测、支集测度有限的函数，且 } 0 \leq h \leq f \right\}.$$

那么存在 h_0 满足以上条件且

$$\int_{\mathbb{R}} f - \int_{\mathbb{R}} h_0 < \frac{\varepsilon}{2}.$$

又由 $\int_{\mathbb{R}} h_0$ 定义知道存在简单函数 η 满足

$$\int_{\mathbb{R}} h_0 - \int_{\mathbb{R}} \eta < \frac{\varepsilon}{2}.$$

那么就有

$$\int_{\mathbb{R}} |f - \eta| = \int_{\mathbb{R}} f - \int_{\mathbb{R}} \eta < \varepsilon,$$

其中 η 是简单函数且支集测度有限.

对于第二问，对于上述 h_0，首先由积分的连续性得到

$$\lim_{n \to \infty} \int_{[-n,n]} h_0 = \int_{\mathbb{R}} h_0.$$

那么存在一个足够大的 n_0，使得

$$\int_{(-\infty,-n_0)\cup(n_0,+\infty)} h_0 = \int_{\mathbb{R}} h_0 - \int_{[-n_0,n_0]} h_0 < \frac{\varepsilon}{4}.$$

考虑 $\int_{[-n_0,n_0]} h_0$. 由 Lusin 定理，存在 $[-n_0, n_0]$ 的子集 F 使得 h_0 在 F 上连续，那么由 Riemann 可积性知道存在 F 上的 step function s 使得 $s \leq h_0$ 且

$$\int_F h_0 - \int_F s < \frac{\varepsilon}{8}.$$

又注意到 h_0 在 $[-n_0,n_0]$ 上有界，设 $|h_0|<M$. 那么可以选取上述集合 F 使得 $m([-n_0,n_0]\sim F)<\frac{\varepsilon}{8M}$ 即可，在 $\mathbb{R}\sim F$ 上令 $s=0$，此时有

$$\begin{aligned}
\int_{\mathbb{R}}|f-s| &= \int_{\mathbb{R}} f-\int_{\mathbb{R}} s \\
&= \int_{\mathbb{R}} f-\int_{\mathbb{R}} h_0+\int_{\mathbb{R}} h_0-\int_{[-n_0,n_0]} h_0+\int_{[-n_0,n_0]} h_0-\int_F s \\
&< \frac{\varepsilon}{2}+\frac{\varepsilon}{4}+\int_{[-n_0,n_0]\sim F} h_0+\int_F h_0-\int_F s \\
&< \frac{\varepsilon}{2}+\frac{\varepsilon}{4}+M\cdot\frac{\varepsilon}{8M}+\frac{\varepsilon}{8} \\
&< \varepsilon.
\end{aligned}$$

对于第三问，考虑上述 h_0,n_0 和 F. 考虑构造连续函数 g，使得在 F 上 $g=h_0$，并将 g 延拓到 $[-n_0,n_0]$，进一步延拓到 $\mathbb{R}$，且有 $g\big([n_0+\frac{\varepsilon}{8M},+\infty)\big)=g\big((-\infty,-n_0-\frac{\varepsilon}{8M}]\big)=\{0\}$. 这时有

$$\int_{\mathbb{R}}|f-g|<\frac{\varepsilon}{2}+\frac{\varepsilon}{4}+\int_{[-n_0,n_0]\sim F}(h_0-g)<\varepsilon.$$

Task 4.45: Let f be integrable over E. Define $\hat{f}$ to be the extension of f to all of $\mathbb{R}$ obtained by setting $\hat{f}\equiv 0$ outside of E. Show that $\hat{f}$ is integrable over $\mathbb{R}$ and $\int_E f=\int_{\mathbb{R}}\hat{f}$. Use this and part (1) and (3) of preceding problem to show that for $\varepsilon>0$, there is a simple function η on E and a continuous function g on E for which $\int_E|f-\eta|<\varepsilon$ and $\int_E|f-g|<\varepsilon$.

Proof: 显然有

$$\begin{aligned}
\int_{\mathbb{R}}\hat{f} &= \int_E\hat{f}+\int_{\mathbb{R}\sim E}\hat{f} \\
&= \int_E f+\int_{\mathbb{R}\sim E}0 \\
&= \int_E f.
\end{aligned}$$

然后直接应用上题的结论即可.

Task 4.46 (Riemann-Lebesgue): Let f be integrable over $\mathbb{R}$. Show that

$$\lim_{n\to\infty}\int_{\mathbb{R}} f(x)\cos nx\,\mathrm{d}x=0.$$

Proof: $\forall\varepsilon>0$. 由 44 题的结论知道存在 step function g 仅在有穷闭区间 F 上非 0 且 $\int_{\mathbb{R}}|f-g|<\frac{\varepsilon}{2m(F)}$. 又知道

$$\begin{aligned}
\left|\int_{\mathbb{R}} f(x)\cos nx\,\mathrm{d}x\right| &\le \int_{\mathbb{R}} |(f(x)-g(x))\cos nx\,\mathrm{d}x| + \left|\int_{\mathbb{R}} g(x)\cos nx\,\mathrm{d}x\right| \\
&\le \int_{\mathbb{R}} |f(x)-g(x)|\ |\cos nx\,\mathrm{d}x| + \left|\int_{\mathbb{R}} g(x)\cos nx\,\mathrm{d}x\right| \\
&\le \int_{\mathbb{R}} |f(x)-g(x)|\,\mathrm{d}x + \left|\int_{\mathbb{R}} g(x)\cos nx\,\mathrm{d}x\right| \\
&\le \frac{\varepsilon}{2} + \left|\int_{\mathbb{R}} g(x)\cos nx\,\mathrm{d}x\right|.
\end{aligned}$$

故只要证明 g 是 step function 的情况即可，且 g 仅在有穷闭区间 F 上非 0. 考虑到 g 在每一段上的取值为常数，不妨设 $g(x)$ 是常数. 那么此时只要证明

$$\lim_{n\to\infty}\left|\int_F \cos nx\,\mathrm{d}x\right| = 0.$$

而注意到它是 Riemann 可积的，直接换元积分得到原函数 $\frac{1}{n}\sin nx$，令 $n\to\infty$ 得到结果.

Task 4.47: Let f be integrable over $\mathbb{R}$.

1. Show that for each t,

$$\int_{\mathbb{R}} f(x)\,\mathrm{d}x = \int_{\mathbb{R}} f(x+t)\,\mathrm{d}x\,.$$

2. Let g be a bounded measurable function on $\mathbb{R}$. Show that

$$\lim_{t\to 0}\int_{\mathbb{R}} g(x)\cdot[f(x)-f(x+t)] = 0.$$

Proof: 对于第一问，容易看出

$$\begin{aligned}
\int_{\mathbb{R}} f(x)\,\mathrm{d}x &= \sup\left\{\int_{\mathbb{R}} h(x)\,\mathrm{d}x \middle| \ h \text{ 是有界、可测、支集测度有限的函数，且 } 0\le h\le f\right\} \\
&= \sup\left\{\int_{\mathbb{R}} h(x+t)\,\mathrm{d}x \middle| \ h \text{ 是有界、可测、支集测度有限的函数，且 } 0\le h\le f\right\} \\
&= \int_{\mathbb{R}} f(x+t)\,\mathrm{d}x\,.
\end{aligned}$$

对于第二问，考虑用第 44 题的结论，用仅在有界闭集合 F 上非零的连续函数 f_1 来近似函数 f，满足 $\int_{\mathbb{R}}|f-f_1|<\varepsilon$ 对 $\forall\varepsilon>0$ 成立. 由于 f_1 在闭集 F 上连续，进而一致连续. 那么对 $\forall\varepsilon>0$，记 $\int_{\mathbb{R}}|g| = G < +\infty$，当 $|t|<\frac{\varepsilon}{2Mm(F)}$ 时有

$$\begin{aligned}
\left|\int_F g(x)\cdot[f_1(x)-f_1(x+t)]\right| &\le \int_F M\cdot|f_1(x)-f_1(x+t)| \\
&\le \int_F M\cdot\frac{\varepsilon}{2Mm(F)} \\
&= \varepsilon.
\end{aligned}$$

于是在 f_1 下的命题得证. 对于 f 下的命题，有

$$
\begin{aligned}
\left|\int_{\mathbb{R}} g(x)\cdot[f(x)-f(x+t)]\right| &\le \int_{\mathbb{R}} M\cdot|f(x)-f(x+t)| \\
&\le \int_{\mathbb{R}} M\cdot|f(x)-f_1(x)| + \int_{\mathbb{R}} M\cdot|f_1(x)-f_1(x+t)| \\
&\quad + \int_{\mathbb{R}} M\cdot|f(x+t)-f_1(x+t)| \\
&\le M\cdot\frac{\varepsilon}{3M}+\frac{\varepsilon}{3}+M\cdot\frac{\varepsilon}{3M} \\
&= \varepsilon.
\end{aligned}
$$

其中倒数第二行由 f_1 下的命题和用 f_1 近似 f 的结论可以得出. 命题得证.

Task 4.48: Let f be integrable over E and g be a bounded measurable function on E. Show that $f\cdot g$ is integrable over E.

Proof: 设 $|g|<M$. 则有

$$\left|\int_E f\cdot g\right| \le \int_E |f|\cdot|g| \le M\int_E |f| < +\infty.$$

Task 4.49: Let f be integrable over $\mathbb{R}$. Show that the following four assertions are equivalent:
1. $f=0$ a.e. on $\mathbb{R}$.
2. $\int_{\mathbb{R}} fg=0$ for every bounded measurable function g on $\mathbb{R}$.
3. $\int_A f=0$ for every measurable set A.
4. $\int_{\mathcal{O}} f=0$ for every open set $\mathcal{O}$.

Proof: $1\Longrightarrow 2$：类似第 48 题可以直接证明.

$2\Longrightarrow 3$：对任意的可测集 A，令可测函数 $g=\chi_A$ 即可.

$3\Longrightarrow 4$：显然.

$4\Longrightarrow 1$：反证法. 设存在正测度集合 A 使得 $\int_A f\neq 0$，不妨设 $\int_A f=k>0$. 构造开集合列 $\{O_n\}$ 使得 $A\subseteq O_n$ 且 $m(O_n\sim A)<\frac{1}{n}$. 那么令 $O=\cap_{n=1}^{\infty}O_n$ 则有 $A\subseteq O$ 且 $m(O)=m(A)$. 此时有

$$\int_O f=\int_A f+\int_{O\sim A} f=k>0,$$

矛盾.

Task 4.50: Let $\mathcal{F}$ be a family of functions, each of which is integrable over E. Show that $\mathcal{F}$ is uniformly integrable over E if and only if for each $\varepsilon>0$, there is a $\delta>0$ such that for each $f\in\mathcal{F}$,

$$\text{if } A\subseteq E \text{ is measurable and } m(A)<\delta, \text{then } \left|\int_A f\right|<\varepsilon.$$

Proof: $\Longrightarrow$ 由 $\left|\int_A f\right|\le\int_A|f|<\varepsilon$ 显然.

$\Longleftarrow$ 对于 $\forall f\in\mathcal{F}$，考虑 f^+ 和 f^- 在 A 上定号.对于任意的可测 $A\subseteq E$ 且 $m(A)<\delta$，有

$$
\begin{aligned}
\int_A |f| &= \int_{A\cap[f\geq 0]} |f| - \int_{A\cap[f\leq 0]} |f| \\
&= \left|\int_{A\cap[f\geq 0]} f\right| + \left|\int_{A\cap[f\leq 0]} f\right| \\
&< \varepsilon.
\end{aligned}
$$

最后一步由题目条件得到.

Task 4.51: Let $\mathcal{F}$ be a family of functions, each of which is integrable over E. Show that $\mathcal{F}$ is uniformly integrable over E if and only if for each $\varepsilon > 0$, there is a $\delta > 0$ such that for all $f \in \mathcal{F}$,

$$\text{if } \mathcal{U} \text{ is open and } m(E\cap\mathcal{U}) < \delta, \text{then } \int_{E\cap\mathcal{U}} |f| < \varepsilon.$$

Proof: $\Longrightarrow$ 显然.

$\Longleftarrow$ 考虑对任意的 $\varepsilon > 0$ 和条件中对应的 $\delta > 0$，令 E 的可测子集 A 满足 $m(A) < \delta$. 那么存在开集 $\mathcal{U}$ 包含 A 且 $m(\mathcal{U} \sim A) < \delta - m(A)$. 那么有

$$m(E\cap\mathcal{U}) = m(A) + m((E\cap\mathcal{U}) \sim A) \leq m(A) + m(\mathcal{U} \cap A) < \delta.$$

从而有

$$\int_A |f| \leq \int_{E\cap\mathcal{U}} |f| < \varepsilon.$$

Task 4.52:

1. Let $\mathcal{F}$ be the family of functions f on $[0,1]$, each of which is integrable over $[0,1]$ and has $\int_0^1 |f| \leq 1$. Is $\mathcal{F}$ uniformly integrable over $[0,1]$?
2. Let $\mathcal{F}$ be the family of functions f on $[0,1]$, each of which is continuous over $[0,1]$ and has $|f| \leq 1$. Is $\mathcal{F}$ uniformly integrable over $[0,1]$?
3. Let $\mathcal{F}$ be the family of functions f on $[0,1]$, each of which is integrable over $[0,1]$ and has $\int_a^b |f| \leq b - a$ for all $[a,b] \subseteq [0,1]$. Is $\mathcal{F}$ uniformly integrable over $[0,1]$?

Proof:

1. 错误. 考虑

$$f_n = \begin{cases} n, 0 \leq x \leq \frac{1}{n} \\ 0, \text{其他情况} \end{cases}$$

   不是一致可积的.
2. 正确. 令 $\delta = \varepsilon$，则对任意 $[0,1]$ 的可测子集 A，且 $m(A) < \delta$，有

$$\int_A |f| \leq \int_A 1 = m(A) < \varepsilon.$$

   是一致可积的.
3. 正确. 容易看出 $F(x) = \int_0^x f(t)\,\mathrm{d}t$ 是一致连续的.

5 | Lebesgue Integration: Further Topics

5.1 | Uniform Integrability and Tightness: A General Vitali Convergence Theorem

Task 5.1: Prove Corollary 2.

Task 5.2: Let $\{f_k\}_{k=1}^n$ be a finite family of functions, each of which is integrable over E. Show that $\{f_k\}_{k=1}^n$ is uniformly integrable and tight over E.

Task 5.3: Let the sequences of functions $\{h_n\}$ and $\{g_n\}$ be uniformly integrable and tight over E. Show that for any α and β, $\{\alpha f_n + \beta g_n\}$ also is uniformly integrable and tight over E.

Task 5.4: Let $\{f_n\}$ be a sequence of meaureable functions on E. Show that $\{f_n\}$ is uniformly integrable and tight over E if and only if for each $\varepsilon > 0$, there is a measurable subset E_0 of E that has finite measure and a $\delta > 0$ such that for each measurable subset A of E and index n,

$$\text{if } m(A \cap E_0) < \delta, \text{then } \int_A |f_n| < \varepsilon.$$

Task 5.5: Let $\{f_n\}$ be a sequence of integrable functions on $\mathbb{R}$. Show that $\{f_n\}$ is uniformly integrable and tight over $\mathbb{R}$ if and only if for each $\varepsilon > 0$, there are positive numbers r and δ such that for each open subset $\mathcal{O}$ of $\mathbb{R}$ and index n,

$$\text{if } m(\mathcal{O} \cap (-r, r)) < \delta, \text{then } \int_{\mathcal{O}} |f_n| < \varepsilon.$$

5.2 | Convergence in Measure

5.3 | Chracterizations of Riemann and Lebesgue Integrability

Task 5.6: Let f and g be bounded functions that are Riemann integrable over $[a, b]$. Show that the product fg also is Riemann integrable over $[a, b]$.

Task 5.7: Let f be a bounded function on $[a, b]$ whose set of discontinuities has measure zero. Show that f is measurable. Then show that the same holds without the assumption of boundedness.

Task 5.8: Let f be a function on $[0, 1]$ that is continuous on $(0, 1]$. Show that it is possible for the sequence $\left\{\int_{[1/n,1]} f\right\}$ to converge and yet f is not Lebesgue integrable over $[0, 1]$. Can this happen if f is nonnegative?