Harmonic Measure TD6

2025 年 04 月 20 日

aytony

Exercise 1: Let μ be a σ-finite measure on a set X, and let $f \in \mathscr{L}_\mu^+$. Consider the sets

$$G_f = \{(x,y) \in X \times [0,\infty] \mid y \leqslant f(x)\} \quad \text{and} \quad G_f' = \{(x,y) \in X \times [0,\infty] \mid y < f(x)\}.$$

Show that the sets G_f and G_f' are measurable with respect to the product measure $\mu \otimes \mathscr{L}^1$, where $\mathscr{L}^1$ is the Lebesgue measure. Deduce that

$$\mu \otimes \mathscr{L}^1(G_f) = \mu \otimes \mathscr{L}^1(G_f') = \int_X f(x)\mu(\mathrm{d}x),$$

and give a geometric interpretation of this result. Give alternative expressions for $\mu \otimes \mathscr{L}^1(G_f)$ and $\mu \otimes \mathscr{L}^1(G_f')$.

Proof: *Define $F(x,y) : X \times [0,\infty] \to \mathbb{R}, (x,y) \mapsto f(x) - y$. Since $f \in \mathscr{L}_\mu^+$, $\{F(x,y) > t\}$ is $\mu \otimes \mathscr{L}^1(G_f)$ measurable, hence G_f, G_f' are measurable. For second problem, we observe that*

$$\int_X f(x)\mu(\mathrm{d}x) = \int_X \int_{[0,\infty]} \mathscr{L}(f > y) \,\mathrm{d}\mathscr{L}^1 \,\mathrm{d}\mu = \mu \otimes \mathscr{L}^1(G_f).$$

□

Exercise 2: Recall that the Lebesgue measure on $\mathbb{R}^N$ is the measure defined for any set $A \subseteq \mathbb{R}^N$ by the formula

$$\mathscr{L}^N(A) = \inf\left\{\sum_{i=1}^{\infty} (2r_i)^N, A \subseteq \bigcup_{i=1}^{\infty} Q(x_i, r_i)\right\},$$

where the infimum is over all coverings of A by open cubes $Q(x,r) = \{y \in \mathbb{R}^N \mid \max_{1 \leqslant i \leqslant N} |x_i - y_i| < r\}$. Show that the Lebesgue measure $\mathscr{L}^N$ is translation invariant, i.e., for any $x \in \mathbb{R}^N$ and any set $A \subseteq \mathbb{R}^N$, we have $\mathscr{L}^N(x + A) = \mathscr{L}^N(A)$, where $x + A = \{x + y \mid y \in A\}$.

Proof: *For $A \subset \mathbb{R}^N$, set $\{Q(x_i, r_i)\}$ such that $A \subset \bigcup_i Q(x_i, r_i)$, then it is easy to inform that $A + x \subset \bigcup_i (Q(x_i, r_i) + x)$, so $\mathscr{L}^N(A) \geqslant \mathscr{L}^N(A + x)$, and the other side of inequality can be proved similarly.* □

Exercise 3: Let $A \subseteq \mathbb{R}^N$ and let $f : A \to \mathbb{R}^N$ be a mapping such that for some constants $c, \alpha > 0$, and for all $x, y \in A$,

$$|f(x) - f(y)| \leqslant c|x - y|^\alpha.$$

Show that for all $s \geqslant 0$, we have $\mathscr{H}^{s/\alpha}(f(A)) \leqslant c^{s/\alpha}\mathscr{H}^s(A)$, where $\mathscr{H}^s$ is the s-dimensional Hausdorff measure. In particular, if f is Lipschitz continuous (i.e., $\alpha = 1$), then $\mathscr{H}^s(f(A)) \leqslant$

$c^s\mathscr{H}^s(A)$. What can you deduce about Hausdorff dimensions? Finally, consider the special case of a similarity transformation of scale factor $\lambda > 0$, i.e., an invertible mapping $f : \mathbb{R}^N \to \mathbb{R}^N$ such that $|f(x) - f(y)| = \lambda|x - y|$.

Proof: *Set $\delta \in (0, \infty]$, prove $\mathscr{H}_\delta^{s/\alpha}(f(A)) \leqslant c^{s/\alpha}\mathscr{H}_\delta^s(A)$ then let $\delta \to 0$. $\forall \varepsilon > 0$, set $\{A_i\}$ s.t. $|A_i| \leqslant \delta$, $A \subset \bigcup A_i$ and $\sum_i |A_i|^s \leqslant \mathscr{H}_\delta^s(A) + \varepsilon$. Then $f(A) \subset \bigcup f(A_i)$, and*

$$\sum_i |f(A_i)|^{s/\alpha} \leqslant c^{s/\alpha} \sum_i |A_i|^s \leqslant c^{s/\alpha}\mathscr{H}_\delta^s(A) + \varepsilon,$$

and the result follows according to the arbitrariness of ε. □

Exercise 4: Let C be the middle-third Cantor set $\bigcap_n C_n$, where $C_0 = [0, 1]$ and C_{n+1} is obtained from C_n as follows: split each interval of C_n into three equal parts and remove the open middle third.

1. Show that C is compact, uncountable, and of Lebesgue measure zero.
2. Show that C has Hausdorff dimension $\log 2/\log 3$.

Proof:

1. *Since C_n is closed for $n \geqslant 1$, then $C = \bigcap_n C_n$ is closed, hence compact. For any $x \in C$, set a_n be the part of C_n x is in (left for 0 and right for 1), then $0.a_1a_2a_3\cdots$ constructs a biject between x and $[0, 1]$. Since $\mathscr{L}(C_n) = (2/3)^n$ we can know $\mathscr{L}(C) = 0$.*
2.

□

Exercise 5: Consider C and C_n as above. Define piecewise linear functions $f_n : [0, 1] \to \mathbb{R}$ and linear functionals I_n on $C_c((0, 1))$ by

$$f_n(x) = \int_0^x (3/2)^n \mathbf{1}_{C_n}(t)dt \quad \text{and} \quad I_n(\phi) = \int_0^1 (3/2)^n \mathbf{1}_{C_n}(t)\phi(t)dt.$$

1. Show that $(f_n)_{n\geqslant 0}$ is a Cauchy sequence in $C([0, 1])$ with the uniform norm. Deduce that this sequence converges to a continuous and monotone function ψ. The function ψ is called the Cantor-Vitali function.
2. Show that the sequence $(I_n(\phi))_{n\geqslant 0}$ converges to $-\int_0^1 \psi(t)\phi'(t)dt$ for any C^1-function ϕ with support contained in $(0, 1)$.
3. Deduce that the mapping $\phi \mapsto -\int_0^1 \psi(t)\phi'(t)dt$ extends as a linear functional on $C_c((0, 1))$, representable by $\phi \mapsto \int_{[0,1]} \phi d\mu$ for some Radon measure μ on $[0, 1]$.
4. Show that μ is the weak limit of the measures $\mu_n = (3/2)^n\mathscr{L}^1 \mid_{C_n}$.
5. Show that supp $\mu \subseteq C$. Deduce that μ is singular with respect to the Lebesgue measure.
6. Show that the sequence $(f_n)_{n\in\mathbb{N}}$ is equicontinuous. Deduce that μ has no atoms.