偏微分方程

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练习 1：证明极坐标公式

$$\int_{B(0,R)} f(x)\mathrm{d}x = \int_0^R \int_{\partial B(0,r)} f(x)\mathrm{d}S(x)\mathrm{d}r.$$

证明：对 $x \in \mathbb{R}^n$ 有极坐标分解

$$\begin{aligned}
&x = (\rho, \theta_1, \cdots, \theta_{n-1}) \in B(0,R) = [0,R] \times [0,\pi]^{n-2} \times [0,2\pi], \\
&x = (\rho, \theta_1, \cdots, \theta_{n-1}) \in \partial B(0,R) = \{R\} \times [0,\pi]^{n-2} \times [0,2\pi], \\
&\mathrm{d}S(x) = \rho^{n-1} \sin^{n-2}\theta_1 \sin^{n-3}\theta_2 \cdots \sin\theta_{n-2},
\end{aligned}$$

那么有

$$\begin{aligned}
\mathrm{LHS} &= \int_0^R \int_0^\pi \cdots \int_0^\pi \int_0^{2\pi} f(\rho, \theta_1, \cdots, \theta_{n-1}) \sin^{n-2}\theta_1 \cdots \sin^{\theta}_{n-2} \rho^{n-1}\, \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_{n-1}\, \mathrm{d}\rho \\
&= \int_0^R \int_{\partial B(0,\rho)} f(\rho, \theta_1, \cdots, \theta_{n-1}) \sin^{n-2}\theta_1 \cdots \sin^{\theta}_{n-2} \rho^{n-1}\, \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_{n-1}\, \mathrm{d}\rho \\
&= \int_0^R \int_{\partial B(0,\rho)} f(\rho, \theta_1, \cdots, \theta_{n-1})\, \mathrm{d}S(x)\, \mathrm{d}\rho\,.
\end{aligned}$$

□

练习 2：证明下列函数都是调和函数：

$$f(x,y) = x^3 - 3xy^2; \quad g(x,y) = \mathrm{sh}\ (ny)\sin(nx),$$

其中 n 为常数.

证明：

$$\frac{\partial f}{\partial x} = 3x^2 - 3y^2,$$
$$\frac{\partial^2 f}{\partial x^2} = 6x,$$
$$\frac{\partial f}{\partial y} = -6xy,$$
$$\frac{\partial^2 f}{\partial y^2} = -6x,$$
$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

$$\frac{\partial g}{\partial x} = n \sinh ny \cos nx,$$
$$\frac{\partial^2 g}{\partial x^2} = -n^2 \sinh ny \sin nx,$$
$$\frac{\partial g}{\partial y} = n \cosh ny \sin nx,$$
$$\frac{\partial^2 g}{\partial y^2} = n^2 \sinh ny \sin nx,$$
$$\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = 0.$$

□

练习 3: 证明用极坐标表示的函数 $r \ln r \sin\theta + r\theta \cos\theta$ 满足 Laplace 方程.

证明: 对坐标变换

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

分别对 x, y 求偏导得到

$$\begin{cases} 1 = -r\sin\theta \cdot \theta_x + \cos\theta \cdot r_x \\ 0 = r\cos\theta \cdot \theta_x + \sin\theta \cdot r_x \end{cases}$$

和

$$\begin{cases} 0 = -r\sin\theta \cdot \theta_y + \cos\theta \cdot r_y \\ 1 = r\cos\theta \cdot \theta_y + \sin\theta \cdot r_y \end{cases}$$

分别解出

$$\begin{cases}\theta_x = -\sin\theta/r \\ r_x = \cos\theta\end{cases}, \quad \begin{cases}\theta_y = \cos\theta/r \\ r_y = \sin\theta\end{cases},$$

那么

$$\begin{aligned}
f_x &= f_\theta \theta_x + f_r r_x \\
&= (r\ln r\cos\theta + r\cos\theta - r\theta\sin\theta)\frac{-\sin\theta}{r} + (\ln r\sin\theta + \sin\theta + \theta\cos\theta)\cos\theta \\
&= \theta, \\
f_{xx} &= \theta_x \\
&= -\frac{\sin\theta}{r}, \\
f_y &= f_\theta \theta_y + f_r r_y \\
&= (r\ln r\cos\theta + r\cos\theta - r\theta\sin\theta)\frac{\cos\theta}{r} + (\ln r\sin\theta + \sin\theta + \theta\cos\theta)\sin\theta \\
&= 1 + \ln r, \\
f_{yy} &= \frac{1}{r} r_y \\
&= \frac{\sin\theta}{r},
\end{aligned}$$

那么自然有 $f_{xx} + f_{yy} = 0$. □