Mathematics Courses
Tasks Answers Series

Real Analysis: Tasks Answers

Fourth Edition,

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Task 4.40: Let f be integrable over $\mathbb{R}$. Show that the function F defined by

$$F(x) = \int_{-\infty}^{x} f \text{ for all } x \in \mathbb{R}$$

is properly defined and continuous. Is it necessarily Lipschitz?

Task 4.41: Show that Proposition 25 is false if $E = \mathbb{R}$.

Task 4.42: Show that Theorem 26 is false without the assumption that the h_n's are nonnegative.

Proof: 考虑函数列

$$h_{n(x)} = \begin{cases} n, 0 \leq x \leq \frac{1}{2^n} \\ -n, \frac{1}{2^n} < x \leq \frac{1}{2^{n-1}} \\ 0, \text{其他情况} \end{cases}$$

此时 $\lim \int_{[0,1]} h_n = 0$，但是在 $[0,1]$ 上不一致可积.

考虑函数列

Task 4.43: Let the sequences of functions $\{h_n\}$ and $\{g_n\}$ be uniformly integrable over E. Show that for any α and β, the sequence of linear combinations $\alpha f_n + \beta g_n$ also is uniformly integrable over E.

Proof: 任取 $\varepsilon > 0$. 设 $\delta_1, \delta_2 > 0$ 满足 $\{h_n\}, \{g_n\}$ 在 $\frac{\varepsilon}{2\,|\alpha|}, \frac{\varepsilon}{2\,|\beta|}$ 下对应的一致可积条件. 那么令 $\delta = \min\{\delta_1, \delta_2\}$. 对任意 E 的可测子集 A，且 $m(A) < \delta$，则有

$$
\begin{aligned}
\int_A |\alpha f_n + \beta g_n| &\leq |\alpha| \int_A |f_n| + |\beta| \int_A |g_n| \\
&= |\alpha| \cdot \frac{\varepsilon}{2\ |\alpha|} + |\beta| \cdot \frac{\varepsilon}{2\ |\beta|} \\
&< \varepsilon.
\end{aligned}
$$

Task 4.44: Let f be integrable over $\mathbb{R}$ and $\varepsilon > 0$. Establish the following three approximation properties.

1. There is a simple function η on $\mathbb{R}$ which has finite support and $\int_{\mathbb{R}} |f - \eta| < \varepsilon$.
2. There is a step function s on $\mathbb{R}$ which vanished outside a closed, bounded interval and $\int_{\mathbb{R}} |f - s| < \varepsilon$.
3. There is a continuous function g on $\mathbb{R}$ which vanished outside a bounded set and $\int_{\mathbb{R}} |f - g| < \varepsilon$.

Proof: 不妨假设 f 在 $\mathbb{R}$ 上非负，否则只要对 f 非负和 f 为负的部分分别应用结论即可.

对于第一问，由定义知道

$$
\int_{\mathbb{R}} f = \sup\left\{ \int_{\mathbb{R}} h \mid h \text{ 是有界、可测、支集测度有限的函数，且 } 0 \leq h \leq f \right\}.
$$

那么存在 h_0 满足以上条件且

$$
\int_{\mathbb{R}} f - \int_{\mathbb{R}} h_0 < \frac{\varepsilon}{2}.
$$

又由 $\int_{\mathbb{R}} h_0$ 定义知道存在简单函数 η 满足

$$
\int_{\mathbb{R}} h_0 - \int_{\mathbb{R}} \eta < \frac{\varepsilon}{2}.
$$

那么就有

$$
\int_{\mathbb{R}} |f - \eta| = \int_{\mathbb{R}} f - \int_{\mathbb{R}} \eta < \varepsilon,
$$

其中 η 是简单函数且支集测度有限.

对于第二问，对于上述 h_0，首先由积分的连续性得到

$$
\lim_{n \to \infty} \int_{[-n,n]} h_0 = \int_{\mathbb{R}} h_0.
$$

那么存在一个足够大的 n_0，使得

$$
\int_{(-\infty,-n_0)\cup(n_0,+\infty)} h_0 = \int_{\mathbb{R}} h_0 - \int_{[-n_0,n_0]} h_0 < \frac{\varepsilon}{4}.
$$

考虑 $\int_{[-n_0,n_0]} h_0$. 由 Lusin 定理，存在 $[-n_0, n_0]$ 的子集 F 使得 h_0 在 F 上连续，那么由 Riemann 可积性知道存在 F 上的 step function s 使得 $s \leq h_0$ 且

$$
\int_F h_0 - \int_F s < \frac{\varepsilon}{8}.
$$

又注意到 h_0 在 $[-n_0, n_0]$ 上有界，设 $|h_0| < M$. 那么可以选取上述集合 F 使得 $m([-n_0, n_0] \sim F) < \frac{\varepsilon}{8M}$ 即可，在 $\mathbb{R} \sim F$ 上令 $s = 0$，此时有

$$
\begin{aligned}
\int_{\mathbb{R}} |f-s| &= \int_{\mathbb{R}} f - \int_{\mathbb{R}} s \\
&= \int_{\mathbb{R}} f - \int_{\mathbb{R}} h_0 + \int_{\mathbb{R}} h_0 - \int_{[-n_0,n_0]} h_0 + \int_{[-n_0,n_0]} h_0 - \int_F s \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \int_{[-n_0,n_0]\sim F} h_0 + \int_F h_0 - \int_F s \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + M \cdot \frac{\varepsilon}{8M} + \frac{\varepsilon}{8} \\
&< \varepsilon.
\end{aligned}
$$

对于第三问，考虑上述 h_0, n_0 和 F. 考虑构造连续函数 g，使得在 F 上 $g = h_0$，并将 g 延拓到 $[-n_0, n_0]$，进一步延拓到 $\mathbb{R}$，且有 $g\big([n_0 + \frac{\varepsilon}{8M}, +\infty)\big) = g\big((-\infty, -n_0 - \frac{\varepsilon}{8M}]\big) = \{0\}$. 这时有

$$
\int_{\mathbb{R}} |f-g| < \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \int_{[-n_0,n_0]\sim F} (h_0 - g) < \varepsilon.
$$

Task 4.45: Let f be integrable over E. Define $\hat{f}$ to be the extension of f to all of $\mathbb{R}$ obtained by setting $\hat{f} \equiv 0$ outside of E. Show that $\hat{f}$ is integrable over $\mathbb{R}$ and $\int_E f = \int_{\mathbb{R}} \hat{f}$. Use this and part (1) and (3) of preceding problem to show that for $\varepsilon > 0$, there is a simple function η on E and a continuous function g on E for which $\int_E |f-\eta| < \varepsilon$ and $\int_E |f-g| < \varepsilon$.

Proof: 显然有

$$
\begin{aligned}
\int_{\mathbb{R}} \hat{f} &= \int_E \hat{f} + \int_{\mathbb{R}\sim E} \hat{f} \\
&= \int_E f + \int_{\mathbb{R}\sim E} 0 \\
&= \int_E f.
\end{aligned}
$$

然后直接应用上题的结论即可.

Task 4.46 (Riemann-Lebesgue): Let f be integrable over $\mathbb{R}$. Show that

$$
\lim_{n\to\infty} \int_{\mathbb{R}} f(x) \cos nx \,\mathrm{d}x = 0.
$$

Proof: $\forall \varepsilon > 0$. 由 44 题的结论知道存在 step function g 仅在有穷闭区间 F 上非 0 且 $\int_{\mathbb{R}} |f - g| < \frac{\varepsilon}{2m(F)}$. 又知道

$$
\begin{aligned}
\left|\int_{\mathbb{R}} f(x)\cos nx\,\mathrm{d}x\right| &\le \int_{\mathbb{R}} |(f(x)-g(x))\cos nx\,\mathrm{d}x| + \left|\int_{\mathbb{R}} g(x)\cos nx\,\mathrm{d}x\right| \\
&\le \int_{\mathbb{R}} |f(x)-g(x)|\ |\cos nx\,\mathrm{d}x| + \left|\int_{\mathbb{R}} g(x)\cos nx\,\mathrm{d}x\right| \\
&\le \int_{\mathbb{R}} |f(x)-g(x)|\,\mathrm{d}x + \left|\int_{\mathbb{R}} g(x)\cos nx\,\mathrm{d}x\right| \\
&\le \frac{\varepsilon}{2} + \left|\int_{\mathbb{R}} g(x)\cos nx\,\mathrm{d}x\right|.
\end{aligned}
$$

故只要证明 g 是 step function 的情况即可，且 g 仅在有穷闭区间 F 上非 0. 考虑到 g 在每一段上的取值为常数，不妨设 $g(x)$ 是常数. 那么此时只要证明

$$
\lim_{n\to\infty}\left|\int_F \cos nx\,\mathrm{d}x\right| = 0.
$$

而注意到它是 Riemann 可积的，直接换元积分得到原函数 $\frac{1}{n}\sin nx$，令 $n\to\infty$ 得到结果.

Task 4.47: Let f be integrable over $\mathbb{R}$.

1. Show that for each t,

$$
\int_{\mathbb{R}} f(x)\,\mathrm{d}x = \int_{\mathbb{R}} f(x+t)\,\mathrm{d}x\,.
$$

2. Let g be a bounded measurable function on $\mathbb{R}$. Show that

$$
\lim_{t\to 0}\int_{\mathbb{R}} g(x)\cdot[f(x)-f(x+t)] = 0.
$$

Proof: 对于第一问，容易看出

$$
\begin{aligned}
\int_{\mathbb{R}} f(x)\,\mathrm{d}x &= \sup\left\{\int_{\mathbb{R}} h(x)\,\mathrm{d}x \mid h \text{ 是有界、可测、支集测度有限的函数，且 } 0\le h\le f\right\} \\
&= \sup\left\{\int_{\mathbb{R}} h(x+t)\,\mathrm{d}x \mid h \text{ 是有界、可测、支集测度有限的函数，且 } 0\le h\le f\right\} \\
&= \int_{\mathbb{R}} f(x+t)\,\mathrm{d}x\,.
\end{aligned}
$$

对于第二问，考虑用第 44 题的结论，用仅在有界闭集合 F 上非零的连续函数 f_1 来近似函数 f，满足 $\int_{\mathbb{R}}|f-f_1|<\varepsilon$ 对 $\forall\varepsilon>0$ 成立. 由于 f_1 在闭集 F 上连续，进而一致连续. 那么对 $\forall\varepsilon>0$，记 $\int_{\mathbb{R}}|g| = G < +\infty$，当 $|t| < \frac{\varepsilon}{2Mm(F)}$ 时有

$$
\begin{aligned}
\left|\int_F g(x)\cdot[f_1(x)-f_1(x+t)]\right| &\le \int_F M\cdot|f_1(x)-f_1(x+t)| \\
&\le \int_F M\cdot\frac{\varepsilon}{2Mm(F)} \\
&= \varepsilon.
\end{aligned}
$$

于是在 f_1 下的命题得证. 对于 f 下的命题，有

$$
\begin{aligned}
\left|\int_{\mathbb{R}} g(x)\cdot[f(x)-f(x+t)]\right| &\le \int_{\mathbb{R}} M\cdot|f(x)-f(x+t)| \\
&\le \int_{\mathbb{R}} M\cdot|f(x)-f_1(x)|+\int_{\mathbb{R}} M\cdot|f_1(x)-f_1(x+t)| \\
&\quad +\int_{\mathbb{R}} M\cdot|f(x+t)-f_1(x+t)| \\
&\le M\cdot\frac{\varepsilon}{3M}+\frac{\varepsilon}{3}+M\cdot\frac{\varepsilon}{3M} \\
&=\varepsilon.
\end{aligned}
$$

其中倒数第二行由 f_1 下的命题和用 f_1 近似 f 的结论可以得出. 命题得证.

Task 4.48: Let f be integrable over E and g be a bounded measurable function on E. Show that $f\cdot g$ is integrable over E.

Proof: 设 $|g|<M$. 则有

$$\left|\int_E f\cdot g\right| \le \int_E |f|\cdot|g| \le M\int_E |f| < +\infty.$$

Task 4.49: Let f be integrable over $\mathbb{R}$. Show that the following four assertions are equivalent:
1. $f=0$ a.e. on $\mathbb{R}$.
2. $\int_{\mathbb{R}} fg=0$ for every bounded measurable function g on $\mathbb{R}$.
3. $\int_A f=0$ for every measurable set A.
4. $\int_{\mathcal{O}} f=0$ for every open set $\mathcal{O}$.

Proof: $1\Longrightarrow 2$：类似第 48 题可以直接证明.

$2\Longrightarrow 3$：对任意的可测集 A，令可测函数 $g=\chi_A$ 即可.

$3\Longrightarrow 4$：显然.

$4\Longrightarrow 1$：反证法. 设存在正测度集合 A 使得 $\int_A f\neq 0$，不妨设 $\int_A f=k>0$. 构造开集合列 $\{O_n\}$ 使得 $A\subseteq O_n$ 且 $m(O_n\sim A)<\frac{1}{n}$. 那么令 $O=\cap_{n=1}^{\infty} O_n$ 则有 $A\subseteq O$ 且 $m(O)=m(A)$. 此时有

$$\int_O f=\int_A f+\int_{O\sim A} f=k>0,$$

矛盾.

Task 4.50: Let $\mathcal{F}$ be a family of functions, each of which is integrable over E. Show that $\mathcal{F}$ is uniformly integrable over E if and only if for each $\varepsilon>0$, there is a $\delta>0$ such that for each $f\in\mathcal{F}$,

$$\text{if } A\subseteq E \text{ is measurable and } m(A)<\delta, \text{then } \left|\int_A f\right|<\varepsilon.$$

Proof: $\Longrightarrow$ 由 $\left|\int_A f\right|\le\int_A|f|<\varepsilon$ 显然.

$\Longleftarrow$ 对于 $\forall f\in\mathcal{F}$，考虑 f^+ 和 f^- 在 A 上定号.对于任意的可测 $A\subseteq E$ 且 $m(A)<\delta$，有

$$
\begin{aligned}
\int_A |f| &= \int_{A\cap[f\geq 0]} |f| - \int_{A\cap[f\leq 0]} |f| \\
&= \left|\int_{A\cap[f\geq 0]} f\right| + \left|\int_{A\cap[f\leq 0]} f\right| \\
&< \varepsilon.
\end{aligned}
$$

最后一步由题目条件得到.

Task 4.51: Let $\mathcal{F}$ be a family of functions, each of which is integrable over E. Show that $\mathcal{F}$ is uniformly integrable over E if and only if for each $\varepsilon > 0$, there is a $\delta > 0$ such that for all $f \in \mathcal{F}$,

$$
\text{if } \mathcal{U} \text{ is open and } m(E\cap\mathcal{U}) < \delta, \text{then} \int_{E\cap\mathcal{U}} |f| < \varepsilon.
$$

Proof: $\Longrightarrow$ 显然.

$\Longleftarrow$ 考虑对任意的 $\varepsilon > 0$ 和条件中对应的 $\delta > 0$，令 E 的可测子集 A 满足 $m(A) < \delta$. 那么存在开集 $\mathcal{U}$ 包含 A 且 $m(\mathcal{U} \sim A) < \delta - m(A)$. 那么有

$$
m(E\cap\mathcal{U}) = m(A) + m((E\cap\mathcal{U}) \sim A) \leq m(A) + m(\mathcal{U}\cap A) < \delta.
$$

从而有

$$
\int_A |f| \leq \int_{E\cap\mathcal{U}} |f| < \varepsilon.
$$

Task 4.52:

1. Let $\mathcal{F}$ be the family of functions f on $[0, 1]$, each of which is integrable over $[0, 1]$ and has $\int_0^1 |f| \leq 1$. Is $\mathcal{F}$ uniformly integrable over $[0, 1]$?
2. Let $\mathcal{F}$ be the family of functions f on $[0, 1]$, each of which is continuous over $[0, 1]$ and has $|f| \leq 1$. Is $\mathcal{F}$ uniformly integrable over $[0, 1]$?
3. Let $\mathcal{F}$ be the family of functions f on $[0, 1]$, each of which is integrable over $[0, 1]$ and has $\int_a^b |f| \leq b - a$ for all $[a, b] \subseteq [0, 1]$. Is $\mathcal{F}$ uniformly integrable over $[0, 1]$?

Proof:

1. 错误. 考虑

$$
f_n = \begin{cases} n, 0 \leq x \leq \frac{1}{n} \\ 0, \text{其他情况} \end{cases}
$$

   不是一致可积的.
2. 正确. 令 $\delta = \varepsilon$，则对任意 $[0, 1]$ 的可测子集 A，且 $m(A) < \delta$，有

$$
\int_A |f| \leq \int_A 1 = m(A) < \varepsilon.
$$

   是一致可积的.
3. 正确. 容易看出 $F(x) = \int_0^x f(t)\,\mathrm{d}t$ 是一致连续的.

5 | Lebesgue Integration: Further Topics

5.1 | Uniform Integrability and Tightness: A General Vitali Convergence Theorem

Task 5.1: Prove Corollary 2.

Proof: $\Longleftarrow$ 由 Vitali 收敛定理显然.

$\Longrightarrow$ 先证紧性. 任取 $\varepsilon > 0$. 由 $\int_E h_n \to 0$ 可以取一个足够大的自然数 N 使得对 $\forall n > n_0$ 有 $\int_E h_n < \varepsilon$. 那么对于 $n > N$，可以直接取 $E_n = \emptyset$ 满足

$$\int_{E \sim E_n} |h_n| = \int_E h_n - \int_{E_n} h_n < \varepsilon.$$

对于 $1 \leq n \leq N$ 的每个 h_n，一定可以找到一个测度有限的集合 $E_n \subseteq E$ 使得

$$\int_{E \sim E_n} h_n < \varepsilon.$$

令 $E_0 = \cup_{n=1}^N E_n$ 仍是测度有限的集合，那么 E_0 就满足

$$\int_{E \sim E_0} h_n < \int_{E \sim E_n} h_n < \varepsilon \quad \text{对 } \forall n \in \mathbb{N}^* \text{ 成立.}$$

从而 $\{h_n\}$ 是紧的. 对于上述 E_0，考虑 $\{h_n\}$ 在 E_0 上的限制，应用定理 4.26 直接得出 $\{h_n\}$ 在 E_0 上面一致可积. 考虑拓展到 E 上的情况.

对任意 $\varepsilon > 0$，存在上述 E_0 使得 $\int_{E \sim E_0} h_n < \varepsilon/2$ 对任意 $n \in \mathbb{N}^*$ 成立. 由 $\{h_n\}$ 在 E_0 上一致可积知道存在 $\delta > 0$ 使得对任意 E_0 的测度小于 δ 的子集 A 有 $\int_A |h_n| < \frac{\varepsilon}{2}$. 那么考虑任意 E 上测度小于 δ 的子集 B，一定有

$$\int_B h_n = \int_{B \cap E_0} h_n + \int_{B \sim E_0} h_n \leq \frac{\varepsilon}{2} + \int_{E \sim E_0} h_n < \varepsilon.$$

从而 $\{h_n\}$ 在 E 上一致可积.

Task 5.2: Let $\{f_k\}_{k=1}^n$ be a finite family of functions, each of which is integrable over E. Show that $\{f_k\}_{k=1}^n$ is uniformly integrable and tight over E.

Proof: 先证 $\{f_k\}_{k=1}^n$ 是紧的. 对于 $1 \leq k \leq n$ 的每个 f_k，一定可以找到一个测度有限的集合 $E_k \subseteq E$ 使得

$$\int_{E \sim E_k} |f_k| < \varepsilon.$$

令 $E_0 = \cup_{k=1}^n E_k$ 仍是测度有限的集合，那么 E_0 就满足

$$\int_{E \sim E_0} |f_k| < \int_{E \sim E_k} |f_k| < \varepsilon \quad \text{对 } \forall k = 1, 2, ..., n \text{ 成立.}$$

从而 $\{f_k\}_{k=1}^n$ 是紧的.

再证 $\{f_k\}(k=1)^n$ 是一致可积的. 任取 $\varepsilon > 0$，由于 $\{f_k\}$ 是紧的，存在 E 的有限测度子集 E_0 满足

$$\int_{E\sim E_0}|f_k|<\frac{\varepsilon}{2}.$$

对于每一个 $1\leq k\leq n$，存在 δ_k 满足对于 E_0 的任意子集 A 且 $m(A)<\delta_k$，一定有

$$\int_A|f_k|<\frac{\varepsilon}{2}.$$

令 $\delta=\min\{\delta_1,\delta_2,...,\delta_n\}$，则对于任意 E 的子集 A 且 $m(A)<\delta$，一定有

$$\int_A|f_k|=\int_{A\cap E_0}|f_k|+\int_{A\sim E_0}|f_k|<\frac{\varepsilon}{2}+\int_{E\sim E_0}|f_k|<\varepsilon.$$

从而 $\{f_k\}$ 是一致可积的.

Task 5.3: Let the sequences of functions $\{h_n\}$ and $\{g_n\}$ be uniformly integrable and tight over E. Show that for any α and β, $\{\alpha f_n+\beta g_n\}$ also is uniformly integrable and tight over E.

Proof: 不妨设 $\alpha,\beta\neq 0$.

先证 $\{\alpha f_n+\beta g_n\}$ 是紧的. 对于 $\forall\varepsilon>0$，一定可以找到测度有限的集合 $E_1,E_2\subseteq E$ 使得

$$\int_{E\sim E_1}|f|<\frac{\varepsilon}{2\ |\alpha|}\quad \text{对 } \forall f\in\{f_n\} \text{ 成立}$$

且

$$\int_{E\sim E_2}|g|<\frac{\varepsilon}{2\ |\beta|}\quad \text{对 } \forall g\in\{g_n\} \text{ 成立.}$$

令 $E_0=E_1\cup E_2$ 仍是测度有限的集合，那么 E_0 同时满足上式中 E_1,E_2 的性质，此时有

$$\begin{aligned}\int_{E\sim E_0}|\alpha f+\beta g|&\leq|\alpha|\int_{E\sim E_0}|f|+|\beta|\int_{E\sim E_0}|g|\\&<|\alpha|\cdot\frac{\varepsilon}{2\ |\alpha|}+|\beta|\cdot\frac{\varepsilon}{2\ |\beta|}\\&=\varepsilon\quad \text{对 } \forall f\in\{f_n\},g\in\{g_n\} \text{ 成立.}\end{aligned}$$

从而 $\{\alpha f_n+\beta g_n\}$ 是紧的.

再证 $\{\alpha f_n+\beta g_n\}$ 是一致可积的. 任取 $\varepsilon>0$，由于 $\{\alpha f_n+\beta g_n\}$ 是紧的，存在 E 的有限测度子集 E_0 满足

$$\int_{E\sim E_0}(\alpha f_n+\beta g_n)<\frac{\varepsilon}{2}.$$

对于任意的 $\alpha f+\beta g\in\{\alpha f_n+\beta g_n\}$，存在 δ_1,δ_2 满足对于 E_0 的任意子集 A_1,A_2 且 $m(A_1)<\delta_1,m(A_2)<\delta_2$，一定有

$$\int_{A_1}|f|<\frac{\varepsilon}{4\ |\alpha|}\quad \text{且}\quad \int_{A_2}|g|<\frac{\varepsilon}{4\ |\beta|}$$

令 $\delta=\min\{\delta_1,\delta_2$ 仍满足上式中 δ_1,δ_2 性质. 则对于任意 E 的子集 A 且 $m(A)<\delta$，一定有

$$\begin{aligned}\int_A |\alpha f_n + \beta g_n| &\leq |\alpha| \int_{A\cap E_0} |f| + |\beta| \int_{A\cap E_0} |g| + \int_{A\sim E_0} |\alpha f_n + \beta g_n| \\ &< |\alpha| \cdot \frac{\varepsilon}{4\,|\alpha|} + |\beta| \cdot \frac{\varepsilon}{4\,|\beta|} + \frac{\varepsilon}{2} \\ &= \varepsilon \quad \text{对 } \forall f \in \{f_n\}, g \in \{g_n\} \text{ 成立}.\end{aligned}$$

从而 $\{\alpha f_n + \beta g_n\}$ 是一致可积的.

Task 5.4: Let $\{f_n\}$ be a sequence of meaureable functions on E. Show that $\{f_n\}$ is uniformly integrable and tight over E if and only if for each $\varepsilon > 0$, there is a measurable subset E_0 of E that has finite measure and a $\delta > 0$ such that for each measurable subset A of E and index n,

$$\text{if } m(A \cap E_0) < \delta, \text{then} \int_A |f_n| < \varepsilon.$$

Proof: $\Longrightarrow$ 任取 $\varepsilon > 0$，则由紧性知道存在有限测度集合 $E_0 \subseteq E$ 使得

$$\int_{E\sim E_0} |f| < \frac{\varepsilon}{2} \quad \text{对 } \forall f \in \{f_n\} \text{ 成立}.$$

又由于 E_0 是有限测度的，从而应用一致可积性知道在 E_0 上存在 $\delta > 0$ 使得对任意 $A_0 \subseteq E_0$ 且 $m(A_0) < \delta$，有

$$\int_{A_0} |f_n| < \frac{\varepsilon}{2}.$$

从而对任意的 E 上的子集 A，若满足 $m(A \cap E_0) < \delta$，则有

$$\int_A |f_n| = \int_{A\cap E_0} |f_n| + \int_{A\sim E_0} |f_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\Longleftarrow$ 由题意条件，令 $A = E \sim E_0$ 得到

$$\int_{E\sim E_0} |f| < \varepsilon.$$

再令 A 为任意 E_0 的子集，且满足 $m(A) < \delta$ 得到

$$\int_A |f| < \varepsilon.$$

从而直接得出紧性和一致可积性.

Task 5.5: Let $\{f_n\}$ be a sequence of integrable functions on $\mathbb{R}$. Show that $\{f_n\}$ is uniformly integrable and tight over $\mathbb{R}$ if and only if for each $\varepsilon > 0$, there are positive numbers r and δ such that for each open subset $\mathcal{O}$ of $\mathbb{R}$ and index n,

$$\text{if } m(\mathcal{O} \cap (-r, r)) < \delta, \text{then} \int_{\mathcal{O}} |f_n| < \varepsilon.$$

Proof: $\Longrightarrow$ 类似上一题可以证明.

$\Longleftarrow$ 任取 $\varepsilon > 0$. 令 $E_0 = (-r, r)$ 且 $\mathcal{O} = \mathbb{R} \sim (-r, r)$. 则有 $m(\mathcal{O} \cap (-r, r)) = m(\emptyset) < \delta$ 一定成立. 此时成立

$$\int_{E\sim E_0} |f| < \varepsilon \quad \text{对 } \forall f \in \{f_n\} \text{ 成立.}$$

从而 $\{f_n\}$ 是紧的.

任取 $\varepsilon > 0$，则存在题目条件中的 $\delta > 0$ 和 $E_0 = (-r, r)$ 使对应条件成立. 考虑任意 $(-r, r)$ 中的子集 A，满足 $m(A) < \delta$. 则一定存在开集 $\mathcal{O}$ 使得 $A \subseteq \mathcal{O}$ 且 $m(\mathcal{O} \sim A) < \delta - m(A)$. 那么这时有

$$m(\mathcal{O} \cap (-r, r)) < m(\mathcal{O}) = m(\mathcal{O} \cap A) + m(\mathcal{O} \sim A) < m(A) + \delta - m(A) = \delta.$$

从而应用题目条件得到

$$\int_A |f| \le \int_{\mathcal{O}} |f| < \varepsilon \quad \text{对 } \forall f \in \{f_n\} \text{ 成立,}$$

从而 $\{f_n\}$ 是一致可积的.

5.2 | Convergence in Measure

Task 5.6: Let $\{f_n\} \to f$ in measure on E and g be a measurable function on E that is finite a.e. on E. Show that $\{f_n\} \to g$ in measure on E if and only if $f = g$ a.e. on E.

Proof: $\Longrightarrow$ 假设 $\{f_n\} \to g$ 依测度收敛. 那么对 $\forall \varepsilon > 0$ 有

$$\begin{aligned}
&m\{x \in E \,||f(x) - g(x)| > \varepsilon\} \\
&\le \lim_{n\to\infty} m\{x \in E \,||f_n(x) - g(x)| + |f_n(x) - f(x)| > \varepsilon\} \\
&= 0.
\end{aligned}$$

从而有

$$\begin{aligned}
m\{x \in E \,|\, f(x) \neq g(x)\} &= m\left(\bigcup_{k=1}^{\infty} \left\{x \in E \,\middle|\, |f(x) - g(x)| > \frac{1}{k}\right\}\right) \\
&\le \sum_{k=1}^{\infty} m\left\{x \in E \,\middle|\, |f(x) - g(x)| > \frac{1}{k}\right\} \\
&= 0.
\end{aligned}$$

$\Longleftarrow$ 显然.

Task 5.7: Let E have finite measure, $\{f_n\} \to f$ in measure on E and g be a measurable function on E that is finite a.e. on E. Prove that $\{f_n \cdot g\} \to f \cdot g$ in measure, and use this to show that $\{f_n^2\} \to f^2$ in measure. Infer from this that if $\{g_n\} \to g$ in measure, then $\{f_n \cdot g_n\} \to f \cdot g$ in measure.

Proof: $g = 0$ 的时候命题显然成立，下假设 g 在 E 上非零. 任取 $\eta > 0$，只要证 $\lim_{n\to\infty} m(\{x \in E \,||g(x)f_n(x) - g(x)f(x)| < \eta\}) = 0$，那么只要证明

$$\lim_{n\to\infty} m\left(\left\{x \in E \,\middle|\, |f_n(x) - f(x)| < \frac{\eta}{|g(x)|} < \frac{\eta}{M}\right\}\right) = 0,$$

而这是显然的，其中 M 是 g 的一个界限.（因为 g 在 E 上几乎处处有限，那么存在 $E_0 \subset E$ 使得 g 在 E_0 上有界，且 $E \sim E_0$ 可以任意小）. 后两问略.

Task 5.8: Show that Fatou's Lemma, the Monotone Convergence Theorem, the Lebesgue Dominated Convergence Theorem, and the Vitali Convergence Theorem remain valid if "pointwise convergence a.e." is replaced by "convergence in measure".

Task 5.9: Show that Proposition 3 does not necessarily hold for sets E for infinite measure.

Proof: 令 $f_n = \chi_{[n,n+1]}, f = 0, E = \mathbb{R}$ 即可.

Task 5.10: Show that linear combinations of sequences that converge in measure on a set of finite measure also converge in measure.

Task 5.11: Assume E has finite measure. Let $\{f_n\}$ be a sequence of measurable functions on E and f a measurable function on E for which f and each f_n is finite a.e. on E. Prove that $\{f_n\} \to f$ in measure on E if and only if for every subsequence of $\{f_n\}$ has in turn a further subsequence that converges to f pointwise a.e. on E.

! TODO !

这道题值得做

Task 5.12: Show that a sequence $\{a_j\}$ of real numbers converges to a real number if $|a_{j+1} - a_j| < 1/2^j$ for all j by showing that the sequence $\{a_j\}$ must be Cauchy.

Proof: 显然. 令 N 足够大，使得 $1/(2^{N+1}) < \varepsilon$ 即可.

Task 5.13: A sequence $\{f_n\}$ of measurable functions on E is said to be **Cauchy in measure** provided given $\eta > 0$ and $\varepsilon > 0$ there is an index N such that for all $m, n \geq N$,

$$m\{x \in E \,||f_n(x) - f_m(x)| \geq \eta\} < \varepsilon.$$

Show that if $\{f_n\}$ is Cauchy in measure, then there is a measurable function f on E to which the sequence $\{f_n\}$ converges in measure.

Proof: 只要证明 $m\big(\{x \in E \,|\, f_n(x) \text{ 为柯西列}\}\big)$ 可以任意接近于 $m(E)$. 而题目条件等价于 $m\big(\{x \in E \,|\, f_n(x) \text{ 不为柯西列}\}\big)$ 可以任意小，容易看出这两者等价.

Task 5.14: Assume $m(E) < +\infty$. For two measurable functions g and h on E, define

$$\rho(g, h) = \int_E \frac{|g-h|}{1+|g-h|}.$$

Shwo that $\{f_n\} \to f$ in measure on E if and only if $\lim_{n\to\infty} \rho(f_n, f) = 0$.

Proof: 不失一般性，令 $f = 0$.

$\Longrightarrow$ 当 n 足够大时，有 $m(\{x \in E \,||f_n(x)| \geq \varepsilon/(2m(E))\}) < \varepsilon/2$. 这时有

$$\begin{aligned}
\rho(f_n, f) &= \int_E \frac{|f_n|}{1+|f_n|} \\
&= \int_{[f_n > \varepsilon/(2m(E))]} \frac{|f_n|}{1+|f_n|} + \int_{[f_n \leq \varepsilon/(2m(E))]} \frac{|f_n|}{1+|f_n|} \\
&\leq 1 \cdot \frac{\varepsilon}{2} + \frac{\varepsilon}{2m(E)} \cdot m(E) \\
&\leq \varepsilon.
\end{aligned}$$

$\Longleftarrow$ 反证法. 假设对任意的 $\varepsilon, \eta > 0$，总是存在足够大的 n 使得 $m(\{x \in E \,||f_n(x)| \geq \eta\}) > \varepsilon$. 那么令这个集合为 E'，则有

$$\int_{E'} \rho(f_n, f) \geq \int_{E'} \frac{\eta}{1+\eta} = \frac{\eta\varepsilon}{1+\eta} > 0,$$

与条件矛盾.

5.3 | Chracterizations of Riemann and Lebesgue Integrability

Task 5.15: Let f and g be bounded functions that are Riemann integrable over $[a, b]$. Show that the product fg also is Riemann integrable over $[a, b]$.

Task 5.16: Let f be a bounded function on $[a, b]$ whose set of discontinuities has measure zero. Show that f is measurable. Then show that the same holds without the assumption of boundedness.

Task 5.17: Let f be a function on $[0, 1]$ that is continuous on $(0, 1]$. Show that it is possible for the sequence $\left\{\int_{[1/n,1]} f\right\}$ to converge and yet f is not Lebesgue integrable over $[0, 1]$. Can this happen if f is nonnegative?