Harmonic Measure TD8

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Exercise 1: If u is subharmonic in a domain $\Omega \subset \mathbb{R}^d$ and $c > 0$, then $u_+ = \max\{u, 0\}$ and e^{cu} are subharmonic in Ω. What is the analogue for superharmonic functions?

Proof: *Since u subharmonic, 0 subharmonic, u_+ is subharmonic, and cu is subharmonic. According to the continuity of* $\exp$, *e^{cu} is lsc, which is same as cu. Thus*

$$\exp(cu(x)) \leqslant \exp\left(c \fint_{B(x,r)} u(y)\,\mathrm{d}y\right) \leqslant \fint_{B(x,r)} \exp(cu(y))\,\mathrm{d}y\,.$$

Analogue is for superharmonic function u and $c > 0$, u^- and $\log(cu)$ *are superharmonic.* □

Exercise 2: Let Ω be a domain in $\mathbb{R}^d$ and $u \in C^2(\Omega)$.

1. Using Green's formula, show that

$$\int_{B(x,r)} \Delta u(x)\,\mathrm{d}x = \frac{d\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}+1\right)} r^{d-1} \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=r} \fint_{\partial B(x,s)} u(\zeta)\,\mathrm{d}\sigma(\zeta).$$

2. Show that u is superharmonic if and only if $\Delta u \leqslant 0$ on Ω.

Proof:

1.

$$\begin{aligned}
\int_{B(x,r)} \Delta u(x)\,\mathrm{d}x &= \int_{\partial B(x,r)} \nabla u(x) \cdot \nu\,\mathrm{d}\sigma(x) \\
&= r^{d-1} \int_{\partial B(0,1)} \nabla u(x+\zeta r) \cdot \zeta\,\mathrm{d}\sigma(\zeta) \\
&= r^{d-1} \int_{\partial B(0,1)} \nabla u(x+\zeta r) \cdot \zeta\,\mathrm{d}\sigma(\zeta) \\
&= r^{d-1} \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=r} \int_{\partial B(0,1)} u(x+\zeta s)\,\mathrm{d}\sigma(\zeta) \\
&= \frac{d\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}+1\right)} r^{d-1} \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=r} \fint_{\partial B(x,s)} u(\zeta)\,\mathrm{d}\sigma(\zeta).
\end{aligned}$$

2. $\Longleftarrow$ *Suppose $\Delta u \leqslant 0$ on Ω. Then for any $B(x,r) \subset \Omega$,*

$$0 \geqslant \int_{B(x,r)} \Delta u(x)\,\mathrm{d}x = \frac{d\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}+1\right)} r^{d-1} \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=r} \fint_{\partial B(x,s)} u(\zeta)\,\mathrm{d}\sigma(\zeta).$$

i.e.

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=r} \fint_{\partial B(x,s)} u(\zeta)\,\mathrm{d}\sigma(\zeta) \leqslant 0,$$

so

$$\fint_{\partial B(x,r)} u(\zeta)\,\mathrm{d}\sigma(\zeta) \leqslant \lim_{r\to 0^+} \fint_{\partial B(x,r)} u(\zeta)\,\mathrm{d}\sigma(\zeta) = u(x),$$

Hence

$$\fint_{B(x,r)} u(\zeta)\,\mathrm{d}\zeta \leqslant u(x)$$

holds for all $B(x,r) \in \Omega$.
$\Longrightarrow$ *Since*

$$\fint_{\partial B(x,s)} u(\zeta)\,\mathrm{d}\sigma(\zeta) \leqslant u(x),$$

then

$$\int_{B(x,r)} \Delta u(x)\,\mathrm{d}x \leqslant 0$$

holds for all $B(x,r) \in \Omega$, *hence* $\Delta u \leqslant 0$ *in* Ω.

□

Exercise 3: Let Ω be a bounded domain in $\mathbb{R}^d$. Then, every real-valued continuous function f on $\partial\Omega$ can be uniformly approximated on $\partial\Omega$ by the difference of the restrictions to $\partial\Omega$ of two functions continuous on $\overline{\Omega}$ and superharmonic in Ω.

Proof: *Since f can be uniformly approximated by polynomial functions on $\partial\Omega$, we can suppose $f_n \to f$ uniformly, where f_n are polynomials on $\partial\Omega$. Denote $M_n = \sup_{\partial\Omega}|\Delta f_n|$, and set $v(x) = M_n|x|^2$, $w = f_n + v$. Then it is easy to varify that v, w are superharmonic functions needed.* □

Exercise 4:
1. Show that if $\Omega, \tilde{\Omega}$ are domains in $\mathbb{C}$ and $f : \Omega \to \tilde{\Omega}$ is analytic, then $u \circ f$ is harmonic in Ω for every harmonic function u in $\tilde{\Omega}$. What can we conclude if f is conformal and $\tilde{\Omega} = f(\Omega)$?
2. Show that if $\Omega, \tilde{\Omega}$ are domains in $\mathbb{C}$ and $f : \Omega \to \tilde{\Omega}$ is conformal and onto $\tilde{\Omega}$, then $u \circ f$ is superharmonic in Ω whenever $u : \tilde{\Omega} \to \mathbb{R}$ is superharmonic in $\tilde{\Omega}$.

Exercise 5: Find a bounded harmonic function h in the upper half-plane $\mathbb{H} \subset \mathbb{R}^2$, continuous on $\{z\,;\mathrm{Im}(z) \geqslant 0\} \setminus \{0\}$, such that

$$\lim_{\mathbb{H}\ni y\to x\in\mathbb{R}^*} h(y) = \mathrm{sgn}(x) \quad \text{for all} \quad x \in \mathbb{R}^*.$$

Exercise 6: Determine the harmonic measure of the upper half-plane $\mathbb{H} \subset \mathbb{R}^2$ with respect to a point $x \in \mathbb{R} \times \mathbb{R}_+^*$.

Exercise 7:
1. Let $R > r > 0$ be two positive real numbers and consider the domain $\Omega = B(0, R) \setminus \overline{B(0, r)}$. For $x \in \Omega$, determine the harmonic measure $\omega_x(\partial B(0, r))$.
2. Determine the harmonic measure of the upper half-plane $\mathbb{H} \subset \mathbb{R}^2$ with respect to a point $x \in \mathbb{H}$.
3. Determine the harmonic measure of $\mathbb{R}_+^* \times \mathbb{R}_+^*$ with respect to a point $x \in \mathbb{R}_+^* \times \mathbb{R}_+^*$. (Note: Items 2 and 3 involve unbounded domains but can still be addressed⋯)

Exercise 8: Let Ω be a domain in $\mathbb{R}^d$ and ω its harmonic measure. Show that a set $E \subset \partial\Omega$ has zero harmonic measure if and only if there exists a positive superharmonic function u, not identically $+\infty$ on Ω, such that

$$\lim_{\Omega \ni y \to x} u(y) = +\infty \quad \text{for every} \quad x \in E.$$

Remark: The reference point for the harmonic measure is irrelevant.