Mathematics Courses
Tasks Answers Series

Differential Geometry

of Curves & Surfaces,

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1.2 | Parametrized Curves

Task 1.2.1: Find a parametrized curve $\alpha(t)$ whose trace is the circle $x^2 + y^2 = 1$ such that $\alpha(t)$ runs clockwise around the circle with $\alpha(0) = (0, 1)$.

Solution: $\alpha(t) = (-\sin t, \cos t)$ follows the condition.

Task 1.2.2: Let $\alpha(t)$ be a parametrized curve which does not pass through the origin. If $\alpha(t_0)$ is a point of the trace of α closest to the origin and $\alpha'(t_0) \neq 0$, show that the position vector $\alpha(t_0)$ is orthogonal to $\alpha'(t_0)$.

Proof: Since $\alpha(t)$ is parametrized curve, $|\alpha(t)| = \sqrt{\alpha(t) \cdot \alpha(t)}$ is differentiable.

$$\begin{aligned}\frac{\mathrm{d}|\alpha|}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\sqrt{\alpha \cdot \alpha}\right) = \frac{\mathrm{d}\alpha}{\mathrm{d}t}\frac{1}{2\sqrt{\alpha \cdot \alpha}} \cdot 2(\alpha' \cdot \alpha) \\ &= \frac{\alpha' \cdot \alpha}{|\alpha|}\alpha'\end{aligned}$$

Since $\alpha(t_0)$ is the minimal value of α, we know that

$$\left.\frac{\mathrm{d}|\alpha|}{\mathrm{d}t}\right|_{t=t_0} = \frac{\alpha'(t_0) \cdot \alpha(t_0)}{|\alpha(t_0)|}\alpha'(t_0) = 0.$$

Since $\alpha'(t_0) \neq 0$, we can figure out that $\alpha'(t_0) \cdot \alpha(t_0) = 0$, i.e. $\alpha(t_0)$ is orthogonal to $\alpha'(t_0)$.

Task 1.2.3: A parametrized curve $\alpha(t)$ has the property that its second derivatie $a''(t)$ is identically zero. What can be said about α?

Solution: $\alpha''(t) = (x''(t), y''(t), z''(t)) \equiv 0$, so we can denote by calculus that

$$\alpha(t) = (x_0 + x_1 t, y_0 + y_1 t, z_0 + z_1 t).$$

Task 1.2.4: Let $\alpha : I \to \mathbb{R}^3$ be a parametrized curve and let $v \in \mathbb{R}^3$ be a fixed vector. Assume that $\alpha'(t)$ is orthogonal to v for all $t \in I$ and that $\alpha(0)$ is also orthogonal to v. Prove that $\alpha(t)$ is orthogonal to v for all $t \in I$.

Proof: According to condition, $\alpha'(t) \cdot v = 0$ holds for all $t \in I$. Apply integral to both side and we have $\alpha(t) \cdot v \equiv C_0$, where C_0 is constant. Plug $\alpha(0) = 0$ we can infer that $C_0 = 0$, thus $\alpha(t) \cdot v \equiv 0$.

Task 1.2.5: Let $\alpha : I \to \mathbb{R}^3$ be a parametrized curve, with $\alpha'(t) \neq 0$ for all $t \in I$. Show that $|\alpha(t)|$ is a nonzero constant if and only if $\alpha(t)$ is orthogonal to $\alpha'(t)$ for all $t \in I$.

Proof: $\Longrightarrow$ Denote $|\alpha(t)| = C$. Then we have

$$\frac{\mathrm{d}|\alpha|}{\mathrm{d}t} = \frac{\alpha' \cdot \alpha}{|\alpha|}\alpha' = 0.$$

Plug in that $\alpha' \neq 0$ we know $\alpha' \cdot \alpha = 0$ for all $t \in I$.

$\Longleftarrow$ Since α is orthogonal to α', $\alpha \cdot \alpha' = 0$ always holds. Then $\frac{\mathrm{d}|\alpha|}{\mathrm{d}t} = \frac{\alpha' \cdot \alpha}{|\alpha|}\alpha' = 0$ holds. That is $|\alpha| = C$.

1.3 | Regular Curves; Arc Length

Task 1.3.1: Show that the tangent lines to the regular parameterized curve $\alpha(t) = (3t, 3t^2, 2t^3)$ make a constant angle with the line $y = 0, z = x$.

Proof: The line $y = 0, z = x$ can be denoted as the direction of vector $v = (1, 0, 1)$. And we know that $\alpha'(t) = (3, 6t, 6t^2)$. Thus we have

$$\frac{\alpha'(t) \cdot v}{|\alpha'(t)|} = \frac{6t^2 + 3}{\sqrt{36t^4 + 36t^2 + 9}} = 1$$

is a constant.

Task 1.3.2: A circular disk of radius 1 in the plane xy rolls without slipping along the x axis. The figure described by a point of the circumference of the disk is called a cycloid.

1. Obtain a parametrized curve $\alpha : \mathbb{R} \to \mathbb{R}^2$ the trace of which is the cycloid, and determine its singular points.
2. Compute the arc length of the cycloid correpoinding to a complete rotation of the disk.

Solution: For subtask 1, we consider $\alpha(t) = (t - \sin t, 1 - \cos t), \quad t \in \mathbb{R}$. $\alpha'(t) = (1 - \cos t, \sin t)$. When $\alpha'(t) = 0$, we can find singular point $t = 2k\pi, k \in \mathbb{Z}$.

For subtask 2, consider $t \in [0, 2\pi]$.

$$\begin{aligned} s &= \int_0^{2\pi} |\alpha'(t)| \,\mathrm{d}t = \int_0^{2\pi} \sqrt{1 + 2\cos t + \cos^2 t + \sin^2 t} \,\mathrm{d}t \\ &= \int_0^{2\pi} \sqrt{2 + 2\cos t} = 8. \end{aligned}$$

Task 1.3.3: Let $OA = 2a$ be the diameter of a circle S^1 and Oy and AV be the tangents to S^1 at O and A, respectively. A half-line r is drawn from O which meets the circle S^1 at C and line AV at B. On OB mark off the segment $Op = CB$. If we rotate r about O, the point p will describe a curve called the *cissoid of Diocles*. By taking OA as the x axis and OY as the y axis, prove that

1. The trace of

   $$\alpha(t) = \left(\frac{2at^2}{1 + t^2}, \frac{2at^3}{1 + t^2} \right), \quad t \in \mathbb{R},$$

   is the cissoid of Diocles ($t = \tan\theta$).
2. The origin $(0, 0)$ is a singular point of the cissoid.
3. As $t \to \infty$, $\alpha(t)$ approaches the line $x = 2a$, and $\alpha'(t) \to (0, 2a)$. Thus, as $t \to \infty$, the curve and its tangent approach the line $x = 2a$; we say that $x = 2a$ is an *asymptote* to the cissoid.

Proof: For subtask 1, let $t = \tan\theta$. Then $OC = OA\cos\theta = 2a\cos\theta$, and $OB = 2a\sec\theta$. Then $Op = CB = OB - OC = 2a(\sec\theta - \cos\theta)$. Thus,

$$\alpha(t) = (Op\cos\theta, Op\sin\theta) = (2a\sin^2\theta, 2a(\tan\theta - \sin\theta\cos\theta)) = \left(\frac{2at^2}{1 + t^2}, \frac{2at^3}{1 + t^2} \right).$$

For subtask 2, we calculate the derivative of $\alpha(t)$.

$$\alpha'(t) = \left(-\frac{2t}{(1+t^2)^2}, -\frac{2t^2}{(1+t^2)^2} + \frac{2at^2}{1+t^2}\right).$$

It is obvious that $\alpha(0) = (0,0)$ and $\alpha'(0) = 0$, i.e. the origin is a singular point.

For subtask 3, we let $t \to \infty$, and the statement is trivial.

Task 1.3.4: Let $\alpha : (0, \pi) \to \mathbb{R}^2$ be given by

$$\alpha(t) = \left(\sin t, \cos t + \log \tan \frac{t}{2}\right),$$

where t is the angle that the y axis make with the vector $\alpha'(t)$. The trace of α is called the *tractrix*. Show that

1. α is a differentiable parametrized curve, regular except at $t = \pi/2$.
2. The length of the segment of the tangent of the tractrix between the point of tangency and the y axis is constantly equal to 1.

Proof: For subtask 1, we can simply differentiate the α.

$$\alpha'(t) = (\cos t, -\sin t + \csc t), \quad t \in (0, \pi).$$

Hence $t = \pi/2$ is the only point such taht $\alpha'(t) = 0$.

For subtask 2, we consider the geometry intuitive, and can infer that the length of that segment is equal to the x value of $\alpha(t)$ times $\sec t$, i.e. $\cos t \cdot \sec t = 1$.

Task 1.3.5: Let $\alpha : (-1, +\infty) \to \mathbb{R}$ be given bt

$$\alpha(t) = \left(\frac{3at}{1+t^3}, \frac{3at^2}{1+t^3}\right).$$

Prove that:

1. For $t = 0$, α is tangent to the x axis.
2. As $t \to +\infty$, $\alpha(t) \to (0,0)$ and $\alpha'(t) \to (0,0)$.
3. Take the curve with the opposite orientation. Now, as $t \to -1$, the curve and its tangent approach the line $x + y + a = 0$.

The figure obtained by completing the trace of α in such a way that it becomes symmetric relative to the line $y = x$ is called the *folium of Descartes*.

Proof: For subtask 1, we first calculate

$$\alpha'(t) = \left(\frac{3a(-2t^3+1)}{(1+t^3)^2}, \frac{3at(-t^3+2)}{(1+t^3)^2}\right)$$

and hence $\alpha(0) = (0,0)$, $\alpha'(0) = (3a, 0)$.

For subtask 2, let $t \to +\infty$ and the statement is trivial.

For subtask 3, now

$$\alpha(t) = \left(-\frac{3at}{1-t^3}, \frac{3at^2}{1-t^3}\right), \quad t \in (-\infty, 1),$$
$$\alpha'(t) = \left(\frac{3a(1-2t^3)}{(1-t^3)^2}, \frac{3at(t^3+2)}{(1-t^3)^2}\right).$$

As $t \to 1$, we have

$$x + y + a = -\frac{3at}{1+t+t^2} + a \to 0,$$
$$\frac{y'}{x'} = \frac{t(t^3+2)}{1-2t^3} \to -1.$$

Task 1.3.6: Let $\alpha(t) = \left(ae^{bt}\cos t, ae^{bt}\sin t\right), t \in \mathbb{R}$, a and b constants, $a > 0$, $b < 0$, be a parametrized curve.

1. Show that as $t \to +\infty$, $\alpha(t)$ approaches the origin O, spiraling around it (because of this, the trace of α is called the *logarithmic spiral*).
2. Show that $\alpha'(t) \to (0,0)$ as $t \to +\infty$ and that

$$\lim_{t\to+\infty} \int_{t_0}^{t} |\alpha'(t)| \,\mathrm{d}t$$

is finite; that is, α has finite ar length in $[t_0, +\infty)$.

Proof: For subtask 1, as $t \to +\infty$, $|\alpha(t)| = ae^{bt} \to 0$.

For subtask 2,

$$\alpha'(t) = \left(ae^{bt}(b\cos t - \sin t), ae^{bt}(b\sin t + \cos t)\right).$$

As $t \to +\infty$, we can see that $\alpha'(t) \to (0,0)$. And we have

$$|\alpha'(t)| = a(b^2+1)e^{bt},$$
$$\int_{t_0}^{+\infty} |\alpha'(t)| \,\mathrm{d}t = a(b^2+1)\int_{t_0}^{+\infty} e^{bt} \,\mathrm{d}t = \frac{a(b^2+1)e^{bt_0}}{b} < +\infty.$$

1.4 | The Vector Product in $\mathbb{R}^3$

Task 1.4.1: Check whether the following bases are positive:

1. The basis $\{(1,3),(4,2)\}$ in $\mathbb{R}^2$.
2. The bases $\{(1,3,5),(2,3,7),(4,8,3)\}$ in $\mathbb{R}^3$.

Solution: For subtask 1,

$$\begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix} = -10 < 0.$$

For subtask 2,

$$\begin{vmatrix} 1 & 3 & 5 \\ 2 & 3 & 7 \\ 4 & 8 & 3 \end{vmatrix} = 39 > 0.$$

Task 1.4.2: A plane P contained in $\mathbb{R}^3$ is given by the equation $ax + by + cz + d = 0$. Show that the vector $v = (a, b, c)$ is perpendicular to the plane and that $|d| / \sqrt{a^2 + b^2 + c^2}$ measures the distance from the plane to the origin $(0, 0, 0)$.

Proof: Consider two points $(x_1, y_1, z_1), (x_2, y_2, z_2)$ on the plane. Then the vector in plane $u = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$ holds that $(a, b, c) \cdot u = v \cdot u = 0$, i.e. v is perpendicular to any vector u parallel to plane.

Consider $w = kv$ is on the plane. That is, $k(a^2 + b^2 + c^2) + d = 0$. Then we have

$$|w| = |k||v| = \frac{|d|}{a^2 + b^2 + c^2} \cdot \sqrt{a^2 + b^2 + c^2} = \frac{|d|}{\sqrt{a^2 + b^2 + c^2}}.$$

Task 1.4.3: Determine the angle of intersection of the two planes $5x + 3y + 2z - 4 = 0$ and $3x + 4y - 7z = 0$.

Solution:

$$\cos\theta = \frac{u \cdot v}{|u||v|} = \frac{13}{2\sqrt{19}\sqrt{39}}.$$

Task 1.4.4:

Task 1.4.5:

Task 1.4.6:

Task 1.4.7:

Task 1.4.8: Prove that the distance ρ between the nonparrallel lines

$$\begin{aligned} x - x_0 &= u_1 t, \quad y - y_0 = u_2 t, \quad z - z_0 = u_3 t, \\ x - x_1 &= v_1 t, \quad y - y_1 = v_2 t, \quad z - z_1 = v_3 t \end{aligned}$$

is given by

$$\rho = \frac{|(u \times v) \cdot r|}{|u \times v|},$$

where $u = (u_1, u_2, u_3), v = (v_1, v_2, v_3), r = (x_0 - x_1, y_0 - y_1, z_0 - z_1)$.

Proof: $u \times v$ is perpendicular to u and v. Span the plane that is perpendicular to $u \times v$ from (x_0, y_0, z_0) and (x_1, y_1, z_1) respectively. Then what we need to calculate is the distance between two planes, and statement is trivial.

Task 1.4.9:

Task 1.4.10:

Task 1.4.11:

Task 1.4.12:

Task 1.4.13:

Task 1.4.14:

1.5 | The Local Theorey of Curves Parametrized by Arc Length