Harmonic Measure TD7

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Exercise 1: Let $\Omega \subset \mathbb{R}^d$ be an open set and $f : \Omega \to \overline{\mathbb{R}}$. Show that

1. f is continuous iff it is usc and lsc.
2. f is lsc iff for all $x \in \Omega$ and all $\varepsilon > 0$, there exists $\eta > 0$ such that
$$\|x - y\| < \eta \Longrightarrow f(x) - f(y) < \varepsilon.$$
3. f is usc iff for all $x \in \Omega$ and all $\varepsilon > 0$, there exists $\eta > 0$ such that
$$\|x - y\| < \eta \Longrightarrow f(x) - f(y) > -\varepsilon.$$
4. f is lsc iff the epigraph $\{(x,t) \mid t \geqslant f(x)\}$ is closed in $\Omega \times \mathbb{R}$.
5. f is usc iff the hypograph $\{(x,t) \mid t \leqslant f(x)\}$ is closed in $\Omega \times \mathbb{R}$.
6. If f is lsc on a compact set, then there exists an increasing sequence of continuous functions (φ_j) converging to f.
7. If f is usc on a compact set, then there exists a decreasing sequence of continuous functions (φ_j) converging to f.

Proof:

1. $$f(x) = \lim_{y \to x} f(y) \Longleftrightarrow f(x) \leqslant \liminf_{y \to x} f(y) \leqslant \limsup_{y \to x} f(y) \leqslant f(x).$$
2. *Just apply the definition of* $\liminf_{y \to x} f(y)$.
3. *Omit.*
4. *Pick a sequence* $\{(x_n, t_n)\} \to (x,t)$ *such that* $t_n \geqslant f(x_n)$ *holds for all* n, *we need to prove* $t \geqslant f(x)$. *It is because* $t = \lim t_n \geqslant \liminf f(x_n) \overset{\text{lsc}}{\geqslant} f(x)$.
5. *Omit.*
6. $$\liminf_{y \to x} f(y) \geqslant \liminf_{y \to x} \varphi_j(y) = \lim_{y \to x} \varphi_j(y) = \varphi_j(x)$$
holds for all $j \in \mathbb{N}^+$, *hence* $\liminf_{y \to x} f(y) \geqslant f(x)$.
7. *Omit.*

□

Exercise 2: Show that the collection of usc and the collection of lsc functions on an open set Ω are stable by addition and by multiplication by a positive scalar.

Proof: *Suppose* u, v *are lsc on domain* Ω. *Then*

$$\liminf_{y \to x}(u(y) + v(y)) \geqslant \liminf u(y) + \liminf v(y) \geqslant u(x) + v(x).$$

Let $c > 0$ *and*

$$\liminf_{y \to x} cu(y) = c \liminf_{y \to x} u(y) \geqslant cu(x).$$

Cases of usc are similar. □

Exercise 3: Let $A \subset \mathbb{R}^d$ and $\mathbb{1}_A$ its characteristic function.

1. $\mathbb{1}_A$ is lsc if and only if A is open.
2. $\mathbb{1}_A$ is usc if and only if A is closed.

Proof: *Only prove 1.*

$$\forall x \in A, 1 \geqslant \liminf_{y \to x} \mathbb{1}_A(y) \geqslant \mathbb{1}_A(x) = 1 \Longleftrightarrow y \in A, \forall y \in B(x, \delta).$$ □

Exercise 4: Let μ be a non-negative regular positive measure on $\mathbb{R}^d$ and $f \in L^1(\mu)$. Then

$$\int_{\partial\Omega} f \,\mathrm{d}\mu = \inf\left\{\int_{\partial\Omega} h \,\mathrm{d}\mu \,\middle|\, h \text{ lsc on } \partial\Omega, f \leqslant h\right\}$$

and

$$\int_{\partial\Omega} f \,\mathrm{d}\mu = \sup\left\{\int_{\partial\Omega} h \,\mathrm{d}\mu \,\middle|\, h \text{ usc on } \partial\Omega, f \geqslant h\right\}.$$

Exercise 5: If u and v are superharmonic in the domain $\Omega \subset \mathbb{R}^d$ and $u(x) = v(x)$ for almost every $x \in \Omega$, then u and v are identically equal in Ω. Same if u, v are subharmonic.

Proof: *Suppose not, and we omit the cases that functions reach $\pm\infty$. Pick $x_0 \in \Omega$ s.t. $u(x_0) \neq v(x_0)$. Without lost of generality, we can assume that $u(x_0) > v(x_0)$. Set m such that $v(x_0) < m < u(x_0)$. Since u, v are lsc and $u = v$ a.e., $A = \{x \mid u(x) \geqslant m\}, B = \{x \mid v(x) \geqslant m\}$ are closed, and $\lambda(A \Delta B) = 0$. Observed that $x_0 \in A$, we can deduce that there exists $\{x_n \in A \cap B\}$ s.t. $x_n \to x_0$, then $x_0 \in B$, which causes a contradiction.* □

Exercise 6: Suppose that u is lower-semicontinuous in the domain $\Omega \subset \mathbb{R}^d$. Then the following are equivalent:

1. u is superharmonic in Ω.
2. For every $B(x_0, r) \subset \Omega$ and every v harmonic in $B(x_0, r)$,

$$\liminf_{B(x_0,r) \ni y \to x} (u(y) - v(y)) \geqslant 0 \quad \forall x \in \partial B(x_0, r) \Longrightarrow u \geqslant v \ \text{ in } \ B(x_0, r).$$

Give the analogous assertion for subharmonic functions.

Proof: $1 \Longrightarrow 2$. *Suppose u superharmonic. Set $B(x_0, r) \subset \Omega$ and v harmonic in $B(x_0, r)$, then $u - v$ is superharmonic in $B(x_0, r)$. Thus $u - v \geqslant 0$ can be deduced according to the Maximum Principle of superharmonic function.*

$2 \Longrightarrow 1$. *Set $B(x, r) \subset \Omega$, our goal is to prove $u(x) \geqslant \fint_{\partial B(x,r)} u(y) \,\mathrm{d}y$. Since u is lsc, there exists a sequence of continuous function defined on ∂B such that $f_n \uparrow u|_{\partial B}$. Thus*

$$v_n = \mathrm{PI}(f_n, B) \uparrow \mathrm{PI}(u|_{\partial B}, B) = v \quad \text{for all } n,$$

where PI *stands for Possion Integral. Since* v_n *are harmonic for all* n*,* v *is either harmonic or explicitly* $+\infty$*.* u, v_n *suit the condition, and* $u \geqslant v_n$ *in* $B(x,r)$ *holds for all* n*. So* $u \geqslant v$ *in* $B(x,r)$*. Hence*

$$u(x) \geqslant v(x) = \fint_{\partial B(x,r)} v(y)\,\mathrm{d}y = \fint_{\partial B(x,r)} u(y)\,\mathrm{d}y$$

and the result follows. □

Exercise 7: Prove the equality

$$\fint_{B(x,r)} f(y)\,\mathrm{d}y = \frac{d}{r^d} \int_0^r \left(\fint_{\partial B(x,s)} f(\zeta)\,\mathrm{d}\sigma(\zeta) \right) s^{d-1}\,\mathrm{d}s,$$

for f measurable and positive in $\mathbb{R}^d$.

Proof: *Set* $y = \zeta s$*, where* $s \in (0,r)$ *and* $\zeta \in \partial B(x,s)$*. Then* $\mathrm{d}y = s^{d-1}\,\mathrm{d}\sigma(\zeta)\,\mathrm{d}s$*. Then*

$$\begin{aligned}
&\fint_{B(x,r)} f(y)\,\mathrm{d}y \\
&= \frac{d}{\kappa_d r^d} \int_{B(x,r)} f(y)\,\mathrm{d}y \\
&= \frac{d}{\kappa_d r^d} \int_0^r \left(\int_{\partial B(x,s)} f(\zeta)\,\mathrm{d}\sigma(\zeta) \right) s^{d-1}\,\mathrm{d}s \\
&= \frac{d}{r^d} \int_0^r \left(\fint_{\partial B(x,s)} f(\zeta)\,\mathrm{d}\sigma(\zeta) \right) s^{d-1}\,\mathrm{d}s,
\end{aligned}$$

where κ_d *stands for the measure of surface of unit ball in* $\mathbb{R}^d$*.* □

Exercise 8: Suppose that u is lower-semicontinuous in the domain $\Omega \subset \mathbb{R}^d$. Then u is superharmonic in Ω if and only if, for every $x \in \Omega$ and all $r < d(x, \partial\Omega)$,

$$\fint_{\partial B(x,r)} u(\zeta)\,\mathrm{d}\sigma(\zeta) \leqslant u(x).$$

Proof: *First prove sufficiency, i.e. to prove* u *is superharmonic. It is relatively trivial since*

$$\begin{aligned}
&\fint_{B(x,r)} u(\zeta)\,\mathrm{d}\sigma(\zeta) \\
&= \frac{d}{r^d} \int_0^r \left(\fint_{\partial B(x,s)} f(\zeta)\,\mathrm{d}\sigma(\zeta) \right) s^{d-1}\,\mathrm{d}s \\
&\leqslant u(x) \cdot \frac{d}{r^d} \int_0^r s^{d-1}\,\mathrm{d}s \\
&= u(x).
\end{aligned}$$

Second prove necessity. Assume u be superharmonic. Suppose not, i.e. there exists $x_0 \in \Omega$ and $B(x_0, r_0) \subset \Omega$ such that

$$\fint_{\partial B(x_0,r_0)} u(\zeta)\,\mathrm{d}\sigma(\zeta) > u(x_0).$$

Set function

$$v(x) = \begin{cases} \mathrm{PI}(u, B(x_0, r_0))(x), & x \in B(x_0, r_0) \\ u(x), & \text{otherwise in } \Omega \end{cases}$$

where $\mathrm{PI}(u, B(x_0, r_0))(x)$ stands for Possion Integral of u over $B(x_0, r_0)$. Hence v is a harmonic lift of u, and is superharmonic in Ω, and harmonic in $B(x_0, r_0)$. Thus

$$\begin{aligned} v(x_0) = \fint_{B(x_0,r_0)} v(\zeta)\,\mathrm{d}\zeta \leqslant \fint_{B(x_0,r_0)} u(\zeta)\,\mathrm{d}\zeta \leqslant u(x_0) < \fint_{\partial B(x_0,r_0)} u(\zeta)\,\mathrm{d}\sigma(\zeta) \\ = \fint_{\partial B(x_0,r_0)} v(\zeta)\,\mathrm{d}\sigma(\zeta), \end{aligned}$$

For all $r > r_0$, we have

$$\begin{aligned} v(x_0) &\geqslant \fint_{B(x_0,r)} v(\zeta)\,\mathrm{d}\zeta \\ &= \frac{d}{r^d}\int_0^r s^{d-1}\left(\fint_{\partial B(x_0,s)} v(\zeta)\,\mathrm{d}\sigma(\zeta)\right)\mathrm{d}s \\ &= \frac{d}{r^d}\int_{r_0}^r s^{d-1}\left(\fint_{\partial B(x_0,s)} v(\zeta)\,\mathrm{d}\sigma(\zeta)\right)\mathrm{d}s \quad (\text{since } v \text{ is harmonic in } B(x_0, r_0)). \end{aligned}$$

Let

$$\varphi(r) = \frac{d}{r^d}\int_{r_0}^r s^{d-1}\left(\fint_{\partial B(x_0,s)} v(\zeta)\,\mathrm{d}\sigma(\zeta)\right)\mathrm{d}s,$$

and it is easy to observe that $\varphi(r)$ is continuous, and $\varphi(r_0) = 0, \varphi'(r_0) < 0$. Hence there exists r, which is very near to r_0, such that $\varphi(r_0) < 0$, and

$$v(x_0) > \frac{d}{r^d}\int_{r_0}^r s^{d-1} v(x_0)\,\mathrm{d}s = v(x_0),$$

which causes contradiction. □