Mathematics Courses
Tasks Answers Series

Differential Geometry

of Curves & Surfaces,

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Task 1.2.1: Find a parametrized curve $\alpha(t)$ whose trace is the circle $x^2 + y^2 = 1$ such that $\alpha(t)$ runs clockwise around the circle with $\alpha(0) = (0, 1)$.

Solution: $\alpha(t) = (-\sin t, \cos t)$ follows the condition.

Task 1.2.2: Let $\alpha(t)$ be a parametrized curve which does not pass through the origin. If $\alpha(t_0)$ is a point of the trace of α closest to the origin and $\alpha'(t_0) \neq 0$, show that the position vector $\alpha(t_0)$ is orthogonal to $\alpha'(t_0)$.

Proof: Since $\alpha(t)$ is parametrized curve, $|\alpha(t)| = \sqrt{\alpha(t) \cdot \alpha(t)}$ is differentiable.

$$\begin{aligned}\frac{\mathrm{d}|\alpha|}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\sqrt{\alpha \cdot \alpha}\right) = \frac{\mathrm{d}\alpha}{\mathrm{d}t}\frac{1}{2\sqrt{\alpha \cdot \alpha}} \cdot 2(\alpha' \cdot \alpha) \\ &= \frac{\alpha' \cdot \alpha}{|\alpha|}\alpha'\end{aligned}$$

Since $\alpha(t_0)$ is the minimal value of α, we know that

$$\left.\frac{\mathrm{d}|\alpha|}{\mathrm{d}t}\right|_{t=t_0} = \frac{\alpha'(t_0) \cdot \alpha(t_0)}{|\alpha(t_0)|}\alpha'(t_0) = 0.$$

Since $\alpha'(t_0) \neq 0$, we can figure out that $\alpha'(t_0) \cdot \alpha(t_0) = 0$, i.e. $\alpha(t_0)$ is orthogonal to $\alpha'(t_0)$.

Task 1.2.3: A parametrized curve $\alpha(t)$ has the property that its second derivatie $a''(t)$ is identically zero. What can be said about α?

Solution: $\alpha''(t) = (x''(t), y''(t), z''(t)) \equiv 0$, so we can denote by calculus that

$$\alpha(t) = (x_0 + x_1 t, y_0 + y_1 t, z_0 + z_1 t).$$

Task 1.2.4: Let $\alpha : I \to \mathbb{R}^3$ be a parametrized curve and let $v \in \mathbb{R}^3$ be a fixed vector. Assume that $\alpha'(t)$ is orthogonal to v for all $t \in I$ and that $\alpha(0)$ is also orthogonal to v. Prove that $\alpha(t)$ is orthogonal to v for all $t \in I$.

Proof: According to condition, $\alpha'(t) \cdot v = 0$ holds for all $t \in I$. Apply integral to both side and we have $\alpha(t) \cdot v \equiv C_0$, where C_0 is constant. Plug $\alpha(0) = 0$ we can infer that $C_0 = 0$, thus $\alpha(t) \cdot v \equiv 0$.

Task 1.2.5: Let $\alpha : I \to \mathbb{R}^3$ be a parametrized curve, with $\alpha'(t) \neq 0$ for all $t \in I$. Show that $|\alpha(t)|$ is a nonzero constant if and only if $\alpha(t)$ is orthogonal to $\alpha'(t)$ for all $t \in I$.

Proof: $\Longrightarrow$ Denote $|\alpha(t)| = C$. Then we have

$$\frac{\mathrm{d}|\alpha|}{\mathrm{d}t} = \frac{\alpha' \cdot \alpha}{|\alpha|}\alpha' = 0.$$

Plug in that $\alpha' \neq 0$ we know $\alpha' \cdot \alpha = 0$ for all $t \in I$.

$\Longleftarrow$ Since α is orthogonal to α', $\alpha \cdot \alpha' = 0$ always holds. Then $\frac{\mathrm{d}|\alpha|}{\mathrm{d}t} = \frac{\alpha' \cdot \alpha}{|\alpha|}\alpha' = 0$ holds. That is $|\alpha| = C$.

1.3 | Regular Curves; Arc Length

Task 1.3.1: Show that the tangent lines to the regular parameterized curve $\alpha(t) = (3t, 3t^2, 2t^3)$ make a constant angle with the line $y = 0, z = x$.

Proof: The line $y = 0, z = x$ can be denoted as the direction of vector $v = (1, 0, 1)$. And we know that $\alpha'(t) = (3, 6t, 6t^2)$. Thus we have

$$\frac{\alpha'(t) \cdot v}{|\alpha'(t)|} = \frac{6t^2 + 3}{\sqrt{36t^4 + 36t^2 + 9}} = 1$$

is a constant.

Task 1.3.2: A circular disk of radius 1 in the plane xy rolls without slipping along the x axis. The figure described by a point of the circumference of the disk is called a cycloid.

1. Obtain a parametrized curve $\alpha : \mathbb{R} \to \mathbb{R}^2$ the trace of which is the cycloid, and determine its singular points.
2. Compute the arc length of the cycloid correpoinding to a complete rotation of the disk.

Solution: For subtask 1, we consider $\alpha(t) = (t - \sin t, 1 - \cos t), \quad t \in \mathbb{R}$. $\alpha'(t) = (1 - \cos t, \sin t)$. When $\alpha'(t) = 0$, we can find singular point $t = 2k\pi, k \in \mathbb{Z}$.

For subtask 2, consider $t \in [0, 2\pi]$.

$$\begin{aligned} s &= \int_0^{2\pi} |\alpha'(t)| \,\mathrm{d}t = \int_0^{2\pi} \sqrt{1 + 2\cos t + \cos^2 t + \sin^2 t} \,\mathrm{d}t \\ &= \int_0^{2\pi} \sqrt{2 + 2\cos t} = 8. \end{aligned}$$

Task 1.3.3: Let $OA = 2a$ be the diameter of a circle S^1 and Oy and AV be the tangents to S^1 at O and A, respectively. A half-line r is drawn from O which meets the circle S^1 at C and line AV at B. On OB mark off the segment $Op = CB$. If we rotate r about O, the point p will describe a curve called the *cissoid of Diocles*. By taking OA as the x axis and OY as the y axis, prove that

1. The trace of
$$\alpha(t) = \left(\frac{2at^2}{1 + t^2}, \frac{2at^3}{1 + t^2}\right), \quad t \in \mathbb{R},$$
is the cissoid of Diocles ($t = \tan\theta$).
2. The origin $(0, 0)$ is a singular point of the cissoid.
3. As $t \to \infty$, $\alpha(t)$ approaches the line $x = 2a$, and $\alpha'(t) \to (0, 2a)$. Thus, as $t \to \infty$, the curve and its tangent approach the line $x = 2a$; we say that $x = 2a$ is an *asymptote* to the cissoid.

Proof: For subtask 1, let $t = \tan\theta$. Then $OC = OA\cos\theta = 2a\cos\theta$, and $OB = 2a\sec\theta$. Then $Op = CB = OB - OC = 2a(\sec\theta - \cos\theta)$. Thus,

$$\alpha(t) = (Op\cos\theta, Op\sin\theta) = (2a\sin^2\theta, 2a(\tan\theta - \sin\theta\cos\theta)) = \left(\frac{2at^2}{1 + t^2}, \frac{2at^3}{1 + t^2}\right).$$

For subtask 2, we calculate the derivative of $\alpha(t)$.

$$\alpha'(t) = \left(-\frac{2t}{\left(1+t^2\right)^2}, -\frac{2t^2}{\left(1+t^2\right)^2} + \frac{2at^2}{1+t^2}\right).$$

It is obvious that $\alpha(0) = (0,0)$ and $\alpha'(0) = 0$, i.e. the origin is a singular point.

For subtask 3, we let $t \to \infty$, and the statement is trivial.