

Mathematics Courses
Tasks Answers Series

Real Analysis: Tasks Answers

Fourth Edition,
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Preliminaries on Sets, Mappings, and Relations

1 | The Real Numbers: Sets, Sequences, and Functions

1.1 | The Field, Positivity, and Completeness Axioms

1.2 | The Natural and Rational Numbers

1.3 | Countable and Uncountable Sets

1.4 | Open Sets, Closed Sets, and Borel Sets of Real Numbers

1.5 | Sequences of Real Numbers

1.6 | Continuous Real-valued Functions of a Real Variable

2 | Lebesgue Measure

2.1 | Intruduction

2.2 | Lebesgue Outer Measure

2.3 | The σ -Algebra of Lebesgue Measurable Sets

2.4 | Outer and Inner Approximation of Lebesgue Measurable Sets

2.5 | Countable Additivity, Continuity, and the Borel-Cantelli Lemma

2.6 | Nonmeasurable Sets

2.7 | The Cantor Set and the Cantor-Lebesgue Function

3 | Lebesgue Measurable Functions

3.1 | Sums, Products, and Compositions

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3.3 | Littlewood's Three Principles, Egoroff's Theorem, and Lusin's Theorem

4 | Lebesgue Integration

4.1 | The Riemann Integral

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4.2 | The Lebesgue Integral of a Bounded Measurable Function over a Set of Finite Measure

4.3 | The Lebesgue Integral of a Measurable Nonnegative Function

4.4 | The General Lebesgue Integral

4.5 | Countable Additivity and Continuity of Integration

4.6 | Uniform Integrability: The Vitali Convergence Theorem

Task 4.40: Let f be integrable over \mathbb{R} . Show that the function F defined by

$$F(x) = \int_{-\infty}^x f \text{ for all } x \in \mathbb{R}$$

is properly defined and continuous. Is it necessarily Lipschitz?

Task 4.41: Show that Proposition 25 is false if $E = \mathbb{R}$.

Task 4.42: Show that Theorem 26 is false without the assumption that the h_n 's are nonnegative.

Proof: 考虑函数列

$$h_{n(x)} = \begin{cases} n, 0 \leq x \leq \frac{1}{2^n} \\ -n, \frac{1}{2^n} < x \leq \frac{1}{2^{n-1}} \\ 0, \text{其他情况} \end{cases}$$

此时 $\lim \int_{[0,1]} h_n = 0$, 但是在 $[0, 1]$ 上不一致可积.

考虑函数列

Task 4.43: Let the sequences of functions $\{h_n\}$ and $\{g_n\}$ be uniformly integrable over E . Show that for any α and β , the sequence of linear combinations $\alpha f_n + \beta g_n$ also is uniformly integrable over E .

Proof: 任取 $\varepsilon > 0$. 设 $\delta_1, \delta_2 > 0$ 满足 $\{h_n\}, \{g_n\}$ 在 $\frac{\varepsilon}{2|\alpha|}, \frac{\varepsilon}{2|\beta|}$ 下对应的一致可积条件. 那么令 $\delta = \min\{\delta_1, \delta_2\}$. 对任意 E 的可测子集 A , 且 $m(A) < \delta$, 则有

$$\begin{aligned}
\int_A |\alpha f_n + \beta g_n| &\leq |\alpha| \int_A |f_n| + |\beta| \int_A |g_n| \\
&= |\alpha| \cdot \frac{\varepsilon}{2|\alpha|} + |\beta| \cdot \frac{\varepsilon}{2|\beta|} \\
&< \varepsilon.
\end{aligned}$$

Task 4.44: Let f be integrable over \mathbb{R} and $\varepsilon > 0$. Establish the following three approximation properties.

1. There is a simple function η on \mathbb{R} which has finite support and $\int_{\mathbb{R}} |f - \eta| < \varepsilon$.
2. There is a step function s on \mathbb{R} which vanished outside a closed, bounded interval and $\int_{\mathbb{R}} |f - s| < \varepsilon$.
3. There is a continuous function g on \mathbb{R} which vanished outside a bounded set and $\int_{\mathbb{R}} |f - g| < \varepsilon$.

Proof: 不妨假设 f 在 \mathbb{R} 上非负, 否则只要对 f 非负和 f 为负的部分分别应用结论即可.

对于第一问, 由定义知道

$$\int_{\mathbb{R}} f = \sup \left\{ \int_{\mathbb{R}} h \mid h \text{ 是有界、可测、支集测度有限的函数, 且 } 0 \leq h \leq f \right\}.$$

那么存在 h_0 满足以上条件且

$$\int_{\mathbb{R}} f - \int_{\mathbb{R}} h_0 < \frac{\varepsilon}{2}.$$

又由 $\int_{\mathbb{R}} h_0$ 定义知道存在简单函数 η 满足

$$\int_{\mathbb{R}} h_0 - \int_{\mathbb{R}} \eta < \frac{\varepsilon}{2}.$$

那么就有

$$\int_{\mathbb{R}} |f - \eta| = \int_{\mathbb{R}} f - \int_{\mathbb{R}} \eta < \varepsilon,$$

其中 η 是简单函数且支集测度有限.

对于第二问, 对于上述 h_0 , 首先由积分的连续性得到

$$\lim_{n \rightarrow \infty} \int_{[-n, n]} h_0 = \int_{\mathbb{R}} h_0.$$

那么存在一个足够大的 n_0 , 使得

$$\int_{(-\infty, -n_0) \cup (n_0, +\infty)} h_0 = \int_{\mathbb{R}} h_0 - \int_{[-n_0, n_0]} h_0 < \frac{\varepsilon}{4}.$$

考虑 $\int_{[-n_0, n_0]} h_0$. 由 Lusin 定理, 存在 $[-n_0, n_0]$ 的子集 F 使得 h_0 在 F 上连续, 那么由 Riemann 可积性知道存在 F 上的 step function s 使得 $s \leq h_0$ 且

$$\int_F h_0 - \int_F s < \frac{\varepsilon}{8}.$$

又注意到 h_0 在 $[-n_0, n_0]$ 上有界, 设 $|h_0| < M$. 那么可以选取上述集合 F 使得 $m([-n_0, n_0] \setminus F) < \frac{\varepsilon}{8M}$ 即可, 在 $\mathbb{R} \setminus F$ 上令 $s = 0$, 此时有

$$\begin{aligned}
\int_{\mathbb{R}} |f - s| &= \int_{\mathbb{R}} f - \int_{\mathbb{R}} s \\
&= \int_{\mathbb{R}} f - \int_{\mathbb{R}} h_0 + \int_{\mathbb{R}} h_0 - \int_{[-n_0, n_0]} h_0 + \int_{[-n_0, n_0]} h_0 - \int_F s \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \int_{[-n_0, n_0] \sim F} h_0 + \int_F h_0 - \int_F s \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + M \cdot \frac{\varepsilon}{8M} + \frac{\varepsilon}{8} \\
&< \varepsilon.
\end{aligned}$$

对于第三问, 考虑上述 h_0, n_0 和 F . 考虑构造连续函数 g , 使得在 F 上 $g = h_0$, 并将 g 延拓到 $[-n_0, n_0]$, 进一步延拓到 \mathbb{R} , 且有 $g([n_0 + \frac{\varepsilon}{8M}, +\infty)) = g((-\infty, -n_0 - \frac{\varepsilon}{8M}]) = \{0\}$. 这时有

$$\int_{\mathbb{R}} |f - g| < \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \int_{[-n_0, n_0] \sim F} (h_0 - g) < \varepsilon.$$

Task 4.45: Let f be integrable over E . Define \hat{f} to be the extension of f to all of \mathbb{R} obtained by setting $\hat{f} \equiv 0$ outside of E . Show that \hat{f} is integrable over \mathbb{R} and $\int_E f = \int_{\mathbb{R}} \hat{f}$. Use this and part (1) and (3) of preceding problem to show that for $\varepsilon > 0$, there is a simple function η on E and a continuous function g on E for which $\int_E |f - \eta| < \varepsilon$ and $\int_E |f - g| < \varepsilon$.

Proof: 显然有

$$\begin{aligned}
\int_{\mathbb{R}} \hat{f} &= \int_E \hat{f} + \int_{\mathbb{R} \sim E} \hat{f} \\
&= \int_E f + \int_{\mathbb{R} \sim E} 0 \\
&= \int_E f.
\end{aligned}$$

然后直接应用上题的结论即可.

Task 4.46 (Riemann-Lebesgue): Let f be integrable over \mathbb{R} . Show that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} f(x) \cos nx \, dx = 0.$$

Proof: $\forall \varepsilon > 0$. 由 44 题的结论知道存在 step function g 仅在有限闭区间 F 上非 0 且 $\int_{\mathbb{R}} |f - g| < \frac{\varepsilon}{2m(F)}$. 又知道

$$\begin{aligned}
\left| \int_{\mathbb{R}} f(x) \cos nx \, dx \right| &\leq \int_{\mathbb{R}} |(f(x) - g(x)) \cos nx \, dx| + \left| \int_{\mathbb{R}} g(x) \cos nx \, dx \right| \\
&\leq \int_{\mathbb{R}} |f(x) - g(x)| |\cos nx \, dx| + \left| \int_{\mathbb{R}} g(x) \cos nx \, dx \right| \\
&\leq \int_{\mathbb{R}} |f(x) - g(x)| \, dx + \left| \int_{\mathbb{R}} g(x) \cos nx \, dx \right| \\
&\leq \frac{\varepsilon}{2} + \left| \int_{\mathbb{R}} g(x) \cos nx \, dx \right|.
\end{aligned}$$

故只要证明 g 是 step function 的情况即可, 且 g 仅在有穷闭区间 F 上非 0. 考虑到 g 在每一段上的取值为常数, 不妨设 $g(x)$ 是常数. 那么此时只要证明

$$\lim_{n \rightarrow \infty} \left| \int_F \cos nx \, dx \right| = 0.$$

而注意到它是 Riemann 可积的, 直接换元积分得到原函数 $\frac{1}{n} \sin nx$, 令 $n \rightarrow \infty$ 得到结果.

Task 4.47: Let f be integrable over \mathbb{R} .

1. Show that for each t ,

$$\int_{\mathbb{R}} f(x) \, dx = \int_{\mathbb{R}} f(x+t) \, dx.$$

2. Let g be a bounded measurable function on \mathbb{R} . Show that

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}} g(x) \cdot [f(x) - f(x+t)] \, dx = 0.$$

Proof: 对于第一问, 容易看出

$$\begin{aligned}
\int_{\mathbb{R}} f(x) \, dx &= \sup \left\{ \int_{\mathbb{R}} h(x) \, dx \mid h \text{ 是有界、可测、支集测度有限的函数, 且 } 0 \leq h \leq f \right\} \\
&= \sup \left\{ \int_{\mathbb{R}} h(x+t) \, dx \mid h \text{ 是有界、可测、支集测度有限的函数, 且 } 0 \leq h \leq f \right\} \\
&= \int_{\mathbb{R}} f(x+t) \, dx.
\end{aligned}$$

对于第二问, 考虑用第 44 题的结论, 用仅在有界闭集合 F 上非零的连续函数 f_1 来近似函数 f , 满足 $\int_{\mathbb{R}} |f - f_1| < \varepsilon$ 对 $\forall \varepsilon > 0$ 成立. 由于 f_1 在闭集 F 上连续, 进而一致连续. 那么对 $\forall \varepsilon > 0$, 记 $\int_{\mathbb{R}} |g| = G < +\infty$, 当 $|t| < \frac{\varepsilon}{2Mm(F)}$ 时有

$$\begin{aligned}
\left| \int_F g(x) \cdot [f_1(x) - f_1(x+t)] \, dx \right| &\leq \int_F M \cdot |f_1(x) - f_1(x+t)| \, dx \\
&\leq \int_F M \cdot \frac{\varepsilon}{2Mm(F)} \, dx \\
&= \varepsilon.
\end{aligned}$$

于是在 f_1 下的命题得证. 对于 f 下的命题, 有

$$\begin{aligned}
\left| \int_{\mathbb{R}} g(x) \cdot [f(x) - f(x+t)] \right| &\leq \int_{\mathbb{R}} M \cdot |f(x) - f(x+t)| \\
&\leq \int_{\mathbb{R}} M \cdot |f(x) - f_1(x)| + \int_{\mathbb{R}} M \cdot |f_1(x) - f_1(x+t)| \\
&\quad + \int_{\mathbb{R}} M \cdot |f(x+t) - f_1(x+t)| \\
&\leq M \cdot \frac{\varepsilon}{3M} + \frac{\varepsilon}{3} + M \cdot \frac{\varepsilon}{3M} \\
&= \varepsilon.
\end{aligned}$$

其中倒数第二行由 f_1 下的命题和用 f_1 近似 f 的结论可以得出. 命题得证.

Task 4.48: Let f be integrable over E and g be a bounded measurable function on E . Show that $f \cdot g$ is integrable over E .

Proof: 设 $|g| < M$. 则有

$$\left| \int_E f \cdot g \right| \leq \int_E |f| \cdot |g| \leq M \int_E |f| < +\infty.$$

Task 4.49: Let f be integrable over \mathbb{R} . Show that the following four assertions are equivalent:

1. $f = 0$ a.e. on \mathbb{R} .
2. $\int_{\mathbb{R}} fg = 0$ for every bounded measurable function g on \mathbb{R} .
3. $\int_A f = 0$ for every measurable set A .
4. $\int_{\mathcal{O}} f = 0$ for every open set \mathcal{O} .

Proof: $1 \Rightarrow 2$: 类似第 48 题可以直接证明.

$2 \Rightarrow 3$: 对任意的可测集 A , 令可测函数 $g = \chi_A$ 即可.

$3 \Rightarrow 4$: 显然.

$4 \Rightarrow 1$: 反证法. 设存在正测度集合 A 使得 $\int_A f \neq 0$, 不妨设 $\int_A f = k > 0$. 构造开集合列 $\{O_n\}$ 使得 $A \subseteq O_n$ 且 $m(O_n \setminus A) < \frac{1}{n}$. 那么令 $O = \bigcap_{n=1}^{\infty} O_n$ 则有 $A \subseteq O$ 且 $m(O) = m(A)$. 此时有

$$\int_O f = \int_A f + \int_{O \setminus A} f = k > 0,$$

矛盾.

Task 4.50: Let \mathcal{F} be a family of functions, each of which is integrable over E . Show that \mathcal{F} is uniformly integrable over E if and only if for each $\varepsilon > 0$, there is a $\delta > 0$ such that for each $f \in \mathcal{F}$,

$$\text{if } A \subseteq E \text{ is measurable and } m(A) < \delta, \text{ then } \left| \int_A f \right| < \varepsilon.$$

Proof: \Rightarrow 由 $\left| \int_A f \right| \leq \int_A |f| < \varepsilon$ 显然.

\Leftarrow 对于 $\forall f \in \mathcal{F}$, 考虑 f^+ 和 f^- 在 A 上定号. 对于任意的可测 $A \subseteq E$ 且 $m(A) < \delta$, 有

$$\begin{aligned}
\int_A |f| &= \int_{A \cap [f \geq 0]} |f| - \int_{A \cap [f \leq 0]} |f| \\
&= \left| \int_{A \cap [f \geq 0]} f \right| + \left| \int_{A \cap [f \leq 0]} f \right| \\
&< \varepsilon.
\end{aligned}$$

最后一步由题目条件得到.

Task 4.51: Let \mathcal{F} be a family of functions, each of which is integrable over E . Show that \mathcal{F} is uniformly integrable over E if and only if for each $\varepsilon > 0$, there is a $\delta > 0$ such that for all $f \in \mathcal{F}$,

$$\text{if } \mathcal{U} \text{ is open and } m(E \cap \mathcal{U}) < \delta, \text{ then } \int_{E \cap \mathcal{U}} |f| < \varepsilon.$$

Proof: \Rightarrow 显然.

\Leftarrow 考虑对任意的 $\varepsilon > 0$ 和条件中对应的 $\delta > 0$, 令 E 的可测子集 A 满足 $m(A) < \delta$. 那么存在开集 \mathcal{U} 包含 A 且 $m(\mathcal{U} \setminus A) < \delta - m(A)$. 那么有

$$m(E \cap \mathcal{U}) = m(A) + m((E \cap \mathcal{U}) \setminus A) \leq m(A) + m(\mathcal{U} \setminus A) < \delta.$$

从而有

$$\int_A |f| \leq \int_{E \cap \mathcal{U}} |f| < \varepsilon.$$

Task 4.52:

1. Let \mathcal{F} be the family of functions f on $[0, 1]$, each of which is integrable over $[0, 1]$ and has $\int_0^1 |f| \leq 1$. Is \mathcal{F} uniformly integrable over $[0, 1]$?
2. Let \mathcal{F} be the family of functions f on $[0, 1]$, each of which is continuous over $[0, 1]$ and has $|f| \leq 1$. Is \mathcal{F} uniformly integrable over $[0, 1]$?
3. Let \mathcal{F} be the family of functions f on $[0, 1]$, each of which is integrable over $[0, 1]$ and has $\int_a^b |f| \leq b - a$ for all $[a, b] \subseteq [0, 1]$. Is \mathcal{F} uniformly integrable over $[0, 1]$?

Proof:

1. 错误. 考虑

$$f_n = \begin{cases} n, & 0 \leq x \leq \frac{1}{n} \\ 0, & \text{其他情况} \end{cases}$$

不是一致可积的.

2. 正确. 令 $\delta = \varepsilon$, 则对任意 $[0, 1]$ 的可测子集 A , 且 $m(A) < \delta$, 有

$$\int_A |f| \leq \int_A 1 = m(A) < \varepsilon.$$

是一致可积的.

3. 正确. 容易看出 $F(x) = \int_0^x f(t) dt$ 是一致连续的.

5 | Lebesgue Integration: Further Topics

5.1 | Uniform Integrability and Tightness: A General Vitali Convergence Theorem

Task 5.1: Prove Corollary 2.

Proof: \Leftarrow 由 Vitali 收敛定理显然.

\Rightarrow 先证紧性. 任取 $\varepsilon > 0$. 由 $\int_E h_n \rightarrow 0$ 可以取一个足够大的自然数 N 使得对 $\forall n > n_0$ 有 $\int_E h_n < \varepsilon$. 那么对于 $n > N$, 可以直接取 $E_n = \emptyset$ 满足

$$\int_{E \sim E_n} |h_n| = \int_E h_n - \int_{E_n} h_n < \varepsilon.$$

对于 $1 \leq n \leq N$ 的每个 h_n , 一定可以找到一个测度有限的集合 $E_n \subseteq E$ 使得

$$\int_{E \sim E_n} h_n < \varepsilon.$$

令 $E_0 = \cup_{n=1}^N E_n$ 仍是测度有限的集合, 那么 E_0 就满足

$$\int_{E \sim E_0} h_n < \int_{E \sim E_n} h_n < \varepsilon \quad \text{对 } \forall n \in \mathbb{N}^* \text{ 成立.}$$

从而 $\{h_n\}$ 是紧的. 对于上述 E_0 , 考虑 $\{h_n\}$ 在 E_0 上的限制, 应用定理 4.26 直接得出 $\{h_n\}$ 在 E_0 上面一致可积. 考虑拓展到 E 上的情况.

对任意 $\varepsilon > 0$, 存在上述 E_0 使得 $\int_{E \sim E_0} h_n < \varepsilon/2$ 对任意 $n \in \mathbb{N}^*$ 成立. 由 $\{h_n\}$ 在 E_0 上一致可积知道存在 $\delta > 0$ 使得对任意 E_0 的测度小于 δ 的子集 A 有 $\int_A |h_n| < \frac{\varepsilon}{2}$. 那么考虑任意 E 上测度小于 δ 的子集 B , 一定有

$$\int_B h_n = \int_{B \cap E_0} h_n + \int_{B \sim E_0} h_n \leq \frac{\varepsilon}{2} + \int_{E \sim E_0} h_n < \varepsilon.$$

从而 $\{h_n\}$ 在 E 上一致可积.

Task 5.2: Let $\{f_k\}_{k=1}^n$ be a finite family of functions, each of which is integrable over E . Show that $\{f_k\}_{k=1}^n$ is uniformly integrable and tight over E .

Proof: 先证 $\{f_k\}_{k=1}^n$ 是紧的. 对于 $1 \leq k \leq n$ 的每个 f_k , 一定可以找到一个测度有限的集合 $E_k \subseteq E$ 使得

$$\int_{E \sim E_k} |f_k| < \varepsilon.$$

令 $E_0 = \cup_{k=1}^n E_k$ 仍是测度有限的集合, 那么 E_0 就满足

$$\int_{E \sim E_0} |f_k| < \int_{E \sim E_k} |f_k| < \varepsilon \quad \text{对 } \forall k = 1, 2, \dots, n \text{ 成立.}$$

从而 $\{f_k\}_{k=1}^n$ 是紧的.

再证 $\{f_k\}_{k=1}^n$ 是一致可积的. 任取 $\varepsilon > 0$, 由于 $\{f_k\}$ 是紧的, 存在 E 的有限测度子集 E_0 满足

$$\int_{E \sim E_0} |f_k| < \frac{\varepsilon}{2}.$$

对于每一个 $1 \leq k \leq n$, 存在 δ_k 满足对于 E_0 的任意子集 A 且 $m(A) < \delta_k$, 一定有

$$\int_A |f_k| < \frac{\varepsilon}{2}.$$

令 $\delta = \min\{\delta_1, \delta_2, \dots, \delta_n\}$, 则对于任意 E 的子集 A 且 $m(A) < \delta$, 一定有

$$\int_A |f_k| = \int_{A \cap E_0} |f_k| + \int_{A \sim E_0} |f_k| < \frac{\varepsilon}{2} + \int_{E \sim E_0} |f_k| < \varepsilon.$$

从而 $\{f_k\}$ 是一致可积的.

Task 5.3: Let the sequences of functions $\{h_n\}$ and $\{g_n\}$ be uniformly integrable and tight over E . Show that for any α and β , $\{\alpha f_n + \beta g_n\}$ also is uniformly integrable and tight over E .

Proof: 不妨设 $\alpha, \beta \neq 0$.

先证 $\{\alpha f_n + \beta g_n\}$ 是紧的. 对于 $\forall \varepsilon > 0$, 一定可以找到测度有限的集合 $E_1, E_2 \subseteq E$ 使得

$$\int_{E \sim E_1} |f| < \frac{\varepsilon}{2|\alpha|} \quad \text{对 } \forall f \in \{f_n\} \text{ 成立}$$

且

$$\int_{E \sim E_2} |g| < \frac{\varepsilon}{2|\beta|} \quad \text{对 } \forall g \in \{g_n\} \text{ 成立}.$$

令 $E_0 = E_1 \cup E_2$ 仍是测度有限的集合, 那么 E_0 同时满足上式中 E_1, E_2 的性质, 此时有

$$\begin{aligned} \int_{E \sim E_0} |\alpha f + \beta g| &\leq |\alpha| \int_{E \sim E_0} |f| + |\beta| \int_{E \sim E_0} |g| \\ &< |\alpha| \cdot \frac{\varepsilon}{2|\alpha|} + |\beta| \cdot \frac{\varepsilon}{2|\beta|} \\ &= \varepsilon \quad \text{对 } \forall f \in \{f_n\}, g \in \{g_n\} \text{ 成立}. \end{aligned}$$

从而 $\{\alpha f_n + \beta g_n\}$ 是紧的.

再证 $\{\alpha f_n + \beta g_n\}$ 是一致可积的. 任取 $\varepsilon > 0$, 由于 $\{\alpha f_n + \beta g_n\}$ 是紧的, 存在 E 的有限测度子集 E_0 满足

$$\int_{E \sim E_0} (\alpha f_n + \beta g_n) < \frac{\varepsilon}{2}.$$

对于任意的 $\alpha f + \beta g \in \{\alpha f_n + \beta g_n\}$, 存在 δ_1, δ_2 满足对于 E_0 的任意子集 A_1, A_2 且 $m(A_1) < \delta_1, m(A_2) < \delta_2$, 一定有

$$\int_{A_1} |f| < \frac{\varepsilon}{4|\alpha|} \quad \text{且} \quad \int_{A_2} |g| < \frac{\varepsilon}{4|\beta|}$$

令 $\delta = \min\{\delta_1, \delta_2\}$ 仍满足上式中 δ_1, δ_2 性质. 则对于任意 E 的子集 A 且 $m(A) < \delta$, 一定有

$$\begin{aligned}
\int_A |\alpha f_n + \beta g_n| &\leq |\alpha| \int_{A \cap E_0} |f| + |\beta| \int_{A \cap E_0} |g| + \int_{A \sim E_0} |\alpha f_n + \beta g_n| \\
&< |\alpha| \cdot \frac{\varepsilon}{4|\alpha|} + |\beta| \cdot \frac{\varepsilon}{4|\beta|} + \frac{\varepsilon}{2} \\
&= \varepsilon \quad \text{对 } \forall f \in \{f_n\}, g \in \{g_n\} \text{ 成立.}
\end{aligned}$$

从而 $\{\alpha f_n + \beta g_n\}$ 是一致可积的.

Task 5.4: Let $\{f_n\}$ be a sequence of measurable functions on E . Show that $\{f_n\}$ is uniformly integrable and tight over E if and only if for each $\varepsilon > 0$, there is a measurable subset E_0 of E that has finite measure and a $\delta > 0$ such that for each measurable subset A of E and index n ,

$$\text{if } m(A \cap E_0) < \delta, \text{ then } \int_A |f_n| < \varepsilon.$$

Proof: \Rightarrow 任取 $\varepsilon > 0$, 则由紧性知道存在有限测度集合 $E_0 \subseteq E$ 使得

$$\int_{E \sim E_0} |f| < \frac{\varepsilon}{2} \quad \text{对 } \forall f \in \{f_n\} \text{ 成立.}$$

又由于 E_0 是有限测度的, 从而应用一致可积性知道在 E_0 上存在 $\delta > 0$ 使得对任意 $A_0 \subseteq E_0$ 且 $m(A_0) < \delta$, 有

$$\int_{A_0} |f_n| < \frac{\varepsilon}{2}.$$

从而对任意的 E 上的子集 A , 若满足 $m(A \cap E_0) < \delta$, 则有

$$\int_A |f_n| = \int_{A \cap E_0} |f_n| + \int_{A \sim E_0} |f_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

\Leftarrow 由题意条件, 令 $A = E \sim E_0$ 得到

$$\int_{E \sim E_0} |f| < \varepsilon.$$

再令 A 为任意 E_0 的子集, 且满足 $m(A) < \delta$ 得到

$$\int_A |f| < \varepsilon.$$

从而直接得出紧性和一致可积性.

Task 5.5: Let $\{f_n\}$ be a sequence of integrable functions on \mathbb{R} . Show that $\{f_n\}$ is uniformly integrable and tight over \mathbb{R} if and only if for each $\varepsilon > 0$, there are positive numbers r and δ such that for each open subset \mathcal{O} of \mathbb{R} and index n ,

$$\text{if } m(\mathcal{O} \cap (-r, r)) < \delta, \text{ then } \int_{\mathcal{O}} |f_n| < \varepsilon.$$

Proof: \Rightarrow 类似上一题可以证明.

\Leftarrow 任取 $\varepsilon > 0$. 令 $E_0 = (-r, r)$ 且 $\mathcal{O} = \mathbb{R} \sim (-r, r)$. 则有 $m(\mathcal{O} \cap (-r, r)) = m(\emptyset) < \delta$ 一定成立. 此时成立

$$\int_{E \sim E_0} |f| < \varepsilon \quad \text{对 } \forall f \in \{f_n\} \text{ 成立.}$$

从而 $\{f_n\}$ 是紧的.

任取 $\varepsilon > 0$, 则存在题目条件中的 $\delta > 0$ 和 $E_0 = (-r, r)$ 使对应条件成立. 考虑任意 $(-r, r)$ 中的子集 A , 满足 $m(A) < \delta$. 则一定存在开集 \mathcal{O} 使得 $A \subseteq \mathcal{O}$ 且 $m(\mathcal{O} \sim A) < \delta - m(A)$. 那么这时有

$$m(\mathcal{O} \cap (-r, r)) < m(\mathcal{O}) = m(\mathcal{O} \cap A) + m(\mathcal{O} \sim A) < m(A) + \delta - m(A) = \delta.$$

从而应用题目条件得到

$$\int_A |f| \leq \int_{\mathcal{O}} |f| < \varepsilon \quad \text{对 } \forall f \in \{f_n\} \text{ 成立,}$$

从而 $\{f_n\}$ 是一致可积的.

5.2 | Convergence in Measure

5.3 | Chracterizations of Riemann and Lebesgue Integrability

Task 5.6: Let f and g be bounded functions that are Riemann integrable over $[a, b]$. Show that the product fg also is Riemann integrable over $[a, b]$.

Task 5.7: Let f be a bounded function on $[a, b]$ whose set of discontinuities has measure zero. Show that f is measurable. Then show that the same holds without the assumption of boundedness.

Task 5.8: Let f be a function on $[0, 1]$ that is continuous on $(0, 1]$. Show that it is possible for the sequence $\left\{ \int_{[1/n, 1]} f \right\}$ to converge and yet f is not Lebesgue integrable over $[0, 1]$. Can this happen if f is nonnegative?