Harmonic Measure TD5

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Exercise 1: Let X be a set and $A \subseteq X$. Two sets $E, F \subseteq X$ are said to be *separated by A* if $E \subseteq A$ and $F \subseteq X \setminus A$, or if $F \subseteq A$ and $E \subseteq X \setminus A$. Let μ be a measure on X.

1. Show that A is μ-measurable if and only if for all sets $E, F \subseteq X$ separated by A, we have $\mu(E \cup F) = \mu(E) + \mu(F)$.
2. Show that A is μ-measurable if for all sets $E, F \subseteq X$ separated by A and such that $\mu(E)$ and $\mu(F)$ are finite, we have $\mu(E \cup F) \geqslant \mu(E) + \mu(F)$.
3. Give an example of a measure μ on X such that all subsets of X are μ-measurable.

Proof:

1. $m(E \cup F) = \mu((E \cup F) \cap A) + \mu((E \cup F) \cap A^c) = \mu(E) + \mu(F)$.

2. *Since* $\mu(E) + \mu(F) \leqslant \mu(E \cup F)$ *we can confirm.*

3. $\mu(A) = 0$ *for all* $A \subset X$.

□

Exercise 2: Let X be a set. A *premeasure* on X is a function $\tau : \mathscr{C} \to [0, \infty]$ defined on a collection $\mathscr{C}$ of subsets of X, such that $\varnothing \in \mathscr{C}$ and $\tau(\varnothing) = 0$. For any set $A \subseteq X$, define

$$\mu(A) = \inf\left\{ \sum_{i=1}^{\infty} \tau(C_i) \mid C_i \in \mathscr{C}, A \subseteq \bigcup_{i=1}^{\infty} C_i \right\}.$$

1. Show that μ is a measure on X.
2. Show that any measure ν on X is also a premeasure on X. Show that the measure μ constructed as above from the premeasure ν equals ν.

Proof:

1. *Only prove the subadditivity. Set sequence* $\{A_i \subset X\}$ *and* $\varepsilon > 0$. *Pick* $\{C_{i,j}\}$ *such that*

$$\sum_j \tau(C_{i,j}) \leqslant \mu(A_i) + \frac{\varepsilon}{2^i}$$

holds for all i. *Then*

$$\sum_{i,j} \tau(C_{i,j}) \leqslant \sum_i \mu(A_i) + \varepsilon, \quad \text{for all } \varepsilon > 0,$$

hence

$$\sum_{i,j} \tau(C_{i,j}) \leqslant \sum_i \mu(A_i).$$

Since

$$\bigcup_i A_i \subset \bigcup_{i,j} C_{i,j}$$

the result follows.

2. *Former is trivial, only prove latter.*

$$\mu(A) \leqslant \inf\left\{\sum_{i=1}^{\infty} \tau(C_i) \mid C_i \in \mathscr{C}, A \subseteq \bigcup_{i=1}^{\infty} C_i\right\}$$

is trivial according to monotonicity, and other side is also easy to reach, only need to set $\{C_i\} = \{A, \varnothing, \varnothing, \cdots\}$. □

Exercise 3: Let $\nu : \mathscr{A} \to [0, \infty]$ be a countably additive set function such that $\nu(\varnothing) = 0$, where $\mathscr{A}$ is a σ-algebra on X.

1. Show that ν is a premeasure on X. Let μ be the measure constructed as in Exercise 2 from the premeasure ν. Show that for any subset A of X, we have:
$$\mu(A) = \inf\{\nu(B) \mid B \in \mathscr{A}, B \supseteq A\}.$$
2. Deduce that all sets in $\mathscr{A}$ are μ-measurable and that $\mu(A) = \nu(A)$ for all $A \in \mathscr{A}$.
3. Show that μ is a regular measure on X, meaning that for any set $A \subseteq X$, there exists a set $B \in \mathscr{M}_\mu$ such that $B \supseteq A$ and $\mu(B) = \mu(A)$.

Proof:

1. *Former is trivial, only prove latter. Apparently, for all* $B \in \mathscr{A}$ *there exists* $\{B_i \in \mathscr{A}\}$ *such that* $B = \bigcup_i B_i$, *and the result follows.*
2. *Omit.*
3. *For* $A \subset X$, *set* $\{B_i \in \mathscr{A}\}$ *such that* $\nu(B_i) \downarrow \mu(A)$, *then it is easy to deduce that*
$$\mathscr{A} \ni B = \bigcup_i B_i \supset A \quad \text{and} \quad \mu(B) = \mu(A).$$

□

Exercise 4: Show that if a function $f : \mathbb{R} \to \mathbb{R}$ is monotone, then it is Borel-measurable.

Proof: *Only need to prove* $\{f > t\}$ *is Borel-measurable for* $t \in \mathbb{R}$. *Set* $x = \sup\{x \mid f(x) \leqslant t\}$, *then* $\{f > t\}$ *is either* $[x, \infty)$ *or* (x, ∞), *which depends on* $f(x)$. □

Exercise 5: Let μ be a measure on a set X, and $f_n, g_n, f, g \in L^1_\mu$.

1. Assume the following three properties:
   - μ-almost everywhere, $f_n \to f$ and $g_n \to g$ as $n \to \infty$
   - $|f_n| \leqslant g_n$ for all $n \geqslant 1$
   - $\int_X g_n \,\mathrm{d}\mu \to \int_X g \,\mathrm{d}\mu$ as $n \to \infty$

   Show that $\int_X f_n \,\mathrm{d}\mu \to \int_X f \,\mathrm{d}\mu$ as $n \to \infty$.
2. Assume that μ-almost everywhere, $f_n \to f$ as $n \to \infty$. Show that
$$\int_X |f_n - f| \,\mathrm{d}\mu \to 0 \quad \text{as} \quad n \to \infty$$
   if and only if

$$\int_X |f_n| \,\mathrm{d}\mu \to \int_X |f| \,\mathrm{d}\mu \quad \text{as} \quad n \to \infty.$$

Proof:

1.

$$\begin{aligned}
\int_X (g_n + g) \,\mathrm{d}\mu &= \int_X \liminf (g_n + g - |f_n - f|) \,\mathrm{d}\mu \\
&\leqslant \liminf \int_X (g_n + g - |f_n - f|) \,\mathrm{d}\mu \\
&= \int_X (g_n + g) \,\mathrm{d}\mu - \limsup \int_X |f_n - f| \,\mathrm{d}\mu\,.
\end{aligned}$$

2. $\Longrightarrow |f_n - f| \geqslant ||f_n| - |f||$. $\Longleftarrow$ *Set* $g_n = |f_n| + |f|$ *and use previous result.*

□

Exercise 6: Interpret the monotone convergence theorem, Fatou's lemma, and the dominated convergence theorem in the context of the counting measure on ℕ, and deduce the corresponding statements about series.

Proof: *Monotone convergence theorem: consider* $\{a_{i,j} \in \mathbb{R}\}$ *such that* $\sum_j a_{i,j} < \infty$ *for all* $i \geqslant 1$ *and* $a_{i,j} \leqslant a_{i+1,j}$ *for all* j*, then*

$$\sum_j a_{i,j} \uparrow_i \sum_j \lim_i a_{i,j}.$$

Fatou's lemma and dominated convergence theorem can be similar sentenced. □

Exercise 7: Let μ be the Lebesgue measure on $X = [0, 1]$, let ν be the counting measure on X, and let $D = \{(x, x) \mid x \in X\}$ be the diagonal in $X \times X$. Show that the three integrals

$$\int_X \left(\int_X \mathbf{1}_D(x, y) \,\mathrm{d}\nu(y)\right) \mathrm{d}\mu(x), \quad \int_X \left(\int_X \mathbf{1}_D(x, y) \,\mathrm{d}\mu(x)\right) \mathrm{d}\nu(y), \quad \text{and} \quad \int_{X\times X} \mathbf{1}_D \,\mathrm{d}(\mu \times \nu)$$

are all unequal.

Proof:

$$\begin{aligned}
\int_X \left(\int_X \mathbf{1}_D(x, y) \,\mathrm{d}\nu(y)\right) \mathrm{d}\mu(x) &= \int_X 1 \,\mathrm{d}\mu(x) = 1, \\
\int_X \left(\int_X 1_D(x, y) \,\mathrm{d}\mu(x)\right) \mathrm{d}\nu(y) &= \int_X 0 \,\mathrm{d}\nu(y) = 0, \\
\int_{X\times X} 1_D \,\mathrm{d}(\mu \times \nu) &= \infty.
\end{aligned}$$

□