

**Mathematics Courses**  
**Tasks Answers Series**

# **Real Analysis: Tasks Answers**

**Fourth Edition,**  
by *H.L. Royden* and *P.M. Fitzpatrick*

aytony

2024 -- 04 -- 27



# Contents

<b>Preliminaries on Sets, Mappings, and Relations</b>	<b>3</b>
<b>1   The Real Numbers: Sets, Sequences, and Functions</b>	<b>4</b>
1.1 The Field, Positivity, and Completeness Axioms .....	4
1.2 The Natural and Rational Numbers .....	4
1.3 Countable and Uncountable Sets .....	4
1.4 Open Sets, Closed Sets, and Borel Sets of Real Numbers .....	4
1.5 Sequences of Real Numbers .....	4
1.6 Continuous Real-valued Functions of a Real Variable .....	4
<b>2   Lebesgue Measure</b>	<b>5</b>
2.1 Intruduction .....	5
2.2 Lebesgue Outer Measure .....	5
2.3 The $\sigma$ -Algebra of Lebesgue Measurable Sets .....	5
2.4 Outer and Inner Approximation of Lebesgue Measurable Sets .....	5
2.5 Countable Additivity, Continuity, and the Borel-Cantelli Lemma .....	5
2.6 Nonmeasurable Sets .....	5
2.7 The Cantor Set and the Cantor-Lebesgue Function .....	5
<b>3   Lebesgue Measurable Functions</b>	<b>6</b>
3.1 Sums, Products, and Compositions .....	6
3.2 Sequential Pointwise Limits and Simple Approximation .....	6
3.3 Littlewood's Three Principles, Egoroff's Theorem, and Lusin's Theorem .....	6
<b>4   Lebesgue Integration</b>	<b>7</b>
4.1 The Rimemann Integral .....	7
4.2 The Lebesgue Integral of a Bounded Measurable Function over a Set of Finite Measure .....	8
4.3 The Lebesgue Integral of a Measurable Nonnegatice Function .....	8
4.4 The General Lebesgue Integral .....	8
4.5 Contable Addativity and Continuity of Integration .....	8
4.6 Uniform Integrability: The Vitali Convergence Theorem .....	8
<b>5   Lebesgue Integration: Further Topics</b>	<b>14</b>
5.1 Uniform Integrability and Tightness: A General Vitali Convergence Theorem .....	14
5.2 Convergence in Measure .....	16
5.3 Chracterizations of Riemann and Lebesgue Integrability .....	16

---

## Preliminaries on Sets, Mappings, and Relations

# 1 | The Real Numbers: Sets, Sequences, and Functions

## 1.1 | The Field, Positivity, and Completeness Axioms

## 1.2 | The Natural and Rational Numbers

## 1.3 | Countable and Uncountable Sets

## 1.4 | Open Sets, Closed Sets, and Borel Sets of Real Numbers

## 1.5 | Sequences of Real Numbers

## 1.6 | Continuous Real-valued Functions of a Real Variable

## 2 | Lebesgue Measure

### 2.1 | Intruduction

### 2.2 | Lebesgue Outer Measure

### 2.3 | The $\sigma$ -Algebra of Lebesgue Measurable Sets

### 2.4 | Outer and Inner Approximation of Lebesgue Measurable Sets

### 2.5 | Countable Additivity, Continuity, and the Borel-Cantelli Lemma

### 2.6 | Nonmeasurable Sets

### 2.7 | The Cantor Set and the Cantor-Lebesgue Function

### **3 | Lebesgue Measurable Functions**

#### **3.1 | Sums, Products, and Compositions**

#### **3.2 | Sequential Pointwise Limits and Simple Approximation**

#### **3.3 | Littlewood's Three Principles, Egoroff's Theorem, and Lusin's Theorem**

## 4 | Lebesgue Integration

### 4.1 | The Riemann Integral

Task 4.1 :

Task 4.2 :

Task 4.3 :

Task 4.4 :

Task 4.5 :

Task 4.6 :

Task 4.7 :

Task 4.8 :

Task 4.9 :

Task 4.10 :

Task 4.11 :

Task 4.12 :

Task 4.13 :

Task 4.14 :

Task 4.15 :

Task 4.16 :

Task 4.17 :

Task 4.18 :

Task 4.19 :

Task 4.20 :

Task 4.21 :

Task 4.22 :

Task 4.23 :

Task 4.24 :

Task 4.25 :

Task 4.26 :

Task 4.27 :

Task 4.28 :

Task 4.29 :

Task 4.30 :

Task 4.31 :

Task 4.32:

Task 4.33:

Task 4.34:

Task 4.35:

Task 4.36:

Task 4.37:

Task 4.38:

Task 4.39:

## 4.2 | The Lebesgue Integral of a Bounded Measurable Function over a Set of Finite Measure

## 4.3 | The Lebesgue Integral of a Measurable Nonnegative Function

## 4.4 | The General Lebesgue Integral

## 4.5 | Countable Additivity and Continuity of Integration

## 4.6 | Uniform Integrability: The Vitali Convergence Theorem

**Task 4.40:** Let  $f$  be integrable over  $\mathbb{R}$ . Show that the function  $F$  defined by

$$F(x) = \int_{-\infty}^x f \text{ for all } x \in \mathbb{R}$$

is properly defined and continuous. Is it necessarily Lipschitz?

**Task 4.41:** Show that Proposition 25 is false if  $E = \mathbb{R}$ .

**Task 4.42:** Show that Theorem 26 is false without the assumption that the  $h_n$ 's are nonnegative.

**Proof:** 考虑函数列

$$h_{n(x)} = \begin{cases} n, & 0 \leq x \leq \frac{1}{2^n} \\ -n, & \frac{1}{2^n} < x \leq \frac{1}{2^{n-1}} \\ 0, & \text{其他情况} \end{cases}$$

此时  $\lim \int_{[0,1]} h_n = 0$ , 但是在  $[0, 1]$  上不一致可积.

考虑函数列

**Task 4.43:** Let the sequences of functions  $\{h_n\}$  and  $\{g_n\}$  be uniformly integrable over  $E$ . Show that for any  $\alpha$  and  $\beta$ , the sequence of linear combinations  $\alpha f_n + \beta g_n$  also is uniformly integrable over  $E$ .

**Proof:** 任取  $\varepsilon > 0$ . 设  $\delta_1, \delta_2 > 0$  满足  $\{h_n\}, \{g_n\}$  在  $\frac{\varepsilon}{2|\alpha|}, \frac{\varepsilon}{2|\beta|}$  下对应的一致可积条件. 那么令  $\delta = \min\{\delta_1, \delta_2\}$ . 对任意  $E$  的可测子集  $A$ , 且  $m(A) < \delta$ , 则有



$$\begin{aligned}
\int_A |\alpha f_n + \beta g_n| &\leq |\alpha| \int_A |f_n| + |\beta| \int_A |g_n| \\
&= |\alpha| \cdot \frac{\varepsilon}{2|\alpha|} + |\beta| \cdot \frac{\varepsilon}{2|\beta|} \\
&< \varepsilon.
\end{aligned}$$

**Task 4.44:** Let  $f$  be integrable over  $\mathbb{R}$  and  $\varepsilon > 0$ . Establish the following three approximation properties.

1. There is a simple function  $\eta$  on  $\mathbb{R}$  which has finite support and  $\int_{\mathbb{R}} |f - \eta| < \varepsilon$ .
2. There is a step function  $s$  on  $\mathbb{R}$  which vanished outside a closed, bounded interval and  $\int_{\mathbb{R}} |f - s| < \varepsilon$ .
3. There is a continuous function  $g$  on  $\mathbb{R}$  which vanished outside a bounded set and  $\int_{\mathbb{R}} |f - g| < \varepsilon$ .

**Proof:** 不妨假设  $f$  在  $\mathbb{R}$  上非负, 否则只要对  $f$  非负和  $f$  为负的部分分别应用结论即可.

对于第一问, 由定义知道

$$\int_{\mathbb{R}} f = \sup \left\{ \int_{\mathbb{R}} h \mid h \text{ 是有界、可测、支集测度有限的函数, 且 } 0 \leq h \leq f \right\}.$$

那么存在  $h_0$  满足以上条件且

$$\int_{\mathbb{R}} f - \int_{\mathbb{R}} h_0 < \frac{\varepsilon}{2}.$$

又由  $\int_{\mathbb{R}} h_0$  定义知道存在简单函数  $\eta$  满足

$$\int_{\mathbb{R}} h_0 - \int_{\mathbb{R}} \eta < \frac{\varepsilon}{2}.$$

那么就有

$$\int_{\mathbb{R}} |f - \eta| = \int_{\mathbb{R}} f - \int_{\mathbb{R}} \eta < \varepsilon,$$

其中  $\eta$  是简单函数且支集测度有限.

对于第二问, 对于上述  $h_0$ , 首先由积分的连续性得到

$$\lim_{n \rightarrow \infty} \int_{[-n, n]} h_0 = \int_{\mathbb{R}} h_0.$$

那么存在一个足够大的  $n_0$ , 使得

$$\int_{(-\infty, -n_0) \cup (n_0, +\infty)} h_0 = \int_{\mathbb{R}} h_0 - \int_{[-n_0, n_0]} h_0 < \frac{\varepsilon}{4}.$$

考虑  $\int_{[-n_0, n_0]} h_0$ . 由 Lusin 定理, 存在  $[-n_0, n_0]$  的子集  $F$  使得  $h_0$  在  $F$  上连续, 那么由 Riemann 可积性知道存在  $F$  上的 step function  $s$  使得  $s \leq h_0$  且

$$\int_F h_0 - \int_F s < \frac{\varepsilon}{8}.$$

又注意到  $h_0$  在  $[-n_0, n_0]$  上有界, 设  $|h_0| < M$ . 那么可以选取上述集合  $F$  使得  $m([-n_0, n_0] \setminus F) < \frac{\varepsilon}{8M}$  即可, 在  $\mathbb{R} \setminus F$  上令  $s = 0$ , 此时有

$$\begin{aligned}
\int_{\mathbb{R}} |f - s| &= \int_{\mathbb{R}} f - \int_{\mathbb{R}} s \\
&= \int_{\mathbb{R}} f - \int_{\mathbb{R}} h_0 + \int_{\mathbb{R}} h_0 - \int_{[-n_0, n_0]} h_0 + \int_{[-n_0, n_0]} h_0 - \int_F s \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \int_{[-n_0, n_0] \sim F} h_0 + \int_F h_0 - \int_F s \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + M \cdot \frac{\varepsilon}{8M} + \frac{\varepsilon}{8} \\
&< \varepsilon.
\end{aligned}$$

对于第三问, 考虑上述  $h_0, n_0$  和  $F$ . 考虑构造连续函数  $g$ , 使得在  $F$  上  $g = h_0$ , 并将  $g$  延拓到  $[-n_0, n_0]$ , 进一步延拓到  $\mathbb{R}$ , 且有  $g([n_0 + \frac{\varepsilon}{8M}, +\infty)) = g((-\infty, -n_0 - \frac{\varepsilon}{8M}]) = \{0\}$ . 这时有

$$\int_{\mathbb{R}} |f - g| < \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \int_{[-n_0, n_0] \sim F} (h_0 - g) < \varepsilon.$$

**Task 4.45:** Let  $f$  be integrable over  $E$ . Define  $\hat{f}$  to be the extension of  $f$  to all of  $\mathbb{R}$  obtained by setting  $\hat{f} \equiv 0$  outside of  $E$ . Show that  $\hat{f}$  is integrable over  $\mathbb{R}$  and  $\int_E f = \int_{\mathbb{R}} \hat{f}$ . Use this and part (1) and (3) of preceding problem to show that for  $\varepsilon > 0$ , there is a simple function  $\eta$  on  $E$  and a continuous function  $g$  on  $E$  for which  $\int_E |f - \eta| < \varepsilon$  and  $\int_E |f - g| < \varepsilon$ .

**Proof:** 显然有

$$\begin{aligned}
\int_{\mathbb{R}} \hat{f} &= \int_E \hat{f} + \int_{\mathbb{R} \sim E} \hat{f} \\
&= \int_E f + \int_{\mathbb{R} \sim E} 0 \\
&= \int_E f.
\end{aligned}$$

然后直接应用上题的结论即可.

**Task 4.46 (Riemann-Lebesgue):** Let  $f$  be integrable over  $\mathbb{R}$ . Show that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} f(x) \cos nx \, dx = 0.$$

**Proof:**  $\forall \varepsilon > 0$ . 由 44 题的结论知道存在 step function  $g$  仅在有限闭区间  $F$  上非 0 且  $\int_{\mathbb{R}} |f - g| < \frac{\varepsilon}{2m(F)}$ . 又知道

$$\begin{aligned}
\left| \int_{\mathbb{R}} f(x) \cos nx \, dx \right| &\leq \int_{\mathbb{R}} |(f(x) - g(x)) \cos nx| \, dx + \left| \int_{\mathbb{R}} g(x) \cos nx \, dx \right| \\
&\leq \int_{\mathbb{R}} |f(x) - g(x)| |\cos nx| \, dx + \left| \int_{\mathbb{R}} g(x) \cos nx \, dx \right| \\
&\leq \int_{\mathbb{R}} |f(x) - g(x)| \, dx + \left| \int_{\mathbb{R}} g(x) \cos nx \, dx \right| \\
&\leq \frac{\varepsilon}{2} + \left| \int_{\mathbb{R}} g(x) \cos nx \, dx \right|.
\end{aligned}$$

故只要证明  $g$  是 step function 的情况即可, 且  $g$  仅在有穷闭区间  $F$  上非 0. 考虑到  $g$  在每一段上的取值为常数, 不妨设  $g(x)$  是常数. 那么此时只要证明

$$\lim_{n \rightarrow \infty} \left| \int_F \cos nx \, dx \right| = 0.$$

而注意到它是 Riemann 可积的, 直接换元积分得到原函数  $\frac{1}{n} \sin nx$ , 令  $n \rightarrow \infty$  得到结果.

**Task 4.47:** Let  $f$  be integrable over  $\mathbb{R}$ .

1. Show that for each  $t$ ,

$$\int_{\mathbb{R}} f(x) \, dx = \int_{\mathbb{R}} f(x+t) \, dx.$$

2. Let  $g$  be a bounded measurable function on  $\mathbb{R}$ . Show that

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}} g(x) \cdot [f(x) - f(x+t)] \, dx = 0.$$

**Proof:** 对于第一问, 容易看出

$$\begin{aligned}
\int_{\mathbb{R}} f(x) \, dx &= \sup \left\{ \int_{\mathbb{R}} h(x) \, dx \mid h \text{ 是有界、可测、支集测度有限的函数, 且 } 0 \leq h \leq f \right\} \\
&= \sup \left\{ \int_{\mathbb{R}} h(x+t) \, dx \mid h \text{ 是有界、可测、支集测度有限的函数, 且 } 0 \leq h \leq f \right\} \\
&= \int_{\mathbb{R}} f(x+t) \, dx.
\end{aligned}$$

对于第二问, 考虑用第 44 题的结论, 用仅在有界闭集合  $F$  上非零的连续函数  $f_1$  来近似函数  $f$ , 满足  $\int_{\mathbb{R}} |f - f_1| < \varepsilon$  对  $\forall \varepsilon > 0$  成立. 由于  $f_1$  在闭集  $F$  上连续, 进而一致连续. 那么对  $\forall \varepsilon > 0$ , 记  $\int_{\mathbb{R}} |g| = G < +\infty$ , 当  $|t| < \frac{\varepsilon}{2Mm(F)}$  时有

$$\begin{aligned}
\left| \int_F g(x) \cdot [f_1(x) - f_1(x+t)] \, dx \right| &\leq \int_F M \cdot |f_1(x) - f_1(x+t)| \, dx \\
&\leq \int_F M \cdot \frac{\varepsilon}{2Mm(F)} \, dx \\
&= \varepsilon.
\end{aligned}$$

于是在  $f_1$  下的命题得证. 对于  $f$  下的命题, 有

$$\begin{aligned}
\left| \int_{\mathbb{R}} g(x) \cdot [f(x) - f(x+t)] \right| &\leq \int_{\mathbb{R}} M \cdot |f(x) - f(x+t)| \\
&\leq \int_{\mathbb{R}} M \cdot |f(x) - f_1(x)| + \int_{\mathbb{R}} M \cdot |f_1(x) - f_1(x+t)| \\
&\quad + \int_{\mathbb{R}} M \cdot |f(x+t) - f_1(x+t)| \\
&\leq M \cdot \frac{\varepsilon}{3M} + \frac{\varepsilon}{3} + M \cdot \frac{\varepsilon}{3M} \\
&= \varepsilon.
\end{aligned}$$

其中倒数第二行由  $f_1$  下的命题和用  $f_1$  近似  $f$  的结论可以得出. 命题得证.

**Task 4.48:** Let  $f$  be integrable over  $E$  and  $g$  be a bounded measurable function on  $E$ . Show that  $f \cdot g$  is integrable over  $E$ .

**Proof:** 设  $|g| < M$ . 则有

$$\left| \int_E f \cdot g \right| \leq \int_E |f| \cdot |g| \leq M \int_E |f| < +\infty.$$

**Task 4.49:** Let  $f$  be integrable over  $\mathbb{R}$ . Show that the following four assertions are equivalent:

1.  $f = 0$  a.e. on  $\mathbb{R}$ .
2.  $\int_{\mathbb{R}} fg = 0$  for every bounded measurable function  $g$  on  $\mathbb{R}$ .
3.  $\int_A f = 0$  for every measurable set  $A$ .
4.  $\int_{\mathcal{O}} f = 0$  for every open set  $\mathcal{O}$ .

**Proof:**  $1 \Rightarrow 2$ : 类似第 48 题可以直接证明.

$2 \Rightarrow 3$ : 对任意的可测集  $A$ , 令可测函数  $g = \chi_A$  即可.

$3 \Rightarrow 4$ : 显然.

$4 \Rightarrow 1$ : 反证法. 设存在正测度集合  $A$  使得  $\int_A f \neq 0$ , 不妨设  $\int_A f = k > 0$ . 构造开集合列  $\{O_n\}$  使得  $A \subseteq O_n$  且  $m(O_n \setminus A) < \frac{1}{n}$ . 那么令  $O = \bigcap_{n=1}^{\infty} O_n$  则有  $A \subseteq O$  且  $m(O) = m(A)$ . 此时有

$$\int_O f = \int_A f + \int_{O \setminus A} f = k > 0,$$

矛盾.

**Task 4.50:** Let  $\mathcal{F}$  be a family of functions, each of which is integrable over  $E$ . Show that  $\mathcal{F}$  is uniformly integrable over  $E$  if and only if for each  $\varepsilon > 0$ , there is a  $\delta > 0$  such that for each  $f \in \mathcal{F}$ ,

$$\text{if } A \subseteq E \text{ is measurable and } m(A) < \delta, \text{ then } \left| \int_A f \right| < \varepsilon.$$

**Proof:**  $\Rightarrow$  由  $\left| \int_A f \right| \leq \int_A |f| < \varepsilon$  显然.

$\Leftarrow$  对于  $\forall f \in \mathcal{F}$ , 考虑  $f^+$  和  $f^-$  在  $A$  上定号. 对于任意的可测  $A \subseteq E$  且  $m(A) < \delta$ , 有

$$\begin{aligned}
\int_A |f| &= \int_{A \cap [f \geq 0]} |f| - \int_{A \cap [f \leq 0]} |f| \\
&= \left| \int_{A \cap [f \geq 0]} f \right| + \left| \int_{A \cap [f \leq 0]} f \right| \\
&< \varepsilon.
\end{aligned}$$

最后一步由题目条件得到.

**Task 4.51:** Let  $\mathcal{F}$  be a family of functions, each of which is integrable over  $E$ . Show that  $\mathcal{F}$  is uniformly integrable over  $E$  if and only if for each  $\varepsilon > 0$ , there is a  $\delta > 0$  such that for all  $f \in \mathcal{F}$ ,

$$\text{if } \mathcal{U} \text{ is open and } m(E \cap \mathcal{U}) < \delta, \text{ then } \int_{E \cap \mathcal{U}} |f| < \varepsilon.$$

**Proof:**  $\Rightarrow$  显然.

$\Leftarrow$  考虑对任意的  $\varepsilon > 0$  和条件中对应的  $\delta > 0$ , 令  $E$  的可测子集  $A$  满足  $m(A) < \delta$ . 那么存在开集  $\mathcal{U}$  包含  $A$  且  $m(\mathcal{U} \setminus A) < \delta - m(A)$ . 那么有

$$m(E \cap \mathcal{U}) = m(A) + m((E \cap \mathcal{U}) \setminus A) \leq m(A) + m(\mathcal{U} \setminus A) < \delta.$$

从而有

$$\int_A |f| \leq \int_{E \cap \mathcal{U}} |f| < \varepsilon.$$

**Task 4.52:**

1. Let  $\mathcal{F}$  be the family of functions  $f$  on  $[0, 1]$ , each of which is integrable over  $[0, 1]$  and has  $\int_0^1 |f| \leq 1$ . Is  $\mathcal{F}$  uniformly integrable over  $[0, 1]$ ?
2. Let  $\mathcal{F}$  be the family of functions  $f$  on  $[0, 1]$ , each of which is continuous over  $[0, 1]$  and has  $|f| \leq 1$ . Is  $\mathcal{F}$  uniformly integrable over  $[0, 1]$ ?
3. Let  $\mathcal{F}$  be the family of functions  $f$  on  $[0, 1]$ , each of which is integrable over  $[0, 1]$  and has  $\int_a^b |f| \leq b - a$  for all  $[a, b] \subseteq [0, 1]$ . Is  $\mathcal{F}$  uniformly integrable over  $[0, 1]$ ?

**Proof:**

1. 错误. 考虑

$$f_n = \begin{cases} n, & 0 \leq x \leq \frac{1}{n} \\ 0, & \text{其他情况} \end{cases}$$

不是一致可积的.

2. 正确. 令  $\delta = \varepsilon$ , 则对任意  $[0, 1]$  的可测子集  $A$ , 且  $m(A) < \delta$ , 有

$$\int_A |f| \leq \int_A 1 = m(A) < \varepsilon.$$

是一致可积的.

3. 正确. 容易看出  $F(x) = \int_0^x f(t) dt$  是一致连续的.

## 5 | Lebesgue Integration: Further Topics

### 5.1 | Uniform Integrability and Tightness: A General Vitali Convergence Theorem

**Task 5.1:** Prove Corollary 2.

**Proof:**  $\Leftarrow$  由 Vitali 收敛定理显然.

$\Rightarrow$  先证紧性. 任取  $\varepsilon > 0$ . 由  $\int_E h_n \rightarrow 0$  可以取一个足够大的自然数  $N$  使得对  $\forall n > n_0$  有  $\int_E h_n < \varepsilon$ . 那么对于  $n > N$ , 可以直接取  $E_n = \emptyset$  满足

$$\int_{E \sim E_n} |h_n| = \int_E h_n - \int_{E_n} h_n < \varepsilon.$$

对于  $1 \leq n \leq N$  的每个  $h_n$ , 一定可以找到一个测度有限的集合  $E_n \subseteq E$  使得

$$\int_{E \sim E_n} h_n < \varepsilon.$$

令  $E_0 = \cup_{n=1}^N E_n$  仍是测度有限的集合, 那么  $E_0$  就满足

$$\int_{E \sim E_0} h_n < \int_{E \sim E_n} h_n < \varepsilon \quad \text{对 } \forall n \in \mathbb{N}^* \text{ 成立.}$$

从而  $\{h_n\}$  是紧的. 对于上述  $E_0$ , 考虑  $\{h_n\}$  在  $E_0$  上的限制, 应用定理 4.26 直接得出  $\{h_n\}$  在  $E_0$  上面一致可积. 考虑拓展到  $E$  上的情况.

对任意  $\varepsilon > 0$ , 存在上述  $E_0$  使得  $\int_{E \sim E_0} h_n < \varepsilon/2$  对任意  $n \in \mathbb{N}^*$  成立. 由  $\{h_n\}$  在  $E_0$  上一致可积知道存在  $\delta > 0$  使得对任意  $E_0$  的测度小于  $\delta$  的子集  $A$  有  $\int_A |h_n| < \frac{\varepsilon}{2}$ . 那么考虑任意  $E$  上测度小于  $\delta$  的子集  $B$ , 一定有

$$\int_B h_n = \int_{B \cap E_0} h_n + \int_{B \sim E_0} h_n \leq \frac{\varepsilon}{2} + \int_{E \sim E_0} h_n < \varepsilon.$$

从而  $\{h_n\}$  在  $E$  上一致可积.

**Task 5.2:** Let  $\{f_k\}_{k=1}^n$  be a finite family of functions, each of which is integrable over  $E$ . Show that  $\{f_k\}_{k=1}^n$  is uniformly integrable and tight over  $E$ .

**Proof:** 先证  $\{f_k\}_{k=1}^n$  是紧的. 对于  $1 \leq k \leq n$  的每个  $f_k$ , 一定可以找到一个测度有限的集合  $E_k \subseteq E$  使得

$$\int_{E \sim E_k} |f_k| < \varepsilon.$$

令  $E_0 = \cup_{k=1}^n E_k$  仍是测度有限的集合, 那么  $E_0$  就满足

$$\int_{E \sim E_0} |f_k| < \int_{E \sim E_k} |f_k| < \varepsilon \quad \text{对 } \forall k = 1, 2, \dots, n \text{ 成立.}$$

从而  $\{f_k\}_{k=1}^n$  是紧的.

再证  $\{f_k\}_{k=1}^n$  是一致可积的. 任取  $\varepsilon > 0$ , 由于  $\{f_k\}$  是紧的, 存在  $E$  的有限测度子集  $E_0$  满足

$$\int_{E \sim E_0} |f_k| < \frac{\varepsilon}{2}.$$

对于每一个  $1 \leq k \leq n$ , 存在  $\delta_k$  满足对于  $E_0$  的任意子集  $A$  且  $m(A) < \delta_k$ , 一定有

$$\int_A |f_k| < \frac{\varepsilon}{2}.$$

令  $\delta = \min\{\delta_1, \delta_2, \dots, \delta_n\}$ , 则对于任意  $E$  的子集  $A$  且  $m(A) < \delta$ , 一定有

$$\int_A |f_k| = \int_{A \cap E_0} |f_k| + \int_{A \sim E_0} |f_k| < \frac{\varepsilon}{2} + \int_{E \sim E_0} |f_k| < \varepsilon.$$

从而  $\{f_k\}$  是一致可积的.

**Task 5.3:** Let the sequences of functions  $\{h_n\}$  and  $\{g_n\}$  be uniformly integrable and tight over  $E$ . Show that for any  $\alpha$  and  $\beta$ ,  $\{\alpha f_n + \beta g_n\}$  also is uniformly integrable and tight over  $E$ .

**Proof:** 不妨设  $\alpha, \beta \neq 0$ .

先证  $\{\alpha f_n + \beta g_n\}$  是紧的. 对于  $\forall \varepsilon > 0$ , 一定可以找到测度有限的集合  $E_1, E_2 \subseteq E$  使得

$$\int_{E \sim E_1} |f| < \frac{\varepsilon}{2|\alpha|} \quad \text{对 } \forall f \in \{f_n\} \text{ 成立}$$

且

$$\int_{E \sim E_2} |g| < \frac{\varepsilon}{2|\beta|} \quad \text{对 } \forall g \in \{g_n\} \text{ 成立}.$$

令  $E_0 = E_1 \cup E_2$  仍是测度有限的集合, 那么  $E_0$  同时满足上式中  $E_1, E_2$  的性质, 此时有

$$\begin{aligned} \int_{E \sim E_0} |\alpha f + \beta g| &\leq |\alpha| \int_{E \sim E_0} |f| + |\beta| \int_{E \sim E_0} |g| \\ &< |\alpha| \cdot \frac{\varepsilon}{2|\alpha|} + |\beta| \cdot \frac{\varepsilon}{2|\beta|} \\ &= \varepsilon \quad \text{对 } \forall f \in \{f_n\}, g \in \{g_n\} \text{ 成立}. \end{aligned}$$

从而  $\{\alpha f_n + \beta g_n\}$  是紧的.

再证  $\{\alpha f_n + \beta g_n\}$  是一致可积的. 任取  $\varepsilon > 0$ , 由于  $\{\alpha f_n + \beta g_n\}$  是紧的, 存在  $E$  的有限测度子集  $E_0$  满足

$$\int_{E \sim E_0} (\alpha f_n + \beta g_n) < \frac{\varepsilon}{2}.$$

对于任意的  $\alpha f + \beta g \in \{\alpha f_n + \beta g_n\}$ , 存在  $\delta_1, \delta_2$  满足对于  $E_0$  的任意子集  $A_1, A_2$  且  $m(A_1) < \delta_1, m(A_2) < \delta_2$ , 一定有

$$\int_{A_1} |f| < \frac{\varepsilon}{4|\alpha|} \quad \text{且} \quad \int_{A_2} |g| < \frac{\varepsilon}{4|\beta|}$$

令  $\delta = \min\{\delta_1, \delta_2\}$  仍满足上式中  $\delta_1, \delta_2$  性质. 则对于任意  $E$  的子集  $A$  且  $m(A) < \delta$ , 一定有

$$\begin{aligned}
\int_A |\alpha f_n + \beta g_n| &\leq |\alpha| \int_{A \cap E_0} |f| + |\beta| \int_{A \cap E_0} |g| + \int_{A \sim E_0} |\alpha f_n + \beta g_n| \\
&< |\alpha| \cdot \frac{\varepsilon}{4|\alpha|} + |\beta| \cdot \frac{\varepsilon}{4|\beta|} + \frac{\varepsilon}{2} \\
&= \varepsilon \quad \text{对 } \forall f \in \{f_n\}, g \in \{g_n\} \text{ 成立.}
\end{aligned}$$

从而  $\{\alpha f_n + \beta g_n\}$  是一致可积的.

**Task 5.4:** Let  $\{f_n\}$  be a sequence of measurable functions on  $E$ . Show that  $\{f_n\}$  is uniformly integrable and tight over  $E$  if and only if for each  $\varepsilon > 0$ , there is a measurable subset  $E_0$  of  $E$  that has finite measure and a  $\delta > 0$  such that for each measurable subset  $A$  of  $E$  and index  $n$ ,

$$\text{if } m(A \cap E_0) < \delta, \text{ then } \int_A |f_n| < \varepsilon.$$

**Proof:**

**! TODO !**  
证明

**Task 5.5:** Let  $\{f_n\}$  be a sequence of integrable functions on  $\mathbb{R}$ . Show that  $\{f_n\}$  is uniformly integrable and tight over  $\mathbb{R}$  if and only if for each  $\varepsilon > 0$ , there are positive numbers  $r$  and  $\delta$  such that for each open subset  $\mathcal{O}$  of  $\mathbb{R}$  and index  $n$ ,

$$\text{if } m(\mathcal{O} \cap (-r, r)) < \delta, \text{ then } \int_{\mathcal{O}} |f_n| < \varepsilon.$$

**Proof:**

**! TODO !**  
证明

## 5.2 | Convergence in Measure

## 5.3 | Characterizations of Riemann and Lebesgue Integrability

**Task 5.6:** Let  $f$  and  $g$  be bounded functions that are Riemann integrable over  $[a, b]$ . Show that the product  $fg$  also is Riemann integrable over  $[a, b]$ .

**Task 5.7:** Let  $f$  be a bounded function on  $[a, b]$  whose set of discontinuities has measure zero. Show that  $f$  is measurable. Then show that the same holds without the assumption of boundedness.

**Task 5.8:** Let  $f$  be a function on  $[0, 1]$  that is continuous on  $(0, 1]$ . Show that it is possible for the sequence  $\left\{ \int_{[1/n, 1]} f \right\}$  to converge and yet  $f$  is not Lebesgue integrable over  $[0, 1]$ . Can this happen if  $f$  is nonnegative?