Harmonic Measure TD2

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Exercise 1: Let $\Omega \subset \mathbb{R}^n$ be a nonempty bounded open set with C^1 boundary $\partial\Omega$, and let $F : \overline{\Omega} \to \mathbb{R}^n$ be a vector field that is continuously differentiable in Ω and continuous up to the boundary, i.e., $F \in C^1(\Omega) \cap C\left(\overline{\Omega}\right)$. Then the divergence theorem asserts

$$\int_\Omega \nabla \cdot F \,\mathrm{d}x = \int_{\partial\Omega} F \cdot \nu \,\mathrm{d}S,$$

where ν is the outward pointing unit normal to the boundary $\partial\Omega$.

As a warmup, let $u \in C^1(\Omega) \cap C\left(\overline{\Omega}\right)$ and $\varphi \in C^2(\Omega) \cap C^1\left(\overline{\Omega}\right)$, use the divergence theorem to prove the following identities:

1. **Green's zeroth identity**

$$\int_\Omega \Delta\varphi \,\mathrm{d}x = \int_{\partial\Omega} \partial_\nu \varphi \,\mathrm{d}S\,.$$

2. **Green's first identity**

$$\int_\Omega (\nabla u \cdot \nabla\varphi + u\Delta\varphi) \,\mathrm{d}x = \int_{\partial\Omega} u\partial_\nu \varphi \,\mathrm{d}S\,.$$

3. **Green's second identity**

$$\int_\Omega (u\Delta\varphi - \varphi\Delta u) \,\mathrm{d}x = \int_{\partial\Omega} (u\partial_\nu \varphi - \varphi\partial_\nu u) \,\mathrm{d}S\,.$$

Proof:

1. Set $F = \nabla\varphi$ and

$$\mathrm{LHS} = \int_\Omega \nabla \cdot F \,\mathrm{d}x = \int_{\partial\Omega} F \cdot \nu \,\mathrm{d}S = \int_{\partial\Omega} \nabla\varphi \cdot \nu \,\mathrm{d}S = \mathrm{RHS}.$$

2.

$$\mathrm{LHS} = \int_\Omega \nabla \cdot (u\nabla\varphi) \,\mathrm{d}x = \int_{\partial\Omega} u\nabla\varphi \cdot \nu \,\mathrm{d}S = \mathrm{RHS}.$$

3.

$$\begin{aligned}\mathrm{LHS} &= \int_\Omega (u\Delta\varphi + \nabla u \cdot \nabla\varphi) \,\mathrm{d}x - \int_\Omega (\varphi\Delta u + \nabla u \cdot \nabla\varphi) \,\mathrm{d}x \\ &= \int_{\partial\Omega} u\partial_\nu\varphi \,\mathrm{d}S - \int_{\partial\Omega} \varphi\partial_\nu u \,\mathrm{d}S \\ &= \mathrm{RHS}.\end{aligned}$$

□

Exercise 2: The inhomogeneous version of the Laplace equation is scrled the Poisson equation

$$\Delta u = f, \quad \text{in } \Omega.$$

The common boundary conditions considered include

$$au + b\partial_\nu u = g, \quad \text{on } \partial\Omega,$$

with various choices of the functions a and b on the boundary. The case $a \equiv 1$ and $b \equiv 0$ is scrled the *Dirichlet*, $a \equiv 0$ and $b \equiv 1$ the *Neumann*, $a \equiv 1$ and $b > 0$ the *Robin*, and $a \equiv 1$ and $b < 0$ the *Steklov* boundary conditions.

Here we consider $C^2(\Omega) \cap C^1\left(\overline{\Omega}\right)$ solutions:

1. Prove uniqueness of solutions to the Dirichlet problem for the Poisson equation.
2. Prove that any two solutions of the Neumann problem for the Poisson equation differ by a constant.
3. Prove a uniqueness theorem for the Robin problem for the Poisson equation. What if one specifies a Dirichlet condition on one part of the boundary, and a Neumann condition on the rest?

Proof:

1. *Set u, v be two solutions to the Dirichlet problem for the Poisson equation. Therefore, $u - v$ is harmonic in Ω since $\Delta(u - v) = \Delta u - \Delta v = 0$ and $u - v = 0$ on $\partial\Omega$, hence $u - v \equiv 0$ on Ω according to maximum principle of harmonic function.*
2. *Set u_1, u_2 be two solutions to the Neumann problem for the Poisson equation, and let $v = u_1 - u_2$. Only need to prove $\nabla v = 0$. That's because*

$$0 = \int_\Omega v\Delta v \,\mathrm{d}x = \int_{\partial\Omega} v\partial_\nu v \,\mathrm{d}S - \int_\Omega \nabla v \cdot \nabla v \,\mathrm{d}x,$$

since $\partial_\nu v = 0$ on $\partial\Omega$, first term is always 0, so $\Delta v = 0$ holds for $x \in \Omega$ a.e..

3. *Set u_1, u_2, v similarly.*

$$0 = \int_\Omega v\Delta v \,\mathrm{d}x = \int_{\partial\Omega} v\partial_\nu v \,\mathrm{d}S - \int_\Omega |\nabla v|^2 \,\mathrm{d}x = -\int_{\partial\Omega} \frac{v^2}{b} \,\mathrm{d}S - \int_\Omega |\nabla v|^2 \,\mathrm{d}x,$$

Hence $v^2 = 0$ for $x \in \partial\Omega$ a.e., and use the uniqueness of Dirichlet problem. For next question, suppose Dirichlet condition holds on $\Omega_1 \subset \partial\Omega$ and Neumann holds on $\Omega_2 = \partial\Omega \setminus \Omega_1$. Similarly we can deduce

$$0 = \int_{\partial\Omega} v\partial_\nu v \,\mathrm{d}S - \int_\Omega |\nabla v|^2 \,\mathrm{d}x = \int_{\Omega_1} 0 \cdot \partial_\nu v \,\mathrm{d}S + \int v \cdot 0 \,\mathrm{d}S - \int_\Omega |\nabla v|^2 \,\mathrm{d}x,$$

since first two terms vanish, v is a constant in Ω, therefore $v = 0$ since $v = 0$ on Ω_1.

$\square$

Exercise 3: A **fundamental solution** (or elementary solution) of the Laplacian in n dimensions is a locally integrable function $E \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ satisfying

$$\int_{\mathbb{R}^n} E(y)\Delta\varphi(y)dy = \varphi(0),$$

for all $\varphi \in \mathscr{D}\left(\mathbb{R}^n\right)$, where $\mathscr{D}\left(\mathbb{R}^n\right)$ is the space of compactly supported smooth functions on $\mathbb{R}^n$.

1. Prove that if a fundamental solution E is smooth on an open set Ω that does not contain the origin, then E must be harmonic on Ω.
2. Check the function $E(x) = \phi(|x|)$ with

$$\phi(r) = \begin{cases} C_n r^{2-n} \text{ if } & n \neq 2, \\ C_2 \log r \text{ if } & n = 2, \end{cases}$$

   satisfies $\Delta E = 0$ in $\mathbb{R}^n \setminus \{0\}$.
3. Prove that the constants C_n can be tuned so that E is indeed a fundamental solution, and find the values of C_n.

Proof:
1. For any $\varphi \in \mathscr{D}(\Omega)$,

$$\begin{aligned} 0 = \varphi(0) &= \int_{\mathbb{R}^n} E\Delta\varphi = \int_\Omega E\Delta\varphi \\ &= \int_{\partial\Omega} E\partial_\nu\varphi - \int_\Omega \nabla E \cdot \nabla\varphi \\ &= -\int_\Omega \nabla E \cdot \nabla\varphi \\ &= \int_\Omega \varphi\Delta E - \int_{\partial\Omega} \varphi\partial_\nu E \\ &= \int_\Omega \varphi\Delta E. \end{aligned}$$

For any $x \in \Omega$, we can easily choose $\varphi_n \uparrow \delta(x)$, where $\delta(x)$ stands for Dirac function. And conclution can be deduced according to monotone convergence theorem.
2. Omit.
3. Omit.

□

Exercise 4: In this problem, we look into the possibility of using fundamental solutions of the Laplace operator as one of the functions in Green's second identity.

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with C^1 boundary. Let $y \in \Omega$, and we put $E_y(x) := E(x - y)$, where E is the special fundamental solution above.

For $u \in C^2(\Omega) \cap C^1\left(\overline{\Omega}\right)$, prove

$$u(y) = \int_\Omega E_y \Delta u \,\mathrm{d}x + \int_{\partial\Omega} u\partial_\nu E_y \,\mathrm{d}S - \int_{\partial\Omega} E_y \partial_\nu u \,\mathrm{d}S\,.$$

Proof: *Without lost of generality, we can assume $y = 0$ and prove*

$$u(0) = \int_{\Omega} \Phi \Delta u \, \mathrm{d}x + \int_{\partial\Omega} u \partial_\nu \Phi \, \mathrm{d}S - \int_{\partial\Omega} \Phi \partial_\nu u \, \mathrm{d}S$$

where Φ denotes the fundamental solution. Since

$$\int_{\partial\Omega} \Phi \partial_\nu u \, \mathrm{d}S = \int_{\Omega} \nabla \cdot (\Phi \nabla u) \, \mathrm{d}x = \int_{\Omega} (\nabla \Phi \cdot \nabla u + \Phi \Delta u) \, \mathrm{d}x,$$
$$\int_{\partial\Omega} u \partial_\nu \Phi \, \mathrm{d}S = \int_{\Omega} \nabla \cdot (u \nabla \Phi) \, \mathrm{d}x = \int_{\Omega} (\nabla u \cdot \nabla \Phi + u \Delta \Phi) \, \mathrm{d}x = u(y) + \int_{\Omega} \nabla u \cdot \nabla \Phi \, \mathrm{d}x,$$

The result can be reached after calculation. □

Exercise 5: Let Ω and u be as in Green's formula, and in addition, let $\Delta u = 0$ in Ω. Then by using Green's formula, prove that $u \in C^\infty(\Omega)$ with

$$\sup_{y \in K} |\partial^\alpha u(y)| \leqslant C \left(\sup_{\Omega} |u| + \sup_{\Omega} |\nabla u| \right),$$

for all multi-indices α and compact sets $K \subset \Omega$, with the constant C possibly depending on α and K.

Proof: *Use induction. For $|\alpha| = 0$, it is trivial. Assume condition holds for $|\alpha| = k$, we prove the condition $\partial_i \partial^\alpha$ still holds. Indeed, for every $y \in K$, set $B(y, \delta) \subset \Omega$, and*

$$|\partial_i \partial^\alpha u(y)| \leqslant \frac{C_i}{\delta} \sup_{B(y,\delta)} |\partial^\alpha u(y)| \leqslant \frac{C_i}{\delta} C_\alpha \left(\sup_{\Omega} |u| + \sup_{\Omega} |\nabla u| \right).$$

□

Exercise 6: Let $u \in C^2(\Omega)$ be harmonic in a bounded domain Ω. By using the Harnack Inequality show that unless u is constant, it cannot achieve its extremums in Ω.

Proof: *Without lost of generality, we can assume $\inf_\Omega u = 0$. Suppose $u(x_0) = 0, x_0 \in \Omega$, then we only need to prove $u \equiv 0$ in Ω. It is easy, because $\forall y \in \Omega$, we have $u(y) \leqslant C_{x_0,y} u(x_0) = 0$ according to Harnack Inequality.* □