

In this document, the omitted proofs in the main paper are provided.

A Proof of Lemma 2

Lemma 2. *Let h be an (α, β) -admissible noise probability density function on \mathbb{R} with the general form in Theorem 1. Define h^+ and h^- as*

$$h^+(z) = \begin{cases} 2 \cdot h(z) & (z \geq 0) \\ 0 & (z < 0) \end{cases} \quad \text{and} \quad h^-(z) = \begin{cases} 0 & (z \geq 0) \\ 2 \cdot h(z) & (z < 0) \end{cases}.$$

Let z^+ and z^- be random variables derived from h^+ and h^- , respectively. Set $\alpha = \alpha(\epsilon)$ and $\beta = \beta(\frac{\epsilon}{m})$. For a function $f : D^n \rightarrow \mathbb{R}^m$, let $S^{is} : D^n \rightarrow \mathbb{R}$ be a direction-oriented β -smooth upper bound on the direction-oriented local sensitivity of f for each $i \in [m]$ and $s \in \{+, -\}$. Thereafter, the algorithm

$$\forall i : \quad A(x)_i = f(x)_i + \begin{cases} \frac{S^{i+}(x)}{\alpha} \cdot z^+ & (\text{with a probability of } 1/2) \\ \frac{S^{i-}(x)}{\alpha} \cdot z^- & (\text{with a probability of } 1/2) \end{cases} \quad (1)$$

is ϵ -differentially private, when the following relation holds:

$$\forall x, y \in D^n, d(x, y) = 1 \text{ and } \forall i : S^{i-}(x) \leq e^\beta \cdot S^{i+}(y) \wedge S^{i+}(x) \leq e^\beta \cdot S^{i-}(y).$$

Proof. It is sufficient to show that for all x, y such that $d(x, y) = 1$,

$$\forall T \in \mathbb{R}^m : \Pr[A(x) = T] \leq e^\epsilon \cdot \Pr[A(y) = T].$$

From (1),

$$\begin{aligned} \Pr[A(x) = T] &= \prod_{i \in [m]} \Pr[A(x)_i = T_i] \\ \text{and } \Pr[A(x)_i = T_i] &= \begin{cases} \frac{1}{2} \cdot \Pr\left[z^+ = \frac{T_i - f(x)_i}{N^{i+}(x)}\right] & (T_i \geq f(x)_i) \\ \frac{1}{2} \cdot \Pr\left[z^- = \frac{T_i - f(x)_i}{N^{i-}(x)}\right] & (T_i < f(x)_i) \end{cases}, \end{aligned} \quad (2)$$

where $N^{i\pm}(x) := \frac{S^{i\pm}(x)}{\alpha}$. Hereafter, the difference between $\Pr[A(x)_i = T_i]$ and $\Pr[A(y)_i = T_i]$ is evaluated.

(I) $f(x)_i \leq f(y)_i$

(i) $T_i < f(x)_i$

$$\begin{aligned} \Pr[A(x)_i = T_i] &= \frac{1}{2} \cdot \Pr\left[z^- = \frac{T_i - f(x)_i}{N^{i-}(x)}\right] \\ &\leq \frac{1}{2} \cdot e^{\frac{\epsilon}{2m}} \cdot \Pr\left[z^- = \frac{T_i - f(x)_i}{N^{i-}(y)}\right] \left[\cdot \frac{N^{i-}(x)}{N^{i-}(y)} \leq e^{\beta(\frac{\epsilon}{m})}\right] \\ &\leq \frac{1}{2} \cdot e^{\frac{\epsilon}{2m}} \cdot e^{\frac{f(y)_i - f(x)_i}{2 \cdot S^{i-}(y)} \cdot \epsilon} \cdot \Pr\left[z^- = \frac{T_i - f(y)_i}{N^{i-}(y)}\right] \\ &= \left[\cdot \frac{f(x)_i - f(y)_i}{N^{i-}(y)} = -\alpha(\epsilon) \cdot \frac{f(y)_i - f(x)_i}{S^{i-}(y)} = -\alpha\left(\frac{f(y)_i - f(x)_i}{S^{i-}(y)} \cdot \epsilon\right)\right] \\ &= e^{\frac{\epsilon}{2m}} \cdot e^{\frac{f(y)_i - f(x)_i}{2 \cdot S^{i-}(y)} \cdot \epsilon} \cdot \Pr[A(y)_i = T_i]. \end{aligned}$$

(ii) $f(x)_i \leq T_i < f(y)_i$

$$\begin{aligned}
\Pr[A(x)_i = T_i] &= \frac{1}{2} \cdot \Pr \left[z^+ = \frac{T_i - f(x)_i}{N^{i+}(x)} \right] \\
&\leq \frac{1}{2} \cdot e^{\frac{f(y)_i - f(x)_i}{2 \cdot S^{i+}(x)} \cdot \epsilon} \cdot e^{\frac{\epsilon}{2m}} \cdot \Pr \left[z^+ = \frac{f(y)_i - T_i}{N^{i-}(y)} \right] \\
&\quad \left[\frac{f(y)_i + f(x)_i - 2T_i}{N^{i+}(x)} \leq \alpha(\epsilon) \cdot \frac{f(y)_i - f(x)_i}{S^{i+}(x)} = \alpha \left(\frac{f(y)_i - f(x)_i}{S^{i+}(x)} \cdot \epsilon \right), \right. \\
&\quad \left. \frac{N^{i+}(x)}{N^{i-}(y)} \leq e^{\beta(\frac{\epsilon}{m})} \right] \\
&= \frac{1}{2} \cdot e^{\frac{f(y)_i - f(x)_i}{2 \cdot S^{i+}(x)} \cdot \epsilon} \cdot e^{\frac{\epsilon}{2m}} \cdot \Pr \left[z^- = \frac{T_i - f(y)_i}{N^{i-}(y)} \right] \\
&= e^{\frac{f(y)_i - f(x)_i}{2 \cdot S^{i+}(x)} \cdot \epsilon} \cdot e^{\frac{\epsilon}{2m}} \cdot \Pr[A(y)_i = T_i].
\end{aligned}$$

(iii) $f(y)_i \leq T_i$

$$\begin{aligned}
\Pr[A(x)_i = T_i] &= \frac{1}{2} \cdot \Pr \left[z^+ = \frac{T_i - f(x)_i}{N^{i+}(x)} \right] \\
&\leq \frac{1}{2} \cdot \Pr \left[z^+ = \frac{T_i - f(y)_i}{N^{i+}(x)} \right] \quad [\because 0 \leq T_i - f(y)_i < T_i - f(x)_i] \\
&\leq \frac{1}{2} \cdot e^{\frac{\epsilon}{2m}} \cdot \Pr \left[z^+ = \frac{T_i - f(y)_i}{N^{i+}(y)} \right] \quad \left[\because \frac{N^{i+}(x)}{N^{i+}(y)} \leq e^{\beta(\frac{\epsilon}{m})} \right] \\
&= e^{\frac{\epsilon}{2m}} \cdot \Pr[A(y)_i = T_i].
\end{aligned}$$

(II) $f(y)_i < f(x)_i$

The argument is similar to that in (I).

Consequently,

$$\begin{aligned}
&\Pr[A(x)_i = T_i] \\
&\leq e^{\frac{\epsilon}{2m}} \cdot e^{\max \left\{ \frac{f(y)_i - f(x)_i}{2 \cdot \min\{S^{i+}(x), S^{i-}(y)\}}, \frac{f(x)_i - f(y)_i}{2 \cdot \min\{S^{i-}(x), S^{i+}(y)\}} \right\} \cdot \epsilon} \cdot \Pr[A(y)_i = T_i].
\end{aligned}$$

Here, the following equality holds:

$$\begin{aligned}
&\sum_{i \in [m]} \max \left\{ \frac{f(y)_i - f(x)_i}{2 \cdot \min\{S^{i+}(x), S^{i-}(y)\}}, \frac{f(x)_i - f(y)_i}{2 \cdot \min\{S^{i-}(x), S^{i+}(y)\}} \right\} \leq \frac{1}{2}. \\
&\left[\because \text{When } \forall i : f(x)_i \leq f(y)_i \text{ and } \min \left\{ \min_l S^{l+}(x), \min_l S^{l-}(y) \right\} = S^{j+}(x), \right. \\
&\quad \left. \sum_{i \in [m]} \max \left\{ \frac{f(y)_i - f(x)_i}{2 \cdot \min\{S^{i+}(x), S^{i-}(y)\}}, \frac{f(x)_i - f(y)_i}{2 \cdot \min\{S^{i-}(x), S^{i+}(y)\}} \right\} \right. \\
&= \sum_{i \in [m]} \frac{f(y)_i - f(x)_i}{2 \cdot \min\{S^{i+}(x), S^{i-}(y)\}} \leq \sum_{i \in [m]} \frac{f(y)_i - f(x)_i}{2 \cdot \min\{\min_l S^{l+}(x), \min_l S^{l-}(y)\}}
\end{aligned}$$

$$\begin{aligned}
&= \frac{\sum_{i \in [m]} (f(y)_i - f(x)_i)}{2 \cdot S^{j+}(x)} = \frac{(f(y)_j - f(x)_j) + \sum_{l \in [m] \setminus \{j\}} |f(x)_l - f(y)_l|}{2 \cdot S^{j+}(x)} \\
&\leq \frac{LS_f^{j+}(x)}{2 \cdot S^{j+}(x)} \leq \frac{1}{2}.
\end{aligned}$$

The same argument holds true also in the other cases. \square

From the above discussion and the relation (2),

$$\begin{aligned}
\Pr[A(x) = T] &= \prod_{i \in [m]} \Pr[A(x)_i = T_i] \\
&\leq e^{\frac{\epsilon}{2}} \cdot e^{\frac{\epsilon}{2}} \cdot \prod_{i \in [m]} \Pr[A(y)_i = T_i] = e^\epsilon \cdot \Pr[A(y) = T]
\end{aligned}$$

holds for any $T \in \mathbb{R}^m$. \square

B Proof of Theorem 1

Theorem 1. For any $k > 1$ and $l > 0$, the distribution with density $h(z) \propto \frac{1}{|z|^{k+l}}$ is

$$\left(\frac{l^{\frac{1}{k}} \cdot \epsilon}{2 \cdot (k-1)^{\frac{k-1}{k}}}, \frac{\epsilon}{2(k-1)} \right)\text{-admissible.}$$

Proof. We set

$$\alpha = \frac{l^{\frac{1}{k}} \cdot \epsilon}{2 \cdot (k-1)^{\frac{k-1}{k}}}, \quad \text{and} \quad \beta = \frac{\epsilon}{2(k-1)}.$$

First, we analyze the sliding property. From Definition 6, it is sufficient to show that $\ln \left(\frac{h(z)}{h(z+\Delta)} \right)$ is at most $\frac{\epsilon}{2}$ when $|\Delta| \leq \alpha$. Where $\phi(x) = \ln(x^k + l)$, the following equalities hold:

$$\ln \left(\frac{h(z)}{h(z+\Delta)} \right) = \ln \left(\frac{|z+\Delta|^k + l}{|z|^k + l} \right) = \phi(|z+\Delta|) - \phi(|z|).$$

Here, there exists $\zeta > 0$ such that $\phi(|z+\Delta|) - \phi(|z|) \leq |\Delta| \cdot |\phi'(\zeta)|$. Because

$$\phi'(\zeta) = \frac{k \cdot \zeta^{k-1}}{\zeta^k + l} = \frac{k}{\zeta + l \cdot \zeta^{1-k}} \leq \frac{((k-1) \cdot l)^{\frac{k-1}{k}}}{l} = \frac{(k-1)^{\frac{k-1}{k}}}{l^{\frac{1}{k}}},$$

$$\ln \left(\frac{h(z)}{h(z+\Delta)} \right) \leq |\Delta| \cdot \frac{(k-1)^{\frac{k-1}{k}}}{l^{\frac{1}{k}}}. \quad \text{Thus, when } |\Delta| \leq \alpha, \ln \left(\frac{h(z)}{h(z+\Delta)} \right) \leq \frac{\epsilon}{2}.$$

Thereafter, we analyze the dilation property. Similar to the above, it is sufficient to show that $\ln \left(\frac{h(z)}{e^\lambda \cdot h(e^\lambda \cdot z)} \right)$ is at most $\frac{\epsilon}{2}$ when $|\lambda| \leq \beta$. Because

$$\ln \left(\frac{h(z)}{e^\lambda \cdot h(e^\lambda \cdot z)} \right) = \ln \left(\frac{1}{e^\lambda} \cdot \frac{e^{\lambda k} \cdot |z|^k + l}{|z|^k + l} \right),$$

when $\lambda \geq 0$,

$$\ln \left(\frac{h(z)}{e^\lambda \cdot h(e^\lambda \cdot z)} \right) \leq \ln \left(\frac{1}{e^\lambda} \cdot \frac{e^{\lambda k} \cdot |z|^k + e^{\lambda k} \cdot l}{|z|^k + l} \right) = \lambda(k-1),$$

and when $\lambda < 0$,

$$\ln \left(\frac{h(z)}{e^\lambda \cdot h(e^\lambda \cdot z)} \right) \leq \ln \left(\frac{1}{e^\lambda} \right) = -\lambda = |\lambda|.$$

Thus, when $|\lambda| \leq \beta$, $\ln \left(\frac{h(z)}{e^\lambda \cdot h(e^\lambda \cdot z)} \right) \leq \frac{\epsilon}{2}$. \square

C Proof of Theorem 2

Theorem 2. For all $i \in [m]$ and $s \in \{+, -\}$, $S_{f,\beta}^{*is}$ is a direction-oriented β -smooth upper bound on LS_f^{is} . In addition, for all $x \in D^n$, $S_{f,\beta}^{*is}(x) \leq S^{is}(x)$ holds for every direction-oriented β -smooth upper bound S^{is} on LS_f^{is} .

Proof.

$$\begin{aligned} S_{f,\beta}^{*is}(x) &= \max_{y \in D^n} \left(LS_f^{is}(y) \cdot e^{-\beta \cdot d(y,x)} \right). \\ &= \max \left\{ LS_f^{is}(x), \max_{y \neq x} \left(LS_f^{is}(y) \cdot e^{-\beta \cdot d(y,x)} \right) \right\} \geq LS_f^{is}(x). \end{aligned}$$

Next we fix $x, y \in D^n$ with $d(x, y) = 1$. Let $S_{f,\beta}^{*is}(x) = LS_f^{is}(x') \cdot e^{-\beta \cdot d(x',x)}$ for some $x' \in D^n$. Then,

$$\begin{aligned} S_{f,\beta}^{*is}(y) &\geq LS_f^{is}(x') \cdot e^{-\beta \cdot d(x',y)} \\ &\geq LS_f^{is}(x') \cdot e^{-\beta \cdot (d(x',x)+1)} \quad [\cdot: d(x',y) \leq d(x',x) + d(x,y)] \\ &= e^{-\beta} \cdot LS_f^{is}(x') \cdot e^{-\beta \cdot d(x',x)} = e^{-\beta} \cdot S_{f,\beta}^{*is}(x). \end{aligned}$$

Furthermore, we show that $\forall x \in D^n : S_{f,\beta}^{*is}(x) \leq S^{is}(x)$. From the definition of $S_{f,\beta}^{*is}(x)$, it is sufficient to show that $\forall x, y \in D^n : S^{is}(x) \geq LS_f^{is}(y) \cdot e^{-\beta \cdot d(y,x)}$.

As the base case, $S^{is}(x) \geq LS_f^{is}(x)$ holds from the definition of S^{is} .

For the induction step, we suppose that $\forall x', y \in D^n$ s.t. $d(x', y) = k : S^{is}(x') \geq LS_f^{is}(y) \cdot e^{-\beta \cdot d(y,x')}$. Here, for all $x \in D^n$ such that $d(x, y) = k+1$, there exists some x' satisfying $d(x, x') = 1 \wedge d(x', y) = k$. Therefore,

$$\begin{aligned} S^{is}(x) &\geq e^{-\beta} \cdot S^{is}(x') \quad [\cdot: \text{the definition of } S^{is}] \\ &\geq e^{-\beta} \cdot LS_f^{is}(y) \cdot e^{-\beta \cdot d(y,x')} \\ &= LS_f^{is}(y) \cdot e^{-\beta \cdot (d(y,x')+1)} = LS_f^{is}(y) \cdot e^{-\beta \cdot d(y,x)}. \end{aligned}$$

Consequently, $\forall x, y \in D^n : S^{is}(x) \geq LS_f^{is}(y) \cdot e^{-\beta \cdot d(y,x)}$ holds. \square

D Proof of Theorem 3

Theorem 3. For all $x \in D^n$ and $i \in [m]$ and $s \in \{+, -\}$, $S_{f,\beta}^{*is}(x) \leq S_{f,\beta}^*(x)$ holds.

Proof.

$$\begin{aligned}
LS_f(x) &= \max_{y:d(x,y)=1} \|f(x) - f(y)\|_1 \\
&= \max_{y:d(x,y)=1} \left(\sum_{j \in [m]} |f(x)_j - f(y)_j| \right) \\
&= \max \left\{ \max_{y:d(x,y)=1} \left((f(y)_i - f(x)_i) + \sum_{j \in [m] \setminus \{i\}} |f(x)_j - f(y)_j| \right), \right. \\
&\quad \left. \max_{y:d(x,y)=1} \left((f(x)_i - f(y)_i) + \sum_{j \in [m] \setminus \{i\}} |f(x)_j - f(y)_j| \right) \right\} \\
&\geq LS_f^{is}(x)
\end{aligned}$$

holds for any $i \in [m]$ and $s \in \{+, -\}$. Therefore,

$$\begin{aligned}
S_{f,\beta}^{*is}(x) &= \max_y \left(LS_f^{is}(y) \cdot e^{-\beta \cdot d(y,x)} \right) \\
&\leq \max_y \left(LS_f(y) \cdot e^{-\beta \cdot d(y,x)} \right) = S_{f,\beta}^*(x).
\end{aligned}$$

□

E Proof of Theorem 4

Theorem 4. For all $\beta > 0$ and $i \in [m]$ and $s \in \{+, -\}$,

$$\begin{aligned}
S_{f,\beta}^{*is}(x) &\left(= \max_{y \in D^n} \left(LS_f^{is}(y) \cdot e^{-\beta \cdot d(y,x)} \right) \right) \\
&= \max_{y: d(y,x) \leq \frac{1}{\beta} \cdot \ln \left(\frac{GS_f}{LS_f^{is}(x)} \right)} \left(LS_f^{is}(y) \cdot e^{-\beta \cdot d(y,x)} \right). \tag{3}
\end{aligned}$$

Proof. When $d(y,x) > \frac{1}{\beta} \cdot \ln \left(\frac{GS_f}{LS_f^{is}(x)} \right)$,

$$\begin{aligned}
LS_f^{is}(y) \cdot e^{-\beta \cdot d(y,x)} &< LS_f^{is}(y) \cdot \frac{LS_f^{is}(x)}{GS_f} \\
&\leq LS_f^{is}(x). \quad [\because LS_f^{is}(y) \leq LS_f(y) \leq GS_f]
\end{aligned}$$

From this and that $\forall x : S_{f,\beta}^{*is}(x) \geq LS_f^{is}(x)$, the equality (3) holds. □

F Proof of Theorem 5

Theorem 5. *For all β satisfying*

$$e^\beta \geq \max_{x, x': d(x, x')=1} \frac{LS_f^{is}(x')}{LS_f^{is}(x)} \quad (4)$$

for any $i \in [m]$ and $s \in \{+, -\}$, the following equality holds:

$$\forall x \in D^n : S_{f, \beta}^{*is}(x) = LS_f^{is}(x).$$

Proof. We show that when $e^\beta \geq \max_{x, x': d(x, x')=1} \frac{LS_f^{is}(x')}{LS_f^{is}(x)}$,

$$\forall y \in D^n : LS_f^{is}(x) \geq LS_f^{is}(y) \cdot e^{-\beta \cdot d(y, x)}.$$

(I) When $d(y, x) = 1$,

$$\begin{aligned} LS_f^{is}(x) &\geq LS_f^{is}(y) \cdot e^{-\beta} \quad [\cdot (4)] \\ &= LS_f^{is}(y) \cdot e^{-\beta \cdot d(y, x)}. \end{aligned}$$

(II) We assume that $LS_f^{is}(x) \geq LS_f^{is}(y) \cdot e^{-\beta \cdot d(y, x)}$ holds for all $y \in D^n$ satisfying $d(y, x) = k$. Here, for any y' satisfying $d(y', x) = k + 1$,

$$\exists y : d(y', y) = 1 \wedge d(y, x) = k.$$

Therefore,

$$\begin{aligned} LS_f^{is}(x) &\geq LS_f^{is}(y) \cdot e^{-\beta \cdot d(y, x)} \geq (LS_f^{is}(y') \cdot e^{-\beta}) \cdot e^{-\beta \cdot d(y, x)} \quad [\cdot (4)] \\ &= LS_f^{is}(y') \cdot e^{-\beta \cdot (k+1)} = LS_f^{is}(y') \cdot e^{-\beta \cdot d(y', x)}. \end{aligned}$$

Consequently,

$$S_{f, \beta}^{*is}(x) = \max_{y \in D^n} (LS_f^{is}(y) \cdot e^{-\beta \cdot d(y, x)}) = LS_f^{is}(x)$$

holds for all $x \in D^n$. □