In this document, the omitted proofs in the main paper are provided.

A Proof of Lemma 2

Lemma 2. *Let h be an (α, β)-admissible noise probability density function on $\mathbb{R}$ with the general form in Theorem 1. Define h^+ and h^- as*

$$h^+(z) = \begin{cases} 2 \cdot h(z) & (z \geq 0) \\ 0 & (z < 0) \end{cases} \quad \text{and} \quad h^-(z) = \begin{cases} 0 & (z \geq 0) \\ 2 \cdot h(z) & (z < 0) \end{cases} .$$

Let z^+ and z^- be random variables derived from h^+ and h^-, respectively. Set $\alpha = \alpha(\epsilon)$ and $\beta = \beta\left(\frac{\epsilon}{m}\right)$. For a function $f : D^n \to \mathbb{R}^m$, let $S^{i\,s} : D^n \to \mathbb{R}$ be a direction-oriented β-smooth upper bound on the direction-oriented local sensitivity of f for each $i \in [m]$ and $s \in \{+, -\}$. Thereafter, the algorithm

$$\forall i: \quad A(x)_i = f(x)_i + \begin{cases} \frac{S^{i\,+}(x)}{\alpha} \cdot z^+ & \text{(with a probability of 1/2)} \\ \frac{S^{i\,-}(x)}{\alpha} \cdot z^- & \text{(with a probability of 1/2)} \end{cases} \tag{1}$$

is ϵ-differentially private.

Proof. It is sufficient to show that for all x, y such that $d(x, y) = 1$,

$$\forall T \in \mathbb{R}^m : \ \Pr[A(x) = T] \leq e^{\epsilon} \cdot \Pr[A(y) = T].$$

From (1),

$$\Pr[A(x) = T] = \prod_{i \in [m]} \Pr[A(x)_i = T_i] \tag{2}$$

$$\text{and} \quad \Pr[A(x)_i = T_i] = \begin{cases} \frac{1}{2} \cdot \Pr\left[z^+ = \frac{T_i - f(x)_i}{N^{i\,+}(x)}\right] & (T_i \geq f(x)_i) \\ \frac{1}{2} \cdot \Pr\left[z^- = \frac{T_i - f(x)_i}{N^{i\,-}(x)}\right] & (T_i < f(x)_i) \end{cases} ,$$

where $N^{i\,\pm}(x) := \frac{S^{i\,\pm}(x)}{\alpha}$. Hereafter, the difference between $\Pr[A(x)_i = T_i]$ and $\Pr[A(y)_i = T_i]$ is evaluated.

(I) $f(x)_i \leq f(y)_i$

(i) $T_i < f(x)_i$

$$\begin{aligned} \Pr[A(x)_i = T_i] &= \frac{1}{2} \cdot \Pr\left[z^- = \frac{T_i - f(x)_i}{N^{i\,-}(x)}\right] \\ &\leq \frac{1}{2} \cdot e^{\frac{\epsilon}{2m}} \cdot \Pr\left[z^- = \frac{T_i - f(x)_i}{N^{i\,-}(y)}\right] \quad \left[\because \frac{N^{i\,-}(x)}{N^{i\,-}(y)} \leq e^{\beta\left(\frac{\epsilon}{m}\right)}\right] \\ &\leq \frac{1}{2} \cdot e^{\frac{\epsilon}{2m}} \cdot e^{\frac{f(y)_i - f(x)_i}{2 \cdot S^{i-}(y)} \cdot \epsilon} \cdot \Pr\left[z^- = \frac{T_i - f(y)_i}{N^{i\,-}(y)}\right] \end{aligned}$$

$$\left[\because \frac{f(x)_i - f(y)_i}{N^{i\,-}(y)} = -\alpha(\epsilon) \cdot \frac{f(y)_i - f(x)_i}{S^{i\,-}(y)} = -\alpha\left(\frac{f(y)_i - f(x)_i}{S^{i\,-}(y)} \cdot \epsilon\right)\right]$$

$$= e^{\frac{\epsilon}{2m}} \cdot e^{\frac{f(y)_i - f(x)_i}{2 \cdot S^{i-}(y)} \cdot \epsilon} \cdot \Pr[A(y)_i = T_i].$$

(ii) $f(x)_i \leq T_i < f(y)_i$

$$\begin{aligned}\Pr[A(x)_i = T_i] &= \frac{1}{2} \cdot \Pr\left[z^+ = \frac{T_i - f(x)_i}{N^{\,i+}(x)}\right] \\ &\leq \frac{1}{2} \cdot e^{\frac{f(y)_i - f(x)_i}{2\cdot\min\{S^{i+}(x), S^{i-}(y)\}}\cdot\epsilon} \cdot \Pr\left[z^+ = \frac{f(y)_i - T_i}{N^{\,i-}(y)}\right]\end{aligned}$$

$$\left[\because 0 \leq \frac{T_i - f(x)_i}{N^{\,i+}(x)} < \alpha(\epsilon) \cdot \frac{f(y)_i - f(x)_i}{S^{\,i+}(x)} = \alpha\left(\frac{f(y)_i - f(x)_i}{S^{\,i+}(x)} \cdot \epsilon\right),\right.$$

$$\left. 0 < \frac{f(y)_i - T_i}{N^{\,i-}(y)} \leq \alpha(\epsilon) \cdot \frac{f(y)_i - f(x)_i}{S^{\,i-}(y)} = \alpha\left(\frac{f(y)_i - f(x)_i}{S^{\,i-}(y)} \cdot \epsilon\right)\right]$$

$$\begin{aligned}&= \frac{1}{2} \cdot e^{\frac{f(y)_i - f(x)_i}{2\cdot\min\{S^{i+}(x), S^{i-}(y)\}}\cdot\epsilon} \cdot \Pr\left[z^- = \frac{T_i - f(y)_i}{N^{\,i-}(y)}\right] \\ &= e^{\frac{f(y)_i - f(x)_i}{2\cdot\min\{S^{i+}(x), S^{i-}(y)\}}\cdot\epsilon} \cdot \Pr[A(y)_i = T_i].\end{aligned}$$

(iii) $f(y)_i \leq T_i$

$$\begin{aligned}\Pr[A(x)_i = T_i] &= \frac{1}{2} \cdot \Pr\left[z^+ = \frac{T_i - f(x)_i}{N^{\,i+}(x)}\right] \\ &\leq \frac{1}{2} \cdot \Pr\left[z^+ = \frac{T_i - f(y)_i}{N^{\,i+}(x)}\right] \quad [\because 0 \leq T_i - f(y)_i < T_i - f(x)_i] \\ &\leq \frac{1}{2} \cdot e^{\frac{\epsilon}{2m}} \cdot \Pr\left[z^+ = \frac{T_i - f(y)_i}{N^{\,i+}(y)}\right] \quad \left[\because \frac{N^{\,i+}(x)}{N^{\,i+}(y)} \leq e^{\beta\left(\frac{\epsilon}{m}\right)}\right] \\ &= e^{\frac{\epsilon}{2m}} \cdot \Pr[A(y)_i = T_i].\end{aligned}$$

(II) $f(y)_i < f(x)_i$

The argument is similar to that in (I).

Consequently,

$$\begin{aligned}&\Pr[A(x)_i = T_i] \\ &\leq e^{\frac{\epsilon}{2m}} \cdot e^{\max\left\{\frac{f(y)_i - f(x)_i}{2\cdot\min\{S^{i+}(x),\, S^{i-}(y)\}},\, \frac{f(x)_i - f(y)_i}{2\cdot\min\{S^{i-}(x),\, S^{i+}(y)\}}\right\}\cdot\epsilon} \cdot \Pr[A(y)_i = T_i].\end{aligned}$$

Here, the following equality holds:

$$\sum_{i\in[m]} \max\left\{\frac{f(y)_i - f(x)_i}{2\cdot\min\{S^{\,i+}(x),\, S^{\,i-}(y)\}},\, \frac{f(x)_i - f(y)_i}{2\cdot\min\{S^{\,i-}(x),\, S^{\,i+}(y)\}}\right\} \leq \frac{1}{2}.$$

$$\left[\because \text{When } \forall i: f(x)_i \leq f(y)_i \text{ and } \min\left\{\min_l S^{\,l+}(x),\, \min_l S^{\,l-}(y)\right\} = S^{\,j+}(x),\right.$$

$$\sum_{i\in[m]} \max\left\{\frac{f(y)_i - f(x)_i}{2\cdot\min\{S^{\,i+}(x),\, S^{\,i-}(y)\}},\, \frac{f(x)_i - f(y)_i}{2\cdot\min\{S^{\,i-}(x),\, S^{\,i+}(y)\}}\right\}$$

$$= \sum_{i\in[m]} \frac{f(y)_i - f(x)_i}{2\cdot\min\{S^{\,i+}(x),\, S^{\,i-}(y)\}} \leq \sum_{i\in[m]} \frac{f(y)_i - f(x)_i}{2\cdot\min\{\min_l S^{\,l+}(x),\, \min_l S^{\,l-}(y)\}}$$

$$= \frac{\sum_{i\in[m]}(f(y)_i - f(x)_i)}{2\cdot S^{j+}(x)} = \frac{(f(y)_j - f(x)_j) + \sum_{l\in[m]\setminus\{j\}} |f(x)_l - f(y)_l|}{2\cdot S^{j+}(x)}$$
$$\leq \frac{LS_f^{j+}(x)}{2\cdot S^{j+}(x)} \leq \frac{1}{2}.$$

The same argument holds true also in the other cases. ⌋

From the above discussion and the relation (2),

$$\Pr[A(x) = T] = \prod_{i\in[m]} \Pr[A(x)_i = T_i]$$
$$\leq e^{\frac{\epsilon}{2}} \cdot e^{\frac{\epsilon}{2}} \cdot \prod_{i\in[m]} \Pr[A(y)_i = T_i] = e^{\epsilon} \cdot \Pr[A(y) = T]$$

holds for any $T \in \mathbb{R}^m$. □

B Proof of Theorem 1

Theorem 1. *For any $k > 1$ and $l > 0$, the distribution with density $h(z) \propto \frac{1}{|z|^k + l}$ is*

$$\left(\frac{l^{\frac{1}{k}}\cdot\epsilon}{2\cdot(k-1)^{\frac{k-1}{k}}},\quad \frac{\epsilon}{2(k-1)}\right)\text{-}admissible.$$

Proof. We set

$$\alpha = \frac{l^{\frac{1}{k}}\cdot\epsilon}{2\cdot(k-1)^{\frac{k-1}{k}}},\quad \text{and}\quad \beta = \frac{\epsilon}{2(k-1)}.$$

First, we analyze the sliding property. From Definition 6, it is sufficient to show that $\ln\left(\frac{h(z)}{h(z+\Delta)}\right)$ is at most $\frac{\epsilon}{2}$ when $|\Delta| \leq \alpha$. Where $\phi(x) = \ln(x^k + l)$, the following equalities hold:

$$\ln\left(\frac{h(z)}{h(z+\Delta)}\right) = \ln\left(\frac{|z+\Delta|^k + l}{|z|^k + l}\right) = \phi(|z+\Delta|) - \phi(|z|).$$

Here, there exists $\zeta > 0$ such that $\phi(|z+\Delta|) - \phi(|z|) \leq |\Delta| \cdot |\phi'(\zeta)|$. Because

$$\phi'(\zeta) = \frac{k\cdot\zeta^{k-1}}{\zeta^k + l} = \frac{k}{\zeta + l\cdot\zeta^{1-k}} \leq \frac{((k-1)\cdot l)^{\frac{k-1}{k}}}{l} = \frac{(k-1)^{\frac{k-1}{k}}}{l^{\frac{1}{k}}},$$

$\ln\left(\frac{h(z)}{h(z+\Delta)}\right) \leq |\Delta| \cdot \frac{(k-1)^{\frac{k-1}{k}}}{l^{\frac{1}{k}}}$. Thus, when $|\Delta| \leq \alpha$, $\ln\left(\frac{h(z)}{h(z+\Delta)}\right) \leq \frac{\epsilon}{2}$.

Thereafter, we analyze the dilation property. Similar to the above, it is sufficient to show that $\ln\left(\frac{h(z)}{e^{\lambda}\cdot h(e^{\lambda}\cdot z)}\right)$ is at most $\frac{\epsilon}{2}$ when $|\lambda| \leq \beta$. Because

$$\ln\left(\frac{h(z)}{e^{\lambda}\cdot h(e^{\lambda}\cdot z)}\right) = \ln\left(\frac{1}{e^{\lambda}} \cdot \frac{e^{\lambda k}\cdot|z|^k + l}{|z|^k + l}\right),$$

when $\lambda \geq 0$,

$$\ln\left(\frac{h(z)}{e^{\lambda} \cdot h(e^{\lambda} \cdot z)}\right) \leq \ln\left(\frac{1}{e^{\lambda}} \cdot \frac{e^{\lambda k} \cdot |z|^k + e^{\lambda k} \cdot l}{|z|^k + l}\right) = \lambda(k-1),$$

and when $\lambda < 0$,

$$\ln\left(\frac{h(z)}{e^{\lambda} \cdot h(e^{\lambda} \cdot z)}\right) \leq \ln\left(\frac{1}{e^{\lambda}}\right) = -\lambda = |\lambda|.$$

Thus, when $|\lambda| \leq \beta$, $\ln\left(\frac{h(z)}{e^{\lambda} \cdot h(e^{\lambda} \cdot z)}\right) \leq \frac{\epsilon}{2}$. □

C Proof of Theorem 2

Theorem 2. *For all $i \in [m]$ and $s \in \{+,-\}$, $S^{*\,i\,s}_{f,\beta}$ is a direction-oriented β-smooth upper bound on $LS^{i\,s}_f$. In addition, for all $x \in D^n$, $S^{*\,i\,s}_{f,\beta}(x) \leq S^{i\,s}(x)$ holds for every direction-oriented β-smooth upper bound $S^{i\,s}$ on $LS^{i\,s}_f$.*

Proof.

$$\begin{aligned} S^{*\,i\,s}_{f,\beta}(x) &= \max_{y \in D^n}\left(LS^{i\,s}_f(y) \cdot e^{-\beta \cdot d(y,x)}\right). \\ &= \max\left\{LS^{i\,s}_f(x),\ \max_{y \neq x}\left(LS^{i\,s}_f(y) \cdot e^{-\beta \cdot d(y,x)}\right)\right\} \geq LS^{i\,s}_f. \end{aligned}$$

Next we fix $x, y \in D^n$ with $d(x,y) = 1$. Let $S^{*\,i\,s}_{f,\beta}(x) = LS^{i\,s}_f(x') \cdot e^{-\beta \cdot d(x',x)}$ for some $x' \in D^n$. Then,

$$\begin{aligned} S^{*\,i\,s}_{f,\beta}(y) &\geq LS^{i\,s}_f(x') \cdot e^{-\beta \cdot d(x',y)} \\ &\geq LS^{i\,s}_f(x') \cdot e^{-\beta \cdot (d(x',x)+1)} \quad [\because d(x',y) \leq d(x',x) + d(x,y)] \\ &= e^{-\beta} \cdot LS^{i\,s}_f(x') \cdot e^{-\beta \cdot d(x',x)} = e^{-\beta} \cdot S^{*\,i\,s}_{f,\beta}(x). \end{aligned}$$

Furthermore, we show that $\forall x \in D^n : S^{*\,i\,s}_{f,\beta}(x) \leq S^{i\,s}(x)$. From the definition of $S^{*\,i\,s}_{f,\beta}(x)$, it is sufficient to show that $\forall x, y \in D^n : S^{i\,s}(x) \geq LS^{i\,s}_f(y) \cdot e^{-\beta \cdot d(y,x)}$.

As the base case, $S^{i\,s}(x) \geq LS^{i\,s}_f(x)$ holds from the definition of $S^{i\,s}$.

For the induction step, we suppose that $\forall x', y \in D^n$ s.t. $d(x',y) = k$: $S^{i\,s}(x') \geq LS^{i\,s}_f(y) \cdot e^{-\beta \cdot d(y,x')}$. Here, for all $x \in D^n$ such that $d(x,y) = k+1$, there exists some x' satisfying $d(x,x') = 1 \wedge d(x',y) = k$. Therefore,

$$\begin{aligned} S^{i\,s}(x) &\geq e^{-\beta} \cdot S^{i\,s}(x') \quad [\because \text{the definition of } S^{i\,s}] \\ &\geq e^{-\beta} \cdot LS^{i\,s}_f(y) \cdot e^{-\beta \cdot d(y,x')} \\ &= LS^{i\,s}_f(y) \cdot e^{-\beta \cdot (d(y,x')+1)} = LS^{i\,s}_f(y) \cdot e^{-\beta \cdot d(y,x)}. \end{aligned}$$

Consequently, $\forall x, y \in D^n : S^{i\,s}(x) \geq LS^{i\,s}_f(y) \cdot e^{-\beta \cdot d(y,x)}$ holds. □

D Proof of Theorem 3

Theorem 3. *For all $x \in D^n$ and $i \in [m]$ and $s \in \{+,-\}$, $S^{*\,i\,s}_{f,\beta}(x) \le S^*_{f,\beta}(x)$ holds.*

Proof.

$$\begin{aligned}
LS_f(x) &= \max_{y:d(x,y)=1} \|f(x)-f(y)\|_1 \\
&= \max_{y:d(x,y)=1} \left(\sum_{j\in[m]} |f(x)_j - f(y)_j| \right) \\
&= \max\Bigg\{ \max_{y:d(x,y)=1} \left((f(y)_i - f(x)_i) + \sum_{j\in[m]\setminus\{i\}} |f(x)_j - f(y)_j| \right), \\
&\qquad\qquad \max_{y:d(x,y)=1} \left((f(x)_i - f(y)_i) + \sum_{j\in[m]\setminus\{i\}} |f(x)_j - f(y)_j| \right) \Bigg\} \\
&\ge LS^{i\,s}_f(x)
\end{aligned}$$

holds for any $i \in [m]$ and $s \in \{+,-\}$. Therefore,

$$\begin{aligned}
S^{*\,i\,s}_{f,\beta}(x) &= \max_y \left(LS^{i\,s}_f(y) \cdot e^{-\beta\cdot d(y,x)} \right) \\
&\le \max_y \left(LS_f(y) \cdot e^{-\beta\cdot d(y,x)} \right) = S^*_{f,\beta}(x).
\end{aligned}$$

□

E Proof of Theorem 4

Theorem 4. *For all $\beta > 0$ and $i \in [m]$ and $s \in \{+,-\}$,*

$$\begin{aligned}
S^{*\,i\,s}_{f,\beta}(x) &\left(= \max_{y\in D^n} \left(LS^{i\,s}_f(y) \cdot e^{-\beta\cdot d(y,x)} \right) \right) \\
&= \max_{y:d(y,x)\le \frac{1}{\beta}\cdot\ln\left(\frac{GS_f}{LS^{i\,s}_f(x)}\right)} \left(LS^{i\,s}_f(y) \cdot e^{-\beta\cdot e^{d(y,x)}} \right). \qquad (3)
\end{aligned}$$

Proof. When $d(y,x) > \frac{1}{\beta} \cdot \ln\left(\frac{GS_f}{LS^{i\,s}_f(x)}\right)$,

$$\begin{aligned}
LS^{i\,s}_f(y) \cdot e^{-\beta\cdot d(y,x)} &< LS^{i\,s}_f(y) \cdot \frac{LS^{i\,s}_f(x)}{GS_f} \\
&\le LS^{i\,s}_f(x). \quad [\because LS^{i\,s}_f(y) \le LS_f(y) \le GS_f]
\end{aligned}$$

From this and that $\forall x : S^{*\,i\,s}_{f,\beta}(x) \ge LS^{i\,s}_f(x)$, the equality (3) holds. □

F Proof of Theorem 5

Theorem 5. *For all β satisfying*

$$e^{\beta} \geq \max_{x,x':d(x,x')=1} \frac{LS_f^{i\,s}(x')}{LS_f^{i\,s}(x)} \tag{4}$$

for any $i \in [m]$ and $s \in \{+,-\}$, the following equality holds:

$$\forall x \in D^n : \; S_{f,\beta}^{*\,i\,s}(x) = LS_f^{i\,s}(x).$$

Proof. We show that when $e^{\beta} \geq \max_{x,x':d(x,x')=1} \frac{LS_f^{i\,s}(x')}{LS_f^{i\,s}(x)}$,

$$\forall y \in D^n : \; LS_f^{i\,s}(x) \geq LS_f^{i\,s}(y) \cdot e^{-\beta \cdot d(y,x)}.$$

(I) When $d(y,x) = 1$,

$$\begin{aligned} LS_f^{i\,s}(x) &\geq LS_f^{i\,s}(y) \cdot e^{-\beta} \quad [\because (4)] \\ &= LS_f^{i\,s}(y) \cdot e^{-\beta \cdot d(y,x)}. \end{aligned}$$

(II) We assume that $LS_f^{i\,s}(x) \geq LS_f^{i\,s}(y) \cdot e^{-\beta \cdot d(y,x)}$ holds for all $y \in D^n$ satisfying $d(y,x) = k$. Here, for any y' satisfying $d(y',x) = k+1$,

$$\exists y : \; d(y',y) = 1 \wedge d(y,x) = k.$$

Therefore,

$$\begin{aligned} LS_f^{i\,s}(x) \geq LS_f^{i\,s}(y) \cdot e^{-\beta \cdot d(y,x)} &\geq \left(LS_f^{i\,s}(y') \cdot e^{-\beta}\right) \cdot e^{-\beta \cdot d(y,x)} \quad [\because (4)] \\ &= LS_f^{i\,s}(y') \cdot e^{-\beta \cdot (k+1)} = LS_f^{i\,s}(y') \cdot e^{-\beta \cdot d(y',x)}. \end{aligned}$$

Consequently,

$$S_{f,\beta}^{*\,i\,s}(x) = \max_{y \in D^n} \left(LS_f^{i\,s}(y) \cdot e^{-\beta \cdot d(y,x)}\right) = LS_f^{i\,s}(x)$$

holds for all $x \in D^n$. $\square$