Assignment - 1

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1. Prove that there is no positive integer n such that $n^2 + n^3 = 100$: We can prove this by contradiction. Assume there exists a positive integer n such that $n^2 + n^3 = 100$.

$$\begin{aligned} n^2 + n^3 &= 100 \\ n^3 &= 100 - n^2 \\ n^3 &= (10 - n)(10 + n) > 0 \\ \text{Thus, } &-10 < n < 10 \\ \implies &1 < n < 10 \text{ as } n \text{ is positive.} \end{aligned}$$

But there does not exist any value of n in the given range for which the above equation holds true, which contradicts the assumption. Hence, there is no positive integer n satisfying the equation.

2. Prove that $n^2+1 \geq 2n$ when n is a positive integer with $1 \leq n \leq 4$: We can prove this by induction.

- **Base case** $(n = 1)$:
$$1^2 + 1 = 2 \geq 2 \times 1$$

- **Inductive step:** Assume the inequality holds for some positive integer k such that $1 \leq k \leq 4$:
$$k^2 + 1 \geq 2k$$
  Now, we need to prove it for $k + 1$:
$$\begin{aligned} (k+1)^2 + 1 &= k^2 + 2k + 1 + 1 \\ &= (k^2 + 1) + 2k + 1 \\ &\geq (2k) + 2k + 1 \quad \text{(by the inductive hypothesis)} \\ &= 2(k + 1) \end{aligned}$$
  Thus, by induction, the inequality holds for all positive integers n with $1 \leq n \leq 4$.

3. Find a compound proposition involving the propositional variables p, q, r, and s that is true when exactly three of these propositional variables are true and is false otherwise:

A compound proposition that satisfies the given condition is:

$$(p \land q \land r \land \neg s) \lor (p \land q \land \neg r \land s) \lor (p \land \neg q \land r \land s) \lor (\neg p \land q \land r \land s)$$

4. Let $P(x)$ and $Q(x)$ be propositional functions. Show that

$$\exists x(P(x) \to Q(x)) \quad \text{and} \quad \forall x P(x) \to \exists x Q(x)$$

always have the same truth value.

To show that $\exists x(P(x) \to Q(x))$ and $\forall x P(x) \to \exists x Q(x)$ always have the same truth value, we need to prove that they are logically equivalent.

Case 1: $\exists x(P(x) \to Q(x))$ is true:

This means there exists an x such that $P(x) \to Q(x)$ is true.

If $P(x)$ is true for this particular x, then $Q(x)$ must also be true for x. Thus, $\exists x Q(x)$ is true.

Case 2: $\exists x(P(x) \to Q(x))$ is false:

This means for all x, $P(x) \to Q(x)$ is false.

If $\forall x P(x)$ is true, then for all x, $P(x)$ must be true. However, since $P(x) \to Q(x)$ is false for all x, there exists no x such that $Q(x)$ is true, and hence $\exists x Q(x)$ is false.

Therefore, we've shown that both statements have the same truth value in all cases, and hence they are logically equivalent.

5. Suppose that A and B are sets such that the power set of A is a subset of the power set of B. Does it follow that $A \subseteq B$?

To prove that if $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, then $A \subseteq B$, we'll use the definition of set inclusion and properties of power sets.

Let's assume $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. This means that every subset of A is also a subset of B.

Now, let's consider an element x in A. By definition, $\{x\} \subseteq A$. Since $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, we have $\{x\} \subseteq B$.

This means that every element of A is also an element of B. Therefore, $A \subseteq B$.

Hence, if $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, then $A \subseteq B$.

6. Let A and B be sets. Show that $A \subseteq B$ if and only if $A \cap B = A$.

To prove that $A \subseteq B$ if and only if $A \cap B = A$, we need to show both directions of the implication:

1. If $A \subseteq B$, then $A \cap B = A$:

Assume $A \subseteq B$.

First, we'll prove $A \cap B \subseteq A$:

Let $x \in A \cap B$. This means $x \in A$ and $x \in B$. Since $A \subseteq B$, x must also be in B. Therefore, $x \in A$.

Now, we'll prove $A \subseteq A \cap B$:

Let $x \in A$. Since $A \cap B = \{x \in A \,|\, x \in B\}$, and $x \in A$, it follows that $x \in A \cap B$.

Hence, $A \cap B = A$.

2. If $A \cap B = A$, then $A \subseteq B$:

Assume $A \cap B = A$.

Let $x \in A$. Since $A \cap B = A$, $x \in A \cap B$, which means $x \in B$ (since $x \in A$ and $A \cap B = \{x \in A \,|\, x \in B\}$).

Since x was an arbitrary element of A, this implies that every element of A is also an element of B. Therefore, $A \subseteq B$.

Since we have proved both directions, we can conclude that $A \subseteq B$ if and only if $A \cap B = A$.