Deutsch-Jozsa Algorithm

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Problem Statement

- **Objective:** You are given a black-box function f(x) that takes a n-bit input and returns a single-bit output (either 0 or 1).
- The function is either:
  - **Constant** (outputs the same value, either 0 or 1, for all inputs), or
  - **Balanced** (outputs 0 for half of the inputs and 1 for the other half).
- **Task:** Determine whether the function is constant or balanced with the fewest possible evaluations of f(x).

Classical Approach

To determine if the function is constant or balanced, you would need to check the output for at least half of the inputs:

If all inputs give the same output, it's constant.

If half give 0 and half give 1, it's balanced.

Time Complexity:

For n-bit input, in the worst case, you need to evaluate 2^(n-1) + 1 inputs.

Limitations

As n grows, the number of evaluations grows exponentially, making the classical approach inefficient for large inputs.

Quantum Solution Deutsch-Jozsa Algorithm

Hadamard gate

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$H\left|0\right\rangle = \frac{1}{\sqrt{2}}(\left|0\right\rangle + \left|1\right\rangle) = \left|+\right\rangle$$

$$H\left|1\right\rangle = \frac{1}{\sqrt{2}}(\left|0\right\rangle - \left|1\right\rangle) = \left|-\right\rangle$$

$$H\left|+\right\rangle = \frac{1}{\sqrt{2}}(H\left|0\right\rangle + H\left|1\right\rangle)$$

$$H\left|+\right\rangle = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(\left|0\right\rangle + \left|1\right\rangle + \left|0\right\rangle - \left|1\right\rangle)\right) = \left|0\right\rangle$$

When Hadamard gate is applied to any state in superposition there is possibility of interference. Depending upon the phase of the states, interference can be constructive (as for state 0 in given example) or destructive (as for state 1 in given example).

Hadamard gate

$$H|00\rangle = \frac{1}{\sqrt{2}}(|+\rangle \otimes |+\rangle) = \frac{1}{2}\Big(|00\rangle + |01\rangle + |10\rangle + |11\rangle\Big)$$

$$H|01\rangle = \frac{1}{\sqrt{2}}(|+\rangle \otimes |-\rangle) = \frac{1}{2}\Big(|00\rangle - |01\rangle + |10\rangle - |11\rangle\Big)$$

In General the operation of Hadamard gate is given by :

$$H|x\rangle = \frac{1}{\sqrt{2^n}} \sum_{z=0}^{2^n-1} (-1)^{x.z} |z\rangle$$

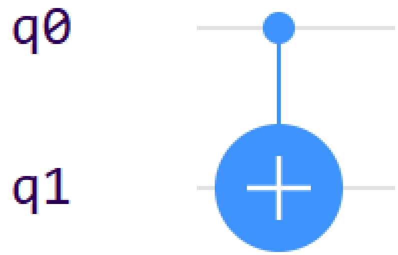
Here $x = x_1x_2x_3...x_n$, *i.e* decimal equivalent of binary state

The Key observation here is that :

- When Hadamard is applied on a particular state we get the superposition of all possible states that could be represented by those qubits
- When Hadamard gate s applied on any state the phase associate with the state $|0\rangle^{\otimes n}$ is always 0.

CX Gate as XOR operator

$$Cx = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$



input controrl	input target	output target
0	0	0
0	1	1
1	0	1
1	1	0

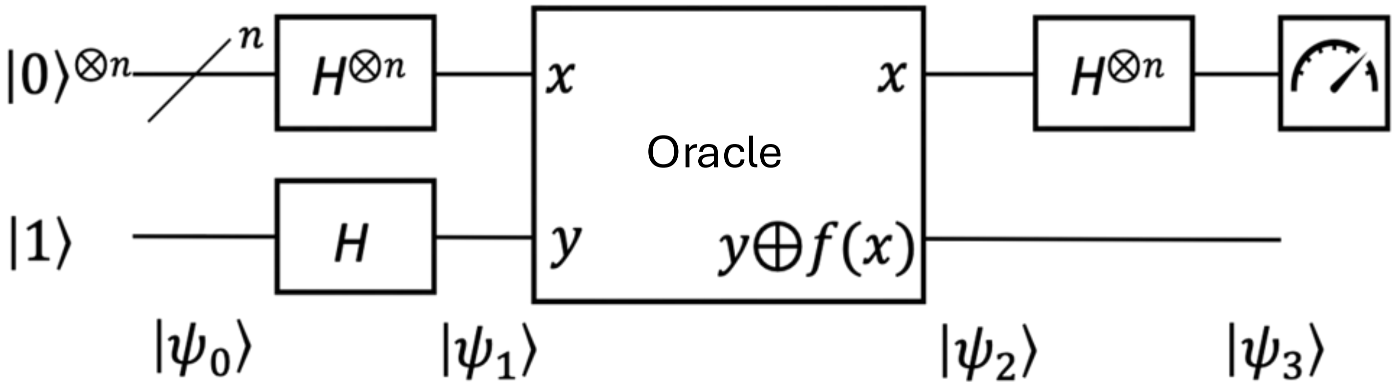
The Key observation here is that :

- When the CX operator is applied on any 2 qubit the output of the target qubit is equivalent to the XOR of the 2 inputs. So, we can say the state of the target qubit after the application of CX is equal to (q_0 XOR q_1).

Deutsch-Jozsa algorithm

Quantum Solution Deutsch-Jozsa Algorithm

The circuit for Deutsch-Jozsa algorithm is given by



- The oracle is a black box function and there is no information about the internal working of this oracle.
- Only the input and the output is known to us.

Deutsch-Jozsa algorithm

Quantum Solution Deutsch-Jozsa Algorithm

The initial state of the system of the algorithm is given by:

$$|\psi_0\rangle = |0\rangle^{\otimes n} \otimes |1\rangle$$

This state is bought to superposition by Hadamard gate which is expressed by :

$$|\psi_1\rangle = H^{\otimes n}\,|0\rangle^{\otimes n} \otimes H\,|1\rangle$$

Where H represent Hadamard gate

$$|\psi_1\rangle = \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle \otimes |-\rangle$$

, Where $|-\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle - |1\rangle\Big)$

Note: x represents the decimal equivalent of binary state of qubits

Deutsch-Jozsa algorithm

Quantum Solution Deutsch-Jozsa Algorithm

Now an operator is applied which takes two inputs x and y and gives two outputs x and y XOR $f(x)$. The state of the system after this operator is applied is :

$$|\psi_2\rangle = \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle^{\otimes n} \otimes |f(x) \oplus -\rangle$$

Now $|f(x) \oplus -\rangle$ can be rewritten as $(-1)^{f(x)} |-\rangle$

Because,

$$f(x) = 0 \rightarrow \ |f(x) \oplus -\rangle = |-\rangle \quad (0 \oplus z = z)$$

$$f(x) = 1 \rightarrow then \ \ |f(x) \oplus -\rangle = \frac{1}{\sqrt{2}}\left(|1 \oplus 0\rangle - |1 \oplus 1\rangle\right) = -|-\rangle$$

Deutsch-Jozsa algorithm

Quantum Solution Deutsch-Jozsa Algorithm

Now if the given function is constant, i.e. $f(x) = 0$ *or* 1 then the state of the system would be represented as :

$$|\psi_2\rangle = (-1)^{f(x)} \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle^{\otimes n} \otimes |-\rangle$$

Which is equivalent to : $|\psi_2\rangle = (-1)^{f(x)} H^{\otimes n} |0\rangle^{\otimes n} \otimes H|1\rangle$

After the application of Hadamard gate the final state of the system is represented as :

$$|\psi_3\rangle = (-1)^{f(x)} \left(H^{\otimes n}\ H^{\otimes n} |0\rangle^{\otimes n} \otimes H|1\rangle\right)$$

As Hadamard gate is self-inverse, if the first *n* qubit is measured, all the qubits is measured as 0.

Deutsch-Jozsa algorithm

Quantum Solution Deutsch-Jozsa Algorithm

Now if the given function is balanced, i.e. $f(x) = 0$ for half of the inputs and equals *1* for the other half of the inputs. The state of the system is represented as:

$$|\psi_2\rangle = \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} (-1)^{f(x)} |x\rangle \otimes |-\rangle$$

This state can be represented as

$$|\psi_2\rangle = \frac{1}{\sqrt{2^n}} \left((-1)^{f(0)} |0\rangle + (-1)^{f(1)} |1\rangle + (-1)^{f(2)} |2\rangle + \ldots + (-1)^{f(2^n-1)} |2^n - 1\rangle \right) \otimes |-\rangle$$

Deutsch-Jozsa algorithm

When the Hadamard gate is applied on the initial n qubits.
The state of the system becomes:

$$|\psi_3\rangle = \frac{1}{\sqrt{2^n}}\left((-1)^{f(0)}H^{\otimes n}|0\rangle + (-1)^{f(1)}H^{\otimes n}|1\rangle + (-1)^{f(2)}H^{\otimes n}|2\rangle + \ldots + (-1)^{f(2^n-1)}H^{\otimes n}|2^n-1\rangle\right) \otimes |-\rangle$$

Now if we just calculated the amplitude of state $|0\rangle^{\otimes n}$, we get

$$amplitude = \frac{1}{2^n}\sum_{x=0}^{2^n-1}(-1)^{f(x)}$$

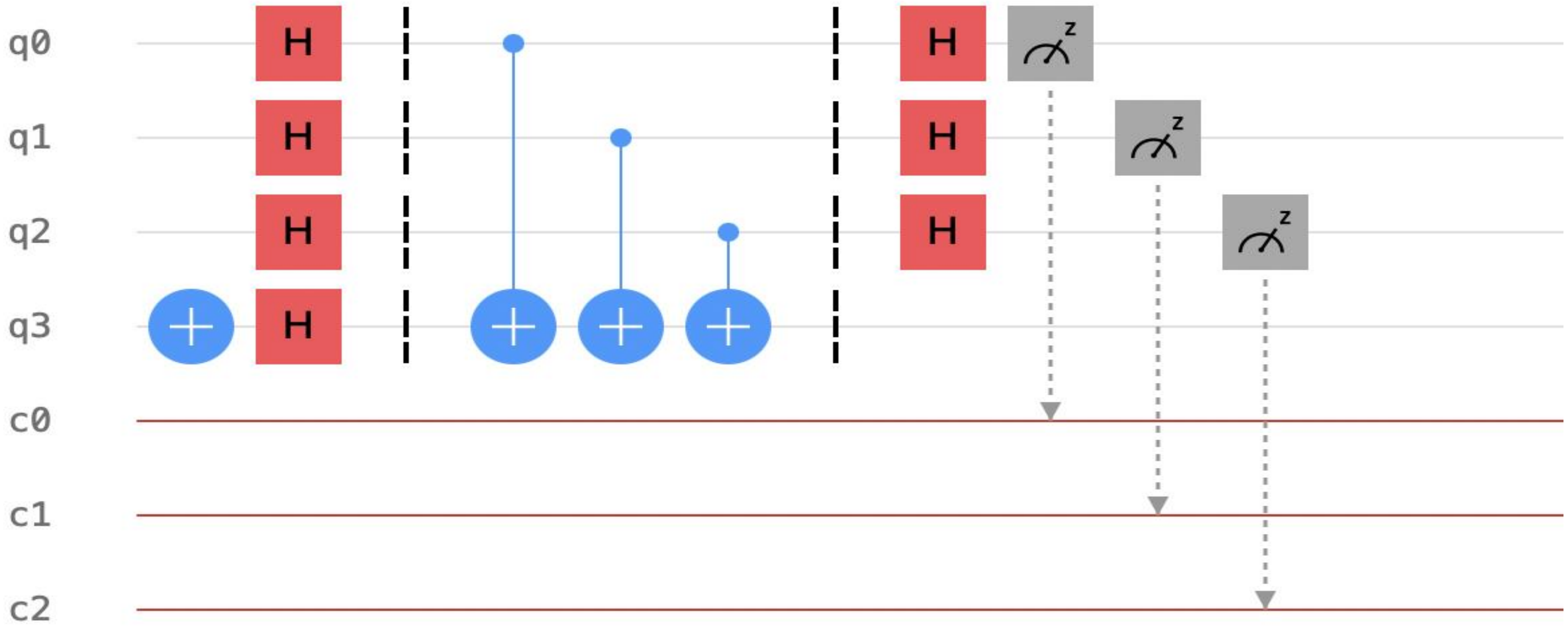
Since $f(x)$ is a balanced function, we have equal number of *1* and
-1.
Hence we get the final probability of measuring $|0\rangle^{\otimes n}$ comes out to be zero.

Conclusion

- After applying Deutsch-Jozsa Algorithm if $|0\rangle^{\otimes n}$ is measured it is concluded that the function is constant.

- If any state other than $|0\rangle^{\otimes n}$ is measured it is concluded that the function is balanced.

Circuit for Balanced Function



Circuit for Constant Function