Building Blocks of Quantum Computing

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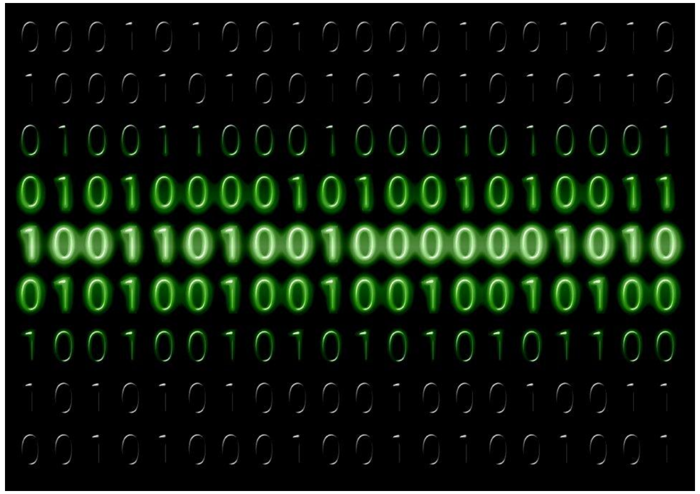
BITS



0 ↔ 1

Boolean Logic

QUBITS

Use of Quantum Mechanics

The state of qubit is a unit vector in two-dimensional complex vector space. (generally called Hilbert space)

Linear Algebra

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$\alpha\,,\quad \beta \;\in \mathbb{C} \qquad |\alpha|^2 + |\beta|^2 = 1$$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

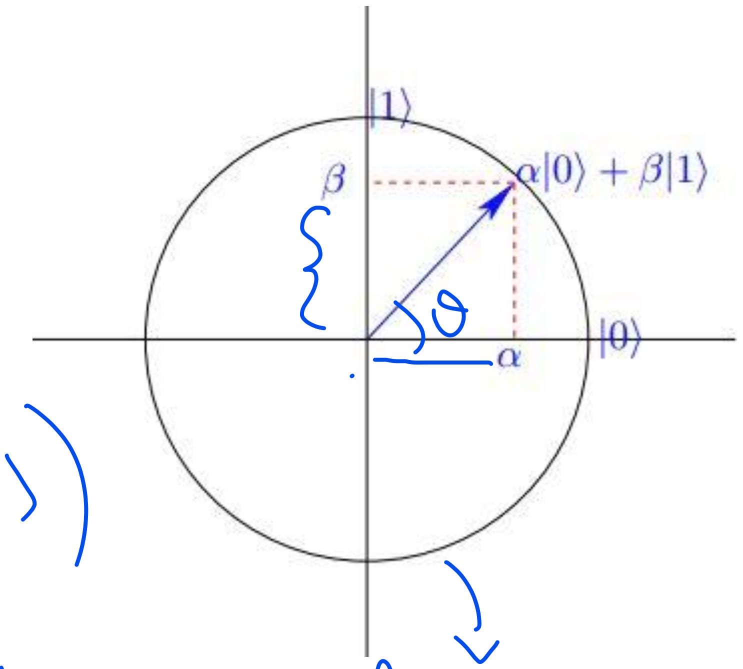
$$= r_1 e^{i\theta_1}|0\rangle + r_2 e^{i\theta_2}|1\rangle$$

$$\left\{ r_1^2 + r_2^2 = 1 \;;\; \theta_2 = \theta_1 + \varphi \right\}$$

$$\Rightarrow |\psi\rangle = \cancel{e^{i\theta_1}}\left(\cos(\theta/2)|0\rangle + \sin(\theta/2)e^{i\varphi}|1\rangle\right)$$

Global Phase

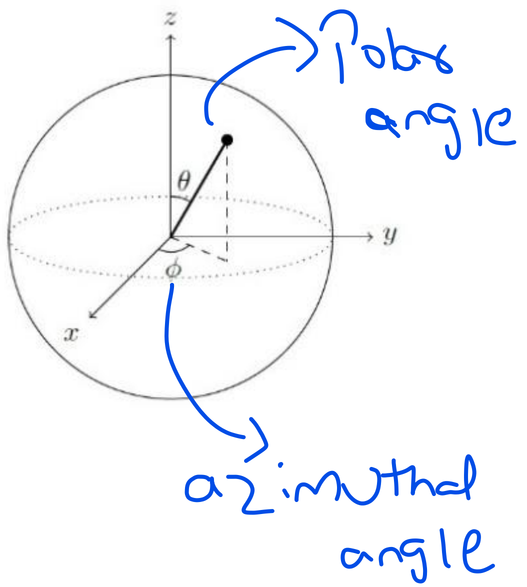
$$= \cos(\theta/2)|0\rangle + \sin(\theta/2)e^{i\varphi}|1\rangle$$



for real amplitudes

$$|\psi\rangle = \cos(\theta/2)\,|0\rangle + e^{i\phi}\sin(\theta/2)\,|1\rangle$$

can be represented on
a Bloch sphere



$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$\alpha, \quad \beta \in \mathbb{C} \qquad\qquad |\alpha|^2 + |\beta|^2 = 1$$

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle$$

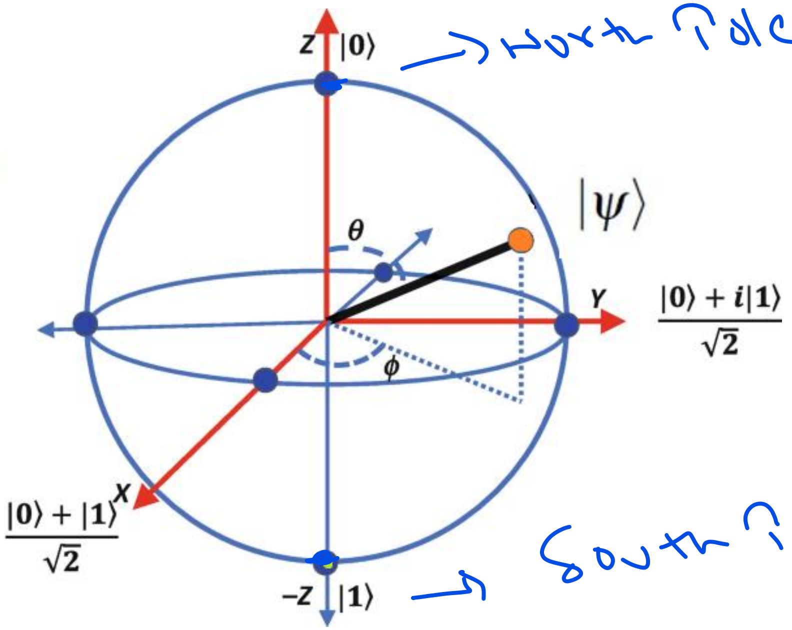
$$0 \leq \theta \leq \pi \text{ and } 0 \leq \phi < 2\pi$$

BLOCH SPHERE

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$\alpha = \cos\left(\frac{\theta}{2}\right), \quad \beta = e^{i\phi}\sin\left(\frac{\theta}{2}\right) \qquad 0 \le \theta \le \pi \text{ and } 0 \le \phi < 2\pi$$

$$\underbrace{\vec{r} = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)}_{\text{Bloch vector}}$$

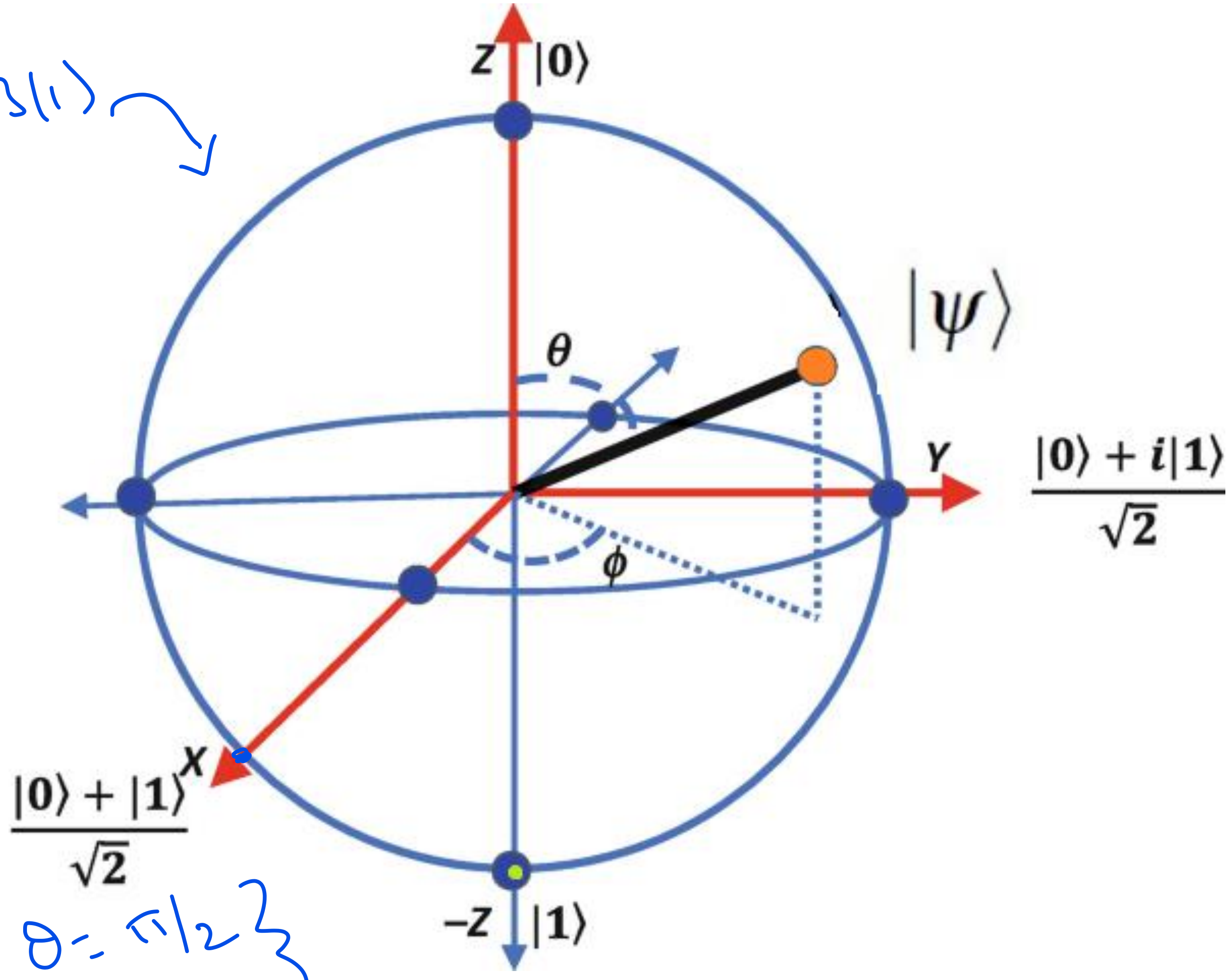


→ North Pole ↔ |0⟩

→ South Pole ↔ |1⟩

Any qubit state

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$



$|+\rangle =$

$\left\{ \begin{matrix} \theta = \pi/2 \\ \phi = 0 \end{matrix} \right\}$

VECTORS

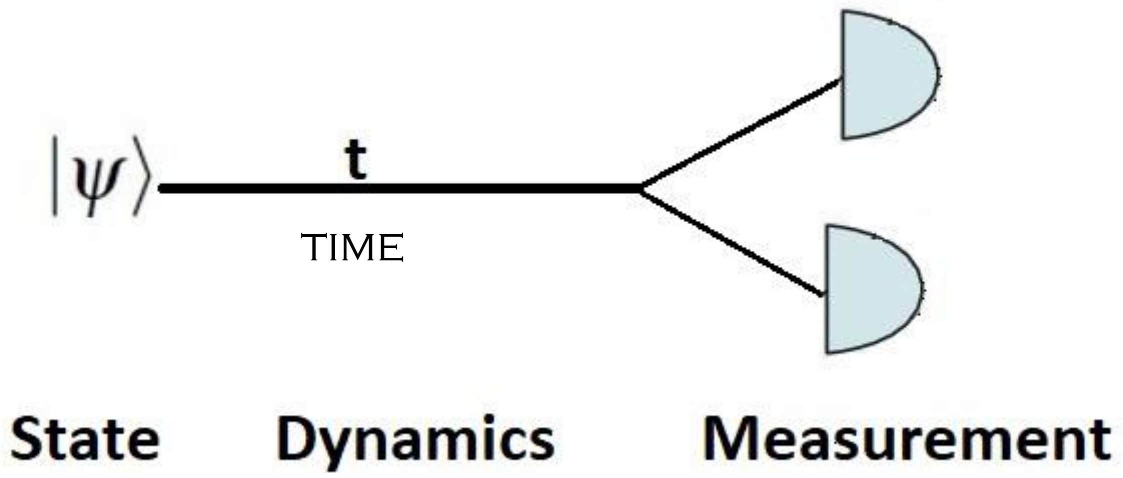
Quantum phenomena do not occur in a Hilbert space. They occur in a laboratory- Asher Peres

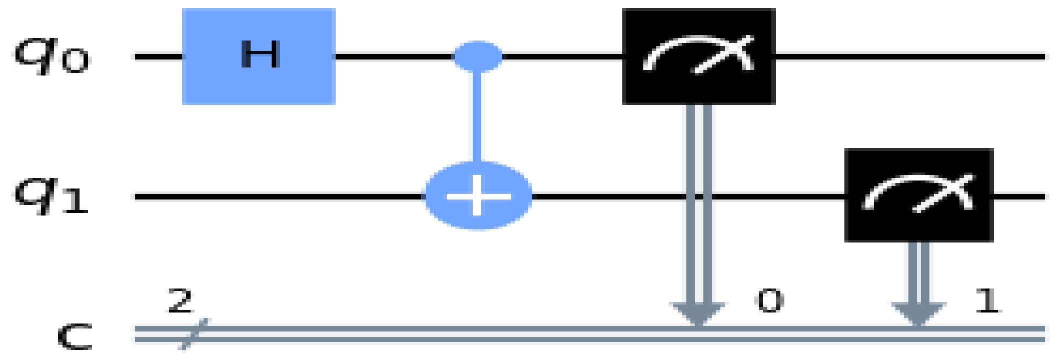
Vector representation:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \; ; \; |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

QUANTUM STEPS





TIME EVOLUTION

The state $|\psi_1\rangle$ of a closed quantum system at time t_1 is related to its state $|\psi_2\rangle$ at a later time t_2 by a unitary operator U, which depends only on t_1 and t_2:

$$|\psi_2\rangle = U|\psi_1\rangle$$

$$|\psi_1\rangle \xrightarrow{U} |\psi_2\rangle$$

$$\underline{U^\dagger U} = \underline{UU^\dagger} = I$$

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} ; \quad U^\dagger = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$$

QUANTUM GATES

$$I|0\rangle = |0\rangle,$$
$$I|1\rangle = |1\rangle.$$

$$X|0\rangle = |1\rangle, \qquad Z|0\rangle = |0\rangle,$$
$$X|1\rangle = |0\rangle. \qquad Z|1\rangle = -|1\rangle.$$

$$Y|0\rangle = i|1\rangle,$$
$$Y|1\rangle = -i|0\rangle.$$

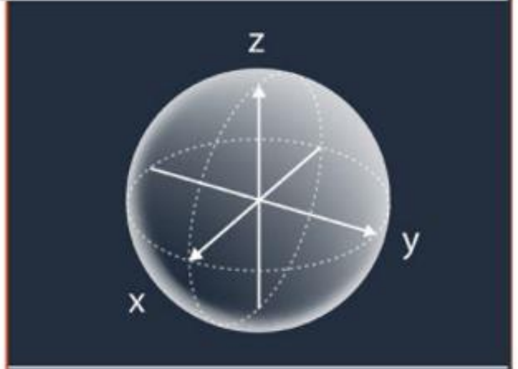
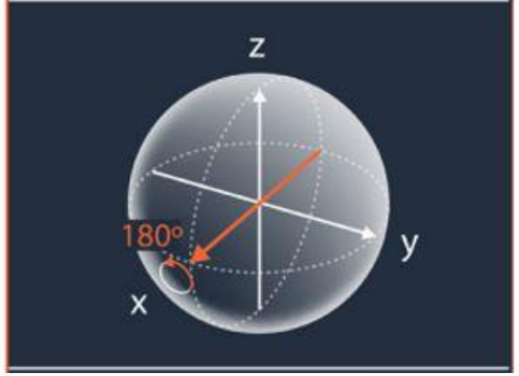
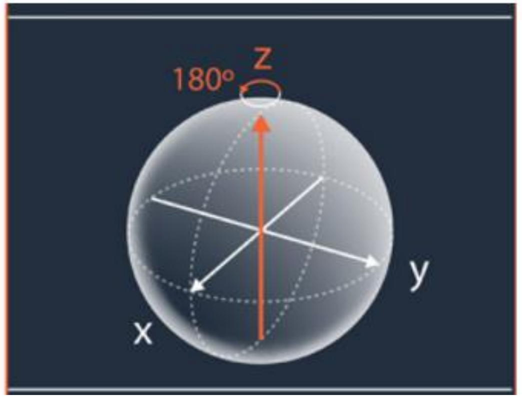
Operator	Gate(s)		Matrix
Pauli-X (X)	$\boxed{X}$	$\oplus$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
Pauli-Y (Y)	$\boxed{Y}$		$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$
Pauli-Z (Z)	$\boxed{Z}$		$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Hadamard (H)	$\boxed{H}$		$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
Phase (S, P)	$\boxed{S}$		$\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$
$\pi/8$ (T)	$\boxed{T}$		$\begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle,$$

$$H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = |-\rangle.$$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

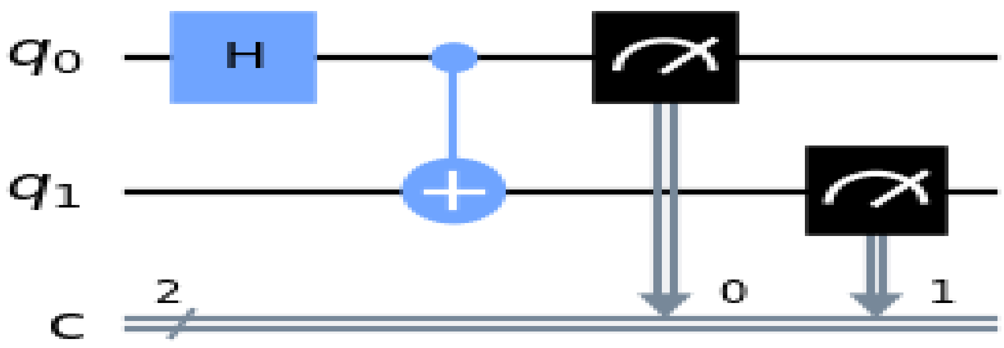
$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

GATE	CIRCUIT REPRESENTATION	MATRIX REPRESENTATION	TRUTH TABLE	BLOCH SPHERE
I Identity-gate: no rotation is performed.	$-\boxed{I}-$	$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	Input / Output $\|0\rangle$ / $\|0\rangle$ $\|1\rangle$ / $\|1\rangle$	
X gate: rotates the qubit state by π radians (180º) about the x-axis.	$-\boxed{X}-$	$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	Input / Output $\|0\rangle$ / $\|1\rangle$ $\|1\rangle$ / $\|0\rangle$	
Z gate: rotates the qubit state by π radians (180º) about the z-axis.	$-\boxed{Z}-$	$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	Input / Output $\|0\rangle$ / $\|0\rangle$ $\|1\rangle$ / $-\|1\rangle$	

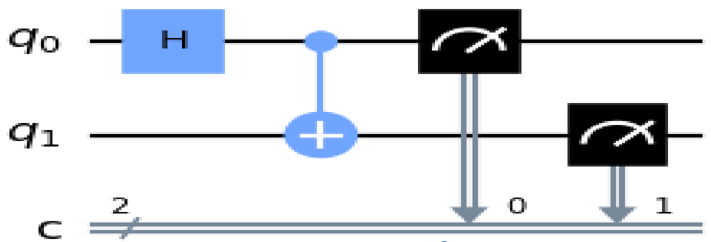
$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle.$

$X|\psi\rangle = \alpha|1\rangle + \beta|0\rangle$

- A quantum circuit contains logic gates connected by straight lines,which indicates the direction of logic flow with time, earlier time being to the left.
- Inputs to the circuit are qubits, as are the outputs.



QUANTUM MEASUREMENT



$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$|\alpha|^2 + |\beta|^2 = 1$$

after measurement

Probability of measuring $|0\rangle \longrightarrow |\alpha|^2 \longrightarrow |0\rangle$

$|1\rangle \longrightarrow |\beta|^2 \longrightarrow |1\rangle$

TWO QUBITS

$$|\psi_1\rangle = \alpha_1|0\rangle + \beta_1|1\rangle$$

$$|\psi_2\rangle = \alpha_2|0\rangle + \beta_2|1\rangle$$

$$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$$

Entangled state (maximally)

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

Can't be decomposed into product states

QUANTUM GATES

Operator	Gate(s)		Matrix
Pauli-X (X)	X	$\oplus$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
Pauli-Y (Y)	Y		$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$
Pauli-Z (Z)	Z		$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Hadamard (H)	H		$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
Phase (S, P)	S		$\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$
$\pi/8$ (T)	T		$\begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$
Controlled Not (CNOT, CX)			$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

$$I|0\rangle = |0\rangle, \qquad X|0\rangle = |1\rangle,$$
$$I|1\rangle = |1\rangle. \qquad X|1\rangle = |0\rangle.$$

$$Y|0\rangle = i|1\rangle, \qquad Z|0\rangle = |0\rangle,$$
$$Y|1\rangle = -i|0\rangle. \qquad Z|1\rangle = -|1\rangle.$$

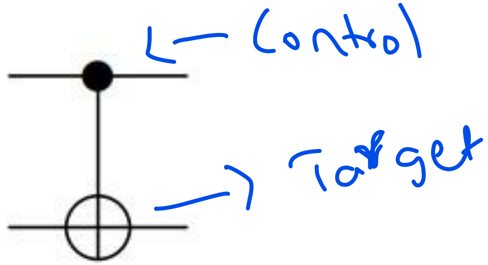
$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle,$$

$$H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = |-\rangle.$$

$$\text{CNOT}|00\rangle = |00\rangle,$$
$$\text{CNOT}|01\rangle = |01\rangle,$$
$$\text{CNOT}|10\rangle = |11\rangle,$$
$$\text{CNOT}|11\rangle = |10\rangle.$$

in standard basis

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$



BELL STATES (EPR)

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

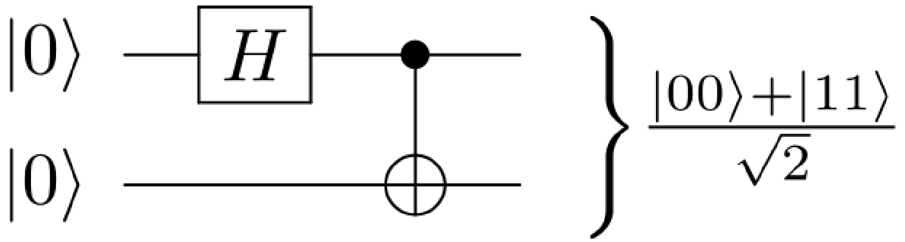
$$\begin{aligned} \mathrm{CNOT}|00\rangle &= |00\rangle, \\ \mathrm{CNOT}|01\rangle &= |01\rangle, \\ \mathrm{CNOT}|10\rangle &= |11\rangle, \\ \mathrm{CNOT}|11\rangle &= |10\rangle. \end{aligned}$$

$$\boxed{|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle),}$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle),$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle),$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$



$$\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)|0\rangle \rightarrow \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$